

Lecture 3:

Compactness of Kahler-Ricci solitons

based on works of

G. - Phong - Song - Sturm

Phong - Song - Sturm

& Z. Zhang

• X compact Kähler manifold, $\dim_{\mathbb{C}} X = n$

• X Fano, i.e. $c_1(X) > 0$, $-K_X$ ample

• a metric $\omega = \omega_{KS} \in C_1(X)$ is a (shrinking)

Kähler-Ricci soliton, if

$$\lambda = \begin{cases} +1 & \text{shrinking} \\ 0 & \text{steady} \\ -1 & \text{expanding} \end{cases}$$

$$\left\{ \begin{array}{l} \text{Ric}(\omega) = +\lambda\omega + L_V\omega \end{array} \right.$$

V a holomorphic vector field

• let $u = u_\omega$ be the Ricci potential of $\omega = \omega_{KS}$
normalized,

ie. $\int_X e^{-u} \omega^n = \int \omega^n = V_g$

$$\text{Ric}(\omega) = \omega - i\partial\bar{\partial}u_\omega$$

$$g \longleftrightarrow \omega$$

$$V = - \nabla_g u$$

soliton equation

$$\left\{ \begin{array}{l} \text{Ric}_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}} \end{array} \right. ,$$

$$\left\{ \begin{array}{l} u_{ij} = u_{\bar{i}\bar{j}} = 0 \end{array} \right. , \text{ie } V \text{ is holomorphic.}$$

- the Futaki invariant w.r.t. V is

$$\begin{aligned} \text{Fut}_X(V) &= \frac{1}{V_g} \int_X |V|_g^2 \omega^n \\ &= \frac{1}{V_g} \int_X |\nabla u|^2 \omega^n \end{aligned}$$

- one goal: $\text{Fut}_X(V)$ is uniformly bounded,
 $\forall (X, \omega, V) \in \text{KR}(n)$.

$$\text{KR}(n) = \left\{ (X, \omega, V) \mid \begin{array}{l} \dim X = n \quad c_1(X) > 0 \\ \omega \text{ is a KR soliton w/} \\ \text{soliton vector field } V \end{array} \right\}$$

Theorem : $\exists C = C(n) > 0$, s.t. $\forall (X, \omega, \nu) \in \mathcal{KR}(n)$

$$\left\{ \begin{array}{l} \text{Fut}_X(\nu) \leq C \\ \|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R_\omega\|_{C^0} \leq C. \\ \text{diam}(X, g) \leq C \end{array} \right.$$

RK: this is trivial if $\nu=0$

$$\frac{1}{C(n)} \leq \frac{\text{Vol}_g(B_g(p, r))}{r^{2n}} \leq C(n)$$

RK: Ivey, Z. Zhang $R_\omega > 0$

Corollary: let $(X_i, \omega_i, V_i) \in \text{KR}(n)$, up to subsequence.

$$(X_i, g_i) \xrightarrow{\text{dGH}} (X_\infty, d_\infty) \quad \leftarrow \begin{array}{l} \text{a compact metric} \\ \text{length space} \end{array}$$

① X_∞^{sing} a closed subset of $\text{codim} \geq 4$ +H.S.M. $X_\infty = X_\infty^{\text{reg}} \cup X_\infty^{\text{sing}}$

$$\textcircled{2} (X_i, g_i, V_i) \xrightarrow{C_{\text{loc}}^\infty(X_\infty^{\text{reg}})} (X_\infty^{\text{reg}}, g_\infty, V_\infty)$$

(g_∞, V_∞) a Kähler-Ricci soliton on X_∞^{reg}

$$\textcircled{3} \overline{(X_\infty^{\text{reg}}, g_\infty)} \stackrel{\text{isometric}}{=} (X_\infty, d_\infty)$$

X_∞ normal projective variety w/ Klt sing

$|V_\infty|_{g_\infty} \leq C$ & V_∞ extends to a hol v.f. on X_∞

• Preliminaries :

Fact: ① smooth Fano manifolds of dim n have only finitely many deformation types, hence

$$c_1(X)^n = \int_X \omega^n = (-K_X)^n \in C(n).$$

② $\frac{1}{2\pi} c_1(X)$ is integral, so

$$c_1(X)^n \geq c(n) > 0$$

\Rightarrow the total volume

$$0 < c(n) \leq V_g \leq C(n) < \infty$$

Definition: (α -invariant), $0 < \omega \in C_1(X)$

$$\alpha(X) = \alpha_\omega(X) = \sup \left\{ \alpha > 0 \mid \int_X e^{-\alpha(\varphi - \sup_X \varphi)} \omega^n \leq C_\alpha \right. \\ \left. \exists C_\alpha > 0 \text{ s.t.} \right.$$

$$\forall \varphi \in \text{PSH}(X, \omega) \cap C^\infty(X)$$

Rk: $\alpha_\omega(X)$ is independent of the choice of $\omega \in C_1(X)$
& holomorphic invariant

• $\alpha(X) > 0$ by Hörmander, Tian.

Def; (log canonical).

X proj normal, $\Delta \geq 0$ effective \mathbb{Q} -Cartier divisor w/
coefficients ≤ 1

the pair (X, Δ) is log canonical (l.c.) if

\exists a log resolution of (X, Δ) ; $\pi: Y \rightarrow X$

$\pi^{-1}(\Delta) \cup \text{exc}(\pi)$ is a divisor of normal crossing

$$K_Y = \pi^*(K_X + \Delta) + \sum a_i F_i$$

$$\mathbb{Q} \ni a_i \geq -1$$

(if $a_i > -1$, klt singularity)

Def: X , proj normal variety. D \mathbb{Q} -Cartier div.

log canonical threshold of D is

$$\text{lct}(X, D) = \sup \{ t \mid (X, tD) \text{ is log canonical} \}$$

Theorem: ① (Demailly) X Fano

$$\alpha(X) = \text{lct}(X, -K_X)$$

$$= \inf_{m \in \mathbb{Z}_{>0}} \sup_{D \in |-mK_X|} \text{lct}(X, \frac{1}{m}D)$$

$$\frac{D}{m} \in |-K_X|$$

$$\left. \begin{array}{l} \Rightarrow \alpha(X) \\ \geq \delta(n) \\ > 0 \end{array} \right\}$$

② (Birka) $\exists \delta(n) > 0$. s.t.

$$\text{lct}(X, -K_X) \geq \delta(n)$$

Def: greatest Ricci lower bound, X Fano manifold.

$$R(X) = \sup_{t \in (0, 1]} \left\{ t > 0 \mid \exists \omega \in C_1(X), \text{ s.t. } \left. \begin{array}{l} \text{Ric}(\omega) > t \omega \end{array} \right\}$$

Lemma: $R(X) \geq \min\left(1, \frac{n+1}{n} \alpha(X)\right)$ if $\alpha(X) > \frac{n}{n+1}$ then $R(X) = 1$, i.e. $\exists KE$

outline of the proof: consider the "continuity method"

Fix a metric $\omega_0 \in C_1(X)$

$$(MA_t) \begin{cases} (\omega_0 + i\partial\bar{\partial}\varphi_t)^n = e^{-t\varphi_t + F_0} \omega_0^n \\ \text{Ric}(\omega_0) - \omega_0 = i\partial\bar{\partial}F_0 \end{cases}$$

① $(MA)_t$ can be solved up to $\min\left(\frac{n+1}{n}\alpha(x), 1\right)$

$$P_{iz}(W_t) = t W_t + (1-t) W_0, \quad W_t = W_0 + i \bar{\sigma} \bar{\sigma}_t^P$$

(see Tian's ^{$t W_t$} 1987 paper) $\Rightarrow R(x) \geq \min\left(1, \frac{n+1}{n}\alpha(x)\right)$

② (Szekelyhidi) $[0, R(x))$ maximal existence interval
of $(MA)_t$

$$\Rightarrow R(x) \geq \min\left(\frac{n+1}{n}\alpha(x), 1\right)$$

□

$$\alpha(x) \geq \delta(n) \Rightarrow R(x) \geq \varepsilon(n)$$

Corollary ; $\exists \varepsilon(n) > 0$, s.t. \forall Fano manifold X ,

$\exists \hat{\omega} \in C_1(X)$, s.t.

$$\text{Ric}(\hat{\omega}) \geq \varepsilon(n) \hat{\omega}.$$

Prf: let $G \subset \text{Aut}(X)$ be any compact subgp.

averaging $\hat{\omega}$ over G , we may take $\hat{\omega}$ to

be G -invariant.

In particular, $\hat{\omega}$ is $\text{Im}V$ -invariant

if $(X, \omega, V) \in \text{KR}(n)$.

Perelman's W- μ -functionals:

$$W_x(g, f) = \frac{1}{V_g} \int_x (R_g + |\nabla f|^2 - f - n) e^{-f} dV_g$$

$$\left[f \in C^\infty(x) \quad \frac{1}{V_g} \int_x e^{-f} dV_g = 1 \right]$$

μ -functional

$$\mu_x(g) = \min \left\{ W_x(g, f) \mid \begin{array}{l} f \in C^\infty \\ \int_x e^{-f} dV_g = V_g \end{array} \right\}$$

Well-known fact:

$\mu_x(g_t)$ is \nearrow along (normalized) KR flow,

Lemma : $\exists A(n) > 0$, s.t. $\mu_x(\hat{g}) \geq -A(n)$.

Proof : Recall $\text{Ric}(\hat{g}) \geq \varepsilon(n)\hat{g}$.

• Myers' theorem : $D = \text{diam}(X, \hat{g}) \leq \frac{2n-1}{\sqrt{\varepsilon(n)}} = C(n)$

• $C(n) \leq \text{Vol}(X, \hat{g}) \leq C(n)$

• Bishop-Gromov volume comparison

$$\frac{\text{Vol}_{\hat{g}}(B_{\hat{g}}(p, r))}{r^{2n}} \geq \frac{\text{Vol}_{\hat{g}}(B(p, D))}{D^{2n}} \geq K(n) > 0$$

\uparrow
noncollapsing

• Croke's theorem: Sobolev constant $\sqrt{C_S}$ of (X, \hat{g}) is bounded by $C(n)$

• $\forall f \in C^\infty, \int_X e^{-f} dV_g = V_g.$

Write $\phi = e^{-f/2}, \quad \frac{1}{V_g} \int \phi^2 dV_g = 1$

by Jensen's inequality

$$\frac{1}{V_g} \int_X \phi^2 \log \phi^{\frac{2}{n-1}} dV_g \leq \log \frac{1}{V_g} \int_X \phi^{2 + \frac{2}{n-1}} dV_g$$

$$\leq \log \left[C_S \left(\int_X |\nabla \phi|^2 + \phi^2 \right) \right]$$

$$\leq \frac{4}{n-1} \frac{1}{V_g} \int_X |\nabla \phi|^2 dV_g + C(n)$$

$$\Rightarrow \frac{1}{V_g} \int_X \phi^2 \log \phi^2 dV_g \leq \frac{4}{V_g} \int |\nabla \phi|^2 dV_g + C(n)$$

$$\begin{aligned} \Rightarrow W(\hat{g}, f) &= \frac{1}{V_g} \int_X \left(\underbrace{R_{\hat{g}} \phi^2}_{\geq 0} + 4|\nabla \phi|^2 - \phi^2 \log \phi^2 \right) dV_{\hat{g}} - n \\ &\geq -C(n) \end{aligned}$$

□

Recall: $R_{\hat{u}}(\hat{u}) \geq \varepsilon(n) \hat{u}$

$$\Rightarrow R_{\hat{g}} \geq n \varepsilon(n) > 0.$$

Corollary: $\forall (X, \omega, V) \in KR(n)$.

$$\mu_X(g) \geq -A(n)$$

Proof: ^① we consider the KR flow

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \omega_t \\ \omega_t|_{t=0} = \hat{\omega} \in G(X) \end{cases}$$

$\hat{\omega}$ is ImV invariant

- since $\omega = \omega_{KS}$ is a KR soliton, by smooth convergence of $\eta_t^* \omega_t$, $\eta_t = \exp(tV)$.

$$\eta_t^* \omega_t \xrightarrow{C^\infty} \omega'_{KS}$$

$$\textcircled{2} \quad \mu_x(g_t) = \mu_x(\tau_t^* g) \nearrow \text{in } t$$

&

$$\lim_{t \rightarrow \infty} \mu_x(\tau_t^* g) = \underbrace{\mu_x(g'_{KS}) = \mu_x(g_{KS})}_{\substack{\text{by uniqueness of KR solitons} \\ \text{Tian-Zhu}}} \text{ ie } g'_{KS} = \sigma^* g_{KS}$$

$$\Rightarrow \mu_x(g_{KS}) \geq \mu_x(\hat{g}) \geq -A(n)$$

□

Lemma : $\forall (X, \omega, \nu) \in \mathcal{KR}(n)$, $\exists C(n) > 0$

$$\begin{cases} \|u\|_{C^0} + \|\nabla u\|_{C^0(\omega)} + \|R_\omega\|_{C^0} \leq C \\ \text{diam}(X, \omega) \leq C \end{cases}$$

\Rightarrow Theorem 1.

Recall : $V = -\nabla u$

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}}$$

$$u_{i\bar{j}} = u_{\bar{i}j} = 0$$

$$\int e^{-u} \omega^n = \int \omega^n.$$

• Recall the well-known identities :

(g, ν) Kähler-Ricci soliton. then

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}} \quad \& \quad u_{i\bar{j}} = 0$$

$$\textcircled{1} \quad R = n - \Delta u \quad \checkmark$$

$$\textcircled{2} \quad R + |\nabla u|_g^2 - u = \text{const.}$$

Pf of ② : $R_{i\bar{j},j} = g_{i\bar{j},j} - u_{i\bar{j}j} = - (u_{i\bar{j}j} - u_k R_{i\bar{k}j\bar{j}})$

// Bianchi

$R_{,i}$

$$= u_k R_{i\bar{k}}$$

$$= u_k (g_{i\bar{k}} - u_{i\bar{k}})$$

$$\Rightarrow \nabla_i (R + |\nabla u|^2 - u) = 0$$

$$= u_i - \nabla_i |\nabla u|^2$$

Integrating ② against $\frac{1}{V_g} e^{-u} \omega^n$ $\frac{1}{V_g} \int e^{-u} \omega^n = 1$

$$\begin{aligned} \text{const} &= \frac{1}{V_g} \int (R + |\nabla u|^2 - u) e^{-u} \omega^n \\ &= \frac{1}{V_g} \int (R + \Delta u - u) e^{-u} \omega^n \\ &= n - \frac{1}{V_g} \int u e^{-u} \omega^n \end{aligned}$$

⇒ $R + |\nabla u|^2 - u = n - \frac{1}{V_g} \int_X u e^{-u} \omega^n$

entropy of e^{-u}

Fact: If $(\bar{\omega}^g, V)$ is a KR soliton, u is a minimizer of $W(g, f)$

$$\Rightarrow \mu_x(g) = W_x(g, u)$$

$$dV_g = \omega^n$$

$$= \frac{1}{V_g} \int_X (R + |u|^2 + u - n) e^{-u} dV_g$$

$$= \frac{1}{V_g} \int_X 2u e^{-u} dV_g - \frac{1}{V_g} \int_X u e^{-u} dV_g$$

$$= \frac{1}{V_g} \int_X u e^{-u} dV_g$$

$$\mu_x(g) \geq -A(n)$$

$$\Rightarrow \frac{1}{V_g} \int_x u e^{-u} \omega^n \geq -A(n).$$

the identity

$$R + |\nabla u|^2 = u - \frac{1}{V_g} \int u e^{-u} dV_g + n$$

$$\Rightarrow \min_x u \geq -A(n) - n$$

- use Perelman's uniform estimates (Seum-Tian)

Recall: $\omega = \omega_{KS}$ is a Kähler soliton, it can be viewed as a solution to a Kähler flow.

$$\text{let } \sigma_t = \exp(-tV), \quad \omega_t = \sigma_t^* \omega_{KS}$$

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \omega_t \\ \omega_t|_{t=0} = \omega_{KS}. \end{cases}$$

$$u_{\omega_t} = u_{\omega} \circ \sigma_t = \sigma_t^* u_{\omega}, \quad \text{normalized Ricci potential.}$$

$$\sqsubseteq u_t$$

Apply Maximum principle to the functions as in Perelman

$$\frac{|\nabla u_t|_{\omega_t}^2}{u_t - \min_x u_t + 1} \leq C(n), \quad \frac{-\Delta_{\omega_t} u_t}{u_t - \min_x u_t + 1} \leq C(n)$$

these two functions are uniformly bounded.

then from Perelman's techniques & $\mu_X(g_t) \geq -A(n)$

$$\Rightarrow \max_x u_t \leq C(n) \quad \& \quad \text{diam}(X, g_t) \leq C(n).$$

$$\Rightarrow |\nabla u_t|^2 \leq C(n)$$

$$\& \quad -\Delta_{\omega_t} u_t \leq C(n), \quad \square$$

$$"R_{\omega_t} = n$$

• Gromov-Hausdorff (pre) compactness : (Z. Zhang),

idea : conformal transformations.

let $(X, \omega, V) \in KR(n)$, consider

$$\tilde{g} = e^{-\frac{u}{n-1}} g$$

u the Ricci potential
as before

\tilde{g} a Riemannian metric (not Kähler)

since u & $|\nabla u|$ are bounded

\tilde{g} is equivalent to g in $C^1(X, g)$

$$C(n)g \leq \tilde{g} \leq C(n)g$$

well-known formula: the Ricci curvatures of \tilde{g} & g are related by $(i, j = 1, 2, \dots, 2n)$
real coordinates indices)

$$\tilde{R}_{ij} = \underbrace{R_{ij} + \nabla_i \nabla_j u}_{= g_{ij} \text{ by soliton equation.}} + \frac{1}{2(n-1)} \frac{u_i u_j}{g} - \frac{1}{2(n-1)} \underbrace{(|\nabla u|_g^2 - \Delta_g u)}_{\text{bounded terms}} g_{ij}$$

$\Rightarrow \exists C(n) > 0, \epsilon > 0$

$$|\tilde{\text{Ric}}|_{\tilde{g}} \leq C(n)$$

$$\text{diam}(X, \tilde{g}) \leq C(n), \quad \frac{1}{C(n)} \leq \text{vol}(X, \tilde{g}) \leq C(n)$$

so applying Cheeger-Colding-Naber's theorem to (X_i, \tilde{g}_i)

& the C^1 equivalence of \tilde{g}_i & g_i , we conclude the GH seq. convergence of (X_i, g_i) & structure of the limit space

In sum: $(X_i, g_i) \xrightarrow{d_{GH}} (X_\infty, d_\infty)$

• $\text{codim } X_\infty^{\text{sing}} \geq 4$, X_∞^{reg} open & dense

• $g_i \xrightarrow{C_{\text{loc}}^\infty(X_\infty^{\text{reg}})} g_\infty$, g_∞ a KR soliton on X_∞^{reg}

• $\overline{(X_\infty^{\text{reg}}, g_\infty)} \stackrel{\text{isometric}}{=} (X_\infty, d_\infty)$

- claim: X_∞ is a normal projective variety.
(Donaldson - Sun)

main tool: partial C^0 estimates.

- $k \in \mathbb{Z}_{>0}$. $(X, \omega) \in \text{KR}(n)$, $\omega \in C_1(X)$

- Fix an hermitian metric h on $-K_X$ s.t.

$$\textcircled{H}_h = -i\partial\bar{\partial} \log h = \omega \in C_1(-K_X)$$

(ω is a KR soliton)
 $h = e^{-u} \omega^n$)

- denote

$$\left\{ \begin{array}{l} -K_X^\# = (-K_X)^{\otimes k} \\ h^\# = h^{\otimes k} \\ \omega^\# = k\omega \\ L^{2,\#} = L^2(X, \omega^\#) \end{array} \right.$$

$$\textcircled{H}_{h^\#} = k\omega = \omega^\#$$

define an inner product on $H^0(X, (-K_X)^\#) \ni s, \tilde{s}$

$$(s, \tilde{s}) = \int_X \langle s, \tilde{s} \rangle_{h^\#} (\omega^\#)^n = k^n \int_X \langle s, \tilde{s} \rangle_{h \otimes k} \omega^n$$

the Bergman kernel (diagonal) is defined

$$\rho_{X,k}(z) = \sum_{j=0}^{N_k} |s_j(z)|_{h^\#}^2 = \sup \left\{ |s|_{h^\#}^2(z) \mid \|s\|_{L^{2,\#}}^2 = 1 \right\}$$

where $\{s_j\}_{j=0}^{N_k}$ is an orthonormal basis of $H^0(X, (-K_X)^\#)$

$\rho_{X,k}$ well-defined, indep of the choice of o.b.

Theorem (partial C°):

$$\exists k = k(n) \in \mathbb{Z}_{>0}, \quad \varepsilon(n) > 0, \quad s-t$$

$\forall (X, \omega, V) \in \mathcal{KR}(n)$, the Bergman kernel of $H^0(X, -kK_X)$

$$\inf_{z \in X} P_{X,k}(z) \geq \varepsilon(n) > 0$$

- main idea: the structure of tangent cone of (X_∞, d_∞)
& Hörmander L^2 -technique to solve $\bar{\partial}$ -equations.

One key step is to show mean value type inequality
for $s \in H^0(X, -kK_X)$

Prop : $\exists C = C(n) > 0$, s.t. $\forall (X, g) \in KR(n)$
 $s \in H^0(X, -kK_X)$

①

$$\|s\|_{L^\infty, \#} \leq C \|s\|_{L^2, \#}$$

② $\|\nabla s\|_{L^\infty, \#} \leq C \|s\|_{L^2, \#}$

③ $\lambda_1(\Delta_{\bar{\partial}, u}^\#) \geq \frac{1}{C(n)} > 0$

where $\Delta_{\bar{\partial}, u}^\# = \bar{\partial} \bar{\partial}_u^* + \bar{\partial}_u^* \bar{\partial} : (-K_X)^{\otimes k} \otimes \Omega^{0,1}(X) \rightarrow$

$\bar{\partial}_u^*$ adjoint of $\bar{\partial}$. w.r.t. $\frac{1}{V_g} \int_X |\sigma|_{\omega^\#, h^\#}^2 e^{-u} (\omega^\#)^n$

Proof: $\forall (x, g) \in \mathcal{KR}(n)$

observe: $\tilde{g} = e^{-\frac{u}{n-1}} g$

$C_S(\tilde{g})$ uniformly bounded. $\Rightarrow C_S(g) \leq C(n)$.

↑
Sobolev constant of \tilde{g}

Sobolev constant scaling invariance $\Rightarrow C_S(g^\#) \leq C(n)$

For a fixed k , omit $\#$ in $h^\#, g^\# \dots$

$$\textcircled{1} \quad s \in H^0(X, -kK_X), \quad \Delta = \Delta_{\omega^\#}$$

$$\Delta |s|_{h^\#}^2 = |\nabla s|_{h^\#}^2 - |s|_{h^\#}^2$$

$$\text{Kato inequality} \Rightarrow \quad \Delta |s|_{h^\#} \geq -|s|_{h^\#}$$

$$\text{Moser iteration} \Rightarrow$$

$$\|s\|_{L^{\infty, \#}} \leq C(n) \|s\|_{L^{2, \#}}$$

□

② from the Bochner formula

$$\begin{aligned} \Delta |\nabla s|^2 &= |\nabla \nabla s|^2 - 2|\nabla s|^2 + n|s|^2 + \text{Ric}(\nabla s, \overline{\nabla s}) \\ &= |\nabla \nabla s|^2 - |\nabla s|^2 + n|s|^2 - \underbrace{i\partial\bar{\partial}u(\nabla s, \overline{\nabla s})} \end{aligned}$$

Multiplying both sides by $(|\nabla s|^2)^p$ $p \geq 1$

& running the usual process of Moser iteration.
unbounded term

$$\begin{aligned} &\int -i\partial\bar{\partial}u(\nabla s, \overline{\nabla s})_{h^\#} |\nabla s|^{2p} \omega^n \\ &= \int u_{\bar{j}} [(\nabla_j s \overline{\nabla_i s})_{h^\#} |\nabla s|^{2p}]_i \omega^n \sim \int \nabla \nabla s * (\nabla s)^{2p+1} \\ &\quad + (\nabla s)^{2p+2} \quad \square \end{aligned}$$

③ Recall

$$(\sigma, \sigma)_u = \int_X |\sigma|_{h^\#, \omega^\#}^2 e^{-u} (\omega^\#)^n$$

$$\sigma \in (-kK_X) \otimes \Omega^{0,1}(X)$$

let D_u be the covariant derivative on $(-kK_X) \otimes \Omega^{0,1}$

w.r.t. $g^\#, & h^\# e^{-u}$

$$\begin{cases} \Delta_{\bar{\partial}, u} = \bar{\partial} \bar{\partial}_u^* + \bar{\partial}_u^* \bar{\partial} \\ \Delta_{D_u} = - (D_u)^{\bar{j}} (D_u)_{\bar{j}} \end{cases}$$

Bochner-Kodaira formula

$$\begin{aligned} (\Delta_{\bar{\partial}, u} \sigma)_{\bar{j}} &= (\Delta_{D_u} \sigma)_{\bar{j}} + \frac{1}{k} \sigma_{\bar{j}} (g^{\#})^{i\bar{i}} (g^{\#}_{i\bar{j}} + u_{i\bar{j}} + \text{Ric}(g)_{i\bar{j}}) \\ &= (\Delta_{D_u} \sigma)_{\bar{j}} + (1 + \frac{1}{k}) \sigma_{\bar{j}} \end{aligned}$$

$$\begin{aligned} \Rightarrow (\Delta_{\bar{\partial}, u} \sigma, \sigma)_u &\geq (1 + \frac{1}{k}) \int |\sigma|^2 e^{-u} (\omega^{\#})^n \\ &\geq c(n) \|\sigma\|_{L^2, \#}^2 \end{aligned}$$

Bergman projection, $\forall \sigma \in -kK_X$. □

$s = \sigma - \tau$ is hol. $\tau = \bar{\partial}_u^* (\Delta_{\bar{\partial}, u})^{-1} \bar{\partial} \sigma$ □
w/ uniform estimates,