PERIODS OF AUTOMORPHIC FORMS ASSOCIATED TO STRONGLY TEMPERED SPHERICAL VARIETIES

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Abstract. In this paper, we compute the local relative characters for 10 strongly tempered spherical varieties in the unramified case. We also study the local multiplicity for these models. By proving a geometric multiplicity formula, we show that the summation of the multiplicities is always equal to 1 over each local tempered Vogan L-packet defined on the pure inner forms of the strongly tempered spherical varieties. Finally, we formulate the Ichino–Ikeda type conjecture on a relation between the period integrals and the central values of certain automorphic L-functions for those strongly tempered spherical varieties.

Contents

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1. Introduction and main results

Let k be a number field and $\mathbb{A}$ its ring of adeles. Let G be a reductive group defined over k, and H a closed connected subgroup of G. We say (G, H) is a spherical pair if $X = H\backslash G$ is a spherical G-variety (i.e., a Borel subgroup of G has a dense orbit in X). We assume that (G, H) is a spherical pair for the rest of this paper. We say the spherical pair (G, H) is reductive if H is reductive. Let Z_G be the center of G and let $Z_{G,H} = Z_G \cap H$. If (G, H) is reductive, for a cuspidal automorphic form ϕ on $G(\mathbb{A})$ whose central character is trivial on $Z_{G,H}(\mathbb{A})$, we define the period integral $\mathcal{P}_H(\phi)$ to be [1]

$$\mathcal{P}_H(\phi) := \int_{H(k)Z_{G,H}(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\, \mathrm{d}h.$$

Besides the reductive cases, one can also study the case when the spherical pair (G, H) is the Whittaker induction of a reductive spherical pair (G_0, H_0) (we refer the reader to Definition 2.2 for the definition of Whittaker induction). In this case, we have $H = H_0 \ltimes U$ where U is the unipotent radical of H and is also the unipotent radical of a parabolic

[1] In general if we allow ϕ to have nontrivial central characters, then we can also put some characters on H

subgroup of G, and the period integral is defined to be

$$\mathcal{P}_H(\phi) := \int_{H(k)Z_{G,H}(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\xi(h)^{-1}\,\mathrm{d}h$$

where $\xi = \Pi_v \xi_v$ is a generic character on $U(k)\backslash U(\mathbb{A})$, extended to $H(\mathbb{A})$ trivially on the reductive part $H_0(\mathbb{A})$. We refer the reader to Definition 2.1 for the definition of generic characters.

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ whose central character is trivial on $Z_{G,H}(\mathbb{A})$. One of the most fundamental problems in the relative Langlands program is to establish the relation between $\mathcal{P}_H|_\pi$-the period integral restricted to the space of π, and special values of some automorphic L-functions $L(s_0,\pi,\rho_X)$ of π. For example, if $G = \mathrm{SO}_{n+1} \times \mathrm{SO}_n$ and $H = \mathrm{SO}_n$, then (G,H) is the famous Gross–Prasad model defined in [GP1], [GP2] and its period integrals are related to the central value of the tensor L-function $L(1/2,\pi_1\times\pi_2)$ (here $\pi = \pi_1\otimes\pi_2$ is a cuspidal automorphic representation of $\mathrm{SO}_{n+1}(\mathbb{A})\times\mathrm{SO}_n(\mathbb{A})$, and the representation ρ_X is the standard tensor product representation of ${}^LG$). This point of view was most systematically put forward by Sakellaridis [Sa12], and Sakellaridis-Venkatesh [SV17]. As in [SV17], the spherical varieties under the consideration in this paper have no Type N spherical root and are wavefront. We refer the reader to Sections 2.1 and 3.1 of [SV17] for the definitions of wavefront and spherical roots.

In general, in order to find the L-functions related to the period integral $\mathcal{P}_H(\phi)$ for $\phi = \otimes_v\phi_v \in \otimes_v\pi_v$, one needs to compute the local relative character $I_{H_v}(\phi_v)$ for the spherical pair $(G_v,H_v) := (G(k_v),H(k_v))$ over unramified places $v \in |k|$. If the model (G,H) is strongly tempered (see Section 2.1 for the definition of strongly tempered) or is the Whittaker induction of a strongly tempered pair (G_0,H_0), the local relative character $I_{H_v}(\phi_v)$ is defined to be the integration of the matrix coefficients over $H(k_v)$, i.e.

$$I_{H_v}(\phi_v) = \int_{Z_{G,H}(k_v)\backslash H(k_v)} \langle \pi_v(h)\phi_v,\phi_v\rangle \xi_v(h)^{-1}\,\mathrm{d}h. \tag{1.1}$$

Note that if (G,H) is the Whittaker induction of a strongly tempered pair, the integral above needs to be regularized (see Section 2.4 for details). In general, if the model (G,H) is not strongly tempered, the local relative character $I_{H_v}(\phi_v)$ is defined via the Plancherel formula. For details, see Section 17.3 of [SV17].

For each spherical pair (G,H), one expects that the local relative character $I_{H_v}(\phi_v)$ equals the quotient of some special values of some local L-functions $\frac{L(s_0,\pi_v,\rho_X)}{L(1,\pi_v,\mathrm{Ad})}$ times a product of certain special values of local zeta functions (denoted by Δ_{X_v}) over all the unramified places. For instance, for the orthogonal Gross–Prasad model (which is strongly tempered), the local relative character was computed by Ichino–Ikeda [II], which is equal to

$$\frac{L(\frac{1}{2},\pi_{1,v}\times\pi_{2,v})}{L(1,\pi_v,\mathrm{Ad})}\cdot\Delta_{\mathrm{SO}_{n+1},v}(1).$$

Here for any reductive group G defined over k that is split over an unramified extension, we use $\Delta_G(s)=\Pi_{v\in|k|}\Delta_{G,v}(s)$ to denote the L-function of the dual $M^\vee$ to the motive M associated to G introduced by Gross in [G].

In [Sa], Sakellaridis developed a general method to compute the local relative characters at unramified places under certain conditions. He showed that the L-function $L(s,\pi,\rho_X)$ is determined by the so-called "virtual colors" of the spherical variety X and the extra factor Δ_{X_v} is related to the volume of $X(\mathcal{O}_v)$ ($\mathcal{O}_v$ is the ring of integers of k_v). He also explicitly computed the virtual colors of many spherical varieties and hence the L-functions $L(s,\pi,\rho_X)$ (see Page 1379 of [Sa]).

In this paper, following the method of Sakellaridis, we explicitly compute the local relative characters for all the strongly tempered reductive spherical varieties without Type N spherical root. We also compute the local relative characters for 7 non-reductive spherical varieties that are the Whittaker inductions of the trilinear GL_2 model $(\mathrm{GL}_2^3,\mathrm{GL}_2)$. Our computation shows that the period integrals for these strongly tempered spherical varieties are always related to the central value of some L-functions of symplectic type, i.e. $s_0=\frac{1}{2}$ and ρ_X is a self-dual representation of ${}^L(G/Z_{G,H})$ of symplectic type. Moreover, we show that the extra factors Δ_{X_v} is equal to $\Delta_{G,v}(1)/\Delta_{H_0/Z_{G,H},v}(1)$ for all the models under consideration (we would like point out that this is only true in the strongly tempered case). Note if H is reductive we just let $H=H_0$ and $U=1$.

In addition, we study the local multiplicities for all the models considered in this paper (except for the E_7 case). By proving a geometric multiplicity formula, we show that the summation of the multiplicities is always equal to 1 over each local tempered Vogan L-packet defined on the pure inner forms of these spherical varieties. In other words, our results indicate that all these strongly tempered spherical varieties enjoy the same local and global properties with the Gan–Gross–Prasad models.

Finally, combining our formulas of the local relative characters and our results for the local multiplicities, we are able to formulate the Ichino–Ikeda type conjectures for these models.

1.1. **The local relative character.** By the classification of split reductive spherical pairs in [BP] (here we say the spherical pair (G,H) is split if both G and H are split), it is easy to show that a split strongly tempered reductive spherical pair is either one of the following 4 cases

$$(1.2)\qquad (\mathrm{GL}_{n+1}\times \mathrm{GL}_n, \mathrm{GL}_n),\ (\mathrm{SO}_{n+1}\times \mathrm{SO}_n, \mathrm{SO}_n),$$
$$(\mathrm{GL}_4\times \mathrm{GL}_2, \mathrm{GL}_2\times \mathrm{GL}_2),\ (\mathrm{GSp}_6\times \mathrm{GSp}_4, (\mathrm{GSp}_4\times \mathrm{GSp}_2)^0),$$

or it is a split symmetric pair (recall that we say a symmetric pair is split if the real form associated to it is split, e.g. $(\mathrm{GL}_n,\mathrm{SO}_n)$, $(\mathrm{Sp}_{2n},\mathrm{GL}_n)$). Here $(\mathrm{GSp}_4\times \mathrm{GSp}_2)^0=\{(g,h)\in \mathrm{GSp}_4\times \mathrm{GSp}_2 \mid l(g)=l(h)\}$ where l is the similitude character of GSp. We refer the reader to Section 3.1 for the explicit description of the embeddings. By the classification of spherical root system in [BP], all the split symmetric pairs have Type N spherical root unless G only has one simple root (i.e. G is of Type A_1). If G only has one simple root, then split symmetric pair (G,H) is essentially the model $(\mathrm{PGL}_2,\mathrm{GL}_1)$. So we only need to consider the 4 models in (1.2).

Remark 1.1. *For each model in* (1.2)*, we can always modify the groups up to some central elements and some finite isogeny, which will give us some other models with the same root systems (this will also preserve the strongly tempered property). For example, the model* $(\mathrm{GSp}_6\times \mathrm{GSp}_4, (\mathrm{GSp}_4\times \mathrm{GSp}_2)^0)$ *and the model* $(\mathrm{Sp}_6\times \mathrm{Sp}_4, \mathrm{Sp}_4\times \mathrm{Sp}_2)$ *have the same root systems. In this paper, we will always choose the spherical pairs* (G,H) *so that over the local field* k_v*, there is only one open Borel orbit in* $G(k_v)/H(k_v)$*. For example, the model* $(\mathrm{GSp}_6\times \mathrm{GSp}_4, (\mathrm{GSp}_4\times \mathrm{GSp}_2)^0)$ *we choose indeed has only one open Borel orbit (see Section 3.1) while the model* $(\mathrm{Sp}_6\times \mathrm{Sp}_4, \mathrm{Sp}_4\times \mathrm{Sp}_2)$ *has* $|k_v^\times/(k_v^\times)^2|$*-many open Borel orbits.*

We also want to point out that for a fixed root system, we may have more than one models with this root system and such that there is a unique open Borel orbit over every local field. An easy example would be the models $(\mathrm{SO}_4\times \mathrm{SO}_3, \mathrm{SO}_3)$ *and* $((\mathrm{PGL}_2)^3, \mathrm{PGL}_2)$*. Another example is* $(\mathrm{GL}_{n+1}\times \mathrm{GL}_n, \mathrm{GL}_n)$ *and* $(U_{n+1}\times U_n, U_n)$*.*

The first one $(\mathrm{GL}_{n+1}\times \mathrm{GL}_n, \mathrm{GL}_n)$ is the model for the Rankin-Selberg integral of $\mathrm{GL}_{n+1}\times \mathrm{GL}_n$. There is also an analogue of this model for unitary groups, which is call the unitary Gan–Gross–Prasad model. The local relative characters have been computed by R. Neal Harris [H] for both the general linear case and the unitary case. In the general linear case (resp. unitary case), ρ_X is the standard tensor product representation of ${}^LG$ (resp. the standard product representation of base change). The second one $(\mathrm{SO}_{n+1}\times \mathrm{SO}_n, \mathrm{SO}_n)$ is the Gross–Prasad model for special orthogonal groups and the local relative characters have been computed by Ichino and Ikeda [II]. In this case, ρ_X is the standard tensor product representation of ${}^LG$. For these three models, the period integrals are related to the central values of the tensor L-functions.

In this paper, we give an explicit formula for the local relative characters over unramified places for the remaining two cases $(\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$ and $(\mathrm{GSp}_6\times\mathrm{GSp}_4, (\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$, as well as the analogue of the model $(\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$ for unitary groups. We also computed 7 non-reductive cases that are the Whittaker inductions of the trilinear GL_2-model $(\mathrm{GL}_2^3, \mathrm{GL}_2)$ (which is strongly tempered).

To be specific, we consider the following table where (G, H) is the spherical pair and ρ_X is a representation of the L-group of $G/Z_{G,H}$.

№	G	H	ρ_X	$\Delta_{X,v} = \Delta_{G,v}(1)/\Delta_{H_0/Z_{G,H},v}(1)$
1	$\mathrm{GL}_4\times\mathrm{GL}_2$	$\mathrm{GL}_2\times\mathrm{GL}_2$	$(\wedge^2\otimes\mathrm{std}_2)\oplus\mathrm{std}_4\oplus\mathrm{std}_4^\vee$	$\zeta_v(1)\zeta_v(3)\zeta_v(4)$
2	$\mathrm{GU}_4\times\mathrm{GU}_2$	$(\mathrm{GU}_2\times\mathrm{GU}_2)^0$	$(\wedge^2\otimes\mathrm{std}_2)\oplus\mathrm{std}_4\oplus\mathrm{std}_4^\vee$	$*$
3	$\mathrm{GSp}_6\times\mathrm{GSp}_4$	$(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0$	$\mathrm{Spin}_7\otimes\mathrm{Spin}_5$	$\zeta_v(1)^2\zeta_v(4)\zeta_v(6)$
4	GL_6	$\mathrm{GL}_2\ltimes U$	$\wedge^3$	$\zeta_v(1)\zeta_v(3)\zeta_v(4)\zeta_v(5)\zeta_v(6)$
5	GU_6	$\mathrm{GU}_2\ltimes U$	$\wedge^3$	$**$
6	GSp_{10}	$\mathrm{GL}_2\ltimes U$	Spin_{11}	$\zeta_v(1)\zeta_v(4)\zeta_v(6)\zeta_v(8)\zeta_v(10)$
7	$\mathrm{GSp}_6\times\mathrm{GL}_2$	$\mathrm{GL}_2\ltimes U$	$\mathrm{Spin}_7\otimes\mathrm{std}_2$	$\zeta_v(1)\zeta_v(2)\zeta_v(4)\zeta_v(6)$
8	$\mathrm{GSO}_8\times\mathrm{GL}_2$	$\mathrm{GL}_2\ltimes U$	$\mathrm{HSpin}_8\otimes\mathrm{std}_2$	$\zeta_v(1)^2\zeta_v(2)\zeta_v(4)^2\zeta_v(6)$
9	GSO_{12}	$\mathrm{GL}_2\ltimes U$	HSpin_{12}	$\zeta_v(1)\zeta_v(4)\zeta_v(6)^2\zeta_v(8)\zeta_v(10)$
10	E_7	$\mathrm{PGL}_2\ltimes U$	ω_7	$\zeta_v(6)\zeta_v(8)\zeta_v(10)\zeta_v(12)\zeta_v(14)\zeta_v(18)$

TABLE 1

Here std_n is the standard representation of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{std}_n^\vee$ is its dual representation, Spin_{2n+1} is the Spin representation of $\mathrm{Spin}_{2n+1}(\mathbb{C})$, HSpin_{2n} is a half-Spin representation of $\mathrm{Spin}_{2n}(\mathbb{C})$, ω_7 is the 56 dimensional representation of E_7, and

$$* = \zeta_v(1)^2\zeta_v(4)L(1,\eta_{k_v'/k_v})L(3,\eta_{k_v'/k_v}),$$

$$** = \zeta_v(1)\zeta_v(4)\zeta_v(6)L(1,\eta_{k_v'/k_v})L(3,\eta_{k_v'/k_v})L(5,\eta_{k_v'/k_v})$$

where η_{k_v'/k_v} is the quadratic character for the quadratic extension k_v'/k_v. We refer the reader to Section 6 for more details about the representation ρ_X for Models 2 and 5.

Theorem 1.2. *For all the spherical pairs in Table 1, assume that all the data are unramified over v. Then*

$$I_{H_v}(\phi_v) = \frac{\Delta_{G,v}(1)}{\Delta_{H_0/Z_{G,H},v}(1)} \cdot \frac{L(\frac{1}{2}, \pi_v, \rho_X)}{L(1, \pi_v, \mathrm{Ad})} \tag{1.3}$$

where ρ_X is a self-dual symplectic representation of ${}^L(G/Z_{G,H})$ given in Table 1.

Remark 1.3. *In* (1.1)*, we choose the local Haar measure* $\mathrm{d}h$ *such that* $\mathrm{vol}(H(\mathcal{O}_v), \mathrm{d}h) = 1$. *If we replace it by Weil's canonical measure* $\mathrm{d}_{\mathrm{can}}h = \Delta_{H_0/Z_{G,H},v}(1)\mathrm{d}h$ *(*[Weil, Chapter 2]*), then the constant* $\Delta_{H_0/Z_{G,H},v}(1)$ *in the above theorem will disappear.*

In Section 2, we will explain our strategies of the proof of this theorem. We will also give the formulas of the Whittaker–Shintani functions of these 10 spherical pairs in Propositions 2.13 and 2.31.

For the rest of this subsection, we explain how we derive the non-reductive models in Table 1. Model 4 was introduced by Ginzburg–Rallis in [GR] and Model 5 is an analogue of Model 4 for similitude unitary groups. Model 9 and 10 are inspired by one row of the Magic Triangle introduced by Deligne and Gross in [DG] (which is a generalization of the the Freudenthal's Magic Square). We recall the following row in the Magic Triangle in [DG, Table 1], a series of algebraic groups of type:

$$A_1 \subset A_1^3 := A_1 \times A_1 \times A_1 \subset C_3 \subset A_5 \subset D_6 \subset E_7.$$

In this sequence, we observe the spherical pair of type (A_1^3, A_1) corresponding to the trilinear GL_2-model. And the algebraic groups G of types A_5, D_6 and E_7 have a parabolic subgroup $P = LU$ such that the Levi subgroup L is of type A_1^3 and the stabilizer H_0 of the generic characters ξ of U is of type A_1. This gives us the Whittaker inductions of the trilinear GL_2-model for these 3 groups, which are the Models 4, 9, and 10 respectively. Meanwhile, the group of type C_3 does not have a Levi subgroup of type A_1^3, but it can be fixed by considering the product $C_3 \times A_1$. This explains Model 7.

In addition, these non-reductive models are also related to the degenerated Whittaker models of smooth admissible representations (we refer the reader to [GZ] for more details.) For instance, consider the degenerated Whittaker model $Wh_\xi(\pi)$ of an irreducible representation π of GSO_{12} with respect to (U, ξ) in Model 9. Here (U, ξ) is arisen from a nilpotent orbit of partition $[6, 6]$ in the Lie algebra of GSO_{12} and $Wh_\xi(\pi)$ is considered as an H_0-module in sense of [GZ]. (Note that the partition $[6,6]$ is used to label two distinct stable nilpotent orbits. However, the corresponding models have no essential differences as explained in Section 8.3.) The distinguished problem in Model 9 is equivalent to determine when the trivial representation of H_0 is a quotient representation in $Wh_{\xi}(\pi)$. By using the theta correspondence, Gomez and Zhu in [GZ] showed that the H_0-module $Wh_{\xi}(\pi)$ is isomorphic to the degenerated Whittaker model of certain representations of GSp_{10} as an H_0-module, arisen from the nilpotent orbit of the partition $[5,5]$ in the Lie algebra of GSp_{10}. Hence, following [GZ], Model 6 and Model 9 are directly bridged by the theta correspondence. Similarly, Model 7 and Model 8 are also bridged by the theta correspondence.

Finally, Model 8 can be viewed as a reduced model of Model 9. To be specific, we can choose a parabolic subgroup of GSO_{12} in Model 9 whose Levi subgroup is isomorphic to $\mathrm{GSO}_8 \times \mathrm{GL}_2$ such that the intersection of the Levi subgroup with the subgroup H of GSO_{12} in Model 9 is exactly the subgroup H of $\mathrm{GSO}_8 \times \mathrm{GL}_2$ in Model 8. Under this point of view, we can also view Model 7 as a reduced model of Model 6, view Model 4 as a reduced model of Model 9 and view Model 9 as a reduced model of Model 10. This explains all the non-reductive models in Table 1. We summarize the relations among these models in the following diagram:

$$\begin{tikzcd}
 & (\mathrm{GL}_6, \mathrm{GL}_2 \ltimes U) \arrow[r, "\text{outer form}"'] & (\mathrm{GU}_6, \mathrm{GU}_2 \ltimes U) \\
(E_7, \mathrm{PGL}_2 \ltimes U) \arrow[r, "\text{reduced}"'] & (\mathrm{GSO}_{12}, \mathrm{GL}_2 \ltimes U) \arrow[u, "\text{reduced}"'] \arrow[r, "\text{reduced}"'] \arrow[d, leftrightarrow, "\theta\text{-correspondence}"] & (\mathrm{GSO}_8 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes U) \arrow[d, leftrightarrow, "\theta\text{-correspondence}"] \\
 & (\mathrm{GSp}_{10}, \mathrm{GL}_2 \ltimes U) \arrow[r, "\text{reduced}"'] & (\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes U)
\end{tikzcd}$$

Remark 1.4. *Besides the 7 non-reductive cases in the table above, there are another three more non-reductive spherical pairs that are the Whittaker induction of strongly tempered reductive spherical pairs without Type N spherical root:*

(1) The Whittaker models for quasi-split reductive groups.
(2) The non-reductive Gan–Gross–Prasad models for the general linear groups, the unitary groups, or the orthogonal groups. They are the Whittaker inductions of the reductive Gan–Gross–Prasad models.

(3) *The model* $(\mathrm{GSO}_{10}, (\mathrm{GL}_2 \times \mathrm{GL}_1) \ltimes U)$ *introduced by Ginzburg* [Gi] *in his study of the Spin L-function of* GSO_{10}. *This is the Whittaker induction of the model* $(\mathrm{GL}_3 \times \mathrm{GL}_2, \mathrm{GL}_2)$.

The local relative characters of the Whittaker models have been computed by Lapid-Mao in [LM] *and the local relative characters of the non-reductive Gan–Gross–Prasad models have been computed by Liu in* [L]. *The period integral of the model* $(\mathrm{GSO}_{10}, (\mathrm{GL}_2 \times \mathrm{GL}_1) \ltimes U)$ *has been studied by Ginzburg in* [Gi] *and its local relative character can be computed by the same method as in this paper. The local relative characters over unramified places for these models are also of the form* (1.3) *as our models in Table 1. The representation* ρ_X *is the tensor representation for the non-reductive Gan–Gross–Prasad models, and the Spin representation of* $\mathrm{GSpin}_{10}(\mathbb{C})$ *for the model* $(\mathrm{GSO}_{10}, (\mathrm{GL}_2 \times \mathrm{GL}_1) \ltimes U)$. *For the Whittaker model, the numerator L-function* $L(\frac{1}{2}, \pi, \rho_X)$ *is just 1.*

In general, by a tedious case by case argument (i.e. we checked all the parabolic subgroups of all the reductive groups) which we will not include in this paper, we believe that any spherical pair that are the Whittaker induction of a strongly tempered spherical pair without Type N spherical root must be one of the 10 cases above (7 in Table 1 and 3 in this remark). Hence the local relative character of a spherical pair that is either strongly tempered or the Whittaker induction of a strongly tempered spherical pair should always be the form (1.3) *over unramified places.*

1.2. **The local multiplicity.** Let (G, H) be one of the models in Table 1. If H is reductive, take χ to be the trivial character of $H(k_v)$; if $H = H_0 \ltimes U$ is non-reductive, take χ to be the character $1 \otimes \xi_v$ of $H(k_v) = H_0(k_v) \ltimes U(k_v)$ where ξ_v is the generic character of $U(k_v)$. Let π_v be an irreducible admissible representation of $G(k_v)$ whose central character is trivial on $Z_{G,H}(k_v)$. Define the multiplicity

$$m(\pi_v) := \dim \mathrm{Hom}_{H(k_v)}(\pi_v, \chi_v).$$

In Section 9, for all the models in Table 1 except the E_7 case, we will prove a multiplicity formula $m(\pi_v) = m_{geom}(\pi_v)$ for all the tempered representations over non-archimedean fields or complex field. In the real case, we can prove the multiplicity formula for Models 1–4. Then by using the multiplicity formula, together with the character identity in the local Langlands conjecture, we can show that the summation of the multiplicities is always equal to 1 over every local tempered Vogan L-packet (i.e. strong multiplicity one over the L-packet). Moreover, we will also show that the unique distinguished element in the L-packet corresponds to a character of the component group (note the the component group for some cases in Table 1 is not necessarily abelian). We refer the reader to Section 9 for more details.

Remark 1.5. *The local multiplicity of some models in Table 1 has already been studied in our previous works. More specifically, Model 4 has been studied by the first author (*[Wan15], [Wan16], [Wan17]*), Model 5 has been studied in our previous paper* [WZ]*, and Model 1 has been studied in* [PWZ19].

Remark 1.6. *Like in the Gan–Gross–Prasad model case (Section 17 of* [GGP]*), one can also formulate an explicit conjecture about the unique distinguished element in the L-packet using the local epsilon factor $\epsilon(s,\pi_v,\rho_X)$ (i.e. the epsilon dichotomy conjecture). We will discuss this in our next paper* [WZ1].

1.3. **The Ichino–Ikeda type conjecture.** Combining the results in the previous two subsections, we can now formulate the Ichino–Ikeda type conjectures for all the models in Table 1. Let (G,H) be one of these models. Since we assume that the central character is trivial on $Z_{G,H}$, we are actually working with the model $(G/Z_{G,H},H/Z_{G,H})$. Following the definition in Section 16.5 of [SV17], the pure inner forms of the spherical varieties are parameterized by the set $H^1(k,H/Z_{G,H})$. For all the models in Table 1 except Model 2, there is a natural bijection between the set $H^1(k,H/Z_{G,H})$ and the set of quaternion algebras D over k. For each quaternion algebra D/k (or for each $D\in H^1(k,H/Z_{G,H})$ in the case of Model 2), we can define an analogue of the model (G,H) associated to D, which will be denoted by (G_D,H_D). We can also define the period integral $\mathcal{P}_{H_D}(\phi_D)$ and the local relative character $I_{H_{D,v}}(\phi_{D,v})$ where ϕ_D is a cuspidal automorphic form on $G_D(\mathbb{A})$. We refer the reader to later sections for the detailed descriptions of (G_D,H_D) for each spherical variety in Table 1. Remark that in our cases G_D and H_D are not the pure inner forms of G and H in general. But after module the central part $Z_{G,H}$, they become pure inner forms of $G/Z_{G,H}$ and $H/Z_{G,H}$, respectively.

We fix a global tempered cuspidal L-packet $\Pi_\phi=\cup_D\Pi_\phi(G_D)$ of $G(\mathbb{A})$ whose central character is trivial on $Z_{G,H}(\mathbb{A})$. For each $\pi_D\in\Pi_\phi(G_D)$ in the L-packet, as in Section 17.4 of [SV17], let $\nu:\pi_D\to\mathcal{A}_{cusp}(G_D(\mathbb{A}))$ be an embedding such that the period integral is identically zero on the orthogonal complement of $\nu(\pi_D)$ in the π_D-isotypic component $\mathcal{A}_{cusp}(G_D(\mathbb{A}))_{\pi_D}$. This embedding is not unique if the multiplicity of π_D in $\mathcal{A}_{cusp}(G_D(\mathbb{A}))$ is greater than 1, but it does not affect the global conjecture.

We first consider all the models in Table 1 except the first one. For those models, the center of $H/Z_{G,H}$ is anisotropic.

Conjecture 1.7. *Let D/k be a quaternion algebra that may be split (or $D\in H^1(k,H/Z_{G,H})$ if we are in the case of Model 2), $\pi_D\in\Pi_\phi(G_D)$ and $\phi_D\in\nu(\pi_D)$. We have*

$$\begin{aligned}|\mathcal{P}_{H_D}(\phi_D)|^2 &= \frac{1}{|S_\phi|}\cdot\frac{C_{H/Z_{G,H}}}{\Delta_{H_0/Z_{G,H}}(1)^S}\cdot\lim_{s\to 1}\frac{\Delta_G(s)^S}{L(1,\Pi_\phi,Ad)^S}\\ &\cdot L(1/2,\Pi_\phi,\rho_X)^S\cdot\Pi_{v\in S}I_{H_{D,v}}(\phi_{D,v})\end{aligned}$$

where

- *S is a finite subset of $|k|$ such that ϕ is unramified outside S, and $\Delta_{H/Z_{G,H}}(1)^S,\Delta_G(s)^S$, $L(1/2,\Pi_\phi,\rho_X)^S,L(1,\Pi_\phi,Ad)^S$ are the partial L-functions.*
- *$C_{H/Z_{G,H}}$ is the Haar measure constant of $H/Z_{G,H}$ defined in Section 1 of* [II] *(see also Section 1 of* [L]*), and the period integral $\mathcal{P}_{H_D}$ is defined by the Tamagawa measure on $Z_{G_D,H_D}(\mathbb{A})\backslash H_D(\mathbb{A})$.*
- *S_ϕ is the conjectural global component group associated to the L-packet Π_ϕ. We refer the reader to Section 3.2 of* [LM] *for details.*

Then we consider the first model $(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2)$ in Table 1. In this case, we have $Z_H/Z_{G,H}\cong\mathrm{GL}_1$ and

$$(G/Z_{G,H},H/Z_{G,H})=(\mathrm{GL}_4\times\mathrm{GL}_2/\mathrm{GL}_1^{diag},\ \mathrm{GL}_2\times\mathrm{GL}_2/\mathrm{GL}_1^{diag}).$$

Conjecture 1.8. *Under the above notation, we have*

$$\begin{aligned}|\mathcal{P}_{H_D}(\phi_D)|^2 &= \frac{1}{|S_\phi|}\cdot\frac{C_{H/Z_H}}{\Delta_{H_0/Z_H}(1)^S}\cdot\lim_{s\to 1}\frac{\Delta_G(s)^S}{L(1,\Pi_\phi,Ad)^S}\\ &\cdot L(1/2,\Pi_\phi,\rho_X)^S\cdot\Pi_{v\in S}\zeta_v(1)I_{H_{D,v}}(\phi_{D,v}).\end{aligned}$$

Note that we have the extra factor $\zeta_v(1)$ due to $Z_H/Z_{G,H}=\mathrm{GL}_1$. This point of view has been discussed in Section 17.5 of [SV17].

In particular, we have the following weak global conjecture, which is a direct consequence of the conjectures above and the multiplicity-one theorem on the local Vogan packets.

Conjecture 1.9. *The following are equivalent:*

(1) *$L(\frac{1}{2},\Pi_\phi,\rho_X)\neq 0$;*
(2) *There exists a quaternion algebra D/k (or $D\in H^1(k,H/Z_{G,H})$ if we are in the case of Model 2) such that the period integral $\mathcal{P}_{H_D}(\phi_D)$ is nonzero for some $\phi_D\in\nu(\pi_D)$ and $\pi_D\in\Pi_\phi(G_D)$.*

Moreover, if the above conditions hold, there exist a unique D and a unique $\pi_D \in \Pi_\phi(G_D)$ that satisfy Condition (2).

When D/k is split, one direction of Conjecture 1.9 has been proved for Models 1 and 4 in joint works of the first author with Pollack and Zydor ([PWZ18], [PWZ19]).

Finally, similar to Gan–Gross–Prasad models as discussed in Section 27 of [GGP], one expects that the central value of L-functions in Models 2 and 3 of Table 1 are related to the arithmetic geometry of the cycles of the certain Shimura varieties. In Model 2, $\mathrm{GU}_4 \times \mathrm{GU}_2$ and $(\mathrm{GU}_2 \times \mathrm{GU}_2)^0$ can be associated with Shimura varieties of dimensions 5 and 2 (resp. 3 and 1). In Model 3, $\mathrm{GSp}_6 \times \mathrm{GSp}_4$ and $(\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0$ can be associated with Shimura varieties of dimensions 9 and 4. Then predicted by Beilinson–Bloch Conjecture, the order of $L(s, \pi, \rho_X)$ at $s = 1/2$ should be related to the rank of the Chow groups of the corresponding cycles, which are all in the middle degree. Like in Gross–Prasad models, Conjecture 1.7 would help one to relate the height pairing against the cycles to the first derivatives $L'(1/2, \pi, \rho_X)$.

1.4. **Organization of the paper.** In Section 2, we explain the strategy of our computation of the local relative characters. In Sections 3 and 4, we compute the local relative characters for the two split reductive cases in Table 1. In Sections 5 and 7, we study the non-reductive cases for GL_6 and E_7, respectively. In Section 6, we deal with the non-split models $(\mathrm{GU}_4 \times \mathrm{GU}_2, (\mathrm{GU}_2 \times \mathrm{GU}_2)^0)$ and $(\mathrm{GU}_6, \mathrm{GL}_2 \ltimes U)$ in Table 1. In Section 8, we compute the formulas for the remaining 4 models. Finally, in Section 9, we will study the local multiplicity for all these models.

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2. The strategy

In this section, we will explain the strategy of our computation. In the reductive cases, we closely follow the method developed by Sakellaridis in [Sa]. For the Whittaker induction cases, due to the non-trivial unipotent radical of H, the local characters $I_{H_v}(\phi_v)$ in (1.1) in these cases are not absolutely convergent. To overcome this convergent issue, we modify the method by regularizing the unipotent integrals. Then for all cases, we can reduce the computation of local relative characters to evaluate the local integrals associated to each simple root of G and verify certain combinatorial identities. We refer the reader to the detailed strategies in Section 2.3.1 for the reductive case and in Section 2.5.1 for the non-reductive case.

More precisely, in Section 2.1 we discuss some notation and conventions of spherical varieties. Then we discuss the strategies for the reductive cases in Sections 2.2 and 2.3, and for the non-reductive cases in Sections 2.4 and 2.5, respectively.

In Sections 2–8, we only consider the non-archimedean places v such that all data are unramified. Denote by $F = k_v$ a p-adic field. Let $\mathcal{O}_F$ be its ring of integers. Fix a uniformizer ϖ, and denote by $\mathbb{F}_q$ the residue field of F with cardinality q and of characteristic p with $p \neq 2$. Fix a nontrivial unramified additive character $\psi : F \to \mathbb{C}^\times$ of F.

2.1. **Notation.** Let G be a connected reductive group defined over F, and Z_G be the center of G. We fix a maximal open compact subgroup K of $G(F)$ and let $\mathrm{d}g$ be the Haar measure on $G(F)$ such that the volume of K is equal to 1. Denote by W_G the Weyl group of $G(F)$.

Definition 2.1. *Let $P = LU$ be a proper parabolic subgroup of G defined over F. For a character $\xi : U(F) \to \mathbb{C}^\times$ of $U(F)$, denote by L_ξ the neutral component of the stabilizer of ξ in L (under the adjoint action).*

A character ξ is called a generic character of $U(F)$ if $\dim(L_\xi)$ is minimal, i.e. $\dim(L_\xi) \leq \dim(L_{\xi'})$ for any character ξ' of $U(F)$. It is easy to see that if ξ is a generic character, so is ${}^l\xi$ for all $l \in L(F)$, where ${}^l\xi$ is the character of $U(F)$ defined by ${}^l\xi(n) = \xi(l^{-1}nl)$.

Moreover, there are finitely many generic characters of $U(F)$ up to $L(F)$-conjugation, which are in bijection with the open $L(F)$-orbits in $\mathfrak{u}(F)/[\mathfrak{u}(F), \mathfrak{u}(F)]$ induced by the adjoint action on the Lie algebra $\mathfrak{u}(F)$ of $U(F)$.

Let $H \subset G$ be a connected closed subgroup also defined over F. We say that H is a spherical subgroup if there exists a Borel subgroup B of G (not necessarily defined over F since $G(F)$ may not be quasi-split) such that BH is Zariski open in G. Such a Borel subgroup is unique up to $H(\bar{F})$-conjugation. Then, (G,H) is called a spherical pair and $X=G/H$ is the corresponding spherical variety of G.

From now on, we assume that H is a spherical subgroup. We say the spherical pair (G,H) is reductive if H is reductive.

Definition 2.2. *A spherical pair (G,H) is called a Whittaker induction of a reductive spherical pair (G_0,H_0) if there exists a parabolic subgroup $P=LU$ of G, and a generic character $\xi: U(F)\to\mathbb{C}^\times$ such that $H=H_0\ltimes U$ where $G_0\cong L$ and $H_0\cong L_\xi\subset L$ is the neutral component of the stabilizer of ξ in L.*

Alternatively, we say that (G,H) is the Whittaker induction of the triple (G_0,H_0,ξ). For convenience, we also consider a reductive spherical pair (G,H) as the Whittaker induction of $(G,H,1)$.

Remark 2.3. *In general the stabilizer of a generic character is not necessarily a reductive or spherical subgroup of L. For instance, if we take $G=\mathrm{GL}_3$ and a parabolic subgroup with Levi subgroup $L\cong\mathrm{GL}_2\times\mathrm{GL}_1$, then L_ξ is isomorphic to the Borel subgroup of GL_2, which is not reductive; if we take $G=\mathrm{GL}_9$ and a parabolic subgroup with Levi subgroup $L\cong\mathrm{GL}_3\times\mathrm{GL}_3\times\mathrm{GL}_3$, then $L_\xi\cong\mathrm{GL}_3$ is not a spherical subgroup of L.*

Finally, for a reductive spherical pair (G,H), we say it is *strongly tempered* if all the tempered matrix coefficients of $G(F)$ are absolutely convergent on $H(F)/Z_{G,H}(F)$. If the spherical pair (G,H) is the Whittaker induction of a reductive spherical pair (G_0,H_0), we say (G,H) is strongly tempered if (G_0,H_0) is strongly tempered.

In the rest of this section, we assume that G is split (this is true for all the models in Table 1 except the $\mathrm{GU}_4\times\mathrm{GU}_2$ and GU_6 cases). The computation for the quasi-split case is slightly different from the split case. We refer the reader to Section 6 for details.

2.2. **The reductive case: some reduction.** Let (G,H) be a reductive strongly tempered spherical pair with $G(F)$ split. Assume that it does not have Type N spherical root. Let $B=TN$ be a Borel subgroup of G defined over F, T the maximal split torus in B and N the unipotent radical of B, and $\bar{B}=T\bar{N}$ be its opposite. There exists a unique open Borel orbit $B(F)\eta H(F)$ (note that for each root system, we already choose suitable representatives (G,H) so that it has unique open Borel orbit, see Remark 1.1). For all the four models in (1.2), it is easy to verify $H(F)\cap\eta^{-1}B(F)\eta=Z_{G,H}(F)$, i.e. the stabilizer of the open Borel orbit belongs to the center of G.

Remark 2.4. *This is not true if the spherical pair has a Type N spherical root. For example, for the model* $(\mathrm{GL}_3, \mathrm{SO}_3)$*, the stabilizer of the open orbit is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$ *and does not belong to the center of* G*.*

Our goal is to compute the local relative character

$$I(\phi_\theta) = \int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\,\mathrm{d}h$$

where ϕ_θ is the unramified matrix coefficient of $I_B^G(\theta)$ normalized by $\phi_\theta(1) = 1$, θ is a unitary unramified character of $T(F)$, and $I_B^G(\cdot)$ is the normalized induced representation from the Borel subgroup B. The integral is absolutely convergent since (G, H) is strongly tempered. We follow the method in Sections 6-7 of [Sa].

Let f_θ be the unramified vector in $I_B^G(\theta)$ with $f_\theta(1) = 1$. Then the normalized unramified matrix coefficient ϕ_θ is given by $\phi_\theta(g) = \int_K f_\theta(kg)\,\mathrm{d}k$. This implies that

$$\begin{aligned} I(\phi_\theta) = \int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\,\mathrm{d}h &= \int_{H(F)/Z_{G,F}(F)} \int_K f_\theta(kh)\,\mathrm{d}k\,\mathrm{d}h \\ &= \int_K \int_{H(F)/Z_{G,F}(F)} f_\theta(kh)\,\mathrm{d}h\,\mathrm{d}k. \end{aligned}$$

Note that since the integral is convergent if we replace θ by its absolute value (which changes f_θ to $f_{|\theta|} = |f_\theta|$), the above double integral is absolutely convergent. In particular, the integral

$$\int_{H(F)/Z_{G,F}(F)} f_\theta(kh)\,\mathrm{d}h \tag{2.1}$$

is absolutely convergent for almost all $k \in K$. As a function on $k \in G$, this integral is right $H(F)$-invariant and left $(B(F), \delta_B^{1/2}\theta)$-invariant, where δ_B is the modular character of B. Since $B(F)\eta H(F)$ is open in $G(F)$, we have the integral (2.1) is absolutely convergent for all $k \in B(F)\eta H(F)$.

On the other hand, consider the function $\mathcal{Y}_\theta$ on $G(F)$ satisfying the following conditions:

(1) $\mathcal{Y}_\theta$ is supported on the open orbit $B(F)\eta H(F)$ with $\mathcal{Y}_\theta(\eta) = 1$;
(2) $\mathcal{Y}_\theta$ is right $H(F)$-invariant and left $(B(F), \theta^{-1}\delta_B^{1/2})$-invariant.

For $g \in B(F)\eta H(F)$, $\mathcal{Y}_{\theta^{-1}}(g)$ is proportional to (2.1) and then

$$\int_{H(F)/Z_{G,F}(F)} f_\theta(gh)\,\mathrm{d}h = \int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h \cdot \mathcal{Y}_{\theta^{-1}}(g).$$

In consequence, since the complementary set of $B(F)\eta H(F)$ has measure zero, we have

$$\begin{aligned} I(\phi_\theta) &= \int_K \int_{H(F)/Z_{G,F}(F)} f_\theta(kh)\,\mathrm{d}h\,\mathrm{d}k \\ &= \int_K \mathcal{Y}_{\theta^{-1}}(k)\,\mathrm{d}k \times \int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h. \end{aligned}$$

To obtain a formula of $I(\phi_\theta)$, it suffices to compute

$$\int_K \mathcal{Y}_{\theta^{-1}}(k)\,\mathrm{d}k \quad \text{and} \quad \int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h.$$

To evaluate the integral $\int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h$, we need the following lemma.

Lemma 2.5. *Under the above notation, for $f \in C_c^\infty(G(F))$, we have*

$$\int_G f(g)\,\mathrm{d}g = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)} \zeta(1)^{-rk(G)} \int_{H(F)/Z_{G,H}(F)} \int_{B(F)} f(b\eta h)\,\mathrm{d}b\,\mathrm{d}h,$$

where $rk(G)$ is the F-rank of G.

Proof. Without loss of generality, it is sufficient to consider the case $\eta = 1$, that is, $H(F)B(F)$ is an open dense subset of $G(F)$. Denote by $\mathrm{d}_{\mathrm{can}}g$, $\mathrm{d}_{\mathrm{can}}b$, and $\mathrm{d}_{\mathrm{can}}h$ the Weil's canonical measures on the smooth varieties G, B and $H/Z_{G,H}$, respectively. Since $B \cap H = Z_{G,H}$ and BH is open dense in G, by [Weil, Chapter 2] we have

$$\int_{G(F)} f(g)\,\mathrm{d}_{\mathrm{can}}g = \int_{H(F)/Z_{G,H}(F)} \int_{B(F)} f(bh)\,\mathrm{d}_{\mathrm{can}}b\,\mathrm{d}_{\mathrm{can}}h.$$

By [Weil, Chapter 2], since the smooth varieties X under consideration are smooth over $\mathcal{O}_F$ and have good reduction over $\mathbb{F}_q$, we have

$$vol(X(\mathcal{O}_F), \mathrm{d}_{\mathrm{can}}x) = \frac{|X(\mathbb{F}_q)|}{q^{\dim X}}.$$

This implies that

$$\mathrm{d}_{\mathrm{can}}g = \frac{|G(\mathbb{F}_q)|}{q^{\dim G}}\,\mathrm{d}g, \quad \mathrm{d}_{\mathrm{can}}b = \frac{|B(\mathbb{F}_q)|}{q^{\dim B}}\,\mathrm{d}b,\ \mathrm{d}_{\mathrm{can}}h = \frac{|H/Z_{G,H}(\mathbb{F}_q)|}{q^{\dim(H)-\dim(Z_{G,H})}}\,\mathrm{d}h.$$

Since $B \cap H = Z_{G,H}$, we have

$$\int_G f(g)\,\mathrm{d}g = \frac{|B(\mathbb{F}_q)| \cdot |H/Z_{G,H}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} \int_{H(F)/Z_{G,H}(F)} \int_{B(F)} f(bh) dbdh.$$

Now the lemma follows from the following equation which is a consequence of (3.1) and (5.1) of [G]

$$\frac{|B(\mathbb{F}_q)|\cdot|H/Z_{G,H}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|}=\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}.$$

□

By Lemma 2.5, $\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k=\int_{G(F)} 1_K(g)\mathcal{Y}_\theta(g)\,\mathrm{d}g$ is equal to

$$\begin{aligned}&\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\int_{H(F)/Z_{G,H}(F)}\int_{B(F)} 1_K(b\eta h)\theta^{-1}\delta^{\frac{1}{2}}(b)\,\mathrm{d}b\,\mathrm{d}h\\ =\;&\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h,\end{aligned}$$

where 1_K is the characteristic function on K. As a result, we have proved the following proposition, which reduces to evaluate the integral $\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k$.

Proposition 2.6. *The local relative character $I(\phi_\theta)$ is equal to*

$$\begin{aligned}&\int_K \mathcal{Y}_{\theta^{-1}}(k)dk\times\int_{H(F)/Z_{G,H}(F)} f_\theta(\eta h)\,\mathrm{d}h\\ =\;&\frac{\Delta_{H/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\int_K \mathcal{Y}_{\theta^{-1}}(k)\,\mathrm{d}k\times\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k.\end{aligned}$$

In the next subsection 2.3, we will explain how to compute the integral $\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k$.

Proposition 2.7. *Let Φ^+ be the set of positive roots of G. There is a decomposition of the weights of a representation ρ_X of $\hat{G}$, denoted by $\Theta=\Theta^+\cup\Theta^-$, such that*

$$\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k=\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\cdot\beta(\theta), \tag{2.2}$$

where

$$\beta(\theta)=\frac{\prod_{\alpha\in\Phi^+}1-q^{-1}e^{\alpha^\vee}}{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}}(\theta).$$

Moreover, we have

$$\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}(\theta^{-1})=\prod_{\gamma^\vee\in\Theta^-}1-q^{-\frac{1}{2}}e^{\gamma^\vee}(\theta). \tag{2.3}$$

Here for $\alpha\in\Phi^+$, we use $e^{\alpha^\vee}(\theta)$ to denote $\theta(e^{\alpha^\vee}(\varpi))$. For $\gamma^\vee\in\Theta^+$, we can identify it with a co-weight of G and we let $e^{\gamma^\vee}$ be the associated homomorphism from GL_1 to T. We define $e^{\gamma^\vee}(\theta)=\theta(e^{\gamma^\vee}(\varpi))$.

Remark 2.8. *For all the models in Table 1, the representation ρ_X in the proposition above (or Proposition 2.25 for the non-reductive cases) is just the representation ρ_X listed in Table 1. In Theorem 7.2.1 of* [Sa]*, for general (not necessarily strongly tempered) spherical varieties, Sakellaridis proved the identities* (2.2) *and* (2.3) *for a W_X-invariant set Θ of weights of $\hat{G}$. Here $W_X \subset W$ is the little Weyl group of X and we have $W = W_X$ if the model is strongly tempered. Later in Corollary 7.3.3 of* [SW]*, Sakellaridis-Wang proved that in the case when (G, H) over functional fields is strongly tempered and H is reductive, Θ must be the set of weights of a representation ρ_X of $\hat{G}$. Our computation in later sections shows that for all the non-reductive cases in Table 1, Θ is also the set of weights of a representation ρ_X of $\hat{G}$.*

Combining Propositions 2.6 and 2.7, we have

$$I(\phi_\theta) = \frac{\Delta_{H/Z_{G,H}}(1)}{\Delta_G(1)} \zeta(1)^{rk(G)} \int_K \mathcal{Y}_{\theta^{-1}}(k)\, \mathrm{d}k \times \int_K \mathcal{Y}_\theta(k)\, \mathrm{d}k$$

$$= \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)} \zeta(1)^{-rk(G)} \cdot \beta(\theta) \cdot \beta(\theta^{-1}) = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)} \cdot \frac{L(1/2, \pi, \rho_X)}{L(1, \pi, \mathrm{Ad})}.$$

This finishes the computation. The L-functions $\frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}$ is just the L_X, L-function of the spherical variety $X = G/H$, defined in Definition 7.2.3 of [Sa].

2.3. **The computation of S_θ.** Set

$$S_\theta(g) = \int_K \mathcal{Y}_\theta(kg^{-1})\, \mathrm{d}g \text{ for } g \in G(F),$$

which is the Whittaker-Shintani function. (See [KMS03] for instance.) In this section, our goal is to prove Proposition 2.7, i.e. compute $S_\theta(1)$. Here, we follow the arguments and the notation in [KMS03]. The unexplained notations and more details are referred to [KMS03, Mac, C80].

Let $\mathcal{I} = B(\mathcal{O}_F)\bar{N}(\varpi\mathcal{O}_F)$ be the Iwahori subgroup of $G(F)$. For all the strongly tempered models in the introduction, we can choose a representative η in the open double coset of $B(F)\backslash G(F)/H(F)$ so that it satisfies the following lemma (we will check this lemma for each model in the later sections).

Lemma 2.9. *Then there exists a representative η for the open double coset of $B(F)\backslash G(F)/H(F)$ such that $\eta \in K$ and $\bar{N}(\varpi\mathcal{O}_F)\eta \subset T(\mathcal{O}_F)N(\varpi\mathcal{O}_F)\eta H(\mathcal{O}_F)$.*

For $w \in W$ (W is the Weyl group of G), let $\Phi_w = 1_{\mathcal{I}w\mathcal{I}}$ be the characteristic function of $\mathcal{I}w\mathcal{I}$. Then 1_K is equal to $\sum_{w\in W} \Phi_w$. (See [Iwa66] for instance.) Let α be a simple root and w_α be the corresponding reflection in W. We would need to compute

$$I_\alpha(\theta) = vol(\mathcal{I})^{-1} \int_{G(F)} \mathcal{Y}_\theta(x\eta)(\Phi_1(x) + \Phi_{w_\alpha}(x))\, \mathrm{d}x$$

for all simple roots α.

First, by Lemma 2.9, we have $\mathcal{I}\eta \subset B(\mathcal{O}_F)\eta H(\mathcal{O}_F)$. Hence $\mathcal{Y}_\theta(x\eta) = 1$ for all $x \in \mathcal{I}$. This implies that

$$vol(\mathcal{I})^{-1} \int_{G(F)} \mathcal{Y}_\theta(x\eta)\Phi_1(x)\, \mathrm{d}x = 1. \tag{2.4}$$

For each root $\alpha \in \Phi_G$ of G, let (note that all the root spaces are one dimensional since we have assumed that G is split)

$$u_\alpha \colon a \in F \mapsto u_\alpha(a) \in N(F) \tag{2.5}$$

be the one-parameter unipotent subgroup of $G(F)$ associated to the root α.

Lemma 2.10. *We have*

$$I_\alpha(\theta) = 1 + q \int_{\mathcal{O}_F} (\theta^{-1}\delta_B^{\frac{1}{2}})(e^{\alpha^\vee}(a^{-1}))\mathcal{Y}_\theta(u_{-\alpha}(a^{-1})\eta)\, \mathrm{d}a, \tag{2.6}$$

where δ_B is the modular character of B.

Proof. This proof is similar to the one of Lemma 8.4 in [KMS03]. It is sufficient to compute the integral

$$vol(\mathcal{I})^{-1} \int_{\mathcal{I}w_\alpha\mathcal{I}} \mathcal{Y}_\theta(x\eta)\, \mathrm{d}x.$$

First, let us evaluate $\mathcal{Y}_\theta(\cdot)$ on the set $\mathcal{I}w_\alpha\mathcal{I}\eta$. Referring to [Mac, Chapter 2], one has

$\mathcal{I}w_\alpha\mathcal{I} = B(\mathcal{O}_F)w_\alpha U_\alpha(\mathcal{O}_F)\bar{N}(\varpi\mathcal{O}_F)$ and $U_\alpha(\mathcal{O}_F) = \{u_\alpha(a) \mid a \in \mathcal{O}_F\}$,

By Lemma 2.9,

$$\mathcal{I}w_\alpha\mathcal{I}\eta \subset B(\mathcal{O}_F)w_\alpha U_\alpha(\mathcal{O}_F)\eta H(\mathcal{O}_F).$$

As $vol(\mathcal{I}w_\alpha\mathcal{I}) = q \cdot vol(\mathcal{I})$, it follows that

$$\begin{aligned} \int_{\mathcal{I}w_\alpha\mathcal{I}} \mathcal{Y}_\theta(x\eta)\, \mathrm{d}x &= vol(\mathcal{I}w_\alpha\mathcal{I}) \int_{\mathcal{O}_F} \mathcal{Y}_\theta(w_\alpha u_\alpha(a)\eta)\, \mathrm{d}a \\ &= q \cdot vol(\mathcal{I}) \int_{\mathcal{O}_F} \mathcal{Y}_\theta(w_\alpha u_\alpha(a)\eta)\, \mathrm{d}a. \end{aligned}$$

Then, since $w_\alpha u_\alpha(a) = u_\alpha(a^{-1})t_0 \cdot e^{\alpha^\vee}(a^{-1})u_{-\alpha}(a^{-1})$ for some $t_0 \in T(\mathcal{O}_F)$, we have

$$\mathcal{Y}_\theta(w_\alpha u_\alpha(a)\eta) = (\theta^{-1}\delta_B^{\frac{1}{2}})(e^{\alpha^\vee}(a^{-1}))\mathcal{Y}_\theta(u_{-\alpha}(a^{-1})\eta).$$

This proves the lemma. □

Then for each model, by an explicit matrix computation, we will show that there exists $\beta_\alpha^\vee \in \Theta$ such that $-\beta_\alpha^\vee + \alpha^\vee \in \Theta$ (here we view $\alpha^\vee$ as a weight on the dual group) and

$$\mathcal{Y}_\theta(u_{-\alpha}(a^{-1})\eta) = \theta(e^{\beta_\alpha^\vee}(1+a^{-1})) \cdot |1+a^{-1}|^{-1/2}. \tag{2.7}$$

This implies that

(2.8)

$$\begin{aligned}
I_\alpha(\theta) =& 1 + q\int_{\mathcal{O}_F} (\theta^{-1}\delta^{\frac{1}{2}})(e^{\alpha^\vee}(a^{-1}))\mathcal{Y}_\theta(u_{-\alpha}(a^{-1})\eta)\,\mathrm{d}a \\
=& 1 + q\int_{\mathcal{O}_F} (\theta^{-1}\delta^{\frac{1}{2}})(e^{\alpha^\vee}(a^{-1}))\theta(e^{\beta_\alpha^\vee}(1+a^{-1})) \cdot |1+a^{-1}|^{-1/2}\,\mathrm{d}a \\
=& 1 + q\int_{\mathcal{O}_F} \theta(e^{\alpha^\vee}(a)) \cdot |a|^{-1} \cdot \theta(e^{\beta_\alpha^\vee}(1+a^{-1})) \cdot |1+a^{-1}|^{-1/2}\,\mathrm{d}a \\
=& 1 + q\int_{\mathcal{O}_F} (\theta(e^{\beta_\alpha^\vee}) \cdot |\ |^{-1/2})(1+a) \cdot (\theta(e^{\alpha^\vee-\beta_\alpha^\vee}) \cdot |\ |^{-1/2})(a)\,\mathrm{d}a \\
=& (q-1) \cdot \frac{1-q^{-1}e^{\alpha^\vee}(\theta)}{(1-q^{-1/2}e^{\beta_\alpha^\vee}(\theta))(1-q^{-1/2}e^{-\beta_\alpha^\vee+\alpha^\vee}(\theta))}.
\end{aligned}$$

Here we use the fact that for unitary unramified characters χ_1, χ_2 of $F^\times$, the integral

$$q\int_{\mathcal{O}_F} (\chi_1 \cdot |\ |^{-1/2})(1+a) \cdot (\chi_2 \cdot |\ |^{-1/2})(a)da \tag{2.9}$$

is absolutely convergent and is equal to

$$q - 2 + (q-1) \cdot \frac{q^{-1/2}\chi_1(\varpi) + q^{-1/2}\chi_2(\varpi) - 2q^{-1}\chi_1\chi_2(\varpi)}{(1-q^{-1/2}\chi_1(\varpi))(1-q^{-1/2}\chi_2(\varpi))}.$$

The proof of this identity is similar to (and easier than) the two identities in Section 6.2. We omit the details here.

Remark 2.11. *The set $\{\beta_\alpha^\vee,\ \alpha^\vee - \beta_\alpha^\vee \mid \alpha \in \Delta(G)\}$ is the set of virtual weighted colors of $X = G/H$ defined in Section 7.1 of* [Sa]*. There is another way to compute the virtual weighted colors using the Luna diagram of $X = G/H$. In* [Lu]*, Luna computed the Luna diagram for all the split reductive spherical varieties of Type A. The Luna diagram of all the split reductive spherical varieties was computed in* [BP]*. In*

Section 3, we will use the model $(\mathrm{GSp}_6\times\mathrm{GSp}_4, (\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ *as an example to explain how to use the Luna diagram to compute the virtual weighted colors. We refer the reader to Remark 3.6 for details.*

Definition 2.12. *Let* Θ^+ *be the unique subset of* Θ *satisfying the following condition:*

- *For every simple root* α*, we have* $\Theta^+ - w_\alpha\Theta^+ = \{\beta_\alpha^\vee,\ \alpha^\vee-\beta_\alpha^\vee\}$.

Recall that for all the models in Table 1, Θ *is the set of weights of the representation* ρ_X *of* $\hat{G}$ *listed in Table 1. We define*

$$\beta(\theta)=\frac{\prod_{\alpha\in\Phi^+}1-q^{-1}e^{\alpha^\vee}}{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}}(\theta)\ \text{and}\ c_{WS}(\theta)=\frac{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+}1-e^{\alpha^\vee}}(\theta).$$

For a Weyl element $w\in W$, the intertwining operator $T_w\colon I_B^G(\theta)\to I_B^G(w\theta)$ is defined by

$$T_w(f)(g)=\int_{N(F)\cap wN(F)w^{-1}\backslash N(F)} f(w^{-1}ng)\,\mathrm{d}n,\ f\in I_B^G(\theta).$$

It is absolutely convergent when θ is positive enough and admits a meromorphic continuation (see Theorem IV.1.1 of [W03]).

By Theorem 1.2.1 of [Sa08], the space $\mathrm{Hom}_{H(F)}(I_B^G(\theta),1)$ is one-dimensional for almost all θ in the unitary line. In fact, for all the cases in Table 1, the little Weyl group W_X of the spherical variety is equal to the Weyl group W. This implies that the factor $(\mathcal{N}_W(\delta^{-1/2}A_X^*):W_X)$ in loc. cit. is equal to 1. Moreover, since the spherical variety has a unique open Borel orbit, the factor $|H^1(k,A_X)|$ in loc. cit. is also equal to 1 (see Lemma 3.4.1 of [Sa08]). This implies that the Hom-space is one dimensional. In Section 9, we will prove a multiplicity formula of the dimension of this Hom space for all the tempered representations which will imply that the Hom-space $\mathrm{Hom}_{H(F)}(I_B^G(\theta),1)$ is actually one dimensional for all unitary characters. But we don't need this result in our computation.

By the definition of $\mathcal{Y}_\theta$, we can define an element ℓ_θ in the Hom-space $\mathrm{Hom}_{H(F)}(I_B^G(\theta),1)$ to be

$$\ell_\theta(\mathcal{P}_\theta(f))=\int_{G(F)} f(g)\mathcal{Y}_\theta(g)\,\mathrm{d}g,\ \text{for}\ f\in\mathcal{C}_c^\infty(G),\tag{2.10}$$

where $\mathcal{P}_\theta(f)=\int_{B(F)}(\theta^{-1}\delta_B^{\frac{1}{2}})(b)f(bg)\,\mathrm{d}b$ is the canonical $G(F)$-equivariant map from $\mathcal{C}_c^\infty(G(F))$ to $I_B^G(\theta)$. Since the Hom-space is one dimensional for almost all θ, for each simple reflection $w_\alpha\in W$ associated to a simple root α, there exists a rational function $b_{w_\alpha}(\theta)$ on θ such that

$$\ell_{w_\alpha\theta}\circ T_{w_\alpha}=c_\alpha(\theta)b_{w_\alpha}(\theta)\ell_\theta.\tag{2.11}$$

Here $c_\alpha(\theta)=\frac{1-q^{-1}e^{\alpha^\vee}}{1-e^{\alpha^\vee}}(\theta)$ is the c-function defined in [C80].

Our goal is to obtain a formula of $b_{w_\alpha}(\theta)$. Similar to the proof of Theorem 10.5 in [KMS03], to obtain $b_{w_\alpha}(\theta)$, we evaluate both sides of equation (2.11) at $\mathcal{P}_\theta(\Phi_1+\Phi_{w_\alpha})$. Note that

$$T_{w_\alpha}(\mathcal{P}_\theta(\Phi_1+\Phi_{w_\alpha}))=c_\alpha(\theta)\mathcal{P}_{w_\alpha\theta}(\Phi_1+\Phi_{w_\alpha}).$$

Then, under the choice (2.10) of ℓ_θ, we have

$$vol(\mathcal{I})I_\alpha(\theta)=\ell_\theta(\mathcal{P}_\theta\circ R(\eta)(\Phi_1+\Phi_{w_\alpha})),$$

where R is the right translation of $G(F)$. On the other hand,

$$\begin{aligned}\ell_{w_\alpha\theta}\circ T_{w_\alpha}(\mathcal{P}_\theta\circ R(\eta)(\Phi_1+\Phi_{w_\alpha}))&=c_\alpha(\theta)b_{w_\alpha}(\theta)\ell_{w_\alpha\theta}(\mathcal{P}_{w_\alpha\theta}\circ R(\eta)(\Phi_1+\Phi_{w_\alpha}))\\&=c_\alpha(\theta)vol(\mathcal{I})I_\alpha(w_\alpha\theta).\end{aligned}$$

Following (2.11), we obtain

$$b_{w_\alpha}(\theta)=\frac{I_\alpha(w_\alpha\theta)}{I_\alpha(\theta)}. \tag{2.12}$$

Recall that $S_\theta(g)=\ell_\theta(R(g)\mathcal{P}_\theta(1_K))$. Plugging $T_{w_\alpha}(\mathcal{P}_{w_\alpha\theta}(1_K))=c_\alpha(\theta)\mathcal{P}_\theta(1_K)$ into the left hand side of (2.11), we have

$$\begin{aligned}S_{w_\alpha\theta}(g)&=\ell_{w_\alpha\theta}(R(g)\mathcal{P}_{w_\alpha\theta}(1_K))\\&=c_\alpha(\theta)^{-1}(\ell_{w_\alpha\theta}\circ T_{w_\alpha})(R(g)\mathcal{P}_\theta(1_K))=b_{w_\alpha}(\theta)S_\theta(g).\end{aligned}$$

Thus for all simple roots α of G, we have

$$\frac{S_{w_\alpha\theta}(g)}{S_\theta(g)}=b_{w_\alpha}(\theta)=\frac{I_\alpha(w_\alpha\theta)}{I_\alpha(\theta)}=\frac{\beta(w_\alpha\theta)}{\beta(\theta)}.$$

This implies that $S_\theta(g)/\beta(\theta)$ is W-invariant as a function of θ.

Proposition 2.13. *Let $T(F)^+=\{t\in T(F)|\ t^{-1}N(\mathcal{O}_F)t\subset N(\mathcal{O}_F)\}$ be the positive chamber of $T(F)$. Then*

$$S_\theta(\eta^{-1}t)/\beta(\theta)=q^{l(W)}vol(\mathcal{I})\sum_{w\in W}c_{WS}(w\theta)((w\theta)^{-1}\delta_B^{\frac12})(t^{-1}),\ \textit{for } t\in T(F)^+,$$

where $l(W)$ is the length of the longest Weyl element in W.

Proof. First, we show that

$$S_\theta(\eta^{-1}t)=vol(\mathcal{I}t(\lambda)\mathcal{I})^{-1}R(1_{\mathcal{I}t\mathcal{I}})S_\theta(\eta^{-1}), \tag{2.13}$$

where R is the right convolution defined by

$$R(1_{\mathcal{I}t\mathcal{I}})S_\theta(\eta^{-1})=\int_{G(F)}1_{\mathcal{I}t\mathcal{I}}(x)S_\theta(\eta^{-1}x)\,\mathrm{d}x=\int_{\mathcal{I}t\mathcal{I}}S_\theta(\eta^{-1}x)\,\mathrm{d}x.$$

Now, it is enough to show that

$$\eta^{-1}\mathcal{I}t\mathcal{I}\subset H(\mathcal{O}_F)\eta^{-1}t\mathcal{I}\subset H(\mathcal{O}_F)\eta^{-1}tK,$$

which follows from Lemma 2.9.

Similar to Proposition 1.10 of [KMS03], there exists a basis $\{f_w : w \in W\}$ of $I_B^G(\theta)^{\mathcal{I}}$ such that

$$R(1_{\mathcal{I}t\mathcal{I}})f_w = vol(\mathcal{I}t^{-1}\mathcal{I})(w\theta)^{-1}\delta_B^{\frac{1}{2}}(t^{-1})f_w, \tag{2.14}$$

$$f_1 = \mathcal{P}_\theta(\Phi_1),\ \mathcal{P}_\theta(1_K) = q^{l(w_\ell)}\sum_{w\in W} c_w(\theta)f_w$$

for $t \in T(F)^+$ where $c_w(\theta) = \prod_{\alpha>0, w\alpha>0} c_\alpha(\theta)$.

Recall that $S_\theta(g) = \ell_\theta(R(g)\mathcal{P}_\theta(1_K))$. Substituting (2.14) into S_θ, we have

$$S_\theta(\eta t)/\beta(\theta) = q^{l(w_\ell)}\beta(\theta)^{-1}\sum_{w\in W} c_w(\theta)\ell_\theta(R(\eta)f_w)\cdot(w\theta)^{-1}\delta^{\frac{1}{2}}(t^{-1}).$$

By (2.4) and $f_1 = \mathcal{P}_\theta(\Phi_1)$, we have the coefficient of $(w\theta)^{-1}\delta^{\frac{1}{2}}(t^{-1})$ for $w = 1$ in $S_\theta(\eta t)/\beta(\theta)$ is equal to

$$q^{l(w_\ell)}\beta(\theta)^{-1}c_1(\alpha) = q^{l(w_\ell)}vol(\mathcal{I})c_{WS}(\theta),$$

where $c_1(\theta) = \prod_{\alpha\in\Phi^+}\frac{1-q^{-1}e^{\alpha^\vee}}{1-e^{\alpha^\vee}}(\theta)$. Since $S_\theta(\eta t)/\beta(\theta)$ is W-invariant, by the linear independence of the characters $(w\theta)^{-1}\delta^{\frac{1}{2}}(t^{-1})$ for generic θ, we obtain

$$S_\theta(\eta^{-1}t)/\beta(\theta) = q^{l(w_\ell)}vol(\mathcal{I})\sum_{w\in W} c_{WS}(w\theta)(w\theta)^{-1}\delta^{\frac{1}{2}}(t^{-1}).$$

□

Since $\eta \in K$, we have $S_\theta(1) = S_\theta(\eta^{-1})$. Combining with the proposition above, we have

$$S_\theta(1)/\beta(\theta) = q^{l(W)}vol(\mathcal{I})\sum_{w\in W} c_{WS}(w\theta).$$

Since $vol(\mathcal{I}) = \Delta_G(1)\zeta(1)^{-rk(G)}\cdot q^{-l(W)}$, we have

$$S_\theta(1)/\beta(\theta) = \Delta_G(1)\zeta(1)^{-rk(G)}\sum_{w\in W} c_{WS}(w\theta).$$

Hence in order to prove Proposition 2.7, it is enough to prove the following lemma.

Lemma 2.14. *The summation* $\sum_{w\in W} c_{WS}(w\theta)$ *is independent of the choice of* θ *and is equal to* $\frac{1}{\Delta_{H/Z_{G,H}}(1)}$.

Proof. Since the spherical varieties for the reductive cases are affine, the first part of this statement follows from Theorem 7.2.1 of [Sa]. For the second part, since the summation is independent of the choice of θ, we can compute it by plugging in some special θ. We will compute it for each of our models in later sections. □

Remark 2.15. *For all reductive cases, if we set $\theta = \delta_B^{1/2}$ as in Lemma 4.2.3 of* [Sa] *(which is used in the proof of* [Sa, Theorem 7.2.1]*), then the only nonvanishing term in the summation is the term corresponding to the longest Weyl element, which is equal to $\frac{1}{\Delta_{H/Z_{G,H}}(1)}$. We will show this for all the reductive models in Table 1 in later sections.*

2.3.1. *The summary.* By the discussion in the previous two subsections, in order to compute the relative character in the reductive case, we just need to perform the following steps:

(1) Show that the double cosets $B(F)\backslash G(F)/H(F)$ have a unique open orbit $B(F)\eta G(F)$ and the representative η can be chosen to satisfy Lemma 2.9.
(2) Verify the identity (2.7) by expressing the product $u_{-\alpha}(a)\eta$ in terms of the decomposition $B(F)\eta H(F)$. This gives us the set of virtual weighted colors of X.
(3) Compute the subset Θ^+ of Θ and show that it satisfies (2.3).
(4) Compute the constant $\sum_{w\in W} c_{WS}(w\theta)$, i.e. Lemma 2.14. This computation is easy for the reductive case, see Remark 2.15.

2.3.2. *The trilinear model.* To end this subsection, we use the trilinear GL_2 model as an example to explain the method. This example also appeared in Section 7.2.4 of [Sa] and we will use it for the non-reductive cases in Table 1 , which are the Whittaker inductions of the trilinear GL_2-model. Let $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ and $H = \mathrm{GL}_2$ diagonally embedded into G. Let B be the upper triangular Borel subgroup of G and $\eta_0 = (I_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$. It is easy to see that $B(F)\eta_0 H(F)$ is the unique open orbit and η_0 satisfies Lemma 2.9.

Let Θ be the set of weights of the tensor product representation of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. We can write it as $\{e_i + e_j' + e_k'' \mid 1 \leq i, j, k \leq 2\}$. Let α_i $(1 \leq i \leq 3)$ be the simple root of the i-th copy of

GL_2. We have

$$u_{-\alpha_1}(a)\eta_0=(\begin{pmatrix}\frac{1}{1-a}&0\\0&1\end{pmatrix},\begin{pmatrix}1&\frac{-a}{1-a}\\0&\frac{1}{1-a}\end{pmatrix},\begin{pmatrix}\frac{1}{1-a}&\frac{-a}{1-a}\\0&1\end{pmatrix})\cdot\eta_0\cdot\begin{pmatrix}1-a&0\\a&1\end{pmatrix}^{\mathrm{diag}},$$

$$u_{-\alpha_2}(b)\eta_0=(\begin{pmatrix}1&-\frac{b}{1-b}\\0&\frac{1}{1-b}\end{pmatrix},\begin{pmatrix}\frac{1}{1-b}&0\\0&1\end{pmatrix},\begin{pmatrix}\frac{1}{1-b}&0\\0&1\end{pmatrix})\cdot\eta_0\cdot\begin{pmatrix}1&b\\0&1-b\end{pmatrix}^{\mathrm{diag}},$$

$$u_{-\alpha_3}(c)\eta_0=(\begin{pmatrix}1&0\\0&\frac{1}{1+c}\end{pmatrix},\begin{pmatrix}\frac{1}{1+c}&0\\0&1\end{pmatrix},\begin{pmatrix}\frac{1}{1+c}&0\\0&1\end{pmatrix})\cdot\eta_0\cdot\begin{pmatrix}1&0\\0&1+c\end{pmatrix}^{\mathrm{diag}}.$$

This proves (2.7) and implies that (note that the representation has trivial central character)

$$\beta_{\alpha_1}^\vee=e_1+e_2'+e_1'',\ \alpha_1^\vee-\beta_{\alpha_1}^\vee=e_1+e_1'+e_2'',\beta_{\alpha_2}^\vee=e_2+e_1'+e_1'',$$

$$\alpha_2^\vee-\beta_{\alpha_2}^\vee=e_1+e_1'+e_2'',\beta_{\alpha_3}^\vee=e_2+e_1'+e_1'',\ \alpha_3^\vee-\beta_{\alpha_3}^\vee=e_1+e_2'+e_1''.$$

Then Θ^+ will be the smallest subset of Θ satisfying the following two conditions:

- $e_1+e_1'+e_2'',e_1+e_2'+e_1'',e_2+e_1'+e_1''\in\Theta^+$.
- $\Theta^+-w_{\alpha_1}\Theta^+=\{e_1+e_1'+e_2'',e_1+e_2'+e_1''\}$, $\Theta^+-w_{\alpha_2}\Theta^+=\{e_1+e_1'+e_2'',e_2+e_1'+e_1''\}$, $\Theta^+-w_{\alpha_3}\Theta^+=\{e_1+e_2'+e_1'',e_2+e_1'+e_1''\}$.

As a result, we have

$$\Theta^+=\{e_1+e_1'+e_1'',e_1+e_1'+e_2'',e_1+e_2'+e_1'',e_2+e_1'+e_1''\}.$$

It is easy to see that Θ^+ satisfies (2.3).

Finally, if we let $\theta=\delta_B^{1/2}$, it is easy to see that for $w\in W$, $c_{WS}(w\theta)=0$ unless w is the longest Weyl element. If w is the longest Weyl element, we have $c_{WS}(w\theta)=1-q^{-2}=\zeta(2)^{-1}=\frac{1}{\Delta_{H/Z_{G,H}}(1)}$. This proves Lemma 2.9. In conclusion, we have proved that the local relative character in this case is equal to

$$\zeta(1)^3\zeta(2)\cdot\frac{L(1/2,\pi_1\times\pi_2\times\pi_2)}{L(1,\pi,\mathrm{Ad})}$$

where $\pi=\pi_1\otimes\pi_2\otimes\pi_3$ is an unramified representation of $G(F)$.

2.4. **The Whittaker induced case: some reductions.** In this subsection, we consider the Whittaker induced case. Let (G,H) be a Whittaker induction of a strongly tempered model (G_0,H_0). In other words, there exists a parabolic subgroup $P=LU$ of G and a generic character ξ of $U(F)$ such that $G_0\simeq L$ and H_0 is the neutral component of the stabilizer of ξ in M. Note that for all the cases we considered in Table 1, the model (G_0,H_0) is essentially the trilinear GL_2-model.

Let $B_0 = TN_0$ be a Borel subgroup of G_0, $\bar{B}_0 = T\bar{N}_0$ be its opposite, and $\bar{P} = L\bar{U}$ be the opposite parabolic subgroup of P. Let $N = N_0U$ and $\bar{N} = \bar{N}_0\bar{U}$. Then $B = TN$ is a Borel subgroup of G and $\bar{B} = T\bar{N}$ is its opposite.

For all of our cases (as well as all the other Whittaker induced cases in Remark 1.4), there exists a Weyl element w_0 such that the w_0-conjugation map

- induces an isomorphism between U and $\bar{U}$,
- stabilizes L and fixes $H_0 \subset L$.

Also there exists a homomorphism $\lambda : U(F) \to F$ such that $\xi(u) = \psi(\lambda(u))$ for all $u \in U(F)$. We extend λ to $H(F)$ by making it trivial on $H_0(F)$. We also have a map $a : \mathrm{GL}_1 \to Z_L$ such that

(2.15)
$$w_0^{-1}a(t)w_0 = a(t)^{-1}, \text{ and } \lambda(a(t)ua(t)^{-1}) = t\lambda(u), \text{ for } t \in F^\times, u \in U(F).$$

Let $B_0(F)\eta_0H_0(F)$ be the unique open Borel orbit of the model (G_0, H_0), and let $\eta = \eta_0w_0$. Then $B(F)\eta H(F)$ is the unique open Borel orbit of the model (G, H) and the stabilizer of this orbit is $Z_{G,H} = H \cap Z_G$. Note that we always assume (G_0, H_0) does not have Type N spherical root. The equation (2.15) implies that $\eta^{-1}a(t)\eta = a(t)^{-1}$.

We want to compute the local relative character

$$I(\phi_\theta) = \int_{H_0(F)/Z_{G,H}(F)} \int_{U(F)} \phi_\theta(hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h \tag{2.16}$$

where ϕ_θ is the unramified matrix coefficient of $I_B^G(\theta)$ with $\phi_\theta(1) = 1$, and θ is a unitary unramified character of $T(F)$. The general idea of the computation is the same as the reductive case, the only difference is some convergent issue. Unlike the reductive case, the integral above is not absolutely convergent because of the extra unipotent integral. Hence we need to regularize the unipotent integral.

There are three (equivalent) ways to regularize the unipotent integral. The first one is using the fact the the unipotent integral is stable, i.e. there exists a compact open subgroup $\mathcal{U}$ of $U(F)$ such that

$$\int_{\mathcal{U}'-\mathcal{U}} \phi_\theta(hu)\xi(u)^{-1}\,\mathrm{d}u = 0$$

for all compact open subgroup $\mathcal{U}'$ of $\mathcal{U}(F)$ with $\mathcal{U} \subset \mathcal{U}'$. Hence we can replace the integral over $U(F)$ by an integral over the compact subgroup $\mathcal{U}$. This regularization has been used by Lapid–Mao [LM] in their computation for the Whittaker model, and used by Liu [L] in his computation of the non-reductive Gan–Gross–Prasad models. The advantage of this regularization is that it works for general matrix coefficients, not just the tempered ones. (Of course, in order for the integral of $H_0(F)$ to be convergent, we still need the character θ to be close to the unitary line.)

The rest two regularizations are only for the tempered case. It uses a critical fact that although the integral (2.16) is not absolutely convergent, the integral will become absolutely convergent if we replace $U(F)$ by $U'(F) = \{u \in U(F) \mid \lambda(u) = 0\}$. (For all the models considered in this paper, this can be proved by the same argument as Lemma 4.3.1 of [Wan17b].) As a result, we only need to regularize the integral over $\lambda(u) \in F$.

Remark 2.16. *In particular, the integral*

$$\int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\Phi(\lambda(h))\,\mathrm{d}h$$

is absolutely convergent for all $\Phi \in C_c^\infty(F)$.

There are two ways to regularize the integral over $\lambda(u)$. The first way is to replace the the integral over $U(F)$ by the integral over

$$U_n(F) = \{u \in U(F) \mid |\lambda(u)| \leq q^n\}.$$

Then one can show that for every matrix coefficient ϕ of $I_B^G(\theta)$, there exists $N > 0$ (depends on the level of ϕ, i.e. the open compact subgroup $K' \subset G(F)$ where ϕ is bi-K'-invariant) such that the integral

$$I_n(\phi_\theta) = \int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} \phi_\theta(hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

is independent of n for $n > N$, i.e. the unipotent integral is stable on the sequence $U_n(F)$ (the difference between this regularization and the previous one is that the unipotent groups $U_n(F)$ we used here is not compact). Hence we can just replace the integral over $U(F)$ in the definition of $I(\phi_\theta)$ by the integral over $U_n(F)$ for some large n (in fact, as ϕ_θ is unramified, one can easily show that we can just replace $U(F)$ by $U_1(F)$). This regularization has been used by Waldspurger in Lemma 5.1 of [W12] for the orthogonal Gan–Gross–Prasad model. The same arguments work for all the Whittaker induction cases in this paper.

Another way is to replace the character $\xi(u)^{-1} = \psi(\lambda(u))^{-1}$ by some Schwartz function $\varphi_n(\lambda(u))$ ($n \geq 0$) of $\lambda(u)$ where

$$\varphi_0 = \varphi = 1_{\mathcal{O}_F} - \frac{1}{q-1} \cdot 1_{\varpi^{-1}\mathcal{O}_F^\times}$$

is the Fourier transform of $\frac{1}{vol(\mathcal{O}_F^\times)} \cdot 1_{\mathcal{O}_F^\times}$ and φ_n is the Fourier transform of the function $\frac{1}{vol(1+\varpi^n\mathcal{O}_F^\times)} 1_{1+\varpi^n\mathcal{O}_F^\times}$ for $n \geq 1$. One can show that for every matrix coefficient ϕ of $I_B^G(\theta)$, there exists $N > 0$ (depends on the level of ϕ) such that the integral

$$I^n(\phi_\theta) = \int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} \phi_\theta(hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

is independent of n for $n > N$.

This regularization has been used by Beuzart-Plessis (Proposition 7.1.1 of [B15]) for the unitary Gan–Gross–Prasad model (note that the group $1+\varpi^n\mathcal{O}_F^\times$ is just the group K_a in loc. cit.) and by the first author (Proposition 5.1 of [Wan16]) for the Ginzburg–Rallis model. The same argument works for all the Whittaker induction cases in this paper. In the unramified case, we may just take $n = 0$ and the regularized integral is given by the formula

$$I(\phi_\theta) = \int_{H(F)/Z_{G,H}(F)} \phi_\theta(h)\varphi_0(\lambda(h))\,\mathrm{d}h.$$

In order to compute this regularized integral, we need another two regularized integrals.

Lemma 2.17. *For $f \in I_B^G(\theta)$, the integrals*

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} f(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

and

$$\int_{H(F)} f(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h$$

are absolutely convergent for all n.

Moreover, there exists $N \geq 0$ (depends on the level of f, i.e. the open compact subgroup $K' \subset G(F)$ where f is right K'-invariant) such that both integrals are equal to each other and are independent of n for $n > N$.

Definition 2.18. *We use $\int^*_{H(F)/Z_{G,H}(F)} f(\eta h)\xi(h)^{-1}\,\mathrm{d}h$ to denote the regularized integral*

$$\lim_{n\to\infty} \int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} f(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$
$$= \lim_{n\to\infty} \int_{H(F)/Z_{G,H}(F)} f(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h.$$

Proof. We first prove the convergence. By replacing f by $|f|$ and θ by $|\theta|$ we may assume that f is a non-negative real valued function. Let f_θ be the unramified vector in $I_B^G(\theta)$ with $f_\theta(1) = 1$. Then the matrix coefficient of f and f_θ is given by

$$\phi_{f,f_\theta}(g) = \int_K f(kg)\,\mathrm{d}g.$$

By the discussion above, we know that the integral

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} \phi_{f,f_\theta}(hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

is absolutely convergent for all n. This implies that (note that f is a non-negative real valued function) the triple integral

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} \int_K f(khu)\xi(u)^{-1}\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}h$$

is absolutely convergent. In particular, the integral

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} f(khu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

is absolutely convergent for almost all $k \in K$. But as a function on $k \in K$, this integral is left $B(\mathcal{O}_F)$ and right $H(\mathcal{O}_F)$ invariant. Combining with the fact that $\eta \in K$ and $B\eta H$ is Zariski open in G, we know that the integral

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} f(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

is absolutely convergent for all n. This also implies that the integral

$$\int_{H(F)} f(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h$$

is absolutely convergent for all n since φ_n is a compactly supported function.

Now we prove the second part of the theorem. We first prove the following statement

(1) there exists $N \geq 0$ such that for all $n > N$, we have

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta hu_0u)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h = 0$$

for all $u_0 \in U(F) - U_n(F)$.

In fact, for n large, we have the function $f(g)$ is right $a(t)$-invariant for all $t \in 1+\varpi^n \mathcal{O}_F^\times$. It is also left $a(t)$-invariant for all $t \in 1+\varpi^n \mathcal{O}_F^\times$ since θ is an unramified character. Then for $u_0 \in U(F) - U_n(F)$, there exists $t_0 \in 1+\varpi^n \mathcal{O}_F^\times$ such that $(t_0-1)\lambda(u_0) \in \varpi^{-1}\mathcal{O}_F^\times$ (in particular, $\psi((t_0-1)\lambda(u_0)) \neq 1)$. This implies that

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h$$

$$= \int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(a(t_0^{-1})\eta h u_0 u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h$$

$$= \int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 (u_0^{-1} a(t_0) u_0) u a(t_0)^{-1})\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h$$
$$= \int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 (u_0^{-1} a(t_0) u_0 a(t_0)^{-1}) u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h.$$

Here we use the fact that since ψ is unramified, $\xi(u) = \xi(a(t)ua(t)^{-1})$ for $u \in U_1(F)$ and $t \in 1+\varpi^n \mathcal{O}_F$. By our choice of u_0 and t_0, we know that $u_0^{-1}a(t_0)u_0 a(t_0)^{-1} \in U_1(F)$, then an easy change of variable shows that $\int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h$ is equal to

$$\xi(u_0^{-1}a(t_0)u_0 a(t_0)^{-1}) \int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h.$$

Since $\xi(u_0^{-1}a(t_0)u_0 a(t_0)^{-1}) = \psi((t_0-1)\lambda(u_0)) \neq 1$, we have

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_1(F)} f(\eta h u_0 u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h = 0.$$

This proves (1).

Now we are ready to prove the theorem. By (1), we know that for all $n > N$, we have

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_{n+1}(F)-U_n(F)} f(\eta h u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h = 0.$$

In particular, the integral

$$\int_{H_0(F)/Z_{G,H}(F)} \int_{U_n(F)} f(\eta h u)\xi(u)^{-1} \,\mathrm{d}u \,\mathrm{d}h$$

is independent of n for $n > N$.

For the second integral, choose n large so that the function $f(g)$ is right $a(t)$-invariant for all $t\in 1+\varpi^n\mathcal{O}_F^\times$. Then we have

$$\begin{aligned}
&\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f(\eta h u)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\\
=&\int_{1+\varpi^n\mathcal{O}_F^\times}\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f(a(t)\eta h u a(t))\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}t\\
=&\int_{1+\varpi^n\mathcal{O}_F^\times}\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f(\eta h a(t)^{-1} u a(t))\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}t\\
=&\int_{1+\varpi^n\mathcal{O}_F^\times}\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f(\eta h u)\psi(t\lambda(u))^{-1}\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}t\\
=&\int_{H_0(F)/Z_{G,H}(F)}\int_{U(F)}\int_{1+\varpi^n\mathcal{O}_F^\times} f(\eta h u)\psi(t\lambda(u))^{-1}1_{\varpi^{-n}\mathcal{O}_F}(\lambda(u))\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}h.
\end{aligned}$$

Here the measure $\mathrm{d}t$ on $1+\varpi^n\mathcal{O}_F^\times$ is chosen so that the total volume is equal to 1. The function

$$\begin{aligned}
x\mapsto &\int_{1+\varpi^n\mathcal{O}_F^\times}\psi(tx)^{-1}1_{\varpi^{-n}\mathcal{O}_F}(x)\,\mathrm{d}t\\
&=1_{\varpi^{-n}\mathcal{O}_F}(x)\cdot\int_F\frac{1}{vol(1+\varpi^n\mathcal{O}_F^\times)}1_{1+\varpi^n\mathcal{O}_F^\times}(t)\psi(tx)^{-1}\,\mathrm{d}y
\end{aligned}$$

is just $1_{\varpi^{-n}\mathcal{O}_F}\cdot\varphi_n$ (recall that φ_n is the Fourier transform of the function $\frac{1}{vol(1+\varpi^n\mathcal{O}_F^\times)}1_{1+\varpi^n\mathcal{O}_F^\times}$). A direct computation shows that the function φ_n is supported on $\varpi^{-n}\mathcal{O}_F$, hence $1_{\varpi^{-n}\mathcal{O}_F}\cdot\varphi_n=\varphi_n$. As a result, we have

$$\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f(\eta h u)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h=\int_{H(F)} f(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h.$$

This proves the lemma. □

Remark 2.19. *As long as f is right $T(\mathcal{O}_F)$-invariant (for example when f is unramified or when f is an Iwahori fixed vector), we can just take $N=0$. We can also show that the integral*

$$\int_{H(F)} f(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h$$

is independent of n for $n\geq 0$.

Let $\mathcal{Y}_{\theta,\xi}$, $\mathcal{Y}_{\theta,\xi,n}$, $\mathcal{Y}_{\theta,\xi}^n$ be the function on $G(F)$ satisfying the following conditions:

- $\mathcal{Y}_{\theta,\xi}$, $\mathcal{Y}_{\theta,\xi,n}$, $\mathcal{Y}_{\theta,\xi}^n$ are supported on the open orbit $B(F)\eta H(F)$.

- For $b \in B(F)$ and $h \in H(F)$, we have

$$\mathcal{Y}_{\theta,\xi}(b\eta h) = \theta^{-1}\delta_B^{1/2}(b)\xi(h),\ \mathcal{Y}_{\theta,\xi,n}(b\eta h) = \theta^{-1}\delta_B^{1/2}(b)\xi(h)1_{\varpi^{-n}\mathcal{O}_F}(\lambda(h)),$$

$$\mathcal{Y}_{\theta,\xi}^n(b\eta h) = \theta^{-1}\delta_B^{1/2}(b)\varphi_n(\lambda(h)).$$

.

Lemma 2.20. *For $\Phi \in C_c^\infty(G(F))$, the integrals*

$$\int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi,n}(g)\,\mathrm{d}g \text{ and } \int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi}^n(g)\,\mathrm{d}g$$

are absolutely convergent for all n. Moreover, there exists $N \geq 0$ (depends on the level of f, i.e. the open compact subgroup $K' \subset G(F)$ where f is right K'-invariant) such that both integrals are equal to each other and are independent of n for $n > N$.

Definition 2.21. *We use $\int^*_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi}(g)\,\mathrm{d}g$ to denote the regularized integral*

$$\lim_{n\to\infty}\int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi,n}(g)\,\mathrm{d}g = \lim_{n\to\infty}\int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi}^n(g)\,\mathrm{d}g.$$

Proof. By the same arguments as Lemma 2.5, we can prove the following statement

- For $\Phi \in C_c^\infty(G(F))$, we have

(2.17)
$$\int_{G(F)} \Phi(g)\,\mathrm{d}g = \frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}\int_{H(F)/Z_{G,H}(F)}\int_{B(F)} \Phi(b\eta h)\,\mathrm{d}b\,\mathrm{d}h.$$

This implies that the integral $\int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi,n}(g)\,\mathrm{d}g$ is equal to the product of $\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}$ with

$$\begin{aligned}&\int_{H_0(F)Z_{G,H}(F)}\int_{U_n(F)}\int_{B(F)} \Phi(b\eta hu)\theta^{-1}\delta^{1/2}(b)\xi(u)^{-1}\,\mathrm{d}b\,\mathrm{d}u\,\mathrm{d}h\\ =&\int_{H_0(F)Z_{G,H}(F)}\int_{H_0(F)/Z_{G,H}(F)}\int_{U_n(F)} f_\Phi(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\end{aligned}$$

and the integral $\int_{G(F)} \Phi(g)\mathcal{Y}_{\theta,\xi}^n(g)\,\mathrm{d}g$ is equal to the product of

$$\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-rk(G)}$$

with

$$\int_{H(F)/Z_{G,H}(F)}\int_{B(F)}\Phi(b\eta h)\theta^{-1}\delta^{1/2}(b)\varphi_n(\lambda(h))\,\mathrm{d}b\,\mathrm{d}h$$
$$=\int_{H_0(F)Z_{G,H}(F)}\int_{H(F)/Z_{G,H}(F)} f_\Phi(\eta h)\varphi_n(\lambda(h))\,\mathrm{d}h.$$

Then the lemma just follows from the lemma above. □

Remark 2.22. *If Φ is right $T(\mathcal{O}_F)$-invariant, then we can just take $N=0$. We can also show that the integral*

$$\int_{G(F)}\Phi(g)\mathcal{Y}^n_{\theta,\xi}(g)\,\mathrm{d}g$$

is independent of n for $n\geq 0$. For any open compact subset $K'\subset G(F)$, we let

$$\int^*_{K'}\mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k := \int^*_{G(F)} 1_{K'}(g)\mathcal{Y}_{\theta,\xi}(g)\,\mathrm{d}g.$$

Recall that f_θ is the unramified vector in $I_B^G(\theta)$ with $f_\theta(1)=1$. We have

$$\phi_\theta(g)=\int_K f_\theta(kg)\,\mathrm{d}k,\ I(\phi_\theta)=\int_{H(F)/Z_{G,H}(F)}\phi_\theta(h)\varphi(\lambda(h))\,\mathrm{d}h$$
$$=\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)}\phi_\theta(hu)\xi(u)^{-1}\,\mathrm{d}u.$$

This implies that

$$I(\phi_\theta)=\int_K\int_{H(F)/Z_{G,H}(F)} f_\theta(kh)\varphi(\lambda(h))\,\mathrm{d}h\,\mathrm{d}k \tag{2.18}$$
$$=\int_K\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(khu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}k.$$

The next lemma follows from the proof of Lemma 2.17.

Lemma 2.23. *For $u_0\in U(F)-U_1(F)$, we have*

$$\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(\eta u_0 hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h=0.$$

Corollary 2.24. *We have*

$$I(\phi_\theta)=\frac{\Delta_{H_0/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\cdot\int^*_K\mathcal{Y}_{\theta^{-1},\xi}(k)\,\mathrm{d}k\cdot\int^*_K\mathcal{Y}_{\theta,\xi^{-1}}(k)\,\mathrm{d}k.$$

Proof. We have

$$I(\phi_\theta)=\int_{K\cap B(F)\eta H_0(F)U_1(F)}\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(khu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}k.$$

The function

$$k\mapsto\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(khu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h$$

on $K\cap B(F)\eta H_0(F)U_1(F)$ is a scalar of the restriction of the function $\mathcal{Y}_{\theta^{-1},\xi}$ to $K\cap B(F)\eta H_0(F)U_1(F)$ and the scalar is equal to

$$\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h.$$

This implies that

$$\begin{aligned} I(\phi_\theta) &= \int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h \\ &\quad\cdot\int_{K\cap B(F)\eta H_0(F)U_1(F)} \mathcal{Y}_{\theta^{-1},\xi}(k)\,\mathrm{d}k.\end{aligned}$$

By Lemma 2.20 and Remark 2.22, we have

$$\int_{K\cap B(F)\eta H_0(F)U_1(F)} \mathcal{Y}_{\theta^{-1},\xi}(k)\,\mathrm{d}k=\int_K^* \mathcal{Y}_{\theta^{-1},\xi}(k)\,\mathrm{d}k.$$

Hence it remains to show that

$$\begin{aligned}&\frac{\Delta_{H_0/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\cdot\int_K^*\mathcal{Y}_{\theta,\xi^{-1}}(k)\,\mathrm{d}k \\ &=\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h.\end{aligned}\tag{2.19}$$

By Lemma 2.20, Remark 2.22 and (2.17), we have

$$\begin{aligned}&\frac{\Delta_{H_0/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\cdot\int_K^*\mathcal{Y}_{\theta,\xi^{-1}}(k)\,\mathrm{d}k \\ =&\frac{\Delta_{H_0/Z_{G,H}}(1)}{\Delta_G(1)}\zeta(1)^{rk(G)}\cdot\int_{K\cap B(F)\eta H_0(F)U_1(F)}\mathcal{Y}_{\theta,\xi^{-1}}(k)\,\mathrm{d}k \\ =&\int_{H_0(F)}\int_{U_1(F)}\int_{B(F)} 1_K(b\eta hu)\theta^{-1}\delta_B^{1/2}(b)\xi(u)^{-1}du \\ =&\int_{H_0(F)/Z_{G,H}(F)}\int_{U_1(F)} f_\theta(\eta hu)\xi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h.\end{aligned}$$

This proves (2.19) and finishes the proof of the lemma. □

In the next subsection, we will explain how to compute the regularized integral $\int_K^* \mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k$. The result is summarized in the proposition below.

Proposition 2.25. *Let Φ^+ be the set of positive roots of G. There is a decomposition of weights of a representation ρ_X of $\hat{G}$ (denoted by $\Theta = \Theta^+ \cup \Theta^-$) such that*

$$\int_K^* \mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k = \frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-rk(G)} \cdot \beta(\theta)$$

where

$$\beta(\theta) = \frac{\prod_{\alpha\in\Phi^+} 1 - q^{-1}e^{\alpha^\vee}}{\prod_{\gamma^\vee\in\Theta^+} 1 - q^{-\frac{1}{2}}e^{\gamma^\vee}}(\theta).$$

Also (2.3) *still holds.*

The proposition above implies that $I(\phi_\theta)$ is equal to

$$\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-rk(G)} \cdot \beta(\theta)\cdot\beta(\theta^{-1}) = \frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)} \cdot \frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}.$$

This finishes the computation.

2.5. The computation of $\int_K^* \mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k$.

Let

$$S_\theta(g) := \int_K^* \mathcal{Y}_{\theta,\xi}(kg^{-1})\,\mathrm{d}k = \int_{G(F)}^* 1_K(g'g)\mathcal{Y}_{\theta,\xi}(g')\,\mathrm{d}g'.$$

Our goal is to prove Proposition 2.25, i.e. compute

$$S_\theta(1) = \int_K^* \mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k = \int_K \mathcal{Y}^0_{\theta,\xi}(k)\,\mathrm{d}k.$$

Let $\mathcal{I} = B(\mathcal{O}_F)\bar{N}(\varpi\mathcal{O}_F)$ (resp. $\mathcal{I}_0 = B_0(\mathcal{O}_F)\bar{N}_0(\varpi\mathcal{O}_F)$) be the Iwahori subgroup of $G(F)$ (resp. $G_0(F) = L(F)$). As in Lemma 2.9, we can choose η_0 so that

$$\bar{N}_0(\varpi\mathcal{O}_F)\eta_0 \subset T(\mathcal{O}_F)N_0(\varpi\mathcal{O}_F)\eta_0 H_0(\mathcal{O}_F). \tag{2.20}$$

Lemma 2.26. *We have $\eta\in K$ and*

$$\bar{N}(\varpi\mathcal{O}_F)\eta \subset T(\mathcal{O}_F)N(\varpi\mathcal{O}_F)\eta H_0(\mathcal{O}_F)U(\varpi\mathcal{O}_F).$$

Proof. Since $\eta_0, w_0 \in K$, we have $\eta = \eta_0 w_0 \in K$. For $\bar{n}\in\bar{N}(\varpi\mathcal{O}_F)$, we write it as $\bar{n}'\bar{u}$ with $\bar{n}'\in\bar{N}_0(\varpi\mathcal{O}_F)$ and $\bar{u}\in\bar{U}(\varpi\mathcal{O}_F)$. Since $\eta=\eta_0 w_0$ and $\eta_0\in L(\mathcal{O}_F)$, we have $\eta^{-1}\bar{u}\eta\in U(\varpi\mathcal{O}_F)$. Hence it is enough to consider the case when $\bar{n}\in\bar{N}_0(\varpi\mathcal{O}_F)$. Then the lemma follows from (2.20) and the fact that H_0 commutes with w_0. $\square$

For $w\in W$, let $\Phi_w=1_{\mathcal{I}w\mathcal{I}}$. Let α be a simple root and w_α be the corresponding element in W. As in the reductive case, we would need to compute

$$\begin{aligned}I_\alpha(\theta)&=vol(\mathcal{I})^{-1}\int_{G(F)}^{*}\mathcal{Y}_{\theta,\xi}(x)(\Phi_1(x\eta^{-1})+\Phi_{w_\alpha}(x\eta^{-1}))\,\mathrm{d}x\\&=vol(\mathcal{I})^{-1}\int_{G(F)}\mathcal{Y}_{\theta,\xi}^0(x\eta)(\Phi_1(x)+\Phi_{w_\alpha}(x))\,\mathrm{d}x.\end{aligned}$$

First, by the lemma above, we have $\mathcal{I}\eta\subset B(\mathcal{O}_F)\eta H_0(\mathcal{O}_F)U(\varpi\mathcal{O}_F)$. Hence $\mathcal{Y}_{\theta,\xi}^0(x\eta)=1$ for all $x\in\mathcal{I}$. This implies that

$$vol(\mathcal{I})^{-1}\int_{G(F)}\mathcal{Y}_{\theta,\xi}^0(x\eta)\Phi_1(x)\,\mathrm{d}x=1.$$

The next lemma follows from the same arguments as in the reductive case.

Lemma 2.27. *Let $u_\alpha:F\to N(F)$ be the homomorphism whose image is the root space of α (the root space is one dimensional since we assume that the group is split). Then*

$$I_\alpha(\theta)=1+q\int_{\mathcal{O}_F}(\theta^{-1}\delta^{\frac{1}{2}})(e^{\alpha^\vee}(a^{-1}))\mathcal{Y}_{\theta,\xi}^0(u_{-\alpha}(a^{-1})\eta)\,\mathrm{d}a,$$

Then for each model, by an explicit matrix computation, we will show that for $\alpha\in\Delta(G_0)$, there exists $\beta_\alpha^\vee\in\Theta$ such that $-\beta_\alpha^\vee+\alpha^\vee\in\Theta$ and

$$\mathcal{Y}_{\theta,\xi}^0(u_{-\alpha}(a^{-1})\eta h^{-1})=\varphi_0(\lambda(h))\cdot\theta(e^{\beta_\alpha^\vee}(1+a^{-1}))\cdot|1+a^{-1}|^{-1/2}. \tag{2.21}$$

As in the reductive case, this will imply that

$$\alpha=(q-1)\cdot\frac{1-q^{-1}e^{\alpha^\vee}(\theta)}{(1-q^{-1/2}e^{\beta_\alpha^\vee}(\theta))(1-q^{-1/2}e^{-\beta_\alpha^\vee+\alpha^\vee}(\theta))}. \tag{2.22}$$

Remark 2.28. *In fact, by our choice of $\eta=\eta_0w_0$, we only need to verify the identity for the reductive model (G_0,H_0) and we know that $\beta_a^\vee$ is just the color associated to α for the reductive model (G_0,H_0). For all of our cases in Table 1, since it is induced from the trilinear GL_2-model, we can just use the computations in Section 2.3.2.*

On the other hand, if $\alpha\in\Delta(G)-\Delta(G_0)$, we will show that

$$\mathcal{Y}_{\theta,\xi}^0(u_{-\alpha}(a^{-1})\eta)=\varphi_0(a^{-1}). \tag{2.23}$$

This implies that

$$I_\alpha(\theta)=1+q\int_{\mathcal{O}_F}\theta(e^{\alpha^\vee}(a))\cdot|a|^{-1}\cdot\varphi_0(a^{-1})da=q(1-q^{-1}e^{\alpha^\vee}(\theta)). \tag{2.24}$$

Remark 2.29. *Note that if we don't regularize the unipotent integral, the integral we get here will be*

$$I_\alpha(\theta)=1+q\int_{\mathcal{O}_F}\theta(e^{\alpha^\vee}(a))\cdot|a|^{-1}\cdot\psi(a^{-1})\,\mathrm{d}a.$$

This is not absolutely convergent (which is also the reason why the original unipotent integral is not absolutely convergent). There are two ways to regularize this integral which correspond to the two ways to regularize the unipotent integral.

The first way is to use the fact that $\int_{\varpi^n\mathcal{O}_F^\times}\theta(e^{\alpha^\vee}(a))\cdot|a|^{-1}\cdot\psi(a^{-1})\,\mathrm{d}a=0$ *for* $n>1$*, and regularize the integral as*

$$I_\alpha(\theta)=1+q\int_{\mathcal{O}_F^\times\cup\varpi^{-1}\mathcal{O}_F^\times}\theta(e^{\alpha^\vee}(a))\cdot|a|^{-1}\cdot\psi(a^{-1})\,\mathrm{d}a,$$

which is equal to $q(1-q^{-1}e^{\alpha^\vee}(\theta))$*. The second way is to replace* $\psi(a^{-1})$ *by* $\varphi_0(a^{-1})$ *as we did above which gives the same answer.*

Definition 2.30. *Let* Θ^+ *be the unique subset of* Θ *satisfying the following two conditions:*

- *For every simple root* $\alpha\in\Delta_{G_0}$*, we have* $\Theta^+-w_\alpha\Theta^+=\{\beta_\alpha^\vee,\ \alpha^\vee-\beta_\alpha^\vee\}$*;*
- *For every simple root* $\alpha\in\Delta_G-\Delta_{G_0}$*,* Θ^+ *is stable under* w_α*.*

We then define

$$\beta(\theta)=\frac{\prod_{\alpha\in\Phi^+}1-q^{-1}e^{\alpha^\vee}}{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}}(\theta)\ \text{and}\ c_{WS}(\theta)=\frac{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+}1-e^{\alpha^\vee}}(\theta).$$

Now by the exactly same arguments as in the reductive case (the only difference is that for the definition of l_θ in (2.10), we replace the integral $\int_{G(F)}$ by the regularized integral $\int_{G(F)}^*$), we can prove the following proposition.

Proposition 2.31. *Let* $T(F)^+=\{t\in T(F)|\ t^{-1}N(\mathcal{O}_F)t\subset N(\mathcal{O}_F)\}$ *be the positive chamber of* $T(F)$*. Then*

$$S_\theta(\eta^{-1}t)/\beta(\theta)=q^{l(W)}vol(\mathcal{I})\sum_{w\in W}c_{WS}(w\theta)(w\theta)^{-1}\delta^{\frac{1}{2}}(t^{-1}),\ \text{for}\ t\in T(F)^+,$$

where $l(W)$ *is the length of the longest Weyl element in* W*.*

Since $\eta\in K$, we have $S_\theta(1)=S_\theta(\eta^{-1})$. Combining with the proposition above, we have

$$S_\theta(1)/\beta(\theta)=q^{l(W)}vol(\mathcal{I})\sum_{w\in W}c_{WS}(w\theta).$$

Since $vol(\mathcal{I}) = \Delta_G(1)\zeta(1)^{-rk(G)} \cdot q^{-l(W)}$, we have

$$S_\theta(1)/\beta(\theta) = \Delta_G(1)\zeta(1)^{-rk(G)} \sum_{w\in W} c_{WS}(w\theta).$$

Hence in order to prove Proposition 2.25, it is enough to prove the following lemma.

Lemma 2.32. *The summation $\sum_{w\in W} c_{WS}(w\theta)$ is independent of the choice of θ and is equal to $\frac{1}{\Delta_{H_0/Z_{G,H}}(1)}$.*

This lemma is much more difficult than the reductive case for two reasons. First, Theorem 7.2.1 of [Sa] only works for the reductive case, so we can not use it to imply that the summation is a constant. Secondly, in the reductive case, if we set $\theta = \delta_B^{1/2}$, then the only nonvanishing term in the summation is the term corresponding to the longest Weyl element, and it will be equal to $\frac{1}{\Delta_{H_0/Z_{G,H}}(1)}$. But for all the non-reductive cases in Table 1, this is not true and actually it is impossible to choose a θ so that all the terms in the summation are equal to 0 except one. We believe that this is related to the fact that for the models in Table 1, the Type T spherical roots and Type (U, ψ) spherical roots will sometimes interlace each other. For example, for the model $(\mathrm{GL}_6, \mathrm{GL}_2 \ltimes U)$, the roots $\alpha_i = e_i - e_{i+1}$ is of Type T when $i = 1, 3, 5$ and is of Type (U, ψ) when $i = 2, 4$.

As a result, for each of these cases, we will prove this lemma by a direct computation. Our computation is based on some reductive steps. For example, for the model $(\mathrm{GSO}_{12}, \mathrm{GL}_2 \ltimes U)$, we will prove the identity by proving another identity which allows us to reduce to the identity for the model $(\mathrm{GL}_6, \mathrm{GL}_2 \ltimes U)$; for the model $(\mathrm{GL}_6, \mathrm{GL}_2 \ltimes U)$, we will prove the identity by proving another identity which allows us to reduce to an identity related to the group $\mathrm{GL}_4 \times \mathrm{GL}_2$.

2.5.1. *The summary.* By the discussion in the previous two subsections, in order to compute the local relative character in the non-reductive case, we just need to do the following steps:

(1) Define the Weyl element w_0 so that the w_0-conjugation map
   - induces an isomorphism between U and $\bar{U}$,
   - stabilizes L and fixes $H_0 \subset L$.

(2) Define the map $a : \mathrm{GL}_1 \to Z_L$ so that it satisfies (2.15).

(3) Show that the double coset $B_0(F)\backslash G_0(F)/H_0(F)$ has a unique open Borel $B_0(F)\eta_0 G_0(F)$ and the representative η_0 can be chosen to satisfy (2.20). Since all the cases in Table 1 are Whittaker inductions of the trilinear GL_2-model, this step has already been verified in Section 2.3.2.

(4) Verify the identity (2.21) and (2.23) by expressing the product $u_{-\alpha}(a)\eta$ in terms of the decomposition $B(F)\eta H(F)$. Since all the cases in Table 1 are Whittaker induction of the trilinear GL_2-model, the identity (2.21) and the colors have already been computed in Section 2.3.2 (see Remark 2.28), so we only need to verify (2.23).
(5) Compute the subset Θ^+ of Θ and show that it satisfies (2.3).
(6) Compute the constant $\sum_{w\in W} c_{WS}(w\theta)$, i.e. Lemma 2.32. This is the most technical part of the computation.

A final remark for the spherical roots. In Table 1, if a model is reductive, then all the simple roots of the spherical variety are of Type T, and our computation of $I_\alpha(\theta)$ in (2.8) confirms Statement 6.3.1 of [Sa]; If a model is non-reductive, the Whittaker induction of the trilinear model (G_0, H_0), then for a simple root α of the spherical variety, α is of Type T if α is a simple root of G_0 (recall that G_0 is embedded as the Levi subgroup of G) and the remaining simple roots are of Type (U,ψ). In such case, our computation of $I_\alpha(\theta)$ in (2.22) and (2.24) also confirms Statement 6.3.1 of [Sa].

3. The model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$

In this section, we compute the local relative character for the model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$. We closely follow the four steps in Section 2.3.1. In Section 3.1, we will define this model and verify Step (1), i.e. there is only one open orbit under the action of the Borel subgroup. Then in Section 3.2, we will first study the matrix identities of the product $u_{-\alpha}(x)\eta$ to get the set of virtual weighted colors (Step (2)). Then we will compute the set Θ^+ (Step (3)) and finally we will compute the constant $\sum_{w\in W} c_{WS}(w\theta)$ (Step (4)).

3.1. **The model and some orbit computation.**

Define the split symplectic similitude group

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid g^t J_{2n} g = l(g) J_{2n}\}$$

where $J_{2n} = \begin{pmatrix} 0 & -w_n \\ w_n & 0 \end{pmatrix}$ and $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. Here l is the similitude character. Let B_{2n} be the Borel subgroup of GSp_{2n} consisting of all upper triangular matrices. Set $G = \mathrm{GSp}_6 \times \mathrm{GSp}_4$ and

$$H = (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0 := \{(h_1,h_2) \in \mathrm{GSp}_4 \times \mathrm{GSp}_2 \mid l(h_1) = l(h_2)\}$$

embeds into G via the map

$$(h_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0 = H$$

$$\mapsto (\begin{pmatrix} a & 0 & b \\ 0 & h_1 & 0 \\ c & 0 & d \end{pmatrix}, h_1) \in \mathrm{GSp}_6 \times \mathrm{GSp}_4 = G.$$

For the non-split version of this model, let D/F be a quaternion algebra. Let

$$\mathrm{GSp}_n(D) = \{g \in \mathrm{GL}_n(D) \mid \bar{g}^t w_n g = l(g) w_n\}$$

where $\bar{g}$ is the conjugation map on $\mathrm{GL}_n(D)$ induced by the conjugation map on D. Let $G_D(F) = \mathrm{GSp}_3(D) \times \mathrm{GSp}_2(D)$ and $H_D(F) = (\mathrm{GSp}_2(D) \times \mathrm{GSp}_1(D))^0 = \{(h_1, h_2) \in \mathrm{GSp}_2(D) \times \mathrm{GSp}_1(D) \mid l(h_1) = l(h_2)\}$ embeds into $G_D(F)$ via the map

$$(h_1, h_2) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, h_2) \in (\mathrm{GSp}_2(D) \times \mathrm{GSp}_1(D))^0$$

$$\mapsto (\begin{pmatrix} a & 0 & b \\ 0 & h_2 & 0 \\ c & 0 & d \end{pmatrix}, h_1) \in \mathrm{GSp}_3(D) \times \mathrm{GSp}_2(D).$$

Set $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $\eta^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Proposition 3.1. *The double cosets $B(F)\backslash G(F)/H(F)$ contain a unique open orbit $B(F)(\eta, I_4)H(F)$. Here $B(F) = B_6(F) \times B_4(F)$.*

Let $H'(F) = \{h_1 \times h_2 \in (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0 = H \mid h_1 \in B_4(F)\}$ be a subgroup of $\mathrm{GSp}_6(F)$. Let $X(F) = GSp_6(F)/B_6(F)$ be the flag variety associated to $\mathrm{GSp}_6(F)$. We have a natural action of $\mathrm{GSp}_6(F)$ on $X(F)$ which induces an action of $H'(F)$ on $X(F)$.

Let $W_6 = Span\{w, w_1, w_2, w_2^\perp, w_1^\perp, w^\perp\}$ be the six dimensional symplectic space defining GSp_6 where $\{w, w_1, w_2, w_2^\perp, w_1^\perp, w^\perp\}$ is the standard basis induced by B_6, i.e.

$$w = (1, 0, 0, 0, 0, 0)^T, w_1 = (0, 1, 0, 0, 0, 0)^T, \cdots$$

Then $X(F)$ is characterized by

$X'(F) = \{(v_1, v_2, v_3) \mid \langle v_i, v_j \rangle = 0,\ v_1, v_2, v_3$ are linearly independent$\}$.

More specifically, $X(F)=\{Span\{v_1\}, Span\{v_1,v_2\}, Span\{v_1,v_2,v_3\} \mid (v_1,v_2,v_3)\in X'(F)\}$. The $\mathrm{GSp}_6(F)$-action is just

$$g\cdot(v_1,v_2,v_3)=(gv_1,gv_2,gv_3).$$

Note that

$$\eta^{-1}\cdot(w,w_1,w_2)=(w+w_1^{\perp}, w_1+w^{\perp}+w_2^{\perp}, w_2+w_1^{\perp}+w_2^{\perp}).$$

Hence in order to prove the proposition, it is enough to prove the following lemma.

Lemma 3.2. *The $H'(F)$-action on $X(F)$ contains a unique open orbit represented by $(w+w_1^{\perp}, w_1+w^{\perp}+w_2^{\perp}, w_2+w_1^{\perp}+w_2^{\perp})$.*

Proof. First, we assume that (v_1,v_2,v_3) belongs to the Zariski open subset such that

- v_1 has nonzero projection to the subspaces $Span\{w,w^{\perp}\}$ and $Span\{w_1^{\perp}\}$;
- The projections of v_1 and v_2 to $Span\{w,w^{\perp}\}$ and $Span\{w_2^{\perp},w_1^{\perp}\}$ are linearly independent;
- The projections of v_1,v_2 and v_3 to $Span\{w_2,w_2^{\perp},w_1^{\perp}\}$ are linearly independent;
- The projections of v_1,v_2 and v_3 to $Span\{w,w^{\perp},w_2^{\perp}\}$ are linearly independent.

Up to the $H'(F)$-action and because of the first condition we may assume that $v_1=w+w_1^{\perp}$. Then by the second condition and since $\langle v_1,v_2\rangle=0$, we may assume $v_2=w_1+w^{\perp}+aw_2^{\perp}+bw_2+cw_1^{\perp}$ with $a,b,c\in F$ and $a\neq 0$. Up to the action of an element

$$diag(I_2,\begin{pmatrix} t & x\\ 0 & t^{-1}\end{pmatrix},I_2)\in H'(F),$$

we may assume that $v_2=w_1+w^{\perp}+w_2^{\perp}+cw_1^{\perp}$. Note that such an element fixes v_1. Now let h be the element in $H'(F)$ that fixes $w,w_1,w_2,w_2^{\perp},w_1^{\perp}$ and maps $w^{\perp}$ to $w^{\perp}+cw$. Then

$$hv_1=v_1,\ hv_2=cv_1+(w_1+w^{\perp}+w_2^{\perp}).$$

Hence we may assume that $v_2=w_1+w^{\perp}+w_2^{\perp}$.

Finally, because $(v_1,v_2)=(w+w_1^{\perp},w_1+w^{\perp}+w_2^{\perp})$ and by the third condition, we may assume that $v_3=w_2+aw_1^{\perp}+bw_1+cw_2^{\perp}$. Since $\langle v_1,v_3\rangle=\langle v_2,v_3\rangle=0$, we have $a=1,b=0$. Hence $v_3=w_2+w_1^{\perp}+cw_2^{\perp}$. By the fourth condition, we know that $c\neq 0$.

Now consider the element $h_0=diag(1,c^{-1},1,c^{-1},1,c^{-1})\in H'(F)$. We have

$$h_0v_1=v_1,\ h_0v_2=c^{-1}v_2,\ h_0v_3=w_2+w_1^{\perp}+w_2^{\perp}.$$

This proves the lemma. ☐

Now if we let $\bar{N}_6$ (resp. $\bar{N}_4$) be the lower triangular unipotent subgroup of GSp_6 (resp. GSp_4) and we embed $\bar{N}_4$ into $\bar{N}_6$ via the embedding of GSp_4 to GSp_6. We also let T_6 (resp. T_4) be the diagonal torus of GSp_6 (resp. GSp_4). The following lemma is a direct consequence of the proof of the previous lemma.

Lemma 3.3. *For all $n \in \bar{N}_6(\varpi\mathcal{O}_F)$ and $n' \in \bar{N}_4(\varpi\mathcal{O}_F)$, we have*

$$n\eta n' \in B_6(F)\eta N_6(\varpi\mathcal{O}_F)T_6(\mathcal{O}_F)N'(\varpi\mathcal{O}_F)$$

where N' is the lower triangular unipotent subgroup of GSp_2 via as a subgroup of GSp_6.

We need a stronger result.

Lemma 3.4. *For all $n \in \bar{N}_6(\varpi\mathcal{O}_F)$ and $n' \in \bar{N}_4(\varpi\mathcal{O}_F)$, we have*

$$n\eta n' \in T_6(\mathcal{O}_F)N_6(\varpi\mathcal{O}_F)\eta N_6(\varpi\mathcal{O}_F)T_6(\mathcal{O}_F)N'(\varpi\mathcal{O}_F).$$

Proof. The above lemma implies that

$$n\eta n' \in T_6(\mathcal{O}_F)N_6(\mathcal{O}_F)\eta N_6(\varpi\mathcal{O}_F)T_6(\mathcal{O}_F)N'(\varpi\mathcal{O}_F).$$

We just need to prove the element in $N_6(\mathcal{O}_F)$ actually belongs to $N_6(\varpi\mathcal{O}_F)$, which is equivalent to show that this element preserves the sets V_4, V_5, V_6 where

$$V_i = \{(a_1, a_2, a_3, a_4, a_6, a_6)^T \mid a_i \in \mathcal{O}_F^{\times}, a_j \in \varpi\mathcal{O}_F \text{ for all } j \neq i\}.$$

But this just follows from the fact that these three sets are fixed by

$$\eta, \eta^{-1}, T_6(\mathcal{O}_F), \bar{N}_6(\varpi\mathcal{O}_F)\bar{N}_4(\varpi\mathcal{O}_F), N_6(\varpi\mathcal{O}_F), N'(\varpi\mathcal{O}_F).$$

This proves the lemma. ☐

The above lemma implies the following proposition which is Lemma 2.9 for the current case.

Proposition 3.5. *For all $n \in \bar{N}_6(\varpi\mathcal{O}_F)$ and $n' \in \bar{N}_4(\varpi\mathcal{O}_F)$, we have*

$$(n, n')(\eta, I_4) \in T(\mathcal{O}_F)N(\varpi\mathcal{O}_F)(\eta, I_4)H(\mathcal{O}_F)$$

with $B = TN$.

3.2. **The computation.** Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = 2\varepsilon_3$ be the simple roots of GSp_6 and $\alpha_1' = \varepsilon_1' - \varepsilon_2', \alpha_2' = 2\varepsilon_2'$ be the simple roots of GSp_4. We want to compute the virtual weighted colors associated to these simple roots. Set

$$u_{-\alpha_1}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 \end{pmatrix},\ u_{-\alpha_2}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$u_{-\alpha_3}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also let Θ the weights of the 32-dimensional representation $\mathrm{Spin}_7 \otimes \mathrm{Spin}_5$ of $\mathrm{GSpin}_7(\mathbb{C}) \times \mathrm{GSpin}_5(\mathbb{C})$. We can write it as

$$\Theta = \{\frac{\pm e_1 \pm e_2 \pm e_3}{2}\} + \{\frac{\pm e_1' \pm e_2'}{2}\}.$$

For α_1, we have

$$(u_{-\alpha_1}(x)\eta, I_4) = (b, h^{-1}) \cdot (\eta, I_4) \cdot (g, h),\ (b, h^{-1}) \in B(F), (g, h) \in H(F) \tag{3.1}$$

where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1+x} & \frac{-x}{1+x} & 0 & \frac{x}{1+x} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1+x} & \frac{x}{1+x} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{x}{1+x} & 0 & 0 & 0 & 0 & \frac{1}{1+x} \end{pmatrix}, h = \begin{pmatrix} \frac{1}{1+x} & \frac{-x}{1+x} & 0 & \frac{x}{1+x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{1+x} & \frac{x}{1+x} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x+1 & 0 & 0 & -x & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x+1 \end{pmatrix}.$$

This implies that (recall that $\beta_{\alpha_1}^\vee$ is defined by the equation (2.7), also note that the representation has trivial central character)

$$\beta_{\alpha_1}^\vee=\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \alpha_1^\vee-\beta_{\alpha_1}^\vee=\frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2}.$$

For α_2, we have have

(3.2)
$$(u_{-\alpha_2}(x)\eta,I_4)=(b,h^{-1})\cdot(\eta,I_4)\cdot(g,h),\ (b,h^{-1})\in B(F),(g,h)\in H(F)$$

where

$$g=\begin{pmatrix}\frac{1}{1-x}&0&0&0&0&\frac{-x}{1-x}\\0&1&0&0&0&0\\0&0&1&\frac{x}{1-x}&0&0\\0&0&0&\frac{1}{1-x}&0&0\\0&0&0&0&\frac{1}{1-x}&0\\0&0&0&0&0&1\end{pmatrix},h=\begin{pmatrix}1&0&0&0\\0&1&\frac{x}{1-x}&0\\0&0&\frac{1}{1-x}&0\\0&0&0&\frac{1}{1-x}\end{pmatrix},$$

$$b=\begin{pmatrix}1-x&x&0&0&0&x\\0&1&0&0&0&0\\0&0&1-x&-x&0&0\\0&0&0&1&0&0\\0&0&0&0&1-x&-x\\0&0&0&0&0&1\end{pmatrix}.$$

This implies that

$$\beta_{\alpha_2}^\vee=\frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2},\ \alpha_2^\vee-\beta_{\alpha_2}^\vee=\frac{e_1+e_2-e_3}{2}+\frac{-e_1'-e_2'}{2}.$$

For α_3, we have

(3.3) $(u_{-\alpha_3}(x)\eta,I_4)=(g,h)\cdot(\eta,I_4)\cdot(g^{-1},h^{-1}),\ (g,h)\in B(F)\cap H(F)$

where $h=diag(1-x,1,1-x,1)$ and $g=diag(1,1-x,1,1-x,1,1-x)$. This implies that

$$\beta_{\alpha_3}^\vee=\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \alpha_3^\vee-\beta_{\alpha_3}^\vee=\frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2}.$$

For α_1', we can reduce to the root α_2 (because the open orbit is represented by the element (η,I_4)) but we need to change $u_{-\alpha_2}(x)\eta$ to $\eta u_{-\alpha_2}(x)^{-1}=\eta u_{-\alpha_2}(-x)$. We have

(3.4)
$$(\eta u_{-\alpha_2}(-x),I_4)=(b,h^{-1})\cdot(\eta,I_4)\cdot(g,h),\ (b,h^{-1})\in B(F),(g,h)\in H(F)$$

where

$$g=\begin{pmatrix} \frac{1}{1-x} & 0 & 0 & 0 & 0 & \frac{-x}{1-x} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-x} & \frac{-x}{1-x} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, h=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-x} & \frac{-x}{1-x} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{1-x} \end{pmatrix},$$

$$b=\begin{pmatrix} 1-x & x & 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 1-x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-x & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that

$$\beta_{\alpha_1'}^{\vee}=\frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2},\ \alpha_1'^{\vee}-\beta_{\alpha_1'}^{\vee}=\frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2}.$$

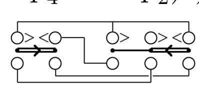
For α_2', we can reduce to the root α_3 but we need to change $u_{-\alpha_3}(x)\eta$ to $\eta u_{-\alpha_3}(x)^{-1}=\eta u_{-\alpha_3}(-x)$. We have

(3.5)
$$(\eta u_{-\alpha_3}(-x),I_4)=(g,h)\cdot(\eta,I_4)\cdot(g^{-1},h^{-1}),\ (g,h)\in B(F)\cap H(F)$$

where $h=diag(1+x,1,1+x,1)$ and $g=diag(1,1+x,1,1+x,1,1+x)$. This implies that

$$\beta_{\alpha_2'}^{\vee}=\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \alpha_2'^{\vee}-\beta_{\alpha_2'}^{\vee}=\frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2}.$$

Remark 3.6. *In this remark, we explain how to use the Luna diagram to compute the virtual colors. We recall the following Luna diagram of the model* $(\mathrm{GSp}_6\times\mathrm{GSp}_4,(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ *in Case (48) of* [BP]*:*



The middle row is the Dynkin diagram of G, from left to right we have the simple roots $\alpha_2',\alpha_1',\alpha_1,\alpha_2,\alpha_3$. For each simple root, there are two colors associated to it (represented by the two $\circ$ above and below the simple root). There is a line connecting two colors if and only if they are equal to each other. For $\alpha=\alpha_1,\alpha_3,\alpha_2'$ (resp. $\alpha=\alpha_2,\alpha_1'$), we use $\beta_\alpha^\vee$ to denote the color above (resp. below) α in the Luna diagram and we use $\alpha^\vee-\beta_\alpha^\vee$ to denote the color below (resp. above) α in the Luna diagram. The Luna diagram above implies that

$$\beta_{\alpha_1}^{\vee}=\beta_{\alpha_3}^{\vee}=\beta_{\alpha_2'}^{\vee},\ \beta_{\alpha_1'}^{\vee}=\alpha_3^{\vee}-\beta_{\alpha_3}^{\vee},\ \alpha_1'^{\vee}-\beta_{\alpha_1'}^{\vee}=\alpha_1^{\vee}-\beta_{\alpha_1}^{\vee},\ \alpha_2'^{\vee}-\beta_{\alpha_2'}^{\vee}=\beta_{\alpha_2}^{\vee}.$$

Combining the first three equations, we have

$\alpha_1'^{\vee}=\beta_{\alpha_1'}^{\vee}+(\alpha_1'^{\vee}-\beta_{\alpha_1'}^{\vee})=(\alpha_3^{\vee}-\beta_{\alpha_3})+(\alpha_1^{\vee}-\beta_{\alpha_1}^{\vee})=\alpha_1^{\vee}+\alpha_3^{\vee}-2\beta_{\alpha_1}^{\vee}.$

This implies that

$$\beta_{\alpha_1}^{\vee}=\beta_{\alpha_3}^{\vee}=\beta_{\alpha_2'}^{\vee}=\frac{\alpha_1^{\vee}+\alpha_3^{\vee}-\alpha_1'^{\vee}}{2}=\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2}.$$

Combining with the last three equations, we have

$$\beta_{\alpha_1'}^{\vee}=\alpha_3^{\vee}-\beta_{\alpha_3}^{\vee}=\frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2},$$
$$\alpha_1'^{\vee}-\beta_{\alpha_1'}^{\vee}=\alpha_1^{\vee}-\beta_{\alpha_1}^{\vee}=\frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2},$$
$$\alpha_2'^{\vee}-\beta_{\alpha_2'}^{\vee}=\beta_{\alpha_2}^{\vee}=\frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2},$$
$$\alpha_2^{\vee}-\beta_{\alpha_2}^{\vee}=\frac{e_1+e_2-e_3}{2}+\frac{-e_1'-e_2'}{2}.$$

This recovers the above computation of colors using matrix identities.

Proposition 3.7. Θ^+ *is consisting of the following 16 elements:*

(3.6)
$$\frac{e_1+e_2\pm e_3}{2}+\frac{\pm e_1'\pm e_2'}{2},\ \frac{e_1-e_2+e_3}{2}+\frac{e_1'\pm e_2'}{2},\ \frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},$$
$$\frac{\pm(e_1-e_2-e_3)}{2}+\frac{e_1'\pm e_2'}{2},\ \frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2}.$$

Proof. By the computations of the virtual weighted colors above, we know that Θ^+ is the smallest subset of Θ satisfies the following 6 conditions:

(1) $\{\frac{e_1+e_2-e_3}{2}+\frac{-e_1'-e_2'}{2}\ \frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2},\ \frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2},\ \frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2}\}\subset\Theta^+$.
(2) $\Theta^+-(\Theta^+\cap w_{\alpha_1}\Theta^+)=\{\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2}\}$.
(3) $\Theta^+-(\Theta^+\cap w_{\alpha_2}\Theta^+)=\{\frac{e_1+e_2-e_3}{2}+\frac{-e_1'-e_2'}{2},\ \frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2}\}$.
(4) $\Theta^+-(\Theta^+\cap w_{\alpha_3}\Theta^+)=\{\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2}\}$.
(5) $\Theta^+-(\Theta^+\cap w_{\alpha_1'}\Theta^+)=\{\frac{-e_1+e_2+e_3}{2}+\frac{e_1'-e_2'}{2},\ \frac{e_1-e_2-e_3}{2}+\frac{e_1'-e_2'}{2}\}$.
(6) $\Theta^+-(\Theta^+\cap w_{\alpha_2'}\Theta^+)=\{\frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \frac{-e_1+e_2-e_3}{2}+\frac{e_1'+e_2'}{2}\}$.

It is clear that the set in the statement satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these 6 conditions. Let Θ'^+ be another subset of Θ satisfies these 6 conditions. Then the set $\Theta^+\cap\Theta'^+$ also satisfies these 6 conditions. This implies that $\Theta^+-(\Theta^+\cap\Theta'^+)$ and $\Theta'^+-(\Theta^+\cap\Theta'^+)$ are W-invariant subsets of Θ (recall that W is the Weyl group of $\hat{G}$). But the only W-invariant subsets of Θ are Θ and the empty set. Hence we must have $\Theta^+ = \Theta'^+$. This proves the proposition. $\square$

It is clear that Θ^+ satisfies (2.3). So the last thing remains to prove Lemma 2.14 for the current case.

Lemma 3.8. *With the notation above, we have*

$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H/Z_{G,H}}(1)} = \frac{1}{\zeta(2)^2\zeta(4)} = (1-q^{-2})^2(1-q^{-4}).$$

Recall that $c_{WS}(\theta) = \frac{\prod_{\gamma^\vee\in\Theta^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+} 1-e^{\alpha^\vee}}(\theta)$.

Proof. Since the summation is independent of θ (see Lemma 2.14), we set $\theta = \delta_B^{1/2}$. The lemma follows from the following two claims:

(1) $c_{WS}(w\theta)$ is zero unless w is the longest Weyl element.
(2) If w is the longest Weyl element, we have $c_{WS}(w\theta) = (1-q^{-2})^2(1-q^{-4})$.

The second claim is easy to prove so we will focus on the first one. Let $w = (s,s') \in W$ so that $c_{WS}(w\theta)$ is nonzero. Here s is a Weyl element of GSp_6 and s' is a Weyl element of GSp_4. By abuse of language, we can also view $w = (s,s')$ as a Weyl element of the dual group $\mathrm{GSpin}_7(\mathbb{C}) \times \mathrm{GSpin}_5(\mathbb{C})$. Then

$$e^{\gamma^\vee}(w\theta) = e^{w^{-1}\gamma^\vee}(\theta) \neq q^{1/2},\ \gamma^\vee \in \Theta^+. \tag{3.7}$$

The values of the modular character $\delta_{B_6}^{1/2}$ on the weights

$$\frac{e_1+e_2+e_3}{2}, \frac{e_1+e_2-e_3}{2}, \frac{e_1-e_2+e_3}{2}, \frac{-e_1+e_2+e_3}{2}, \frac{e_1-e_2-e_3}{2},$$
$$\frac{-e_1+e_2-e_3}{2}, \frac{-e_1-e_2+e_3}{2}, \frac{-e_1-e_2-e_3}{2}$$

are equal to

$$q^3, q^2, q, 1, 1, q^{-1}, q^{-2}, q^{-3}.$$

The values of the modular character $\delta_{B_4}^{1/2}$ on the weights

$$\frac{e_1'+e_2'}{2}, \frac{e_1'-e_2'}{2}, \frac{-e_1'+e_2'}{2}, \frac{-e_1'-e_2'}{2}$$

are equal to

$$q^{3/2}, q^{1/2}, q^{-1/2}, q^{-3/2}.$$

Apply (3.7) to the first eight weights in Θ^+, we know that

$$s^{-1}(\frac{e_1+e_2\pm e_3}{2}) \in \{\frac{e_1+e_2+e_3}{3}, \frac{-e_1-e_2\pm e_3}{2}\}.$$

Since s is a Weyl element, we must have $s^{-1}(\frac{e_1+e_2\pm e_3}{2})=\{\frac{-e_1-e_2\pm e_3}{2}\}$. This implies that

$$\{s^{-1}(e_1),s^{-1}(e_2)\}=\{-e_1,-e_2\},\ s(e_3)\in\{\pm e_3\}.$$

If $s^{-1}(e_1)=-e_2,s^{-1}(e_2)=-e_1$ and $s^{-1}(e_3)=e_3$, then s^{-1} fixes $\frac{e_1-e_2+e_3}{2}$ and $\frac{\pm(e_1-e_2-e_3)}{2}$. Combining with (3.7) and the fact that Θ^+ contains the following 7 elements

$$\frac{e_1-e_2+e_3}{2}+\frac{e_1'\pm e_2'}{2},\ \frac{e_1-e_2+e_3}{2}+\frac{-e_1'+e_2'}{2},\ \frac{\pm(e_1-e_2-e_3)}{2}+\frac{e_1'\pm e_2'}{2},$$

we know that

$$s'^{-1}\{\frac{e_1'\pm e_2'}{2},\frac{-e_1'+e_2'}{2}\}=\{\frac{e_1'\pm e_2'}{2},\frac{-e_1'-e_2'}{2}\},$$

$$s'^{-1}(\frac{e_1'\pm e_2'}{2})\in\{\frac{\pm e_1'+e_2'}{2},\frac{-e_1'-e_2'}{2}\}.$$

It is easy to see that such an s' does not exist, so we get a contradiction. Similarly we can also get a contradiction when $s^{-1}(e_1)=-e_2,s^{-1}(e_2)=-e_1,s^{-1}(e_3)=-e_3$ or $s^{-1}(e_1)=-e_1,s^{-1}(e_2)=-e_2,s^{-1}(e_3)=e_3$.

Now the only case left is when $s^{-1}(e_1)=-e_1,s^{-1}(e_2)=-e_2$ and $s^{-1}(e_3)=-e_3$. In this case, we have $s^{-1}(\alpha^\vee)=-\alpha^\vee$ for all $\alpha^\vee\in\{\frac{\pm e_1\pm e_2\pm e_3}{2}\}$. By the same argument as in the previous cases, we know that

$$s'^{-1}\{\frac{e_1'\pm e_2'}{2},\frac{-e_1'+e_2'}{2}\}=\{\frac{-e_1'\pm e_2'}{2},\frac{e_1'-e_2'}{2}\},$$

$$s'^{-1}(\frac{e_1'\pm e_2'}{2})\in\{\frac{\pm e_1'+e_2'}{2},\frac{-e_1'-e_2'}{2}\}.$$

This implies that $s'^{-1}(e_1')=-e_1'$ and $s'^{-1}(e_2')=-e_2'$. In particular $w=(s,s')$ is the longest Weyl element. This proves the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\cdot\frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}=\zeta(1)^2\zeta(4)\zeta(6)\frac{L(1/2,\pi,\mathrm{Spin}_7\otimes\mathrm{Spin}_5)}{L(1,\pi,\mathrm{Ad})}$$

where $\pi=\pi_1\otimes\pi_2$ is an unramified representation of $\mathrm{GSp}_6(F)\times\mathrm{GSp}_4(F)$.

4. The model $(\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$

In this section, we compute the local relative character for the model $(\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$. We again closely follow four steps in Section 2.3.1.

Let $G=\mathrm{GL}_4\times\mathrm{GL}_2$ and $H=\mathrm{GL}_2\times\mathrm{GL}_2$ embed into G via the map $(a,b)\mapsto(diag(a,b),b)$. Similarly, we can also define the quaternion version of this model. Let D/F be a quaternion algebra, and let $G_D(F)=\mathrm{GL}_2(D)\times\mathrm{GL}_1(D)$ and $H_D(F)=\mathrm{GL}_1(D)\times\mathrm{GL}_1(D)$ embed into G_D via the map $(a,b)\mapsto(diag(a,b),b)$.

We let $\Theta=\Theta_1\cup\Theta_2\cup\Theta_3$ with Θ_1 being the weights of the representation $\wedge^2\otimes\mathrm{std}$ of $\mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})$, Θ_2 being the weights of the standard representation of $\mathrm{GL}_4(\mathbb{C})$ and Θ_3 being the weights of the dual of the standard representation of $\mathrm{GL}_4(\mathbb{C})$. We can write Θ_i as

$$\Theta_1=\{e_i+e_j+e_k' \mid 1\le i<j\le 4, 1\le k\le 2\},$$

$$\Theta_2=\{e_i\mid 1\le i\le 4\},\ \Theta_3=\{-e_i\mid 1\le i\le 4\}.$$

Set $\eta=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\-1&1&-1&1\end{pmatrix}$ and $\eta^{-1}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\1&0&1&1\end{pmatrix}$. The proofs of the following two lemmas are similar to the $(\mathrm{GSp}_6\times\mathrm{GSp}_4,(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ case, and we will skip them here.

Lemma 4.1. *The double cosets $B(F)\backslash G(F)/H(F)$ contain a unique open orbit $B(F)(\eta,I_2)H(F)$. Here $B(F)=B_4(F)\times B_2(F)$ is the upper triangular Borel subgroup.*

Lemma 4.2. *For all $n\in\bar{N}_4(\varpi\mathcal{O}_F)$ and $n'\in\bar{N}_2(\varpi\mathcal{O}_F)$, we have*

$$(n,n')(\eta,I_2)\in T(\mathcal{O}_F)N(\varpi\mathcal{O}_F)(\eta,I_2)H(\mathcal{O}_F)$$

with $B=TN$.

Then we compute the colors for this case. Let $\alpha_1=\varepsilon_1-\varepsilon_2,\alpha_2=\varepsilon_2-\varepsilon_3,\alpha_3=\varepsilon_3-\varepsilon_4$ be the simple roots of GL_4 and $\alpha'=\varepsilon_1'-\varepsilon_2'$ be the simple root of GL_2. Set

$$u_{-\alpha_1}(x)=\begin{pmatrix}1&0&0&0\\x&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ u_{-\alpha_2}(x)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&x&1&0\\0&0&0&1\end{pmatrix},$$

$$u_{-\alpha_3}(x)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&x&1\end{pmatrix}.$$

We first study α_1, we have

(4.1)

$$(u_{-\alpha_1}(x)\eta,I_2)=(b,h^{-1})\cdot(\eta,I_2)\cdot(g,h),\ (b,h^{-1})\in B(F),(g,h)\in H(F)$$

where

$$g=\begin{pmatrix}1&0&0&0\\\frac{x}{x+1}&\frac{1}{x+1}&0&0\\0&0&\frac{1}{x+1}&\frac{x}{x+1}\\0&0&0&1\end{pmatrix},h=\begin{pmatrix}\frac{1}{x+1}&\frac{x}{x+1}\\0&1\end{pmatrix},$$

$$b=\begin{pmatrix}1&0&0&0\\0&x+1&0&0\\0&0&1&-x\\0&0&0&x+1\end{pmatrix}.$$

This implies that $\beta_{\alpha_1}^\vee=e_1+e_3+e_2'$ and $\alpha_1^\vee-\beta_{\alpha_1}^\vee=e_1+e_4+e_1'$ (note that the representation has trivial central character).

For α_2, we have

(4.2)

$$(u_{-\alpha_2}(x)\eta,I_2)=(b,h^{-1})\cdot(\eta,I_2)\cdot(g,h),\ (b,h^{-1})\in B(F),(g,h)\in H(F)$$

where

$$g=\begin{pmatrix}\frac{1}{1-x}&\frac{-x}{1-x}&0&0\\0&1&0&0\\0&0&\frac{1}{1-x}&0\\0&0&0&\frac{1}{1-x}\end{pmatrix},h=\begin{pmatrix}\frac{1}{1-x}&0\\0&\frac{1}{1-x}\end{pmatrix},$$

$$b=\begin{pmatrix}1-x&x&0&0\\0&1&0&0\\0&0&1-x&0\\0&0&0&1-x\end{pmatrix}.$$

This implies that $\beta_{\alpha_2}^\vee=e_2$ and $\alpha_2^\vee-\beta_{\alpha_2}^\vee=-e_3$.

For α_3, we have

(4.3)

$$(u_{-\alpha_3}(x)\eta,I_2)=(b,h^{-1})\cdot(\eta,I_2)\cdot(g,h),\ (b,h^{-1})\in B(F),(g,h)\in H(F)$$

where

$$g=\begin{pmatrix}\frac{1}{1-x}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{1-x}\end{pmatrix},h=\begin{pmatrix}1&0\\0&\frac{1}{1-x}\end{pmatrix},b=\begin{pmatrix}1-x&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1-x\end{pmatrix}.$$

This implies that $\beta_{\alpha_3}^\vee = e_2+e_3+e_1'$ and $\alpha_3^\vee - \beta_{\alpha_3}^\vee = e_1+e_3+e_2'$.

For the root α' on GL_2, we can reduce to the root α_3 on GL_4 but we need to change $u_{-\alpha_3}(x)\eta$ to $\eta u_{-\alpha_3}(-x)$. We have

$$(\eta u_{-\alpha_3}(-x), I_2) = (b,h^{-1})\cdot(\eta, I_2)\cdot(g,h),\ (b,h^{-1})\in B(F), (g,h)\in H(F) \tag{4.4}$$

where

$$g=\begin{pmatrix} \frac{1}{1+x} & \frac{x}{1+x} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{1+x}\end{pmatrix}, h=\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+x}\end{pmatrix}, b=\begin{pmatrix} 1+x & -x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+x\end{pmatrix}.$$

This implies that $\beta_{\alpha'}^\vee = e_2+e_3+e_1'$ and $\alpha'^\vee - \beta_{\alpha'}^\vee = e_1+e_4+e_1'$.

Proposition 4.3. *Θ^+ is consisting of the following 10 elements:*

$$e_1+e_i+e_j', 2\leq i\leq 3, 1\leq j\leq 2;\ e_1+e_4+e_1', e_2+e_3+e_1';\ e_1, e_2, -e_3, -e_4.$$

Proof. By the computation of colors above, we know that Θ^+ is the smallest subset of Θ satisfying the following 5 conditions:

(1) $e_1+e_3+e_2', e_1+e_4+e_1', e_2+e_3+e_1', e_2, -e_3\in\Theta^+$.
(2) $\Theta^+ - (\Theta^+\cap w_{\alpha_1}\Theta^+) = \{e_1+e_3+e_2', e_1+e_4+e_1'\}$.
(3) $\Theta^+ - (\Theta^+\cap w_{\alpha_2}\Theta^+) = \{e_2, -e_3\}$.
(4) $\Theta^+ - (\Theta^+\cap w_{\alpha_3}\Theta^+) = \{e_1+e_3+e_2', e_2+e_3+e_1'\}$.
(5) $\Theta^+ - (\Theta^+\cap w_{\alpha_1'}\Theta^+) = \{e_2+e_3+e_1', e_1+e_4+e_1'\}$.

It is clear that the set in the statement satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6\times\mathrm{GSp}_4, (\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. □

It is clear that Θ^+ satisfies (2.3). The last thing remains to prove Lemma 2.14 for the current case.

Lemma 4.4. *With the notation above, we have*

$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H/Z_{G,H}}(1)} = \frac{1}{\zeta(1)\zeta(2)^2} = (1-q^{-1})(1-q^{-2})^2.$$

Proof. Since the summation is independent of θ, we set $\theta=\delta_B^{1/2}$. The lemma follows from the following two claims:

(1) $c_{WS}(w\theta)$ is zero unless w is the longest Weyl element.
(2) If w is the longest Weyl element, we have $c_{WS}(w\theta) = (1-q^{-1})(1-q^{-2})^2$.

The second claim is easy to prove so we will focus on the first one. For $w=(s,s')\in W=S_4\times S_2$, we know that $c_{WS}(w\theta)$ is nonzero if and only if

$$1-q^{-1/2}\theta_{s(i)},\ 1-q^{-1/2}\theta_{s(1)}\theta_{s(j)}\theta'_{s'(k)},\tag{4.5}$$

$$1-q^{-1/2}\theta_{s(1)}\theta_{s(4)}\theta'_{s'(1)},\ 1-q^{-1/2}\theta_{s(2)}\theta_{s(3)}\theta'_{s'(1)}$$

are nonzero for $1\leq i\leq 4, 2\leq j\leq 3, 1\leq k\leq 2$ where $\theta_1=q^{3/2},\theta_2=\theta'_1=q^{1/2},\theta_3=\theta'_2=q^{-1/2},\theta_4=q^{-3/2}$.

Using the four terms $1-q^{-1/2}\theta_{s(i)}$ in (4.5), we have $s(1),s(2)\in\{1,3,4\}$, $s(3),s(4)\in\{1,2,4\}$. This implies that $\{s(1),s(2)\}$ is equal to $\{1,3\}$ or $\{3,4\}$. If it is equal to $\{1,3\}$, then $\theta_{s(1)}\theta_{s(2)}=q$. Hence $\theta_{s(1)}\theta_{s(2)}\theta'_{s'(1)}$ or $\theta_{s(1)}\theta_{s(2)}\theta'_{s'(2)}$ is equal to $q^{1/2}$. This is a contradiction. So we must have $\{s(1),s(2)\}=\{3,4\}$.

If $s(1)=3$, then $\theta_{s(1)}\theta_{s(3)}$ is equal to 1 or q (depends on whether $s(3)=2$ or $s(3)=1$). In both cases, we have $\theta_{s(1)}\theta_{s(3)}\theta'_{s'(1)}$ or $\theta_{s(1)}\theta_{s(3)}\theta'_{s'(2)}$ is equal to $q^{1/2}$. This is a contradiction. So we must have $s(1)=4$ and $s(2)=3$.

Now if $s(3)=1$, then $\theta_{s(1)}\theta_{s(3)}=1$, which implies that $\theta_{s(1)}\theta_{s(3)}\theta'_{s'(1)}$ or $\theta_{s(1)}\theta_{s(3)}\theta'_{s'(2)}$ is equal to $q^{1/2}$. This is a contradiction. So we must have $s(3)=2$ and $s(4)=1$.

Finally, using the fact that $1-q^{-1/2}\theta_{s(1)}\theta_{s(4)}\theta'_{s'(1)}\neq 0$ we know that $s'(1)=2$ and $s'(2)=1$. Hence w is the longest Weyl element. This proves the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)\zeta(3)\zeta(4)\frac{L(1/2,\pi,\wedge^2\otimes \mathrm{std}_2)L(1/2,\pi_1,\mathrm{std}_4)L(1/2,\pi_1,\mathrm{std}_4^\vee)}{L(1,\pi,\mathrm{Ad})}$$

where $\pi=\pi_1\otimes\pi_2$ is an unramified representation of $\mathrm{GL}_4(F)\times\mathrm{GL}_2(F)$.

5. The model $(\mathrm{GL}_6,\mathrm{GL}_2\ltimes U)$

In this section, we compute the local relative character for the model $(\mathrm{GL}_6,\mathrm{GL}_2\ltimes U)$. We closely follow the six steps in Section 2.5.1.

Let $G=\mathrm{GL}_6$, $H=H_0\ltimes U$ with

$$H_0=\{diag(h,h,h)\mid h\in\mathrm{GL}_2\},$$

$$U=\{u(X,Y,X)=\begin{pmatrix} I_2 & X & Z\\ 0 & I_2 & Y\\ 0 & 0 & I_2\end{pmatrix}\mid X,Y,Z\in Mat_{2\times 2}\}.$$

Let $P = LU$ be the parabolic subgroup of G with $L = \{(h_1, h_2, h_3) \mid h_i \in \mathrm{GL}_2\}$. We define a generic character ξ on $U(F)$ to be $\xi(u(X, Y, Z)) = \psi(\lambda(u(X, Y, Z)))$ where $\lambda(u(X, Y, Z)) = \mathrm{tr}(X)+\mathrm{tr}(Y)$. It is easy to see that H_0 is the stabilizer of this character and (G, H) is the Whittaker induction of the trilinear GL_2 model (L, H_0, ξ). The model (G, H, ξ) is the so called Ginzburg–Rallis model introduced by Ginzburg and Rallis in [GR].

We can also define the quaternion version of this model. Let D/F be a quaternion algebra, and let $G_D(F) = \mathrm{GL}_3(D)$, $H_D = H_{0,D} \ltimes U_D$ with

$$H_{0,D}(F) = \{diag(h, h, h) \mid h \in \mathrm{GL}_1(D)\},$$
$$U_D(F) = \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \mid X, Y, Z \in D\}.$$

Like the split case, we can define the character ξ_D on $U_D(F)$ by replacing the trace map of $Mat_{2\times 2}$ by the trace map of D.

Let $w_0 = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix}$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a : \mathrm{GL}_1 \to Z_L$ to be

$$a(t) = (tI_2, I_2, t^{-1}I_2).$$

This clearly satisfies (2.15). For the open Borel orbit, let

$$\eta_0 = diag(I_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$$

be the representative of the open Borel orbit for the model (L, H_0) as in Section 2.3.2, and $\eta = \eta_0 w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Now we compute the set of colors and also the set Θ^+. Let Θ be the weights of the exterior cube representation of $\mathrm{GL}_6(\mathbb{C})$. We can write it as

$$\Theta = \{e_i + e_j + e_k \mid 1 \leq i < j < k \leq 6\}.$$

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ be the simple roots for $1 \leq i \leq 5$. By the computation of the trilinear GL_2-model in Section 2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we get the set of colors for this case:

$$\beta_{\alpha_1}^\vee = e_1 + e_4 + e_5,\ \alpha_1^\vee - \beta_{\alpha_1}^\vee = e_1 + e_3 + e_6,$$
$$\beta_{\alpha_3}^\vee = e_2 + e_3 + e_5,\ \alpha_3^\vee - \beta_{\alpha_3}^\vee = e_1 + e_3 + e_6,$$

$$\beta_{\alpha_5}^\vee = e_2 + e_3 + e_5,\ \alpha_5^\vee - \beta_{\alpha_3}^\vee = e_1 + e_4 + e_5.$$

Then we verify (2.23) for α_2 and α_4.

For α_2, let $u_{-\alpha_2}(a) = (x_{ij})_{1\le i,j\le 6}$ with $x_{ii} = 1,\ x_{32} = a$ and $x_{ij} = 0$ for all the other (i, j). We have

$$u_{-\alpha_2}(a)\eta = \eta \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & X \\ 0 & 0 & I_2 \end{pmatrix},\ X = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

This proves (2.23) for α_2.

For α_4, let $u_{-\alpha_4}(a) = (x_{ij})_{1\le i,j\le 6}$ with $x_{ii} = 1,\ x_{54} = a$ and $x_{ij} = 0$ for all the other (i, j). We have

$$u_{-\alpha_4}(a)\eta = \eta \begin{pmatrix} I_2 & X & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix},\ X = \begin{pmatrix} -a & 0 \\ a & 0 \end{pmatrix}.$$

This proves (2.23) for α_4. Next, we compute the set Θ^+.

Proposition 5.1. Θ^+ *is consisting of the following 10 elements:*

$$e_1 + e_2 + e_i, e_1 + e_3 + e_j, e_1 + e_4 + e_5, e_2 + e_3 + e_4, e_2 + e_3 + e_5$$

where $3 \le i \le 6$ *and* $4 \le j \le 6$.

Proof. By the computations above, we know that Θ^+ is the smallest subset of Θ satisfying the following 5 conditions:

(1) $e_1 + e_4 + e_5, e_1 + e_3 + e_6, e_2 + e_3 + e_5 \in \Theta^+$.
(2) $\Theta^+ - (\Theta^+ \cap w_{\alpha_1}\Theta^+) = \{e_1 + e_4 + e_5, e_1 + e_3 + e_6\}$.
(3) $\Theta^+ - (\Theta^+ \cap w_{\alpha_3}\Theta^+) = \{e_2 + e_3 + e_5, e_1 + e_3 + e_6\}$.
(4) $\Theta^+ - (\Theta^+ \cap w_{\alpha_5}\Theta^+) = \{e_2 + e_3 + e_5, e_1 + e_4 + e_5\}$.
(5) Θ^+ is stable under w_{α_2} and w_{α_4}.

It is clear that the set in the statement satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. $\square$

It is clear that Θ^+ satisfies (2.3). The last thing remains to prove Lemma 2.32 for the current case.

Lemma 5.2. *With the notation above, we have*

$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H_0/Z_{G,H}}(1)} = \frac{1}{\zeta(2)} = (1 - q^{-2}).$$

Proof. Recall that $W = S_6$ is the permutation group of 6 variables. The goal is to show that

$$\sum_{s\in S_6} \frac{\Pi_{e_i+e_j+e_k\in\Theta^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\leq i<j\leq 6}(1-\theta_{s(i)}/\theta_{s(j)})} = 1-q^{-2}.$$

Here θ_i are arbitrary variables satisfying the equation $\Pi_{i=1}^6\theta_i = 1$. We define the subset Θ_0^+ of Θ^+ to be

$$\Theta_0^+ = \{e_1+e_i+e_j, e_1+e_4+e_5, e_2+e_3+e_5 \mid 2\leq i\leq 3, 5\leq j\leq 6\}.$$

It contains those weights in the $\wedge^2\otimes \mathrm{std}_2$ representation of $\mathrm{GL}_4\times\mathrm{GL}_2$. We need a lemma.

Lemma 5.3. *We embed $S_4\times S_2$ into S_6 by letting S_4 act on the first four elements and S_2 acts on the last two elements. Then*

$$\sum_{s\in S_4\times S_2} \frac{\Pi_{e_i+e_j+e_k\in\Theta_0^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{\{(i,j)|1\leq i<j\leq 4 \text{ or } 5\leq i<j\leq 6\}}(1-\theta_{s(i)}/\theta_{s(j)})} = 1-q^{-2}.$$

Proof. By the identity for the triple product in Section 2.3.2, we have (here we embed $S_2\times S_2$ into S_4 by letting the first S_2-copy act on the first two elements and the second S_2-copy act on the last two elements)

$$\sum_{s\in S_2\times S_2\times S_2} \frac{\Pi_{e_i+e_j+e_k\in\Theta_1^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{(1-\theta_{s(1)}/\theta_{s(2)})(1-\theta_{s(3)}/\theta_{s(4)})(1-\theta_{s(5)}/\theta_{s(6)})} = 1-q^{-2},$$

where $\Theta_1^+ = \{e_1+e_3+e_5, e_1+e_3+e_6, e_1+e_4+e_5, e_2+e_3+e_5\}$. Hence in order to prove the lemma, it is enough to show that

$$\sum_{s\in S_4/S_2\times S_2} \frac{(1-q^{-1/2}\theta_{s(1)}\theta_{s(2)}\theta_5)(1-q^{-1/2}\theta_{s(1)}\theta_{s(2)}\theta_6)}{\Pi_{1\leq i\leq 2, 3\leq j\leq 4}(1-\theta_{s(i)}/\theta_{s(j)})} = 1.$$

This follows from an easy computation. $\square$

By the lemma above, we have $\sum_{s\in S_6}\frac{\Pi_{e_i+e_j+e_k\in\Theta^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\leq i<j\leq 6}(1-\theta_{s(i)}/\theta_{s(j)})}$ is equal to

$$(1-q^{-2})\cdot \sum_{s\in S_6/S_4\times S_2} \frac{\Pi_{e_i+e_j+e_k\in\Theta^+-\Theta_0^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\leq i\leq 4, 5\leq j\leq 6}(1-\theta_{s(i)}/\theta_{s(j)})}.$$

So it is enough to show that

$$\sum_{s\in S_6/S_4\times S_2} \frac{\Pi_{e_i+e_j+e_k\in\Theta^+-\Theta_0^+}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\leq i\leq 4, 5\leq j\leq 6}(1-\theta_{s(i)}/\theta_{s(j)})} = 1. \tag{5.1}$$

The set $\Theta^+ - \Theta_0^+$ is equal to $\{e_i + e_j + e_k \mid 1 \le i < j < k \le 4\}$. It is easy to see that the constant coefficient of the left hand side of (5.1) is equal to 1. So we just need to show that the $q^{-1/2}, q^{-1}, q^{-3/2}, q^{-2}$-coefficients are equal to 0. For this, we can replace the summation over $S_6/S_4 \times S_2$ by the summation over S_6, and we need to show that the $q^{-1/2}, q^{-1}, q^{-3/2}, q^{-2}$-coefficients of

$$\sum_{s\in S_6} \frac{\Pi_{1\le i<j<k\le 4}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\le i\le 4, 5\le j\le 6}(1-\theta_{s(i)}/\theta_{s(j)})}$$

are equal to 0. We can rewrite the function in the summation as

$$\begin{aligned} & \frac{\Pi_{1\le i<j<k\le 4}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\le i\le 4, 5\le j\le 6}(1-\theta_{s(i)}/\theta_{s(j)})} \\ = & \frac{\theta_{s(5)}^4\theta_{s(6)}^4 \cdot \Pi_{1\le i<j<k\le 4}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)})}{\Pi_{1\le i\le 4, 5\le j\le 6}(\theta_{s(j)}-\theta_{s(i)})} \\ = & \frac{\theta_{s(5)}^4\theta_{s(6)}^4 \cdot \Pi_{\{(i,j)\mid 1\le i<j\le 4 \text{ or } 5\le i<j\le 6\}}(\theta_{s(j)}-\theta_{s(i)})}{\Pi_{1\le i<j\le 6}(\theta_{s(j)}-\theta_{s(i)})} \\ & \cdot \Pi_{1\le i<j<k\le 4}(1-q^{-1/2}\theta_{s(i)}\theta_{s(j)}\theta_{s(k)}). \end{aligned}$$

Since the denominator is (S_6, sgn)-invariant (sgn is the sign character of S_6), we just need to show that the (S_6, sgn)-summation of the $q^{-1/2}, q^{-1}, q^{-3/2}, q^{-2}$-coefficients of

$$\theta_5^4\theta_6^4 \cdot \Pi_{\{(i,j)\mid\, 1\le i<j\le 4 \text{ or } 5\le i<j\le 6\}}(\theta_j - \theta_i) \cdot \Pi_{1\le i<j<k\le 4}(1-q^{-1/2}\theta_i\theta_j\theta_k) \tag{5.2}$$

are equal to 0. A direct computation shows that $\theta_5^4\theta_6^4 \cdot \Pi_{\{(i,j)\mid\, 1\le i<j\le 4 \text{ or } 5\le i<j\le 6\}}(\theta_j - \theta_i)$ is consisting of elements of the form

$$\Pi_i \theta_i^{a_i},\ \{a_1, a_2, a_3, a_4\} = \{0, 1, 2, 3\},\ \{a_5, a_6\} = \{4, 5\}.$$

Then any term $\Pi_i \theta_i^{b_i}$ appeared in the $q^{-1/2}$-coefficient of (5.2) must satisfy the condition $\{b_5, b_6\} = \{4, 5\}$, and also satisfies at least one of the following two conditions

- $b_i = 4$ for some $1 \le i \le 4$;
- $b_i = b_j$ for some $1 \le i < j \le 4$.

In both cases, we have $b_i = b_j$ for some $i \ne j$. This implies that the (S_6, sgn)-summation of the $q^{-1/2}$-coefficient is equal to 0.

Moreover, any term $\Pi_i \theta_i^{b_i}$ appeared in the $q^{-1}, q^{-3/2}, q^{-2}$-coefficients of (5.2) must satisfy the following two conditions

- $b_i \in \{4, 5\}$ for some $1 \le i \le 4$;
- $\{b_5, b_6\} = \{4, 5\}$.

This implies that $b_i = b_j$ for some $i \neq j$. This implies that the (S_6, sgn)-summation of the $q^{-1}, q^{-3/2}, q^{-2}$-coefficients are equal to 0. This finishes the proof of the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)\zeta(3)\zeta(4)\zeta(5)\zeta(6)\frac{L(1/2,\pi,\wedge^3)}{L(1,\pi,\mathrm{Ad})}$$

where π is an unramified representation of $\mathrm{GL}_6(F)$.

6. The models $(\mathrm{GU}_6, \mathrm{GU}_2 \ltimes U)$ and $(\mathrm{GU}_4 \times \mathrm{GU}_2, (\mathrm{GU}_2 \times \mathrm{GU}_2)^0)$

6.1. **The models.** Let $E = F(\sqrt{\epsilon})$ be a quadratic extension of F, $\eta_{E/F}$ be the quadratic character associated to E, $N_{E/F}$ (resp. $\mathrm{tr}_{E/F}$) be the norm map (resp. trace map), and $x \to \bar{x}$ be the Galois action on E. Denote w_n to be the longest Weyl element of GL_n. Define the quasi-split even unitary similitude group $\mathrm{GU}_{n,n}(F)$ to be

$$\mathrm{GU}_{n,n}(F) = \{g \in \mathrm{GL}_{2n}(E) \colon {}^t\bar{g}w_{2n}g = l(g)w_{2n}\} \tag{6.1}$$

where $l(g) \in F^\times$ is the similitude factor of g.

We first define the model $(\mathrm{GU}_6, \mathrm{GU}_2 \ltimes U)$. Let $G = \mathrm{GU}_{3,3}$, and $P = LU$ be the standard parabolic subgroup of G with $(g^* = w_2{}^t\bar{g}^{-1}w_2)$

$$L(F) = \{m(g,h) = \begin{pmatrix} g & & \\ & h & \\ & & l(h)g^* \end{pmatrix} \mid g \in \mathrm{GL}_2(E),\ h \in \mathrm{GU}_{1,1}(F)\},$$

$$U(F) = \{u(X,Y) = \begin{pmatrix} I_2 & X & Y \\ & I_2 & X' \\ & & I_2 \end{pmatrix} \mid X, Y \in Mat_{2\times 2}(E),$$

$$X' = -w_2{}^tXw_2, w_2Y + {}^tYw_2 + {}^tX'w_2X' = 0\}.$$

Let ξ be a generic character of $U(F)$ given by

$$\xi(u(X,Y)) = \psi(\lambda(u(X,Y))),\ \lambda(u(X,Y)) = \mathrm{tr}_{E/F}(\mathrm{tr}(X)).$$

Then the stabilizer of ξ under the adjoint action of $L(F)$ is

$$H_0(F) := \{m(h,h) \mid h \in \mathrm{GU}_{1,1}(F)\} = \{diag(h,h,h) \mid h \in \mathrm{GU}_{1,1}(F)\}.$$

Let $H = H_0 \ltimes U$ and we extend the character ξ to H by making it trivial on H_0. The model (G, H, ξ) is the analogue of the Ginzburg–Rallis model in the previous section for unitary similitude group. We can also define the quaternion (non quasi-split) version of this model by letting G_D be the non quasi-split unitary similitude group (in the archimedean case $G_D = \mathrm{GU}_{4,2}$).

Now we define the model $(\mathrm{GU}_4\times\mathrm{GU}_2,(\mathrm{GU}_2\times\mathrm{GU}_2)^0)$. Let $G=\mathrm{GU}_{2,2}\times\mathrm{GU}_{1,1}$ and $H=(\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1})^0=\{(h_1,h_2)\in\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1}\mid l(h_1)=l(h_2)\}$. We can embed H into G via the map

$$(h_1,h_2)\in H\mapsto\begin{pmatrix}a&0&b\\0&h_1&0\\c&0&d\end{pmatrix},h_1)\in G,\ h_2=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

For the pure inner forms of this model, we use $\mathrm{GU}_{2,0}=\mathrm{GU}_{0,2}$ to denote the non quasi-split unitary similitude group of rank 2, and we use $\mathrm{GU}_{3,1}$ to denote the non quasi-split unitary similitude group of rank 4 and split rank 1 (we use these notation in order to be compatible with the standard notation in the archimedean case). In the p-adic case, the pure inner forms are $(\mathrm{GU}_{2,2}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{0,2})^0)$, $(\mathrm{GU}_{3,1}\times\mathrm{GU}_{1,1},(\mathrm{GU}_{1,1}\times\mathrm{GU}_{2,0})^0)$, $(\mathrm{GU}_{3,1}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{1,1})^0)$. In the archimedean case, there is an extra compact pure inner form $(\mathrm{GU}_{4,0}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{2,0})^0)$.

The goal of this section is to compute the local relative character $I(\phi_\theta)$ for these two models. As we mentioned in Section 2, the difference between these models and all the other models is that since G is not split, the root space maybe two-dimensional. In the next subsection, we will prove two identities that will be used in our computation. Then we will compute the relative character in the last two subsections. From now on, we assume that E/F is unramified and $\epsilon\in\mathcal{O}_F^\times$.

6.2. **Two identities.**

Lemma 6.1. *Let η (resp. σ) be a unitary unramified character of $E^\times$ (resp. $F^\times$). We have*

(6.2)
$$1+q^2\int_{\mathcal{O}_F^2}\sigma(x^2-\epsilon y^2-x)\eta(x+y\sqrt{\epsilon})\,\mathrm{d}x\,\mathrm{d}y=\frac{q^2(1-q^{-1})(1-q^{-4}\sigma^2\eta(\varpi))}{(1-q^{-1}\sigma(\varpi))(1-q^{-2}\sigma\eta(\varpi))}.$$

This integral is an analogy of (2.9). We need this identity when the root space is two dimensional. To compute it, we need the following lemma.

Lemma 6.2. *The equation $x^2-\epsilon y^2-x=0$ has q nonzero solutions in $\mathbb{F}_q\times\mathbb{F}_q$.*

Proof. The equation is equivalent to $(2x-1)^2-\epsilon(2y)^2=1$. So the number of solutions is equal to $|U_1(\mathbb{F}_q)|=q+1$. In particular, there are q nonzero solutions. □

Now we prove Lemma 6.1. Set $X = \mathcal{O}_F \times \mathcal{O}_F^\times \cup \mathcal{O}_F^\times \times \mathcal{O}_F$. Then for $k \geq 0$, we have

$$\varpi^k X = \{(x,y) \in \mathcal{O}_F^2 |\ max\{|x|,|y|\} = q^{-k}\}.$$

This implies that $\mathcal{O}_F \times \mathcal{O}_F$ is a disjoint union of $\varpi^k X$ for $k \geq 0$. Also for $(x,y) \in \varpi^k X$, we have

$$x + y\sqrt{\epsilon} \in \varpi^k \mathcal{O}_E^\times.$$

As a result, the left hand side of (6.2) is equal to

$$\begin{aligned}
&1 + \sum_{k\geq 0} q^2 \int_{\varpi^k X} \sigma(x^2 - \epsilon y^2 - x)\eta(x + y\sqrt{\epsilon})\,\mathrm{d}x\,\mathrm{d}y \\
=& 1 + \sum_{k\geq 0} q^{2-2k} \int_X \sigma(\varpi^{2k}x^2 - \varpi^{2k}\epsilon y^2 - \varpi^k x)\eta^k(\varpi)\,\mathrm{d}x\,\mathrm{d}y \\
=& 1 + \sum_{k\geq 0} q^{2-2k}\eta^k\sigma^k(\varpi) \int_X \sigma(\varpi^k x^2 - \varpi^k \epsilon y^2 - x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Now we study the integral $\int_X \sigma(\varpi^k x^2 - \varpi^k \epsilon y^2 - x)\,\mathrm{d}x\,\mathrm{d}y$. When $k > 0$, by Theorem 10.2.1 in [I00], the integral is equal to

$$\begin{aligned}
&q^{-2}(((q^2-1)-(q-1)) + (q-1)\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}) \\
=& q^{-2}((q^2-q) + (q-1)\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}).
\end{aligned}$$

When $k = 0$, by Theorem 10.2.1 in [I00] and the lemma above, the integral is equal to

$$\begin{aligned}
&q^{-2}(((q^2-1)-(q)) + q\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}) \\
=& q^{-2}((q^2-q) + (q-1)\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}) - q^{-2} + q^{-2}\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}.
\end{aligned}$$

This implies that the left hand side of (6.2) is equal to

$$\begin{aligned}
&\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)} + \sum_{k\geq 0} \eta^k\sigma^k(\varpi)q^{-2k} \cdot ((q^2-q) + (q-1)\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}) \\
=&\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)} + \frac{1}{1-\eta\sigma(\varpi)q^{-2}}((q^2-q) + (q-1)\frac{(1-q^{-1})\sigma(\varpi)}{1-q^{-1}\sigma(\varpi)}) \\
=&\frac{q^2(1-q^{-1})(1-q^{-4}\sigma^2\eta(\varpi))}{(1-q^{-1}\sigma(\varpi))(1-q^{-2}\sigma\eta(\varpi))}.
\end{aligned}$$

This proves Lemma 6.1. We also need the following identity.

Lemma 6.3. *We have (recall that* $\varphi=\varphi_0=1_{\mathcal{O}_F}-\frac{1}{q-1}\cdot 1_{\varpi^{-1}\mathcal{O}_F^\times}$*)*

$$1+q^2\int_{\mathcal{O}_F^2}\eta(x+\sqrt{\varepsilon}y)\cdot|x^2-\varepsilon y^2|^{-1}\cdot\varphi(\frac{2x}{x^2-\varepsilon y^2})\,\mathrm{d}x\,\mathrm{d}y=q^2(1-q^{-2}\eta(\varpi)).$$

Proof. A direct computation shows that

$$\int_X \eta(x+\sqrt{\varepsilon}y)\cdot|x^2-\varepsilon y^2|^{-1}\cdot\varphi(\frac{2x}{x^2-\varepsilon y^2})\,\mathrm{d}x\,\mathrm{d}y=\frac{q^2-1}{q^2},$$

$$\int_{\varpi X} \eta(x+\sqrt{\varepsilon}y)\cdot|x^2-\varepsilon y^2|^{-1}\cdot\varphi(\frac{2x}{x^2-\varepsilon y^2})\,\mathrm{d}x\,\mathrm{d}y=-\frac{\eta(\varpi)}{q^2},$$

$$\int_{\varpi^k X} \eta(x+\sqrt{\varepsilon}y)\cdot|x^2-\varepsilon y^2|^{-1}\cdot\varphi(\frac{2x}{x^2-\varepsilon y^2})\,\mathrm{d}x\,\mathrm{d}y=0,\ k\geq 2.$$

This proves the lemma. $\square$

6.3. **The computation for** $(\mathrm{GU}_6,\mathrm{GU}_2\ltimes U)$**.** In this subsection, we compute the local relative character for the model $(\mathrm{GU}_6,\mathrm{GU}_2\ltimes U)$. First, all the arguments in Section 2.4 still work for the current case, the only exception is that the equation (2.17) will become

$$\int_{G(F)}\Phi(g)\,\mathrm{d}g=\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta_E(1)^{-3}\zeta(1)^{-1}\cdot\int_{H(F)/Z_{G,H}(F)}\int_{B(F)}\Phi(b\eta h)\,\mathrm{d}b\,\mathrm{d}h. \tag{6.3}$$

This is because in the split case, $|T(\mathbb{F}_q)|=(q-1)^{\dim(T)}$; for our current model, $|T(\mathbb{F}_q)|=(q^2-1)^3(q-1)$. This implies that

$$I(\phi_\theta)=\frac{\Delta_{H_0/Z_{G,H}}(1)}{\Delta_G(1)}\zeta_E(1)^3\zeta(1)\cdot\int_K^*\mathcal{Y}_{\theta^{-1},\xi}(k)\,\mathrm{d}k\cdot\int_K^*\mathcal{Y}_{\theta,\xi^{-1}}(k)\,\mathrm{d}k.$$

Now we compute the integral $\int_K^*\mathcal{Y}_{\theta,\xi}(k)\,\mathrm{d}k$. Let $\alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3$ and $\alpha_3=2\varepsilon_3$ be the simple roots of $G(F)$. Note that the root spaces of α_1 and α_2 are two dimensional and the root space of α_3 is one dimensional. Let $w_0=\begin{pmatrix}0&0&I_2\\0&I_2&0\\I_2&0&0\end{pmatrix}$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a:\mathrm{GL}_1\to Z_L$ to be $a(t)=(tI_2,I_2,t^{-1}I_2)$. This clearly satisfies (2.15).

For the open Borel orbit, let

$$\eta_0=diag(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix},I_2,\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix})$$

be the representative of the open Borel orbit for the model (L, H_0), and $\eta = \eta_0 w_0$. The relation (2.20) can be easily verified as in the trilinear GL_2-model case in Section 2.3.2.

Now we compute the colors. Let Θ be the weights of the exterior cube representation of $\hat{G}(\mathbb{C})$. We can write it as

$$\Theta = \{e_i, -e_i, \frac{\pm e_1 \pm e_2 \pm e_3}{2} \mid 1 \leq i \leq 3\}.$$

The weight spaces of $e_i, -e_i$ are two dimensional and the weight spaces of $\frac{\pm e_1 \pm e_2 \pm e_3}{2}$ are one dimensional. More precisely, the exterior representation of the L-group ${}^L\mathrm{GU}_6$ of GU_6 is explicated in Section 3.1 [Z]. More details on the exterior cubic L-function of GU_6 are also given there.

For α_1, as in the split case, we let $I_{\alpha_1}(\theta) = vol(\mathcal{I})^{-1} \int_{G(F)} \mathcal{Y}_{\theta,\xi}^0(x\eta)(\Phi_1(x) + \Phi_{w_{\alpha_1}}(x))\,\mathrm{d}x$. Since the root space is two dimensional, the same argument in the split case implies that

$$\begin{aligned} I_{\alpha_1}(\theta) \quad &= \quad 1 + q^2 \int_{\mathcal{O}_F^2} (\theta^{-1}\delta^{1/2})(e^{\alpha_1^\vee}((x + y\sqrt{\varepsilon})^{-1})) \\ &\qquad\qquad \mathcal{Y}_{\theta,\xi}^0(u_{-\alpha_1}((x + y\sqrt{\varepsilon})^{-1})\eta)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Here $u_{-\alpha_1}(a) = diag(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, I_2, \begin{pmatrix} 1 & 0 \\ -\bar{a} & 1 \end{pmatrix})$. Meanwhile, a direct computation shows that $u_{-\alpha_1}(x + y\sqrt{\epsilon})\eta$ is equal to

$$diag(\begin{pmatrix} \frac{1}{x+1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{-y\sqrt{\epsilon}}{x+1} \\ 0 & \frac{1}{x+1} \end{pmatrix}, \begin{pmatrix} \frac{1}{x+1} & 0 \\ 0 & 1 \end{pmatrix}) \cdot \eta$$

$$\cdot diag(\begin{pmatrix} 1 & y\sqrt{\epsilon} \\ 0 & x+1 \end{pmatrix}, \begin{pmatrix} 1 & y\sqrt{\epsilon} \\ 0 & x+1 \end{pmatrix}, \begin{pmatrix} 1 & y\sqrt{\epsilon} \\ 0 & x+1 \end{pmatrix}).$$

Since $\frac{1}{x+y\sqrt{\epsilon}} = \frac{x}{x^2-y^2\epsilon} - \frac{y\sqrt{\epsilon}}{x^2-y^2\epsilon}$ and $1 + \frac{x}{x^2-y^2\epsilon} = \frac{x+x^2-y^2\epsilon}{x^2-y^2\epsilon}$, we have

$$I_{\alpha_1}(\theta) = 1 + q^2 \int_{\mathcal{O}_F^2} \sigma(x^2 - \varepsilon y^2 - x)\eta(x + y\sqrt{\varepsilon})\,\mathrm{d}x\,\mathrm{d}y$$

where $\eta = \theta(e^{\alpha_1^\vee}) \cdot (|\,|^{-1} \cdot \sigma^{-1}) \circ N_{E/F}$ and $\sigma = \theta(e^{\beta_{\alpha^\vee}})|_{F^\times} \cdot |\,|^{-1/2}$ with $\beta_{\alpha_1}^\vee = \frac{e_1-e_2-e_3}{2}$. Combing with Lemma 6.1, we have

$$I_{\alpha_1}(\theta) = q^2(1 - q^{-1}) \cdot \frac{1 - q^{-2}e^{\alpha_1^\vee}(\theta)}{(1 - q^{-1/2}e^{\beta_{\alpha_1}^\vee}(\theta))(1 - q^{-1/2}e^{\alpha_1^\vee - \beta_{\alpha_1}^\vee}(\theta))}$$

with $\beta_{\alpha_1}^\vee = \frac{e_1-e_2-e_3}{2}$, $\alpha_1^\vee - \beta_{\alpha_1}^\vee = \frac{e_1-e_2+e_3}{2}$.

For α_2, as in the Ginzburg–Rallis model case, it is easy to see that

$$\mathcal{Y}^0_{\theta,\xi}(u_{-\alpha_2}(a)\eta)=\varphi(a+\bar{a}), u_{-\alpha_2}(a)=diag(1,\begin{pmatrix}1&0\\a&1\end{pmatrix},\begin{pmatrix}1&0\\-\bar{a}&1\end{pmatrix},1).$$

Then we have (note that the root space in this case is also 2-dimensional)

$$I_{\alpha_2}(\theta)=1+q^2\int_{\mathcal{O}_F^2}\theta(e^{\alpha_2^\vee}(x+\sqrt{\varepsilon}y))\cdot|x^2-\varepsilon y^2|^{-1}\cdot\varphi(\frac{2x}{x^2-\varepsilon y^2})\,\mathrm{d}x\,\mathrm{d}y.$$

By Lemma 6.3, we know that

$$I_{\alpha_2}(\theta)=q^2\cdot(1-q^{-2}e^{\alpha_2^\vee}(\theta)).$$

For α_3, the root space is one dimensional, so we have the identity

$$I_{\alpha_3}(\theta)=1+q\int_{\mathcal{O}_F}(\theta^{-1}\delta^{1/2})(e^{\alpha_3^\vee}(a^{-1}))\mathcal{Y}^0_{\theta,\xi}(u_{-\alpha_3}(a^{-1})\eta)\,\mathrm{d}a$$

where $u_{-\alpha_3}(x)=diag(I_2,\begin{pmatrix}1&0\\x\sqrt{\epsilon}&1\end{pmatrix},I_2)$. On the other hand, $u_{-\alpha_3}(x)\eta$ is equal to

$$diag(\begin{pmatrix}1+x\sqrt{\epsilon}&-x\sqrt{\epsilon}\\0&1-x\sqrt{\epsilon}\end{pmatrix},\begin{pmatrix}1&-x\sqrt{\epsilon}\\0&1-x^2\epsilon\end{pmatrix},\begin{pmatrix}1-x\sqrt{\epsilon}&-x\sqrt{\epsilon}\\0&1+x\sqrt{\epsilon}\end{pmatrix})\cdot\eta$$
$$\times\, diag(\begin{pmatrix}\frac{1}{1-x^2\epsilon}&\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}\\\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}&\frac{1}{1-x^2\epsilon}\end{pmatrix},\begin{pmatrix}\frac{1}{1-x^2\epsilon}&\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}\\\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}&\frac{1}{1-x^2\epsilon}\end{pmatrix},\begin{pmatrix}\frac{1}{1-x^2\epsilon}&\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}\\\frac{x\sqrt{\epsilon}}{1-x^2\epsilon}&\frac{1}{1-x^2\epsilon}\end{pmatrix}).$$

This implies that (note that all the characters are unramified and hence their values at $a+\sqrt{\varepsilon},\ a-\sqrt{\varepsilon}$ are equal to 1 for all $a\in\mathcal{O}_F$)

$$I_{\alpha_3}(\theta)=q+1=(q+1)\cdot\frac{1-q^{-1}e^{\alpha_3^\vee}(\theta)}{1-q^{-1}e^{\beta_{\alpha_3}^\vee}(\theta)},$$

with $\beta_{\alpha_3}^\vee=\alpha_3^\vee-\beta_{\alpha_3}^\vee=e_3$. Then we compute the set Θ^+.

Lemma 6.4. *Let $W=S_3\ltimes(\mathbb{Z}/2\mathbb{Z})^3$ be the Weyl group of G and let Θ^+ be the smallest subset of Θ satisfying the following two conditions:*
- *(1) $\frac{e_1-e_2\pm e_3}{2}, e_3\in\Theta^+$.*
- *(2) $\Theta^+-(\Theta^+\cap w_{\alpha_1}\Theta^+)=\{\frac{e_1-e_2\pm e_3}{2}\}$, $\Theta^+=w_{\alpha_2}\Theta^+$, $\Theta^+-(\Theta^+\cap w_{\alpha_3}\Theta^+)=\{e_3\}$.*

Then we have $\Theta^+=\{e_1,e_2,e_3,\frac{e_1\pm e_2\pm e_3}{2}\}$.

Proof. It is clear that the set $\{e_1,e_2,e_3,\frac{e_1\pm e_2\pm e_3}{2}\}$ satisfies both conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6\times\mathrm{GSp}_4,(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. $\square$

Now we decompose Θ as $\Theta_1\cup\Theta_2$ and $\Phi=\Phi_1\cup\Phi_2$ where Θ_i,Φ_i contain the weights/roots whose weight spaces/root spaces are i dimensional. More specifically,

$$\Phi_1=\{\pm 2e_i\},\ \Phi_2=\{\pm e_i\pm e_j\},\ \Theta_1=\{\frac{\pm e_1\pm e_2\pm e_3}{2}\},$$
$$\Theta_2=\{\pm e_i\},\ 1\leq i,j\leq 3, i\neq j.$$

Similarly, we can define Φ_i^+ and Θ_i^+ for $i=1,2$. Set

$$\beta(\theta)=\frac{\Pi_{i\in\{1,2\}}\Pi_{\alpha\in\Phi_i^+}1-q^{-i}e^{\alpha^\vee}}{\Pi_{i\in\{1,2\}}\Pi_{\gamma^\vee\in\Theta_i^+}1-q^{-i/2}e^{\gamma^\vee}}.$$

Then it is clear that

$$\zeta(1)^{-1}\zeta_E^{-3}(1)\beta(\theta)\beta(\theta^{-1})=\frac{L(1/2,\pi,\wedge^3)}{L(1,\pi,\mathrm{Ad})}.$$

The next lemma is an analogue of Lemma 2.32 for the current case.

Lemma 6.5. *Set* $c_{WS}(\theta)=\frac{\Pi_{i\in\{1,2\}}\Pi_{\gamma^\vee\in\Theta_i^+}1-q^{-i/2}e^{\gamma^\vee}}{\Pi_{i\in\{1,2\}}\Pi_{\alpha\in\Phi_i^+}1-e^{\alpha^\vee}}(\theta)$. *Then*

$$\sum_{w\in W}c_{WS}(w\theta)$$

is independent of θ *and is equal to* $\frac{1}{\Delta_{H_0/Z_{G,H}}(1)}=\zeta(2)^{-1}=1-q^{-2}$.

Proof. Our goal is to show that (θ_i are arbitrary variables)

$$\sum_{w\in W}w\big(\frac{\Pi_{\varepsilon_1,\varepsilon_2\in\{\pm1\}}(1-q^{-1/2}\sqrt{\theta_1\theta_2^{\varepsilon_1}\theta_3^{\varepsilon_2}})\cdot\Pi_{i=1}^3(1-q^{-1}\theta_i)}{(1-\frac{\theta_1}{\theta_2})(1-\frac{\theta_1}{\theta_3})(1-\frac{\theta_2}{\theta_3})(1-\theta_1\theta_2)(1-\theta_1\theta_3)(1-\theta_2\theta_3)}$$
$$\cdot\frac{1}{(1-\theta_1)(1-\theta_2)(1-\theta_3)}\big)$$

is equal to $\zeta(2)^{-1}=1-q^{-2}$. Multiplying both the denominator and the numerator by $\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{-1/2}$, the denominator will be (W,sgn)-invariant. Hence it is enough to show that

$$\sum_{w\in W}\mathrm{sgn}(w)\cdot w\big(\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{-1/2}\cdot\Pi_{i=1}^3(1-q^{-1}\theta_i)$$
$$\cdot\Pi_{\varepsilon_1,\varepsilon_2\in\{\pm1\}}(1-q^{-1/2}\sqrt{\theta_1\theta_2^{\varepsilon_1}\theta_3^{\varepsilon_2}})\big)$$

is equal to $1-q^{-2}$ times

$$\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{-1/2}(1-\frac{\theta_1}{\theta_2})(1-\frac{\theta_1}{\theta_3})(1-\frac{\theta_2}{\theta_3})\tag{6.4}$$
$$(1-\theta_1\theta_2)(1-\theta_1\theta_3)(1-\theta_2\theta_3)(1-\theta_1)(1-\theta_2)(1-\theta_3).$$

We need to study the $q^{-k/2}$-coefficients $(0\leq q\leq 10)$ of

$$\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{-1/2}\cdot \Pi_{\varepsilon_1,\varepsilon_2\in\{\pm 1\}}(1-q^{-1/2}\sqrt{\theta_1\theta_2^{\varepsilon_1}\theta_3^{\varepsilon_2}})\cdot \Pi_{i=1}^3(1-q^{-1}\theta_i).$$

For $k=1,3,5,7,9$, the $q^{-k/2}$-coefficients are combinations of

$$\theta_1^{a_1}\theta_2^{a_2}\theta_3^{a_3},\ a_1,a_2\in\{0,-1,-2\},a_3\in\{1,0,-1\}.$$

For any such triple (a_1,a_2,a_3), we either have $a_i=\pm a_j$ for some $i\neq j$ or we have $a_i=0$ for some i. Hence the (W,sgn)-summation of the $q^{-k/2}$-coefficients are all equal to 0 for $k=1,3,5,7,9$.

The q^0-coefficient is equal to $\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{-1/2}$, and the (W,sgn)-summation of it is equal to the denominator (6.4).

The q^{-5}-coefficient is equal to $-\theta_1^3\theta_2\theta_3$, and the (W,sgn)-summation of it is equal to zero since the powers of θ_2 and θ_3 are equal.

The q^{-1}-coefficient is equal to

$$\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{-1/2}+\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{1/2}+\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{-1/2}+\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{-3/2}$$
$$-\theta_1^{-5/2}\theta_2^{-1/2}\theta_3^{-1/2}+\theta_1^{-3/2}\theta_2^{-5/2}\theta_3^{-1/2}-\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{1/2}.$$

The (W,sgn)-summation of all the terms except the last two are equal to zero because either two of the powers are equal to other or two of the powers are opposite to each other. The (W,sgn)-summation of $\theta_1^{-3/2}\theta_2^{-5/2}\theta_3^{-1/2}$ and $\theta_1^{-5/2}\theta_2^{-3/2}\theta_3^{1/2}$ are both equal to -1 times the denominator (6.4). As a result, the (W,sgn)-summation of the q^{-1}-coefficient is equal to 0.

The q^{-4}-coefficient is equal to

$$\theta_1^{1/2}\theta_2^{-1/2}\theta_3^{-1/2}+\theta_1^{1/2}\theta_2^{-3/2}\theta_3^{1/2}-\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{1/2}-\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{-1/2}$$
$$-\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{1/2}-\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{3/2}-\theta_1^{-1/2}\theta_2^{1/2}\theta_3^{1/2}.$$

The (W,sgn)-summation of all the terms is equal to zero because either two of the powers are equal to other or two of the powers are opposite to each other. As a result, the (W,sgn)-summation of the q^{-4}-coefficient is equal to 0.

The q^{-2}-coefficient is equal to

$$-\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{-1/2}-\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{1/2}-\theta_1^{-3/2}\theta_2^{1/2}\theta_3^{-1/2}-\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{3/2}$$
$$-2\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{1/2}-\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{-1/2}-\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{1/2}+\theta_1^{-5/2}\theta_2^{-1/2}\theta_3^{1/2}$$
$$-\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{-1/2}-2\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{-1/2}-\theta_1^{-1/2}\theta_2^{-5/2}\theta_3^{-1/2}$$
$$-\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{-3/2}-\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{-3/2}-\theta_1^{-3/2}\theta_2^{-5/2}\theta_3^{1/2}.$$

The (W,sgn)-summation of all the terms except the last term is equal to zero because either two of the powers are equal to other or two of the powers are opposite to each other. The (W,sgn)-summation of $\theta_1^{-3/2}\theta_2^{-5/2}\theta_3^{1/2}$ is equal to the denominator (6.4).

The q^{-3}-coefficient is equal to

$$\theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{1/2}+\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{-1/2}+\theta_1^{-1/2}\theta_2^{-5/2}\theta_3^{1/2}+\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{-3/2}$$
$$+2\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{-1/2}+\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{1/2}+\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{-1/2}-\theta_1^{1/2}\theta_2^{-3/2}\theta_3^{-1/2}$$
$$+\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{1/2}+2\theta_1^{-1/2}\theta_2^{-1/2}\theta_3^{1/2}+\theta_1^{-3/2}\theta_2^{1/2}\theta_3^{1/2}$$
$$+\theta_1^{-3/2}\theta_2^{-1/2}\theta_3^{3/2}+\theta_1^{-1/2}\theta_2^{1/2}\theta_3^{-1/2}+\theta_1^{-1/2}\theta_2^{-3/2}\theta_3^{3/2}.$$

The (W,sgn)-summation of all the terms is equal to zero because either two of the powers are equal to other or two of the powers are opposite to each other. Hence the (W,sgn)-summation of the q^{-3}-coefficient is equal to 0.

This finishes the proof of the lemma. □

Now by a very similar argument as in Section 2.3, our computation of the colors and the lemma above implies that

$$\int_K^* \mathcal{Y}_\theta(k)\,\mathrm{d}k=\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}}\zeta(1)^{-1}\zeta_E(1)^{-3}\cdot\beta(\theta).$$

There are only two differences

- The c-function function for GU_6 is defined to be
$$c_\alpha(\theta)=\frac{1-q^{-1}e^{\alpha^\vee}}{1-e^{\alpha^\vee}}(\theta)$$
if the root space of α is one dimensional and is defined to be
$$c_\alpha(\theta)=\frac{1-q^{-2}e^{\alpha^\vee}}{1-e^{\alpha^\vee}}(\theta)$$
if the root space of α is two dimensional. This matches our definition of $\beta(\theta)$ and $c_{WS}(\theta)$ for this case.
- The volume of Iwahori subgroup of GU_6 is equal to
$$\Delta_G(1)\zeta(1)^{-1}\zeta_E(1)^{-3}\cdot q^{-l(W)}.$$
This is why we get $\zeta(1)^{-1}\zeta_E(1)^{-3}$ instead of $\zeta(1)^{-rk(G)}$ for this case.

This implies that

$$I(\phi_\theta)=\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\zeta(1)^{-1}\zeta_E(1)^{-3}\cdot\beta(\theta)\cdot\beta(\theta^{-1})$$
$$=\frac{\Delta_G(1)}{\Delta_{H_0/Z_{G,H}}(1)}\cdot\frac{L(1/2,\pi,\wedge^3)}{L(1,\pi,\mathrm{Ad})}.$$

6.4. **The computation for** $(\mathrm{GU}_4\times\mathrm{GU}_2,(\mathrm{GU}_2\times\mathrm{GU}_2)^0)$**.** In this subsection, we compute the local relative character for the model $(\mathrm{GU}_4\times\mathrm{GU}_2,(\mathrm{GU}_2\times\mathrm{GU}_2)^0)$. We first study the open Borel orbit. Let B_{2n} be the upper triangular Borel subgroup of $\mathrm{GU}_{n,n}$ and $B=B_4\times B_2$ be a Borel subgroup of G. We write $B=TN$ and let $\bar{B}=T\bar{N}$ be the opposite Borel subgroup.

Set $\eta^{-1}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&0&1&0\\1&-1&1&1\end{pmatrix}$ and $\eta=\begin{pmatrix}1&0&0&0\\1&1&0&0\\-1&0&1&0\\1&1&-1&1\end{pmatrix}$. The proofs of the following two lemmas are similar to the $(\mathrm{GSp}_6\times\mathrm{GSp}_4,(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$ case, and we will skip them here.

Lemma 6.6. *The double cosets $B(F)\backslash G(F)/H(F)$ contain a unique open orbit $B(F)(\eta,I_2)H(F)$.*

Lemma 6.7. *For all $n\in\bar{N}(\varpi\mathcal{O}_F)$, we have*

$$n(\eta,I_2)\in T(\mathcal{O}_F)N(\varpi\mathcal{O}_F)(\eta,I_2)H(\mathcal{O}_F).$$

Now all the arguments in Section 2.2 still work for the current case, the only exception is that the equation in Lemma 2.5 will become

$$\int_{G(F)}\Phi(g)\,\mathrm{d}g=\frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta_E(1)^{-3}\zeta(1)^{-2}\cdot\int_{H(F)/Z_{G,H}(F)}\int_{B(F)}\Phi(b\eta h)\,\mathrm{d}b\,\mathrm{d}h. \tag{6.5}$$

This implies that

$$I(\phi_\theta)=\frac{\Delta_{H/Z_{G,H}}(1)}{\Delta_G(1)}\zeta_E(1)^3\zeta(1)^2\cdot\int_K\mathcal{Y}_{\theta^{-1}}(k)\,\mathrm{d}k\cdot\int_K\mathcal{Y}_\theta(k)\,\mathrm{d}k.$$

Next we compute the colors. For this model, since the representation π of $G(F)$ is of trivial central character, the associated L-parameter factors through the L-group of $\mathrm{GU}_4\times\mathrm{GU}_2/(Res_{E/F}\mathrm{GL}_1)^{diag}$, which is a subgroup of the L-group of GU_6. The 20-dimensional representation $\rho_X=\wedge^2\otimes\mathrm{std}_2\oplus\mathrm{std}_4\oplus\mathrm{std}_4^\vee$ in this case is the restriction of the 20 dimensional exterior cube representation of ${}^L\mathrm{GU}_6$ to ${}^L(\mathrm{GU}_4\times\mathrm{GU}_2/(Res_{E/F}\mathrm{GL}_1)^{diag})$. Let Θ be the weights of the representation $\wedge^2\otimes\mathrm{std}_2\oplus\mathrm{std}_4\oplus\mathrm{std}_4^\vee$. We can write it as

$$\Theta=\{\frac{\pm e_1\pm e_2\pm e_1'}{2},\ \pm e_1',\ \pm e_i\mid 1\le i\le 2\}.$$

The weight spaces of $\pm e_1'$, $\pm e_i$ are two dimensional and the weight spaces of $\frac{\pm e_1\pm e_2\pm 2e_1'}{2}$ are one dimensional.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = 2\varepsilon_2$ and $\alpha' = 2\varepsilon_1'$ be the simple roots of G. We can define $I_{\alpha_1}, I_{\alpha_2}$ and $I_{\alpha'}$ as in the previous case. For α_1, the root space is two dimensional and we have the matrix identity

$$(u_{-\alpha_1}(x+y\sqrt{\varepsilon})\eta, I_2) = (b, h^{-1}) \cdot (\eta, I_2) \cdot (g, h) \tag{6.6}$$

with $(b, h^{-1}) \in B(F), (g, h) \in H(F)$ where

$$u_{-\alpha_1}(x+y\sqrt{\varepsilon}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x+y\sqrt{\varepsilon} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x+y\sqrt{\varepsilon} & 1 \end{pmatrix},\ h = \begin{pmatrix} 1 & y\sqrt{\varepsilon} \\ 0 & 1+x \end{pmatrix},$$

$$g = \begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1 & y\sqrt{\varepsilon} & 0 \\ 0 & 0 & 1+x & 0 \\ -y\sqrt{\varepsilon} & 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} \frac{1}{1+x} & 0 & 0 & 0 \\ 0 & 1 & \frac{-y\sqrt{\varepsilon}}{1+x} & 0 \\ 0 & 0 & \frac{1}{1+x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the same argument as in the α_1 case in the previous subsection, we have

$$I_{\alpha_1}(\theta) = q^2(1-q^{-1}) \cdot \frac{1-q^{-2}e^{\alpha_1^\vee}(\theta)}{(1-q^{-1/2}e^{\beta_{\alpha_1}^\vee}(\theta))(1-q^{-1/2}e^{\alpha_1^\vee-\beta_{\alpha_1}^\vee}(\theta))}$$

with $\beta_{\alpha_1}^\vee = \frac{e_1-e_2-e_1'}{2}$ and $\alpha_1^\vee - \beta_{\alpha_1}^\vee = \frac{e_1-e_2+e_1'}{2}$.

For α_2, the root space is one dimensional and we have the matrix identity

$$(u_{-\alpha_2}(x)\eta, I_2) = (b, h^{-1}) \cdot (\eta, I_2) \cdot (g, h) \tag{6.7}$$

with $(b, h^{-1}) \in B(F), (g, h) \in H(F)$ where

$$u_{-\alpha_2}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x\sqrt{\varepsilon} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ g = \begin{pmatrix} 1 & 0 & 0 & x\sqrt{\varepsilon} \\ 0 & 1+x\sqrt{\varepsilon} & 0 & 0 \\ 0 & 0 & 1+x\sqrt{\varepsilon} & 0 \\ x\sqrt{\varepsilon} & 0 & 0 & 1 \end{pmatrix},$$

$$h = (1+x\sqrt{\varepsilon})I_2,\ b = \frac{1}{1-x^2\varepsilon} \begin{pmatrix} 1-x\sqrt{\varepsilon} & x\sqrt{\varepsilon} & -x\sqrt{\varepsilon} & -x\sqrt{\varepsilon} \\ 0 & 1 & -x\sqrt{\varepsilon} & -x\sqrt{\varepsilon} \\ 0 & 0 & 1-x^2\varepsilon & -x^2\varepsilon+x\sqrt{\varepsilon} \\ 0 & 0 & 0 & 1-x\sqrt{\varepsilon} \end{pmatrix}.$$

By the same argument as in the α_3 case in the previous subsection (use the fact that all the unramified characters have value 1 at $a \pm \sqrt{\varepsilon}$ for $a \in \mathcal{O}_F$), we have

$$I_{\alpha_2}(\theta) = q+1 = (q+1) \cdot \frac{1-q^{-1}e^{\alpha_2^\vee}(\theta)}{1-q^{-1}e^{\beta_{\alpha_2}^\vee}(\theta)},$$

with $\beta_{\alpha_2}^\vee = \alpha_2^\vee - \beta_{\alpha_2}^\vee = e_2$.

For α', the root space is one dimensional and it can be reduced to α_2 but we need to change $u_{-\alpha_2}(x)\eta$ to $\eta u_{-\alpha_2}(-x)$. We have the matrix identity

$$(\eta u_{-\alpha_2}(-x), I_2) = (b, h^{-1}) \cdot (\eta, I_2) \cdot (g, h) \tag{6.8}$$

with $(b, h^{-1}) \in B(F), (g, h) \in H(F)$ where

$$g = \begin{pmatrix} \frac{1}{1-x\sqrt{\varepsilon}} & 0 & 0 & \frac{-x\sqrt{\varepsilon}}{1-x\sqrt{\varepsilon}} \\ 0 & 1+x\sqrt{\varepsilon} & \frac{x\sqrt{\varepsilon}}{1-x\sqrt{\varepsilon}} & 0 \\ 0 & 0 & \frac{1}{1-x\sqrt{\varepsilon}} & 0 \\ \frac{-x\sqrt{\varepsilon}}{1-x\sqrt{\varepsilon}} & 0 & 0 & \frac{1}{1-x\sqrt{\varepsilon}} \end{pmatrix}, h = \begin{pmatrix} 1-x\sqrt{\varepsilon} & \frac{-x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} \\ 0 & \frac{1}{1+x\sqrt{\varepsilon}} \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & \frac{-x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} & \frac{x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} & \frac{x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} \\ 0 & \frac{1-x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} & 0 & \frac{x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} \\ 0 & 0 & \frac{1-x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} & \frac{-x\sqrt{\varepsilon}}{1+x\sqrt{\varepsilon}} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the same argument as in the α_3 case in the previous subsection, we have

$$I_{\alpha'}(\theta) = q+1 = (q+1) \cdot \frac{1-q^{-1}e^{\alpha'^\vee}(\theta)}{1-q^{-1}e^{\beta_{\alpha'}^\vee}(\theta)},$$

with $\beta_{\alpha'}^\vee = \alpha'^\vee - \beta_{\alpha'}^\vee = e_1'$. Then we compute the set Θ^+.

Lemma 6.8. *Let $W = (S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2) \times (\mathbb{Z}/2\mathbb{Z})$ be the Weyl group of G and let Θ^+ be the smallest subset of Θ satisfying the following two conditions:*

(1) $\frac{e_1-e_2\pm e_1'}{2}, e_2, e_1' \in \Theta^+$.

(2) $\Theta^+ - (\Theta^+ \cap w_{\alpha_1}\Theta^+) = \{\frac{e_1-e_2\pm e_1'}{2}\}$, $\Theta^+ - (\Theta^+ \cap w_{\alpha_2}\Theta^+) = \{e_2\}$, $\Theta^+ - (\Theta^+ \cap w_{\alpha'}\Theta^+) = \{e_1'\}$.

Then we have $\Theta^+ = \{e_1, e_2, e_1', \frac{e_1\pm e_2\pm e_1'}{2}\}$.

Proof. It is clear that the set $\{e_1, e_2, e_1', \frac{e_1\pm e_2\pm e_1'}{2}\}$ satisfies the two conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. □

Now as in the previous case, we decompose Θ as $\Theta_1 \cup \Theta_2$ and $\Phi = \Phi_1 \cup \Phi_2$ where Θ_i, Φ_i contain the weights/roots whose weight spaces/root spaces are i dimensional:

$$\Phi_1 = \{\pm 2e_i, \pm 2e_1'\},\ \Phi_2 = \{\pm e_1 \pm e_2\},$$

$$\Theta_1 = \{\frac{\pm e_1 \pm e_2 \pm e_1'}{2}\},\ \Theta_2 = \{\pm e_i, \pm e_1'\},\ 1 \leq i \leq 2.$$

Similarly, we can define Φ_i^+ and Θ_i^+ for $i = 1, 2$. Set

$$\beta(\theta) = \frac{\Pi_{i\in\{1,2\}}\Pi_{\alpha\in\Phi_i^+} 1 - q^{-i}e^{\alpha^\vee}}{\Pi_{i\in\{1,2\}}\Pi_{\gamma^\vee\in\Theta_i^+} 1 - q^{-i/2}e^{\gamma^\vee}}.$$

Then it is clear that

$$\zeta(1)^{-2}\zeta_E^{-3}(1)\beta(\theta)\beta(\theta^{-1}) = \frac{L(1/2, \pi, \rho_X)}{L(1, \pi, \mathrm{Ad})}.$$

The next lemma is an analogue of Lemma 2.14 for the current case.

Lemma 6.9. *Set* $c_{WS}(\theta) = \frac{\Pi_{i\in\{1,2\}}\Pi_{\gamma^\vee\in\Theta_i^+} 1 - q^{-i/2}e^{\gamma^\vee}}{\Pi_{i\in\{1,2\}}\Pi_{\alpha\in\Phi_i^+} 1 - e^{\alpha^\vee}}(\theta)$. *Then*

$$\sum_{w\in W} c_{WS}(w\theta)$$

is independent of θ *and is equal to*

$$\frac{1}{\Delta_{H/Z_{G,H}}(1)} = \zeta(2)^{-2}L(1, \eta_{E/F})^{-1} = (1 - q^{-2})^2(1 + q^{-1}).$$

Proof. Since H is reductive, Theorem 7.2.1 of [Sa] implies that the summation is independent of θ. Now we let $\theta = \delta_B^{1/2}$. The lemma follows from the following two claims:

(1) $c_{WS}(w\theta)$ is zero unless w is the longest Weyl element.
(2) If w is the longest Weyl element, we have $c_{WS}(w\theta) = (1 - q^{-2})^2(1 + q^{-1})$.

The second claim is easy to prove so we will focus on the first one. Let $w = (s, s') \in W$ with $s \in S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$ and $s' \in \mathbb{Z}/2\mathbb{Z}$ so that $c_{WS}(w\theta)$ is nonzero.

The factor $1-q^{-1}e^{e_1'}(w\theta)$ in the numerator of $c_{WS}(w\theta)$ forces s' to be the longest Weyl element of $\mathrm{GU}_{1,1}$. The factors $1-q^{-1}e^{e_i}(w\theta),\ i = 1, 2$ in the numerator force $s(e_1), s(e_2) \in \{\pm e_1, -e_2\}$. Hence there are four possibilities of s: $s(e_1) = \pm e_1, s(e_2) = -e_2$ or $s(e_2) = \pm e_1, s(e_1) = -e_2$. If $s(e_2) = \pm e_1, s(e_1) = -e_2$ or $s(e_1) = e_1, s(e_2) = -e_2$, one of the factors $1 - q^{-1/2}e^{e_1\pm e_2+e_1'}(w\theta)$ in the numerator is equal to 0. Hence we must have $s(e_1) = -e_1, s(e_2) = -e_2$, i.e. w is the longest Weyl element. This proves the lemma. □

As in the previous case, our computation of the colors and the lemma above imply that

$$\int_K \mathcal{Y}_\theta(k)\,\mathrm{d}k = \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}}\zeta(1)^{-2}\zeta_E(1)^{-3}\cdot\beta(\theta).$$

This implies that

$$\begin{aligned} I(\phi_\theta) &= \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\zeta(1)^{-2}\zeta_E(1)^{-3}\cdot\beta(\theta)\cdot\beta(\theta^{-1}) \\ &= \frac{\Delta_G(1)}{\Delta_{H/Z_{G,H}}(1)}\cdot\frac{L(1/2,\pi,\rho_X)}{L(1,\pi,\mathrm{Ad})}. \end{aligned}$$

7. The model $(E_7, \mathrm{PGL}_2 \ltimes U)$

In this section, we compute the local relative character of the model $(E_7, \mathrm{PGL}_2 \ltimes U)$. We closely follow the six steps in Section 2.5.1.

To define this model, we recall a description of the adjoint group of type E_7, following notation in [P20]. Let $H_3(\mathbb{H})$ be the degree three central simple Jordan algebra over k. Here $\mathbb{H}$ is a quaternion algebra over k and denote by N its norm map, tr the trace, and $x \mapsto x^*$ its conjugation. More precisely, one may realize $H_3(\mathbb{H})$ as the vector space of all 3×3 Hermitian symmetric matrices over $\mathbb{H}$, which are of form

$$J = \begin{pmatrix} a & z & y^* \\ z^* & b & x \\ y & x^* & c \end{pmatrix}, \tag{7.1}$$

where $x, y, z \in \mathbb{H}$ and $a, b, c \in k$. The Jordan algebra on $H_3(\mathbb{H})$ is defined by the composition $J_1 \circ J_2 := \frac{1}{2}(J_1J_2 + J_2J_1)$ for $J_1, J_2 \in H_3(\mathbb{H})$, where J_1J_2 and J_2J_1 are under the matrix multiplications. The cubic norm det on $H_3(\mathbb{H})$ is defined by

$$\det(J) := abc - aN(x) - bN(y) - cN(z) + \mathrm{tr}(xyz), \tag{7.2}$$

and the adjoint map $\sharp$ is

$$J^\sharp := \begin{pmatrix} bc - N(x) & y^*x^* - cz & zx - by^* \\ xy - cz^* & ac - N(y) & z^*y^* - ax \\ x^*z^* - by & yz - ax^* & ab - N(z) \end{pmatrix}.$$

Denote by $(\cdot,\cdot,\cdot)$ the symmetric trilinear form corresponding to the cubic norm det with $(A, A, A) = \det(A)$ for $A \in H_3(\mathbb{H})$.

In [R97], Rumelhart constructed the Lie algebra $\mathfrak{g}(H_3(\mathbb{H}))$ through a $\mathbb{Z}_3$-grading. (Here we following the notation in [P20, Section 4.2].) More precisely, define

$$\mathfrak{g} = \mathfrak{sl}_3 \oplus \mathfrak{m}^0 \oplus V_3 \otimes H_3(\mathbb{H}) \oplus V_3^\vee \otimes H_3(\mathbb{H})^\vee \tag{7.3}$$

where V_3 and $V_3^\vee$ are the standard representation of $\mathfrak{sl}_3$ and its dual representation, respectively. Here let $\mathfrak{m}^0$ be the Lie algebra consisting of all linear transformations ϕ on $H_3(\mathbb{H})$ such that

$$(\phi(z_1), z_2, z_3) + (z_1, \phi(z_2), z_3) + (z_1, z_2, \phi(z_3)) = 0$$

for all $z_1, z_2, z_3 \in H_3(\mathbb{H})$. And we refer the reader to Section 4.2.1 in [P20] for the description of the Lie bracket on $\mathfrak{g}(H_3(\mathbb{H}))$.

Now, let us consider the identity component of the automorphism group $\mathrm{Aut}(\mathfrak{g}(H_3(\mathbb{H})))$, which is the quaternionic adjoint group of type E_7. In particular, if $\mathbb{H}$ is split, then it is the split adjoint group of E_7, denoted by G. If $\mathbb{H}$ is not split, then we denote it by G_D, which is of type $E_{7,4}$ and of k-rank 4.

Next, let us explicate this model for the split case. In this case, the quaternion $\mathbb{H}$ is split and take $\mathbb{H} = M_{2\times 2}(F)$ with

$$x^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \mathrm{tr}(x) = a + d, \quad N(x) = \det(x) = ad - bc,$$

for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{H}$. We may identify $H_3(\mathbb{H})$ to $\{A \in M_{6\times 6}(F) \colon A = \Gamma A^t \Gamma^{-1}\}$ as follows

$$\begin{pmatrix} a & z & y^* \\ z^* & b & x \\ y & x^* & c \end{pmatrix} \mapsto \begin{pmatrix} aI_2 & z & y^* \\ z^* & bI_2 & x \\ y & x^* & cI_2 \end{pmatrix} \in M_{6\times 6}(F),$$

where $\Gamma = \mathrm{diag}\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. Then the cubic norm det in (7.2) on $H_3(\mathbb{H})$ is given by $\det(A) = \mathrm{Pf}(\Gamma A)$ where Pf is the Pfaffian of the skew-symmetric matrices.

The Lie algebra $\mathfrak{m}^0(F)$ is isomorphic to $\mathfrak{sl}_6(F)$ via the action of $\mathfrak{sl}_6(F)$ on $H_3(\mathbb{H})$ given by $A \cdot X := AX + XA^*$ for $X \in H_3(\mathbb{H})$ where $A^* = \Gamma^t A \Gamma^{-1}$. Consider V_3 and $V_3^\vee$ in (7.3) as the 3-dimensional vector spaces of column vectors. The action of $\mathfrak{sl}_3(F)$ on V_3 and $V_3^\vee$ are given by: for $v \in V_3$, $\delta \in V_3^\vee$, and $\phi \in \mathfrak{sl}_3(F)$,

$$\phi(v) = \phi v \text{ and } \phi(\delta) = -{}^t\phi\delta \tag{7.4}$$

where the products in the right hand sides are the matrix multiplications.

For $A \in \mathrm{GL}_6(F)$, define $\Phi_A \in End(H_3(\mathbb{H}))$ by

$$\Phi_A(X) := AXA^* \text{ and } \Phi_A^\vee(X) := (A^*)^{-1}XA^{-1}.$$

Write

$$(\mathrm{GL}_3 \times \mathrm{GL}_6 \times \mathrm{GL}_1)^0 = \{(a, g, \lambda) \in \mathrm{GL}_3 \times \mathrm{GL}_6 \times \mathrm{GL}_1 \mid \lambda^3 \mathrm{Det}(g)\mathrm{Det}(a) = 1\}$$

where Det is the usual determinant of GL_n. Define the map ι from $(\mathrm{GL}_3 \times \mathrm{GL}_6 \times \mathrm{GL}_1)^0$ to $\mathrm{GL}(\mathfrak{g})$ as

$$\begin{aligned}\iota\colon \phi &\mapsto a\phi a^{-1}\\ A &\mapsto gAg^{-1}\\ v\otimes X &\mapsto (av)\otimes \lambda\Phi_A(X)\\ \delta\otimes\gamma &\mapsto (a^t)^{-1}\delta\otimes\lambda^{-1}\Phi_A^\vee(\gamma).\end{aligned}$$

By a straightforward computation, we have the image of ι lies in G. Moreover, the kernel of ι is $\ker\iota = \{(wI_3, zI_6, (wz^2)^{-1}) \mid w, z \in F^\times\} \cong F^\times \times F^\times$.

We take the unipotent subgroup U of Lie algebra $\mathfrak{u}$ consisting of elements

$$\{\begin{pmatrix}0 & v_1 & v_3\\ & 0 & v_2\\ & & 0\end{pmatrix} \in \mathfrak{sl}_3\} \oplus \{\begin{pmatrix}0_{2\times 2} & x & y\\ & 0_{2\times 2} & z\\ & & 0_{2\times 2}\end{pmatrix} \in \mathfrak{sl}_6\}$$

$$\oplus Fw_1 \otimes H_3(\mathbb{H}) \oplus Fw_2 \otimes H_3(\mathbb{H}) \oplus Fw_3 \otimes H_3(\mathbb{H})^\vee,$$

where $\{w_1, w_2, w_3\}$ is the standard basis of F^3. Then its corresponding Levi subgroup L is given by the image

$$\iota(\{(\begin{pmatrix}a & & \\ & b & \\ & & c\end{pmatrix}, \begin{pmatrix}g_1 & & \\ & g_2 & \\ & & g_3\end{pmatrix}, \lambda) \mid \mathrm{Det}(g_1)\mathrm{Det}(g_2)\mathrm{Det}(g_3)abc = \lambda^{-3}\}).$$

For $u \in \mathfrak{u}$, define the character ξ of U by

$$\xi(\exp(u)) = \psi(v_1 + \mathrm{Tr}(x) + \mathrm{Tr}(z) + e)$$

where e is the entry corresponding to the simple root $\alpha_3 = e_2 - e_1$, i.e. the coefficient of $w_2 \otimes (E_{5,5} + E_{6,6})$. ($E_{i,j}$ are the elementary matrices in $M_{6\times 6}(F)$.) The stabilizer H_0 of ξ is given by the image

$$\iota(\{(\begin{pmatrix}a & & \\ & a & \\ & & c\end{pmatrix}, \begin{pmatrix}g & & \\ & g & \\ & & g\end{pmatrix}, \lambda) \mid a\lambda\mathrm{Det}(g) = 1,\ \mathrm{Det}(g)^3a^2c = \lambda^{-3}\})$$

$$= \iota(\{(aI_3, \begin{pmatrix}g & & \\ & g & \\ & & g\end{pmatrix}, \lambda) \mid a\lambda\mathrm{Det}(g) = 1\}),$$

which is isomorphic to $\mathrm{PGL}_2(F)$. Let $H = H_0 \ltimes U$ and we extend the character ξ to H by making it trivial on H_0. The model (G, H, ξ) is the Whittaker induction of the trilinear GL_2 model (L, H_0, ξ). We can also define the quaternion (non-split) version of this model by letting G_D be of type $E_{7,4}$. In the non-split case, $L_D \rtimes U_D$ is a minimal parabolic subgroup of G_D defined over F and ξ_D is a generic character of U_D. Then the stabilizer $H_{0,D}$ of ξ_D in L_D is isomorphic to $PD^\times$. Thus we obtain the quaternion (non-split) version (G_D, H_D, ξ_D) with $H_D = H_{0,D} \rtimes U_D$.

Define the Weyl element w_0 of E_7 by

$$\begin{aligned} w_0\colon \phi \in \mathfrak{sl}_3 &\mapsto -\phi^t \in \mathfrak{sl}_3 \\ A \in \mathfrak{sl}_6 &\mapsto -A^* \in \mathfrak{sl}_6 \\ v\otimes X \in V_3 \otimes H_6 &\mapsto v \otimes X \in V_3^\vee \otimes H_6^\vee \\ \delta\otimes\gamma \in V_3^\vee \otimes H_6^\vee &\mapsto \delta\otimes\gamma \in V_3\otimes H_6. \end{aligned}$$

Then $w_0^2 = 1$ and w_0 sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a : \mathrm{GL}_1 \to Z_L$ to be

$$a(t) = \iota(\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-4} \end{pmatrix}, \begin{pmatrix} tI_2 & & \\ & I_2 & \\ & & t^{-1}I_2 \end{pmatrix}, t).$$

This clearly satisfies (2.15). For the open Borel orbit, let $\eta_0 = w'\gamma_0$ where

$$w' = \iota(I_3, \begin{pmatrix} I_2 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}, 1),\ \gamma_0 = \iota(I_3, \begin{pmatrix} I_4 & & \\ & 1 & 1 \\ & 0 & 1 \end{pmatrix}, 1),$$

be the representative of the open Borel orbit for the model (L, H_0) as in Section 2.3.2, and $\eta = \eta_0 w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Now we compute the set of colors and also the set Θ^+. Following the notation in [B02], let $\alpha_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_8) - \frac{1}{2}\sum_{i=2}^{7}\varepsilon_i$, $\alpha_2 = \varepsilon_1+\varepsilon_2$ and $\alpha_{i+1} = \varepsilon_i - \varepsilon_{i-1}$ for $3 \le i \le 6$ be the simple roots. Let Θ be the weights of the 56-dimensional irreducible representation of $E_7(\mathbb{C})$, corresponding to the 7-th fundamental weight ω_7, where $\omega_7 = e_6 + \frac{1}{2}(e_8 - e_7)$. We can write it as

$$\begin{aligned} \Theta = \{&\pm e_i \pm \frac{1}{2}(e_8 - e_7) \mid 1 \le i \le 6\} \\ &\cup \{\frac{1}{2}\sum_{i=1}^{6} a_i e_i \mid \#\{i\colon a_i = 1\} \text{ is even and } a_i = \pm 1\}. \end{aligned}$$

By the computation of the trilinear GL_2-model in Section 2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we get the set of colors for this case:

$$\beta_{\alpha_7}^{\vee}=\frac{e_1+e_2+e_3-e_4-e_5+e_6}{2},$$
$$\alpha_7^{\vee}-\beta_{\alpha_1}^{\vee}=\frac{-e_1-e_2-e_3+e_4-e_5+e_6}{2},$$
$$\beta_{\alpha_5}^{\vee}=\frac{e_1+e_2-e_3+e_4+e_5-e_6}{2}$$
$$\alpha_5^{\vee}-\beta_{\alpha_3}^{\vee}=\frac{-e_1-e_2-e_3+e_4-e_5+e_6}{2},$$
$$\beta_{\alpha_2}^{\vee}=\frac{e_1+e_2-e_3+e_4+e_5-e_6}{2},$$
$$\alpha_2^{\vee}-\beta_{\alpha_5}^{\vee}=\frac{e_1+e_2+e_3-e_4-e_5+e_6}{2}.$$

Then we verify (2.23) for α_1, α_3, α_4 and α_6. Let

$$u_{-\alpha_1}(a)=\iota(\begin{pmatrix}1&&\\a&1&\\&&1\end{pmatrix},I_6,1),\ u_{-\alpha_3}(a)=Id+a\cdot\mathrm{ad}_{w_2\otimes(E_{5,5}+E_{6,6})^{\vee}},$$
$$u_{-\alpha_4}(a)=\iota(I_3,I_6+aE_{5,4},1),\ u_{-\alpha_6}(a)=\iota(I_3,I_6+aE_{3,2},1).$$

We have the following 4 identities

$$u_{-\alpha_1}(a)\eta=\eta u_{\alpha_1}(a),\ u_{-\alpha_3}(a)\eta=\eta u_{\alpha_3}(a),$$
$$u_{-\alpha_4}(a)\eta=\eta u_{\alpha_4}(-a)u_{e_3+e_1}(-a),\ u_{-\alpha_6}(a)\eta=\eta u_{e_6-e_4}(-a).$$

This proves (2.23) for α_1, α_3, α_4 and α_6. In addition, we label the type of each simple root in the following weighted Dynkin Diagram:

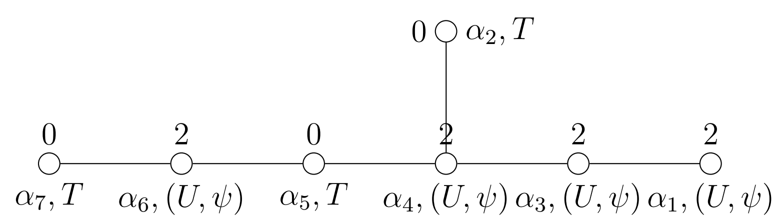


FIGURE 1. Weighted Dynkin Diagram of E_7

Note that this weighted Dynkin Diagram is associated to the special nilpotent stable orbit of Balar-Carter label E_6. Its corresponding unipotent stable orbit is the maximal unipotent orbit with a non-empty intersection with the unipotent subgroup U.

Next, we compute the set Θ^+.

Proposition 7.1. Θ^+ *is consisting of the following 28 elements:*

$$\frac{\sum_{l=1}^{6} e_l - 2e_i - 2e_j}{2}, \frac{-\sum_{l=1}^{6} e_l + 2e_{i'} + 2e_{j'}}{2}, \tag{7.5}$$

$$\frac{e_1+e_2+e_3+e_4+e_5+e_6}{2}, \frac{-e_1+e_2+e_3+e_4+e_5+e_6-2e_k}{2}, \tag{7.6}$$

$$\pm e_m + \frac{e_8-e_7}{2}, \text{ for } 1 \leq m \leq 6, \tag{7.7}$$

where $(i,j) \in \{(23),(24),(34),(25),(35),(45),(26),(36)\}$, $2 \leq k \leq 6$ *and* $(i',j') \in \{(56),(46)\}$.

Proof. By the computation of the colors, we know that Θ^+ is the smallest subset of Θ satisfying the following 5 conditions:

(1) $\frac{1}{2}(e_1+e_2+e_3-e_4-e_5+e_6)$, $\frac{1}{2}(-e_1-e_2-e_3+e_4-e_5+e_6)$, $\frac{1}{2}(e_1+e_2-e_3+e_4+e_5-e_6) \in \Theta^+$.
(2) $\Theta^+ - (\Theta^+ \cap w_{\alpha_7}\Theta^+) = \{\frac{1}{2}(e_1+e_2+e_3-e_4-e_5+e_6),\ \frac{1}{2}(-e_1-e_2-e_3+e_4-e_5+e_6)\}$.
(3) $\Theta^+ - (\Theta^+ \cap w_{\alpha_5}\Theta^+) = \{\frac{1}{2}(e_1+e_2-e_3+e_4+e_5-e_6),\ \frac{1}{2}(-e_1-e_2-e_3+e_4-e_5+e_6)\}$.
(4) $\Theta^+ - (\Theta^+ \cap w_{\alpha_2}\Theta^+) = \{\frac{1}{2}(e_1+e_2-e_3+e_4+e_5-e_6),\ \frac{1}{2}(e_1+e_2+e_3-e_4-e_5+e_6)\}$.
(5) Θ^+ is stable under w_{α_6}, w_{α_4}, w_{α_3} and w_{α_1}.

It is clear that the set in the statement satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. □

It is clear that Θ^+ satisfies (2.3). The last thing remains is to prove Lemma 2.32 for the current case. Denote by Θ_1^+ the subset of Θ^+ consisting of the 12 weights in (7.7) and Θ_2^+ the complement of Θ_1^+ in Θ^+, that is, consisting of the 16 weights in (7.5) and (7.6). Then Θ_2^+ corresponds to the weights of the GSO_{12} model in Proposition 8.4. We also decompose the set of positive roots Φ^+ as $\Phi_1^+ \cup \Phi_2^+$ where $\Phi_2^+ = \{e_j \pm e_i \mid 1 \leq i < j \leq 6\}$ is the set of the roots contained in GSO_{12}, and Φ_1^+ consists of the remaining positive roots, that is,

$$e_8 - e_7,\ \frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{a_i}e_i) \text{ with } \sum_{i=1}^{6} a_i \text{ odd.}$$

Denote by $W(D_6)$ the Wely group of the Levi subgroup of type D_6, generated by the simple reflections w_{α_i} for $i \neq 1$. We embed $W(D_6)$ into the Weyl group W.

Lemma 7.2. *With the notation above, we have*
$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H_0}(1)} = \frac{1}{\zeta(2)} = (1-q^{-2}).$$

Proof. By the identity for the GSO_{12} model case proved in Lemma 8.5, we have
$$\sum_{w\in W} c_{WS}(w\theta) = \sum_{w\in W} \frac{\prod_{\gamma^\vee\in\Theta^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+} 1-e^{\alpha^\vee}}(w\theta)$$
$$= (1-q^{-2})\cdot \sum_{w\in W/W(D_6)} \frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta).$$
Hence it is enough to show that
$$\sum_{w\in W/W(D_6)} \frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta) = 1.$$
It is easy to see that the constant coefficient of the above summation is equal to 1, so it is enough to show that all the $q^{-i/2}$-coefficients are equal to 0 for $1\le i\le 12$. We can replace the summation on $W/W(D_6)$ by the summation on W and rewrite the function inside the summation (θ_i are arbitrary variables):
$$\frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta)$$
$$= \frac{e^{-\rho^\vee}\prod_{\alpha\in\Phi_2^+} 1-e^{\alpha^\vee}\cdot\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{e^{-\rho^\vee}\prod_{\alpha\in\Phi^+} 1-e^{\alpha^\vee}}(w\theta)$$
$$= w\Big(\frac{\theta_7^{-\frac{17}{2}}\prod_{i=2}^6\theta_i^{-i+1}\prod_{1\le i<j\le 6}(1-\theta_j\theta_i^{-1})(1+\theta_j\theta_i)}{\theta_7^{-\frac{17}{2}}\prod_{i=2}^6\theta_i^{-i+1}\cdot(1-\theta_7)\prod_{\sum a_i\ \mathrm{odd}}(1-\theta_7^{\frac{1}{2}}\prod_{i=1}^6\theta_i^{(-\frac{1}{2})^{a_i}})}$$
$$\cdot\frac{\prod_{i=1}^6(1-q^{-1/2}\cdot\theta_i\theta_7^{\frac{1}{2}})(1-q^{-1/2}\cdot\theta_i^{-1}\theta_7^{\frac{1}{2}})}{\prod_{1\le i<j\le 6}(1-\theta_j\theta_i^{-1})(1+\theta_j\theta_i)}\Big)$$
where $e^{\rho^\vee}(\theta)=\theta_7^{\frac{17}{2}}\prod_{i=2}^6\theta_i^{i-1}$.

Then the denominator becomes (W,sgn)-invariant, so it is enough to show that the (W,sgn)-summation of the $q^{-i/2}$-coefficient of
$$\theta_7^{-\frac{17}{2}}\prod_{i=2}^6\theta_i^{-i+1}\prod_{1\le i<j\le 6}(1-\theta_j\theta_i^{-1})(1+\theta_j\theta_i) \tag{7.8}$$

$$\cdot \prod_{i=1}^{6}(1-q^{-1/2}\cdot \theta_i\theta_7^{\frac{1}{2}})(1-q^{-1/2}\cdot \theta_i^{-1}\theta_7^{\frac{1}{2}})$$

is equal to 0 for $1\leq i\leq 12$. We need the following claim which follows from the Weyl Denominator formula of type D_6.

Claim : the product

$$\prod_{i=2}^{6}\theta_i^{-i+1}(1-\theta_j\theta_i^{-1})(1+\theta_j\theta_i)=\sum_{w\in W(D_6)} sgn(w)w(\prod_{i=2}^{6}\theta_i^{i-1})$$

is consisting of terms of the form

$$\prod_{i=1}^{6}\theta_i^{a_i},\ \{|a_1|,|a_2|,|a_3|,|a_4|,|a_5|,|a_6|\}=\{5,4,3,2,1,0\}.$$

Now we can study the coefficients of $q^{-i/2}$. For the $q^{-1/2}$-coefficient, the above claim implies that any term $\prod_{i=1}^{7}\theta_i^{b_i}$ appears in the $q^{-1/2}$-coefficient of (7.8) satisfies $b_7=-8$ and one of the following two conditions

- $b_i=\pm b_j$ for some $1\leq i\neq j\leq 6$;
- $\{|b_1|,|b_2|,|b_3|,|b_4|,|b_5|,|b_6|\}=\{6,4,3,2,1,0\}$.

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W,sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$\theta_7^{-8}\theta_6^{6}\prod_{i=2}^{5}\theta_i^{i-1}=e^{-8(e_8-e_7)+\sum_{i=2}^{5}(i-1)e_i+6e_6}(\theta).$$

However by changing variable θ_7^{-1} to θ_7, the weight

$$8(e_8-e_7)+\sum_{i=2}^{5}(i-1)e_i+6e_6$$

is orthogonal to $\alpha_1^{\vee}$. This implies that the (W,sgn)-summation of the $q^{-1/2}$-coefficient is equal to 0.

For the q^{-1}-coefficient, the above claim implies that any term $\prod_{i=1}^{7}\theta_i^{b_i}$ appears in the q^{-1}-coefficient of (7.8) satisfies $b_7=-\frac{15}{2}$ and one of the following two conditions

- $b_i=\pm b_j$ for some $1\leq i\neq j\leq 6$;
- $\{|b_1|,|b_2|,|b_3|,|b_4|,|b_5|,|b_6|\}=\{6,5,3,2,1,0\}$ or $\{5,4,3,2,1,0\}$.

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W,sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$e^{-\frac{15}{2}(e_8-e_7)+\sum_{i=2}^{6}(i-1)e_i}(\theta)$$

or

$$e^{-\frac{15}{2}(e_8-e_7)+\sum_{i=2}^{4}(i-1)e_i+5\theta_5+6e_6}(\theta).$$

By changing variable $\theta_7^{-1}\to\theta_7$, the weight

$$\frac{15}{2}(e_8-e_7)+\sum_{i=2}^{6}(i-1)e_i$$

is orthogonal to $\alpha_1^\vee$. And the weight

$$w_{\alpha_1}(\frac{15}{2}(e_8-e_7)+\sum_{i=2}^{4}(i-1)e_i+5\theta_5+6e_6)$$

$$=8(e_8-e_7)+\frac{1}{2}(e_1+e_2+3e_3+5e_4+9e_5+11e_6)$$

is orthogonal to $\alpha_3^\vee$. This implies that the (W,sgn)-summation of the q^{-1}-coefficient is equal to 0.

For the $q^{-3/2}$-coefficient, the above claim implies that any term $\prod_{i=1}^{7}\theta_i^{b_i}$ appears in the $q^{-1/2}$-coefficient of (7.8) satisfies $b_7=-7$ and one of the following two conditions

- $b_i=\pm b_j$ for some $1\le i\neq j\le 6$;
- $\{|b_1|,|b_2|,|b_3|,|b_4|,|b_5|,|b_6|\}=\{6,4,3,2,1,0\}$ or $\{6,5,4,2,1,0\}$.

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W,sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$e^{-7(e_8-e_7)+\sum_{i=2}^{5}(i-1)e_i+6e_6}(\theta)$$

or

$$e^{-7(e_8-e_7)+\sum_{i=2}^{3}(i-1)e_i+\sum_{i=4}^{5}ie_i}(\theta).$$

By changing variable θ_7^{-1} to θ_7, the weight

$$w_{\alpha_1}(7(e_8-e_7)+\sum_{i=2}^{5}(i-1)e_i+6e_6)$$

$$=\frac{15}{2}(e_8-e_7)+\frac{1}{2}(e_1+e_2+3e_3+5e_4+7e_5+11e_6)$$

is orthogonal to e_2-e_1; and the weight

$$w_{\alpha_3}w_{\alpha_1}(7(e_8-e_7)+\sum_{i=2}^{3}(i-1)e_i+\sum_{i=4}^{5}ie_i)$$

$$= 8(e_8 - e_7) + e_2 + e_3 + 3e_4 + 4e_5 + 5e_6$$

is orthogonal to $e_3 - e_2$. This implies that the (W, sgn)-summation of the $q^{-3/2}$-coefficient is equal to 0.

For the q^{-2}-coefficient, the above claim implies that any term $\prod_{i=1}^{7} \theta_i^{b_i}$ appears in the $q^{-1/2}$-coefficient of (7.8) satisfies $b_7 = -\frac{13}{2}$ and one of the following two conditions

- $b_i = \pm b_j$ for some $1 \leq i \neq j \leq 6$;
- $\{|b_1|, |b_2|, |b_3|, |b_4|, |b_5|, |b_6|\} = \{5, 4, 3, 2, 1, 0\}$, $\{6, 5, 3, 2, 1, 0\}$ or $\{6, 5, 4, 3, 1, 0\}$.

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W, sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$e^{-\frac{13}{2}(e_8 - e_7) + \sum_{i=2}^{6}(i-1)e_i}(\theta),$$

$$e^{-\frac{13}{2}(e_8 - e_7) + \sum_{i=2}^{4}(i-1)e_i + \sum_{i=5}^{6} ie_i}(\theta),$$

or

$$e^{-\frac{13}{2}(e_8 - e_7) + e_2 + \sum_{i=3}^{6} ie_i}(\theta).$$

By changing variable θ_7^{-1} to θ_7, the weight

$$w_{\alpha_1}(-\frac{13}{2}(e_8 - e_7) + \sum_{i=2}^{6}(i-1)e_i)$$

$$= 7(e_8 - e_7) + \frac{1}{2}(e_1 + e_2 + 3e_3 + 5e_4 + 7e_5 + 9e_6)$$

is orthogonal to $e_2 - e_1$; the weight

$$w_{\alpha_3} w_{\alpha_1}(\frac{13}{2}(e_8 - e_7) + \sum_{i=2}^{4}(i-1)e_i + \sum_{i=5}^{6} ie_i)$$

$$= \frac{15}{2}(e_8 - e_7) + e_2 + e_3 + 2e_4 + 4e_5 + 5e_6$$

is orthogonal to $e_3 - e_2$; the weight

$$w_{\alpha_3} w_{\alpha_1}(\frac{13}{2}(e_8 - e_7) + e_2 + \sum_{i=3}^{6} ie_i)$$

$$= 8(e_8 - e_7) + \frac{1}{2}(-e_1 + 3e_2 + 3e_3 + 5e_4 + 7e_5 + 9e_6$$

is orthogonal to $e_3 - e_2$. This implies that the (W, sgn)-summation of the q^{-2}-coefficient is equal to 0.

For the $q^{-5/2}$-coefficient, the above claim implies that any term $\prod_{i=1}^{7}\theta_i^{b_i}$ appears in the $q^{-1/2}$-coefficient of (7.8) satisfies $b_7=-6$ and one of the following two conditions

- $b_i=\pm b_j$ for some $1\leq i\neq j\leq 6$;
- $\{|b_1|,|b_2|,|b_3|,|b_4|,|b_5|,|b_6|\}=\{6,4,3,2,1,0\}$, $\{6,5,4,2,1,0\}$, or $\{6,5,4,3,2,0\}$.

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W,sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$e^{-6(e_8-e_7)+\sum_{i=2}^{5}(i-1)e_i+6e_6}(\theta),$$

$$e^{-6(e_8-e_7)+\sum_{i=2}^{3}(i-1)e_i+\sum_{i=4}^{6}ie_i}(\theta)$$

or

$$e^{-6(e_8-e_7)+\sum_{i=2}^{6}ie_i}(\theta).$$

By changing variable θ_7^{-1} to θ_7, under the action of w_{α_1}, we have

$$\begin{aligned}
&w_{\alpha_1}(0,1,2,3,4,6,-6,6)=(1,0,1,2,3,5,-7,7)\\
&w_{\alpha_1}(0,1,2,4,5,6,-6,6)=(\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2},-\frac{15}{2},\frac{15}{2})\\
&w_{\alpha_1}(0,2,3,4,5,6,-6,6)=(2,0,1,2,3,4,-8,8).
\end{aligned}$$

Here $(b_1,b_2,\dots,b_8)$ corresponds the weight $\sum_{i=1}^{8}b_ie_i$. In particular we have $b_i=\pm b_j$ for some $1\leq i\neq j\leq 6$ which is just the first case. This implies that the (W,sgn)-summation of the $q^{-5/2}$-coefficient is equal to 0.

For the q^{-3}-coefficient, the above claim implies that any term $\prod_{i=1}^{7}\theta_i^{b_i}$ appears in the $q^{-1/2}$-coefficient of (7.8) satisfies $b_7=-\frac{11}{2}$ and one of the following two conditions

- $b_i=\pm b_j$ for some $1\leq i\neq j\leq 6$.
- $\{|b_1|,|b_2|,|b_3|,|b_4|,|b_5|,|b_6|\}$ is equal to

$$\{5,4,3,2,1,0\},\ \{6,5,3,2,1,0\},\ \{6,5,4,3,1,0\},\ \text{or } \{6,5,4,3,2,1\}.$$

In the first case, by using a simple reflection in the Weyl group of D_6, we know that the (W,sgn)-summation of the term is equal to 0. In the second case, up to a Weyl element w_0 action, we may assume that the term is of the form

$$e^{-\frac{11}{2}(e_8-e_7)+\sum_{i=2}^{6}(i-1)e_i}(\theta),$$

$$e^{-\frac{11}{2}(e_8-e_7)+\sum_{i=2}^{4}(i-1)e_i+\sum_{i=5}^{6}ie_i}(\theta),$$

$$e^{-\frac{11}{2}(e_8-e_7)+e_2+\sum_{i=3}^{6}ie_i}(\theta),$$

or

$$e^{-\frac{11}{2}(e_8-e_7)+\sum_{i=2}^{6} ie_i}(\theta).$$

By changing variable θ_7^{-1} to θ_7, under the action of w_{α_1}, we have

$$\begin{aligned}
w_{\alpha_1}(0,1,2,3,4,5,-\frac{11}{2},\frac{11}{2}) &= (1,0,1,2,3,4,-\frac{13}{2},\frac{13}{2})\\
w_{\alpha_1}(0,1,2,3,5,6,-\frac{11}{2},\frac{11}{2}) &= (\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{7}{2},\frac{9}{2},-7,7)\\
w_{\alpha_1}(0,1,3,4,5,6,-\frac{11}{2},\frac{11}{2}) &= (2,-1,1,2,3,4,-\frac{15}{2},\frac{15}{2})\\
w_{\alpha_1}(1,2,3,4,5,6,-\frac{11}{2},\frac{11}{2}) &= (3,0,1,2,3,4,-\frac{15}{2},\frac{15}{2}).
\end{aligned}$$

After the action of w_{α_1}, we have $b_i = \pm b_j$ for some $1 \le i \ne j \le 6$ which is just the first case. This implies that the (W, sgn)-summation of the q^{-3}-coefficient is equal to 0.

Due to symmetry, the remaining $q^{-i/2}$-coefficients for $7 \le i \le 12$ are vanishing by similar arguments and we omit the details here. This finishes the proof of the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(6)\zeta(8)\zeta(10)\zeta(12)\zeta(14)\zeta(18)\frac{L(1/2,\pi,\omega_7)}{L(1,\pi,\mathrm{Ad})}$$

where π is an unramified representation of $E_7(F)$.

8. The remaining models

In this section, we will compute the local relative characters for the remaining 4 models in Table 1. The computations are very similar to the cases in the previous sections.

8.1. The model $(\mathrm{GSp}_{10}, \mathrm{GL}_2 \ltimes U)$. In this subsection, we compute the local relative character for the model $(\mathrm{GSp}_{10}, \mathrm{GL}_2 \ltimes U)$. For simplicity, define

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid {}^t g J'_{2n} g = l(g) J'_{2n}\}, \text{ where } J'_{2n} = \begin{pmatrix} 0 & J'_{2n-2} \\ J_2 & 0 \end{pmatrix}.$$

Note that the skew-symmetric matrix J'_{2n} is different with J_{2n} when $n > 1$ in Section 3.1 and $J_2 = J'_2$. We use J'_{2n} here to simplify the definition and computation. Let $G = \mathrm{GSp}_{10}$, $H = H_0 \ltimes U$ with

$$\begin{aligned}
H_0 &= \{diag(h,h,h,\det(h)h^*,\det(h)h^*) \mid h \in \mathrm{GL}_2,\ h^* = J_2'{}^t h^{-1}(J_2')^{-1}\}\\
&= \{diag(h,h,h,h,h) \mid h \in \mathrm{GL}_2\}
\end{aligned}$$

and U be the unipotent radical of the standard parabolic subgroup $P = LU$ of G where

$$L = \{(h_1, h_2, h_3, \det(h_3)h_2^*, \det(h_3)h_1^*) \mid h_i \in \mathrm{GL}_2\}.$$

We define a generic character ξ on $U(F)$ to be $\xi(u) = \psi(\lambda(u))$ where

$$\lambda(u) = \mathrm{tr}(X) + \mathrm{tr}(Y),\ u = \begin{pmatrix} I_2 & X & * & * & * \\ 0 & I_2 & Y & * & * \\ 0 & 0 & I_2 & * & * \\ 0 & 0 & 0 & I_2 & * \\ 0 & 0 & 0 & 0 & I_2 \end{pmatrix}.$$

It is easy to see that H_0 is the stabilizer of this character and (G, H) is the Whittaker induction of the trilinear GL_2-model (L, H_0, ξ).

We can also define the quaternion version of this model. Let D/F be a quaternion algebra, and let $G_D(F) = \mathrm{GSp}_5(D)$ (the group $\mathrm{GSp}_n(D)$ has been defined in Section 3.1), $H_D = H_{0,D} \ltimes U_D$ with

$$H_{0,D}(F) = \{diag(h, h, h, h, h) \mid h \in \mathrm{GL}_1(D),\ h^* = \bar{h}^{-1}\}$$

and U_D is the unipotent radical of the standard parabolic subgroup $P_D = L_D U_D$ of G_D where

$$L_D(F) = \{(h_1, h_2, h_3, N_{D/F}(h_3)h_2^*, N_{D/F}(h_3)h_1^*) \mid h_i \in \mathrm{GL}_1(D)\}.$$

Here $N_{D/F} : \mathrm{GL}_1(D) \to F^\times$ is the norm map and $x \to \bar{x}$ is the conjugation map on the quaternion algebra. Like the split case, we can define the character ξ_D on $U_D(F)$ by replacing the trace map of $Mat_{2\times 2}$ by the trace map of D.

Let $w_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 & 0 \end{pmatrix}$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a : \mathrm{GL}_1 \to Z_L$ to be

$$a(t) = diag(t^2 I_2, tI_2, I_2, t^{-1}I_2, t^{-2}I_2).$$

This clearly satisfies the equation (2.15). For the open Borel orbit, let

$$\eta_0 = diag(I_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -I_2)$$

be the representative of the open Borel orbit for the model (L, H_0) as in Section 2.3.2, and $\eta = \eta_0 w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Now we compute the set of colors and also the set Θ^+. Let Θ be the weights of the 32-dimensional representation Spin_{11} of $\mathrm{GSpin}_{11}(\mathbb{C})$. We can write it as

$$\Theta = \{\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5}{2}\}.$$

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i \le 4$ and $\alpha_5 = 2\varepsilon_5$ be the simple roots of GSp_{10}. By the computation of the trilinear GL_2-model in Section 2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we have

$$\beta_{\alpha_1}^\vee = \frac{e_1 - e_2 - e_3 + e_4 + e_5}{2},\ \alpha_1^\vee - \beta_{\alpha_1}^\vee = \frac{e_1 - e_2 + e_3 - e_4 - e_5}{2},$$

$$\beta_{\alpha_3}^\vee = \frac{-e_1 + e_2 + e_3 - e_4 + e_5}{2},\ \alpha_3^\vee - \beta_{\alpha_3}^\vee = \frac{e_1 - e_2 + e_3 - e_4 - e_5}{2},$$

$$\beta_{\alpha_5}^\vee = \frac{-e_1 + e_2 + e_3 - e_4 + e_5}{2},\ \alpha_5^\vee - \beta_{\alpha_3}^\vee = \frac{e_1 - e_2 - e_3 + e_4 + e_5}{2}.$$

By a similar argument as in the Ginzburg–Rallis model case in Section 5, we can also verify (2.23) for the roots α_2 and α_4. Next, we compute the set Θ^+.

Proposition 8.1. Θ^+ *is consisting of the following 16 elements:*

$$\frac{e_1 + e_2 \pm e_3 \pm e_4 \pm e_5}{2}, \frac{e_1 - e_2 + e_3 \pm e_4 \pm e_5}{2}, \frac{e_1 - e_2 - e_3 + e_4 + e_5}{2},$$

$$\frac{-e_1 + e_2 + e_3 + e_4 \pm e_5}{2}, \frac{-e_1 + e_2 + e_3 - e_4 + e_5}{2}.$$

Proof. By the computation of the colors, we know that Θ^+ is the smallest subset of Θ satisfying the following 5 conditions:

(1) $\frac{e_1-e_2-e_3+e_4+e_5}{2}, \frac{e_1-e_2+e_3-e_4-e_5}{2}, \frac{-e_1+e_2+e_3-e_4+e_5}{2} \in \Theta^+$.
(2) $\Theta^+ - (\Theta^+ \cap w_{\alpha_1}\Theta^+) = \{\frac{e_1-e_2-e_3+e_4+e_5}{2}, \frac{e_1-e_2+e_3-e_4-e_5}{2}\}$.
(3) $\Theta^+ - (\Theta^+ \cap w_{\alpha_3}\Theta^+) = \{\frac{e_1-e_2+e_3-e_4-e_5}{2}, \frac{-e_1+e_2+e_3-e_4+e_5}{2}\}$.
(4) $\Theta^+ - (\Theta^+ \cap w_{\alpha_5}\Theta^+) = \{\frac{e_1-e_2-e_3+e_4+e_5}{2}, \frac{-e_1+e_2+e_3-e_4+e_5}{2}\}$.
(5) Θ^+ is stable under w_{α_2} and w_{α_4}.

It is clear that the set in the statement satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. $\square$

It is clear that Θ^+ satisfies (2.3). The last thing remains is to prove Lemma 2.32 for the current case. For $i = 1, 2$, we decompose Θ^+ as $\Theta_1^+ \cup \Theta_2^+$ with Θ_1^+ consisting of the following 10 elements:

$$\frac{e_1 + e_2 + e_3 \pm e_4 \pm e_5}{2}, \frac{e_1 + e_2 + e_3 + e_4 \pm e_5 - 2e_i}{2},\ 1 \le i \le 3$$

and Θ_2^+ consisting of the remaining 6 elements. Then Θ_2^+ corresponds to the weights in Lemma 5.3 (here we view $\mathrm{GL}_4\times\mathrm{GL}_2\simeq\mathrm{GL}_4\times\mathrm{GSp}_2$ as a standard Levi subgroup of GSp_{10}). Decompose the set of positive roots Φ^+ as $\Phi_1^+\cup\Phi_2^+$ where $\Phi_2^+=\{e_i-e_j,2e_5\mid 1\leq i<j\leq 4\}$ is the set of the positive roots contained in $\mathrm{GL}_4\times\mathrm{GSp}_2$ and Φ_1^+ contains the remaining positive roots. We also embed the Weyl group $S_4\times S_2$ of $\mathrm{GL}_4\times\mathrm{GL}_2$ into W.

Lemma 8.2. *With the notation above, we have*

$$\sum_{w\in W}c_{WS}(w\theta)=\frac{1}{\Delta_{H_0/Z_{G,H}}(1)}=\frac{1}{\zeta(2)}=(1-q^{-2}).$$

Proof. By the identity in Lemma 5.3, we have

$$\sum_{w\in W}c_{WS}(w\theta)=\sum_{w\in W}\frac{\prod_{\gamma^\vee\in\Theta^+}1-q^{-\frac12}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+}1-e^{\alpha^\vee}}(w\theta)$$

$$=(1-q^{-2})\cdot\sum_{w\in W/S_4\times S_2}\frac{\prod_{\gamma^\vee\in\Theta_1^+}1-q^{-\frac12}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+}1-e^{\alpha^\vee}}(w\theta).$$

Hence it is enough to show that

$$\sum_{w\in W/S_4\times S_2}\frac{\prod_{\gamma^\vee\in\Theta_1^+}1-q^{-\frac12}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+}1-e^{\alpha^\vee}}(w\theta)=1.$$

It is easy to see that the constant coefficient of the above summation is equal to 1, so it is enough to show that all the $q^{-i/2}$-coefficients are equal to 0 for $1\leq i\leq 10$. Like in the previous cases, we can replace the summation on $W/S_4\times S_2$ by the summation on W. We also need to rewrite the function inside the summation $\frac{\prod_{\gamma^\vee\in\Theta_1^+}1-q^{-\frac12}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+}1-e^{\alpha^\vee}}(w\theta)$ as (here θ_i are arbitrary variables):

$$w\Big(\frac{(1-q^{-1/2}\cdot\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5})\cdot\Pi_{i=1}^5(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i})}{\Pi_{1\leq i<j\leq 5}(1-\theta_i\theta_j)\cdot\Pi_{i=1}^4(1-\theta_i)\Pi_{i=1}^4(1-\theta_i/\theta_5)}$$

$$\cdot\Pi_{1\leq i\leq 4}(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i\theta_5})\Big)$$

$$=w\Big(\frac{*\cdot(1-q^{-1/2}\cdot\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5})\cdot\Pi_{i=1}^5(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i})}{\Pi_{1\leq i<j\leq 5}(\theta_i^{-1}-\theta_i-\theta_j^{-1}+\theta_j)\cdot\Pi_{i=1}^5(\theta_i^{-1/2}-\theta_i^{1/2})}$$

$$\cdot\Pi_{1\leq i\leq 4}(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i\theta_5})\Big)$$

where

$$* = \theta_1^{-3/2}\theta_2^{-3/2}\theta_3^{-3/2}\theta_4^{-3/2}(\theta_5^{-1/2}-\theta_5^{1/2})\cdot \Pi_{1\leq i<j\leq 4}(\theta_i^{-1}-\theta_j^{-1}).$$

Then the denominator becomes (W,sgn)-invariant, so it is enough to show that the (W,sgn)-summation of the $q^{-i/2}$-coefficient of

$$(8.1)\qquad *\cdot(1-q^{-1/2}\cdot\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5})\cdot\Pi_{i=1}^{5}(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i})$$
$$\cdot\Pi_{1\leq i\leq 4}(1-q^{-1/2}\cdot\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5}}{\theta_i\theta_5})$$

is equal to 0 for $1\leq i\leq 10$. The product $*$ consists of terms of the form

$$\Pi_{i=1}^{5}\theta_i^{a_i},\ \{a_1,a_2,a_3,a_4\}=\{-9/2,-7/2,-5/2,-3/2\}, a_5=\pm 1/2.$$

Then the q^{-5}-coefficients consisting of terms of the form

$$\Pi_{i=1}^{5}\theta_i^{b_i},\ \{b_1,b_2,b_3,b_4\}=\{-3/2,-1/2,1/2,3/2\},\ b_5=\pm 1/2.$$

The (W,sgn)-summation of these terms is equal to 0 since $b_i=\pm b_j$ for some $i\neq j$.

For the $q^{-1/2}$-coefficient, any term $\Pi_{i=1}^{5}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i=b_j$ for some $1\leq i<j\leq 4$.
- $\{b_1,b_2,b_3,b_4\}=\{-5,-3,-2,-1\}$ or $\{-4,-3,-2,-1\}$ and $b_5\in\{-1,1,0\}$.

In either case, we have $b_i=\pm b_j$ for some $i\neq j$ or $b_5=0$. This implies that the (W,sgn)-summation of the $q^{-1/2}$-coefficient is equal to 0. Similarly, we can also show that the (W,sgn)-summation of the $q^{-9/2}$-coefficient is equal to 0.

For the q^{-1}-coefficient, any term $\Pi_{i=1}^{5}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i=b_j$ for some $1\leq i<j\leq 4$.
- $\{b_1,b_2,b_3,b_4\}$ is equal to $\{-11/2,-5/2,-3/2,-1/2\}$, $\{-9/2,-7/2,-3/2,-1/2\}$, $\{-9/2,-5/2,-3/2,-1/2\}$ or $\{-7/2,-5/2,-3/2,-1/2\}$, and $b_5\in\{\pm 3/2,\pm 1/2\}$.

In either case, we have $b_i=\pm b_j$ for some $i\neq j$. This implies that the (W,sgn)-summation of the q^{-1}-coefficient is equal to 0. Similarly, we can also show that the (W,sgn)-summation of the q^{-4}-coefficient is equal to 0.

For the $q^{-3/2}$-coefficient, any term $\Pi_{i=1}^{5}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i=b_j$ for some $1\leq i<j\leq 4$.

- $b_i = 0$ for some $1 \leq i \leq 4$.

This implies that the (W, sgn)-summation of the $q^{-3/2}$-coefficient is equal to 0. Similarly, we can also show that the (W, sgn)-summation of the $q^{-7/2}$-coefficient is equal to 0.

For the q^{-2}-coefficient, any term $\Pi_{i=1}^5 \theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = \pm b_j$ for some $1 \leq i < j \leq 4$.
- $\{b_1, b_2, b_3, b_4\}$ is equal to $\{-7/2, -5/2, -3/2, -1/2\}$, $\{-9/2, -5/2, -3/2, 1/2\}$, or $\{-7/2, -5/2, -3/2, 1/2\}$, and $b_5 \in \{\pm 5/2, \pm 3/2, \pm 1/2\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the q^{-2}-coefficient is equal to 0. Similarly, we can also show that the (W, sgn)-summation of the q^{-3}-coefficient is equal to 0.

For the $q^{-5/2}$-coefficient, any term $\Pi_{i=1}^5 \theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = \pm b_j$ for some $1 \leq i < j \leq 4$.
- $b_i = 0$ for some $1 \leq i \leq 4$.

This implies that the (W, sgn)-summation of the $q^{-5/2}$-coefficient is equal to 0. This finishes the proof of the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)\zeta(4)\zeta(6)\zeta(8)\zeta(10) \frac{L(1/2, \pi, \mathrm{Spin}_{11})}{L(1, \pi, \mathrm{Ad})}$$

where π is an unramified representation of $\mathrm{GSp}_{10}(F)$.

8.2. **The model** $(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes U)$**.** In this subsection, we compute the local relative character for the model $(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes U)$. Let $G = \mathrm{GSp}_6 \times \mathrm{GL}_2$, $H = H_0 \ltimes U$ with

$$H_0 = \{diag(h, h, h) \times h \mid h \in \mathrm{GL}_2\}$$

and U be the unipotent radical of the standard parabolic subgroup $P = LU$ of GSp_6 embedded into G via the map $u \mapsto (u, I_2)$ where

$$L = \{(h_1, h_2, \det(h_2)h_1^*) | \; h_i \in \mathrm{GL}_2\}.$$

We define a generic character ξ on $U(F)$ to be $\xi(u) = \psi(\lambda(u))$ where

$$\lambda(u) = \mathrm{tr}(X), \; u = \begin{pmatrix} I_2 & X & * \\ 0 & I_2 & * \\ 0 & 0 & I_2 \end{pmatrix}.$$

The model (G,H) is the Whittaker induction of the trilinear GL_2-model $(L\times \mathrm{GL}_2, H_0,\xi)$. As in the previous case, we can also define the quaternion version of this model.

Let $w_0=\begin{pmatrix} 0&0&I_2\\ 0&I_2&0\\ I_2&0&0\end{pmatrix}$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a:\mathrm{GL}_1\to Z_L$ to be

$$a(t)=diag(tI_2,I_2,t^{-1}I_2)\times I_2.$$

This clearly satisfies the equation (2.15). For the open Borel orbit, let

$$\eta_0=diag(I_2,\begin{pmatrix} 0&1\\ 1&0\end{pmatrix},I_2)\times \begin{pmatrix} 0&1\\ 1&0\end{pmatrix}\begin{pmatrix} 1&1\\ 0&1\end{pmatrix}$$

be the representative of the open Borel orbit for the model $(L\times \mathrm{GL}_2,H_0)$ as in Section 2.3.2, and $\eta=\eta_0 w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Let Θ be the weights of the 16-dimensional representation $\mathrm{Spin}_7\times \mathrm{std}_2$ of $\mathrm{GSpin}_7(\mathbb{C})\times \mathrm{GL}_2(\mathbb{C})$. We can write it as

$$\Theta=\{\frac{\pm e_1\pm e_2\pm e_3}{2}+e_i'\mid 1\leq i\leq 2\}.$$

Let $\alpha_i=\varepsilon_i-\varepsilon_{i+1},\ 1\leq i\leq 2$ and $\alpha_3=2\varepsilon_3$ be the simple roots of GSp_6 and $\alpha'=\varepsilon_1'-\varepsilon_2'$ be the simple root of GL_2. By the computation of the trilinear GL_2-model in Section 2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we have

$$\beta_{\alpha_1}^\vee=\frac{e_1-e_2-e_3}{2}+e_1',\ \alpha_1^\vee-\beta_{\alpha_1}^\vee=\frac{e_1-e_2+e_3}{2}+e_2',$$
$$\beta_{\alpha_3}^\vee=\frac{-e_1+e_2+e_3}{2}+e_1',\ \alpha_3^\vee-\beta_{\alpha_3}^\vee=\frac{e_1-e_2+e_3}{2}+e_2',$$
$$\beta_{\alpha'}^\vee=\frac{-e_1+e_2+e_3}{2}+e_1',\ \alpha'^\vee-\beta_{\alpha'}^\vee=\frac{e_1-e_2-e_3}{2}+e_1'.$$

By a similar argument as in the Ginzburg–Rallis model case in Section 5, we can also verify (2.23) for the root α_2. The proof of the following proposition follows from a similar but easier argument as the model $(\mathrm{GSp}_{10},\mathrm{GL}_2\ltimes U)$ in the previous subsection. The only difference is to replace the identity in Lemma 5.3 by the identity in Section 2.3.2 for the trilinear GL_2-model. We will skip it here.

Proposition 8.3. Θ^+ *is consisting of the following 8 elements:*

$$\frac{e_1+e_2\pm e_3}{2}+e_i',\frac{e_1-e_2+e_3}{2}+e_i',\frac{\pm(e_1-e_2-e_3)}{2}+e_1',\ 1\leq i\leq 2.$$

The set Θ^+ satisfies (2.3)*. Moreover, we have*

$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H_0/Z_{G,H}}(1)} = \frac{1}{\zeta(2)} = (1-q^{-2}).$$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)\zeta(2)\zeta(4)\zeta(6)\frac{L(1/2,\pi,\mathrm{Spin}_7\times std_2)}{L(1,\pi,\mathrm{Ad})}$$

where π is an unramified representation of $\mathrm{GSp}_6(F)\times \mathrm{GL}_2(F)$.

8.3. **The model** $(\mathrm{GSO}_{12}, \mathrm{GL}_2\ltimes U)$**.** In this subsection, we compute the local relative character for the model $(\mathrm{GSO}_{12}, \mathrm{GL}_2\ltimes U)$. There are two models in this case (corresponding to the two Siegel parabolic subgroups) and they are differed by the outer automorphism of GSO_{12}. Each of them corresponds to one of the Half-Spin L-function of $\mathrm{GSpin}_{12}(\mathbb{C})$. We will only compute the local relative character of one of the models, the other one can be computed just by applying the outer automorphism to the first one. Let $J_2' = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$. Set $L_4 = \begin{pmatrix} 0 & J_2' \\ -J_2' & 0\end{pmatrix}$ and $L_{4n} = \begin{pmatrix} 0 & 0 & J_2' \\ 0 & L_{4n-4} & 0 \\ -J_2' & 0 & 0\end{pmatrix}$. Define

$$\mathrm{GSO}_{4n} = \{g\in \mathrm{GL}_{4n} \mid g^t L_{4n} g = l(g)L_{4n}\}.$$

Let $G=\mathrm{GSO}_{12}$, $H=H_0\ltimes U$ with

$$H_0 = diag(h,h,h,h,h,h) \mid h\in \mathrm{GL}_2\}$$

and U be the unipotent radical of the standard parabolic subgroup $P=LU$ of G with $(h^* = {J_2'}^t h^{-1}(J_2')^{-1})$

$$L=\{diag(h_1,h_2,h_3,th_3^*,th_2^*,th_1^*) \mid h_i\in \mathrm{GL}_2, t\in \mathrm{GL}_1\}.$$

We define a generic character ξ on $U(F)$ to be $\xi(u)=\psi(\lambda(u))$ where

$$\lambda(u) = \mathrm{tr}(X)+\mathrm{tr}(Y)+\mathrm{tr}(Z),\ u=\begin{pmatrix} I_2 & X & * & * & * & * \\ 0 & I_2 & Y & * & * & * \\ 0 & 0 & I_2 & Z & * & * \\ 0 & 0 & 0 & I_2 & * & * \\ 0 & 0 & 0 & 0 & I_2 & * \\ 0 & 0 & 0 & 0 & 0 & I_2 \end{pmatrix}.$$

It is easy to see that H_0 is the stabilizer of this character and (G,H) is the Whittaker induction of the trilinear GL_2-model (L,H_0,ξ).

We can also define the quaternion version of this model. Let D/F be a quaternion algebra, and let

$$\mathrm{GSO}_{2n}(D)=\{g\in \mathrm{GL}_{2n}(D)\mid {}^t\bar{g}J_{2n'}g=l(g)J_{2n}'\}.$$

Let $G_D(F)=\mathrm{GSO}_6(D)$, $H_D=H_{0,D}\ltimes U_D$ with

$$H_{0,D}(F)=\{diag(h,h,h,h,h,h)\mid h\in \mathrm{GL}_1(D)\}$$

and U_D be the unipotent radical of the standard parabolic subgroup $P_D=L_DU_D$ of G_D where ($h^*=\bar{h}^{-1}$)

$$L_D(F)=\{(h_1,h_2,h_3,th_3^*,th_2^*,th_1^*)|\ h_i\in \mathrm{GL}_1(D),t\in \mathrm{GL}_1(F)\}.$$

Here $x\to\bar{x}$ is the conjugation map on the quaternion algebra. Like the split case, we can define the character ξ_D on $U_D(F)$ by replacing the trace map of $Mat_{2\times 2}$ by the trace map of D.

Let $w_0=\begin{pmatrix} 0&0&0&0&0&I_2\\ 0&0&0&0&I_2&0\\ 0&0&0&I_2&0&0\\ 0&0&I_2&0&0&0\\ 0&I_2&0&0&0&0\\ I_2&0&0&0&0&0\end{pmatrix}$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a:\mathrm{GL}_1\to Z_L$ to be

$$a(t)=diag(t^3I_2,t^2I_2,tI_2,I_2,t^{-1}I_2,t^{-2}I_2).$$

This clearly satisfies the second identity of the equation (2.15). Although it does not satisfy the first equation of (2.15), but the difference between $a(t)^{-1}$ and $w_0^{-1}a(t)w_0$ belongs to the center so all the arguments in Section 2.4 still work (because all the characters are unramified). For the open Borel orbit, let

$$\eta_0=diag(I_2,\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\-1&-1\end{pmatrix},-\begin{pmatrix}0&1\\1&0\end{pmatrix},I_2)$$

be the representative of the open Borel orbit for the model (L,H_0) as in Section 2.3.2, and $\eta=\eta_0w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Now we compute the set of colors and also the set Θ^+. Let Θ be the weights of the 32-dimensional Half-Spin representation HSpin_{12} of $\mathrm{GSpin}_{12}(\mathbb{C})$ given by

$$\Theta=\{\frac{\pm e_1\pm e_2\pm e_3\pm e_4\pm e_5\pm e_6}{6}\mid -\text{ appears odd times}\}.$$

Let $\alpha_i=\varepsilon_i-\varepsilon_{i+1},\ 1\leq i\leq 5$ and $\alpha_6=\varepsilon_5+\varepsilon_6$ be the simple roots of GSO_{12}. By the computation of the trilinear GL_2-model in Section

2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we have

$$\beta_{\alpha_1}^\vee = \frac{e_1-e_2-e_3+e_4+e_5-e_6}{2},$$
$$\alpha_1^\vee - \beta_{\alpha_1}^\vee = \frac{e_1-e_2+e_3-e_4-e_5+e_6}{2},$$
$$\beta_{\alpha_3}^\vee = \frac{-e_1+e_2+e_3-e_4+e_5-e_6}{2},$$
$$\alpha_3^\vee - \beta_{\alpha_3}^\vee = \frac{e_1-e_2+e_3-e_4-e_5+e_6}{2},$$
$$\beta_{\alpha_5}^\vee = \frac{-e_1+e_2+e_3-e_4+e_5-e_6}{2},$$
$$\alpha_5^\vee - \beta_{\alpha_5}^\vee = \frac{e_1-e_2-e_3+e_4+e_5-e_6}{2}.$$

By a similar argument as in the Ginzburg–Rallis model case in Section 5, we can also verify (2.23) for the roots α_2, α_4 and α_6. Next, we compute the set Θ^+.

Proposition 8.4. Θ^+ *is consisting of the following 16 elements:*

$$\frac{e_1+e_2+e_3+e_4+e_5+e_6-2e_l}{2},$$
$$\frac{-e_1-e_2-e_3-e_4-e_5-e_6+2e_i+2e_j+2e_k}{2}$$

with $1 \leq l \leq 6$ *and* (i,j,k) *belongs to the set*

$$\{(123),(124),(125),(126),(134),(135),(136),(145),(234),(235)\}.$$

Proof. By the computation of the colors, we know that Θ^+ is the smallest subset of Θ satisfying the following 5 conditions:

(1) $\frac{e_1-e_2-e_3+e_4+e_5-e_6}{2}, \frac{-e_1+e_2+e_3-e_4+e_5-e_6}{2}, \frac{e_1-e_2+e_3-e_4-e_5+e_6}{2} \in \Theta^+$.
(2) $\Theta^+ - (\Theta^+ \cap w_{\alpha_1}\Theta^+) = \{\frac{e_1-e_2-e_3+e_4+e_5-e_6}{2}, \frac{e_1-e_2+e_3-e_4-e_5+e_6}{2}\}$.
(3) $\Theta^+ - (\Theta^+ \cap w_{\alpha_3}\Theta^+) = \{\frac{e_1-e_2+e_3-e_4-e_5+e_6}{2}, \frac{-e_1+e_2+e_3-e_4+e_5-e_6}{2}\}$.
(4) $\Theta^+ - (\Theta^+ \cap w_{\alpha_5}\Theta^+) = \{\frac{e_1-e_2-e_3+e_4+e_5-e_6}{2}, \frac{-e_1+e_2+e_3-e_4+e_5-e_6}{2}\}$.
(5) Θ^+ is stable under w_{α_2}, w_{α_4} and w_{α_6}.

It is clear that the set in the proposition satisfies these conditions. So we just need to show that the set is the unique subset of Θ satisfying these conditions. The argument is exactly the same as the case $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ in Proposition 3.7. We will skip it here. $\square$

It is clear that Θ^+ satisfies (2.3). The last thing remains to prove Lemma 2.32 for the current case. Let Θ_1^+ (resp. Θ_2^+) be the subset of Θ^+ consisting of elements of the form

$$\frac{e_1+e_2+e_3+e_4+e_5+e_6-2e_l}{2}$$

and let Θ_2^+ be the subset of Θ^+ consisting of elements of the form

$$\frac{e_1+e_2+e_3+e_4+e_5+e_6-2e_i-2e_j-2e_k}{2}.$$

Then Θ_2^+ corresponds to the weights of the Ginzburg–Rallis model discussed in Section 5. We also decompose the set of positive roots Φ^+ as $\Phi_1^+ \cup \Phi_2^+$ where

$$\Phi_2^+ = \{e_i - e_j |\ 1 \leq i < j \leq 6\}$$

is the set of the roots contained in GL_6 and Φ_1^+ contains the remaining positive roots. We also embed the Weyl group S_6 of GL_6 into the Weyl group W.

Lemma 8.5. *With the notation above, we have*

$$\sum_{w\in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H_0/Z_{G,H}}(1)} = \frac{1}{\zeta(2)} = (1-q^{-2}).$$

Proof. By the identity for the Ginzburg–Rallis model case proved in Lemma 5.2, we have

$$\sum_{w\in W} c_{WS}(w\theta) = \sum_{w\in W} \frac{\prod_{\gamma^\vee\in\Theta^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi^+} 1-e^{\alpha^\vee}}(w\theta)$$

$$= (1-q^{-2})\cdot \sum_{w\in W/S_6} \frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta).$$

Hence it is enough to show that

$$\sum_{w\in W/S_6} \frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta) = 1.$$

It is easy to see that the constant coefficient of the above summation is equal to 1, so it is enough to show that all the $q^{-i/2}$-coefficients are equal to 0 for $1 \leq i \leq 6$. Like in the previous cases, we can replace the summation on W/S_6 by the summation on W. We also need to rewrite the function inside the summation (θ_i are arbitrary variables):

$$\frac{\prod_{\gamma^\vee\in\Theta_1^+} 1-q^{-\frac{1}{2}}e^{\gamma^\vee}}{\prod_{\alpha\in\Phi_1^+} 1-e^{\alpha^\vee}}(w\theta) = \frac{\Pi_{i=1}^6(1-q^{-1/2}\cdot w(\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6}}{\theta_i}))}{\Pi_{1\leq i<j\leq 6}(1-w(\theta_i\theta_j))}$$

$$= \frac{\Pi_{1\leq i<j\leq 6} w(\theta_i^{-1}-\theta_j^{-1})\cdot \Pi_{i=1}^6(1-q^{-1/2}\cdot w(\frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6}}{\theta_i}))}{\Pi_{1\leq i<j\leq 6} w(\theta_i^{-1}-\theta_i-\theta_j^{-1}+\theta_j)}.$$

Then the denominator becomes (W, sgn)-invariant, so it is enough to show that the (W, sgn)-summation of the $q^{-i/2}$-coefficient of

$$\Pi_{1\leq i<j\leq 6}(\theta_i^{-1}-\theta_j^{-1})\cdot \Pi_{i=1}^{6}(1-q^{-1/2}\cdot \frac{\sqrt{\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6}}{\theta_i}) \tag{8.2}$$

is equal to 0 for $1 \leq i \leq 6$. The product $\Pi_{1\leq i<j\leq 6}(\theta_i^{-1}-\theta_j^{-1})$ consists of terms of the form

$$\Pi_{i=1}^{6}\theta_i^{a_i},\ \{a_1,a_2,a_3,a_4,a_5,a_6\} = \{-5,-4,-3,-2,-1,0\}.$$

Then any term $\Pi_{i=1}^{6}\theta_i^{b_i}$ appearing in the $q^{-1/2}$-coefficient of (8.2) must satisfy one of the following two conditions

- $b_i = b_j$ for some $i \neq j$.
- $\{b_1,b_2,b_3,b_4,b_5,b_6\} = \{-11/2,-7/2,-5/2,-3/2,-1/2,1/2\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the $q^{-1/2}$-coefficient is equal to 0.

For the q^{-1}-coefficient, any term $\Pi_{i=1}^{6}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = b_j$ for some $i \neq j$.
- $\{b_1,b_2,b_3,b_4,b_5,b_6\} = \{-5,-4,-2,-1,0,1\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the q^{-1}-coefficient is equal to 0.

For the $q^{-3/2}$-coefficient, any term $\Pi_{i=1}^{6}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = b_j$ for some $i \neq j$.
- $\{b_1,b_2,b_3,b_4,b_5,b_6\} = \{-9/2,-7/2,-5/2,-1/2,1/2,3/2\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the $q^{-3/2}$-coefficient is equal to 0.

For the q^{-2}-coefficient, any term $\Pi_{i=1}^{6}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = b_j$ for some $i \neq j$.
- $\{b_1,b_2,b_3,b_4,b_5,b_6\} = \{-4,-3,-2,-1,1,2\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the q^{-2}-coefficient is equal to 0.

For the $q^{-5/2}$-coefficient, any term $\Pi_{i=1}^{6}\theta_i^{b_i}$ appearing in it must satisfy one of the following two conditions

- $b_i = b_j$ for some $i \neq j$.
- $\{b_1,b_2,b_3,b_4,b_5,b_6\} = \{-7/2,-5/2,-3/2,-1/2,1/2,5/2\}$.

In either case, we have $b_i = \pm b_j$ for some $i \neq j$. This implies that the (W, sgn)-summation of the $q^{-3/2}$-coefficient is equal to 0.

Finally, the q^{-3}-coefficients consisting of terms of the form

$$\Pi_{i=1}^{6}\theta_i^{b_i},\ \{b_1,b_2,b_3,b_4,b_5,b_6\}=\{-3,-2,-1,0,1,2\}.$$

The (W,sgn)-summation of these terms is equal to 0. This finishes the proof of the lemma. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)\zeta(4)\zeta(6)^2\zeta(8)\zeta(10)\frac{L(1/2,\pi,\mathrm{HSpin}_{12})}{L(1,\pi,\mathrm{Ad})}$$

where π is an unramified representation of $\mathrm{GSO}_{12}(F)$.

8.4. **The model** $(\mathrm{GSO}_8\times\mathrm{GL}_2,\mathrm{GL}_2\ltimes U)$**.** In this subsection, we compute the local relative character for the model $(\mathrm{GSO}_8\times\mathrm{GL}_2,\mathrm{GL}_2\ltimes U)$. Like the previous case, there are two models in this case and they are differed by the outer automorphism of GSO_8. Each of them corresponds to one of the Half-Spin L-function of $\mathrm{GSpin}_8(\mathbb{C})$. We will only compute the local relative character of one of the models, the other one can be computed just by applying the outer automorphism to the first one.

Let $G=\mathrm{GSO}_8\times\mathrm{GL}_2$, $H=H_0\ltimes U$ with

$$H_0=\{diag(h,h,h,h)\times h\mid h\in\mathrm{GL}_2\}$$

and U be the unipotent radical of the standard parabolic subgroup $P=LU$ of GSO_8 (we embed U into G via the map $u\mapsto u\times I_2$) where

$$L=\{diag(h_1,h_2,th_2^*,th_1^*)|\ h_i\in\mathrm{GL}_2,t\in\mathrm{GL}_1\}.$$

We define a generic character ξ on $U(F)$ to be $\xi(u)=\psi(\lambda(u))$ where

$$\lambda(u)=\mathrm{tr}(X)+\mathrm{tr}(Y),\ u=\begin{pmatrix}I_2&X&*&*\\0&I_2&Y&*\\0&0&I_2&*\\0&0&0&I_2\end{pmatrix}.$$

The model (G,H) is the Whittaker induction of the trilinear GL_2-model $(L\times\mathrm{GL}_2,H_0,\xi)$. Similarly we can also define the quaternion algebra version of this model.

Let $w_0=\begin{pmatrix}0&0&0&I_2\\0&0&I_2&0\\0&I_2&0&0\\I_2&0&0&0\end{pmatrix}\times I_2$ be the Weyl element that sends U to its opposite. It is clear that the w_0-conjugation map stabilizes L and fixes H_0. We define the map $a:\mathrm{GL}_1\to Z_L$ as

$$a(t)=diag(t^2I_2,tI_2,I_2,t^{-1}I_2)\times I_2.$$

This clearly satisfies the second identity of the equation (2.15). Although it does not satisfy the first equation of (2.15), but the difference between $a(t)^{-1}$ and $w_0^{-1}a(t)w_0$ belongs to the center so all the arguments in Section 2.4 still work. For the open Borel orbit, let

$$\eta_0 = diag(I_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_2) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the representative of the open Borel orbit for the model (L, H_0) as in Section 2.3.2, and $\eta = \eta_0 w_0$. The relation (2.20) has already been verified in Section 2.3.2. This finishes the first three steps in Section 2.5.1.

Now we compute the set of colors and also the set Θ^+. Let Θ be the weights of the 16-dimensional Half-Spin representation HSpin_{12} of $\mathrm{GSpin}_{12}(\mathbb{C})$ given by

$$\Theta = \{\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2} + e_i' \mid - \text{ appears even times}, i \in \{1, 2\}\}.$$

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq 3$ and $\alpha_4 = \varepsilon_3 + \varepsilon_4$ be the simple roots of GSO_8 and $\alpha' = \varepsilon_1' - \varepsilon_2'$ be the simple root of GL_2. By the computation of the trilinear GL_2-model in Section 2.3.2 and the discussion in Section 2.5 (in particular, Remark 2.28), we have

$$\beta_{\alpha_1}^\vee = \frac{e_1 - e_2 - e_3 + e_4}{2} + e_1', \ \alpha_1^\vee - \beta_{\alpha_1}^\vee = \frac{e_1 - e_2 + e_3 - e_4}{2} + e_2',$$

$$\beta_{\alpha_3}^\vee = \frac{-e_1 + e_2 + e_3 - e_4}{2} + e_1', \ \alpha_3^\vee - \beta_{\alpha_3}^\vee = \frac{e_1 - e_2 + e_3 - e_4}{2} + e_2',$$

$$\beta_{\alpha'}^\vee = \frac{-e_1 + e_2 + e_3 - e_4}{2} + e_1', \ \alpha'^\vee - \beta_{\alpha'}^\vee = \frac{e_1 - e_2 - e_3 + e_4}{2} + e_1'.$$

By a similar argument as in the Ginzburg–Rallis model case in Section 5, we can also verify (2.23) for the roots α_2 and α_4. The next proposition computes the set Θ^+ and proves Lemma 2.32 for the current case.

Proposition 8.6. Θ^+ *is consisting of the following 8 elements:*

$$\frac{e_1 + e_2 \pm (e_3 + e_4)}{2} + e_i', \frac{e_1 - e_2 + e_3 - e_4}{2} + e_i',$$

$$\frac{\pm(e_1 - e_2 - e_3 + e_4)}{2} + e_1', \ 1 \leq i \leq 2.$$

Moreover, we have

$$\sum_{w \in W} c_{WS}(w\theta) = \frac{1}{\Delta_{H_0/Z_{G,H}}(1)} = \frac{1}{\zeta(2)} = (1 - q^{-2}).$$

Proof. The proof follows from a similar but easier argument as the $(\mathrm{GSO}_{12}, \mathrm{GL}_2 \ltimes U)$ model case in the previous subsection. The only difference is that we need to use Lemma 5.3 instead of Lemma 5.2. We will skip the details here. $\square$

To sum up, we have proved that the local relative character is equal to

$$\zeta(1)^2\zeta(2)\zeta(4)^2\zeta(6)\frac{L(1/2,\pi,\mathrm{HSpin}_8 \times std_2)}{L(1,\pi,Ad)}$$

where π is an unramified representation of $\mathrm{GSO}_8(F) \times \mathrm{GL}_2(F)$.

9. Local multiplicity

In this section we will study the multiplicity for the models in Table 1. Let F be a local field of characteristic 0, (G, H) be one of the models in Table 1, and ξ be the character of $H(F)$ defined in the previous sections (note that ξ is trivial in the reductive case). Let π be an irreducible admissible representation of $G(F)$ whose central character is trivial on $Z_{G,H}(F)$. Recall that the multiplicity is defined by

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \xi).$$

Similarly, if $F \neq \mathbb{C}$, let D/F be the unique quaternion algebra (or $D \in H^1(F, H/Z_{G,H})$ if we are in the case of Model 2 of Table 1), and let (G_D, H_D, ξ_D) be the pure inner form of the model (G, H, ξ) defined in the previous sections. Let π_D be an irreducible representation of $G_D(F)$ whose central character is trivial on $Z_{G_D,H_D}(F)$. We can also define the multiplicity $m(\pi_D) = \dim(\mathrm{Hom}_{H_D(F)}(\pi_D, \xi_D))$.

In this section, we will prove a geometric multiplicity formula of $m(\pi)$ and $m(\pi_D)$ in terms of the Harish-Chandra character. Then by using the geometric multiplicity formula, together with the character identity in the local Langlands correspondence, we will show that for all the tempered L-packets, the summation of the multiplicities is equal to 1 and the unique distinguished element in the packet corresponds to a character of the component group. The proof of all the results in this section is very similar to the Gan–Gross–Prasad model case ([W10], [W12], [B15]) and the Ginzburg–Rallis model case ([Wan15], [Wan16], [Wan17], [WZ]) since similar to the Gan–Gross–Prasad model and the Ginzburg–Rallis model, all the models in Table 1 are strongly tempered without Type N root and has a unique open Borel orbit.

In Section 9.1, we will recall the local Langlands conjecture. In Section 9.2 we will study the reductive models in Table 1 and in Section 9.3 we will study the non-reductive models.

9.1. **The local Langlands conjecture.** In this subsection we recall the local Langlands conjecture in Conjecture E of [K]. Let G be a quasi-split reductive group defined over F and let $\{G_\alpha \mid \alpha \in H^1(F,G)\}$ be the set of pure inner forms of G. Let $\Pi_{irr,temp}(G_\alpha)$ be the set of irreducible tempered representations of $G_\alpha(F)$. The local Langlands conjecture states that $\cup_{\alpha\in H^1(F,G)}\Pi_{irr,temp}(G_\alpha)$ is a disjoint union of finite sets (i.e. the local tempered Vogan L-packets)

$$\cup_\phi \Pi_\phi$$

where ϕ runs over all the tempered L-parameters of G and $\Pi_\phi = \cup_{\alpha\in H^1(F,G)}\Pi_\phi(G_\alpha)$ consists of a finite number of tempered representations with $\Pi_\phi(G_\alpha) \subset \Pi_{irr,temp}(G_\alpha)$ such that the following conditions hold.

- There is a unique generic element in $\Pi_\phi(G)$ with respect to any Whittaker datum of G.
- For given Whittaker datum, there is a bijection between $\hat{S}_\phi$, the set of irreducible representations of the component group $S_\phi = Z_\phi/Z_\phi^\circ$ (Z_ϕ is the centralizer of $Im(\phi)$ in $\hat{G}$) of the Langlands parameter ϕ, and Π_ϕ (denoted by $\pi \leftrightarrow \chi_\pi$) such that
  - the trivial character of S_ϕ corresponds to the unique generic element of $\Pi_\phi(G)$ with respect to the given Whittaker datum.
  - for $\alpha \in H^1(F,G)$, the distribution character $\theta_{\Pi_\phi(G_\alpha)} = \sum_{\pi\in\Pi_\phi(G_\alpha)} \dim(\chi_\pi)\theta_\pi$ is stable. Moreover, $\iota(G_\alpha)\theta_{\Pi_\phi(G_\alpha)}$ is the transfer of $\theta_{\Pi_\phi(G)}$ where $\iota(G_\alpha)$ is the Kottwitz sign.
  - endoscopic identity.

We will not discuss the endoscopic identity of the local Langlands conjecture here since we don't need to use it in this paper, we refer the reader to [K] for more details. In order to prove the multiplicity one of the L-packet for the models in Table 1, we need to assume that the local Langlands conjecture holds for the groups associated to the models. Note that for the group G in Model 3 and Model 6-10 of Table 1, the component group S_ϕ is not necessarily abelian.

9.2. **The reductive case.** In this subsection we assume that H is reductive. The model $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$ has already been considered in the previous paper [PWZ19], so we will focus on the models $(G,H) = (\mathrm{GSp}_6 \times \mathrm{GSp}_4, (\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0)$ and $(G,H) = (\mathrm{GU}_4 \times \mathrm{GU}_2, (\mathrm{GU}_2 \times \mathrm{GU}_2)^0)$.

Let π be an irreducible representation of $G(F)$ with trivial central character and θ_π be its Harish-Chandra character. For a semisimple element $x \in G(F)$, we let $c_\pi(x)$ be the average of the regular germs of

θ_π at x. We refer the reader to Section 4.5 of [B15] for the definition of regular germs. We want to emphasize that $c_\pi(x)$ is zero if the centralizer G_x is not quasi-split. We also let $\mathcal{T}_{ell}(G)$ be a set of representatives of maximal elliptic tori of $G(F)$.

9.2.1. *The model* $(\mathrm{GSp}_6\times\mathrm{GSp}_4, (\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$. We first consider the case $(G,H)=(\mathrm{GSp}_6\times\mathrm{GSp}_4, (\mathrm{GSp}_4\times\mathrm{GSp}_2)^0)$. For $T\in\mathcal{T}_{ell}(\mathrm{GSp}_2)$, let

$$T^{n,0}=\{(t_1,\cdots,t_n)\in T^n|\ \det(t_i)=\det(t_j) \text{ for all } 1\leq i,j\leq n\}.$$

We use ι_n to denote the diagonal embedding from T to $T^{n,0}$. We can view $T^{n,0}$ as a maximal elliptic torus of GSp_{2n}. Moreover, up to GSp_{2n}-conjugation, there are 2^{n-1}-many different embeddings from $T^{n,0}$ to GSp_{2n}.

When $n=2$, there are two embeddings ν_2,ν_2' from $T^{2,0}$ to GSp_4 and the centralizer of the image of $\nu_2\circ\iota_2$ (resp. $\nu_2'\circ\iota_2$) in GSp_4 is the quasi-split (resp. non quasi-split) unitary similitude group of 3 variables. Meanwhile, there are four embeddings from $T^{3,0}$ to $(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0$ and there are two of them whose projection to GSp_4 coincide with ν_2. Compose with the embedding from $(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0$ to GSp_6, we get two embeddings ν_{31},ν_{32} from $T^{3,0}$ to GSp_6. The centralizers of the image of $\nu_{3i}\circ\iota_3$ $(i=1,2)$ in GSp_6 are the two unitary similitude groups of 3 variables (both of them are quasi-split). We use $\nu_{T,i}=(\nu_{3i}\circ\iota_3)\times(\nu_2\circ\iota_2)$ to denote the two embeddings from T to G (both factor through H). It is easy to see that these two embeddings are conjugated to each other in H and we will use ν_T to denote one of it.

Meanwhile, let $\iota_{1,2}$ be the embedding from $T^{2,0}$ to $T^{3,0}$ given by

$$(t_1,t_2)\mapsto(t_1,t_2,t_2).$$

Among the four embeddings from $T^{3,0}$ to GSp_6, there are two of them (denoted by ν_3,ν_3') such that the centralizers in GSp_6 of the image of $\nu_3\circ\iota_{1,2}$ and $\nu_3'\circ\iota_{1,2}$ are quasi-split (the centralizer is the quasi-split unitary similitude group of 2 variables times an abelian group). Up to conjugation we may assume that ν_3,ν_3' factor through $(\mathrm{GSp}_4\times\mathrm{GSp}_2)^0$ and the projection to GSp_4 of $\nu_3\circ\iota_{1,2}$ (resp. $\nu_3'\circ\iota_{1,2}$) is equal to ν_2 (resp. ν_2'). We use

$$\nu_{T^{2,0},1}=(\nu_3\circ\iota_{1,2})\times\nu_2,\ \nu_{T^{2,0},2}=(\nu_3'\circ\iota_{1,2})\times\nu_2'$$

to denote the two embeddings from $T^{2,0}$ to G. Both of them factor through H.

Finally, for $T_1,T_2\in\mathcal{T}_{ell}(\mathrm{GSp}_2)$ with $T_1\neq T_2$ (this will not happen in the archimedean case), let

$$(T_1\times T_2)^0=\{(t_1,t_2)\in T_1\times T_2|\ \det(t_1)=\det(t_2)\}.$$

Similarly, we can define $(T_1\times T_2\times T_2)^0$. Up to conjugation, there is only one embedding from $(T_1\times T_2)^0$ to GSp_4 and there are two embeddings from $(T_1\times T_2\times T_2)^0$ to GSp_6. The two embeddings induce two embeddings from $(T_1\times T_2)^0$ to GSp_6 (we first map T_2 diagonally into $(T_2\times T_2)^0$). We let ν be the embedding such that the centralizer of its image is quasi-split (the centralizer of the other embedding is not quasi-split). Up to conjugation we may assume that ν factors through $(\mathrm{GSp}_4\times \mathrm{GSp}_2)^0$ and its projection to GSp_4 is equal to the embedding from $(T_1\times T_2)^0$ to GSp_4. This gives us an embedding ν_{T_1,T_2} from $(T_1\times T_2)^0$ to G that factors through H.

Define the geometric multiplicity to be ($D^H(\cdot)$ is the Weyl determinant)

$$\begin{aligned} m_{geom}(\pi) \;=\;& c_\pi(1)+\sum_{T\in\mathcal{T}_{ell}(H)}|W(H,T)|^{-1}\int^{*}_{T(F)/Z_{G,H}(F)} D^H(t)\theta_\pi(t)\,\mathrm{d}t\\ &+\frac{1}{2}\sum_{T\in\mathcal{T}_{ell}(\mathrm{GSp}_2)}\Big(\int^{*}_{T(F)/Z_{\mathrm{GL}_2}(F)} D^H(\nu_T(t))c_\pi(\nu_T(t))\,\mathrm{d}t\\ &+\sum_{i\in\{1,2\}}\int^{*}_{T^{2,0}(F)/Z_{\mathrm{GL}_2}(F)} D^H(\nu_{T^{2,0},i}(t))c_\pi(\nu_{T^{2,0},i}(t))\,\mathrm{d}t\Big)\\ &+\frac{1}{4}\sum_{T_1,T_2\in\mathcal{T}_{ell}(\mathrm{GSp}_2),T_1\neq T_2}\int^{*}_{(T_1\times T_2)^0(F)/Z_{\mathrm{GL}_2}(F)^{diag}}\\ &D^H(\nu_{T_1,T_2}(t))c_\pi(\nu_{T_1,T_2}(t))\,\mathrm{d}t. \end{aligned}$$

Here 1 always stands for the identity element of $G(F)$, $W(H,T)$ is the Weyl group, all the Haar measures are chosen so that the total volume is equal to 1 (note that all the integral domains are compact), and the factors $\frac{1}{2},\frac{1}{4}$ come from the cardinality of the Weyl groups $W(\mathrm{GSp}_2,T),W(\mathrm{GSp}_2,T_i)$. Note that if $F=\mathbb{C}$, then $\mathcal{T}_{ell}(H)$ and $\mathcal{T}_{ell}(\mathrm{GL}_2)$ are empty. Hence we have $m_{geom}(\pi)=c_\pi(1)$. When $F=\mathbb{R}$, $\mathcal{T}_{ell}(\mathrm{GSp}_2)$ only contains one element and the term associated to

$$T_1,T_2\in\mathcal{T}_{ell}(\mathrm{GSp}_2),T_1\neq T_2$$

will not appear. We leave it as an excise for the reader to check that our definition of $m_{geom}(\pi)$ matches the definition in [Wan] for general spherical varieties. We refer the reader to [Wan] for a detailed discussion of the geometric multiplicity for general spherical varieties.

Remark 9.1. *Like the unitary Gan–Gross–Prasad model case (Proposition 11.2.1 of* [B15]*), the integrals defining the geometric multiplicity are not necessarily absolutely convergent and they need to be regularized (this is why we write the integral as $\int^*$). The regularization is the same as the unitary Gan–Gross–Prasad model case. To be specific, one replace the Weyl determinant D^H in the integrand by $(D^G)^{1/2}\cdot(\frac{(D^H)^2}{D^G})^{s-1/2}$. By a very similar argument as in Proposition 11.2.1 of* [B15]*, we know that the integral is absolutely convergent when $s>0$ and has a limit as $s\to 0^+$. Then we can define the regularized integral to be this limit. This remark also applies to the model $(\mathrm{GU}_4\times\mathrm{GU}_2,(\mathrm{GU}_2\times\mathrm{GU})^0)$ in the next subsection.*

Similarly, if $F\neq\mathbb{C}$, for the quaternion version of the model, we can also define the embeddings $\nu_{T_D,i},\nu_{T_D^{2,0},i},\nu_{T_{1,D},T_{2,D}}$ for $T_D,T_{1,D},T_{2,D}\in\mathcal{T}_{ell}(\mathrm{GSp}_1(D))=\mathcal{T}_{ell}(\mathrm{GSp}_2)$ with $T_{1,D}\neq T_{2,D}$. We can define the geometric multiplicity $m_{geom}(\pi_D)$ to be

$$\sum_{T_D\in\mathcal{T}_{ell}(H_D)}|W(H_D,T_D)|^{-1}\int^*_{T_D(F)/Z_{G_D,H_D}(F)}D^{H_D}(t)\theta_{\pi_D}(t)\,\mathrm{d}t$$
$$+\frac{1}{2}\sum_{T_D\in\mathcal{T}_{ell}(\mathrm{GSp}_1(D))}\Big(\int^*_{T_D(F)/Z_{\mathrm{GL}_1(D)}(F)}D^{H_D}(\nu_{T_D}(t))c_{\pi_D}(\nu_{T_D}(t))\,\mathrm{d}t$$
$$+\sum_{i\in\{1,2\}}\int^*_{T_D^{2,0}(F)/Z_{\mathrm{GL}_1(D)}(F)}D^{H_D}(\nu_{T_D^{2,0},i}(t))c_{\pi_D}(\nu_{T_D^{2,0},i}(t))\,\mathrm{d}t\Big)$$
$$+\frac{1}{4}\sum_{T_{1,D},T_{2,D}\in\mathcal{T}_{ell}(\mathrm{GSp}_1(D)),T_{1,D}\neq T_{2,D}}\int^*_{(T_{1,D}\times T_{2,D})^0(F)/Z_{\mathrm{GL}_1(D)}(F)^{diag}}$$
$$D^{H_D}(\nu_{T_{1,D},T_{2,D}}(t))c_{\pi_D}(\nu_{T_{1,D},T_{2,D}}(t))\,\mathrm{d}t.$$

The only difference between $m_{geom}(\pi)$ and $m_{geom}(\pi_D)$ is that $m_{geom}(\pi)$ contains the germ at 1 (since $G(F)$ is quasi-split) while $m_{geom}(\pi_D)$ dose not. The following theorem gives a geometric multiplicity formula for the model.

Theorem 9.2. *For all tempered representations π of $G(F)$ (resp. π_D of $G_D(F)$) whose central character is trivial on $Z_{G,H}(F)$ (resp. $Z_{G_D,H_D}(F)$), we have*

$$m(\pi)=m_{geom}(\pi),\ m(\pi_D)=m_{geom}(\pi_D).$$

Proof. This follows from a similar but easier argument as in the Gan–Gross–Prasad model case ([W10], [W12], [B15]) and the Ginzburg–Rallis model case ([Wan15], [Wan16], [Wan17]). The argument is easier for this model because it is reductive and hence there is no need to regularize the integral over H. The only difference is that the proofs in the above papers used the Gelfand pair condition (i.e. $m(\pi) \leq 1$ for all irreducible representation π of $G(F)$) which is not known for this model. But this can be solved by the same argument as the unitary Ginzburg–Rallis model case in our previous paper (Section 6 and Appendix A of [WZ]). We will skip the proof. $\square$

If $F = \mathbb{C}$, then any tempered representation of $G(F)$ is generic and we have $m_{geom}(\pi) = c_\pi(1) = 1$ by the result for Whittaker model in [Mat]. Hence the above theorem implies that $m(\pi) = 1$ for all tempered representations π of $G(F)$ with trivial central character (note that the L-packet only contains one element in the complex case).

If $F \neq \mathbb{C}$, let $\Pi_\phi = \Pi_\phi(G) \cup \Pi_\phi(G_D)$ be a tempered local L-packet of G whose central character is trivial on $Z_{G,H}(F)$. Assume that the local Langlands conjecture holds for $G(F)$. Let

$$\theta_{\Pi_\phi(G)} = \sum_{\pi \in \Pi_\phi(G)} \dim(\chi_\pi)\theta_\pi,\ \theta_{\Pi_\phi(G_D)} = \sum_{\pi_D \in \Pi_\phi(G_D)} \dim(\chi_{\pi_D})\theta_{\pi_D}$$

be the corresponding stable characters (note that $G(F)$ has a unique Whittaker datum). The Kottwitz sign between G and G_D is -1. Hence we have

$$\theta_{\Pi_\phi(G)}(g) = -\theta_{\Pi_\phi(G_D)}(g_D),\ \forall g \in G_{reg}(F), g_D \in G_D(F), g \leftrightarrow g_D.$$

Combining with Proposition 4.5.1 of [B15], we have

$$c_{\theta_{\Pi_\phi(G)}}(\nu_T(t)) = -c_{\theta_{\Pi_\phi(G_D)}}(\nu_{T_D}(t_D)),\ \forall t \in T(F) \leftrightarrow t_D \in T_D(F);$$

$$c_{\theta_{\Pi_\phi(G)}}(\nu_{T^{2,0},i}(t)) = -c_{\theta_{\Pi_\phi(G_D)}}(\nu_{T_D^{2,0},i}(t_D)),\ \forall t \in T^{2,0}(F) \leftrightarrow t_D \in T_D^{2,0}(F);$$

$$c_{\theta_{\Pi_\phi(G)}}(\nu_{T_1,T_2}(t)) = -c_{\theta_{\Pi_\phi(G_D)}}(\nu_{T_{1,D},T_{2,D}}(t_D)),$$
$$\forall t \in (T_1 \times T_2)^0(F) \leftrightarrow t_D \in (T_{1,D} \times T_{2,D})^0(F).$$

Here $g \leftrightarrow g_D$ (resp. $t \leftrightarrow t_D$) means that they have the same characteristic polynomial. Together with the multiplicity formula, we have

$$\sum_{\pi \in \Pi_\phi(G)} \dim(\chi_\pi)m(\pi) + \sum_{\pi_D \in \Pi_\phi(G_D)} \dim(\chi_{\pi_D})m(\pi_D)$$
$$= m_{geom}(\theta_{\Pi_\phi(G)}) + m_{geom}(\theta_{\Pi_\phi(G_D)}) = c_{\theta_{\Pi_\phi(G)}}(1) = 1$$

where the last equality follows from the results for Whittaker model in [Rod81], [Mat] and the fact that there is a unique generic element in the L-packet. In particular, we have proved that the summation of the multiplicities is equal to 1 over every tempered local Vogan L-packet and the unique distinguished element corresponds to a character of the component group.

9.2.2. *The model* $(\mathrm{GU}_4\times\mathrm{GU}_2,(\mathrm{GU}_2\times\mathrm{GU}_2)^0)$. Let $(G,H)=(\mathrm{GU}_{2,2}\times\mathrm{GU}_{1,1},(\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1})^0)$, and

$$(G_1,H_1)=(\mathrm{GU}_{2,2}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{0,2})^0),$$
$$(G_2,H_2)=(\mathrm{GU}_{3,1}\times\mathrm{GU}_{1,1},(\mathrm{GU}_{1,1}\times\mathrm{GU}_{2,0})^0),$$
$$(G_3,H_3)=(\mathrm{GU}_{3,1}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{1,1})^0),$$
$$(G_4,H_4)=(\mathrm{GU}_{4,0}\times\mathrm{GU}_{2,0},(\mathrm{GU}_{2,0}\times\mathrm{GU}_{2,0})^0)$$

be the pure inner forms (the pair (G_4,H_4) only appears in the archimedean case).

Let T_0 be the unique element in $\mathcal{T}_{ell}(\mathrm{GU}_{1,1})=\mathcal{T}_{ell}(\mathrm{GU}_{2,0})$ that is isomorphic to

$$E^{2,0}:=\{(a,b)\in E^\times\times E^\times|\ a\bar{a}=b\bar{b}\}\subset E^\times\times E^\times.$$

For $T\in\mathcal{T}_{ell}(\mathrm{GU}_{1,1})=\mathcal{T}_{ell}(\mathrm{GU}_{2,0})$ with $T\neq T_0$, let

$$(T\times T)^0=\{(t_1,t_2)\in T\times T|\ \lambda(t_1)=\lambda(t_2)\}.$$

Up to conjugation, there is a unique embedding from $(T\times T)^0$ to $(\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1})^0$ (resp. $(\mathrm{GU}_{2,0}\times\mathrm{GU}_{0,2})^0$). Combining with the diagonal embedding from T to $(T\times T)^0$, we get an embedding (denoted by ν_T) from T to G (resp. $G_1(F)$) that factors through H (resp. H_1), and we will denote this embedding by ν_T (resp. $\nu_{1,T}$).

For T_0, in the p-adic case up to conjugation there are two embeddings from $(T_0\times T_0)^0$ to $(\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1})^0$ (resp. $(\mathrm{GU}_{2,0}\times\mathrm{GU}_{0,2})^0$). Combining with the diagonal embedding from T_0 to $(T_0\times T_0)^0$, we get two embeddings from T_0 to G (resp. G_1). The centralizer of the image of one of the embedding is quasi-split (it is isomorphic to $(\mathrm{GU}_{1,1}\times\mathrm{GU}_{1,1})^0$ times a torus), we will denote this embedding by ν_{T_0} (resp. ν_{1,T_0}), while the centralizer of the image of the other embedding is not quasi-split. In the archimedean case, we can define the embedding ν_{T_0} in the same way as in the p-adic case. On the other hand, up to conjugation there is only one embedding from $(T_0\times T_0)^0$ to $(\mathrm{GU}_{2,0}\times\mathrm{GU}_{0,2})^0$ and this defines the embedding ν_{1,T_0}. Note that in this case the centralizer of the image of ν_{1,T_0} is still quasi-split.

Remark 9.3. *For $T\in\mathcal{T}_{ell}(\mathrm{GU}_{1,1})$, we can also define the embeddings to G_2 and G_3 (also G_4 in the archimedean case), but the centralizer of the images will not be quasi-split.*

Meanwhile, consider the following two subgroups of $(T_0\times T_0)^0$ (we identify T_0 with $E^{2,0}=\{(a,b)\in E^\times\times E^\times|\ a\bar{a}=b\bar{b}\}$):

$$T_0'=\{(1,1)\times(1,a)\in(T_0\times T_0)^0|\ a\in E^1\},$$
$$T_0''=\{(1,a)\times(1,b)\in(T_0\times T_0)^0|\ a,b\in E^1\}.$$

The two embeddings from $(T_0\times T_0)^0$ to $(\mathrm{GU}_{1,1}\times \mathrm{GU}_{1,1})^0$ (resp. $(\mathrm{GU}_{1,1}\times \mathrm{GU}_{2,0})^0$) induce two embeddings from T_0' to G (resp. G_2) that are conjugated to each other. Let $\nu_{T_0'}$ (resp. $\nu_{2,T_0'}$) be one of the embedding. Note that the projection of these embeddings to the $\mathrm{GU}_{1,1}$-factor is the trivial map. The centralizers of the image of these embeddings are quasi-split (they are isomorphic to $\mathrm{GU}_3\times \mathrm{GU}_{1,1}\times U_1$).

Remark 9.4. *We can also define embeddings from T_0' to G_1 and G_3 (also G_4 in the archimedean case), but the centralizer of the images will not be quasi-split.*

On the other hand, the two embeddings from $(T_0\times T_0)^0$ to $(\mathrm{GU}_{1,1}\times \mathrm{GU}_{1,1})^0$ induce two embeddings from T_0'' to G. The centralizer of the image of one of the embedding is quasi-split (isomorphic to $\mathrm{GU}_{1,1}$ times some torus, we will denote this embedding by $\nu_{T_0''}$) and the centralizer of the image of the other embedding is not quasi-split. Similarly, we can also define the embeddings $\nu_{i,T_0''}$ from T_0'' to G_i for $1\leq i\leq 3$.

Remark 9.5. *We can also define the embedding from T_0'' to G_4 in the archimedean case but the centralizer of the images will not be quasi-split.*

Now we are ready to define the geometric multiplicity. Let π (resp. π_i) be an irreducible representation of $G(F)$ (resp. $G_i(F)$) with trivial central character. For $T\in\mathcal{T}_{ell}(\mathrm{GU}_{1,1})=\mathcal{T}_{ell}(\mathrm{GU}_{2,0})$, we use $T^*(F)$ to denote $T(F)/Z_{\mathrm{GU}_{1,1}}(F)=T(F)/Z_{\mathrm{GU}_{2,0}}(F)$. Define

$$\begin{aligned} m_{geom}(\pi) \;=\;& c_\pi(1)+\sum_{T\in\mathcal{T}_{ell}(H)}|W(H,T)|^{-1}\int^*_{T(F)/Z_{G,H}(F)} D^H(t)\theta_\pi(t)\,\mathrm{d}t\\ &+\frac{1}{2}\sum_{T\in\mathcal{T}_{ell}(\mathrm{GU}_{1,1})}\int^*_{T^*(F)} D^H(\nu_T(t))c_\pi(\nu_T(t))\,\mathrm{d}t\\ &+\int^*_{T_0'(F)} D^H(\nu_{T_0'}(t))c_\pi(\nu_{T_0'}(t))\,\mathrm{d}t\\ &+\int^*_{T_0''(F)} D^H(\nu_{T_0''}(t))c_\pi(\nu_{T_0''}(t))\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} m_{geom}(\pi_1) &= \sum_{T\in\mathcal{T}_{ell}(H_1)} |W(H_1,T)|^{-1} \int_{T(F)/Z_{G_1,H_1}(F)}^{*} D^{H_1}(t)\theta_{\pi_1}(t)\,\mathrm{d}t \\ &\quad +\frac{1}{2}\sum_{T\in\mathcal{T}_{ell}(\mathrm{GU}_{2,0})} \int_{T^*(F)}^{*} D^{H_1}(\nu_{1,T}(t))c_{\pi_1}(\nu_{1,T}(t))\,\mathrm{d}t \\ &\quad + \int_{T_0''(F)}^{*} D^{H_1}(\nu_{1,T_0''}(t))c_{\pi_1}(\nu_{1,T_0''}(t))\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} m_{geom}(\pi_2) &= \sum_{T\in\mathcal{T}_{ell}(H_2)} |W(H_2,T)|^{-1} \int_{T(F)/Z_{G_2,H_2}(F)}^{*} D^{H_2}(t)\theta_{\pi_2}(t)\,\mathrm{d}t \\ &\quad + \int_{T_0'(F)}^{*} D^{H_2}(\nu_{2,T_0'}(t))c_{\pi_2}(\nu_{2,T_0'}(t))\,\mathrm{d}t \\ &\quad + \int_{T_0''(F)}^{*} D^{H_2}(\nu_{2,T_0''}(t))c_{\pi_2}(\nu_{2,T_0''}(t))\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} m_{geom}(\pi_3) = &\sum_{T\in\mathcal{T}_{ell}(H_3)} |W(H_3,T)|^{-1} \int_{T(F)/Z_{G_3,H_3}(F)}^{*} D^{H_3}(t)\theta_{\pi_3}(t)\,\mathrm{d}t \\ &+ \int_{T_0''(F)}^{*} D^{H_3}(\nu_{3,T_0''}(t))c_{\pi_3}(\nu_{3,T_0''}(t))\,\mathrm{d}t. \end{aligned}$$

If we are in the archimedean case, we also define

$$m_{geom}(\pi_4) = \sum_{T\in\mathcal{T}_{ell}(H_4)} |W(H_4,T)|^{-1} \int_{T(F)/Z_{G_4,H_4}(F)}^{*} D^{H_4}(t)\theta_{\pi_4}(t)\,\mathrm{d}t.$$

Like in the previous case, we always choose the Haar measure so that the total volume is equal to 1 and the extra $\frac{1}{2}$ factor comes from the cardinality of the Weyl group of GU_2. Also the integrals in the geometric multiplicity may not be absolutely convergent and they need to be regularized (see Remark 9.1). We leave it as an excise for the reader to check that our definition of $m_{geom}(\pi)$ matches the definition in [Wan] for general spherical varieties. Like the previous case, by a similar but easier argument as in the Gan–Gross–Prasad model case ([W10], [W12], [B15]) and the Ginzburg–Rallis model case ([Wan15], [Wan16], [Wan17], [WZ]), we can prove the following theorem.

Theorem 9.6. *For all tempered representations π of $G(F)$ (resp. π_i of $G_i(F)$) whose central character is trivial on $Z_{G,H}(F)$ (resp. $Z_{G_i,H_i}(F)$), we have*

$$m(\pi) = m_{geom}(\pi),\ m(\pi_i) = m_{geom}(\pi_i).$$

Now let $\Pi_\phi=\Pi_\phi(G)\cup\Pi_\phi(G_i)$ be a tempered local L-packet whose central character is trivial on $Z_{G,H}(F)$ ($1\leq i\leq 3$ in the p-adic case and $1\leq i\leq 4$ in the archimedean case). We can also define the character $\theta_{\Pi_\phi(G)}$ and $\theta_{\Pi_\phi(G_i)}$ as before (note that the component group is always abelian in this case). The summation $\sum_{\pi\in\Pi_\phi(G)}m(\pi)+\sum_{1\leq i\leq k,\pi_i\in\Pi_\phi(G_i)}m(\pi_i)$ is equal to

$$m_{geom}(\theta_{\Pi_\phi(G)})+\sum_{i=1}^{k}m_{geom}(\theta_{\Pi_\phi(G_i)})=c_{\theta_{\Pi_\phi(G)}}(1)=1$$

where $k=3$ in the p-adic case and $k=4$ in the archimedean case. Here the last equality follows from the results for Whittaker model in [Mat], [MW] and the fact that there is a unique generic element in the L-packet. For the identity

$$m_{geom}(\theta_{\Pi_\phi(G)})+\sum_{i=1}^{k}m_{geom}(\theta_{\Pi_\phi(G_i)})=c_{\theta_{\Pi_\phi(G)}}(1),$$

we just need to apply the following cancellations (the Kottwitz sign between G and G_3 is equal to 1, the Kottwitz sign between G and G_i is equal to -1 for $i=1,2,4$)

- The term $\sum_{T\in\mathcal{T}_{ell}(H)}$ in $m_{geom}(\theta_{\Pi_\phi(G)})$ plus the term $\sum_{T\in\mathcal{T}_{ell}(H_3)}$ in $m_{geom}(\theta_{\Pi_\phi(G_3)})$ can be cancelled with the term $\sum_{T\in\mathcal{T}_{ell}(H_1)}$ in $m_{geom}(\theta_{\Pi_\phi(G_1)})$ plus the term $\sum_{T\in\mathcal{T}_{ell}(H_2)}$ in $m_{geom}(\theta_{\Pi_\phi(G_2)})$ (and also plus the term $\sum_{T\in\mathcal{T}_{ell}(H_4)}$ in $m_{geom}(\theta_{\Pi_\phi(G_4)})$ if we are in the archimedean case).
- The term $\frac{1}{2}\sum_{T\in\mathcal{T}_{ell}(\mathrm{GU}_{1,1})}$ in $m_{geom}(\theta_{\Pi_\phi(G)})$ can be cancelled with the term $\frac{1}{2}\sum_{T\in\mathcal{T}_{ell}(\mathrm{GU}_{2,0})}$ in $m_{geom}(\theta_{\Pi_\phi(G_1)})$.
- The term associated to T_0' in $m_{geom}(\theta_{\Pi_\phi(G)})$ can be cancelled with the term associated to T_0' in $m_{geom}(\theta_{\Pi_\phi(G_2)})$.
- The terms associated to T_0'' in $m_{geom}(\theta_{\Pi_\phi(G)})$ and $m_{geom}(\theta_{\Pi_\phi(G_3)})$ can be cancelled with the terms associated to T_0'' in $m_{geom}(\theta_{\Pi_\phi(G_1)})$ and $m_{geom}(\theta_{\Pi_\phi(G_2)})$.

In particular, we have proved that the summation of the multiplicities is equal to 1 over every tempered local Vogan L-packet.

9.3. **The non-reductive case.** In this subsection we consider the non-reductive cases. Let $(G,H)=(G,H_0\ltimes U)$ be one of the non-reductive models in Table 1. For all the cases, $H_0(F)$ is essentially $\mathrm{GL}_2(F)$ (up to the center). If $F\neq\mathbb{C}$, we let $(G_D,H_{0,D}\ltimes U_D)$ be the quaternion version of the model.

Let $\mathcal{T}_{ell}(H_0)$ (resp. $\mathcal{T}_{ell}(H_{0,D})$) be a set of representatives of maximal elliptic tori of $H_0(F)$ (resp. $H_{0,D}(F)$). Define

$$m_{geom}(\pi) = c_\pi(1) + \sum_{T\in\mathcal{T}_{ell}(H_0)} |W(H_0,T)|^{-1} \int_{T(F)/Z_{G,H}(F)} D^H(t)c_\pi(t)\,\mathrm{d}t,$$

$$\begin{aligned} m_{geom}(\pi_D) \;=\; & \sum_{T_D\in\mathcal{T}_{ell}(H_{0,D})} |W(H_{0,D},T_D)|^{-1} \\ & \cdot \int_{T_D(F)/Z_{G_D,H_D}(F)} D^{H_D}(t)c_{\pi_D}(t)\,\mathrm{d}t \end{aligned}$$

where π (resp. π_D) is an irreducible admissible representation of $G(F)$ (resp. $G_D(F)$) with trivial central character, $W(H_0,T)$ (resp. $W(H_{0,D},T_D)$ is the Weyl group, and all the Haar measure are chosen so that the total volume is equal to 1. Again we leave it as an excise for the reader to check that our definition of $m_{geom}(\pi)$ matches the definition in [Wan] for general spherical varieties.

Theorem 9.7. *Assume that $F\neq\mathbb{R}$, and (G,H) is not the last model $(E_7,\mathrm{PGL}_2\ltimes U)$ in Table 1. For all tempered representations π of $G(F)$ (resp. π_D of $G_D(F)$) whose central character is trivial on $Z_{G,H}(F)$ (resp. $Z_{G_D,H_D}(F)$), we have*

$$m(\pi)=m_{geom}(\pi),\ m(\pi_D)=m_{geom}(\pi_D).$$

Proof. The multiplicity formula for Model 4 in Table 1 has been proved in the previous papers of the first author ([Wan15], [Wan16], [Wan17]), and the multiplicity formula for the Model 5 has been proved in our previous paper [WZ]. The argument for the remaining 4 models is very similar to the Ginzburg–Rallis model case ([Wan15], [Wan16], [Wan17]), we will skip it here. Like the reductive case, the Gelfand pair condition is not known for these models, but it can be solved by the same argument as the unitary Ginzburg-–Rallis model case in our previous paper (Section 6 and Appendix A of [WZ]).

The reason we need to assume that $F\neq\mathbb{R}$ is that in the case when $F=\mathbb{R}$, we don't know how to prove the nonvanishing property of certain explicit intertwining operator is invariant under the parabolic induction because the operator is defined by a normalized integral in the non-reductive case and it is not clear how to study it under the parabolic induction in the real case. In Gan–Gross–Prasad case (Section 7.4 of [B15]), this can be solved by passing to a reductive model of a larger group (e.g. instead of studying $(U_{n+2k+1}\times U_n, U_n\ltimes N)$ one can just study (U_{n+2k+1},U_{n+2k})). But for all the cases in Table 1, we cannot pass it to a reductive model of a larger group simply because such a module does not exist. For Model 4 in Table 1, we solved this issue by using a special property that all the tempered representations of $\mathrm{GL}_6(\mathbb{R})$ are the parabolic induction of some tempered representations of $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$, see Section 5.4 of [Wan16] for details. But this is not true for Models 5–10 of Table 1 (although it is still true in the complex case which is why we can prove the multiplicity formula in the complex case). In general if one can prove that the nonvanishing property of the explicit intertwining operator is invariant under the parabolic induction, then we can also prove the multiplicity formula in the real case.

On the other hand, the reason we exclude the model $(E_7, \mathrm{PGL}_2 \ltimes U)$ is that in the proof of the geometric side of the trace formula, we need to study the slice representation, i.e. the conjugation action of $H(F)$ on the tangent space. We need to show that the regular orbits coincide with the stable conjugacy classes of $G(F)$. For all the other cases, this can be down by computing the characteristic polynomials as in the Gan–Gross–Prasad model case (Section 9 of [W10] and Section 10 of [B15]) and the Ginzburg–Rallis model case (Section 8 of [Wan15]). But this is not possible for the E_7 case since the matrix presentation of E_7 is very complicated. If one can prove this result for the model $(E_7, \mathrm{PGL}_2 \ltimes U)$, then we can also prove the multiplicity formula in this case.

$\square$

As in the reductive cases, combining the multiplicity formulas and the local Langlands conjecture, we can show that for any tempered L-packet $\Pi_\phi = \Pi_\phi(G) \cup \Pi_\phi(G_D)$ of $G(F)$ whose central character is trivial on $Z_{G,H}(F)$, the summation

$$\sum_{\pi \in \Pi_\phi(G)} \dim(\chi_\pi) m(\pi) + \sum_{\pi_D \in \Pi_\phi(G_D)} \dim(\chi_{\pi_D}) m(\pi_D)$$

is equal to

$$m_{geom}(\theta_{\Pi_\phi(G)}) + m_{geom}(\theta_{\Pi_\phi(G_D)}) = c_{\theta_{\Pi_\phi(G)}}(1) = 1.$$

In other words, the summation of the multiplicities is equal to 1 over every tempered local Vogan L-packet and the unique distinguished element corresponds to a character of the component group.

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