POSITIVITY OF WEIL-PETERSSON CURRENTS ON CANONICAL MODELS

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Abstract. We show that the Weil-Petersson current is a global nonnegative closed $(1,1)$-current in the twisted Kähler-Einstein equation on non-general type canonical models.

1. Introduction

Let X be a projective manifold of complex dimension n and let

$$\Phi : X \to Y \tag{1.1}$$

be a projective surjective morphism onto a normal variety Y of complex dimension $0 < m < n$. If the canonical line bundle K_X is the pullback of a $\mathbb{Q}$-line bundle L on Y, i,e.,

$$K_X = \Phi^* L, \tag{1.2}$$

then the general fibre of Φ is a smooth Calabi-Yau manifold of complex dimension $n-m$. If we let Y° be set of smooth points over which Φ is regular, then Y° is a smooth quasi-projective variety and we let $X^\circ = \Phi^{-1}(Y^\circ)$.

A projective normal variety is $\mathbb{Q}$-Gorenstein if its canonical sheaf is a $\mathbb{Q}$-Cartier divisor, in other words, a $\mathbb{Q}$-line bundle. If Y is $\mathbb{Q}$-Gorenstein, the relative canonical divisor is $\mathbb{Q}$-Cartier. Let Ψ be a holomorphic section of the relative canonical bundle K_{X°/Y°. We define the hermitian metric h_{WP} for K_{X°/Y° by

$$|\Psi_y|^2_{h_{WP}} = (\sqrt{-1})^{n-m} \int_{\Phi^{-1}(y)} \Psi_y \wedge \overline{\Psi_y}, \tag{1.3}$$

for each $y \in Y^\circ$. The Weil-Petersson metric ω_{WP} on Y° is defined by

$$\omega_{WP} = Ric(h_{WP}) = -\sqrt{-1}\partial\bar\partial \log h_{WP}. \tag{1.4}$$

We will extend the Weil-Petersson metric ω_{WP} to a closed current globally on Y. Since K_X is the pullback of L on Y, there exists a smooth volume form Ω on X such that

$$\sqrt{-1}\partial\bar\partial \log \Omega = \Phi^*\chi$$

for some closed $(1,1)$-current χ on Y.

Definition 1.1. *We define the Weil-Peterson current on Y for $\Phi : X \to Y$ to be*

$$\omega_{WP} = \sqrt{-1}\partial\bar\partial \log \Omega - \sqrt{-1}\partial\bar\partial \log \Phi_*\Omega, \tag{1.5}$$

where $\Phi_*\Omega$ is the push-forward of Ω on Y defined in (2.2).

Research supported in part by National Science Foundation grants DMS-1710500 and DMS-1711439.

It is well-known that formula (1.5) coincides with formula (1.4) on Y° (cf. Lemma 2.1).

A natural question to ask is if ω_{WP} is a positive closed $(1,1)$-current on the normal variety Y. The smoothness and nonnegativity of ω_{WP} over Y° is well-known and proved in [12, 11]. However, it is not known in general if ω is nonnegative globally on Y and in general, ω_{WP} can have nonvanishing Lelong number and it might charge mass somewhere on $Y \setminus Y^\circ$. The global Weil-Petersson currents appear naturally in the study of canonical Kähler metrics on singular varieties and its positivity will play an important role to understand the analytic and geometric behavior of the canonical metrics near the singularities of Y as well as the singular fibers. The following is our main result.

Theorem 1.1. *If Y is $\mathbb{Q}$-Gorenstein, the Weil-Petersson current ω_{WP} on Y defined in Definition 1.1 is a nonnegative closed $(1,1)$-current, i.e.,*

$$\omega_{WP} \geq 0. \tag{1.6}$$

In general, Y is not necessarily $\mathbb{Q}$-Gorenstein and it is not clear how to define the Ricci current globally for a given Kähler current even if its volume measure behaves well. We will now give a global definition for the Weil-Petersson current ω_{WP} on Y when K_Y is not $\mathbb{Q}$-Cartier. We first resolve singularities of Y and let $\pi_Y : Y' \to Y$ be a log resolution of Y. π_Y induces a birational surjective morphism $\pi_X : X' \to X$ with the following diagram.

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ {\scriptstyle \Phi'}\downarrow & & \downarrow{\scriptstyle \Phi} \\ Y' & \xrightarrow{\pi_Y} & Y \end{array} \tag{1.7}$$

X' is again a Calabi-Yau fibration over Y' and $K_{X'}$ is also the pullback of some line bundle L' on Y'.

Definition 1.2. *Let ω'_{WP} be the Weil-Petersson current of the Calabi-Yau fibration on Y'. We define the Weil-Petersson current ω_{WP} on Y for $\Phi : X \to Y$ to be the push-forward of ω'_{WP} by $\pi_Y : Y' \to Y$.*

Here the pushforward of currents is defined by the standard procedure using smooth test forms. The test forms in this case are given by the restriction of a smooth form on the ambient space via local affine embedding of Y' into certain $\mathbb{C}^N$.

Theorem 1.2. *Let $\Phi : X \to Y$ be a morphism from a projective manifold X to a normal variety satisfying (1.2). The Weil-Petersson current ω_{WP} of Definition 1.2 is a closed nonnegative $(1,1)$-current on Y. Furthermore, ω_{WP} does not depend on the resolution of Y.*

We remark that if Y is $\mathbb{Q}$-Gorenstein and $\pi : Y' \to Y$ is a blow-up of Y, the Weil-Petersson current on Y defined in Definition 1.2 as the push-forward of the Weil-Petersson current on Y' coincides with the usual Weil-Petersson current on Y in Definition 1.1. However, it is not clear if ω_{WP} can be locally represented by $\sqrt{-1}\partial\overline{\partial} f$ for some local plurisubharmonic function f on Y due to the presence of normal singularities of Y unless Y is $\mathbb{Q}$-Gorenstein in the setting of Theorem 1.1.

We will apply Theorem 1.1 and Theorem 1.2 to canonical metrics on non-general type canonical models. When the canonical line bundle K_X is semi-ample, the canonical ring of X is finitely generated and when m is large enough the pluricanonical system $|mK_X|$ induces a unique surjective morphism

$$\Phi : X \to X_{can},$$

where X_{can} is the unique canonical model of X. The Kodaira dimension of X is defined to be

$$\mathrm{kod}(X) = \dim X_{can}$$

and since K_X is assumed to be semi-ample, we always have

$$0 \leq \mathrm{kod}(X) = \dim X_{can} \leq \dim X.$$

For the rest of this section, we will always assume K_X is semi-ample.

When $\mathrm{kod}(X) = \dim X$, X is said to be of general type and there exists a unique Kähler-Einstein current $\omega_{KE} \in [K_{X_{can}}]$ with bounded local potentials ([5, 1]).

When $0 < \mathrm{kod}(X) < \dim X$, $\Phi : X \to X_{can}$ is a fibration of Calabi-Yau manifolds over X_{can} with possible singular fibers. We define X°_{can} to be the set of all smooth points of X_{can} over which Φ is regular. The twisted Kähler-Einstein current ω_{can} on the canonical model X_{can} is defined in [8, 9] by the following curvature equation on X°_{can}

$$Ric(\omega_{can}) = -\omega_{can} + \omega_{WP}, \tag{1.8}$$

where ω_{WP} is the Weil-Petersson current define in (1.5). The existence of ω_{can} is proved in [9] and ω_{can} is closed positive $(1,1)$-current on X_{can} with bounded local potentials. In particular, ω_{can} is smooth on X°_{can} and $\Phi^*\omega_{can} \in [K_X]$. The curvature equation (1.8) is induced by the global complex Monge-Ampère equation and it is only globally defined when X_{can} is $\mathbb{Q}$-Gorenstein. The following corollary shows that indeed equation (1.8) can be globally defined and the Ricci curvature is a global closed $(1,1)$ current with uniform lower bound.

Corollary 1.1. *The Weil-Petersson current ω_{WP} in the twisted Kähler-Einstein equation (1.8) on X_{can} is a nonnegative closed $(1,1)$-current. In particular, the Ricci current $Ric(\omega_{can})$ is a globally defined closed $(1,1)$-current on X_{can} bounded below by $-\omega_{can}$.*

Corollary 1.1 immediately implies that the twisted Kähler-Einstein equation (1.8) is globally defined on the canonical model X_{can}. A special case of Corollary 1.1 was proved in [10] when X_{can} has at most orbifold singularities and it was applied to show that the metric completion of $(X^\circ_{can}, \omega_{can})$ is homeomorphic to X_{can} itself.

When $\mathrm{kod}(X) = 0$, the first Chern class of X must vanish and by Yau's celebrated solution [15] to the Calabi conjecture, there exists a unique Ricci-flat Kähler metric in any Kähler class on X. The degeneration of Ricci-flat Kähler metrics on X is extensively studied in [13, 14]. A special case of the degeneration is given by a projective morphism

$$\Phi : X \to Y$$

induced by the linear system of a $\mathbb{Q}$-ample line bundle L over Y. The morphism Φ gives a holomorphic fibration of Calabi-Yau manifolds over Y. The twisted Kähler-Einstein equation is also defined in [8, 9] for a canonical current $\omega_{can} \in [L]$ on Y° as follows

$$Ric(\omega_{can}) = \omega_{WP}. \tag{1.9}$$

Such ω_{can} always exists and is unique with bounded local potentials. In particular, it is smooth on Y°, the set of all smooth points of Y over which Φ is regular.

Corollary 1.2. *The Weil-Petersson current ω_{WP} in the twisted Calabi-Yau equation (1.9) on Y is a nonnegative closed $(1,1)$-current. In particular, the Ricci current $Ric(\omega_{can})$ is a globally defined nonnegative closed $(1,1)$-current on Y.*

The significance of Corollary 1.1 and Corollary 1.2 lies in the observation that the Ricci curvature of the canonical current ω_{can} is uniformly bounded below by -1 or 0, in particular, the Ricci current of the canonical Kähler-Einstein currents is globally well-defined. This would help us understand the local behavior of ω_{can} near $Y \setminus Y^\circ$, moreover, one can make use of pluripotential theory to study the local Ricci potentials and the geometric blow-up limit of ω_{can} if exists will always have nonnegative Ricci curvature. The anomaly flow introduced in [6] is related to the Fu-Yau equations [2, 7] and shares many properties of the Kähler-Ricci flow and the Weil-Petersson currents might also appear in the long time collapsing solutions.

2. Proof of Theorem 1.1

In this section, we always assume Y is $\mathbb{Q}$-Gorenstein. By our assumption, $K_X = \Phi^* L$ for some $\mathbb{Q}$-line bundle L on Y. Therefore there exist a smooth closed $(1,1)$-form $\chi \in c_1(L)$ on the normal variety Y and a smooth volume form Ω on X satisfying

$$\Phi^*\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega. \tag{2.1}$$

We remark that a smooth real valued function (form) on a normal variety is defined to be the restriction of a smooth real-valued function (form) from its local embedding. We define $\Phi_*\Omega$ to be the push-forward of Ω. For each $y \in Y^\circ$, the push-forward of Ω can be computed by

$$\Phi_*\Omega = \int_{\Phi^{-1}(y)} \Omega \tag{2.2}$$

as integration over fibres.

We will make use of the semi-Ricci flat metrics ω_{SF} introduced in [4, 8, 9]. For any ample line bundle $\mathcal{A}$ on X and a smooth Kähler metric $\omega_{\mathcal{A}} \in c_1(\mathcal{A})$, there exists ψ on X such that

$$\omega_{SF} = \omega_{\mathcal{A}} + \sqrt{-1}\partial\overline{\partial}\psi \tag{2.3}$$

is a Ricci-flat Kähler metric when restricted on each fibre over Y°, i.e.

$$Ric(\omega_{SF}|_{\Phi^{-1}(y)}) = 0$$

for each $y \in Y^\circ$. In fact, $\psi \in C^\infty(X^\circ)$, where $X^\circ = \Phi^{-1}(Y^\circ)$. Without loss of generality by rescaling $\mathcal{A}$, we can assume that

$$[\omega_{SF}|_{\Phi^{-1}(y)}]^{n-m} = [\mathcal{A}]^{n-m} \cdot [\Phi^{-1}(y)] = 1 \tag{2.4}$$

for each $y \in Y^\circ$.

Lemma 2.1. *Let ω_{SF} be a semi-Ricci flat form on X° satisfying the normalization condition (2.4). Then*

$$(\omega_{SF})^{n-m} \wedge \Phi_*\Omega = \Omega. \tag{2.5}$$

In particular, we have

$$\omega_{WP} = \chi - \sqrt{-1}\partial\overline{\partial}\log \Phi_*\Omega. \tag{2.6}$$

Proof. Let

$$f = \frac{(\omega_{SF})^{n-m} \wedge \Phi_*\Omega}{\Omega}.$$

Then f is smooth on X° and

$$\sqrt{-1}\partial\overline{\partial}\log f = \sqrt{-1}\partial\overline{\partial}\log\left((\omega_{SF})^{n-m} \wedge \Phi_*\Omega\right) - \chi.$$

In particular, for each $y \in Y^\circ$,

$$\left(\sqrt{-1}\partial\overline{\partial}\log f\right)|_{\Phi^{-1}(y)} = \left(\sqrt{-1}\partial\overline{\partial}\log(\omega_{SF})^{n-m}\right)|_{\Phi^{-1}(y)} - \chi|_{\Phi^{-1}(y)} = 0.$$

It implies that f is constant on $\Phi^{-1}(y)$ and f must be the pullback of a smooth function on Y°. Now

$$f\left(\Phi_*\Omega\right) = f\int_{\Phi^{-1}(y)}\Omega = \int_{\Phi^{-1}(y)}(\omega_{SF})^{n-m} \wedge \Phi_*\Omega = \left(\int_{\Phi^{-1}(y)}(\omega_{SF})^{n-m}\right)\Phi_*\Omega = \Phi_*\Omega.$$

Therefore $f = 1$ everywhere on Y°. For any holomorphic section Ψ of $K_{X/Y}$,

$$\begin{aligned}\int_{\Phi^{-1}(y)}\Psi \wedge \overline{\Psi} &= \int_{\Phi^{-1}(y)}\left(\frac{\Psi\wedge\overline{\Psi}}{(\omega_{SF})^{n-m}}\right)(\omega_{SF})^{n-m} = \frac{\Psi\wedge\overline{\Psi}}{(\omega_{SF})^{n-m}}\\ &= \frac{\Psi\wedge\overline{\Psi}\wedge\Phi_*\Omega}{(\omega_{SF})^{n-m}\wedge\Phi_*\Omega} = \frac{\Psi\wedge\overline{\Psi}\wedge\Phi_*\Omega}{\Omega},\end{aligned}$$

and so

$$\omega_{WP} = \chi - \sqrt{-1}\partial\overline{\partial}\log\Phi_*\Omega.$$

□

Let Y_{reg} be the smooth part of Y. $Y_{sing} = Y \setminus Y_{reg}$ is a subvariety of complex codimension greater than 1. By definition,

$$Y^\circ \subset Y_{reg}.$$

We can also define a smooth adapted volume measure Ω_Y on Y as in [1] since Y is $\mathbb{Q}$-Gorenstein. More precisely, for any $y \in Y$, there is an open neighborhood U of y such that the pluricanonical sheaf $(\omega_Y)^K$ is a rank one locally free sheaf, for some $K = K_y \in \mathbb{N}$. Let α be a local generator of $(\omega_Y)^K|_U$ if U is sufficiently small. Then α is a nowhere vanishing holomorphic pluricanonical form on U and $(\alpha\wedge\bar{\alpha})^{\frac{1}{K}}$ is a volume measure on U and we can define the adapted volume measure Ω_Y by gluing all such local volume measures using partition of unity. In particular, Ω_Y is a smooth volume form on Y_{reg}. We let

$$\theta = \sqrt{-1}\partial\overline{\partial}\log\Omega_Y$$

and define a real-valued function F on X° by

$$F = -\log\frac{\Omega}{(\omega_{SF})^{n-m}\wedge\Omega_Y}, \tag{2.7}$$

on $X^\circ = \Phi^{-1}(Y^\circ)$. F is smooth on X° satisfying

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}F &= -\sqrt{-1}\partial\overline{\partial}\log\left(\frac{\Omega}{(\omega_{SF})^{n-m}\wedge\Phi_*\Omega}\right)+\sqrt{-1}\partial\overline{\partial}\log\left(\frac{\Omega_Y}{\Phi_*\Omega}\right)\\ &= -\chi+\theta+\omega_{WP}.\end{aligned}$$

In particular,

$$(\sqrt{-1}\partial\overline{\partial}F)|_{\Phi^{-1}(y)}=0$$

for each $y\in Y^\circ$. Therefore F must be a constant along each fibre over Y° and F descends to a smooth function on Y°. Immediately we have the following lemma.

Lemma 2.2. *$F\in C^\infty(Y^\circ)\cap \mathrm{PSH}(Y^\circ,\chi-\theta)$ and it satisfies*

$$F=\log\left(\frac{\Omega_Y}{\Phi_*\Omega}\right) \tag{2.8}$$

and

$$\omega_{WP}=\chi-\theta+\sqrt{-1}\partial\overline{\partial}F \tag{2.9}$$

on Y°.

Our goal is to show that formula (2.9) extends globally to Y, or equivalently, F extends to a $(\chi-\theta)$-plurisubharmonic function on Y. We first show that F can indeed be extended uniquely to Y_{reg}.

Lemma 2.3. *F uniquely extends to a $(\chi-\theta)$-plurisubharmonic function on Y_{reg}, i.e.,*

$$F\in\mathrm{PSH}(Y_{reg},\chi-\theta).$$

Proof. Since Y_{reg} is a smooth open manifold and $F\in\mathrm{PSH}(Y^\circ,\chi-\theta)$, it suffices to show that F is locally bounded above on Y_{reg}, i.e., for any point $p\in Y_{reg}$, there exists an open neighborhood U_p in Y_{reg} such that

$$\sup_{U_p\cap Y^\circ}F<\infty.$$

We will apply the trick in the proof of Proposition 3.2 in [9]. For any $p\in Y_{reg}$, we choose U_p to be an open neighborhood of p such that its closure $\overline{U_p}\subset\subset Y_{reg}$. Then

$$\sup_{\Phi^{-1}(U_p)}\left(\frac{(\omega_{\mathcal{A}})^{n-m}\wedge\Omega_Y}{\Omega}\right)<\infty$$

because Ω and $(\omega_{\mathcal{A}})^{n-m}\wedge\Omega_Y$ are smooth volume forms on $\Phi^{-1}(Y_{reg})$. By the mean value theorem and the fact that for each $y\in Y^\circ$,

$$\int_{\Phi^{-1}(y)}(\omega_{SF})^{n-m}=\int_{\Phi^{-1}(y)}(\omega_{\mathcal{A}})^{n-m},$$

there is a point $q_y\in\Phi^{-1}(y)$ such that

$$(\omega_{SF})^{n-m}|_{q_y}=(\omega_{\mathcal{A}})^{n-m}|_{q_y}.$$

Then

$$\begin{aligned} F(y) &= -\log\left(\left.\frac{\Omega}{(\omega_{SF})^{n-m}\wedge \Omega_Y}\right|_{q_y}\right) \\ &= -\log\left(\left.\frac{\Omega}{(\omega_{\mathcal{A}})^{n-m}\wedge \Omega_Y}\right|_{q_y}\right) \\ &\leq \sup_{\Phi^{-1}(U_p)} \log\left(\frac{(\omega_{\mathcal{A}})^{n-m}\wedge\Omega_Y}{\Omega}\right). \end{aligned}$$

This implies that F is uniformly bounded above in $U_p \cap Y^\circ$ and so by Satz 3 in [3] F extends to a plurisubharmonic function on U_p with respect to $\chi - \theta$ since $\chi - \theta$ is a smooth closed $(1,1)$-form on U_p.

□

Lemma 2.4. *The function F defined in (2.7) on Y° uniquely extends to a $(\chi-\theta)$-plurisubharmonic function on Y, i.e.,*

$$F \in \mathrm{PSH}(Y, \chi - \theta).$$

In particular, F is uniformly bounded above on Y.

Proof. Riemann's removable singularity theorem for normal complex spaces (see Satz 4 in [3]) says that any plurisubharmonic function defined on the regular part of a normal space V can extend uniquely to a plurisubharmonic function everywhere on V through its complex codimensional 2 singularities. Since $F \in \mathrm{PSH}(Y_{reg}, \chi - \theta)$ and $\chi - \theta$ is a smooth closed $(1,1)$-form on Y, F extends uniquely to a $(\chi - \theta)$-plurisubharmonic function on Y with respect to $\chi - \theta$.

□

Lemma 2.4 shows that the nonnegative $(1,1)$-current $\omega_{WP} = \chi - \theta + \sqrt{-1}\partial\overline{\partial}F$ can be extended to Y by extending F and this completes the proof of Theorem 1.1.

3. Generalizations

In this section, we give a second proof of Theorem 1.1 with slight generalization and we prove Theorem 1.2. We consider the case when the base Y is not necessarily $\mathbb{Q}$-Gorenstein. Let Y' and X' be defined as in the diagram (1.7). Since $K_{X'}$ is the pullback of a line bundle on Y', there exists a smooth volume form $\Omega_{X'}$ on X' such that

$$\sqrt{-1}\partial\overline{\partial}\log\Omega_{X'} = (\Phi')^*\chi',$$

where χ' is a smooth closed $(1,1)$-form on Y'. We then let the volume measure on Y'

$$(\Phi')_*\Omega_{X'}$$

be the pushforward of $\Omega_{X'}$. Obviously, away from the exceptional locus of π_X,

$$\sqrt{-1}\partial\overline{\partial}\log\Omega_{X'} - \sqrt{-1}\partial\overline{\partial}\log(\Phi')_*\Omega_{X'} = (\pi_X)^*\left(\sqrt{-1}\partial\overline{\partial}\log\Omega - \sqrt{-1}\partial\overline{\partial}\log\Phi_*\Omega\right) = \omega'_{WP}.$$

Let ω_{SF} be the semi-Ricci flat metric associated to the fibration $\Phi' : X' \to Y'$, as defined in (2.3).

Lemma 3.1.

$$\Omega_{X'} = (\omega_{SF})^{n-m} \wedge (\Phi')_* \Omega_{X'}.$$

Proof. The follows by the same argument in Lemma 2.1.

□

The relative canonical bundle $K_{X'/Y'}$ is also the pullback of a line bundle over Y'. Let Y° be the set of smooth points of Y over which Φ is regular and let $(Y')^\circ = (\pi_Y)^{-1}(Y^\circ)$. Y° is a Zariski open set of Y.

Lemma 3.2. *For any point $p \in Y'$, we let η be a local nowhere vanishing holomorphic section of $K_{X'/Y'}$ in a neighborhood U near p. Then there exists $c > 0$ such that for any $y' \in U \cap (Y')^\circ$, we have*

$$\int_{(\Phi')^{-1}(y')} (\sqrt{-1})^{n-m} \eta \wedge \overline{\eta} \geq c.$$

Proof. We use a trick similarly in [1]. Since X' is projective, there exists a projective embedding $\iota : X' \to \mathbb{CP}^N$ and we let θ be the Fubini-Study metric on $\mathbb{CP}^N$. For any point $p \in Y'$, there exists a local holomorphic section η of $K_{X'/Y'}$ such that η is nowhere vanishing near p. For $q \in (\Phi')^{-1}(p)$, there exists an open neighborhood V_q of q such that $\iota|_{V_q} : V_q \to \mathbb{C}^N$ induces an affine embedding of V_q. Let $z = (z_1, z_2, ..., z_N)$ be the local holomorphic coordinates of $\mathbb{C}^N$. Then for any $(n-m)$-holomorphic form in V_q given by the restriction to V_q of $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge ... dz_{i_{n-m}}$ with $I = (i_1, i_2, ..., i_{n-m})$ and $1 \leq i_1 < i_2 < ... < i_{n-m} \leq N$, there exists a holomorphic function f_I on V_q such that for any y' close to p

$$dz_I \Big|_{V_q \cap (\Phi')^{-1}(y')} = (f_I \eta)\Big|_{V_q \cap (\Phi')^{-1}(y')}.$$

This implies that

$$\eta \wedge \overline{\eta}\Big|_{V_q \cap (\Phi')^{-1}(y')} = \Big(\sum_{I=(i_1,...,i_{n-m})} |f_I|^2 \Big)^{-1} \sum_{I=(i_1,...,i_{n-m})} dz_I \wedge d\overline{z_I}\Big|_{V_q \cap (\Phi')^{-1}(y')}.$$

In particular, there exist $C > 0$ and a neighborhood U of p in Y' such that in $(\Phi')^{-1}(U)$

$$\theta^{n-m} \leq C (\sqrt{-1})^{n-m} \eta \wedge \overline{\eta}.$$

Since $\int_{(\Phi')^{-1}(y')} \theta^{n-m} = [\mathcal{O}_{\mathbb{CP}^N}(1)|_{X'}]^{n-m} \cdot [(\Phi')^{-1}(y')]$ is a topological constant for all $y' \in (Y')^\circ$, there exists $c > 0$ such that for all $y' \in U \cap (Y')^\circ$, we have

$$\int_{(\Phi')^{-1}(y')} (\sqrt{-1})^{n-m} \eta \wedge \overline{\eta} \geq c > 0.$$

□

Recall the Weil-Petersson current ω'_{WP} on Y' is given by

$$\omega'_{WP} = \sqrt{-1}\partial\overline{\partial} \log \Omega_{X'} - \sqrt{-1}\partial\overline{\partial} \log (\Phi')_* \Omega_{X'}.$$

For any $y' \in (Y')^\circ$, we let

$$H(y') = \log \left(\frac{(\omega_{SF}|_{(\Phi')^{-1}(y')})^{n-m}}{(\sqrt{-1})^{n-m} \eta \wedge \overline{\eta}} \right). \tag{3.1}$$

Lemma 3.3. *For any point $p \in Y'$, there exists a neighborhood U of p such that on $U \cap (Y')^\circ$,*

$$\omega'_{WP} = \sqrt{-1}\partial\overline{\partial} H. \tag{3.2}$$

Moreover, H is uniformly bounded above in U.

Proof. By Lemma 3.1,

$$\begin{aligned} \omega'_{WP} &= \sqrt{-1}\partial\overline{\partial}\log\left(\frac{(\omega_{SF})^{n-m}\wedge(\Phi')_*\Omega_{X'}}{(\sqrt{-1})^{n-m}\eta\wedge\overline{\eta}\wedge(\Phi')_*\Omega_{X'}}\right) + \sqrt{-1}\partial\overline{\partial}\log\left((\sqrt{-1})^{n-m}\eta\wedge\overline{\eta}\right) \\ &= \sqrt{-1}\partial\overline{\partial}\log\left(\frac{(\omega_{SF})^{n-m}\wedge(\Phi')_*\Omega_{X'}}{(\sqrt{-1})^{n-m}\eta\wedge\overline{\eta}\wedge(\Phi')_*\Omega_{X'}}\right) \\ &= \sqrt{-1}\partial\overline{\partial} H. \end{aligned}$$

For any $y' \in U \cap (Y')^\circ$, by Lemma 3.2 there exists $C > 0$ such that

$$e^{H(y')} = \frac{(\omega_{SF}|_{(\Phi')^{-1}(y')})^{n-m}}{(\sqrt{-1})^{n-m}\eta\wedge\overline{\eta}} = \frac{\int_{(\Phi')^{-1}(y')}(\omega_{SF})^{n-m}}{\int_{(\Phi')^{-1}(y')}(\sqrt{-1})^{n-m}\eta\wedge\overline{\eta}} \leq C.$$

□

Lemma 3.4. *The Weil-Petersson current ω'_{WP} is a closed nonnegative $(1,1)$-current on Y'.*

Proof. ω'_{WP} is nonnegative on $(Y')^\circ$. For any point $p \in Y'$, there exists an open neighborhood U of p such that $\omega'_{WP} = \sqrt{-1}\partial\overline{\partial}H$ on $U \cap (Y')^\circ$. Therefore H is a plurisubharmonic function on $U \cap (Y')^\circ$. Since H is uniformly bounded above and $Y' \setminus (Y')^\circ$ is a proper analytic subvariety of Y', H uniquely extends to a plurisubharmonic function on U and so ω'_{WP} is closed and nonnegative.

□

Immediately we have the following lemma using push-forward of currents.

Lemma 3.5. *Let ω_{WP} be the push-forward current of ω'_{WP} by $\pi_Y : Y' \to Y$ as in Definition 1.2. Then ω_{WP} is a closed nonnegative $(1,1)$-current on the normal variety Y.*

The next lemma shows that the definition of Weil-Petersson current in Definition 1.2 does not depend on the choice of resolution of singularities for Y.

Lemma 3.6. *Suppose $\pi_1 : Y_1 \to Y$ and $\pi_2 : Y_2 \to Y$ be two resolution of singularities for Y. Let $\omega_{WP,1}$ and $\omega_{WP,2}$ be the corresponding Weil-Petersson currents on Y_1 and Y_2. Then*

$$(\pi_1)_*\omega_{WP,1} = (\pi_2)_*\omega_{WP,2}.$$

Proof. There exists a projective manifold Y' with blow-ups $\pi'_1 : Y' \to Y_1$ and $\pi'_2 : Y' \to Y_2$ such that $\pi_1 \circ \pi'_1 = \pi_2 \circ \pi'_2$. Let ω'_{WP} be the Weil-Petersson current on Y'. Then using (3.1), we have

$$\omega_{WP,i} = (\pi'_i)_*\omega'_{WP}, \ \ i = 1, 2$$

and so

$$\omega_{WP} = (\pi_i \circ \pi'_i)_*\omega'_{WP} = (\pi_i)_*\omega_{WP,i}, \ \ i = 1, 2.$$

□

Theorem 1.2 immediately follows by Lemma 3.5 and Lemma 3.6.

4. Proof of the corollaries

We now prove Corollary 1.1. Let X be an n-dimensional projective manifold of semi-ample canonical bundle and its Kodaira dimension is $0 < m < n$. Then the pluricanonical system induces a unique surjective morphism $\Phi : X \to X_{can}$ from X to its canonical model X_{can} of dimension m and K_X is the pullback of an ample line bundle L on X_{can}.

Let Ω be a smooth volume form on X such that

$$\sqrt{-1}\partial\overline{\partial}\log\Omega = \Phi^*\chi$$

for some positive smooth $(1,1)$-form $\chi \in [L]$. Let $\Phi_*\Omega$ be the push-forward of Ω by Φ. The canonical Monge-Ampère equation associated to the twisted Kähler-Einstein equation on X_{can} is defined by

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^m = e^{\varphi}\Phi_*\Omega \tag{4.1}$$

for $\varphi \in \mathrm{PSH}(X_{can}, \chi)$. It is proved in [8, 9] that $\Phi_*\Omega$ is an L^p-volume measure on X_{can} for some $p > 1$ and there exists a unique $\varphi \in \mathrm{PSH}(X_{can}, \chi) \cap L^\infty(X_{can})$ solving the equation (4.1). Moreover, φ is smooth on X°_{can}. Let $\omega_{can} = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$. Then on X°_{can}, we have

$$\begin{aligned} Ric(\omega_{can}) &= -\sqrt{-1}\partial\overline{\partial}\log(\omega_{can})^m \\ &= -\omega_{can} + \sqrt{-1}\partial\overline{\partial}\log\Omega - \sqrt{-1}\partial\overline{\partial}\log\Phi_*\Omega \\ &= -\omega_{can} + \omega_{WP}. \end{aligned}$$

By Theorem 1.2, the Weil-Petersson metric ω_{WP} extends uniquely to the Weil-Petersson current defined in Definition 1.2 and it is nonnegative. Immediately, the Ricci current of ω_{can} also extends to a closed $(1,1)$-current on X_{can} and it is bounded below by $-\omega_{can}$. This completes the proof of Corollary 1.1. Corollary 1.2 follows by the same argument with very little modification.

Acknowledgements: This paper is dedicated to Professor D.H. Phong on the occasion of his 65th birthday. The authors are grateful to him for his constant support and encouragement.

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