A MULTIPLICITY FORMULA OF K-TYPES

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ABSTRACT. In this paper, by proving a simple local trace formula for real reductive groups, we prove a multiplicity formula of K-types for all irreducible representations of real reductive groups. This multiplicity formula expresses the K-characters in terms of the Harish-Chandra characters.

CONTENTS

1. Introduction

Let G be a connected real reductive group and $K = G^\theta$ be a maximal compact subgroup of G where θ is a Cartan involution. Let π be an irreducible smooth representation (i.e. an irreducible Casselman-Wallach representation) of G and τ be an irreducible representation of K, the goal of this paper is to study the multiplicity of K-types

$$m(\pi, \tau) = \dim(\mathrm{Hom}_K(\pi, \tau)).$$

By Section 8.2 of [10], there exists a continuous linear functional $\Theta_{K,\pi}$ on $C^\infty(K)$, called the K-character, such that

$$\Theta_{K,\pi}(f) = \sum_{\tau \in Irr(K)} m(\pi, \tau) \int_K \theta_\tau(k) f(k) dk$$

where $\theta_\tau(k) = \mathrm{tr}(\tau(k))$ is the character of τ. In particular, we have $m(\pi, \tau) = \Theta_{K,\pi}(\bar{\theta}_\tau)$. By Theorem 8.2.2 of [10], we know that the K-character $\Theta_{K,\pi}$ is equal to the Harish-Chandra character θ_π of π on the regular semisimple elements $K \cap G_{reg}$. In our previous paper [13], we proposed a conjectural multiplicity formula for general spherical pairs. A special case of our conjectural multiplicity formula is a conjectural multiplicity formula for K-types (note that (G, K) is a spherical pair). The main result of this paper is to prove this conjectural multiplicity formula of K-types. This multiplicity formula also implies a formula of the K-character in terms of the Harish-Chandra character.

1.1. **Main results.** Let $\mathcal{S}(G, K)$ be the set of K-conjugacy classes ($x \in K$ is any element in the conjugacy class) such that the pair (G_x, K_x) is a minimal spherical pair. Here G_x (resp. K_x) is the neutral component of the centralizer of x in G (resp. K). We refer the reader to Section 2.6 of [13] for the definition of minimal spherical pair. In the case of K-types, the pair (G_x, K_x) is a minimal spherical pair if and only if G_x is split modulo the center (i.e. the difference between the rank of G_x and the split rank of G_x is equal to the difference between the rank of Z_{G_x} and the split rank of Z_{G_x} where Z_{G_x} is the center of G_x). In Section 3 we will define a measure on the set $\mathcal{S}(G, K)$. Let θ be a quasi-character on G (we refer the reader to Section 2 for the definition of quasi-character) and θ_K be a smooth function on K that is invariant under conjugation, define

$$
\begin{aligned}
m_{geom,G,K}(\theta, \theta_K) \;\; &:= \;\; \int_{\mathcal{S}(G,K)}^* \frac{1}{c(G_t, K \cap G_t, \mathbb{R}) \cdot |Z_K(t) : K \cap G_t|} \\
&\quad \cdot D^G(t)^{1/2} \Delta(t)^{-1/2} \theta_K(t) c_\theta(t) dt
\end{aligned}
$$

where

- $c_\theta(t)$ is the average of the regular germs of θ at t defined in Section 2.4;

- the constant $c(G_t, K \cap G_t, \mathbb{R})$ is the number of connected components of $B_t \cap K \cap G_t$ (B_t is any Borel subgroup of G_t) defined in Section 5 of [13];
- D^G (resp. D^K) is the Weyl determinant of G (resp. K) and $\Delta(t) = D^G(t)D^K(t)^{-2}$;
- $Z_K(t)$ is the centralizer of t in K.

The integral above is not necessarily absolutely convergent and it needs to be regularized. We refer the reader to Section 3 for details. The main theorem of this paper is a multiplicity formula for K-types.

Theorem 1.1. *For any irreducible smooth representation π of G and for any irreducible representation τ of K, we have the multiplicity formula*

$$m(\pi, \tau) = m_{geom,G,K}(\theta_\pi, \bar{\theta}_\tau).$$

Remark 1.2. *The expression $m_{geom,G,K}(\theta_\pi, \bar{\theta}_\tau)$ is called the geometric multiplicity.*

Corollary 1.3. *The K-character $\Theta_{K,\pi}$ is equal to*

$$\Theta_{K,\pi}(f) = m_{geom,G,K}(\theta_\pi, \theta_f), \ f \in C^\infty(K)$$

where $\theta_f(x) = \int_K f(k^{-1}xk)$, $x \in K$ is the orbital integral of f.

1.2. **Organization of the paper and remarks on the proofs.** In this section we will briefly explain the proof of Theorem 1.1. The first step is to show that both the multiplicity and the geometric multiplicity behave nicely under parabolic induction. For the multiplicity, this is a direct consequence of the Iwasawa decomposition and the Frobenius reciprocity. For the geometric multiplicity, this is a direct consequence of Proposition 2.9. Combine this with the Langlands classification, we only need to prove the multiplicity formula for tempered representations. This also proves the multiplicity formula for all the representations when G does not have discrete series ($\iff$ G does not have a maximal elliptic torus).

To prove the multiplicity formula for tempered representations, we follow the method developed by Waldspurger in his proof of the local Gan–Gross–Prasad conjecture ([8], [9]). To be specific, we will prove a local trace formula for the K-types which would imply the multiplicity formula for tempered representations. We would like to point out that Waldspurger's method was later adapted by other people for some other cases including the unitary Gan–Gross–Prasad model ([2], [3]), the Ginzburg–Rallis model ([11], [12]), the Galois model ([4]), the Shalika model ([5]) and some strongly tempered spherical pairs ([14], [15]). Based on these works, the author proposed a conjectural multiplicity formula and a conjectural local trace formula for general spherical varieties in [13].

Compared with all the existing cases above, there are two difficulties in the proof of the local trace formula for K-types. The first difficulty is on the spectral expansion. For all the known cases above, when the spherical pair is strongly tempered (recall that we say a spherical pair (G, H) is strongly

tempered if all the tempered matrix coefficients of G are integrable over H up to modulo the center), the proof of the spectral expansion used the fact that the multiplicity is less or equal to 1 (i.e. the spherical pair is a Gelfand pair). But this is not true for the case of K-types. To solve this issue, we combine the method in the known strongly tempered cases with the proof of the spectral expansion in the Galois model case due to Beuzart-Plessis [4] (which does not use the Gelfand pair condition but is only for discrete series instead of tempered representations). We refer the reader to Section 4 for details.

The second difficulty is on the geometric expansion. The most important step in the proof of the geometric expansion is to descend the test function to the Lie algebra and then take the Fourier transform. However, this only works when the K-type is trivial (i.e. $\theta_K \equiv 1$). In general when the K-type is nontrivial, it is not clear how to take the Fourier transform on the Lie algebra. Instead, we will first prove the geometric expansion when the K-type is trivial. Then we will prove the general case by using the trivial K-type case and the spectral side of the trace formula. We refer the reader to Section 5 and 6 for details.

This paper is organized as follows. In Section 2 we introduce basic notation and conventions used in this paper. We will also discuss some facts about quasi-characters and strongly cuspidal functions. In Section 3 we state the multiplicity formula and the local trace formula. We also show that the trace formula implies the multiplicity formula. We will postpone the proof of a technical lemma to the Appendix. In Section 4 we will prove the spectral side of the trace formula. In Section 5 we will study the analogue of the trace formula for Lie algebra. Finally in Section 6 we will prove the geometric side of the trace formula.

2. Preliminary

2.1. **Notation.** Fix a nontrivial additive character $\psi : \mathbb{R} \to \mathbb{C}^\times$. Let G be a connected real reductive group, K be a maximal compact subgroup of G, K° be the neutral component of K which is a maximal connected compact subgroup of G, θ be the Cartan involution of G with $K = G^\theta$, $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be the Lie algebra of K°. Let Z_G be the center of G, A_G be the maximal split torus of Z_G.

We use G_{ss}, G_{reg} (resp. $\mathfrak{g}_{ss}$, $\mathfrak{g}_{reg}$) to denote the set of semisimple and regular semisimple elements of G (resp. $\mathfrak{g}$) and let $\mathcal{T}(G)$ (resp. $\mathcal{T}_{ell}(G)$) be a set of representatives of maximal tori (resp. maximal elliptic tori) of

G. Here we say $T \subset G$ is a maximal elliptic torus if T is a maximal torus of G and T/Z_G is compact. Note that the set $\mathcal{T}_{ell}(G)$ is non-empty if and only if G has discrete series. For $x \in G_{ss}$ (resp. $X \in \mathfrak{g}_{ss}$), let $Z_G(x)$ (resp. $Z_G(X) = G_X$) be the centralizer of x (resp. X) in G and let G_x be the neutral component of $Z_G(x)$. Similarly, for any abelian subgroup T of G, let $Z_G(T)$ be the centralizer of T in G and let G_T be the neutral component of $Z_G(T)$. We say $x \in G_{ss}$ is elliptic if x is contained in a maximal elliptic torus and we use $G_{ell} \subset G_{ss}$ to denote the set of elliptic elements of G. We also use $G_{ell,reg} = G_{ell} \cap G_{reg}$ to denote the set of regular elliptic elements of G. Similarly we can define $\mathfrak{g}_{ell}$ and $\mathfrak{g}_{ell,reg}$. Let $G_{ss}/conj$ be the set of semisimple conjugacy classes of G. Finally, for $x \in G_{ss}$ (resp. $X \in \mathfrak{g}_{ss}$), let $D^G(x) = |\det(1 - Ad(x))_{|\mathfrak{g}/\mathfrak{g}_x}|$ (resp. $D^G(X) = |\det(Ad(X))_{|\mathfrak{g}/\mathfrak{g}_X}|$) be the Weyl determinant.

We say a subset $\Omega \subset G$ (resp. $\omega \subset \mathfrak{g}$) is G-invariant if it is invariant under the G-conjugation, we say the set is completely G-invariant if it is G-invariant and for any $X \in \Omega$ (resp. $X \in \omega$), the semisimple part of X also belongs to Ω (resp. ω). For any subset $\Omega \subset G$ (resp. $\omega \subset \mathfrak{g}$), we define the G-invariant subset

$$\Omega^G := \{g^{-1}\gamma g \mid g \in G, \gamma \in \Omega\}, \ \omega^G := \{g^{-1}\gamma g \mid g \in G, \gamma \in \omega\}.$$

We say a G-invariant subset Ω of G (resp. ω of $\mathfrak{g}$) is compact modulo conjugation if there exists a compact subset Γ of G (resp. $\mathfrak{g}$) such that $\Omega \subset \Gamma^G$ (resp. $\omega \subset \Gamma^G$). A G-domain on G (resp. $\mathfrak{g}$) is an open subset of G (resp. $\mathfrak{g}$) invariant under the G-conjugation.

We denote by $X(G)$ the group of $\mathbb{R}$-rational characters of G. Define $\mathfrak{a}_G = \mathrm{Hom}(X(G), \mathbb{R})$, and let $\mathfrak{a}_G^* = X(G) \otimes_{\mathbb{Z}} \mathbb{R}$ be the dual of $\mathfrak{a}_G$. We define a homomorphism $H_G : G \to \mathfrak{a}_G$ by $H_G(g)(\chi) = \log(|\chi(g)|)$ for every $g \in G$ and $\chi \in X(G)$. This is a surjective homomorphism.

For a Levi subgroup M of G, let $\mathcal{P}(M)$ be the set of parabolic subgroups of G whose Levi part is M, $\mathcal{L}(M)$ be the set of Levi subgroups of G containing M, and $\mathcal{F}(M)$ be the set of parabolic subgroups of G containing M. We have a natural decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$. We denote by $proj_M^G$ and $proj_G$ the projections of $\mathfrak{a}_M$ to each factors. For each $P \in \mathcal{P}(M)$, we can associate a positive chamber $\mathfrak{a}_P^+ \subset \mathfrak{a}_M$. For each $P = MU$, we can also define a function $H_P : G \to \mathfrak{a}_M$ by $H_P(g) = H_M(m_g)$ where $g = m_g u_g k_g$ is the Iwasawa decomposition of g.

In this paper we shall freely use the notion of log-norms on algebraic varieties as defined in Section 1.2 of [3]. For every algebraic variety X over $\mathbb{R}$, we will fix a log-norm σ_X on it. For $C > 0$, we use $X[< C]$ to denote the set $\{x \in X \mid \sigma_X(x) < C\}$. In particular, we have log-norms σ_G and $\sigma_\mathfrak{g}$ on G and $\mathfrak{g}$ respectively. It will be convenient to assume, as we may, that σ_G is left and right K-invariant. Following Harish-Chandra, we can also define the height function $\|\cdot\|_G$ on G (resp. $\|\cdot\|_\mathfrak{g}$ on $\mathfrak{g}$), taking values in $\mathbb{R}_{\geq 0}$ so that the log-norm σ_G on G (resp. $\sigma_\mathfrak{g}$ on $\mathfrak{g}$) is given by $\sigma_G(g) = \sup(1, \log(\|g\|_G))$ (resp. $\sigma_\mathfrak{g}(X) = \sup(1, \log(\|X\|_\mathfrak{g}))$).

We fix a minimal Levi subgroup (resp. parabolic subgroup) M_0 (resp. $P_0 = M_0 N_0$) of G and let A be a maximal split torus of M_0. Let $\Sigma(A, P_0)$ be the set of roots of A in P_0 and let $A^+ = \{a \in A^\Delta |\ \alpha(a) \geq 1,\ \forall \alpha \in \Sigma(A, P_0)\}$. Here A^Δ is the neutral component of A under the analytic topology (i.e. if $A \simeq (\mathbb{R}^\times)^k$ then $A^\Delta \simeq (\mathbb{R}_{>0})^k$). We have the Cartan decomposition $G = KA^+K$.

Remark 2.1. *Throughout this paper we will always use $^\circ$ to denote the neutral component under the Zariski topology and $^\Delta$ to denote the neutral component under the analytic topology. For example, if $G = \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, then $G^\circ = G$ and $G^\Delta = \mathbb{R}_{>0}$.*

We say a parabolic subgroup of G is standard if it contains P_0. We say a Levi subgroup of G is standard if it is a Levi subgroup of a standard parabolic subgroup and it contains M_0.

For two complex valued functions f and g on a set X with g taking values in $\mathbb{R}_{\geq 0}$, we write that

$$f(x) \ll g(x)$$

and say that f is essentially bounded by g, if there exists a constant $c > 0$ such that for all $x \in X$, we have

$$|f(x)| \leq cg(x).$$

We say f and g are equivalent, which is denoted by

$$f(x) \sim g(x)$$

if f is essentially bounded by g and g is essentially bounded by f.

Let $C^\infty(G)$ be the space of smooth functions on G and let $C_c^\infty(G)$ (resp. $\mathcal{S}(G)$) be the space of smooth compactly supported functions (resp. Schwartz functions) on G. We use $\mathcal{C}(G)$ to denote the Harish-Chandra-Schwartz space of G (see Section 1.5 of [3] for details). On the Lie algebra level, let $C_c^\infty(\mathfrak{g})$ (resp. $\mathcal{S}(\mathfrak{g})$) be the space of smooth compactly supported functions (resp. Schwartz functions) on $\mathfrak{g}$. We also use Ξ^G to denote the Harish-Chandra Ξ-function of G.

Let $C_{c,scusp}^\infty(G) \subset C_c^\infty(G)$ be the subspace of strongly cuspidal functions in $C_c^\infty(G)$. Similarly we can define the spaces $\mathcal{S}_{scusp}(G), \mathcal{C}_{scusp}(G), C_{c,scusp}^\infty(\mathfrak{g})$, and $\mathcal{S}_{scusp}(\mathfrak{g})$. We refer the reader to Section 5 of [3] for the definition and basic properties of strongly cuspidal functions.

2.2. **Measure.** We fix a non-degenerate symmetric bilinear form $< \cdot, \cdot >$ on $\mathfrak{g}$ that is invariant under G-conjugation (i.e. Killing form). For $f \in \mathcal{S}(\mathfrak{g})$, we can define its Fourier transform $f \to \hat{f}$ to be

$$(2.1) \qquad \hat{f}(X) = \int_{\mathfrak{g}} f(Y)\psi(< X, Y >)dY$$

where dY is the selfdual Haar measure on $\mathfrak{g}$ such that $\hat{\hat{f}}(X) = f(-X)$. Then we get a Haar measure on G such that the Jacobian of the exponential map

is equal to 1 at $0 \in \mathfrak{g}$. If H is a subgroup of G such that the restriction of the bilinear form to $\mathfrak{h}$ is also non-degenerate, then we can define the measures on $\mathfrak{h}$ and H by the same method.

Let G be a split real reductive group with trivial center, $K = G^\theta$ be a maximal compact subgroup of G, $\mathfrak{k}$ be the Lie algebra of K° and $\mathfrak{p} = \mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ (i.e. $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition). Let $\mathfrak{t} \subset \mathfrak{p}$ be a maximal abelian subspace. Since G is split, we know that $\mathfrak{t}$ is the Lie algebra of a maximal split torus T of G. Let $B = TN$ be a Borel subgroup of g and $\bar{B} = T\bar{N}$ be its opposite. We have the Iwasawa decomposition $G = BK = \bar{B}K$ and $T \cap K$ is a finite group. The next lemma will be used in the proof of the geometric side of the trace formula.

Lemma 2.2. *For $f \in \mathcal{S}(\mathfrak{p})$, we have ($W(T)$ is the Weyl group)*

$$\int_{\mathfrak{p}} f(X) dX = \frac{1}{|W(T)| \cdot |T \cap K|} \int_{\mathfrak{t}} \int_K D^G(Y)^{1/2} f(k^{-1}Yk) dk dY.$$

Proof. The lemma is a consequence of the following three facts.

- Every element in $\mathfrak{p} \cap \mathfrak{g}_{reg}$ is K-conjugated to an element of $\mathfrak{t}$.
- Two elements in $\mathfrak{p} \cap \mathfrak{g}_{reg}$ are G-conjugated to each other if and only if they are K-conjugated to each other.
- The Jacobian of the map $\mathfrak{p}//K \to \mathfrak{t}//W(T)$ is equal to $D^G(\cdot)^{1/2}$.

The first fact follows from the fact that two different choices of maximal abelian subspace of $\mathfrak{p}$ are K-conjugated to each other. The second fact follows from the first fact and the fact that the normalizer of T in G is contained in TK.

For the third one, let Σ (resp. $\bar{\Sigma}$) be the roots of T in $\mathfrak{n} = Lie(N)$ (resp. $\bar{\mathfrak{n}} = Lie(\bar{N})$). Then $\Sigma \cup \bar{\Sigma}$ is the set of roots of G with Σ (resp. $\bar{\Sigma}$) being the set of positive (resp. negative) roots. For $\alpha \in \Sigma \cup \bar{\Sigma}$, let V_α be the root space of α. Since G is split, we know that V_α is one dimensional and the Cartan involution θ will map V_α onto $V_{-\alpha}$ for all $\alpha \in \Sigma \cup \bar{\Sigma}$. Hence we can fix $0 \neq X_\alpha \in V_\alpha$ such that $\theta(X_\alpha) = X_{-\alpha}$ for all $\alpha \in \Sigma \cup \bar{\Sigma}$. Then $\{X_\alpha + X_{-\alpha} | \alpha \in \Sigma\}$ is a basis of $\mathfrak{k}$ and $\mathfrak{p} = \mathfrak{t} \oplus Span\{X_\alpha - X_{-\alpha} | \alpha \in \Sigma\}$. Combining with the definition of the Weyl determinant we know that the Jacobian is equal to $D^G(\cdot)^{1/2}$. This finishes the proof of the lemma. $\square$

2.3. **Representations.** We say a representation π of G is irreducible (resp. finite length) smmoth if it is an irreducible (resp. finite length) Casselman-Wallach representation of G. We say a finite length smooth representation π of G is an induced representation if there exists a proper parabolic subgroup $P = MN$ of G and a finite length smooth representation τ of M such that $\pi = I_P^G(\tau)$. Here $I_P^G(\cdot)$ is the normalized parabolic induction.

We use $\mathcal{R}(G)$ to denote the Grothendieck group of finite length smooth representations of G, and we use $\mathcal{R}(G)_{ind} \subset \mathcal{R}(G)$ (resp. $\mathcal{R}(G)_{temp} \subset \mathcal{R}(G)$) to denote the subspace of $\mathcal{R}(G)$ generated by induced representations (resp. tempered representations).

Proposition 2.3. *We have $\mathcal{R}(G) = \mathcal{R}(G)_{ind} + \mathcal{R}(G)_{temp}$ (i.e. $\mathcal{R}(G)$ is generated by induced representations and tempered representations). Moreover, if $\mathcal{T}_{ell}(G) = \emptyset$, then $\mathcal{R}(G) = \mathcal{R}(G)_{ind}$.*

Proof. The first part is a direct consequence of the Langlands classification. For the second part, since $\mathcal{T}_{ell}(G) = \emptyset$, G does not have any elliptic representations. This implies that all the tempered representations of G are generated by induced representations. This proves the proposition. $\square$

2.4. **Quasi characters.** Let $Nil(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$ and $Nil_{reg}(\mathfrak{g})$ be the set of regular nilpotent orbits of $\mathfrak{g}$. In particular, the set $Nil_{reg}(\mathfrak{g})$ is empty unless G is quasi-split. For every $\mathcal{O} \in Nil(\mathfrak{g})$ and $f \in \mathcal{S}(\mathfrak{g})$, we use $J_{\mathcal{O}}(f)$ to denote the nilpotent orbital integral of f associated to $\mathcal{O}$. Harish-Chandra proved that there exists a unique smooth function $Y \to \hat{j}(\mathcal{O}, Y)$ on $\mathfrak{g}_{reg}$, which is invariant under G-conjugation and locally integrable on $\mathfrak{g}$, such that for every $f \in \mathcal{S}(\mathfrak{g})$, we have

$$J_{\mathcal{O}}(\hat{f}) = \int_{\mathfrak{g}} f(Y)\hat{j}(\mathcal{O}, Y)dY.$$

We refer the reader to Section 4.2-4.4 of [3] for the definition of quasi-characters. Let θ (resp. θ') be a quasi character on G (resp. $\mathfrak{g}$). Given $x \in G_{ss}$ (resp. $X \in \mathfrak{g}_{ss}$), we have the germ expansions

$$
\begin{aligned}
& D^G(x\exp(X))^{1/2}\theta(x\exp(X)) \\
= {} & D^G(x\exp(X))^{1/2} \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\hat{j}(\mathcal{O}, X) + O(|X|),
\end{aligned}
$$

$$
\begin{aligned}
& D^G(X+Y)^{1/2}\theta'(X+Y) \\
= {} & D^G(X+Y)^{1/2} \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g}_X)} c_{\theta',\mathcal{O}}(X)\hat{j}(\mathcal{O}, Y) + O(|Y|)
\end{aligned}
$$

for $X \in \mathfrak{g}_x$ (resp. $Y \in \mathfrak{g}_X$) close to 0. Here $c_{\theta,\mathcal{O}}(x) \in \mathbb{C}$ (resp. $c_{\theta',\mathcal{O}}(X) \in \mathbb{C}$) are called the regular germs of θ (resp. θ') at x (resp. X).

The most important examples of quasi-characters on G are the Harish-Chandra characters of finite length smooth representations of G. Examples of quasi-characters on $\mathfrak{g}$ are the functions $\hat{j}(\mathcal{O}, \cdot)$ ($\mathcal{O} \in Nil(\mathfrak{g})$) defined above.

We use $QC(G)$ (resp. $QC(\mathfrak{g})$) to denote the set of quasi-characters on G (resp. $\mathfrak{g}$), and we use $QC_c(G)$ (resp. $QC_c(\mathfrak{g})$) to denote the set of quasi-characters on G (resp. $\mathfrak{g}$) whose support is compact modulo conjugation. We also use $SQC(\mathfrak{g})$ to denote the space of Schwartz quasi-character on $\mathfrak{g}$ defined in Section 4.2 of [3]. We have $QC_c(\mathfrak{g}) \subset SQC(\mathfrak{g})$. If Ω (resp. ω) is an open completely G-invariant subset of G (resp. $\mathfrak{g}$), we define $QC(\Omega)$ (resp. $QC(\omega)$) to be the set of quasi-characters on Ω (resp. ω). Similarly we can also define the spaces $QC_c(\Omega)$, $QC_c(\omega)$. We refer the reader to Section 4.2-4.4 of [3] for the topology on these spaces.

For $f \in \mathcal{C}_{scusp}(G)$ (resp. $f \in \mathcal{S}_{scusp}(\mathfrak{g})$), let θ_f be the quasi-character on G (resp. $\mathfrak{g}$) defined via the weighted orbital integrals of f. For $f \in \mathcal{S}_{scusp}(\mathfrak{g})$,

let $\hat{\theta}_f = \theta_{\hat{f}}$ be the Fourier transform of θ_f. We refer the reader to Section 5.2 and 5.6 of [3] for details.

Definition 2.4. *Let θ be a quasi-character on G. For $x \in G_{ss}$, define the average of the regular germs to be*

$$c_\theta(x) = \begin{cases} \frac{1}{|Nil_{reg}(\mathfrak{g}_x)|} \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x) & \text{if} \quad Nil_{reg}(\mathfrak{g}_x) \neq \emptyset; \\ 0 & \text{if} \quad Nil_{reg}(\mathfrak{g}_x) = \emptyset. \end{cases}$$

Remark 2.5.   (1) The set $Nil_{reg}(\mathfrak{g}_x)$ is non-empty if and only if G_x is quasi-split.
   (2) For $x \in G_{reg}$, $c_\theta(x)$ is just $\theta(x)$.

Definition 2.6. *For $\lambda \in \mathbb{R}^\times$, $l \in \mathbb{R}$ and $\theta \in QC(\mathfrak{g})$, define*

$$M_{\lambda,l}(\theta)(X) = |\lambda|^{-l} \theta(\lambda^{-1} X), \; X \in \mathfrak{g}.$$

We also use θ_λ to denote $M_{\lambda,0}(\theta)$, i.e. $\theta_\lambda(X) = \theta(\lambda^{-1} X)$.

Lemma 2.7. *For $\lambda \neq \pm 1$, $l \in \mathbb{Z}_{>0}$ and $\theta \in QC_c(\mathfrak{g})$, the following hold.*
   (1) *If $c_{\theta,\mathcal{O}} = 0$ for all $\mathcal{O} \in Nil_{reg}(\mathfrak{g})$, then there exists $\theta_1, \theta_2 \in QC_c(\mathfrak{g})$ such that $\theta = (M_{\lambda,\delta(G)/2} - 1)^d \theta_1 + \theta_2$ and $0 \notin Supp(\theta_2)$. Here $\delta(G) = \dim(G) - rank(G)$.*
   (2) *For $l > \delta(G)/2$, there exists $\theta_1, \theta_2 \in QC_c(\mathfrak{g})$ such that $\theta = (M_{\lambda,l} - 1)^d \theta_1 + \theta_2$ and $0 \notin Supp(\theta_2)$.*

Proof. The first part is just Proposition 4.6.1(i) of [3]. The second part follows from the same proof as in loc. cit. together with the fact that the function $D^G(\cdot)^{1/2} \hat{j}(\mathcal{O}, \cdot)$ is locally bounded for all $\mathcal{O} \in Nil(\mathfrak{g})$ (Theorem 17 of [7]). $\qquad\square$

Remark 2.8. *If we assume that $0 < \lambda < 1$, then the above lemma is also true when we replace $QC_c(\mathfrak{g})$ by $QC_c(\omega)$ for any complete G-invariant convex neighborhood ω of 0 in $\mathfrak{g}$. Here we say ω is convex if for any $X \in \omega$ and $0 \leq \lambda \leq 1$, we have $\lambda X \in \omega$.*

Let $P = MN$ be a parabolic subgroup of G, θ_M be a quasi-character of M and $\theta = I_P^G(\theta_M)$ (we refer the reader to Section 3.4 and 4.7 of [3] for the definition of parabolic induction of quasi-characters). For all $x \in G_{ss}$, let $\mathcal{X}_M(x)$ be a set of representatives for the M-conjugacy classes of elements in M that are G-conjugated to x. The following proposition was proved in Proposition 4.7.1 of [3] and it tells us the behavior of $c_\theta(x)$ under parabolic induction.

Proposition 2.9. *For all $x \in G_{ss}$, $D^G(x)^{1/2} c_\theta(x)$ is equal to*

$$|Z_G(x) : G_x| \sum_{y \in \mathcal{X}_M(x)} |Z_M(y) : M_y|^{-1} D^M(y)^{1/2} c_{\theta_M}(y).$$

In particular, $c_\theta(x) = 0$ if the set $\mathcal{X}_M(x)$ is empty.

The next lemma is well known (e.g. Lemma 3.2 of [1]) and it will be used in our proof of the geometric side of the trace formula.

Lemma 2.10. *Let G be a quasi-split real reductive group and $\mathcal{O}_1, \mathcal{O}_2$ be two regular nilpotent orbits of $\mathfrak{g}$. Then there exists $g \in Res_{\mathbb{C}/\mathbb{R}}G$ such that the g-conjugation map preserves G and it sends $\mathcal{O}_1$ to $\mathcal{O}_2$.*

Remark 2.11. *Since any two maximal compact subgroups of G are conjugated to each other, we may choose g in the above lemma such that $g^{-1}Kg = K$.*

3. The trace formula and the multiplicity formula

In this section, we will state the trace formula and the multiplicity formula. In Section 3.1, we will define the distribution $I(f, \theta_K)$ in the trace formula. The key is to show that the integral defining $I(f, \theta_K)$ is absolutely convergent. In Section 3.2 and Section 3.3, we will define the spectral and geometric sides of the trace formula. The key is to regularize the integral defining the geometric multiplicity. The trace formula and multiplicity formula will be stated in Section 3.4. In Section 3.5, we will show that the trace formula implies the multiplicity formula.

3.1. **The distribution I.** Let θ_K be a smooth function on K that is invariant under conjugation. For $f \in \mathcal{C}(G)$, define

$$I(f, x, \theta_K) = \int_K f(x^{-1}kx)\theta_K(k)dk.$$

Set $||\theta_K|| = \max_{k \in K} |\theta_K(k)|$. Let Θ_K be a set consists of some smooth K-invariant functions on K with a fixed upper on both the functions and derivatives, i.e. there exists $C > 0$ such that

$$||\theta_K|| < C, \ |\frac{d}{dt}\theta_K(k\exp(tX))| < C \cdot ||X||_{\mathfrak{k}}$$

for all $\theta_K \in \Theta_K, k \in K$ and $X \in \mathfrak{k}$. Our goal is to prove the following proposition.

Proposition 3.1. *For $d > 0$, there exists a norm ν_d on $\mathcal{C}(G)$ such that*

$$|I(f, x, \theta_K)| \leq \nu_d(f)(||\theta_K|| + \sqrt{||\theta_K||})\Xi^G(x)^2\sigma_{G/Z_G}(x)^{-d}$$

for all $\theta_K \in \Theta_K$, $x \in G$ and $f \in \mathcal{C}_{scusp}(G)$. Here Ξ^G is the Harish-Chandra Ξ-function on G.

The proof of the proposition is technical, but it is very similar to Theorem 8.1.1 of [3]. By the Cartan decomposition $G = KA^+K$, it is enough to consider the case when $x = a \in A^+$. For every standard parabolic subgroup $Q = LU_Q$ of G with $A \subset L$ and $\delta > 0$, set ($R(A, U_Q)$ is the set of roots of A in U_Q)

$$A^{Q,+}(\delta) = \{a \in A^+| \ |\alpha(a)| \geq e^{\delta \cdot \sigma_{G/Z_G}(a)}, \ \forall \alpha \in R(A, U_Q)\}.$$

We choose $\delta > 0$ small so that the complement of

$$\cup_{Q \text{ standard}} A^{Q,+}(\delta)$$

in A^+ is compact modulo the center Z_G. Hence we only need to prove the estimate for $a \in A^{Q,+}(\delta)$.

Let $\bar{Q} = L\bar{U}_Q$ be the opposite parabolic subgroup of Q, $K_Q = K \cap Q$. Up to conjugating L by some element in U_Q we may assume that $K_Q \subset L$ and we let $K_L = K_Q$. We can define the function $\theta_{K,L}$ on K_L to be $\theta_{K,L} = \theta_K|_{K_L}$.

Set $K^Q = K_L \ltimes \bar{U}_Q$ and define θ_K^Q on K^Q to be $\theta_K^Q(k_L u_Q) = \theta_{K,L}(k_L)$. We fix the Haar measures on $K^Q, K_Q \simeq K_L$ and $\bar{U}_Q$ so that

$$\int_{K^Q} f(k^Q) dk^Q = \int_{K_L} \int_{\bar{U}_Q} f(k_L u_Q) du_Q dk_L.$$

For $f \in \mathcal{C}(G)$ and $a \in A^+$, we define

$$I^Q(f, a, \theta_K) = \int_{K^Q} f(a^{-1} k^Q a) \theta_K^Q(k^Q) dk^Q.$$

Since $I^Q(f, a, \theta_K) = 0$ if f is strongly cuspidal, we are reduced to prove the following proposition.

Proposition 3.2. *There exists a constant $c > 0$ depends on the choice of Haar measures such that for all $d > 0$, we have*

$$|I(f, a, \theta_K) - cI^Q(f, a, \theta_K)| \leq \nu_d(f)(\|\theta_K\| + \sqrt{\|\theta_K\|}) \Xi^G(a)^2 \sigma_{G/Z_G}(a)^{-d}$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$ and $f \in \mathcal{C}(G)$. Here ν_d is a norm on $\mathcal{C}(G)$ depends on d.

Proof. Let $K' = K \cap P_0 \bar{N}_0$ and let $u : K' \to \bar{N}_0$ be the map sending k to the unique element $u(k)$ in $\bar{N}_0$ so that $ku(k)^{-1} \in P_0$. Recall that $P_0 = M_0 N_0$ is a minimal parabolic subgroup of G and $\bar{P}_0 = M_0 \bar{N}_0$ is its opposite parabolic subgroup. By the Iwasawa decomposition we know this map is submersive at the identity. Hence we can find a relatively compact open neighborhood $\mathcal{U}$ of 1 in $\bar{N}_0$ and an $\mathbb{R}$-analytic section

$$k : \mathcal{U} \to K', \ u \mapsto k(u)$$

to the map $u(\cdot)$ over $\mathcal{U}$ with $k(1) = 1$. Set $\mathcal{U}_Q = \mathcal{U} \cap \bar{U}_Q$ and $\mathcal{K} = K_Q k(\mathcal{U}_Q)$. By the same argument as in (8.1.7) of [3], we have

(1) The map $\iota : K_Q \times \mathcal{U}_Q \to K$, $(k_Q, u) \mapsto k_Q k(u)$ is an $\mathbb{R}$-analytic embedding with image $\mathcal{K}$ and there exists a smooth function j on $\mathcal{U}_Q$ such that

$$\int_{\mathcal{K}} \varphi(k) dk = \int_{K_Q} \int_{\mathcal{U}_Q} \varphi(k_Q k(u)) j(u) du dk_Q$$

for all $\varphi \in L^1(\mathcal{K})$.

By Lemma 1.3.1(ii) of [3], for $\epsilon > 0$ small enough, we have

$$a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1} \subset \mathcal{U}_Q$$

for all $a \in A^{Q,+}(\delta)$. We set

$$K^{<\epsilon,a} = K_Q k(a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1}), \ K^{Q,<\epsilon,a} = K_L a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1},$$

$$I^{<\epsilon}(f,a,\theta_K) = \int_{K^{<\epsilon,a}} f(a^{-1}ka)\theta_K(k)dk,$$

$$I^{Q,<\epsilon}(f,a,\theta_K) = \int_{K^{Q,<\epsilon,a}} f(a^{-1}k^Q a)\theta_K^Q(k^Q)dk^Q$$

for $a \in A^{Q,+}(\delta)$. Let $0 < \delta_0 < \delta/2$ and $c = j(1)$. We only need to prove the following 3 inequalities (ν_d is a norm on $\mathcal{C}(G)$ depends on d)

(3.1) $|I(f,a,\theta_K) - I^{<\epsilon}(f,a,\theta_K)| \leq \nu_d(f)\|\theta_K\|\Xi^G(a)^2\sigma_{G/Z_G}(a)^{-d},$

(3.2) $|I^Q(f,a,\theta_K) - I^{Q,<\epsilon}(f,a,\theta_K)| \leq \nu_d(f)\|\theta_K\|\Xi^G(a)^2\sigma_{G/Z_G}(a)^{-d},$

(3.3) $|I^{<\epsilon}(f,a,\theta_K) - cI^{Q,<\epsilon}(f,a,\theta_K)| < \nu_d(f)\sqrt{\|\theta_K\|}\Xi^G(a)^2 e^{-\delta_0\cdot\sigma_{G/Z_G}(a)}$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$ and $f \in \mathcal{C}(G)$.

Remark 3.3. *The three inequalities above is an analogue of (8.1.8)-(8.1.10) of [3]. In our case, since $K_Q = K_L$ is compact, we only need to use K_Q, K_L instead of $K_Q[< \epsilon\sigma_{G/Z_G}(a)], K_L[< \epsilon\sigma_{G/Z_G}(a)]$ in the definition of $K^{<\epsilon,a}, K^{Q,<\epsilon,a}$.*

By the same argument as in (8.1.11) of [3] (note that Proposition 6.4.1(iii) and Proposition 6.8.1(v) of loc. cit. are trivial for our case since K is compact), we have

(3.4) $\sigma_{G/Z_G}(a) \ll \sigma_{G/Z_G}(a^{-1}k^Q a), \ \sigma_{G/Z_G}(a) \ll \sigma_{G/Z_G}(a^{-1}ka)$

for all $a \in A^{Q,+}(\delta)$, $k \in K - K^{<\epsilon,a}$, and $k^Q \in K^Q - K^{Q,<\epsilon,a}$. As a result, we have

$$|I(f,a,\theta_K) - I^{<\epsilon}(f,a,\theta_K)| \leq \int_{K-K^{<\epsilon,a}} |f(a^{-1}ka)\theta_K(k)|dk \leq \nu_d(f)\|\theta_K\|$$

$$\cdot \int_{K-K^{<\epsilon,a}} \Xi^G(a^{-1}ka)\sigma_G(a^{-1}ka)^{-d}dk \ll \nu_d(f)\|\theta_K\|\sigma_{G/Z_G}(a)^{-d}\cdot\int_K \Xi^G(a^{-1}ka)dk$$

$$= \nu_d(f)\|\theta_K\|\sigma_{G/Z_G}(a)^{-d}\Xi^G(a^{-1})\Xi^G(a) \sim \nu_d(f)\|\theta_K\|\sigma_{G/Z_G}(a)^{-d}\Xi^G(a)^2,$$

and

$$|I^Q(f,a,\theta_K) - I^{Q,<\epsilon}(f,a,\theta_K)| \leq \int_{K^Q-K^{Q,<\epsilon,a}} |f(a^{-1}ka)\theta_K(k)|dk$$

$$\leq \nu_d(f)\|\theta_K\| \cdot \int_{K^Q-K^{Q,<\epsilon,a}} \Xi^G(a^{-1}ka)\sigma_G(a^{-1}ka)^{-3d}dk$$

$$\ll \nu_d(f)\|\theta_K\|\sigma_{G/Z_G}(a)^{-2d} \cdot \int_{K^Q} \Xi^G(a^{-1}ka)\sigma_G(a^{-1}ka)^{-d}dk$$

$$= \nu_d(f)||\theta_K||\sigma_{G/Z_G}(a)^{-2d}\delta_Q(a)^{-1} \cdot \int_{K_L}\int_{\bar{U}_Q} \Xi^G(a^{-1}kau)\sigma_G(a^{-1}kau)^{-d}dk$$

$$\ll \nu_d(f)||\theta_K||\sigma_{G/Z_G}(a)^{-2d}\delta_Q(a)^{-1} \cdot \int_{K_L} \Xi^L(a^{-1}ka)dk$$

$$\sim \nu_d(f)||\theta_K||\sigma_{G/Z_G}(a)^{-2d}\delta_Q(a)^{-1}\Xi^L(a^{-1})\Xi^L(a) \ll \nu_d(f)||\theta_K||\sigma_{G/Z_G}(a)^{-d}\Xi^G(a)^2$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$ and $f \in \mathcal{C}(G)$. Here we have used Proposition 1.5.1 (i), (iv) and (vi) of [3]. The norm ν_d is the above inequalities can be chosen to be $\nu_d(f) = \sup_{g \in G}|\frac{f(g)}{\Xi^G(g)\sigma_G(g)^{-3d}}|$. This proves (3.1) and (3.2).

For (3.3), we have

$$I^{<\epsilon}(f, a, \theta_K) = \int_{K_L}\int_{a\bar{U}_Q[<\epsilon\sigma_{G/Z_G}(a)]a^{-1}} f(a^{-1}k_Lk(u_Q)a)\theta_K(k_Lk(u_Q))j(u_Q)du_Qdk_Q,$$

$$I^{Q,<\epsilon}(f, a, \theta_K) = \int_{K_L}\int_{a\bar{U}_Q[<\epsilon\sigma_{G/Z_G}(a)]a^{-1}} f(a^{-1}k_Lu_Qa)\theta_K(k_L)du_Qdk_L.$$

Fix $\delta' > 0$ with $0 < 2\delta_0 < \delta' < \delta$ and fix $d' > 0$ large. We first show that in order to prove (3.3), it is enough to prove the following statement.

(2) For $\epsilon > 0$ small, we have

$$|f(a^{-1}k_Lk(u_Q)a)\theta_K(k_Lk(u_Q))j(u_Q) - cf(a^{-1}k^Qa)\theta_K(k_L)|$$

$$< \nu_d(f)\sqrt{||\theta_K||}\Xi^G(a^{-1}k^Qa)\sigma_G(a^{-1}k^Qa)^{-d'}e^{-\frac{\delta'}{2}\cdot\sigma_{G/Z_G}(a)}$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$, $u_Q \in a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1}$, $k_L \in K_L$, $f \in \mathcal{C}(G)$, and for some norm ν_d on $\mathcal{C}(G)$. Here we have set $k^Q = k_Lu_Q$.

In fact, if (2) holds, then we have

$$|I^{<\epsilon}(f, a, \theta_K) - cI^{Q,<\epsilon}(f, a, \theta_K)| < \nu_d(f)\sqrt{||\theta_K||}e^{-\frac{\delta'}{2}\cdot\sigma_{G/Z_G}(a)}\int_{K^{Q,<\epsilon,a}} \Xi^G(a^{-1}k^Qa)$$

$$\cdot\sigma_G(a^{-1}k^Qa)^{-d'}dk^Q \leq \nu_d(f)\sqrt{||\theta_K||}e^{-\frac{\delta'}{2}\cdot\sigma_{G/Z_G}(a)}\int_{K^Q} \Xi^G(a^{-1}k^Qa)\sigma_G(a^{-1}k^Qa)^{-d'}dk$$

$$\ll \nu_d(f)\sqrt{||\theta_K||}e^{-\frac{\delta'}{2}\cdot\sigma_{G/Z_G}(a)}\Delta_Q(a)^{-1}\Xi^L(a^{-1})\Xi^L(a) \ll \nu_d(f)\sqrt{||\theta_K||}e^{-\delta_0\cdot\sigma_{G/Z_G}(a)}\Xi^G(a)^2$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$ and $f \in \mathcal{C}(G)$. Here the inequalities in the third line follow from Proposition 1.5.1(i), (iv) and (vi) of [3]. This proves (3.3).

To prove (2), first by Lemma 1.3.1(ii) of [3] together with the facts that k_L belongs to a compact set and $||\theta'_K||, ||\theta_K||$ are bounded, we know that there exists a constant C that depends on the upper bounds of $||\theta_K||$, $||\theta'_K||$, $||j||$, $||j'||$ (in particular, it only depends on the set Θ_K) such that

$$|\theta_K(k_Lk(u_Q))j(u_Q) - c\theta_K(k_L)| = |\theta_K(k_Lk(u_Q))j(u_Q) - \theta_K(k_L)j(1)| \leq C\cdot e^{-\delta'\cdot\sigma_{G/Z_G}(a)},$$

$$|\theta_K(k_Lk(u_Q))j(u_Q) - c\theta_K(k_L)| \leq C \cdot ||\theta_K||$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$, $u_Q \in a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1}$, $k_L \in K_L$. This implies that

$$|\theta_K(k_L k(u_Q))j(u_Q) - c\theta_K(k_L)| \leq C \cdot e^{-\frac{\delta'}{2} \cdot \sigma_{G/Z_G}(a)} \cdot \sqrt{||\theta_K||}$$

for all $\theta_K \in \Theta_K$, $a \in A^{Q,+}(\delta)$, $u_Q \in a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1}$, $k_L \in K_L$. Hence in order to prove (2), it is enough to prove the following statement:

(3) For $\epsilon > 0$ small, we have

$$|f(a^{-1}k_L k(u_Q)a) - f(a^{-1}k^Q a)| < \nu_d(f)e^{-\delta' \cdot \sigma_{G/Z_G}(a)}\Xi^G(a^{-1}k^Q a)\sigma_G(a^{-1}k^Q a)^{-d'}$$

for all $a \in A^{Q,+}(\delta)$, $u_Q \in a\bar{U}_Q[< \epsilon\sigma_{G/Z_G}(a)]a^{-1}$, $k_L \in K_L$, $f \in \mathcal{C}(G)$, and for some norm ν_d on $\mathcal{C}(G)$.

This follows from the same argument as (8.1.17) of [3]. Now we have finished the proof of the proposition. $\qquad\square$

Remark 3.4. *From the proof of the proposition, we know that we can replace* $||\theta_K||$ *by* $||\theta_{K,f}||$ *where*

$$||\theta_{K,f}|| = \max_{k \in K \cap Supp(\theta_f)} |\theta_K(k)|.$$

We can also replace the space $\mathcal{C}_{scusp}(G)$ *by* $\mathcal{C}_{scusp}(G/A_G^\Delta)$ *– the Mellin transform of the space* $\mathcal{C}_{scusp}(G)$ *with respect to the trivial character of* A_G^Δ.

Definition 3.5. *For* $f \in \mathcal{C}_{scusp}(G/A_G^\Delta)$, *define*

$$I(f, \theta_K) = \int_{KA_G^\Delta \backslash G} I(f, x, \theta_K)dx.$$

By Proposition 3.1 above and Proposition 1.5.1(v) of [3]*, we know the integral is absolutely convergent.*

Remark 3.6. *Note that the double integral*

$$\int_{KA_G^\Delta \backslash G} \int_K f(x^{-1}kx)\theta_K(k)dkdx$$

is not absolutely convergent in general.

3.2. **The spectral side.** Let τ be a finite dimensional representation of K and $\theta_\tau(k) = \mathrm{tr}(\tau(k))$, $k \in K$ be its character. For $f \in \mathcal{C}_{scusp}(G/A_G^\Delta)$, we define the spectral side of the trace formula to be

$$I_{spec}(f, \tau) = \int_{\mathcal{X}(G/A_G^\Delta)} D(\pi)\theta_f(\pi)m(\bar{\pi}, \tau^\vee)d\pi.$$

Here $\mathcal{X}(G/A_G^\Delta)$ (resp. $\mathcal{X}_{ell}(G/A_G^\Delta)$) is the set of virtual tempered representations (resp. elliptic representations) of G whose central character is trivial on A_G^Δ defined in Section 2.7 of [3]. The number $D(\pi)$ and the measure $d\pi$ were also defined in Section 2.7 of [3]. As in Section 5.4 of loc. cit., for $\pi \in \mathcal{X}(G/A_G^\Delta)$, we can define a map

$$f \in \mathcal{C}_{scusp}(G/A_G^\Delta) \mapsto \theta_f(\pi) \in \mathbb{C}$$

via the weighted character (this map was denoted by $f \mapsto \hat{\theta}_f(\pi)$ in loc. cit.).

3.3. **The geometric side.** In this section we will define the geometric side of the trace formula and the geometric multiplicity. These have already been defined in [13] when K is connected (i.e. $K = K^\circ$). We just needs to slightly extend the definitions to the non-connected case. We also need to regularize the integral in the geometric multiplicity.

Definition 3.7. *(the support of geometric multiplicity) Let $\mathcal{S}(G, K)$ be the set of K-conjugacy classes $x \in K$ such that the pair (G_x, K_x) is a minimal spherical pair. We refer the reader to Section 2.6 of [13] for the definition of minimal spherical pair. The set $\mathcal{S}(G, K)$ is the support of the geometric multiplicity.*

Remark 3.8.   (1) *The pair (G_x, K_x) is a minimal spherical pair if and only if G_x is split modulo the center.*
   (2) *This definition is just Definition 4.1 of [13]. Here since K is compact, the elliptic condition in loc. cit. is automatic. Also the quasi-split condition in loc. cit. is a direct consequence of the condition that (G_x, K_x) is a minimal spherical pair.*

In order to define the measure on $\mathcal{S}(G, K)$, we will give an equivalent definition of $\mathcal{S}(G, K)$. The next definition is an analog of Definition 4.3 of [13].

Definition 3.9. *Let $\mathcal{T}(G, K)$ be the set of all the closed (not necessarily connected) abelian subgroups T of K (up to K-conjugation) satisfying the following three conditions.*
   (1) *The pair (G_T, K_T) is a minimal spherical pair.*
   (2) *We have $T = Z_{Z_G(T)} \cap K$ where $Z_{Z_G(T)}$ is the center of $Z_G(T)$. In particular, we have $Z_{G,K} = Z_G \cap K \subset T$.*
   (3) *There exists $t \in T$ such that $(G_t, K_t) = (G_T, K_T)$.*

Let $\mathcal{T}(G, K)^\circ = \{T \in \mathcal{T}(G, K) \mid T = T^\circ Z_{G,K}\}$ where T° is the neutral component of T which is a subtorus of K°.

For $T \in \mathcal{T}(G, K)$, there exists a nonempty subset $C(T, K)$ of the component group T/T° satisfying the following two conditions:
   - For $\gamma \in C(T, K)$, $(G_t, K_t) = (G_T, K_T)$ for almost all $t \in \gamma T^\circ$.
   - For $\gamma \in T/T^\circ - C(T, K)$, $(G_t, K_t) \neq (G_T, K_T)$ for all $t \in \gamma T^\circ$.

Definition 3.10. *For $T \in \mathcal{T}(G, K)$, let $T_K = \cup_{\gamma \in C(T,K)} \gamma T^\circ \subset T \subset K$ and $W(K, T) = N_K(T)/Z_K(T)$ where $N_K(T)$ is the normalizer of T in K. Let T'_K be the Zariski open subset of T_K consisting of those elements $t \in T_K$ such that $(G_t, K_t) = (G_T, K_T)$.*

Remark 3.11. *For $T \in \mathcal{T}(G, K)^\circ$, $(G_t, K_t) = (G_T, K_T)$ for almost all $t \in T$ and we have $T_K = T$.*

Lemma 3.12. *The support of the geometric multiplicity $\mathcal{S}(G,K)$ is equal to the set $\cup_{T \in \mathcal{T}(G,K)} T'_K // W(K,T)$.*

Proof. From the definition it is clear that $T'_K // W(K,T)$ belongs to the support $\mathcal{S}(G,K)$ for all $T \in \mathcal{T}(G,K)$. For the other direction, given $t \in \mathcal{S}(G,K)$, let $T = Z_{Z_G(t)} \cap K$. Then it is easy to see that $T \in \mathcal{T}(G,K)$ and $t \in T'_K // W(K,T)$. This proves the lemma. $\square$

The lemma above gives us a natural measure on the set $\mathcal{S}(G,K)$. More specifically, since T_K is a finite union of translations of the subtori T°, the Haar measure on T° induces a measure on T_K (we choose the Haar meausre on T° so that the total volume is equal to 1) such that $T_K - T'_K$ has measure zero (because T'_K is a Zariski open subset of T_K). This gives us a measure on T'_K and hence a measure on the support $\mathcal{S}(G,K)$.

Lemma 3.13. *For $T \in \mathcal{T}(G,K)$, we have $\dim(K_T) = \frac{\delta(G_T)}{2} + \dim(T)$ and*

$$\dim(K) - \dim(K_T) - \frac{\dim(G) - \dim(G_T)}{2} + \frac{\delta(G)}{2} + \dim(T) = \dim(K)$$

Proof. By the definition of $\mathcal{T}(G,K)$, we know that (here B_T is a Borel subgroup of G_T) $\dim(K_T)$ is equal to

$$\dim(G_T) - \dim(B_T) + \dim(K_T \cap Z_{G_T}) = \dim(G_T) - \dim(B_T) + \dim(K \cap Z_{G_T})$$

$$= \dim(G_T) - \dim(B_T) + \dim(T) = \frac{\delta(G_T)}{2} + \dim(T).$$

For the second equation, by the first equation, we have

$$\dim(K) - \dim(K_T) - \frac{\dim(G) - \dim(G_T)}{2} + \frac{\delta(G)}{2} + \dim(T) = \dim(K) - \frac{\delta(G_T)}{2}$$

$$- \frac{\dim(G) - \dim(G_T)}{2} + \frac{\delta(G)}{2} = \dim(K) - \frac{\dim(G) - rank(G_T)}{2} + \frac{\delta(G)}{2} = \dim(K).$$

This proves the lemma. $\square$

Definition 3.14. *For $\theta \in QC(G)$, $\theta_K \in C^\infty(K)^K$ and $s \in \mathbb{C}$ ($C^\infty(K)^K$ is the space of smooth K-invariant functions on K), define*

$$m_{geom,G,K,s}(\theta, \theta_K) \;\; := \;\; \int_{\mathcal{S}(G,K)} \frac{1}{c(G_t, K \cap G_t, \mathbb{R}) \cdot |Z_K(t) : K \cap G_t|}$$
$$\cdot D^G(t)^{1/2} \Delta(t)^{s-1/2} \theta_K(t) c_\theta(t) dt$$

where the constant $c(G_t, K \cap G_t, \mathbb{R})$ is the number of connected components of $B_t \cap (K \cap G_t)$ (B_t is any Borel subgroup of G_t) defined in Section 5 of [13] and $\Delta(t) = D^G(t) D^K(t)^{-2}$.

We also need to define the Lie algebra analogue of the geometric multiplicity. Let $\mathcal{S}_{Lie}(G,K)$ be the union of $\mathfrak{t}$ for $T \in \mathcal{T}(G,K)^\circ$ where $\mathfrak{t}$ is the Lie algebra of T°. The measure on $T^\circ // W(K,T)$ induces a measure on $\mathcal{S}_{Lie}(G,K)$.

Definition 3.15. *For $\theta \in QC_c(\mathfrak{g})$, $\theta_K \in C^\infty(\mathfrak{k})^K$ and $s \in \mathbb{C}$ ($C^\infty(\mathfrak{k})^K$ is the space of smooth K-invariant functions on K), define*

$$m_{geom,G,K,s,Lie}(\theta,\theta_K) := \int_{\mathcal{S}_{Lie}(G,K)} \frac{1}{c(G_X, K \cap G_X, \mathbb{R})}$$
$$\cdot D^G(X)^{1/2} \Delta(X)^{s-1/2} \theta_K(X) c_\theta(X) dX$$

where the constant $c(G_X, K \cap G_X, \mathbb{R})$ is the number of connected components of $B_X \cap (K \cap G_X)$ (B_X is any Borel subgroup of G_X) and $\Delta(X) = D^G(X) D^K(X)^{-2}$.

Proposition 3.16. *The integral defining $m_{geom,G,K,s}(\theta, \theta_K)$ is absolutely convergent when $\frac{1}{2} > Re(s) > 0$. Moreover, the limit*

$$\lim_{s \to 0^+} m_{geom,G,K,s}(\theta, \theta_K)$$

exists and it defines a continuous linear form on $QC(G) \times C^\infty(K)^K$.

Remark 3.17. *The topology on $QC(G)$ was defined in Section 4.4 of [3]. The topology on $C^\infty(K)^K$ is induced from the usual locally convex topology on $C^\infty(K)$.*

Proof. To simplify the notation, we will assume that the center Z_G is trivial. The proof is very similar to Proposition 11.2.1 of [3]. We first need a lemma.

Lemma 3.18. *The integral $\int_{\mathcal{S}(G,K)} \Delta(t)^{s-1/2} dt$ is absolutely convergent when $\frac{1}{2} > Re(s) > 0$. Moreover, there exists $d > 0$ such that*

$$\lim_{s \to 0^+} s^d \int_{\mathcal{S}(G,K)} \Delta(t)^{s-1/2} dt = 0.$$

Proof. This is a technical lemma and we will postpone the proof to the appendix. □

Using the above lemma together with the fact that the function $D^G(t)^{1/2} c_\theta(t)$ is locally bounded (Proposition 4.5.1 of [3]), we know that the integral defining $m_{geom,G,K,s}(\theta, \theta_K)$ is absolutely convergent when $\frac{1}{2} > Re(s) > 0$. It remains to show that the limit $\lim_{s \to 0^+} m_{geom,G,K,s}(\theta, \theta_K)$ exists (once we have proved the limit exists, by the uniform boundedness principal principal in Appendix A.1 of [3] we know that it defines a continuous linear form). We follow the same argument as in Proposition 11.2.1 of [3].

For $x \in K$, $K \cap G_x$ is a maximal compact subgroup of G_x. For a quasi-character θ_x on G_x, a smooth function $\theta_{K,x}$ on $Z_K(x) \cap G_x$ that is invariant under conjugation, and $s \in \mathbb{C}$, let $m_{geom,G,K,x,s}(\theta_x, \theta_{K,x})$ be the analogue of $m_{geom,G,K,s}(\theta)$ for the model $(G_x, K \cap G_x)$. In particular, we know that it is well defined when $\frac{1}{2} > Re(s) > 0$. Let $\Omega_x \subset G_x$ be a G-good open neighborhood of x (we refer the reader to Section 3.2 of [3] for the definition of G-good open neighborhood) and set $\Omega = \Omega_x^G$. If Ω_x is sufficiently small, by the definition of the set $\mathcal{S}(G,K)$, we have $\mathcal{S}(G,K) \cap \Omega = \mathcal{S}(G_x, K \cap G_x) \cap \Omega_x$.

Combining with Proposition 4.5.1.1(iv) of [3], we have the following claim which is an analogue of (11.2.11) of [3].

Claim 1. With the notation above, we have the equality

$$m_{geom,G,K,s}(\theta, \theta_K) = |Z_K(x)/K \cap G_x|^{-1} \cdot m_{geom,G,K,x,s}((\eta_{x,G})^{s-1/2}\theta_{x,\Omega_x}, \eta_{x,K}^{1-2s}\theta_K|_{Z_K(x) \cap G_x})$$

for all $\theta \in QC_c(\Omega)$ and $s \in \mathbb{C}$ with $\frac{1}{2} > Re(s) > 0$. Here θ_{x,Ω_x} is the semisimple descent of θ (which is a quasi-character on G_x) defined in Section 3.2 of [3], $\eta_{x,G}(y) = D^G(y)D^{G_x}(y)^{-1}$ and $\eta_{x,K}(y) = D^K(y)D^{K_x}(y)^{-1}$.

Then by the same argument as in (11.2.12) of [3], the above claim implies the following claim.

Claim 2. The limit $\lim_{s \to 0^+} m_{geom,G,K,s}(\theta, \theta_K)$ exists when $1 \notin Supp(\theta)$ or $1 \notin Supp(\theta_K)$.

Let $\omega \subset \mathfrak{g}$ be a G-excellent open neighborhood of 0 (we refer the reader to Section 3.3 of [3] for the definition of G-excellent open neighborhood) and set $\Omega = \exp(\omega)$. We only need to show that the limit $\lim_{s \to 0^+} m_{geom,G,K,s}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\Omega)$.

In order to prove this, we study the Lie algebra analogue of $m_{geom,G,K,s}(\theta, \theta_K)$. Like the group case, we know that the integral defining $m_{geom,G,K,s,Lie}(\theta, \theta_K)$ is absolutely convergent for $\frac{1}{2} > Re(s) > 0$. The following claim is a direct consequence of the definition of $m_{geom,G,K,s}(\theta, \theta_K)$ and $m_{geom,G,K,s,Lie}(\theta, \theta_K)$.

Claim 3. For $\theta \in QC_c(\Omega)$ and $\theta_K \in C^\infty(K)^K$ with $Supp(\theta_K) \subset \Omega$, we have

$$m_{geom,G,K,s}(\theta, \theta_K) = m_{geom,G,K,s,Lie}((j_G)^{s-1/2}\theta_\omega, (j_K)^{1/2-2s}\theta_{K,\omega})$$

for all $s \in \mathbb{C}$ with $\frac{1}{2} > Re(s) > 0$. Here θ_ω (resp. $\theta_{K,\omega}$) is the descent of θ (resp. θ_K) to the Lie algebra (defined in Section 3.3 of [3]) whose support is contained in ω (resp. $\mathfrak{k} \cap \omega$) and $j_G(X) = D^G(\exp(X))D^G(X)^{-1}$ (resp. $j_K(X) = D^K(\exp(X))D^K(X)^{-1}$).

Combining Claim 2 and Claim 3, we know that the limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\omega)$ and $\theta_K \in C^\infty(\mathfrak{k})^K$ with $0 \notin Supp(\theta)$ or $0 \notin Supp(\theta_K)$. Moreover, in order to prove the proposition, it is enough to show that the limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$.

By Lemma 3.13 and Lemma 3.18, together with the same argument as in the proof of (11.2.18) and (11.2.19) of [3], we have the following two claims.

Claim 4. There exists $d > 0$ such that for all $\theta \in QC_c(\mathfrak{g})$, $\theta_K \in C^\infty(\mathfrak{k})^K$ and $\lambda \in \mathbb{R}^\times$, we have

$$\lim_{s \to 0^+} \sum_{i=0}^{d} (-1)^{d-i} \frac{d!}{i!(d-i)!} m_{geom,G,K,s,Lie}((M_{\lambda,\dim(K)})^i \theta, (M_{\lambda,0})^i \theta_K) = 0.$$

Claim 5. The limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$ with $0 \notin Supp(\theta)$ or $0 \notin Supp(\theta_K)$.

We first consider the case when θ_K is the constant function. In this case, Claim 4 above becomes

Claim 6. There exists $d > 0$ such that for all $\theta \in QC_c(\mathfrak{g})$ and $\lambda \in \mathbb{R}^\times$, we have

$$\lim_{s \to 0^+} m_{geom,G,K,s,Lie}((M_{\lambda,\dim(K)} - 1)^d \theta, \theta_K) = 0.$$

If G is not split, we have $\dim(K) > \delta(G)/2$. By Claim 5, Claim 6 and Proposition 2.7, we know that the limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\mathfrak{g})$.

If G is split, by Claim 5, Claim 6 and Proposition 2.7, we know that the limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K)$ exists for all $\theta \in QC_c(\mathfrak{g})$ such that $c_{\theta,\mathcal{O}} = 0$ for all $\mathcal{O} \in Nil_{reg}(\mathfrak{g})$. Hence it is enough to show that the limit exists when $\theta \in Span\{\hat{j}(\mathcal{O}, \cdot) | \mathcal{O} \in Nil_{reg}(\mathfrak{g})\}$ in a neighborhood of 0.

For $\mathcal{O}_1, \mathcal{O}_2 \in Nil_{reg}(\mathfrak{g})$, if θ is equal to $\hat{j}(\mathcal{O}_1, \cdot) - \hat{j}(\mathcal{O}_2, \cdot)$ in a neighborhood of 0, let $g \in Res_{\mathbb{C}/\mathbb{R}}G$ be an element as in Lemma 2.10 and we can write θ as $\theta_1 - \theta_2$ where θ_i is equal to $\hat{j}(\mathcal{O}_i, \cdot)$ in a neighborhood of 0. We know that $0 \notin Supp(^g\theta_1 - \theta_2)$ where $^g\theta_1(X) = \theta_1(g^{-1}Xg)$. This implies that the limit

$$\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K) = \lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta_1 - \theta_2, \theta_K)$$

$$= \lim_{s \to 0^+} m_{geom,G,K,s,Lie}(^g\theta_1 - \theta_2, \theta_K)$$

exists. As a result, it is enough to show that the limit exists when $\theta = \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g})} \hat{j}(\mathcal{O}, \cdot)$ in a neighborhood of 0. But this follows from the following two facts

- $\theta = \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g})} \hat{j}(\mathcal{O}, \cdot)$ is parabolically induced from a maximal torus of a Borel subgroup of G (Section 3.4 of [3]);
- the intersection of a Borel subgroup of G with K is finite.

Now we consider the case when θ_K is not the constant function. Since we have already proved the limit exists when θ_K is a constant function. We may assume that $\theta_K(0) = 0$. Since $\theta \in QC_c(\mathfrak{g})$, we may also assume that θ_K is compactly supported.

Consider the space $V = QC(\mathfrak{g}) \hat{\otimes} C^\infty(\mathfrak{k})^K$ realized as functions on $\mathfrak{g}_{reg} \times \mathfrak{k}$. Let $V_c \subset V$ be the subspace consisting of functions whose support is compact modulo conjugation. For $\Theta \in QC(\mathfrak{g}) \hat{\otimes} C^\infty(\mathfrak{k})^K$, $\Theta(\cdot, X)$ is a quasi-character on $\mathfrak{g}$ for all $X \in \mathfrak{k}$. Hence we can still define the regular germ $c_\Theta(Y, X) := c_{\Theta(\cdot,X)}(Y)$. For $Re(s) > 0$, we can also define $m_{geom,G,K,s,Lie}(\Theta)$ for $\Theta \in V_c$. Claim 4 implies that

$$\lim_{s \to 0^+} m_{geom,G,K,s,Lie}((M_{\lambda,\dim(K)} - 1)^d \Theta) = 0$$

for all $\Theta \in V_c$. Here $(M_{\lambda,\dim(K)})\Theta(X,Y) = |\lambda|^{-\dim(K)}\Theta(\lambda^{-1}X, \lambda^{-1}Y)$. Claim 5 implies that the limit $\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\Theta)$ exists when $\Theta \in V_c$ and $0 \notin Supp(\Theta)$.

Let $\Theta = \theta \times \theta_K \in V_c$. We need a lemma.

Lemma 3.19. *For $\lambda \in \mathbb{R}^\times$ with $\lambda \neq \pm 1$, we can write Θ as $(M_{\lambda,\dim(K)} - 1)^d \Theta_1 + \Theta_2$ for some $\Theta_1, \Theta_2 \in V_c$ with $0 \notin \text{Supp}(\Theta_2)$.*

Proof. The proof is very similar to Proposition 4.6.1(i) of [3]. We may assume that $|\lambda| > 1$. It is enough to show that the sequence

$$\sum_{i=0}^\infty \frac{(i+d-1)!}{i!}(M_{\lambda,\dim(K)})^i \Theta$$

converges in V. Let L be an invariant subset of $\mathfrak{g} \times \mathfrak{k}$ that is compact modulo conjugation. Let $I(\mathfrak{g} \oplus \mathfrak{k})$ be the space of $G \times K$-invariant differential operators on $\mathfrak{g} \oplus \mathfrak{k}$. We only need to show that

$$\sum_{i=0}^\infty \frac{(i+d-1)!}{i!} q_{L,u}((M_{\lambda,\dim(K)})^i \Theta)$$

converges for all $u \in I(\mathfrak{g} \oplus \mathfrak{k})$ where

$$q_{L,u}\Theta_0 = \sup_{(X,Y) \in L \cap (\mathfrak{g}_{reg} \times \mathfrak{k})} D^G(X)^{1/2}|\partial(u)\Theta_0(X,Y)|, \ \Theta_0 \in V.$$

We may assume that u is homogeneous. It is enough to show that

(3.5) $q_{L,u}((M_{\lambda,\dim(K)})^i \Theta) \ll |\lambda|^{-i}$

for all $i \geq 0$.

If $\deg(u) > 0$, by enlarging L we can assume that $\lambda^{-1}L \subset L$. Then for $i \geq 1$, we have

$$q_{L,u}((M_{\lambda,\dim(K)})^i \Theta)$$

$$= |\lambda|^{-i\deg(u)-i(\dim(K)-\delta(G)/2)} \sup_{(X,Y) \in L \cap (\mathfrak{g}_{reg} \times \mathfrak{k})} D^G(\lambda^{-i}X)^{1/2}|\partial(u)\Theta(\lambda^{-i}X, \lambda^{-i}Y)|$$

$$\leq |\lambda|^{-i\deg(u)} \sup_{(X,Y) \in \lambda^{-i}L \cap (\mathfrak{g}_{reg} \times \mathfrak{k})} D^G(X)^{1/2}|\partial(u)\Theta(X,Y)| \leq |\lambda|^{-i\deg(u)}q_{L,u}(\Theta).$$

This proves (3.5). If $u = 1$, since $\theta_K(0) = 0$, we have

$$D^G(X)^{1/2}\Theta(X,Y) = O(|Y|)$$

for $(X,Y) \in \mathfrak{g}_{reg} \times \mathfrak{k}$ close to 0. Hence we have

$$q_{L,1}((M_{\lambda,\dim(K)})^i \Theta) = |\lambda|^{-i(\dim(K)-\delta(G)/2)} \sup_{(X,Y) \in \lambda^{-i}L \cap (\mathfrak{g}_{reg} \times \mathfrak{k})} D^G(X)^{1/2}|\Theta(X,Y)|$$

$$\leq \sup_{(X,Y) \in \lambda^{-i}L \cap (\mathfrak{g}_{reg} \times \mathfrak{k})} D^G(X)^{1/2}|\Theta(X,Y)| \ll |\lambda|^{-i}$$

for all $i \geq 0$. This finishes the proof of the lemma. $\square$

By the lemma above, we have

$$\lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\theta, \theta_K) = \lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\Theta)$$

$$= \lim_{s \to 0^+} m_{geom,G,K,s,Lie}((M_{\lambda,\dim(K)} - 1)^d \Theta_1) + m_{geom,G,K,s,Lie}(\Theta_2)$$

$$= \lim_{s \to 0^+} m_{geom,G,K,s,Lie}(\Theta_2).$$

In particular, we know that $\lim_{s\to 0^+} m_{geom,G,K,s,Lie}(\theta,\theta_K)$ exists. This finishes the proof of the proposition. $\square$

Definition 3.20. *For $\theta \in QC(G)$ and $\theta_K \in C^\infty(K)^K$, define*

$$m_{geom,G,K}(\theta,\theta_K) = \lim_{s\to 0^+} m_{geom,G,K,s}(\theta,\theta_K).$$

For $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$, define

$$m_{geom,G,K,Lie}(\theta,\theta_K) = \lim_{s\to 0^+} m_{geom,G,K,s,Lie}(\theta,\theta_K).$$

The next Corollary is a direct consequence of the proof of the proposition above.

Corollary 3.21. (1) *Let $x \in K$, $\Omega_x \in G_x$ be a G-good open neighborhood of x and $\Omega = \Omega_x^G$. If Ω_x is sufficiently small, we have*

$$m_{geom,G,K}(\theta,\theta_K) = |Z_K(x)/K\cap G_x|^{-1}\cdot m_{geom,G,K,x}((\eta_{x,G})^{-1/2}\theta_{x,\Omega_x}, \eta_{x,K}\theta_K|_{K\cap G_x})$$

for all $\theta \in QC_c(\Omega)$. Here $m_{geom,G,K,x}(\cdot,\cdot)$ is the analogue of $m_{geom,G,K}(\cdot,\cdot)$ for the model $(G_x, K\cap G_x)$.

(2) *For $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$, we have*

$$m_{geom,G,K,Lie}(\theta,\theta_K) = |\lambda|^{-\dim(K)} m_{geom,G,K,Lie}(\theta_\lambda, \theta_{K,\lambda}).$$

(3) *Let $\omega \subset \mathfrak{g}$ be a G-excellent open neighborhood of 0 and let $\Omega = \exp(\omega)$. Then*

$$m_{geom,G,K}(\theta,\theta_K) = m_{geom,G,K,Lie}((j_G)^{-1/2}\theta_\omega, j_K^{1/2}\theta_{K,\omega})$$

for all $\theta \in QC_c(\Omega)$ and $\theta_K \in C^\infty(K)^K$ with $Supp(\theta_K) \subset \Omega \cap K$.

Definition 3.22. *For $f \in \mathcal{C}_{scusp}(G/A_G^\Delta)$, define*

$$I_{geom}(f,\theta_K) = m_{geom,G,K}(\theta_f,\theta_K).$$

If π is a finite length smooth representation of G, and τ is a finitely dimensional representation of K, define the geometric multiplicity

$$m_{geom}(\pi,\tau) = m_{geom,G,K}(\theta_\pi, \theta_{\tau^\vee}) = m_{geom,G,K}(\theta_\pi, \overline{\theta_\tau})$$

where $\tau^\vee$ is the dual representation of τ.

3.4. **The trace formula and the multiplicity formula.**

Theorem 3.23. *Let θ_K be a smooth function on K that is invariant under conjugation. For $f \in \mathcal{C}_{scusp}(G/A_G^\Delta)$, we have*

$$I(f,\theta_K) = I_{geom}(f,\theta_K).$$

If $\theta_K = \theta_\tau$ where τ is a finite dimensional representation of K, then

$$I(f,\theta_K) = I_{spec}(f,\tau).$$

Theorem 3.24. *Let τ be a finitely dimensional representation of K. The multiplicity formula*

$$m(\pi,\tau) = m_{geom}(\pi,\tau)$$

holds for all smooth finite length representations π of G.

Remark 3.25. *It is enough to prove the multiplicity formula when π and τ are irreducible. Moreover, up to twist π by some character, we only need to prove the multiplicity formula when the central character of π is trivial on A_G^{Δ}.*

To end this subsection, we give some examples for the multiplicity formula. First, when $G = \mathrm{GL}_n(\mathbb{R})$ or when G is a complex reductive group, the multiplicity formula was proved in our previous paper [13]. We refer the reader to Section 9 and 10 of [13] for details. If $G = \mathrm{Sp}_4(\mathbb{R})$, then $K \simeq U_{2,0}(\mathbb{R})$. The group $U_{2,0}(\mathbb{R})$ contains the following 6 types of conjugacy classes (G_t is the neutral component of the centralizer of the conjugacy class in G):

(1) $\pm I_2$, $G_t \simeq \mathrm{Sp}_4(\mathbb{R})$;
(2) $diag(1, -1)$, $G_t \simeq \mathrm{Sp}_2(\mathbb{R}) \times \mathrm{Sp}_2(\mathbb{R})$;
(3) $diag(a, a)$ where $a \in \mathbb{C}^1$ with $a \neq \pm 1$, $G_t \simeq U_{2,0}(\mathbb{R})$;
(4) $diag(a, \bar{a})$ where $a \in \mathbb{C}^1$ with $a \neq \pm 1$, $G_t \simeq U_{1,1}(\mathbb{R})$;
(5) $diag(a, \pm 1)$ where $a \in \mathbb{C}^1$ with $a \neq \pm 1$, $G_t \simeq \mathrm{Sp}_2(\mathbb{R}) \times U_1(\mathbb{R})$;
(6) $diag(a, b)$ where $a, b \in \mathbb{C}^1$ with $a, b \neq \pm 1$ and $a \neq b, \bar{b}$, $G_t \simeq U_1(\mathbb{R}) \times U_1(\mathbb{R})$;

For the 6 types of conjugacy classes above, G_t is split modulo the center for all the cases except Type (3). As a result, the set $\mathcal{S}(G, K)$ consists of the remaining 5 types of conjugacy classes. Type (1) (resp. Type (2)) conjugacy class is discrete and the measure on it is just the discrete measure (i.e. each point has measure 1), and the constant $c(G_t, K \cap G_t, \mathbb{R}) \cdot |Z_K(t) : K \cap G_t|$ is equal to 4. The measure on Type (4) (resp. Type (5)) conjugacy classes is induced from the Haar measure on $\mathbb{C}^1$, and the constant $c(G_t, K \cap G_t, \mathbb{R}) \cdot |Z_K(t) : K \cap G_t|$ is equal to 1 (resp. 2). Finally the measure on Type (6) conjugacy classes is induced from the Haar measure on $\mathbb{C}^1 \times \mathbb{C}^1$ and the constant $c(G_t, K \cap G_t, \mathbb{R}) \cdot |Z_K(t) : K \cap G_t|$ is equal to 1. As a result, the multiplicity formula for $\mathrm{Sp}_4(\mathbb{R})$ is

$$m(\pi, \tau) = m_{geom}(\pi, \tau) = \frac{c_{\theta_\pi}(I_2)\theta_{\bar{\tau}}(I_2) + c_{\theta_\pi}(-I_2)\theta_{\bar{\tau}}(-I_2)}{4}$$

$$+\frac{D^K(diag(1, -1))c_{\theta_\pi}(diag(1, -1))\theta_{\bar{\tau}}(diag(1, -1))}{4} + \frac{1}{2}\int_{\mathbb{C}^1}^* D^K(diag(a, \bar{a}))c_{\theta_\pi}(diag(a, \bar{a}))$$

$$\cdot\theta_{\bar{\tau}}(diag(a, \bar{a}))da + \int_{\mathbb{C}^1}^* \frac{D^K(diag(1, a))c_{\theta_\pi}(diag(1, a))\theta_{\bar{\tau}}(diag(1, a))}{2}$$

$$+\frac{D^K(diag(-1, a))c_{\theta_\pi}(diag(-1, a))\theta_{\bar{\tau}}(diag(-1, a))}{2}da$$

$$+\frac{1}{2}\int_{\mathbb{C}^1 \times \mathbb{C}^1}^* D^K(diag(a, b))\theta_\pi(diag(a, b))\theta_{\bar{\tau}}(diag(a, b))dadb.$$

3.5. **Trace formula implies multiplicity formula.** In this subsection, we will assume that the trace formula in Theorem 3.23 holds, the goal is to prove the multiplicity formula in Theorem 3.24. The key ingredient of the proof is to show that both the multiplicity and the geometric multiplicity behave nicely under parabolic induction.

To be specific, let $P = MN$ be a parabolic subgroup of G. Up to conjugating M we may assume that $P \cap K \subset M$. Let $K_M = P \cap K = M \cap K$. It is a maximal compact subgroup of M. Let σ be a smooth finite length representation of M and $\pi = I_P^G(\sigma)$. Let τ be a finitely dimensional representation of K and $\tau_M = \tau|_{K_M}$. We need to prove the following proposition.

Proposition 3.26. *With the notation above, we have*

$$m(\pi, \tau) = m(\sigma, \tau_M), \ m_{geom}(\pi, \tau) = m_{geom}(\sigma, \tau_M).$$

Proof. The first identity $m(\pi, \tau) = m(\sigma, \tau_M)$ follows from the Iwasawa decomposition $G = PK$ and the reciprocity law. Now we prove the second identity $m_{geom}(\pi, \tau) = m_{geom}(\sigma, \tau_M)$.

We have a natural map from $\iota_{M,G} : M_{ss}/conj$ to $G_{ss}/conj$. For $t_M \in M_{ss}/conj$, it is clear from the definition that

(3.6) $$t_M \in \mathcal{S}(M, K_M) \iff \iota_{M,G}(t_M) \in \mathcal{S}(G, K).$$

For $t_M \in \mathcal{S}(M, K_M)$, let $t = \iota_{M,G}(t_M) \in \mathcal{S}(G, K)$. We fix a representative of the conjugacy class t_M and denote it by t_0. Then t_0 is also a representative of the conjugacy class t. We have

$$D^K(t)D^{K_M}(t_0)^{-1} = |\det(1 - Ad(t_0))_{|\mathfrak{n}/\mathfrak{n}_{t_0}}|.$$

This implies that

(3.7) $$D^K(t)D^G(t)^{-1/2} = D^{K_M}(t_M)D^M(t_M)^{-1/2}.$$

We also need to show that
(3.8)
$$\frac{|Z_G(t_0) : G_{t_0}|}{|Z_K(t_0) : (K \cap G_{t_0})| \cdot c(G_{t_0}, K \cap G_{t_0}, \mathbb{R})} = \frac{|Z_M(t_0) : M_{t_0}|}{|Z_{K_M}(t_0) : (K_M \cap M_{t_0})| \cdot c(M_{t_0}, K_M \cap G_{t_0}, \mathbb{R})}.$$

Since $K \cap G_{t_0}$ (resp. $K_M \cap M_{t_0}$) a maximal compact subgroup of G_{t_0} (resp. M_{t_0}) and $Z_K(t_0)$ (resp. $Z_{K_M}(t_0)$) is a maximal compact subgroup of $Z_G(t_0)$ (resp. $Z_M(t_0)$), we have

$$\frac{|Z_G(t_0) : G_{t_0}|}{|Z_K(t_0) : (K \cap G_{t_0})|} = \frac{|Z_M(t_0) : M_{t_0}|}{|Z_{K_M}(t_0) : (K_M \cap M_{t_0})|}.$$

Hence it is enough to show that $c(G_{t_0}, K \cap G_{t_0}, \mathbb{R}) = c(M_{t_0}, K_M \cap M_{t_0}, \mathbb{R})$. But this just follows from the definition of $c(G, K, \mathbb{R})$ and the fact that M_{t_0} is a Levi subgroup of G_{t_0}.

Then the identity $m_{geom}(\pi, \tau) = m_{geom}(\sigma, \tau_M)$ follows from the definition of the geometric multiplicities, (3.6), (3.7), (3.8), and Proposition 2.9. □

Now we prove the multiplicity formula. By induction, we will assume that the multiplicity formula holds for all proper Levi subgroups of G. Combining with the proposition above, we only need to prove the multiplicity formula for elliptic representations. In particular, this proves the multiplicity formula for all representations of G when G does not admit a maximal elliptic torus ($\iff$ there is no elliptic representation of G).

Assume that G admit a maximal elliptic torus. The multiplicity formula will be a direct consequence of the trace formula and the proposition above. The argument is the same as the GGP case (Proposition 11.5.1 of [3]), we will only give a sketch of the proof. First, as in Proposition 11.5.1(ii) of [3], using the trace formula, the proposition above, and the assumption that the multiplicity formula holds for all proper Levi subgroups, we know that for all strongly cuspidal function $f \in \mathcal{C}_{scusp}(G/A_G^\Delta)$, we have

$$(3.9) \qquad \sum_{\pi \in \mathcal{X}_{ell}(G/A_G^\Delta)} D(\pi)\theta_f(\pi)(m(\bar\pi, \tau^\vee) - m_{geom}(\bar\pi, \tau^\vee)) = 0.$$

Then by Corollary 5.7.2(iv) of [3], for each $\pi \in \mathcal{X}_{ell}(G/A_G^\Delta)$, there exists $f_\pi \in \mathcal{C}_{scusp}(G/A_G^\Delta)$ such that for all $\pi' \in \mathcal{X}_{ell}(G/A_G^\Delta)$, $\theta_f(\pi') \neq 0$ if and only if $\pi = \pi'$. Put f_π in the equation (3.9), we have

$$D(\pi)\theta_{f_\pi}(\pi)(m(\bar\pi, \tau^\vee) - m_{geom}(\bar\pi, \tau^\vee)) = 0,$$

which implies that $m(\bar\pi, \tau^\vee) = m_{geom}(\bar\pi, \tau^\vee)$. This proves the multiplicity formula.

In the next few sections we will prove the trace formula. By the above discussion, we only need to consider the case when G admit a maximal elliptic torus. By induction, we will assume that the trace formula holds for all groups whose dimension is less than G. Also to simplify the notation, we will assume that the center of G is trivial for the rest of the paper.

4. The spectral side

In this section we will prove the spectral side of the trace formula. In Section 4.1 we slightly reformulate the spectral expansion. In Section 4.2 we will discuss some explicit elements in the Hom space. Finally in Section 4.3 we will prove the spectral expansion.

4.1. **A reformulation.** Let $G' = G \times K$ and we diagonally embeds K into G'. For $f' \in \mathcal{C}_{scusp}(G')$, define

$$I_{G'}(f', x) = \int_K f'(x^{-1}kx)dx, \ I_{G'}(f') = \int_{K \backslash G'} I_{G'}(f', x),$$

$$I_{G',spec}(f') = \int_{\mathcal{X}(G')} D(\pi')\theta_{f'}(\pi')m(\bar\pi')d\pi'$$

where $m(\pi') = \dim(\mathrm{Hom}_K(\pi', 1))$. By choosing $f' = f \times \theta_\tau$ with $f \in \mathcal{C}_{scusp}(G)$ we know that in order to prove the spectral expansion, we only

need to show that
$$I_{G'}(f') = I_{G',spec}(f')$$
for all $f' \in \mathcal{C}_{scusp}(G')$.

4.2. **Explicit intertwining operator.**

Lemma 4.1. *Let π' be an irreducible tempered representation of G' and $l \in \mathrm{Hom}_K(\pi', 1)$ be a K-invariant linear form. Then there exists $d > 0$ and a continuous semi-norm ν_d on π' such that*
$$|l(\pi'(x)e)| \leq \nu_d(e)\Xi^{G'}(x)\sigma_{G'}(x)^d$$
for all $e \in \pi'$ and $x \in G'$.

Proof. Let P_{min} be a minimal parabolic subgroup of G' (it is the product of a minimal parabolic subgroup of G with K). We only need to prove the inequalities for $x \in A^+$.

Let $X_1, \cdots, X_p$ be a basis of $\mathfrak{p}_{min}$ and let
$$\Delta_{min} = 1 - (X_1^2 + \cdots + X_p^2) \in \mathcal{U}(\mathfrak{p}_{min}).$$
Let k be an integer greater than $\dim(P_{min}) + 1$. By elliptic regularity, there exists $\varphi_1 \in C_c^{k_1}(P_{min})$ and $\varphi_2 \in C_c^\infty(P_{min})$ with $k_1 = 2k - \dim(P_{min}) - 1$ such that
$$\pi'(\varphi_1)\pi'(\Delta_{min}^k) + \pi'(\varphi_2) = Id_{\pi'}$$
for all π'. Let φ_K be a smooth function on K such that $\int_K \varphi_K(k)dk = 1$. Then we have
$$l(\pi'(a)e) = l(\pi'(\varphi_1)\pi'(\Delta_{min}^k)\pi'(a)e) + l(\pi'(\varphi_2)\pi'(a)e)$$
$$= l(\pi'(\varphi_1)\pi'(a)\pi'(a^{-1}\Delta_{min}^k a)e) + l(\pi'(\varphi_2)\pi'(a)e)$$
$$= l(\pi'(\varphi_K * \varphi_1)\pi'(a)\pi'(a^{-1}\Delta_{min}^k a)e) + l(\pi'(\varphi_K * \varphi_2)\pi'(a)e).$$
It is clear that the map $a \in A^+ \mapsto a^{-1}\Delta_{min}^k a \in \mathcal{U}(\mathfrak{g})$ has bounded image. Moreover, we know that the functions $\varphi_K * \varphi_i$ $(i = 1, 2)$ belong to $C_c^{k_1}(G')$. Hence the lemma follows from the following fact which is (7.3.7) of [3].

- There exists $k_1' \geq 1$ such that for all $\varphi \in C_c^{k_1'}(G')$, there exists a continuous semi-norm ν_φ on π' such that
$$|l(\pi'(\varphi)\pi'(g)e)| \leq \nu_\varphi(e)\Xi^{G'}(g)$$
for all $e \in \pi'$ and $g \in G'$.

$\square$

Corollary 4.2. *For $l \in \mathrm{Hom}_K(\pi', 1), v \in \pi'$ and $f \in \mathcal{C}(G')$, the integral*
$$\int_{G'} f(g)l(\pi'(g)v)dg$$
is absolutely convergent and is equal to $l(\pi'(f)v)$.

Proof. The proof is the same as (7.5.1) and (7.5.4) of [3], we will skip it here. $\square$

Fix a G'-invariant scalar product $(\cdot, \cdot)$ on π'. Define a sesquilinear form

$$\mathcal{B}_{\pi'} : \pi' \times \pi' \to \mathbb{C}, \ \mathcal{B}_{\pi'}(v, v') = \int_K (\pi'(k)v, v')dk.$$

The integral is convergent since K is compact. It factors through $\mathcal{B}_{\pi'}$: $\pi'_K \times \pi'_K \to \mathbb{C}$. where π'_K is the space of K-coinvariants in π'. The next lemma follows from the fact that K is compact.

Lemma 4.3. *The sesquilinear form $\mathcal{B}_{\pi'} : \pi'_K \times \pi'_K \to \mathbb{C}$ is non degenerate and the map*

$$v \in \pi' \mapsto \mathcal{B}_{\pi'}(\cdot, v) \in \operatorname{Hom}_K(\pi', 1)$$

is surjective

Remark 4.4. *The above lemma is highly nontrivial for general strongly tempered spherical pairs (although we expect it to be true), especially when the spherical pair is not a Gelfand pair, i.e. when the multiplicity is greater than 1.*

We also define the linear form $\mathcal{L}_{\pi'} : \operatorname{End}(\pi')^\infty \to \mathbb{C}$ to be (here $\operatorname{End}(\pi')^\infty$ is the space of smooth vectors in $\operatorname{End}(\pi')$ and we refer the reader to Section 2.2 of [3] for the topology on $\operatorname{End}(\pi')^\infty$)

$$\mathcal{L}_{\pi'}(T) = \int_K \operatorname{tr}(\pi'(k)T)dk.$$

For $e, e' \in \pi'$, by abusing of notion, we define

$$\mathcal{L}_{\pi'}(e, e') = \mathcal{L}_{\pi'}(T_{e,e'}) = \int_K (e, \pi'(k)e')dk$$

where $T_{e,e'} \in \operatorname{End}(\pi)^\infty$ is defined to be $T_{e,e'}(e_0) = (e_0, e')e$. The lemma above implies that

$$\mathcal{L}_{\pi'} \neq 0 \iff m(\pi') \neq 0.$$

Since $\mathcal{L}_{\pi'}$ is a continuous sesquilinear form on π', it defines a continuous linear map

$$L_{\pi'} : \pi' \to \overline{\pi'^{-\infty}}, \ e \mapsto \mathcal{L}_{\pi'}(e, \cdot)$$

where $\overline{\pi'^{-\infty}}$ denotes the topological conjugate-dual of π' endowed with the strong topology. The operator $L_{\pi'}$ has image in $\overline{\pi'^{-\infty}}^K = \operatorname{Hom}_K(\overline{\pi'}, 1)$ which is finite dimensional. For $T \in \operatorname{End}(\pi')^\infty$, it extends uniquely to a continuous operator $T : \overline{\pi'^{-\infty}} \to \pi'$. This gives two operators

$$TL_{\pi'} : \pi' \to \pi', \ L_{\pi'}T : \overline{\pi'^{-\infty}} \to \overline{\pi'^{-\infty}},$$

and we have

$$\operatorname{tr}(TL_{\pi'}) = \operatorname{tr}(L_{\pi'}T) = \mathcal{L}_{\pi'}(T).$$

The next corollary is a direct consequence of Lemma 4.3 and it is an analogue of Corollary 7.6.1 of [3].

Corollary 4.5. *Let $\mathcal{K} \subset \mathcal{X}_{temp}(G')$ be a compact subset. Then there exists a section $T \in \mathcal{C}(\mathcal{X}_{temp}(G'), \mathcal{E}(G'))$ such that the restriction of $L_\pi T_\pi$ to $\operatorname{Hom}_K(\bar{\pi}, 1)$ is the identity map for all $\pi \in \mathcal{K}$.*

Here we refer the reader to Section 2.6 of [3] for the definition of $\mathcal{X}_{temp}(G')$, $\mathcal{C}(\mathcal{X}_{temp}(G'), \mathcal{E}(G'))$ and $C^\infty(\mathcal{X}_{temp}(G'), \mathcal{E}(G'))$. The next lemma is an analogue of Lemma 7.2.2 of [3].

Lemma 4.6. (1) *The maps*

$$\pi \in \mathcal{X}_{temp}(G') \mapsto L_\pi \in \mathrm{Hom}(\pi, \overline{\pi^{-\infty}}), \ \pi \in \mathcal{X}_{temp}(G') \mapsto \mathcal{L}_\pi \in \mathrm{End}(\pi)^{-\infty}$$

are smooth in the sense of Lemma 7.2.2(1) of [3].

(2) *For all $S, T \in \mathrm{End}(\pi')^\infty$ such that the restriction of $L_{\pi'}T$ to $\mathrm{Hom}_K(\bar{\pi}', 1)$ is the identity map, we have $SL_{\pi'} \in \mathrm{End}(\pi')^\infty$ and $\frac{\mathcal{L}_{\pi'}(S)\mathcal{L}_{\pi'}(T)}{m(\pi')} = \mathcal{L}_{\pi'}(SL_{\pi'}T)$.*

(3) *For $S, T \in \mathcal{C}(\mathcal{X}_{temp}(G'), \mathcal{E}(G'))$, the section $\pi \mapsto S_\pi L_\pi T_\pi \in \mathrm{End}(\pi)^\infty$ belongs to $C^\infty(\mathcal{X}_{temp}(G'), \mathcal{E}(G'))$.*

(4) *Let $f \in \mathcal{C}(G')$ with compactly supported Fourier transform $\pi \in \mathcal{X}_{temp}(G') \to \pi(f)$. We have*

$$\int_K f(k)dk = \int_{\mathcal{X}_{temp}(G')} \mathcal{L}_\pi(\pi(f))\mu(\pi)d\pi.$$

Here $\mu(\pi)d\pi$ is the Plancherel measure defined in Section 2.6 of [3].

(5) *Let f as in part (iv) and let $f' \in \mathcal{C}(G')$ such that for all π belongs to the support of f, the restriction of $L_\pi \pi(f')$ to $\mathrm{Hom}_K(\bar{\pi}, 1)$ is the identity map. Then we have*

$$\int_{\mathcal{X}_{temp}(G')} \frac{\mathcal{L}_\pi(\pi(f))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}}{m(\pi)}\mu(\pi)d\pi = \int_K \int_K \int_{G'} f(kgk')f'(g)dgdk'dk.$$

Proof. The proof of the first part is the same as Lemma 7.2.2(i) of [3]. For the second part, the proof of $SL_{\pi'} \in \mathrm{End}(\pi')^\infty$ follows from the same argument as in Lemma 7.2.2(ii) of [3]. As for the eqaution, we have $\mathcal{L}_{\pi'}(SL_{\pi'}T) = \mathrm{tr}(L_{\pi'}SL_{\pi'}T)$. Since the images of $L_{\pi'}S$ and $L_{\pi'}T$ are contained in $\mathrm{Hom}_K(\bar{\pi}', 1)$ and the restriction of $L_{\pi'}T$ to $\mathrm{Hom}_K(\bar{\pi}', 1)$ is the identity map, we have

$$\mathcal{L}_{\pi'}(SL_{\pi'}T) = \mathrm{tr}(L_{\pi'}SL_{\pi'}T) = \mathrm{tr}(L_{\pi'}S) = \mathcal{L}_{\pi'}(S) = \mathcal{L}_{\pi'}(S) \cdot \frac{\mathcal{L}_{\pi'}(T)}{m(\pi')}.$$

This proves the second part.

The proof of part (3)-(5) is almost the same as Lemma 7.2.2(iii)-(v) of [3], there is only one difference.

- Compared with Lemma 7.2.2(v) of [3], the last part of the lemma has an extra $m(\pi)$ on the left hand side of the equation, this comes from the extra $m(\pi)$ on the second part of the lemma.

We will skip the details here. $\square$

4.3. **The proof of the spectral expansion.** In this subsection we are going to show that for all $f \in \mathcal{C}_{scusp}(G')$, we have

$$I_{G'}(f) = I_{G',spec}(f).$$

By Lemma 5.4.2 of [3] and Proposition 3.1, both sides of the equation are continuous on $f \in \mathcal{C}_{scusp}(G')$, hence it is enough to prove the equation for all $f \in \mathcal{C}_{scusp}(G')$ whose Fourier transform $\pi \in \mathcal{X}_{temp}(G') \to \pi(f)$ is compactly supported. For the rest of this subsection we will fix a function $f \in \mathcal{C}_{scusp}(G')$ whose Fourier transform is compactly supported.

For $f' \in \mathcal{C}(G')$, define

$$K_{f,f'}^A(g_1, g_2) = \int_{G'} f(g_1^{-1} g g_2) f'(g) dg, \ g_1, g_2 \in G',$$

$$K_{f,f'}^1(g, x) = \int_K K_{f,f'}^A(g, kx) dk, \ g, x \in G',$$

$$K_{f,f'}^2(x, y) = \int_K K_{f,f'}^1(kx, y) dk, \ x, y \in G',$$

$$J_{aux}(f, f') = \int_{K \backslash G'} K_{f,f'}^2(x, x) dx.$$

The next lemma is a direct consequence of Theorem 5.5.1(i)-(ii) of [3] (note that K is compact).

Lemma 4.7.   (1) *Let* $* \in \{1, 2, A\}$. *For all* $d \geq 0$, *there exists* $d' \geq 0$ *such that*

$$|K_{f,f'}^*(g_1, g_2)| \ll \Xi^{G'}(g_1)\sigma_{G'}(g_1)^{-d}\Xi^{G'}(g_2)\sigma_{G'}(g_2)^{d'},$$

$$|K_{f,f'}^*(g_1, g_2)| \ll \Xi^{G'}(g_1)\sigma_{G'}(g_1)^{d'}\Xi^{G'}(g_2)\sigma_{G'}(g_2)^{-d},$$

$$|K_{f,f'}^*(x, x)| \ll \Xi^{G'}(x)^2\sigma_{G'}(x)^{-d}$$

for all $g_1, g_2, x \in G'$.
     (2) *The triple integral defining* $J_{aux}(f, f')$

$$\int_{K \backslash G'} \int_K \int_K K_{f,f'}^A(k_1 x, k_2 x) dk_1 dk_2 dx$$

is absolutely convergent.

Proposition 4.8. *We have*

$$J_{aux}(f, f') = \int_{\mathcal{X}(G')} D(\pi)\theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} d\pi.$$

Proof. We have

$$J_{aux}(f, f') = \int_{K \backslash G'} \int_K \int_K K_{f,f'}^A(k_1 x, k_2 x) dk_1 dk_2 dx$$

$$= \int_{G'} \int_K K_{f,f'}^A(x, kx) dk dx = \int_K \int_{G'} K_{f,f'}^A(x, kx) dx dk$$

$$= \int_K \int_{\mathcal{X}(G')} D(\pi)\theta_f(\pi)\theta_{\bar{\pi}}(R(k)f') d\pi dk = \int_{\mathcal{X}(G')} \int_K D(\pi)\theta_f(\pi)\theta_{\bar{\pi}}(R(k)f') dk d\pi$$

$$= \int_{\mathcal{X}(G')} D(\pi)\theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} d\pi.$$

where the first equality on the third line follows from Theorem 5.5.1(v) of [3]. □

Now we are ready to prove the spectral expansion. Recall that we have fixed a function $f \in \mathcal{C}_{scusp}(G')$ whose Fourier transform is compactly supported. Let $\mathcal{K}$ be the support of the Fourier transform of f. By Lemma 4.6, we have

$$I_{G'}(f, x) = \int_{\mathcal{X}_{temp}(G')} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\mu(\pi)d\pi.$$

By Corollary 4.5, there exists a function $f' \in \mathcal{C}(G')$ such that the restriction of $L_\pi \pi(f')$ to $\mathrm{Hom}_K(\bar{\pi}, 1)$ is the identity map for all $\pi \in \mathcal{K}$. Fix such a f'. We have

$$\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} = m(\pi) = m(\bar{\pi}), \ \pi \in \mathcal{K}.$$

This implies that

$$I_{G'}(f, x) = \int_{\mathcal{X}_{temp}(G')} \frac{\mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}}{m(\pi)}\mu(\pi)d\pi = K^2_{f,f'}(x, x)$$

where the last equation follows from Lemma 4.6. In particular, we have

$$I_{G'}(f) = J_{aux}(f, f') = \int_{\mathcal{X}(G')} D(\pi)\theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi$$

$$= \int_{\mathcal{X}(G')} D(\pi)\theta_f(\pi)m(\bar{\pi})d\pi = I_{G',spec}(f).$$

This finishes the proof of the spectral expansion.

5. The distribution on the Lie algebra

In this section, we will study the analogue of the distribution on the Lie algebra when G is split. Let G be a split real reductive group with trivial center, $K = G^\theta$ be a maximal compact subgroup of G, $\mathfrak{k}$ be the Lie algebra of K° and $\mathfrak{p} = \mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ (i.e. $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition). Let $\mathfrak{t} \subset \mathfrak{p}$ be a maximal abelian subspace. Since G is split, we know that $\mathfrak{t}$ is the Lie algebra of a maximal split torus T of G. Let $B = TN$ be a Borel subgroup of G, $\bar{B} = T\bar{N}$ be its opposite and $W(T)$ be the Weyl group. For $f \in \mathcal{S}_{scusp}(\mathfrak{g})$, define

$$I_{Lie}(f, g) = \int_{\mathfrak{k}} f(g^{-1}kg)dk, \ I_{Lie}(f) = \int_{K\backslash G} I(f, g)dg.$$

As in the group case, we know that the integral defining $I_{Lie}(f)$ is absolutely convergent. The goal of this section is to prove the following theorem.

Theorem 5.1. *For all $f \in \mathcal{S}_{scusp}(\mathfrak{g})$, we have*

$$I_{Lie}(f) = \frac{1}{c(G, K, \mathbb{R}) \cdot |W(T)|} \int_{\mathfrak{t}} D^G(X)^{1/2}\hat{\theta}_f(X)dX$$

where $c(G, K, \mathbb{R}) = |K \cap T|$ is the constant defined in Definition 3.14.

Since both side of the theorem are continuous on f, we only need to prove the identity when the support of $\hat{f}$ is compact modulo conjugation. For the rest of this section, we will assume that the support of $\hat{f}$ is compact modulo conjugation.

The first step is to introduce a sequence of truncation functions $(\kappa_N)_{N>0}$ on $K\backslash G$. Let (recall that T^Δ is the neutral component of T under the analytic topology)

$$T^+ = \{t \in T^\Delta|\ \alpha(t) \geq 1,\ \forall \alpha \in \Delta\},\ T^- = \{t \in T^\Delta|\ \alpha(t) \geq 1,\ \forall \alpha \in \bar{\Delta}\}$$

where Δ (resp. $\bar{\Delta}$) is the set of positive root with respect to $B = TN$ (resp. $\bar{B} = T\bar{N}$). We have the Iwasawa decomposition $G = BK = \bar{B}K$ and the Cartan decomposition $G = KT^+K = KT^-K$. We fix a sequence $(\kappa_N)_{N>1}$ of smooth functions $\kappa_N : G \to [0,1]$ satisfying the following three conditions.

- The function κ_N is bi-K-invariant.
- There exists $C_1 > C_2 > 0$ such that for all $x \in G$, we have

$$\sigma_G(g) \leq C_2 N \Rightarrow \kappa_N(g) = 1,\ \kappa_N(g) \neq 0 \Rightarrow \sigma_G(g) \leq C_1 N.$$

- There exists $C > 0$ such that

$$|\frac{d}{dt}\kappa_N(e^{tX}x)|_{t=0}| \leq C \cdot \|X\|_{\mathfrak{g}}$$

for all $x \in G, X \in \mathfrak{g}$ and $N > 1$.

It is clear that such functions exist. For $N > 1$, define

$$I_{N,Lie}(f) = \int_{K\backslash G} \kappa_N(g) I_{Lie}(f,g) dg.$$

We have $I_{Lie}(f) = \lim_{N\to\infty} I_{N,Lie}(f)$.

Lemma 5.2. *For $f \in \mathcal{S}(\mathfrak{g})$, we have*

$$I_{N,Lie}(f) = \frac{1}{c(G,K,\mathbb{R}) \cdot |W(T)|} \int_{\mathfrak{t}} D^G(X)^{1/2} \int_{T\backslash G} \kappa_{N,T}(g) \hat{f}(g^{-1}Xg) dg dY$$

where $\kappa_{N,T}(g) = \int_T \kappa_N(tg) dt$.

Proof. By the Fourier inversion formula and Lemma 2.2, we have

$$I_{Lie}(f,g) = \int_{\mathfrak{k}} f(g^{-1}kg) dk = \int_{\mathfrak{p}} \hat{f}(g^{-1}Xg) dX$$

$$= \frac{1}{|K \cap T| \cdot |W(T)|} \int_K \int_{\mathfrak{t}} D^G(X)^{1/2} \hat{f}(g^{-1}k^{-1}Xkg) dX dk.$$

This implies that

$$I_{N,Lie}(f) = \frac{1}{|K \cap T| \cdot |W(T)|} \int_{K\backslash G} \kappa_N(g) \int_K \int_{\mathfrak{t}} D^G(X)^{1/2} \hat{f}(g^{-1}kXkg) dY dk$$

$$= \frac{1}{c(G,K,\mathbb{R}) \cdot |W(T)|} \int_{\mathfrak{t}} D^G(X)^{1/2} \int_G \kappa_N(g) \hat{f}(g^{-1}Xg) dg dX$$

$$= \frac{1}{c(G, K, \mathbb{R}) \cdot |W(T)|} \int_{\mathfrak{t}} D^G(X)^{1/2} \int_{T \backslash G} \kappa_{N,T}(g) \hat{f}(g^{-1} X g) dg dX.$$

This proves the lemma. □

Lemma 5.3. *There exists $C > 0$ and $k > 0$ such that*

$$|\kappa_{N,T}(g)| \leq C N^k$$

for all $N > 1$ and $g \in G$.

Proof. This is a direct consequence of the definition of κ_N and $\kappa_{N,T}$. □

For $l > 0$, let $\mathfrak{t}_{\leq l}$ be the set of $X \in \mathfrak{t}$ such that $D^G(X) \leq l$, and let $\mathfrak{t}_{> l}$ be its complement. We define

$$I_{N,l}(f) = \int_{\mathfrak{t}_{> l}} D^G(X)^{1/2} \int_{T \backslash G} 1_{< \log N}(g^{-1} X g) \kappa_{N,T}(g) \hat{f}(g^{-1} X g) dg dY$$

where $1_{< \log N}$ is the characteristic function of the set $\{g \in G | \sigma_G(g) < \log N\}$.

Lemma 5.4. *The following statements hold.*

(1) *There exist $k \in \mathbb{N}$ and $c > 0$ such that $| I_{N,Lie}(f) | \leq cN^k$ for all $N > 1$.*

(2) *There exist $b \geq 1$ and $c > 0$ such that $| I_{N,Lie}(f) - \frac{1}{c(G,K,\mathbb{R}) \cdot |W(T)|} I_{N,N^{-b}}(f) | \leq cN^{-1}$ for all $N > 1$. In particular, we have*

$$I_{Lie}(f) = \lim_{N \to \infty} I_{N,Lie}(f) = \frac{1}{c(G, K, \mathbb{R}) \cdot |W(T)|} \lim_{N \to \infty} I_{N,N^{-b}}(f).$$

Proof. This is a direct consequence of the Lemma above and (1.2.3), (1.2.4) of [3]. □

The next step is to change the truncation function κ_N. Let $\mathfrak{a} = \mathfrak{a}_T$ and $\mathfrak{a}^+$ be the positive chamber with respect to the Borel subgroup $B = TN$. For $Y \in \mathfrak{a}^+$, we have the positive (G, T)-orthogonal set $(Y_P)_{P \in \mathcal{P}(T)}$ as defined in Section 1.9 of [3]. For $g \in G$, we can define the (G, T)-orthogonal set $\mathcal{Y}(g, Y) = (\mathcal{Y}(g, Y)_P)_{P \in \mathcal{P}(T)}$ with $\mathcal{Y}(g, Y)_P = Y_P - H_{\bar{P}}(g)$. As in (10.10.1) of [3], we have the following statement.

(1) There exists $c > 0$ such that for all $g \in G$ and $Y \in \mathfrak{a}^+$ with

$$\sigma_G(g) \leq c \cdot \inf_{\alpha \in \Delta} \delta(Y),$$

the (G, T)-orthogonal set $\mathcal{Y}(g, Y) = (\mathcal{Y}(g, Y)_P)_{P \in \mathcal{P}(T)}$ is positive.

On the other hand, if $\mathcal{Y} = (\mathcal{Y}_P)_{P \in \mathcal{P}(T)}$ is a positive (G, T)-orthogonal set and $Q = LU \in \mathcal{F}(T)$, we will denote by $\sigma_T^Q(\cdot, \mathcal{Y})$ (resp. $\tau_T^Q(\cdot)$) the characteristic function in $\mathfrak{a}$ of the sum of $\mathfrak{a}_L$ with the convex hull of the family $(\mathcal{Y}_P)_{P \subset Q}$ (resp. the characteristic function of $\mathfrak{a}_T^L + \mathfrak{a}_Q^+$). By Lemma 1.8.4(3) of [6], we have

(5.1) $$\sum_{Q \in \mathcal{F}(T)} \sigma_T^Q(\zeta, \mathcal{Y}) \tau_Q(\zeta - \mathcal{Y}_Q) = 1, \ \zeta \in \mathfrak{a}_T.$$

For $Y \in \mathfrak{a}^+$, define the function $\tilde{v}(Y, \cdot)$ on $T \backslash G$ to be

$$\tilde{v}(Y, g) = \int_T \sigma_T^G(H_T(t), \mathcal{Y}(g, Y)) dt.$$

Set

$$J_{Y, N^{-b}}(f) = \int_{\mathfrak{t}_{>N^{-b}}} D^G(X)^{1/2} \int_{T \backslash G} 1_{<\log N}(g^{-1} X g) \tilde{v}(Y, g) \hat{f}(g^{-1} X g) dg dX$$

for all $N > 1$ and $Y \in \mathfrak{a}^+$.

Proposition 5.5. *There exists $c_1, c_2 > 0$ such that*

$$|J_{Y, N^{-b}}(f) - I_{N, N^{-b}}(f)| \ll N^{-1}.$$

for all $N > 1$ and $Y \in \mathfrak{a}^+$ that satisfy the following two inequalities

(5.2) $c_1 \log(N) \leq \inf_{\alpha \in \Delta} \alpha(Y), \quad \sup_{\alpha \in \Delta} \alpha(Y) \leq c_2 N.$

Proof. Let ω_f be a compact subset of $\mathfrak{t}$ such that $Supp(\hat{f}) \cap (\mathfrak{t} - \omega_f)^G = \emptyset$ (recall that $Supp(\hat{f})$ is compact modulo conjugation), and let $\mathcal{A}_N$ be the subset of $\omega_f \times T \backslash G$ consisting of pairs (X, g) such that $D^G(X) > N^{-b}$ and $\sigma_G(g^{-1} X g) < \log N$. Let $Q = LU \in \mathcal{F}(T)$ be a proper parabolic subgroup and $\bar{Q} = L\bar{U}$ be its opposite. Let $\mathcal{B}_N$ be the set of quadruple $(X, l, u, k) \in \mathfrak{t} \times T \backslash L \times \bar{U} \times K$ such that $(X, luk) \in \mathcal{A}_N$. Define

$$\kappa_{N,Q}^Y(g) = \int_T \kappa_N(tg) \sigma_T^Q(H_T(t), \mathcal{Y}(g, Y)) \tau_Q(H_M(t) - \mathcal{Y}(g, Y)_Q) dt$$

for all $N > 1$, $(X, g) \in \mathcal{A}_N$ and $Y \in \mathfrak{a}^+$ that satisfy (5.2). The proposition can be proved by the exact same argument as Proposition 10.10.1 of [3] once we have proved the following lemma which is an analogue of (10.10.10) of [3] for our case.

Lemma 5.6. *For c_1 large enough, we have*

$$|\kappa_{N,Q}^Y(luk) - \kappa_{N,Q}^Y(lk)| \ll N^{-1}$$

for all $N > 1$, $(X, l, u, k) \in \mathcal{B}_N$ and $Y \in \mathfrak{a}^+$ satisfying (5.2).

Proof. By (1.2.2) of [3], we know that there exists $c > 0$ such that for all $N > 1$ and for all $(X, l, u, k) \in \mathcal{B}_N$, up to translating l by an element of T, we have $\sigma_G(l), \sigma_G(u) < c \log(N)$. As a result, we only need to prove the following statement.

   (2) For c_1 large enough, we have

$$|\kappa_{N,Q}^Y(ug) - \kappa_{N,Q}^Y(g)| \ll N^{-1}$$

   for all $N > 1$, $u \in \bar{U}$, $g \in G$ and $Y \in \mathfrak{a}^+$ satisfying (5.2) and the inequalities $\sigma_G(u), \sigma_G(g) < c \log(N)$.

We have

$$|\kappa_{N,Q}^Y(ug) - \kappa_{N,Q}^Y(g)| \leq \int_T |\kappa_N(tug) - \kappa_N(tg)| \sigma_T^Q(H_T(t), \mathcal{Y}(g,Y)) \tau_Q(H_M(t) - \mathcal{Y}(g,Y)_Q) dt.$$

By the definition of κ_N and condition on g, u, we know that there exists $C > 0$ such that

$$\kappa_N(tug) - \kappa_N(tg) \neq 0 \Rightarrow \sigma_G(t) \leq CN.$$

Since the volume of the set $\{t \in T | \sigma_G(t) \leq CN\}$ is bounded by $C'N^k$ for some $C', k > 0$, it is enough to prove the following statement.

(2) For c_1 large enough, the following holds: for all $t \in T$ satisfying

$$\sigma_T^Q(H_T(t), \mathcal{Y}(g,Y)) \tau_Q(H_M(t) - \mathcal{Y}(g,Y)_Q) = 1,$$

we have

$$|\kappa_N(tug) - \kappa_N(tg)| \leq N^{-k-1}$$

for all $N > 1$, $u \in \bar{U}$, $g \in G$ and $Y \in \mathfrak{a}^+$ with $\sigma_G(u), \sigma_G(g) < c \log(N)$ and $c_1 \log(N) \leq \inf_{\alpha \in \Delta} \alpha(Y)$.

Fix a large constant $C_3 > 0$ and let Σ_Q^+ be the set of roots of T in U_Q. By choosing the constant c_1 in (5.2) large enough, we know that for all $t \in T$ satisfying

$$\sigma_T^Q(H_T(t), \mathcal{Y}(g,Y)) \tau_Q(H_M(t) - \mathcal{Y}(g,Y)_Q) = 1,$$

we have

$$< \beta, H_T(t) > \geq C_3 \log(N), \ \forall \beta \in \Sigma_Q^+$$

for all $N > 1$, $u \in \bar{U}$, $g \in G$ and $Y \in \mathfrak{a}^+$ with $\sigma_G(u), \sigma_G(g) < c \log(N)$ and $c_1 \log(N) \leq \inf_{\alpha \in \Delta} \alpha(Y)$. This implies that for any $C_4 > 0$, we can choose c_1 large enough such that the following holds: for all $t \in T$ satisfying

$$\sigma_T^Q(H_T(t), \mathcal{Y}(g,Y)) \tau_Q(H_M(t) - \mathcal{Y}(g,Y)_Q) = 1,$$

we have

$$tut^{-1} \in \exp(B(0, N^{-C_4})), \ B(0, N^{-C_4}) = \{X \in \mathfrak{g} | \ \|X\|_{\mathfrak{g}} < N^{-C_4}\}$$

for all $N > 1$, $u \in \bar{U}$, $g \in G$ and $Y \in \mathfrak{a}^+$ with $\sigma_G(u), \sigma_G(g) < c \log(N)$ and $c_1 \log(N) \leq \inf_{\alpha \in \Delta} \alpha(Y)$. Then (2) follows from the last condition of the truncation function κ_N. This proves the lemma. $\square$

$\square$

Now Theorem 5.1 follows from the proposition above and the exact same argument as in Section 10.11 of [3]. We will skip the details here.

6. The geometric expansion

In this section, we will prove the geometric expansion of the trace formula. In Section 6.1, we will prove some reductions. In Section 6.2 we will prove the geometric expansion for a special θ_K. In Section 6.3, we will prove the geometric expansion for the general case.

6.1. **Some reductions.** In this subsection we will prove some reductions for the geometric expansion. By our inductional hypothesis, we know that the multiplicity formula and trace formula hold for all the proper Levi subgroups of G and for all the groups G_t where $1 \neq t \in G_{ss}$. By the spectral side of the trace formula and the fact that the multiplicity formula holds for all induced representations, we have the following proposition which is an analogue of Proposition 11.5.1 of [3].

Proposition 6.1. *For any $\theta_K \in C^\infty(K)^K$, the distribution $I(f, \theta_K)$ only depends on the quasi-character θ_f. Moreover, there exists a continuous linear form $J(\cdot, \cdot)$ on $QC(G) \times C^\infty(K)^K$ such that*

- *$J(\theta_f, \theta_K) = I(f, \theta_K) - I_{geom}(f, \theta_K)$ for all $f \in \mathcal{C}_{scusp}(G)$ and $\theta_K \in C^\infty(K)^K$;*
- *$J(\cdot, \cdot)$ is supported on $G_{ell} \times K$.*

Proof. The proof is exactly the same as Proposition 11.5.1 of [3]. By the spectral expansion proved in the previous section and the fact that the multiplicity formula holds for all induced representations, we can just define $J(\theta, \theta_K)$ to be

$$J(\theta, \theta_K) = \sum_{\pi \in \mathcal{X}_{ell}(G), \tau \in \mathcal{X}_{ell}(K)} D(\pi) D(\tau) \theta_K(\tau) (m(\bar{\pi}, \bar{\tau}) - m_{geom}(\bar{\pi}, \bar{\tau}))$$

$$\cdot \int_{\Gamma_{ell}(G)} D^G(x) \theta(x) \theta_\pi(x) dx$$

where $\Gamma_{ell}(G)$ is the set of regular elliptic conjugacy classes of G and $\theta_K(\tau) = \mathrm{tr}(\tau(\theta_K))$. It is clear that J is supported on $G_{ell} \times K$. By (2.7.2), Proposition 4.8.1 and Corollary 5.7.2 of [3] we know that J is continuous. This proves the proposition. $\square$

Combining the proposition above with Corollary 5.7.2 of [3], we have the following corollary.

Corollary 6.2. *It is enough to prove the geometric expansion when f is a Schwartz function and $Supp(f)$ is contained in Ω_x^G where Ω_x is a small G-good open neighborhood of G_x for some $x \in G_{ell}$.*

Proposition 6.3. *The geometric expansion holds when $1 \notin Supp(\theta_f)$.*

Proof. The proposition follows from the same semisimple descent argument as in Section 11.6 of [3]. To be specific, we may assume that the test function f is of the form as in the Corollary above with $1 \neq x \in G_{ell}$. By the proposition above, we may also assume that f comes from a strongly cuspidal function f_x of G_x supported on Ω_x under the map in Proposition 5.7.1 of [3]. If x is not conjugated to an element of K, then by choosing Ω_x small enough we have $I(f, \theta_K) = I_{geom}(f, \theta_K) = 0$. It remains to consider the case when $x \in K$.

By Proposition 5.7.1 of [3] and Corollary 3.21(1), we know that (note that $|Z_K(x)/K \cap G_x| = |Z_G(x)/G_x|$)

$$I_{geom}(f, \theta_K) = I_{geom,x}((\eta_{x,G})^{-1/2} f_{x,\Omega_x}, \eta_{x,K} \theta_K|_{K \cap G_x})$$

where $I_{geom,x}$ is the analogue of I_{geom} for the model $(G_x, K \cap G_x)$. Here we have used the identity $I_{geom,x}(f_x, \theta_K|_{K \cap G_x}) = I_{geom,x}(^z f_x, \theta_K|_{K \cap G_x})$ for all $z \in Z_G(x)$ and $f_x \in \mathcal{C}_{scusp}(G_x)$ which is due to the facts that θ_K is invariant under K-conjugation and every element in $Z_G(x)$ can be written as $z = z_1 z_2$ with $z_1 \in Z_K(x)$, $z_2 \in G_x$ (note that $^z f_x(g) := f_x(z^{-1} gz)$). Hence in order to prove the proposition, it is enough to show that

$$I(f, \theta_K) = I_x((\eta_{x,G})^{-1/2} f_{x,\Omega_x}, \eta_{x,K} \theta_K|_{K \cap G_x})$$

where I_x is the analogue of I for the model $(G_x, K \cap G_x)$.

To prove this, by choosing Ω_x small, we have

$$I(f, \theta_K) = \int_{K \backslash G} \int_K {}^g f(k) \theta_K(k) dk dg$$

$$= |Z_K(x)/K \cap G_x|^{-1} \int_{K \backslash G} \int_{(K \cap G_x) \backslash K} \int_{K \cap G_x} \eta_{x,K}(k_x)^{kg} f(k_x) \theta_K(k_x) dk_x dk dg$$

$$= |Z_K(x)/K \cap G_x|^{-1} \int_{K \backslash G} \int_{(K \cap G_x) \backslash K} \int_{K \cap G_x}$$

$$\eta_{x,K}(k_x) \eta_{x,G}(k_x)^{-1/2} (^{kg} f)_{x,\Omega_x}(k_x) \theta_K(k_x) dk_x dk dg$$

$$= |Z_K(x)/K \cap G_x|^{-1} \int_{G_x \backslash G} \int_{(K \cap G_x) \backslash G_x} \int_{K \cap G_x}$$

$$\eta_{x,K}(k_x) \eta_{x,G}(k_x)^{-1/2} (^g f)_{x,\Omega_x}(g_x^{-1} k_x g_x) \theta_K(k_x) dk_x dg_x dg.$$

Here the equation on the second line follows from (3.2.5) of [3] and the equation on the third line follows from the definition of the function f_{x,Ω_x} in Section 3.2 of [3].

Introduce a function α on $Z_G(x) \backslash G$ as in Proposition 5.7.1 of [3]. Then Proposition 5.7.1 of [3] implies that for all $g \in G_x \backslash G$, there exists $z \in Z_G(x)$ such that

$$\int_{(K \cap G_x) \backslash G_x} \int_{K \cap G_x} \eta_{x,K}(k_x) \eta_{x,G}(k_x)^{-1/2} (^g f)_{x,\Omega_x}(g_x^{-1} k_x g_x) \theta_K(k_x) dk_x dg_x$$

is equal to

$$\alpha(g) I_x((\eta_{x,G})^{-1/2} \cdot {}^z f_x, \eta_{x,K} \theta_K|_{K \cap G_x}) = \alpha(g) I_x((\eta_{x,G})^{-1/2} f_x, \eta_{x,K} \theta_K|_{K \cap G_x})$$

where the last equation again follows from the fact that every element in $Z_G(x)$ can be written as $z = z_1 z_2$ with $z_1 \in Z_K(x)$ and $z_2 \in G_x$. This implies that

$$I(f, \theta_K) = |Z_K(x)/K \cap G_x|^{-1} \cdot I_x((\eta_{x,G})^{-1/2} f_{x,\Omega_x}, \eta_{x,K} \theta_K|_{K \cap G_x}) \cdot \int_{G_x \backslash G} \alpha(g) dg$$

$$= I_x((\eta_{x,G})^{-1/2} f_{x,\Omega_x}, \eta_{x,K} \theta_K|_{K \cap G_x}).$$

This finishes the proof of the proposition.   □

6.2. **A special case.** Throughout this subsection, we will use f (resp. F) to denote test function on the group (resp. Lie algebra). For $f \in \mathcal{C}_{scusp}(G)$ and $\theta_K \in C^\infty(K)^K$ (resp. $F \in \mathcal{S}_{scusp}(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$), we have defined

$$I(f, x, \theta_K) = \int_K f(x^{-1}kx)\theta_K(k)dk, \ I(f, \theta_K) = \int_{K \backslash G} I(f, x, \theta_K)dx,$$

$$I_{geom}(f, \theta_K) = m_{geom,G,K}(\theta_f, \theta_K), \ I_{Lie}(F, x, \theta_K) = \int_{\mathfrak{k}} F(x^{-1}Xx)\theta_K(X)dX,$$

$$I_{Lie}(F, \theta_K) = \int_{K \backslash G} I_{Lie}(F, x, \theta_K)dx, \ I_{geom,Lie}(F, \theta_K) = m_{geom,G,K,Lie}(\theta_F, \theta_K).$$

For $\theta \in QC(G)$ (resp. $\theta \in QC_c(\mathfrak{g})$), let

$$I_{geom}(\theta, \theta_K) = m_{geom,G,K}(\theta, \theta_K), \ I_{geom,Lie}(\theta, \theta_K) = m_{geom,G,K,Lie}(\theta, \theta_K).$$

If θ_K is the identity function, we will omit θ_K in the expression, e.g. we will use $I(f)$ to denote $I(f, \theta_K)$ when $\theta_K \equiv 1$.

Let $\omega' \subset \omega \subset \mathfrak{g}$ be two convex G-excellent open neighborhoods of 0 and let $\Omega = \exp(\omega)$, $\Omega' = \exp(\omega')$. By choosing ω' small, we can find a function $\theta_{K,0} \in C^\infty(K)^K$ such that

- $\theta_{K,0}$ is supported on $\Omega \cap K$;
- the restriction of the function $j_K^{1/2} \cdot (\theta_{K,0})_\omega$ to ω' is the identity function.

Lemma 6.4. *For $f \in \mathcal{S}_{scusp}(\Omega)$ and $\theta_K \in C^\infty(K)^K$ with $Supp(\theta_K) \in \Omega$, we have*

$$I(f, \theta_K) = I_{Lie}(j_G^{-1/2}f_\omega, j_K^{1/2} \cdot \theta_{K,\omega}), \ I_{geom}(f, \theta_K) = I_{Lie,geom}(j_G^{-1/2}f_{\omega'}, j_K^{1/2} \cdot \theta_{K,\omega}).$$

Proof. The second identity is a direct consequence of the Corollary 3.21. For the first equation, by (3.3.2) of [3], we have

$$I(f, x, \theta_K) = \int_K f(x^{-1}kx)\theta_K(k)dk = \int_{\omega \cap \mathfrak{k}} j_K^{1/2}(X)f(x^{-1}\exp(X)x)\theta_{K,\omega}(X)dX$$

$$= \int_{\mathfrak{k}} j_K^{1/2}(X)j_G^{-1/2}(X)f_\omega(x^{-1}Xx)\theta_{K,\omega}(X)dX = I_{Lie}(j_G^{-1/2}f_{\omega'}, j_K^{1/2} \cdot \theta_{K,\omega}, x).$$

This proves the lemma.   □

Proposition 6.5.   (1) *There exists a continuous linear form J_{Lie} on $QC_c(\omega')$ such that*

$$I_{Lie}(F) = J_{Lie}(\theta_F)$$

for all $F \in \mathcal{S}_{scusp}(\omega')$.

(2) *We have $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$ for all $\theta \in QC_c(\omega')$ with $0 \notin Supp(f)$.*

(3) *We have $J_{Lie}(\theta_\lambda) = |\lambda|^{\dim(K)}J_{Lie}(\theta)$ for all $\theta \in QC_c(\omega')$ and $\lambda \in \mathbb{R}^\times$ with $\theta_\lambda \in QC_c(\omega')$.*

Proof. For the first part, by Proposition 6.1, there exists a linear form J' on $QC(G)$ such that $J'(\theta_f) = I(f, \theta_{K,0})$ for all $f \in \mathcal{C}_{scusp}(G)$. Using the above lemma, we just need to let $J_{Lie}(\theta) = J'(j_G^{1/2} \cdot \theta_\Omega)$ ($\theta_\Omega \in QC_c(\Omega)$ was defined in Section 3.3 of [3]). The second part follows from Proposition 6.3.

For the third part, by Proposition 5.6.1 of [3], we only need to prove the equation when $\theta = \theta_F$ for some $F \in \mathcal{S}_{scusp}(\omega')$. Since $\theta_\lambda \in QC_c(\omega')$, we may choose F such that $F_\lambda \in \mathcal{S}_{scusp}(\omega')$ where $F_\lambda(X) = F(\lambda^{-1}X)$. We have $I_{Lie}(F) = J_{Lie}(\theta)$ and $I_{Lie}(F_\lambda) = J_{Lie}(\theta_\lambda)$. Then the equation follows from the fact that $I_{Lie}(F_\lambda) = |\lambda|^{\dim(K)} I_{Lie}(F)$. $\qquad\square$

Proposition 6.6.   (1) *If G is not split ($\iff \dim(K) > \delta(G)/2$), then $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$ for all $\theta \in QC_c(\omega')$.*
    (2) *If G is split, there exists $c \in \mathbb{C}$ such that $J_{Lie}(\theta) - I_{geom,Lie}(\theta) = c \cdot c_\theta(0)$ for all $\theta \in QC_c(\omega')$.*

Proof. If G is not split, by Lemma 2.7, there exists $\theta_1, \theta_2 \in QC_c(\omega')$ such that $\theta = (M_{\lambda,\dim(K)}-1)\theta_1+\theta_2$ with $0 \notin Supp(\theta_2)$. By the corollary above, we have $J_{Lie}(\theta_2) = I_{geom,Lie}(\theta_2)$ and $J_{Lie}((M_{\lambda,\dim(K)}-1)\theta_1) = 0$. By Corollary 3.21, we have $I_{geom,Lie}((M_{\lambda,\dim(K)}-1)\theta_1) = 0$. This proves the first part.

If G is split, by the same argument as in the non-split case we know that that there exist constants $c_\mathcal{O}$ for $\mathcal{O} \in Nil_{reg}(\mathfrak{g})$ such that

$$J_{Lie}(\theta) - I_{geom,Lie}(\theta) = \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g})} c_\mathcal{O} \cdot c_{\theta,\mathcal{O}}(0)$$

for all $\theta \in QC_c(\omega')$. For any $\mathcal{O}_1, \mathcal{O}_2 \in Nil_{reg}(\mathfrak{g})$, let $g \in Res_{\mathbb{C}/\mathbb{R}}G$ be an element as in Lemma 2.10. Since $g^{-1}Kg = K$, the linear forms J_{Lie} and $I_{geom,Lie}$ are invariant under g-conjugation. This implies that $c_{\mathcal{O}_1} = c_{\mathcal{O}_2}$. This proves the lemma. $\qquad\square$

Proposition 6.7. *Assume that G is split.*
    (1) *There exists a continuous linear form J_{Lie} on $SQC(\mathfrak{g})$ such that*

$$I_{Lie}(F) = J_{Lie}(\theta_F)$$

    for all $F \in \mathcal{S}_{scusp}(\mathfrak{g})$.
    (2) *We have $J_{Lie}(\theta_\lambda) = |\lambda|^{\dim(K)} J_{Lie}(\theta)$ for all $\lambda \in \mathbb{R}^\times$ and $\theta \in SQC(\mathfrak{g})$.*
    (3) *We can extend the linear form $I_{geom,Lie}(\theta)$ on $QC_c(\mathfrak{g})$ to $SQC(\mathfrak{g})$.*
    (4) *We have $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$ for all $\theta \in SQC(\mathfrak{g})$ with $0 \notin Supp(\theta)$.*
    (5) *There exists $c \in \mathbb{C}$ such that $J_{Lie}(\theta) - I_{geom,Lie}(\theta) = c \cdot c_\theta(0)$ for all $\theta \in SQC(\mathfrak{g})$.*

Proof. For the first part, by Theorem 5.1, we just need to let

$$J_{Lie}(\theta) = \frac{1}{c(G, K, \mathbb{R}) \cdot |W(T)|} \int_\mathfrak{t} D^G(X)^{1/2} \hat{\theta}(X) dX.$$

This also proves the second part (note that $\dim(K) = \delta(G)/2$). The third part follows from Proposition 4.6.1(ii) of [3]. For (4), by the same argument as in Proposition 11.7.1(iv) of [3], we may assume that $\theta \in QC_c(\omega)$. Then

(4) has already been proved in the previous proposition. The last part follows from the the first four parts of the proposition together with the same argument as in the previous proposition. This finishes the proof of the proposition. $\qquad\square$

Proposition 6.8. *Assume that G is split. Then $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$ for all $\theta \in SQC(\mathfrak{g})$.*

Proof. We follow the same argument as in Section 11.9 of [3]. Let θ_0 and $\theta = \hat{\theta}_0$ be the quasi-character defined in Section 11.9 of [3]. We have $c_\theta(0) \neq 0$. In order to show $c = 0$, it is enough to show that $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$. Since θ is supported in $\mathfrak{t}$ and $\mathfrak{t} \cap \mathcal{S}_{Lie}(G,K) = \{0\}$, we have

$$I_{geom,Lie}(\theta) = \frac{1}{c(G,K,\mathbb{R})} c_\theta(0) = \frac{1}{c(G,K,\mathbb{R}) \cdot |W(T)|} \int_{\mathfrak{t}} D^G(X)^{1/2} \theta_0(X) dX$$

where the second equation is just (11.9.4) of [3]. Combining with Theorem 5.1, we have $J_{Lie}(\theta) = I_{geom,Lie}(\theta)$. This finishes the proof of the proposition. $\qquad\square$

6.3. **The proof of the geometric expansion.** In this subsection we will prove the geometric side of the trace formula. By the argument in the previous two subsections, it is enough to prove the geometric expansion

$$I_{Lie}(F, \theta_K) = I_{geom,Lie}(F, \theta_K)$$

on the Lie algebra for $F \in C^\infty_{c,scusp}(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$. We also know that the geometric expansion holds when F and θ_K satisfy one of the following three conditions

- $0 \notin Supp(F)$;
- $0 \notin Supp(\theta_K)$;
- $\theta_K \equiv 1$.

Let $\omega \subset \mathfrak{g}$ be an G-excellent open neighborhood of 0 and let $\Omega = \exp(\omega)$.

Proposition 6.9.   (1) *There exists a continuous linear form J_{Lie} on $QC_c(\omega) \times C^\infty(\omega \cap \mathfrak{k})^K$ such that*

$$I_{Lie}(F, \theta_K) = J_{Lie}(\theta_F, \theta_K)$$

for all $F \in \mathcal{S}_{scusp}(\omega)$ and $\theta_K \in C^\infty(\omega \cap \mathfrak{k})^K$.
   (2) *We have $J_{Lie}(\theta, \theta_K) = I_{geom,Lie}(\theta, \theta_K)$ for all $\theta \in QC_c(\omega)$ and $\theta_K \in C^\infty(\omega \cap \mathfrak{k})^K$ with $0 \notin Supp(\theta)$ or $0 \notin Supp(\theta_K)$.*
   (3) *We have $J_{Lie}(\theta_\lambda, \theta_{K,\lambda}) = |\lambda|^{\dim(K)} J_{Lie}(\theta, \theta_K)$ for all $\theta \in QC_c(\omega)$, $\theta_K \in C^\infty(\omega \cap \mathfrak{k})^K$ and $\lambda \in \mathbb{R}^\times$ with $\theta_\lambda \in QC_c(\omega)$ and $\theta_{K,\lambda} \in C^\infty(\omega \cap \mathfrak{k})^K$.*

Proof. The proof is the same as Proposition 6.5, we will skip it here. $\qquad\square$

By the homogeneous property of the linear form J_{Lie}, we can extend it to all $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$. We get the following corollary.

Corollary 6.10. (1) *There exists a continuous linear form J_{Lie} on $QC_c(\mathfrak{g}) \times C^\infty(\mathfrak{k})^K$ such that*

$$I_{Lie}(F, \theta_K) = J_{Lie}(\theta_F, \theta_K)$$

for all $F \in \mathcal{S}_{scusp}(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$ with $Supp(F)$ being compact modulo conjugation.

(2) *We have $J_{Lie}(\theta, \theta_K) = I_{geom,Lie}(\theta, \theta_K)$ for all $\theta \in QC_c(\mathfrak{g})$ and $\theta_K \in C^\infty(\mathfrak{k})^K$ with $0 \notin Supp(\theta)$ or $0 \notin Supp(\theta_K)$.*

(3) *We have $J_{Lie}(\theta_\lambda, \theta_{K,\lambda}) = |\lambda|^{\dim(K)} J_{Lie}(\theta, \theta_K)$ for all $\theta \in QC_c(\mathfrak{g})$, $\theta_K \in C^\infty(\mathfrak{k})^K$ and $\lambda \in \mathbb{R}^\times$.*

Let $V = QC(\mathfrak{g}) \hat{\otimes} C^\infty(\mathfrak{k})^K$ realized as functions on $\mathfrak{g}_{reg} \times \mathfrak{k}$ and let V_c be the subspace of V consisting of functions whose support is compact modulo conjugation. We can extend the linear forms J_{Lie} and $I_{geom,Lie}$ to V_c. By the above corollary and Corollary 3.21, we have

(1) $J_{Lie}(\Theta) = I_{geom,Lie}(\Theta)$ for all $\Theta \in V_c$ with $0 \notin Supp(\Theta)$;

(2) $J_{Lie}(M_{\lambda,\dim(K)}\Theta) = J_{Lie}(\Theta)$ and $I_{geom,Lie}(M_{\lambda,\dim(K)}\Theta) = I_{geom,Lie}(\Theta)$ for all $\Theta \in V_c$ where $M_{\lambda,\dim(K)}\Theta(X,Y) = |\lambda|^{-\dim(K)}\Theta(\lambda^{-1}X, \lambda^{-1}Y)$.

Our goal is to prove

$$J_{Lie}(\theta, \theta_K) = I_{geom,Lie}(\theta, \theta_K)$$

for all $\theta \in QC_c(\theta)$ and $\theta_K \in C^\infty(\mathfrak{k})^K$. Since we have already proved the case when $\theta_K \equiv 1$. We may assume that $\theta_K(0) = 0$ and it is compactly supported. By Lemma 3.19, we can write $\Theta = \theta \times \theta_K \in V_c$ as $(M_{\lambda,\dim(K)} - 1)\Theta_1 + \Theta_2$ for some $\Theta_1, \Theta_2 \in V_c$ with $0 \notin Supp(\Theta_2)$. By (1) and (2) above, we have

$$J_{Lie}(\Theta) = J_{Lie}(\Theta_2) = I_{geom,Lie}(\Theta_2) = I_{geom,Lie}(\Theta).$$

This finishes the proof of the geometric expansion.

Appendix A. The proof of Lemma 3.18

Our goal is to prove the following lemma.

Lemma A.1. *The integral $\int_{\mathcal{S}(G,K)} \Delta(t)^{s-1/2} dt$ is absolutely convergent when $\frac{1}{2} > Re(s) > 0$. Moreover, there exists $d > 0$ such that*

$$\lim_{s \to 0^+} s^d \int_{\mathcal{S}(G,K)} \Delta(t)^{s-1/2} dt = 0.$$

By induction, we may assume that the lemma holds for all redutive groups whose dimension is less than G. Since $\mathcal{S}(G,K)$ is compact, it is enough to prove the absolute convergence and the inequality near an element $x \in K$. If $x \neq 1$, this follows from the inductional hypothesis (applied to the model $(G_x, K \cap G_x)$). Hence we only need to prove the case when $x = 1$. In other words, we only need to prove the following statement.

(1) For all $T \in \mathcal{T}(G, K)^\circ$, the integral $\int_{T^\circ} \Delta(t)^{s-1/2} dt$ is absolutely convergent when $\frac{1}{2} > s > 0$ and there exists $d > 0$ such that

$$\lim_{s \to 0^+} s^d \int_{T^\circ} \Delta(t)^{s-1/2} dt = 0.$$

If T is not a maximal torus of K, then G_T is not abelian. On the other hand, by the definition of $\mathcal{T}(G, K)$, we know that G_T is split modulo the center. Let T' be a maximal split torus of G_T that is θ-stable, and let L_T be the centralizer of T' in G. Then we know that L_T is a proper θ-stable Levi subgroup of G with $T \subset L$. Then (1) follows from the inductional hypothesis (applied to the model $(L_T, K \cap L_T)$). Hence it remains to prove the following statement.

(2) Let T be a maximal torus of K. Then the integral $\int_T \Delta(t)^{s-1/2} dt$ is absolutely convergent when $\frac{1}{2} > s > 0$ and there exists $d > 0$ such that

$$\lim_{s \to 0^+} s^d \int_T \Delta(t)^{s-1/2} dt = 0.$$

If T is not a maximal torus of G ($\iff$ G does not have any discrete series), we can find a θ-stable proper Levi subgroup L of G such that $T \subset L$. Then (2) follows from the inductional hypothesis (applied to the model $(L, K \cap L)$). From now on, assume that T is a maximal torus of G and let $n = \dim(T) = rank(G)$. We have $\dim(K) \geq \frac{\dim(G) - n}{2}$.

Let ω be a small convex neighborhood of 0 in $\mathfrak{t}$ and let $\omega' \subset \omega$ be a open subset such that

- $0 \notin \overline{\omega'}$;
- $\omega - \{0\} = \cup_{i=0}^\infty \frac{1}{2^i} \cdot \omega'$.

Here $\overline{\omega'}$ is the closure of ω' and $\frac{1}{2^i} \cdot \omega' = \{\frac{1}{2^i} X | X \in \omega'\}$. For example, we can let ω' to be any open subset contains $\omega - \frac{1}{2} \cdot \omega$ and is contained in $\omega - \frac{1}{3} \cdot \omega$. It is enough to show that

(3) the integral $\int_\omega \Delta(X)^{s-1/2} dX$ is absolutely convergent when $\frac{1}{2} > s > 0$ and there exists $d > 0$ such that

$$\lim_{s \to 0^+} s^d \int_\omega \Delta(X)^{s-1/2} dX = 0.$$

By induction (applied to $(G_X, G_X \cap K)$ where X is any semisimple element of ω') we know that

(4) the integral $\int_{\omega'} \Delta(X)^{s-1/2} dX$ is absolutely convergent when $\frac{1}{2} > s > 0$ and there exists $d > 0$ such that

$$\lim_{s \to 0^+} s^d \int_{\omega'} \Delta(X)^{s-1/2} dX = 0.$$

Since the function $D^G(X)$ (resp. $D^K(X)$) is homogeneous of degree $\dim(G) - n$ (resp. $\dim(K) - n$), together with the inequality $\dim(K) \geq$

$\frac{\dim(G)-n}{2}$, we know that $\Delta(t)^{s-1/2}$ is homogeneous of degree greater or equal to $-n + 2ns$ for $0 < s < \frac{1}{2}$. This implies that

$$\int_\omega \Delta(X)^{s-1/2} dX \leq \sum_{i=0}^\infty \int_{\frac{1}{2^i}\cdot\omega'} \Delta(X)^{s-1/2} dX \leq \sum_{i=0}^\infty \frac{1}{2^{2nsi}} \int_{\omega'} \Delta(X)^{s-1/2} dX$$
$$= \frac{2^{ns}}{2^{ns}-1} \cdot \int_{\omega'} \Delta(X)^{s-1/2} dX.$$

Then (3) follows from (4), the inequality above and the fact that

$$\lim_{s\to 0^+} s^2 \cdot \frac{2^{ns}}{2^{ns}-1} = 0.$$

This proves the lemma.

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