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Some variational problems in conformal geometry

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ABSTRACT. In this survey, we give some results related to renormalized volume coefficients $v^{(2k)}(g)$ of a Riemannian metric g, which include σ_k-Yamabe problems, variational characterization of space forms, Kazdan-Warner type identities, the first variational formula for the functional $\int_M v^{(2k)}(g)dv_g$, the second variational formula of $\int_M v^{(6)}(g)dv_g$ and its applications. We note that when (M,g) is locally conformally flat, $v^{(2k)}(g)$ reduces to the $\sigma_k(g)$ curvature up to a scaling constant.

1. σ_k-Yamabe Problems

Let (M^n,g) be a compact Riemannian manifold. The *Schouten* tensor is defined by

$$A_g = \frac{1}{n-2}\Big(Ric - \frac{R}{2(n-1)}g\Big), \tag{1.1}$$

here Ric is the Ricci tensor of the metric, and R is the scalar curvature.

In terms of Schouten tensor, we can express the *Weyl* curvature tensor by

$$W_{ijkl} = R_{ijkl} - (A_{ik}g_{jl} - A_{il}g_{jk} + A_{jl}g_{ik} - A_{jk}g_{il}), \tag{1.2}$$

where R_{ijkl} are the components of the Riemannian curvature tensor of (M^n,g).

Related with the Schouten tensor, one can define the *Cotten* tensor by

$$C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}. \tag{1.3}$$

The following results are well known

THEOREM 1.1. *For a Riemannian manifold (M^n,g), $n \geq 3$, we have*

(i) If $n=3$, then $W_{ijkl} \equiv 0$; (M^3,g) is locally conformally flat if and only if $C_{ijk} \equiv 0$.

(ii) If $n \geq 4$, then (M^n,g) is locally conformally flat if and only if $W_{ijkl} \equiv 0$.

(iii) $\nabla^i W_{ijkl} = -(n-3)C_{jkl}$. In particular, if (M^n,g) is locally conformally flat and $n \geq 4$, then $C_{jkl} \equiv 0$.

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The $\sigma_k(g)$ curvature, introduced by J. Viaclovsky in [**V1**], is defined to be

$$\sigma_k(g^{-1} \cdot A) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.4}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $g^{-1} \cdot A$, i.e., $\sigma_k(g)$ is the k-th elementary symmetric polynomials with respect to the eigenvalues of the matrix $g^{-1} \cdot A_g$.

The σ_k-Yamabe problem is to prove the existence of a conformal metric $g = g_v = v^{\frac{4}{n-2}} g_0$ which solves the equation

$$\sigma_k(g) \equiv \text{constant} \quad \text{on } M. \tag{1.5}$$

Note that

$$\sigma_1(g) = \frac{R}{2(n-1)},$$

so when $k = 1$ the σ_k-Yamabe problem is reduced to the classical Yamabe problem.

When $k \geq 2$, the σ_k-Yamabe problem (1.5) is a fully nonlinear partial differential equation for the function v, which is elliptic if the eigenvalues $\{\lambda_i(A_g)\}_{i=1}^n$ lies in the convex cone Γ_k (or $-\Gamma_k$), given by (see [**GV2**])

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0 \text{ for } j = 1, \cdots, k\} \tag{1.6}$$

Under the assumption that $\lambda(A_g) \in \Gamma_k$, the σ_k-Yamabe problem has been solved in the following cases:

- for $k = 2, n = 4$ by Chang, Gursky and Yang in [**CGY1**] and [**CGY2**],
- for $k = 2, n > 8$ by Y. Ge and G. Wang in [**GeW**],
- for the locally conformally flat manifolds by Y. Y. Li and A. Li in [**LL**] and P. Guan, G. Wang in [**GW**], independently,
- for the case $k \leq n/2$ and $n \geq 2$ under the hypothesis that the problem is variational by W. Sheng, N. Trudinger and X. Wang in [**STW**].

2. Variational Characterizations of space forms

A classical result in Riemannian geometry says that the critical metric of the Hilbert functional

$$\mathcal{F}[g] := \int_M R dv, \tag{2.1}$$

is the Ricci-flat metric. When restricted to metrics with fixed volume, the critical metric of this functional is Einstein.

For the $\sigma_k(g)$ curvature, we can define a family of functionals

$$\mathcal{F}_k[g] := \int_M \sigma_k(g) dv_g, \quad k = 1, 2, \ldots, n. \tag{2.2}$$

Note that $\mathcal{F}_1[g] = \frac{1}{2(n-1)} \mathcal{F}[g]$. In [**V1**], Viaclovsky proved the following statements:

(1) When $k = 1$ or 2, and $2k < n$, $\mathcal{F}_k$ is variational in the conformal class of metrics $g_w \in [g]$ with fixed volume one, i.e., the critical metric for the functional in this class of metrics, when achieved, satisfies the equation

$$\sigma_k(g_w) = \text{constant}.$$

(2) When $k \geq 3$ and $2k < n$, the assertion in statement (1) holds only when (M, g) is locally conformally flat.

(3) When $k = 2$, $n = 4$, $\mathcal{F}_2[g]$ is conformally invariant; while for $k = \frac{n}{2}$ and $k \geq 3$, $\mathcal{F}_k[g]$ is conformally invariant only when the manifold (M, g) is locally conformally flat.

We remark that when $\dim M = 4$, by Gauss-Bonnet-Chern formula,

$$\int_M \Big(\frac{1}{4}|W|^2 + \sigma_2(g)\Big) dv_g = 8\pi\chi(M), \tag{2.3}$$

where $\chi(M)$ is the Euler characteristic of M, W is the Weyl tensor of (M, g), whose L^2 norm $\int_M |W_g|^2 dv_g$ is invariant under conformal change of metrics, hence $\int_M \sigma_2(g) dv_g$ is also conformally invariant.

In [**BG**], Branson and Gover proved that if $3 \leq k \leq n$ and g is not locally conformally flat, then the equation $\sigma_k(g)$ =constant is not the Euler-Lagrange equation of any functional.

For general variation of metrics (that is, variation is not restricted in conformal class $[g]$), J. Viaclovsky and M. Gursky ([**GV1**]) proved that

THEOREM 2.1 ([**GV1**]). *Let M be a 3-dimensional compact manifold. Then a metric g with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature, where*

$$\mathcal{M}_1 = \{g |\, Vol(g) = 1\} \tag{2.4}$$

is the Riemannian metrics with unit volume.

We remark that, for 3-dimensional manifolds, a metric g is of constant curvature if and only if it is Einstein because the Weyl curvature tensor vanishes identically. For the higher dimensional case, when restricted to locally conformally flat manifolds, Hu-Li ([**HL**]) proved that

THEOREM 2.2 ([**HL**]). *Let $M^n (n \geq 5)$ be compact. Then a conformally flat metric g with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature.*

REMARK 2.3. From [**GV1**] or [**HL**], we can check that a metric g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g satisfies

$$(B_g)_{ij} + (n-4)T_{2ij} = \lambda g_{ij}, \qquad \lambda = \frac{(n-2)(n-4)}{n}\sigma_2(g), \tag{2.5}$$

where *Bach* tensor $(B_g)_{ij}$ is defined by (4.6) and T_{2ij} is defined by (6.6).

REMARK 2.4. Similar to the $n = 3$ case in Theorem 2.1, the condition $\mathcal{F}_2[g] \geq 0$ in Theorem 2.2 remains necessary: let $E = Ric - \frac{R}{n} g$ denote the trace-free Ricci tensor, then

$$(n-2)^2 \sigma_2(g) = -\frac{1}{2}|E|^2 + \frac{(n-2)^2 R^2}{8n(n-1)}. \tag{2.6}$$

If g has constant sectional curvature, then $E = 0$ and $\sigma_2(g) = \frac{R^2}{8n(n-1)} \geq 0$. However, there do exist critical metrics with $\mathcal{F}_2 < 0$ (see [**GV1**] or [**HL**]).

Let (M^3, g) be a compact Riemannian three dimensional manifold. We define a functional which is linear combination of $\mathcal{F}_1$ and $\mathcal{F}_2$, i.e.,

$$\mathcal{F}_a[g] = \mathcal{F}_2[g] - a\mathcal{F}_1[g] = \int_M \sigma_2(g) - a\sigma_1(g), \tag{2.7}$$

where a is a nonpositive constant. In [**GL2**], the authors proved that under some assumptions, the critical metric to the functional $\mathcal{F}_a$ is also of constant curvature, more precisely, they showed the following result:

THEOREM 2.5. *Let (M^3, g) be a compact three dimensional Riemannian manifold. Then a critical metric g to $\mathcal{F}_a|_{\mathcal{M}_1}$ satisfies*

$$b := \sigma_2 + a\sigma_1 \equiv \text{constant}. \tag{2.8}$$

Moreover, if $b + \frac{3}{4}a^2 > 0$, the critical metric must be of constant sectional curvature.

REMARK 2.6. The analogue of Theorem 2.5 holds for higher dimensional locally conformally flat manifolds.

3. Locally conformally flat manifolds with $\sigma_k(g) = \text{constant}$

In this section, we will study the global properties of a locally conformally flat Riemannian manifold (M^n, g) with constant $\sigma_k(g)$ for some $k \in \{1, 2, \ldots, n\}$.

Since $\sigma_1(g) = \frac{1}{2(n-1)}R$, $\sigma_1(g)$ is constant if and only if (M, g) has constant scalar curvature R. On the other hand, from the solution of Yamabe problem, we know that every compact conformally flat manifold (M, g) admits a pointwise conformal metric $\tilde{g}$ such that $(M^n, \tilde{g})$ is a conformally flat manifold with constant scalar curvature. The following theorem is well known.

THEOREM 3.1 ([**Ta**], [**We**], [**Ch**]). *Let (M^n, g) be a compact conformally flat Riemannian manifold with constant scalar curvature. If the Ricci tensor is semi-positive definite, then (M^n, g) is either a space form, or a space $S^1 \times N^{n-1}$, where N^{n-1} is an $(n-1)$- dimensional space form.*

On the other hand, for complete conformally flat manifolds with non-negative Ricci curvature, S. H. Zhu ([**Zh**]) established the following theorem.

THEOREM 3.2 ([**Zh**]). *If (M^n, g) is a complete locally conformally flat Riemannian manifold with $Ric(g) \geq 0$, then the universal cover $\tilde{M}$ of M with the pull-back metric is either conformally equivalent to S^n, $\mathbb{R}^n$ or is isometric to $\mathbb{R} \times S^{n-1}(c)$. If M^n itself is compact, then $\tilde{M}$ is either conformally equivalent to $S^n(1)$ or isometric to $\mathbb{R}^n$, $\mathbb{R} \times S^{n-1}(c)$.*

Motivated by Theorems 3.1 and 3.2, one may try to classifiy compact locally conformally flat Riemannian manifolds with constant $\sigma_k(g)$ $(k \geq 2)$. In this respect, Z. Hu, H. Li and U. Simon ([**HLS**]) proved that

THEOREM 3.3 ([**HLS**]). *Let (M^n, g) be a compact locally conformally flat manifold with constant non-zero $\sigma_k(g)$ for some $k \in \{2, 3, \ldots, n\}$. If the tensor A_g is semi-positive definite, then (M^n, g) is a space form of positive sectional curvature.*

In [**GVW**], Guan-Viaclovsky-Wang established the following classification result:

THEOREM 3.4 ([**GVW**]). *Let (M^n, g) be a compact locally conformally flat manifold with nonnegative $\sigma_k(g)$ for some $k \geq \frac{n}{2}$. Then (M^n, g) is conformally equivalent to either a space form or a finite quotient of a Riemannian product $S^{n-1}(c) \times S^1$ for some constant $c > 0$ and $k = \frac{n}{2}$. In particular, if $g \in \Gamma_k^+$, then (M^n, g) is conformally equivalent to a spherical space form.*

If A_g is semi-positive definite, then (M^n, g) has nonnegative scalar curvature and nonnegative Ricci curvature. For the case $k = 2$, Z. Hu, H. Li and U. Simon ([**HLS**]) showed that

THEOREM 3.5 ([**HLS**]). *Let (M^n, g) be a compact locally conformally flat manifold with $\sigma_2(g)$ a non-negative constant. If the Ricci tensor is semi-positive definite, then (M^n, g) is a space form for $n = 3$ and is either a space form or a space $S^1 \times N^{n-1}$ with N^{n-1} a space form for $n \geq 4$.*

For the complete case, similar results are rare. Z. Hu, H. Li and U. Simon proved in [**HLS**]:

THEOREM 3.6 ([**HLS**]). *Let (M^n, g) be a complete locally conformally flat manifold with constant scalar curvature. If the Ricci tensor is semi-positive definite, then the universal cover $\tilde{M}$ of M^n with pull-back metric is isometric to $S^n(c)$, $\mathbb{R}^n$ or $\mathbb{R} \times S^{n-1}(c)$.*

4. Renormalized volume coefficients $v^{(2k)}(g)$

Renormalized volume coefficients, $v^{(2k)}(g)$, of a Riemannian metric g, were introduced in the physics literature in the late 1990's in the context of AdS/CFT correspondence. Given a manifold (X^{n+1}, M^n, g^+) with boundary $\partial X = M$. Let r be a defining function for M^n in X^{n+1} such that $r > 0$ in X and $r = 0$ on M, while $dr|_M \neq 0$.

For any given Riemannian manifold (M^n, h_0), Fefferman-Graham ([**FG**]) proved that there is an extension g^+ of h_0, which is "asymptotically Poincare Einstein" in a neighborhood of M^n, i.e., in $[0, \epsilon) \times M^n$ for some $\epsilon > 0$,

$$Ric(g^+) = -ng^+ \tag{4.1}$$

as $r \to 0$. Locally we can write

$$g^+ = \frac{1}{r^2}(dr^2 + h_{ij}(r, x)dx^i dx^j), \tag{4.2}$$

where $h(0, \cdot) = h_0(\cdot)$ and $h(r, \cdot)$ is a metric defined on $M_c := \{r = c\} \subset X$ when c is sufficiently small, $\{x^i\}$ are local coordinates of M^n.

Suppose the expression of $\frac{\sqrt{\det h_{ij}(r,x)}}{\sqrt{\det h_{ij}(0,x)}}$ near $r = 0$ is given by

$$\frac{\sqrt{\det h_{ij}(r, x)}}{\sqrt{\det h_{ij}(0, x)}} = \sum_{k=0}^{\infty} v^{(k)}(x, h_0) r^k, \tag{4.3}$$

where $v^{(k)}(x, h_0)$ is a curvature invariant of the metric h_0 for $2k \leq n$. The followings are some important properties of $v^{(k)}(g)$:

- Graham ([**G0**]): $v^{(k)}$ vanishes for k odd.
- Graham ([**GJ**]): when (M, g) is locally conformally flat, $v^{(2k)}(g)$ is equal to $\sigma_k(g)$ up to a scaling constant.

Here are some expressions of $v^{(k)}(g)$ for lower k's.

$$v^{(2)}(g) = -\frac{1}{2}\sigma_1(g), \qquad v^{(4)}(g) = \frac{1}{4}\sigma_2(g). \tag{4.4}$$

For $k = 3$, Graham and Juhl ([**GJ**]) listed the following formula for $v^{(6)}(g)$:

$$v^{(6)}(g) = -\frac{1}{8}[\sigma_3(g) + \frac{1}{3(n-4)}(A_g)^{ij}(B_g)_{ij}], \tag{4.5}$$

where

$$(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj} \tag{4.6}$$

is the *Bach* tensor of the metric.

Since $v^{(k)}(g)$ depend on derivatives of the curvature of g, and are defined via an indirect, higher nonlinear, inductive algorithm, they are algebrically complicated and no explicit formula is known for general k.

For even n, similar to J. Viaclovsky's result that $\int_M \sigma_{n/2} dv_g$ is a conformal invariant for n-dimensional compact locally conformally flat Riemannian manifolds, in [**G0**], Graham proved that the functional $\int_M v^{(n)}(g)dv_g$ is a conformal invariant for n-dimensional compact Riemannian manifolds.

5. Kazdan-Warner type identities

In studying the prescribed Gauss curvature in S^n, Kazdan and Warner (Ann. Math. 99 (1974)) found an identity which serves as a global geometric obstruction to the existence of such functions, namely,

$$\int_{S^n} \langle \nabla x_j, \nabla K(x)\rangle dv_g = 0, \quad \text{for } j = 1, \dots, n+1, \tag{5.1}$$

where x_j are the coordinate functions of $S^n \to \mathbb{R}^{n+1}$, $K(x)$ is the prescribed function to be the scalar curvature of a conformal metric g.

Definition 5.1. A vector field X is called a *conformal Killing vector field* if it generates a one-parameter subgroup of the diffeomorphism (ξ_t) which are conformal transformations, i.e., there exists a $\mu \in C^\infty(M)$ such that $\mathcal{L}_X g = \mu g$.

Later, these identities were extended to compact manifolds involving a conformal Killing vector field X by Bourguignon and Ezin [**BE**] and Schoen [**Sc**]

$$\int_M \langle X, \nabla R_g\rangle dv_g = 0, \tag{5.2}$$

where R_g is the scalar curvature of g. Note that ∇x_j generates a conformal Killing vector field on S^n, so (5.2) is a generalization of (5.1).

Kazdan-Warner type identities have played important roles in understanding the blow-up behavior of solutions to PDEs that prescribe the relevant curvatures in a conformal class. For σ_k curvatures, J. Viaclovsky ([**V2**]) and Z. Han ([**H**]) proved

Theorem 5.2 ([**V2**], [**H**]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, $\sigma_k(g^{-1}\cdot A_g)$ be the σ_k curvature of g, and X be a conformal Killing vector field on (M, g). When $k \geq 3$, we also assume that (M, g) is locally conformally flat, then*

$$\int_M \langle X, \nabla\sigma_k(g)\rangle dv_g = 0. \tag{5.3}$$

Remark 5.3. When $k = 1$, (5.3) reduces to (5.2).

Without the constraint (M^n, g) being locally conformally flat, Guo-Han-Li ([**GHL**]) proved similar identities hold for the renormalized volume coefficients $v^{(2k)}(g)$, THEOREM 5.4 ([**GHL**]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, X be a conformal Killing vector field on (M^n, g). For $k \geq 1$, we have*

$$\int_M \langle X, \nabla v^{(2k)}(g)\rangle dv_g = 0, \tag{5.4}$$

where $v^{(2k)}(g)$ is the renormalized volume coefficients defined in section 4.

In the same paper [**GHL**], they found another Kazdan-Warner type identities for the so-called Gauss-Bonnet curvatures $G_{2r}(g)$.

DEFINITION 5.5. The *Gauss-Bonnet curvatures* G_{2r} $(2r \leq n)$, introduced by H. Weyl, is defined to be (also see [**La**])

$$G_{2r}(g) = \delta^{j_1 j_2 \dots j_{2r-1} j_{2r}}_{i_1 i_2 \dots i_{2r-1} i_{2r}} R^{i_1 i_2}{}_{j_1 j_2} \dots R^{i_{2r-1} i_{2r}}{}_{j_{2r-1} j_{2r}} \tag{5.5}$$

where $\delta^{j_1 j_2 \dots j_{2r-1} j_{2r}}_{i_1 i_2 \dots i_{2r-1} i_{2r}}$ is the generalized Kronecker symbol.

Note that $G_2(g) = 2R$, R the scalar curvature. When $2r = n$, $G_{2r}(g)$ is the integrand in the Gauss-Bonnet-Chern formula, up to a multiple of constant. Guo-Han-Li ([**GHL**]) proved:

THEOREM 5.6 ([**GHL**]). *Let (M^n, g) be a compact Riemannian manifold, and X be a conformal Killing vector field. Then for the Gauss-Bonnet curvatures $G_{2r}(g)$ $(2r < n)$, we have*

$$\int_M \langle X, \nabla G_{2r}(g)\rangle dv_g = 0, \tag{5.6}$$

REMARK 5.7. When (M, g) is locally conformally flat, both $v^{(2k)}(g)$ and $G_{2r}(g)$ reduce to the $\sigma_k(g)$ curvature up to a constant. So both Theorems 5.4 and 5.6 are natural generalizations of J. Viaclovsky ([**V2**]) and Z. Han ([**H**])'s result Theorem 5.2.

6. The first and second variational formulas of $\int_M v^{(6)}(g) dv_g$

In [**CF**], Chang-Fang proved the following interesting result

THEOREM 6.1 ([**CF**]). *For any metric g on M^n and $2k \leq n$, define the functional*

$$\mathcal{F}_k(g) = \frac{\int_M v^{(2k)}(g) dv_g}{\Big(Vol(M, g)\Big)^{(n-2k)/n}}, \tag{6.1}$$

then $\mathcal{F}_k$ is variational within the conformal class when $2k < n$; i.e., the critical metric g_w in $[g]$ satisfies the equation

$$v^{(2k)}(g_w) = constant. \tag{6.2}$$

In this section, we aim to study the stability of the critical metric of the functional

$$\mathcal{F}_3[g] = \frac{\int_M v^{(6)}(g) dv_g}{\big(\int_M dv_g\big)^{(n-6)/n}}. \tag{6.3}$$

First we recall Theorem 6.1 of Chang-Fang states that the critical metric g_w of $\mathcal{F}_3$ in the conformal class $[g]$ satisfies the equation

$$v^{(6)}(g_w) \equiv \text{const}. \tag{6.4}$$

The second variational formula of $\mathcal{F}_3[g]$ within the conformal class $[g]$ is computed by Guo-Li ([**GL1**]). Their results are as follows:

THEOREM 6.2 ([**GL1**]). *Let (M^n, g) be an n-dimensional $(n \geq 7)$ compact Riemannian manifold with $v^{(6)}(g)$ =const, then the second variational formula of the functional $\mathcal{F}_3[g_t]$ within its conformal class at g is*

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F}_3[g_t] &= (n-6)V^{-(n-6)/n}\Bigg\{\int\Big[-6v^{(6)}(g)\bar{\phi}^2 \\ &\quad +\Big(\frac{B_{ij}\bar{\phi}_{ij}}{24(n-4)}+\frac{1}{8}T_{2ij}\bar{\phi}_{ij}-\frac{1}{12}A_{ij}C_{ijk}\bar{\phi}_k\Big)\bar{\phi}\Big]dv\Bigg\} \end{aligned} \tag{6.5}$$

where $g_t = e^{2u_t}g$, $\frac{\partial}{\partial t}\big|_{t=0}u_t = \phi$, and $\bar{\phi} = \phi - \frac{\int_M \phi dv_g}{\int_M dv_g}$,

$$T_{2ij} = \sigma_2 g_{ij} - \sigma_1 A_{ij} + A_{ik}A_{kj} \tag{6.6}$$

is one of the Newton transformations associated to A_{ij}, and the Cotton tensor is defined in (1.3), $V = \int_M dv_g$ is the volume of (M, g).

From the second variational formula of $\mathcal{F}_3$, and by using Lichnerowicz and Obata theorem on the lower bound of the first nonzero eigenvalue of the Laplacian on compact manifolds with positive Ricci curvature lower bound, we proved the following stability result:

THEOREM 6.3 ([**GL1**]). *Let (M^n, g) be an n-dimensional $(n \geq 7)$ compact Einstein manifold with positive scalar curvature. Then it is a strict local maximum within its conformal class $[g]$, unless (M^n, g) is isometric to S^n with the standard metric up to some multiple constant.*

REMARK 6.4. When (M^n, g) is locally conformally flat, for the functional $\int_M \sigma_3(g)dv_g$, J. Viaclovsky ([**V1**]) proved that a metric with positive constant sectional curvature is a strict local minimum, unless the manifold is isometric to S^n with the standard metric. Theorem 6.3 coincides with his at the locally conformally flat Einstein metrics, however, it does not need the assumption of locally conformally flatness in Theorem 6.3.

REMARK 6.5. Let (M^n, g) be an n-dimensional Einstein manifold with non-positive scalar curvature, from second variational formula Theorem 6.2, we have that

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F}_3(g_t) \leq 0.$$

REMARK 6.6. Recently, by using a first variational formula of Graham, A. Chang, H. Fang and M. Graham simplify the computation of the second variational formula for $\mathcal{F}_3$, and they can also give the second variational formula for general functionals $\mathcal{F}_k$ and obtain similar stability result for Einstein manifolds as in Theorem 6.3.

For the general first variational formula of $\mathcal{F}_3(g) = \int_M v^{(6)}(g)dv_g$ in arbitrary direction h_{ij}, where h_{ij} is any symmetric 2-tensor, Fang-Guo-Li ([**FGL**]) obtained the following formula

$$\frac{d}{dt}\Big|_{t=0}\int_M v^{(6)}(g_t)dv_{g_t} = \int_M F_{ij}h_{ij}dv, \tag{6.7}$$

where g_t is a family of Riemannian metrics, $\frac{d}{dt}(g_t)_{ij}\Big|_{t=0} = h_{ij}$, and F_{ij} is a symmetric 2-tensor defined by

(6.8)

$$\begin{aligned} F_{ij} \;=\; & -\tfrac{\Delta B_{ij}}{3(n-2)(n-4)} - \tfrac{C_{ikl}C_{jkl}}{6(n-4)} - \tfrac{2A_{kl}C_{(ij)k,l}}{3(n-4)} - \tfrac{\nabla_k\sigma_1 C_{(ij)k}}{3(n-4)} \\ & + \tfrac{C_{kil}C_{ljk}}{3(n-4)} - \tfrac{2B_{kl}W_{ikjl}}{3(n-4)(n-2)} + \tfrac{1}{3(n-4)}\sigma_1 B_{ij} - \tfrac{2}{3(n-2)}A_{km}A_{ml}W_{ikjl} \\ & +(n-6)[-\tfrac{1}{6(n-4)(n-2)}\Big(B_{ik}A_{kj} + B_{jk}A_{ki}\Big) + \tfrac{1}{2(n-2)}v^{(6)}g_{ij} \\ & -\tfrac{1}{6(n-1)(n-4)}(\sigma_2)_{ij} - \tfrac{1}{6(n-2)(n-4)}\Delta T^2_{ij} + \tfrac{1}{6(n-1)(n-4)}\Delta\sigma_2 g_{ij} \\ & +\tfrac{1}{3(n-4)(n-2)}\sigma_1 A_{kl}W_{ikjl} - \tfrac{1}{3(n-2)}T^2_{ik}A_{kj} + \tfrac{n}{6(n-1)(n-4)}\sigma_2\Big(A_{ij} - \tfrac{\sigma_1}{n}g_{ij}\Big)], \end{aligned}$$

where $B_{ij} = (B_g)_{ij}$ is the Bach tensor defined in (4.6), C_{ijk} is the Cotton tensor defined in (1.3), and $C_{(ij)k} := \frac{1}{2}\Big(C_{ijk} + C_{jik}\Big)$ is the symmetrization of the tensor C_{ijk} with respect to the first two indices.

REMARK 6.7. When $n = 6$, the 2-tensor F_{ij} defined in (6.8) coincides with the "obstruction tensor" $\mathcal{O}_{ij}$ defined in [**GrH**], precisely,

$$\begin{aligned} \mathcal{O}_{ij} = -24F_{ij} &= \Delta B_{ij} + 2C_{ikl}C_{jkl} + 8A_{kl}C_{(ij)k,l} + 4\nabla_k\sigma_1 C_{(ij)k} \\ &\quad - 4C_{kil}C_{ljk} - 2B_{kl}W_{kijl} - 4\sigma_1 B_{ij} + 4A_{km}A_{ml}W_{ikjl}. \end{aligned}$$

We note that $\mathcal{O}_{ij}$ has the following important properties when $n = 6$:

(1) $\mathcal{O}_{ij}$ is trace-free and divergence-free, i.e.,

$$\sum_i \mathcal{O}_{ii} = 0, \qquad \sum_i \mathcal{O}_{ij,j} = 0.$$

(2) $\mathcal{O}_{ij}$ is conformally invariant of weight $2-n$; i.e., if $0 < u \in C^\infty(M)$ and $\tilde{g}_{ij} = u^2 g_{ij}$, then $\tilde{\mathcal{O}}_{ij} = u^{2-n}\mathcal{O}_{ij}$.

(3) If g_{ij} is conformal to an Einstein metric, then $\mathcal{O}_{ij} = 0$.

We also note that Bach tensor $(B_g)_{ij}$ has the similar properties when $n = 4$.

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