Kähler-Ricci flow & Kähler-Ricci solitons.

joint Phong, Sturm.

- KE metrics and eigenvalue gap
- convergence of KR flow when KE exists
- compactness of KR solitons.

Lecture 1: KE metrics $\leftrightarrow$ eigenvalue gaps. $\Delta_{\bar\partial}$

§1: Kähler manifolds & complex Monge-Ampère equations.

Ref: Futaki. "KE metrics and integral invariants." lecture Notes in Math, 1314.

- X a compact Kahler manifold.
- ω a Kahler metric $\Leftrightarrow$ $d\omega = 0$

locally $\rightarrow \omega = \sqrt{-1}\, g_{i\bar j}\, dz^i \wedge d\bar{z}^j$

$(g_{i\bar j}) > 0$ Hermitian matrix

$$\frac{\partial g_{i\bar j}}{\partial z_k} = \frac{\partial g_{k\bar j}}{\partial z_i} \quad \forall\, i, j, k$$

- Riemannian curvature tensor of ω

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}$$

$(g^{i\bar{j}}) =$ inverse of $(g_{i\bar{j}})$.

- Ricci tensor of ω.

$$R_{i\bar{j}} = R_{i\bar{j}k\bar{l}}\, g^{k\bar{l}}$$

$$Ric = \sqrt{-1}\, R_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} \qquad \Rightarrow d\, Ric = 0$$

$$= -\sqrt{-1}\, \partial\bar{\partial} \log \det(g_{i\bar{j}}).$$

- $[\mathrm{Ric}(\omega)] \in H^{1,1}(X, \mathbb{R})$.

  $\quad \llcorner\ c_1(X)$ 1st Chern class.

- scalar curvature

$$R_\omega = R = g^{i\bar{j}} R_{i\bar{j}}$$

- "Riemannian" Laplacian of ω. on $f : X \to \mathbb{R}$

$$\Delta_\omega f = g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \qquad \Gamma_{AB}^{C}$$

- <u>$\partial\bar{\partial}$-lemma</u> on (X, ω).

$$[\alpha] = [\beta] \in H^{1,1}(X, \mathbb{R})$$

then $\alpha - \beta = \sqrt{-1}\,\partial\bar{\partial} f$. $\qquad f \in C^\infty(X, \mathbb{R})$.

- Calabi conjecture $\longleftrightarrow$ complex Monge-Ampère equation.

$\forall \gamma \in c_1(X)$, ? $\exists\ \omega$. $\boxed{Ric(\omega) = \gamma}$ (*)

& solved by Yau, 70's.

- fix a Kahler metric ω_0. $[\omega_0] = [\omega]$.

. $\partial\bar\partial$-lemma $\quad \omega = \omega_0 + i\partial\bar\partial\varphi$.

. $[Ric(\omega_0)] = [\gamma] \quad Ric(\omega_0) - \gamma = i\partial\bar\partial F$

(*) $\Longleftrightarrow \quad Ric(\omega) - Ric(\omega_0) = \gamma - Ric(\omega_0) = -i\partial\bar\partial F$

$$= -i\partial\bar\partial \log \frac{\omega^n}{\omega_0^n}$$

$$\Longleftrightarrow \quad i\partial\bar\partial\left(\log \frac{\omega^n}{\omega_0^n} - F\right) = 0$$

$$\Longleftrightarrow \quad \omega^n = e^F \omega_0^n, \qquad \int_X e^F \omega_0^n = \int_X \omega^n = \int_X \omega_0^n$$

(MA) $\omega = \omega_0 + i\partial\bar\partial\varphi > 0$

$$(\omega_0 + i\partial\bar\partial\varphi)^n = e^F \omega_0^n$$

locally

$\Leftrightarrow \det(g_{0 i\bar j} + \varphi_{i\bar j}) = e^F \det(g_{0 i\bar j})$

RK: $\omega = (g_{i\bar j})\, dz_i \wedge d\bar z_j$

$$\omega^n = c_i(\det g_{i\bar j})\, dz_1 \wedge \cdots \wedge d\bar z_n$$

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n\text{-many}}$$

$\frac{1}{n!}\omega^n$ = volume form of ω.

Yau: (MA) admits a unique smooth solution φ.

$(\sup_X \varphi = 0)$

- continuity method
- a priori estimates
  - C^0: → pluripotential theory. Kolodziej. / → Moser iteration. Yau
  - C^2: Maximum principle
  - higher order estimates:

Application: if $c_1(X) = 0$, then in each Kahler class

$\exists!$ $\underbrace{\text{Ricci flat}}_{\text{CY metrics}}$ Kahler metric

§2: $\begin{cases} \text{KE metrics} \longleftrightarrow \text{MA equations} \\ \text{obstructions} \end{cases}$

- if $\underline{\mathrm{Ric}(\omega) = \lambda\omega}$ (KE) $\lambda = \pm 1, 0$,

  ω is called a KE metric.

$$\text{(KE)} \Longleftrightarrow \begin{cases} (\omega_0 + i\partial\bar\partial\varphi)^n = e^{F - \lambda\varphi}\,\omega_0^n. \\ \mathrm{Ric}(\omega_0) - \lambda\omega_0 = i\partial\bar\partial F \end{cases}$$

$$\int e^F \omega_0^n = \int \omega_0^n$$

- $\exists$ KE $\Rightarrow$ $c_1(X) > 0$, $= 0$, < 0

  $\lambda = 1$ $\lambda = 0$ $\lambda = -1$.

- $\lambda = -1$, Yau, Aubin. (indep). $\exists!$ KE metric
- $\lambda = 1$, $c_1(X) > 0 \Longleftrightarrow -K_X$ ample or positive.

- From now on, $c_1(X) > 0$. $\omega = \omega_{KE}$

  look for $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$.

obstructions :

① Matsushima Reductivity of $\mathrm{Aut}(X)$

$$= \{\sigma : X \to X \text{ bihol}\}.$$

② Futaki invariant $\equiv 0$

$$\mathrm{Fut}(\cdot) : H^0(X, TX) \to \mathbb{C}$$

$$\mathrm{Fut}(V) = \pm \int (V \cdot F)\, \omega^n$$

$$\mathrm{Ric}(\omega) - \omega = i\partial\bar{\partial} F$$

③ K-stability $\Leftrightarrow$ KE metrics.

Rk: toric Fano manifolds.

$$\exists KE \Longleftrightarrow Fut(\cdot) = 0$$

by Wang-Zhu.

goal: another analytic condition on $\exists$ of KE

§3: eigenvalue gap & Main Theorem.

X, Fano manifolds, $c_1(X)>0$, $\omega_0 \in c_1(X)$ fixed reference metric.

· Mabuchi K-energy, $\omega = \omega_0 + i\partial\bar\partial\varphi > 0$

$$K_{\omega_0}(\omega) = K_{\omega_0}(\varphi) \qquad \dot\varphi_t = \frac{\partial \varphi_t}{\partial t}$$

$$= -\frac{n}{V}\underbrace{\int_0^1 dt \int_X}_{\text{well-defined}} \dot\varphi_t \left(Ric\,\omega_{\varphi_t} - \omega_{\varphi_t}\right)\wedge \omega_{\varphi_t}^{n-1}$$

$$\begin{cases} (\varphi_t)_{t\in[0,1]} \subset \mathcal{H}(X,\omega_0). \\ \varphi_0 \equiv 0 \\ \varphi_1 = \varphi \end{cases}$$

- $\forall\, \omega \in c_1(X)$, $\lambda_\omega =$ smallest positive eigenvalue of $\Delta_{\bar\partial}$

here $\Delta_{\bar\partial} : \Gamma(TX) \to \Gamma(TX)$,

$$\Delta_{\bar\partial} V = \bar\partial_\omega^* \bar\partial V$$

- $$\lambda_\omega = \inf_{\substack{V \in \Gamma(TX) \\ V \perp_\omega H^0(X, TX)}} \frac{\|\bar\partial V\|^2_{L^2(\omega)}}{\|V\|^2_{L^2(\omega)}}$$

<u>Naive question</u>: $\inf_{\omega \in c_1(X)} \lambda_\omega > 0$

- $\forall A > 0$. define a subset of Kahler metrics in $c_1(X)$.

$$c_1(X;A) = \left\{ \omega \in c_1(X) \;\middle|\; \begin{array}{c} \|u_\omega\|_{C^0} + \|\nabla u_\omega\|_{C^0(\omega)} + \|\Delta_\omega u_\omega\|_{C^0} \le A \\ K_{\omega_0}(\omega) \le A \end{array} \right\}.$$

here $\begin{cases} Ric(\omega) - \omega = -i\partial\bar\partial u_\omega \\ \int e^{-u_\omega}\omega^n = \int \omega^n = \int \omega_0^n \end{cases}$

Motivated by Perelman's results on KR flow.

- <u>eigenvalue gap</u>: for $c_1(X;A)$,

$$\lambda(X;A) = \inf_{\omega \in c_1(X;A)} \lambda_\omega \ge 0.$$

<u>Theorem:</u> (G.-Phong-Sturm).

Suppose X Fano, & $F_{ut}(\cdot) \equiv 0$

then X admits a KE

$\Longleftrightarrow$ $\forall A > 0$, $\lambda(X; A) \geq c(A) > 0$.

<u>RK</u>: "$\Leftarrow$" follows from Perelman's theorem on KR flow combined w/ convergence results of Phong-Song-Sturm-Weinkove & Z. Zhang. (will come back later to this)

§4: Proof of the Theorem: KE $\Rightarrow \lambda(X;A) > 0$.

A NEW proof other than that in our paper.

$\begin{cases} \text{Chen-Cheng} \\ \text{Li-Li-Zhang} \end{cases}$

· Assume X admits a KE, $\omega_{KE} = \omega_0$. $\left(\Rightarrow F_{ut}(\cdot) \equiv 0\right)$

$Ric(\omega_0) = \omega_0$

· Fix $A > 0$

goal: to derive uniform $C^{1,\alpha}(X, \omega_0)$ estimates of $\omega \in C_1(X; A)$.

idea: express ω in terms of certain MA equation.

$$
\begin{cases} \mathrm{Ric}(\omega) - \omega = -i\partial\bar\partial u_\omega & \text{—} \ (*) \\ \int e^{-u_\omega}\omega^n = \int \omega^n \end{cases}
$$

$$
\omega = \omega_0 + i\partial\bar\partial\varphi, \quad \sup_X \varphi = 0
$$

(MA).

$$
(*) \Longleftrightarrow \boxed{(\omega_0 + i\partial\bar\partial\varphi)^n = e^{-\varphi + u_\omega + c_\varphi}\,\omega_0^n}
$$

$c_\varphi \in \mathbb{R}$ a normalizing constant

" $c_\varphi \le 0$ by Jensen inequality "

Three lemmas $\Rightarrow$ $\left.\begin{array}{l} \cdot\ C^{0,\alpha} \text{ of } \varphi, \\ \cdot\ C^1 \text{ of } \varphi \\ \cdot\ C^2 \text{ of } \varphi \end{array}\right\}$ bounded.

<u>Lemma 1</u>: $\forall\, \omega \in c_1(X;A)$, $\exists\, \sigma = \sigma_\omega \in \mathrm{Aut}(X)$

s.t. $\tilde{\omega} = \sigma^*\omega = \omega_0 + i\partial\bar{\partial}\psi$, $\sup_X \psi = 0$

$$\|\psi\|_{C^{0,\alpha}(X,\omega_0)} \le C(A, \omega_0)$$

<u>Convention</u>: say a constant is uniform, if it depends on A, ω_0, n.

<u>Proof</u>: ① Moser-Trudiger inequality $\begin{cases} \text{Phong-Song-Sturm-Weinkove} \\ \text{Darvas-Rubinstein.} \end{cases}$

$\exists\, \varepsilon_0 = \varepsilon_0(\omega_0) > 0$. s.t.

MT-ineq.

$$A \ge \boxed{K_{\omega_0}(\omega) \ge \varepsilon_0 \boxed{\inf_{\sigma \in \mathrm{Aut}(X)} I_{\omega_0}(\sigma^*\omega)} - C}$$

$$\ge \varepsilon_0 I_{\omega_0}(\tilde{\omega}) - C'$$

for some $\sigma \in \mathrm{Aut}$

$$\ge I_{\omega_0}(\sigma^*\omega) - \varepsilon$$

here $I_{\omega_0}(\tilde{\omega}) = \frac{1}{V}\int \psi(\omega_0^n - \omega_\psi^n)$ $(\omega_\psi = \tilde{\omega})$

$$\Rightarrow \underbrace{\int (-\psi)\,\omega_\psi^n}_{\text{self-energy of } \psi \in PSH(X,\omega_0).} \leq C$$

② Apply either Skoda-Zeriahi compactness theorem (pluripotential)

or a Trudinger type inequality by G.-Phong

$$\int_X e^{\beta(-\psi)^{\frac{n+1}{n}}}\omega_0^n \leq C$$ (G.-Phong-Tong)

$$\Rightarrow \underbrace{\int e^{-p\psi}\,\omega_0^n}_{\|e^{-\psi}\|^p_{L^p(\omega_0^n)}} \leq C_p \quad \forall p>1$$

② Recall the MA equation

$$\omega_\psi^n = (\omega_0 + i\partial\bar\partial\psi)^n = e^{-\psi + \underline{u_\psi} + \underline{c_\psi}}\,\omega_0^n \quad \text{(bdd)}$$

$$\leq C e^{-\psi}\,\omega_0^n$$

RHS $\in L^p$ $p>1$

by Hölder continuity estimate of Kolodziej

$$\|\psi\|_{C^{0,\alpha}(X,\omega_0)} \leq C(A).$$

$\square$

<u>RK</u>: $\forall\,\omega \in c_1(X;A)$, consider $\tilde\omega = \sigma^*\omega \in c_1(X;A)$

$\lambda_{\tilde\omega} = \lambda_\omega$.

we may assume

$$\omega = \omega_0 + i\partial\bar\partial\varphi.$$

$$\sup\varphi = 0 \qquad \|\varphi\|_{C^{0,\alpha}} \leq C.$$