Recall:

MA equation of $\omega = \omega_0 + i\partial\bar\partial\varphi \in c_1(X;A)$

$$\begin{cases} (\omega_0 + i\partial\bar\partial\varphi)^n = e^{-\varphi + u_\omega + C}\,\omega_0^n \\ \sup_X \varphi = 0 \\ \|\varphi\|_{C^{0,\alpha}(X,\omega_0)} \le C(\omega_0, A) \end{cases}$$

bdd

$= e^F \omega_0^n$

$F := -\varphi + u_\omega + C$ bdd in C^0

Assumptions:

$$\|u_\omega\| + \|\nabla u_\omega\|_{C^0(\omega)} + \|\Delta_\omega u_\omega\|_{C^0} \le A$$

Goal: to derive uniform $C^{3,\alpha}$ estimates of φ.

Step 1 : $\|\nabla\varphi\|_{C^0(\omega_0)} \leq C$

Step 2 : $\|\Delta_{\omega_0}\varphi\|_{C^0} \leq C$

Step 3 : $\|\varphi\|_{C^{3,\alpha}(X,\omega_0)} \leq C.$

Main difficulty: the RHS is not bdd in $C^1(X,\omega_0)$.

• estimates of Green's function G_ω of ω.

<u>Theorem</u>: (G.-Phong-Sturm) arXiv: 2202.04715

$\exists\ C = C(A, \omega_0) > 0$. s.t

$$\inf_{x\in X} G_\omega(x,\cdot) \geq -C$$

$$\int_X |G_\omega(x,\cdot)|^p \omega^n \leq C_p \qquad \forall\ p < \frac{n}{n-1}$$

$$\int_X |\nabla_y G_\omega(x,y)|_\omega^q \omega^n \leq C_q \qquad \forall\ q < \frac{2n}{2n-1}$$

<u>Recall</u>: Green's formula

$$u(x) = \frac{1}{V}\int_X u\,\omega^n - \int_X G_\omega(x,\cdot)\,\Delta_\omega u\ \omega^n$$

Lemma 2: $\sup_X |\nabla\varphi|^2_{\omega_0} \le C(A, \omega_0)$.

Proof: ① $\forall$ fixed point y_0, take normal coordinates w.r.t. g_0 s.t. $g|_{y_0}$ is diagonal.

Calculations at y_0:

$$\Delta_\omega |\nabla\varphi|^2_{\omega_0} \ge -C_1 (tr_\omega \omega_0) |\nabla\varphi|^2_{\omega_0} + g^{p\bar{p}} \varphi_{\bar{i}p} \varphi_{i\bar{p}}$$

(2.1) $$+ g^{p\bar{p}} \varphi_{ip} \varphi_{\bar{i}\bar{p}} + 2\,\mathrm{Re}(\varphi_i F_{\bar{i}})$$

(see Appendix for a proof)

$-C_1$ lower bound of $Rm(g_0)$

- set $\Phi = -F - \lambda\varphi$, $\lambda > 0$ TBD

- $$\Delta_\omega e^{\Phi} = e^{\Phi}\left(\Delta_\omega \Phi + |\nabla\Phi|^2_\omega\right) \qquad (2.2)$$

$$= e^{\Phi}\left((\lambda-1)\, tr_\omega \omega_0 - C + |\nabla\Phi|^2_\omega\right)$$

By (2.1) & (2.2).

$$\Delta_\omega\left(e^{\Phi}|\nabla\varphi|^2_{\omega_0}\right)$$

$$= e^{\Phi}\Delta_\omega|\nabla\varphi|^2_{\omega_0} + |\nabla\varphi|^2_{\omega_0}\Delta_\omega e^{\Phi} + 2Re\left(\nabla e^{\Phi}, \nabla|\nabla\varphi|^2_{\omega_0}\right)_\omega$$

$$\Rightarrow \Delta_\omega \left(e^{\Phi} |\nabla \varphi|^2_{\omega_0} \right)$$

$$\geq e^{\Phi} \Big(\underbrace{- c_1 (\mathrm{tr}_\omega \omega_0) |\nabla \varphi|^2_{\omega_0}}_{⑥} + g^{p\bar{p}} \varphi_{i\bar{p}} \varphi_{\bar{i}p} + \underbrace{g^{p\bar{p}} \varphi_{\bar{i}\bar{p}} \varphi_{ip}}_{③}$$

$$+ \underbrace{2 \,\mathrm{Re} (\varphi_i F_{\bar{i}})}_{④}$$

$$\underbrace{+ (\lambda - 1) |\nabla \varphi|^2_{\omega_0} \mathrm{tr}_\omega \omega_0}_{⑦} - C |\nabla \varphi|^2_{\omega_0} + \underbrace{|\nabla \varphi|^2_{\omega_0} |\nabla \Phi|^2_\omega}_{②}$$

$$+ 2 \,\mathrm{Re} \Big(\underbrace{g^{p\bar{p}} \Phi_p \varphi_j \varphi_{\bar{j}\bar{p}}}_{①} + \underbrace{g^{p\bar{p}} \Phi_p \varphi_{\bar{j}} \varphi_{jp}}_{⑤} \Big) \Big)$$

By CR-ineq

① + ② + ③ ≥ 0

⑥ + ⑦ ≥ 0 if $\lambda - 1 - c_1 = 1$.

$$④ + ⑤ = 2\,\mathrm{Re}\left(F_i \varphi_{\bar i} + g^{j\bar j} \Phi_j \varphi_{\bar j}\left(g_{j\bar j} - 1\right)\right)$$

$$\Phi = -F - \lambda\varphi$$
$$= -(\lambda - 1)\varphi - u_\omega - c_\omega$$

$$= 2\,\mathrm{Re}\left(F_i \varphi_{\bar i} + \Phi_j \varphi_{\bar j} - g^{j\bar j}\Phi_j \varphi_{\bar j}\right)$$

$$= 2\,\mathrm{Re}\left(-\lambda |\nabla\varphi|^2_{\omega_0} + (\lambda - 1)|\nabla\varphi|^2_{\omega} + \langle \nabla u_\omega, \nabla\varphi\rangle_\omega\right)$$

$$\geq -2\lambda |\nabla\varphi|^2_{\omega_0} - C$$

<u>Hence</u>:

$$\Delta_\omega\Big(\underbrace{\boxed{e^{\Phi}|\nabla\varphi|^2_{\omega_0}}}_{=:H}\Big) \geq -C e^{\Phi}|\nabla\varphi|^2_{\omega_0} - C$$

$$\Delta_\omega H \geq -CH - C$$

② Let $H(x_0) = H_{max}$

$$H(x_0) = \frac{1}{V}\int_X H\,\omega^n - \int_X G(x_0,\cdot)\,\Delta_\omega H\,\omega^n$$

$$\le C + C\int_X G(x_0,\cdot)\cdot H\,\omega^n$$

$$\le C + C\,H_{max}^{1-\eta}\int_X G(x_0,\cdot)\,H^{\eta}\omega^n$$

$$\le C + C\,H_{max}^{1-\eta}\left(\int_X G(x_0,\cdot)^p\omega^n\right)^{1/p}\left(\int_X H^{\eta p^*}\omega^n\right)^{1/p^*}$$

take $\eta = 1/p^*$

here we have used

$$\int_X H\,\omega^n \le C \int |\nabla\varphi|^2_{\omega_0}\,\omega_0^n \le C .$$

Pf:

$$\omega^n - \omega_0^n = i\partial\bar\partial\varphi \wedge (\omega_0^{n-1} + \cdots + \omega^{n-1})$$

$$\int (-\varphi)(\omega^n - \omega_0^n) = \int (e^F - 1)(-\varphi)\,\omega_0^n \text{ bounded}$$

$$= \int (-\varphi)\, i\partial\bar\partial\varphi \wedge (\omega_0^{n-1} + \cdots + \omega^{n-1})$$

$$= \int i\partial\varphi \wedge \bar\partial\varphi \wedge (\omega_0^{n-1} + \cdots + \omega^{n-1})$$

$$\ge \int i\partial\varphi \wedge \bar\partial\varphi \wedge \omega_0^{n-1} = \frac{1}{n}\int_X |\nabla\varphi|^2_{\omega_0}\,\omega_0^n .$$

Lemma 3: C^2 estimate of φ

$$\sup_X \big(\underbrace{n + \Delta_{\omega_0}\varphi}_{= \operatorname{tr}_{\omega_0}\omega}\big) \le C(A, \omega_0)$$

$= \operatorname{tr}_{\omega_0}\omega$, $\omega = \omega_0 + i\partial\bar\partial\varphi$.

Proof: Yau's C^2 estimate.

$$\Delta_\omega(\operatorname{tr}_{\omega_0}\omega) \geq -C_1(\operatorname{tr}_\omega \omega_0)\cdot(\operatorname{tr}_{\omega_0}\omega) + g^{i\bar{i}}g^{j\bar{j}}\varphi_{i\bar{j}k}\varphi_{\bar{i}j\bar{k}}$$

(2.3)

$$+ \Delta_{\omega_0}F - \underbrace{R_{\omega_0}}_{\text{scalar curvature of } \omega_0}$$

as before

$$\Phi = -F - \lambda\varphi$$

$$\Delta_\omega\left(e^{\Phi}\operatorname{tr}_{\omega_0}\omega\right) = e^{\Phi}\Delta_\omega \operatorname{tr}_{\omega_0}\omega + (\operatorname{tr}_{\omega_0}\omega)\,\Delta_\omega e^{\Phi}$$
$$+2\operatorname{Re}\langle \nabla e^{\Phi}, \nabla \operatorname{tr}_{\omega_0}\omega\rangle_\omega$$

$$\geq e^{\Phi}\Big(- c_1(\operatorname{tr}_\omega \omega_0)(\operatorname{tr}_{\omega_0}\omega) + g^{i\bar{i}} g^{j\bar{j}} \varphi_{i\bar{j}k}\varphi_{\bar{i}j\bar{k}}$$
$$+ \Delta_{\omega_0} F - R_{\omega_0}$$

unbounded term

$$+ (\lambda-1)(\operatorname{tr}_{\omega_0}\omega)(\operatorname{tr}_\omega \omega_0) - C\operatorname{tr}_{\omega_0}\omega + (\operatorname{tr}_{\omega_0}\omega)|\nabla\Phi|^2_\omega$$
$$+ 2\operatorname{Re}\langle\nabla\Phi, \nabla\operatorname{tr}_{\omega_0}\omega\rangle_\omega\Big)$$

RED terms ≥ 0 by Cauchy-Schwarz

PINK term > 0 if take $\lambda = 2 + C_1$

Recall $F = -\varphi + u_\omega + c_\omega$, $\Phi = -F - \lambda\varphi$

$$\Delta_{\omega_0} F = -\mathrm{tr}_{\omega_0}\omega + n + \Delta_{\omega_0} u_\omega$$

so

$$\Delta_\omega\Big(\underbrace{e^{\Phi}\,\mathrm{tr}_{\omega_0}\omega}_{=M}\Big) \geq e^{\Phi}\Big(-C\,\mathrm{tr}_{\omega_0}\omega - C + \Delta_{\omega_0} u_\omega\Big)$$

Apply Green's formula to M, at $x_0 \in X$. $M(x_0) = \max\limits_X M$

$$M(x_0) = \frac{1}{V}\int_X M\,\omega^n - \int_X G(x_0,\cdot)\,\Delta_\omega M$$

$$\leq C + \int_X G(x_0,\cdot)\Big(\underbrace{CM}_{①} + C \underbrace{- e^{\Phi}\Delta_{\omega_0} u_\omega}_{②}\Big)\,\omega^n$$

$$① \leq C M(x_0)^{1-\eta}\int_X G(x_0,\cdot)\,M^{\eta}\,\omega^n \leq C M(x_0)^{1-\eta}\Big(\int G(x_0,\cdot)^p\Big)^{1/p}\Big(\int M\omega^n\Big)^{1/p^*}$$

$$\int_X M\omega^n = \int e^{-F-\lambda\varphi}\,\mathrm{tr}_{\omega_0}\omega \cdot e^{F}\omega_0^n$$
$$\le C\int \omega\wedge\omega_0^{n-1} \le C \qquad \checkmark$$

$$② = \int G(x_0,\cdot)\left(-\Delta_{\omega_0}u_\omega\right) e^{-\lambda\varphi}\,\omega_0^n$$
$$\overset{\text{IBP}}{=} \int \langle \nabla G(x_0,\cdot), \nabla u_\omega\rangle_{\omega_0} e^{-\lambda\varphi}\,\omega_0^n - \int G(x_0,\cdot)\langle \nabla u_\omega, \lambda\nabla\varphi\rangle_{\omega_0} e^{-\lambda\varphi}\,\omega_0^n$$
$$\le C\int_X |\nabla G(x_0,\cdot)|_\omega\, |\nabla u_\omega|_\omega \left(\mathrm{tr}_{\omega_0}\omega\right)\omega^n \le C\,M(x_0)^{1-\gamma}$$
$$+\; C\int_X |G(x_0,\cdot)|\,|\nabla u_\omega|_\omega \left(\mathrm{tr}_{\omega_0}\omega\right)^{1/2}\omega^n \le C\,M(x_0)^{1/2}$$

$\square$

Lemma 2 & the MA equation

$$\omega^n = e^F \omega_0^n$$

$$\Rightarrow \quad \frac{1}{C}\omega_0 \le \omega \le C\omega_0 \Rightarrow \lambda(\omega) \ge \frac{1}{C}\lambda(\omega_0)$$

✓

$\square$

<u>Rk</u>: higher order estimates

$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{-\varphi + u_\omega + c_\omega}\omega_0^n = e^F \omega_0^n$$

$$|\nabla e^F|_{\omega_0} \le C$$

By Evans-Krylov type theorem, we get

$$\|\varphi\|_{C^{3,\alpha}(X,\omega_0)} \le C.$$

Appendix ①: Proof of (2.1).

Recall we take normal coordinates at y_0 w.r.t. ω_0 & $\omega|_{y_0}$ is diagonal.

$$\begin{cases} \omega_0|_{y_0} = \delta_{ij} \\ d\omega_0|_{y_0} = 0 \end{cases}$$

at y_0

$$\Delta_\omega |\nabla\varphi|^2_{\omega_0} = g^{p\bar p}\left(g_0^{i\bar j}\varphi_i\varphi_{\bar j}\right)_{p\bar p}$$

$$= g^{p\bar p}\frac{\partial^2}{\partial z_p\partial\bar z_p}\left(g_0^{i\bar j}\right)\varphi_i\varphi_{\bar j} + g^{p\bar p}g_0^{i\bar j}\varphi_{ip}\varphi_{\bar j\bar p} + g^{p\bar p}g_0^{i\bar j}\varphi_{i\bar p}\varphi_{\bar j p}$$

$$+ g^{p\bar p}g_0^{i\bar j}\frac{\partial^2\varphi_i}{\partial z_p\partial\bar z_p}\varphi_{\bar j} + g^{p\bar p}g_0^{i\bar j}\varphi_i\frac{\partial^2\varphi_{\bar j}}{\partial z_p\partial\bar z_p}$$

$$= Rm(g_0)_{j\bar k p\bar p}\, g^{p\bar p}\varphi_k\varphi_{\bar j} + g^{p\bar p}\varphi_{ip}\varphi_{\bar i\bar p} + g^{p\bar p}\varphi_{i\bar p}\varphi_{\bar i p}$$

$$+ g^{p\bar p}\frac{\partial^2\varphi_i}{\partial z_p\partial\bar z_p}\varphi_{\bar i} + g^{p\bar p}\varphi_i\frac{\partial^2\varphi_{\bar i}}{\partial z_p\partial\bar z_p}$$

take $\frac{\partial}{\partial z_i}$ on both sides of $(\omega_0 + i\partial\bar\partial\varphi)^n = e^F \omega_0^n$

$$\Rightarrow \quad g^{p\bar p} \frac{\partial^2 \varphi_i}{\partial z_p \partial \bar z_p} = F_i \quad \text{at } y_0$$

Plugging this, we get (2.1).

· 2nd Topic: convergence of KR flow when $\exists$ KR solitons.

· X. $c_1(X) > 0$, Fano

· (normalized) Kähler-Ricci flow

$$(KRF)\quad \begin{cases} \dfrac{\partial}{\partial t}\omega = -Ric(\omega) + \omega \\ \omega = \omega(t), \quad \omega\big|_{t=0} = \omega_0 \in c_1(X). \end{cases}$$

$[\omega] = [\omega_0] = c_1(X)$.

KRF exists for $\forall\, t \in [0, \infty)$.

Hamilton-Tian conj: (X, ω_t) converges in some "weak" topology to a $\overset{\text{singular}}{\wedge}$ KR soliton.

Chen-Wang, Bamler.

- Kähler-Ricci soliton: $\omega = \omega_{KS}$.

  if $Ric(\omega) - \omega = L_V \omega$

  here V is a holomorphic vector field

- necessity: $Im(V)$ is ω-Killing, ie generating ω-isometries

Theorem: (Perelman, Tian - Zhu)

X, $c_1(X)>0$. assume a ω_{KS} $\exists$

consider: KRF $\begin{cases} \dfrac{\partial \omega}{\partial t} = -Ric(\omega) + \omega \\ \omega|_{t=0} = \omega_0 \in c_1(X) \end{cases}$

$(L_{\mathrm{Im} V}\, \omega_0 = 0)$

$\hookrightarrow L_{\mathrm{Im} V}\, \omega = 0$.

then $\exists\, \eta_t \in Aut(X)$ $\eta_t^* \omega_t$ converges smoothly to a KR soliton ω_{KS}'

RK: slight improvement by G.- Phong- Sturm

can take $\eta_t = \exp(tV)$ holomorphisms generated by V.

Main ideas: $\begin{cases} \text{Moser-Trudinger (by Darvas-Rubinstein for KR soliton)} \\ \text{Skoda-Zeriahi compactness or Trudinger type inequality by G.-Phong.} \end{cases}$

+ classical a priori estimates for parabolic MA equations.

- Notations : Fix an $\omega_0 \in c_1(X)$ $L_{\mathrm{Im} v}\,\omega_0 = 0$.

denote $\mathcal{K}_v = \{\omega \in c_1(X) \mid L_{\mathrm{Im} v}\,\omega = 0\}$

$$\mathcal{H}_v = \mathcal{H}_v(X, \omega_0) = \left\{\varphi \in C^\infty(X) \;\middle|\; \begin{array}{l} \omega_\varphi = \omega_0 + i\partial\bar\partial\varphi \in \mathcal{K}_v \\ \& \ (\mathrm{Im} V)(\varphi) = 0 \end{array}\right\}$$

- given $\omega \in \mathcal{K}_v$, define the "Hamiltonian" $\underline{\theta_{v,\omega} \in C^\infty(X)}$

$$\begin{cases} \iota_v \omega = i\bar\partial\theta_{v,\omega} \\ \int e^{\theta_{v,\omega}}\omega^n = \int \omega^n \end{cases}$$

- Ricci potential of $\omega \in c_1(X)$.

$$\begin{cases} \operatorname{Ric}(\omega) - \omega = -i\partial\bar{\partial} u_\omega \\ \int_X e^{-u_\omega} \omega^n = \int_X \omega^n \end{cases}$$

- modified Ricci potential (w.r.t. V), $\omega \in \mathcal{K}_V$

$$f_{V,\omega} := u_\omega + \theta_{V,\omega}$$

$$\operatorname{Ric}(\omega) - \omega - L_V\omega = -i\partial\bar{\partial} f_{V,\omega}$$

$$\int_X e^{-f_{V,\omega}} e^{\theta_{V,\omega}} \omega^n = \int_X \omega^n$$

<u>Lemma</u> (Zhu) If $\omega = \omega_0 + i\partial\bar\partial\varphi$, $\varphi \in \mathcal{H}_V(X, \omega_0)$

then $\theta_{V,\omega} = \underbrace{\theta_{V,\omega_0}}_{\text{fixed function}} + V\cdot\varphi$

& $\|V\cdot\varphi\|_{C^0} \leq C(\omega_0, V)$.

<u>Pf</u>: Write $V = Y + iJY$, Y real v.f. $(JY)(\varphi) = 0$

① $\omega = \omega_0 + i\partial\bar\partial\varphi > 0 \Rightarrow \omega(Y, JY) > 0$

$\Rightarrow i\partial\bar\partial\varphi(Y, JY) > -\omega_0(Y, JY) = -|Y|^2_{\omega_0}$

$= Y\cdot Y\cdot\varphi$

② σ_t the flow generated by Y. $\begin{cases} \frac{\partial}{\partial t}\sigma_t = Y\circ\sigma_t \\ \sigma_0 = id \end{cases}$

$x \in X$ any point

consider $h(t) = (Y\cdot\varphi)(\sigma_t(x))$

$$h'(t) = \frac{d}{dt}(Y \cdot \varphi)(\sigma_t(x)) = Y \cdot Y \cdot \varphi(\sigma_t(x))$$

$$\geq -|Y|^2_{\omega_0}(\sigma_t(x)) = -k'(t)$$

③ $k(t) = \theta_{V,\omega_0}(\sigma_t(x))$ smooth & bdd function

$$k'(t) = Y \cdot \theta_{V,\omega_0}(\sigma_t(x)) = |Y|^2_{\omega_0}$$

④ $(k+h)(t) \nearrow$ in t

$$\Rightarrow \quad h(0) + k(0) \leq \lim_{t \to +\infty} k(t) + h(t)$$

$$= \lim_{t \to \infty} k(t)$$

b/c $\lim_{t \to \infty} |Y|^2_{\omega_0}(\sigma_t(x)) = 0$.

$\square$

• modified Mabuchi K-energy.

$$\mu_{V,\omega_0} : \mathcal{H}_V \longrightarrow \mathbb{R} \qquad \text{via variational formula}$$

$$\delta \mu_{V,\omega_0} = -\frac{1}{V_{\omega_0}} \int_X (\delta\varphi) \cdot \left(R_{\omega_\varphi} - n - \nabla_j V^j - V(f_{V,\omega_\varphi}) \right) e^{\theta_{V,\omega_\varphi}} \omega_\varphi^n$$

• A KR solition is a critical point for μ_{V,ω_0}

- denote $\mathfrak{aut}_V(X) = \{ Y \in H^0(X, TX) \mid L_V Y = [V, Y] = 0 \}$

  a Lie sub-algebra.

  $Aut_V(X) \subset Aut(X)$, s.t. $Lie(Aut_V(X)) = \mathfrak{aut}_V(X)$.

- $\forall\ \sigma \in Aut_V(X)$, $\exists!\ \sigma.0 \in \mathcal{H}_V$, $\sup_X (\sigma.0) = 0$

  $$\sigma^* \omega_0 = \omega_0 + i\partial\bar{\partial}(\sigma.0)$$

- $\omega \in \mathcal{K}_V$, $\omega = \omega_0 + i\partial\bar{\partial}\varphi$

  $$\sigma^*\omega = \omega_0 + i\partial\bar{\partial}(\sigma.0) + i\partial\bar{\partial}(\varphi \circ \sigma)$$

  $$= \omega_0 + i\partial\bar{\partial}\varphi_\sigma$$

  $\varphi_\sigma = \sigma.0 + \varphi \circ \sigma + C$ s.t. $\sup_X \varphi_\sigma = 0$.

Moser-Trudinger inequality (Darvas-Rubinstein).

Assume $\exists$ a KR soliton w/ V soliton v.f.

then $\exists$ $\varepsilon_0 = \varepsilon_0(X, \omega_0, V) > 0$, $C > 0$ s.t

$$C_0 \geq \mu_{V,\omega_0}(\varphi) \geq \varepsilon_0 \inf_{\sigma \in \mathrm{Aut}_V(X)} I_{\omega_0}(\varphi_\sigma) - C$$

for some $\sigma \in \mathrm{Aut}_V(X)$ $\geq \varepsilon_0 I_{\omega_0}(\varphi_\sigma) - 1 - C$

known fact: $\omega = \omega_0 + i\partial\bar\partial\varphi$, KRF. $\Rightarrow \frac{d}{dt}\mu_{V,\omega_0}(\varphi) \leq 0$

$$\Rightarrow I_{\omega_0}(\varphi_\sigma) \leq C \quad \text{for some } \sigma \in \mathrm{Aut}_V(X)$$

$$\sigma = \sigma(\varphi)$$

- uniform estimates of Perelman (see. Sesum-Tian)

$$\begin{cases} \frac{\partial \omega}{\partial t} = - Ric(\omega) + \omega \\ \omega|_{t=0} = \omega_0 \in K_V. \end{cases}$$

then $\exists\ C = C(X, \omega_0) > 0$. s.t. u_ω = normalized Ricci potential

$$\|u_\omega\|_{C^0} + \|\nabla u_\omega\|_{C^0(\omega)} + \|\Delta_\omega u_\omega\|_{C^0} \leq C$$

$\omega = \omega(t)$, $\forall\ t \in [0, \infty)$.

<u>key lemma</u>: $\forall t \in [0, \infty)$, $\omega = \omega(t) = \omega_0 + i\partial\bar{\partial}\varphi$, KRF

$\exists\ C = C(X, \omega_0, V) > 0$ (indep of t)

$\sigma = \sigma_t \in Aut_V(X)$, s.t. $\|\varphi_\sigma\|_{C^{0,\alpha}(X,\omega_0)} \leq C$.

(Recall $\begin{cases} \sigma^*\omega = \omega_0 + i\partial\bar{\partial}\varphi_\sigma \\ \sup \varphi_\sigma = 0 \end{cases}$) $\alpha \in (0,1)$

<u>Proof</u>: ① We have known $I_{\omega_0}(\varphi_\sigma) = \int \varphi_\sigma(\omega_0^n - \omega_{\varphi_\sigma}^n) \leq C$

$$\Rightarrow \int (-\varphi_\sigma)\,\omega_{\varphi_\sigma}^n \leq C$$

Trudinger type inequality $\Rightarrow \int e^{\beta(-\varphi_\sigma)^{\frac{n+1}{n}}}\omega_0^n \leq C$

$$\Rightarrow \int \left(e^{-\varphi_\sigma}\right)^p \omega_0^n \leq C$$

$p > 1$.

② the MA equation of $\tilde{\omega} = \sigma^*\omega = \omega_0 + i\partial\bar\partial\varphi_\sigma$

$$(\omega_0 + i\partial\bar\partial\varphi_\sigma)^n = e^{-\varphi_\sigma + u_{\tilde{\omega}} - u_{\omega_0} + c}\,\omega_0^n$$

$(c \le 0)$

by definition $\sigma^* u_\omega = u_{\sigma^*\omega} = u_{\tilde{\omega}}$

$\Rightarrow$ $u_{\tilde{\omega}}$ bdd by Perelman's estimates.

$$\Rightarrow \quad (\omega_0 + i\partial\bar\partial\varphi_\sigma)^n \le C e^{-\varphi_\sigma}\,\omega_0^n$$

RHS $\in L^p(X, \omega_0^n)$,

then Kolodziej's Hölder estimate

$$\Rightarrow \quad \|\varphi_\sigma\|_{C^{0,\alpha}(X,\omega_0)} \le C.$$

Observation. $C^{-1}\omega_0^n \le (\omega_0 + i\partial\bar\partial\varphi_\sigma)^n \le C\omega_0^n$ $\square$

· for any FIXED $t \in (0,\infty)$, let $\sigma \in \mathrm{Aut}_r(X)$ be as in lemma

$$\sigma \updownarrow \omega = \omega(t)$$

$$\tilde{\omega} = \sigma^* \omega = \omega_0 + i\partial\bar{\partial}\varphi_\sigma$$

$$\begin{cases} (\omega_0 + i\partial\bar{\partial}\varphi_\sigma)^n = e^{-\varphi_\sigma + u_{\tilde{\omega}} \overbrace{- u_{\omega_0} + C}^{\text{bdd terms}}} \omega_0^n \\ \sup_X \varphi_\sigma = 0 \end{cases}$$

$$\| u_{\tilde{\omega}} \|_{C^0} + \| \nabla u_{\tilde{\omega}} \|_{C^0(\tilde{\omega})} + \| \Delta_{\tilde{\omega}} u_{\tilde{\omega}} \|_{C^0} \le C$$

by the C^1, C^2, $C^{2,\alpha}$ estimates we conclude that

$$\| \varphi_\sigma \|_{C^{k,\alpha}(X,\omega_0)} \le C.$$

uniformly bdd

$$\Rightarrow \quad \lambda_{\widetilde{\omega}} = \lambda_{\omega} \geq c_0 > 0 \quad \text{some uniform } c_0$$

$$\Rightarrow \quad \inf_{t \in [0,\infty)} \lambda_{\omega(t)} \geq c_0 > 0 \qquad (*)$$

---

here

$$\lambda_{\omega} = \inf_{0 \neq Z \perp H^0(X,TX)} \frac{\|\bar{\partial} Z\|^2_{L^2(e^{\theta_{v,\omega}}\omega^n)}}{\|Z\|^2_{L^2(e^{\theta_{v,\omega}}\omega^n)}}$$

$$\|Z\|^2_{L^2(e^{\theta_{v,\omega}}\omega^n)} = \int |Z|^2_{\omega} e^{\theta_{v,\omega}} \omega^n$$

(Recall by Zhu's lemma $\|\theta_{v,\omega}\|_{C^0} \leq C$)

Completion of the proof:

· Tian - Zhu : $\inf_{\varphi \in \mathcal{H}_V} \mu_{\omega_0, V}(\varphi) > -\infty$ (**)

[PSSW]

by Phong - Song - Sturm - Weinkove, the "modified"

KR flow $\quad \eta_t^* \omega(t) \xrightarrow{C^\infty} \omega_{KS}'$

here $\quad \eta_t = \exp(tV) \in \mathrm{Aut}_V(X)$, $\quad \square$

Rough idea in [PSSW]:

the flow $\hat{\omega}_t = \eta_t^* \omega(t)$ satisfies

modified KRF

$$\frac{\partial \hat{\omega}_t}{\partial t} = -Ric(\hat{\omega}_t) + \hat{\omega}_t + L_V \hat{\omega}_t$$

write $\hat{\omega} = \hat{\omega}_t$

consider $$Y_V(t) = \int |\nabla f_{V,\hat{\omega}}|^2_{\hat{\omega}} \, e^{\theta_{V,\hat{\omega}}} \hat{\omega}^n$$

weighted energy of the modified Ricci potential.

$$\frac{d}{dt} Y_V(t) \le -2\lambda_{\hat{\omega}}(t) Y_V(t) - \underbrace{2\lambda^{(t)}_{\hat{\omega}} Fut_V(\pi(\nabla f_{V,\hat{\omega}}))}_{=0}$$

$+$ controlled terms.

$\Rightarrow \frac{d}{dt} Y_V(t) \le -\kappa Y_V(t) \Rightarrow Y_V(t) \le C e^{-\kappa t}$

$\Rightarrow$ exponential decay of $\hat{\omega}_t$

limit = soliton.
by modified K-energy
or W-functional

$\square$