STRONGLY TEMPERED BZSV QUADRUPLES

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Abstract. In this paper, we give a list of strongly tempered BZSV quadruples. This gives a conceptual explanation of many existing Rankin-Selberg integrals and period integrals. It also proposes many new interesting period integrals to study.

1. Introduction

1.1. **BZSV Duality.** In [1], Ben-Zvi, Sakellaridis, and Venkatesh proposed a beautiful relative Langlands duality for spherical varieties (in this paper, we will call it BZSV duality). We briefly recall the datum in the duality. Throughout this paper, k is a global field, $\mathbb{A} = \mathbb{A}_k$, F is a local field, and ψ is a non-trivial additive character of $\mathbb{A}/k$ (resp. F) if we are in the global (resp. local) setting. The BZSV duality concerns a pair of dual data $(\Delta, \hat{\Delta})$ where each side contains 4 datum: $\Delta = (G, H, \rho_H, \iota)$ and $\hat{\Delta} = (\hat{G}, \hat{H}', \rho_{\hat{H}'}, \hat{\iota}')$. Here G is a split reductive group; H is a split reductive subgroup of G; ρ_H is a symplectic representation of H; and ι is a homomorphism from SL_2 into G whose image commutes with H. The map ι induces a homomorphism $H \times \mathrm{SL}_2 \to G$, which will still be denoted by ι. This map induces an adjoint action of $H \times \mathrm{SL}_2$ on the Lie algebra $\mathfrak{g}$ of G and we can decompose it as

$$\oplus_{k\in I} \rho_k \otimes Sym^k$$

where ρ_k is some representation of H and I is a finite subset of $\mathbb{Z}_{\geq 0}$. We let I_{odd} be the subset of I containing all the odd numbers. There are two main requirements for the quadruple (G, H, ρ_H, ι).

(1) The representation $\rho_{H,\iota} = \rho_H \oplus (\oplus_{i\in I_{odd}} \rho_i)$ is a symplectic anomaly-free representation (see Section 5 of [1]) of H.
(2) The Hamiltonian space associated to the quadruple (G, H, ρ_H, ι) (defined in Section 3 of [1]) is hyperspherical ([1, Section 3.5]). In particular, its generic stabilizer is connected.

We refer the reader to [1] for more details. Note that under BZSV duality, the group $\hat{G}$ is the Langlands dual group of G and $\hat{H}' = \hat{G}_\Delta$ can be viewed as the "dual group" of the quadruple Δ (note that the groups H and $\hat{H}'$ are not dual to each other in general, and the nilpotent orbits ι and $\hat{\iota}'$ are also not dual to each other in general). We recall the conjecture about period integrals in the BZSV duality.

Let $\Delta = (G, H, \rho_H, \iota)$ and $\hat{\Delta} = (\hat{G}, \hat{H}', \rho_{\hat{H}'}, \hat{\iota}')$ be two quadruples that are dual to each other under the BZSV duality. We use $\rho_{H,\iota}$ and $\rho_{\hat{H}',\hat{\iota}'}$ to denote the symplectic anomaly-free representations associated to these quadruples. As we explained above, the maps ι and $\hat{\iota}'$ induce adjoint actions of $H \times \mathrm{SL}_2$ (resp. $\hat{H}' \times \mathrm{SL}_2$) on $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) and they can be decomposed

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as

$$\mathfrak{g} = \oplus_{k\in I}\rho_k \otimes Sym^k,\ \hat{\mathfrak{g}} = \oplus_{k\in \hat{I}}\hat{\rho}_k \otimes Sym^k$$

where ρ_k (resp. $\hat{\rho}_k$) are representations of H (resp. $\hat{H}'$). It is clear that the adjoint representation of H (resp. $\hat{H}'$) is a subrepresentation of ρ_0 (resp. $\hat{\rho}_0$).

For an automorphic form ϕ of $G(\mathbb{A})$ (resp. $\hat{G}(\mathbb{A})$), we can define the period integral $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ (resp. $\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)$) of it associated to the quadruple. Let's briefly recall the definition. We have a symplectic representation $\rho_{H,\iota} : H \to \mathrm{Sp}(V)$. Let Y be a maximal isotropic subspace of V and Ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(V)$ on the Schwartz space $\mathcal{S}(Y(\mathbb{A}))$. The anomaly free condition on $\rho_{H,\iota}$ ensures $\widetilde{\mathrm{Sp}}(V)$ splits over $Im(\rho_{H,\iota})$ and Ω_ψ restricts to a representation of $H(\mathbb{A})$ on $\mathcal{S}(Y(\mathbb{A}))$. We define the theta series

$$\Theta_\psi^\varphi(h) = \sum_{X\in Y(k)} \Omega_\psi(h)\varphi(X),\ h \in H(\mathbb{A}), \varphi \in \mathcal{S}(Y(\mathbb{A})),$$

and we can define the period integral to be

$$\mathcal{P}_{H,\iota,\rho_H}(\phi,\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \mathcal{P}_\iota(\phi)(h)\Theta_\psi^\varphi(h)dh.$$

Here $\mathcal{P}_\iota$ is the degenerate Whittaker period associated to ι (we refer the reader to Section 1.2 of [27] for its definition). To simplify the notation, we will omit the Schwartz function in the notion of the period and simply write it as $\mathcal{P}_{H,\iota,\rho_H}(\phi,\varphi)$ [1]. Similarly we can also define the period integral $\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)$. The following conjecture is the main conjecture regarding global periods in BZSV duality.

Conjecture 1.1. *(Ben-Zvi–Sakellaridis–Venkatesh, [1])*

(1) Let π be an irreducible discrete automorphic representation of $G(\mathbb{A})$ and let $\nu : \pi \to L^2(G(k)\backslash G(\mathbb{A}))_\pi$ be an embedding. Then the period integral

$$\mathcal{P}_{H,\iota,\rho_H}(\phi),\ \phi \in Im(\nu)$$

is nonzero only if the Arthur parameter of π factors through $\hat{\iota}' : \hat{H}'(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C})$. If this is the case, π is a lifting of a global tempered Arthur packet Π of $H'(\mathbb{A})$ (the Langlands dual group of $\hat{H}'$). Then we can choose the embedding ν so that

$$\frac{|\mathcal{P}_{H,\iota,\rho_H}(\phi)|^2}{\langle\phi,\phi\rangle} \text{“} = \text{”} \frac{L(1/2,\Pi,\rho_{\hat{H}'}) \cdot \Pi_{k\in\hat{I}} L(k/2+1,\Pi,\hat{\rho}_k)}{L(1,\Pi,Ad)^2},\ \phi \in Im(\nu).$$

Here $\langle,\rangle$ is the L^2-norm, and “ = ” means the equation holds up to some Dedekind zeta functions, some global constant determined by the component group of the global L-packet associated to π, and some finite product over the ramified places (including all the archimedean places).

(2) Let π be an irreducible discrete automorphic representation of $\hat{G}(\mathbb{A})$ and let $\nu : \pi \to L^2(\hat{G}(k)\backslash \hat{G}(\mathbb{A}))_\pi$ be an embedding. Then the period integral

$$\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi),\ \phi \in Im(\nu)$$

[1] when the nilpotent orbit associated to ι is not even, the degenerate Whittaker period $\mathcal{P}_\iota$ is a Fourier-Jacobi coefficient and one also need to include an extra Schwartz function in its definition

is nonzero only if the Arthur parameter of π factors through $\iota: H(\mathbb{C})\times \mathrm{SL}_2(\mathbb{C})\to G(\mathbb{C})$. If this is the case, π is a lifting of a global tempered Arthur packet Π of $\hat{H}(\mathbb{A})$ (the Langlands dual of H). Then we can choose the embedding ν so that

$$\frac{|\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)|^2}{\langle\phi,\phi\rangle}\text{“}=\text{”}\frac{L(1/2,\Pi,\rho_H)\cdot\Pi_{k\in I}L(k/2+1,\Pi,\rho_k)}{L(1,\Pi,Ad)^2},\ \phi\in Im(\nu).$$

Remark 1.2. *The above conjecture is usually called the Ichino-Ikeda type conjecture. To state an explicit identity instead of "$=$", one needs to make two adjustments on the right-hand side of the equation.*

- *In the ramified places, instead of using the local L-function, one needs to use the so-called local relative character defined by the (conjectural) Plancherel decomposition (see Section 17 of* [32] *and Section 9 of* [1]*).*
- *One also needs to add some Dedekind zeta functions on the right-hand side determined by the groups G and H (in all the known examples, those zeta functions are the L-function of the dual $M^\vee$ to the motive M associated to G,H introduced by Gross in* [17]*), as well as some global constant determined by component group of the global L-packet associated to π (see Section 14.6.4 of* [1]*) for these two quadruples.*

Remark 1.3. *In* [1]*, they also formulated many other conjectures for the duality (i.e., local/global geometric conjecture, local conjecture for Plancherel decomposition). The expectation is that those conjectures would uniquely determine the duality. In this paper we will only focus on their conjecture for period integrals. We also want to point out that given a general BZSV quadruple $\Delta=(G,H,\rho_H,\iota)$, at this moment there is no algorithm to compute the dual quadruple $\hat{\Delta}$. The only exception is for the so-called polarized case (i.e., when $\rho_H=0$) where the algorithm is given in Section 4 of* [1] *(most quadruples considered in this paper are not polarized). As a result, given two BZSV quadruples Δ and $\hat{\Delta}$, at this moment one can only provide evidence for the duality between them by studying the various conjectures (i.e., local/global geometric conjecture, local conjecture for Plancherel decomposition, global conjecture for period integrals) in* [1].

1.2. **Strongly tempered BZSV quadruples.**

Definition 1.4. *We say the quadruple $\Delta=(G,H,\rho_H,\iota)$ is strongly tempered if $\hat{G}=\hat{H}'Z_{\hat{G}}$, i.e. the "dual group" of Δ is equal to the dual group of G up to center. We say the quadruple is reductive if ι is trivial.*

If the quadruple $\Delta=(G,H,\rho_H,\iota)$ is strongly tempered, then Conjecture 1.1(1) states that for all global tempered L-packet Π of $G(\mathbb{A})$ [2], there exists $\pi\in\Pi$ and $\nu:\pi\to L^2(G(k)\backslash G(\mathbb{A}))_\pi$ such that

$$\frac{|\mathcal{P}_{H,\iota,\rho_H}(\phi)|^2}{\langle\phi,\phi\rangle}\text{“}=\text{”}\frac{L(1/2,\Pi,\rho_{\hat{H}'})}{L(1,\Pi,Ad)},\ \phi\in Im(\nu). \tag{1.1}$$

In other words, it means that the period integral $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ is essentially equal to the central value of an automorphic L-function on every tempered global L-packet.

[2] when $\hat{G}\neq\hat{H}'$, we need to make some assumptions on the central character of Π so that its Langlands parameter factors through $\hat{H}'$

The most well-known example of strongly tempered quadruple is the Gross-Prasad model $(G,H,\rho_H,\iota)=(\mathrm{SO}_{2n+1}\times\mathrm{SO}_{2n},\mathrm{SO}_{2n},0,1)$. In this case the dual quadruple is given by

$$(\hat{G},\hat{G},\hat{\rho},1)=(\mathrm{Sp}_{2n}\times\mathrm{SO}_{2n},\mathrm{Sp}_{2n}\times\mathrm{SO}_{2n},std_{\mathrm{Sp}_{2n}}\otimes std_{\mathrm{SO}_{2n}},1).$$

In this case, Conjecture 1.1(1) is just the Ichino-Ikeda conjecture in [19] and Conjecture 1.1(2) is just the Rallis inner product formula for the theta correspondence between Sp_{2n} and SO_{2n}.

Remark 1.5. *Conjecturally the quadruple is strongly tempered if and only if the integral*

$$\int_{H(F)}\mathcal{P}_\iota(\phi)(h)\varphi(h)dh \tag{1.2}$$

is absolutely convergent for all tempered matrix coefficient ϕ of $G(F)$. Here $F=k_v$ is a local field for some $v\in|k|$, $\mathcal{P}_\iota$ is the local analogue of the global degenerate Whittaker period, and $\varphi(h)$ is a matrix coefficient of the local Weil representation of $H(F)$ associated to the symplectic representation ρ_H (although the unipotent integral $\mathcal{P}_\iota$ is not necessarily convergent and it needs to be regularized, see examples in [2, 25, 34, 35, 36]*). It is easy to check for all cases in Table 21 – 26, the above integral is absolutely convergent. In these cases, the local relative character in Remark 1.2 is given by the integral* (1.2) *where ϕ is the matrix coefficient of π_v; and π_v is the local component of π at v which is a tempered representation of $G(F)$.*

In [27], we proposed a relative trace formula comparison that relates the periods $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ on G for a BZSV quadruple (G,H,ρ_H,ι) to the periods $\mathcal{P}_{H_0,\iota_0,\rho_{H_0}}(\phi_0)$ on G_0 for a strongly tempered BZSV quadruple $(G_0,H_0,\rho_{H_0},\iota_0)$. Thus it is natural to consider Conjecture 1.1 first for the strongly tempered BZSV quadruples. The goal of this paper is to provide and study a list of strongly tempered BZSV quadruples.

By duality, in order to classify the strongly tempered quadruple Δ, it is enough to classify its dual quadruple

$$\hat{\Delta}=(\hat{G},\hat{H}',\hat{\rho},1).$$

Since $\hat{H}'Z_{\hat{G}}=\hat{G}$, it is enough to classify all the BZSV quadruples of the form

$$(\hat{G},\hat{G},\hat{\rho},1).$$

By [1], a quadruple $\hat{\Delta}=(\hat{G},\hat{G},\hat{\rho},1)$ is a BZSV quadruple if it satisfies the following three conditions.

1. The symplectic representation $\hat{\rho}$ is anomaly-free (see [1, Section 5]).
2. The symplectic representation $\hat{\rho}$ is multiplicity free.
3. The generic stabilizer of the representation $\hat{\rho}$ of $\hat{G}$ is connected.

In [22], Knop gave a classification of multiplicity-free symplectic representations. By [22, Theorem 2.3], the classification is reduced to that of symplectic representations that are *saturated* and multiplicity free, which are listed in Table 1, 2, 11, 12, 22, S of [22]. In this paper we write down the strongly tempered quadruples that are (up to isogeny) the duals of $(\hat{G},\hat{G},\hat{\rho},1)$ when $\hat{\rho}$ is the symplectic representations listed in Knop's tables except for Table S. Currently we use an ad hoc method to determine the data ρ_H, which is why we can not handle the infinite family of representations given by Table S, although the choice of H and ι is systematic and applies to Table S as well (see Property 2.11).

Remark 1.6. *Condition (3) above is related to the Type N spherical root. Whenever this condition fails, we should expect some covering group to appear in the dual quadruple* $\Delta = (G, H, \rho_H, \iota)$. *This is not covered in BZSV's framework at this moment. Nonetheless, for the cases in* [22] *that do not satisfy (3), we are still able to write down a candidate for the dual of the quadruple* $\hat{\Delta}$ *and we can provide evidence for the conclusions in Conjecture 1.1 under the duality.* [3].

1.3. **Statement of main results.** We consider all quadruples $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ satisfy the following two conditions:

(1) The symplectic representation $\hat{\rho}$ is anomaly-free.
(2) The symplectic representation $\hat{\rho}$ appears in Table 1, 2, 11, 12, 22 of [22].

For each of them, we will write down a quadruple $\Delta = (G, H, \rho_H, \iota)$ and claim it is dual to $\hat{\Delta}$ up to isogeny, or more precisely it is dual to $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ where $Z_\Delta = Z_G \cap ker(\rho_H)$ and Z_G is the center of G. To support the claim we provide evidence through the three main theorems below. Our results are summarized in the 6 tables at the end of this paper (Table 21, 22, 23, 24, 25 and 26, the first two tables are for reductive cases while the last four tables are for non-reductive cases).

Theorem 1.7. *For all the reductive cases (Table 21 and 22) except the quadruple* $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$*, and for all quadruples in Table 23 and 24, the local relative character of the period integral* $\mathcal{P}_{H,\rho_H,\iota}$ *is equal to the* $L-$*value in Conjecture 1.1(1) at unramified places, namely equals* $\frac{L(1/2,\Pi,\hat{\rho})}{L(1,\Pi,Ad)}$ *for the unramified representation* Π.

Recall that the local relative character at unramified places is defined in (1.2) with ϕ and φ being unramified matrix coefficients normalized to be 1 at identity, and with suitably chosen Haar measures.

Remark 1.8. *For the quadruple* $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$ *and for all quadruples in Table 25 and 26, as far as we know, their local relative characters have not been computed at unramified places. Although we believe they can be computed by the same method as in* [19] *and* [36].

Theorem 1.9. *For the quadruples in Table 21, 23 and 25, Conjecture 1.1(2) holds, if we assume (when applicable) the global period integral conjectures in* [7, 8, 19] *for Gan-Gross-Prasad models.*

Remark 1.10. *In most cases for Theorem 1.9 and some cases for Theorem 1.7 we utilize the theta correspondence. We summarize the results needed for theta correspondence in Section 2.2.*

Remark 1.11. *In* [8]*, the authors only formulated a global conjecture regarding the non-vanishing of the period integrals for non-tempered Arthur L-packets (Conjecture 9.11 of* [8]*). An Ichino-Ikeda type conjecture for the period is not available in* [8] *because of the difficulty in the definition of local relative character in the non-tempered case (see the last paragraph of Section 9 of* [8]*). Thus strictly speaking, for some cases in Theorem 1.9 we can only claim the*

[3]There are two of such cases in Knop's table: one relates to the symmetric cube representation of SL_2 and the other one relates to the two copies of the symmetric square representation of SL_n, we refer the reader to Section 3.1 and 4.1 for details. In this paper, we will not check the connectedness condition for representations in [22], we will leave it as an exercise for the reader.

nonvanishing part of Conjecture 1.1(2). However the identity in Conjecture 1.1(2) disregards the local factors at bad places, thus to prove it we only need an Ichino-Ikeda type conjecture without specifying the local factors at bad places. The formulation of such a conjecture is well known and we assume this version of the conjecture in Theorem 1.9.

We say duality holds for a quadruple in the tables 21–26, if it is the dual of a quadruple $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ coming from the corresponding entry in Knop's tables. Beside the above two theorems, we provide one further evidence for the duality for all the non-reductive quadruples. In the next section, we will introduce a notion of Whittaker induction and we will show that any non-reductive quadruple is the Whittaker induction of a reductive quadruple. We will also make a conjecture about the BZSV duality under Whittaker induction (i.e. Conjecture 2.10) which generalizes the conjecture in Section 3.4 of [1].

Theorem 1.12. *Any quadruple (G, H, ρ_H, ι) in Table 23, 24, 25 and 26 is a Whittaker induction of a reductive quadruple $(G_0, H, \rho'_H, 1)$ in Table 21 and 22.*

Assume the duality holds for the reductive quadruple $(G_0, H, \rho'_H, 1)$, then Conjecture 2.10 holds for the quadruple $\Delta = (G, H, \rho_H, \iota)$ if and only if the duality holds for Δ.

Remark 1.13. *Most of the quadruples in Table 21 and 22 come from Tables 1, 11, 2, 12, 22 of* [22]*. There are some exceptions; the quadruples given in* (5.5), (6.3), (6.4), (7.7) *and* (7.8) *are strongly tempered and dual to $\hat{\rho}$ from Table S in* [22].

Remark 1.14. *For quadruples in Table 23, 24 and 25, Theorem 1.7 and 1.9 already provide strong evidence for the duality of (G, H, ρ_H, ι). Combining with Theorem 1.12, we get strong evidence of Conjecture 2.10 for quadruples in these three tables.*

Remark 1.15. *Historically Whittaker induction plays an important role in the study of period integrals. Many interesting L-function can be obtained by studying the period integral of the Whittaker induction of some spherical varieties (e.g. the Shalika model, and the models in* [36]*). Most prior examples of the Whittaker inductions are of Bessel type, but in this paper we would also need the Whittaker induction of Fourier-Jacobi type (see next section for definition of Whittaker induction).*

In this paper, we provide the evidence of duality mainly through the period integral aspect, i.e., Conjecture 1.1. As we mentioned in Remark 1.3, there are other ways to justify the duality, for example from the geometric conjectures and local Plancherel conjectures. We will not consider those conjectures in this paper. We just want to remark that Theorem 1.7 provides numerical evidence for the local Plancherel conjecture in Proposition 9.2.1 of [1], but we will not digress in these directions here.

1.4. **Rankin-Selberg integrals and special values of period integrals.** To end this introduction, we would like to point out that the list of strongly tempered quadruples we found in this paper recovers many existing integrals such as the Rankin-Selberg integrals in [3], [4], [5], [6], [9], [10], [11], [12], [20], [21], [28], [29] and the period integrals in [7], [16], [36]. It also produces many new interesting period integrals for studying.

A simple example that leads to a Rankin-Selberg integral is the quadruple (4.1):

$$(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n, T(std_{\mathrm{GL}_n}), 1)$$

which is dual to

$$(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}), 1).$$

The attached period integral is

$$\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} \phi_1(g)\phi_2(g)\Theta^{\Phi}(g)\ dg$$

where $\phi_1 \in \pi_1, \phi_2 \in \pi_2$ are cusp forms in irreducible unitary cuspidal automorphic representations π_1 and π_2 on GL_n and $\Theta^{\Phi}(g)$ is a theta series on GL_n explicitly given by

$$\Theta^{\Phi}(g) = |\det g|^{-\frac{1}{2}} \sum_{\xi\in k^n} \Phi(\xi g).$$

Let $\xi_0 = (0, 0, \ldots, 0, 1)$, then we can identify $\Phi(g)$ with the sum of $|\det g|^{-\frac{1}{2}}\Phi(0)$ and a mirabolic Eisenstein series

$$E^{\Phi}(g) = |\det g|^{-\frac{1}{2}} \sum_{\gamma\in P_0(k)\backslash\mathrm{GL}_n(k)} \Phi(\gamma g)$$

where P_0 is the mirabolic subgroup that fixes ξ_0. This period integral is just the specialization of the well-known Rankin-Selberg integral for tensor product $L-$function [20] evaluated at a specified value.

The theory of Rankin-Selberg integrals is a very successful theory, producing many integral representations to study L-functions. A noted drawback of this theory is that the integrals are mostly developed in an ad hoc way. The list provided in this paper can actually fit many of the Rankin-Selberg integrals into the framework of BZSV duality. To be precise, those Rankin-Selberg integrals (evaluated at certain value) are simply the period integrals attached to some strongly tempered BZSV quadruples whose dual is closely related to the L-functions associated to the Rankin-Selberg integrals. The following is a list of such Rankin-Selberg integrals.

- Integrals for exterior square $L-$functions by Bump-Friedberg [3].
- Integrals for Spin $L-$function by Bump-Ginzburg [4], [5] and [11].
- Integrals for symmetric square $L-$functiosn by Bump-Ginzburg [6] (preceded by Gelbart-Jacquet [13] and Patterson-Piatetski-Shapiro [28], and complemented by Takeda's work [33]).
- Integrals for standard $L-$functions of exceptional groups E_6 and G_2 by Ginzburg [9] and [10].
- Multivariable Rankin-Selberg integrals by Ginzburg-Hundley [12] and Pollack-Shah [29].
- Rankin-Selberg convolution by Jacquet-Piatetski-Shapiro-Shalika [20].
- Integrals for exterior square $L-$functions by Jacquet-Shalika [21].

The above list exhausts all currently known Rankin-Selberg integrals utilizing the mirobolic Eisenstein series. There are also examples above that use the Eisenstein series of other types (e.g., the ones in [12] and [29]).

Our list provides more candidates for Rankin-Selberg integrals. For example, Model 13 of Table 26 suggests considering the following Rankin-Selberg integral of $G = \mathrm{GSO}_8$, which should produce the standard L-function and the Half-Spin L-function. Let π be a generic cuspidal automorphic representation of $\mathrm{GSO}_8(\mathbb{A})$, $\phi \in \pi$ and $P = MN$ be a maximal parabolic subgroup GSO_8 with its Levi subgroup $M = \mathrm{GL}_2\times\mathrm{GSO}_4$. Let $H = S(\mathrm{GL}_2\times\mathrm{GSO}_4)$ be a subgroup of M and let $E(h, s_1, s_2)$ be an automorphic function on H induced from the trivial function on GL_2 and the Borel Eisenstein series of GSO_4 (s_1, s_2 are the parameter of the Eisenstein series). It is easy to see that one can take a Fourier-Jacobi coefficient of

ϕ along the unipotent subgroup N that produces an automorphic function on H. We will denote it by $\mathcal{P}_N(\phi)$. Then, the integral associated to Model 13 of Table 26 is just

$$\int_{H(k)\backslash H(\mathbb{A})/Z_G(\mathbb{A})} \mathcal{P}_N(\phi)(h)E(h,s_1,s_2)dh.$$

In the spirit of Conjecture 1.1, we expect this to be the integral representation of the L-function $L(s_1,\pi,\rho_1)L(s_2,\pi,\rho_2)$ where ρ_1 (resp. ρ_2) is the standard representation (resp. Half-Spin representation) of $\mathrm{Spin}_8(\mathbb{C})$.

Meanwhile the majority of the quadruples in our list have period integrals that cannot be considered as specializations of Rankin-Selberg integrals. In some cases, the identities between the periods and the $L-$values in Conjecture 1.1 are consequences of Gan-Gross-Prasad conjectures [7, 8, 19]) and the Conjectures in [36]. There is also one case where the integral is predicted by the work of Ginzburg-Jiang-Rallis [16] on the central value of symmetric cube $L-$functions. Of more interest are the many cases where the conjectured identity in Conjecture 1.1 is new and unrelated to the conjectures mentioned above. For example each of the quadruple in tables 25 and 26 gives such a new conjecture.

We now list one example from Table 22 that not only provides a new Ichino-Ikeda type conjecture for a strongly tempered quadruple but also can be used to explain the Rankin-Selberg in [12]. The example is Model 4 of Table 22. The quadruple is reductive and is given by

$$\Delta=(G,H,\rho_H)=(\mathrm{GSp}_4\times\mathrm{GSpin}_8\times\mathrm{GL}_2, S(\mathrm{GSpin}_8\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}).$$

Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$, $\phi\in\pi$ and Θ_{ρ_H} be the theta series associated to the symplectic representation ρ_H. Then the period integral is given by

$$\mathcal{P}_\Delta(\phi)=\int_{H(k)\backslash H(\mathbb{A})/Z_\Delta(\mathbb{A})} \phi(h)\Theta_{\rho_H}(h)dh.$$

In the spirit of Conjecture 1.1, we expect the square of this period integral to be equal to

$$\frac{L(1/2,\Pi,\hat{\rho})}{L(1,\Pi,Ad)}$$

where $\hat{\rho}$ is the representation $std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}$ of $\widehat{G/Z_\Delta}(\mathbb{C})$. This is a new period integral that has not been considered before. If we replace the cusp form on GSp_4 and GL_2 by Borel Eisenstein series, then the period integral $\mathcal{P}_\Delta$ becomes the Rankin-Selberg integral in [12].

1.5. **Organization of the paper.** In Section 2, we will explain our strategy for writing down the dual quadruple. In Sections 3-7, we will consider Tables 1, 2, 11, 12, and 22 of [22]. In Section 8 we summarize our findings in six tables.

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2. Our strategy

2.1. **Notation and convention.** In this paper, for a group G of Type A_n (resp. B_n, C_n, D_n, G_2, E_6, E_7), we use std_G to denote the n-dimensional (resp. $2n+1$-dimensional, $2n$-dimensional, $2n$-dimensional, 7-dimensional, 27-dimensional, 56-dimensional) standard representation of G. We use Spin_{2n} (resp. Spin_{2n+1}) to denote the Spin representation of the reductive group of Type D_n (resp. B_n) and we use HSpin_{2n} to denote the Half-Spin representation of reductive group with Type D_n. We use Sym^n (resp. $\wedge^n$) to denote the n-th symmetric power (resp. exterior power) of a reductive group of Type A. We use $\wedge_0^3$ to denote the third fundamental representation of a reductive group of Type C_3. Lastly, for a representation ρ of G, we use $\rho^\vee$ to denote the dual representation and $T(\rho)$ to denote $\rho\oplus\rho^\vee$.

In this paper, we always use l to denote the similitude character of a similitude group. If we have two similitude group GH_1 and GH_2, we let

$$G(H_1\times H_2)=\{(h_1,h_2)\in GH_1\times GH_2|\ l(h_1)=l(h_2)\},$$
$$S(GH_1\times GH_2)=\{(h_1,h_2)\in GH_1\times GH_2|\ l(h_1)l(h_2)=1\}.$$

Similarly we can also define $G(H_1\times\cdots\times H_n)$ and $S(GH_1\times\cdots\times GH_n)$. For example,

$$S(\mathrm{GL}_2^3)=S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2)=\{(h_1,h_2,h_3)\in\mathrm{GL}_2^3|\ \det(h_1h_2h_3)=1\}.$$

All the nilpotent orbits considered in this paper are principal in a Levi subgroup (this is also the case in [1]). As a result, we will use the Levi subgroup or just the root type of the Levi subgroup to denote the nilpotent orbit (the zero nilpotent orbit is denoted by 1). For a split reductive group G, we will use T_G to denote a maximal split torus of G (a minimal Levi subgroup).

For a BZSV quadruple $\hat{\Delta}=(\hat{G},\hat{G},\hat{\rho},1)$, there are many other quadruples that is essentially equal to $\hat{\Delta}$ up to some central isogeny. To be specific, one can take any group $\hat{H}$ of the same root Type as $\hat{G}$ such that the representation $\hat{\rho}$ can also be defined on $\hat{H}$. Then one can choose any group $\hat{G}'$ containing $\hat{H}$ such that $\hat{G}'=\hat{H}Z_{\hat{G}'}$. The quadruple $(\hat{G}',\hat{H},\hat{\rho},1)$ is essentially equal to $\hat{\Delta}$ up to some central isogeny. For example, both $(\mathrm{PGL}_2^3,\mathrm{PGL}_2,0,1)$ and $(\mathrm{GL}_2^3,\mathrm{GL}_2,0,1)$ can be viewed as trilinear GL_2-model. The dual quadruple of them are $(\mathrm{SL}_2^3,\mathrm{SL}_2^3,\hat{\rho},1)$ and $(\mathrm{GL}_2^3,S(\mathrm{GL}_2^3),\hat{\rho},1)$ where $\hat{\rho}$ is the tensor product of SL_2^3 and $S(\mathrm{GL}_2^3)$ respectively, and they are equal to each other up to some central isogeny. While there are various choices of dual quadruples pairs $(\Delta,\hat{\Delta})$ associated to $\hat{\rho}$ due to the isogeny issue, in this paper, for each representation $\hat{\rho}$ in [22], we will only write down one quadruple $\Delta=(G,H,\rho_H,\iota)$ whose dual quadruple $\hat{\Delta}$ is $(\hat{G},\widehat{G/Z_\Delta},\hat{\rho},1)$ where $Z_\Delta=Z_G\cap ker(\rho_H)$.

Remark 2.1. *In our proof of Theorem 1.7, we frequently quote the unramified computation in* [19] *and* [36]*. The settings in* [19] *and* [36] *may actually differ from ours through finite isogeny or central isogeny. It is clear that the computation can be adapted and the results there still apply. For example, in* [19]*, they computed the local relative character for the Gross-Prasad model* $(\mathrm{SO}_{n+1}\times\mathrm{SO}_n,\mathrm{SO}_n)$ *at unramified places. Their results can be also applied to models like* $(\mathrm{GL}_4\times\mathrm{GSp}_4,\mathrm{GSp}_4)$ *(which is essentially the Gross-Prasad model* $(\mathrm{SO}_6\times\mathrm{SO}_5,\mathrm{SO}_5)$ *up to some central isogeny).*

2.2. **Theta correspondence for classical groups.** In this paper we will frequently use theta correspondence for classical groups. We will briefly review it in this subsection. We start with the theta correspondence for the general linear group. Let $n\geq m\geq 1$ and

$G = H_1 \times H_2 = \mathrm{GL}_n \times \mathrm{GL}_m$. We use V to denote the underlying vector space of the representation $\rho = std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_m}$ of G. For $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define the theta function

$$\Theta_\psi^\varphi(g) = \sum_{X \in V(k)} \rho(g)\varphi(X),\ g \in G(\mathbb{A})$$

which is an automorphic function on $G(\mathbb{A}) = H_1 \times H_2(\mathbb{A})$. Let π be a cuspidal automorphic representation of $H_2(\mathbb{A})$. For $\phi \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$, the integral

$$\int_{H_2(k)\backslash H_2(\mathbb{A})} \Theta_\psi^\varphi(h_1, h_2)\phi(h_2)dh_2$$

gives an automorphic function on $H_1(\mathbb{A})$ which will be denoted by $\Theta(\phi)$.

Theorem 2.2. *([26]) We have*

$$\{\Theta(\phi) |\ \phi \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi\} = \{E(\phi', 1) |\ \phi' \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi\}$$

where $E(\phi', 1)$ is the Eisenstein series on $H_1(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})$ induced from ϕ' and the identity function on $\mathrm{GL}_{n-m}(\mathbb{A})$. Moreover, for $\phi_1, \phi_2 \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$, we have the Rallis inner product formula

$$\int_{H_2(k)\backslash H_2(\mathbb{A})/Z_{H_2}(\mathbb{A})} \int_{H_1(k)\backslash H_1(\mathbb{A})} \int_{H_1(k)\backslash H_1(\mathbb{A})} \Theta_\psi^\varphi(h_1, h_2)\Theta_\psi^\varphi(h_1', h_2)E(\phi_1, 1)(h_1)E(\phi_2, 1)(h_1')dh_1dh_1'dh_2$$

$$\text{``}=\text{''}\mathrm{Res}_{s=\frac{n-m}{2}} L(s + \frac{1}{2}, \pi) \cdot \int_{H_2(k)\backslash H_2(\mathbb{A})/Z_{H_2}(\mathbb{A})} \phi_1(h_2)\phi_2(h_2)dh_2.$$

Remark 2.3. *When $m = 1$, the above theorem implies that if we integrate the theta series on GL_n associated to the symplectic representation $T(std_n)$ over the center of GL_n we will get the mirabolic Eisenstein series of GL_n. We will frequently use this fact in later discussions.*

For the unramified computation, we also need the local theta correspondence for unramified representation. Let F be a p-adic local field that is a local place of k. We use $\phi_\rho(h_1, h_2)$ to denote the local spherical matrix coefficient of the Weil representation with $\phi_\rho(1, 1) = 1$. Let π be a tempered unramified representation of $H_2(F)$, ϕ_π (resp. $\phi_{\pi,1}$) be the unramified matrix coefficient of π (resp. $Ind_{\mathrm{GL}_m \times \mathrm{GL}_{n-m}}^{\mathrm{GL}_n}(\pi \otimes 1)$) with $\phi_\pi(1) = \phi_{\pi,1}(1) = 1$.

Theorem 2.4. *([26]) With the notation above, we have*

$$\int_{H_2(F)} \phi_\rho(h_1, h_2)\phi_\pi(h_2)dh_2 = L(\frac{n - m + 1}{2}, \pi) \cdot \phi_{\pi,1}(h_1).$$

Next we study the theta correspondence between SO_{2n} and Sp_{2m} with $n \geq m \geq 1$. Let $G = H_1 \times H_2 = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2m}$ and we use V to denote the underlying vector space of the representation $\rho = std_{\mathrm{SO}_{2n}} \otimes std_{\mathrm{Sp}_{2m}}$ of G. Let Y be a maximal isotropic subspace of V, we can define $\Theta_\psi^\varphi(g)$ an automorphic function on $G(\mathbb{A})$ as in the introduction, for any Schwartz function φ on Y.

Let Π be a cuspidal tempered global Arthur packet of $H_2(\mathbb{A}) = \mathrm{Sp}_{2m}(\mathbb{A})$ and let Π' be its lifting to $H_1(\mathbb{A}) = \mathrm{SO}_{2n}(\mathbb{A})$ under the map $\mathrm{SO}_{2m+1}(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n}(\mathbb{A})$ whose restrict to

SL_2 is the principal embedding from SL_2 to $\mathrm{SO}_{2n-2m-1}$ (if $n > m$ then Π' is a non-tempered Arthur L-packet) [4]. For $\phi \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$, the integral

$$\int_{H_2(k)\backslash H_2(\mathbb{A})} \Theta_\psi^\varphi(h_1,h_2)\phi(h_2)dh_2$$

gives an automorphic function on $H_1(\mathbb{A}) = \mathrm{SO}_{2n}(\mathbb{A})$ which will be denoted by $\Theta(\phi)$. Then the following theorem holds.

Theorem 2.5. *([24, 37, 14]) With the notation above, the representation*

$$\{\Theta(\phi) |\ \phi \in L^2(\mathrm{Sp}_{2m}(k)\backslash \mathrm{Sp}_{2m}(\mathbb{A}))_\Pi\}$$

of $\mathrm{SO}_{2n}(\mathbb{A})$ is a direct sum of some distinct irreducible representations belonging to the Arthur L-packet Π' of $H_1(\mathbb{A}) = \mathrm{SO}_{2n}(\mathbb{A})$. Moreover, for $\phi_1, \phi_2 \in \Pi'$, we have the Rallis inner product formula

$$\int_{H_2(k)\backslash H_2(\mathbb{A})}\int_{H_1(k)\backslash H_1(\mathbb{A})}\int_{H_1(k)\backslash H_1(\mathbb{A})} \Theta_\psi^\varphi(h_1,h_2)\Theta_\psi^\varphi(h_1',h_2)\phi_1(h_1)\phi_2(h_1')dh_1dh_1'dh_2$$

$$\text{“}=\text{”}\mathrm{Res}_{s=\frac{2n-2m-1}{2}} L(s+\frac{1}{2},\Pi')\cdot \int_{H_1(k)\backslash H_1(\mathbb{A})} \phi_1(h_1)\phi_2(h_1)dh_1.$$

For the unramified computation, we also need the local theta correspondence for unramified representation. Let F be a p-adic local field that is a local place of k. We use $\phi_\rho(h_1,h_2)$ to denote the local spherical matrix coefficient of the Weil representation with $\phi_\rho(1,1) = 1$. Let π be a tempered unramified representation of $H_2(F)$ and π' be its lifting to $H_1(F)$ (which is also unramified). Let ϕ_π (resp. $\phi_{\pi'}$) be the unramified matrix coefficient of π (resp. π') with $\phi_\pi(1) = \phi_{\pi'}(1) = 1$.

Theorem 2.6. *([26]) With the notation above, we have*

$$\int_{H_2(F)} \phi_\rho(h_1,h_2)\phi_\pi(h_2)dh_2 = L(n-m,\pi')\cdot \phi_{\pi'}(h_1).$$

The theta correspondence between SO_{2m} and Sp_{2n} (resp. GSO_{2n} and GSp_{2m}, GSO_{2m} and GSp_{2n}) is similar and we will skip it here.

2.3. **Whittaker induction.** Let ι be a map from SL_2 into a split reductive group G and let $\mathcal{O}_\iota$ be the nilpotent orbit of $\mathfrak{g}$ associated to it. Let

$$N = \{g \in G |\ \lim_{t\to 0} \iota(diag(t,t^{-1}))g\iota(diag(t,t^{-1}))^{-1} = 1\}$$

and let M be the centralizer of $Im(\iota(diag(t,t^{-1})))$. Then $P = MN$ is a parabolic subgroup of G.

We start with the Bessel case (namely $\mathcal{O}_\iota$ is even) which is easier. In this case, ι induces a generic character ξ of N (see Section 2 of [13]) and let M_ξ be the stabilizer of ξ under the adjoint action of M. Let $(M,H,\rho,1)$ be a quadruple with $H \subset M_\xi$. Then we say the quadruple (G,H,ρ,ι) is the Whittaker induction of $(M,H,\rho,1)$. A simple example would be the Shalika model $(\mathrm{GL}_{2n},\mathrm{GL}_n,0,\iota)$ which is the Whittaker induction of the group case $(\mathrm{GL}_n\times \mathrm{GL}_n,\mathrm{GL}_n,0,1)$ where ι is the nilpotent orbit of $\mathfrak{gl}_{2n}$ with partition 2^n.

[4] in fact here Π' should be an Arthur packet of $O_{2n}(\mathbb{A})$ which is the union of two Arthur packets of $\mathrm{SO}_{2n}(\mathbb{A})$ differed by the outer automorphism

Next we discuss the Fourier-Jacobi case (namely $\mathcal{O}_\iota$ is not even) which is slightly more complicated. In this case, let M_ι be the centralizer of $\iota(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ in M. By Section 2.3 of [13], we get a symplectic representation ρ_ι of M_ι (when ι is even, M_ι is just M_ξ above and ρ_ι is trivial). Let $(M, H, \rho, 1)$ be a quadruple with $H \subset M_\iota$ and $\rho = \rho_\iota|_H \oplus \rho'$. Then we say the quadruple (G, H, ρ', ι) is the Whittaker induction of $(M, H, \rho, 1)$. An easy example would be the Gan-Gross-Prasad model for $U_n \times U_{n+2k}$ being the Whittaker induction of the Gan-Gross-Prasad model for $U_n \times U_n$.

Remark 2.7. *Here for the notion of Whittaker induction we do not need the quadruple to be a BZSV quadruple. We just need H to commute with $Im(\iota)$ and ρ_H to be a symplectic representation of H.*

Proposition 2.8. *Any quadruple $\Delta = (G, H, \rho_H, \iota)$ is the Whittaker induction of a reductive quadruple.*

Proof. If ι is trivial then Δ is already reductive. If ι is not trivial, by our discussion above, it induces a parabolic subgroup $P = MN$ of G and a symplectic representation $\rho_\iota|_H$ of H (which is nontrivial only when $\mathcal{O}_\iota$ is not even). Then Δ is the Whittaker induction of $(M, H, \rho_H \oplus \rho_\iota|_H, 1)$. This proves the proposition. $\square$

With the notion of Whittaker induction, it is natural to ask what happens to the dual quadruple under the Whittaker induction. In Section 3.4 of [1], Ben-Zvi–Sakellaridis–Venkatesh made a conjecture for this in the Bessel case. Motivated by their conjecture, we make a conjecture here for strongly tempered models in both Bessel and Fourier-Jacobi cases. We first need a definition.

Definition 2.9. *Let M be a Levi subgroup of G and ρ be an irreducible representation of M with the highest weight ϖ_M. There exists a Weyl element w of G such that $w\varpi_M$ is a dominant weight of G [5]. We define $(\rho)_M^G$ to be the irreducible representation of G whose highest weight is $w\varpi_M$. In general, if $\rho = \oplus_i \rho_i$ is a finite-dimensional representation of M with ρ_i irreducible, we define*

$$(\rho)_M^G = \oplus_i (\rho_i)_M^G.$$

Now we are ready to make the conjecture about the BZSV dual of the Whittaker induction of strongly tempered quadruples.

Conjecture 2.10. *Let Δ be a quadruple that is the Whittaker induction of a strongly tempered BZSV quadruple Δ_M, then Δ is a strongly tempered BZSV quadruple. Moreover if $\hat{\Delta}_M = (\hat{M}, \hat{M}', \hat{\rho}_{\hat{M}}, 1)$ be the dual of Δ_M with $\hat{M} = \hat{M}' Z_{\hat{M}}$, then the dual of Δ is given by*

$$\hat{\Delta} = (\hat{G}, \hat{G}', (\hat{\rho}_{\hat{M}})_{\hat{M}}^{\hat{G}}, 1)$$

where $\hat{G}'$ is generated by $\hat{M}'$ and $\{Im(\iota_\alpha) |\ \alpha \in \Delta_{\hat{G}} - \Delta_{\hat{M}}\}$. Here $\Delta_{\hat{G}}$ (resp. $\Delta_{\hat{M}}$) is the set of simple roots of $\hat{G}$ (resp. $\hat{M}$) and $\iota_\alpha : \mathrm{SL}_2 \to \hat{G}$ is the embedding associated to α.

[5]the choice of w is not unique but $w\varpi_M$ is uniquely determined by ϖ_M

2.4. **General strategy.** Let $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ be a quadruple such that $\hat{\rho}$ is an anomaly-free symplectic representation of $\hat{G}$, and it appears in Table 1, 2, 11, 22 of [22]. Our goal is to write down a dual quadruple (up to isogeny) $\Delta = (G, H, \rho_H, \iota)$.

The data in Knop's tables of [22], besides $(\hat{G}, \hat{\rho})$, also contains the following two items: a Levi subgroup $\hat{L}$ of $\hat{G}$ and a Weyl group $\hat{W}_V$ written in the form of $W_{\hat{H}}$ where $\hat{H}$ is the root type (e.g. A_n, B_n, C_n, etc). (In [22] the notations are L, G, W_V in place of $\hat{L}, \hat{G}, \hat{W}_V$ respectively.) Our key observation is that two data (H, ι) of the dual quadruple $\Delta = (G, H, \rho_H, \iota)$ are given by the following properties.

Property 2.11. *(1) The root type of H is dual to the root type of $\hat{W}_V$ in the tables of* [22].

(2) The nilpotent orbit $\mathcal{O}_\iota$ associated to ι is the principal nilpotent orbit of L where L is the dual Levi of $\hat{L}$.

Remark 2.12. *Basically, the Weyl group $\hat{W}_V$ can be viewed as the "little Weyl group" of the quadruple $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$, and $\hat{\mathfrak{l}}$ in tables of* [22] *is an analogue of $\hat{\mathfrak{l}}_X$ in Table 3 of* [23].

As a result, it remains to find out what is ρ_H. We do not have a systematic way to write down ρ_H. Instead we propose a ρ_H in an ad hoc way and then provide evidence for the duality between $\Delta = (G, H, \rho_H, \iota)$ and $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$.

We provide two strong evidences for the duality. The first one is evidence for Conjecture 1.1, i.e., Theorem 1.7 and 1.9. The second evidence is for non-reductive models. For those models, we can explain the duality in terms of Whittaker induction (Theorem 1.12).

In the sections that follow, we will go through Knop's list of representations $\hat{\rho}$. For each ρ we write down a quadruple (G, H, ρ_H, ι). When the quadruple is not reductive, we show it is a Whittaker induction of a reductive quadruple that is dual to another representation $(\hat{M}, \hat{\rho}_M)$ in Knop's list and verify that Theorem 1.12 holds. For cases in Table 21, 22, 23 and 24, we give references where the local relative character is calculated in the unramified places, thus verifying Theorem 1.7. We also verify Theorem 1.9 for the global periods associated to the dual side $\hat{\Delta}$ for cases in Table 21, 23 and 25.

3. Models in Table 1 of [22]

In this section we will consider Table 1 of [22], this is for the case when $\hat{\rho}$ is an irreducible representation of $\hat{G}$. It is easy to check that the representations in (1.2), (1.8), (1.9) and (1.10) of [22] are not anomaly free and the representation in (1.1) of [22] is only anomaly free when $p = 2n$ is even. Hence it remains to consider the following cases. Note that we only write the root type of $\hat{\mathfrak{l}}$ and we write 0 if it is abelian. Also we separate the cases when $\hat{\mathfrak{l}}$ is abelian and when $\hat{\mathfrak{l}}$ is not abelian. These are precisely the cases where the dual quadruple is reductive/non-reductive (see Property 2.11).

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\iota}$
(1.1), p=2m	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2m}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m}})$	D_m	0
(1.1), p=2m+2	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2m+2}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m+2}})$	C_m	0
(1.3), m=2	$(\mathrm{Spin}_5\otimes \mathrm{Spin}_7, \mathrm{Spin}_5\otimes \mathrm{Spin}_7)$	$C_2\times A_1$	0
(1.3), m=3	$(\mathrm{Sp}_6\otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_6}\otimes \mathrm{Spin}_7)$	$C_3\times B_3$	0
(1.3), m=4	$(\mathrm{Sp}_8\otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_8}\otimes \mathrm{Spin}_7)$	$D_4\times B_3$	0
(1.6)	(SL_2, Sym^3)	A_1	0

Table 1. Reductive models in Table 1 of [22]

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\iota}$
(1.1), $p=2n<2m$	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2n}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2n}})$	D_n	C_{m-n}
(1.1), $p=2n>2m+2$	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2n}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2n}})$	C_m	D_{n-m}
(1.3), m=1	$(\mathrm{SL}_2\times \mathrm{Spin}_7, std_{\mathrm{SL}_2}\otimes \mathrm{Spin}_7)$	A_1	A_2
(1.3), $m>4$	$(\mathrm{Sp}_{2m}\otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_{2m}}\otimes \mathrm{Spin}_7)$	$D_4\times B_3$	C_{m-4}
(1.4)	$(\mathrm{SL}_2\times \mathrm{Spin}_9, std_{\mathrm{SL}_2}\otimes \mathrm{Spin}_9)$	$A_1\times A_1$	A_2
(1.5), n=11	$(\mathrm{Spin}_{11}, \mathrm{Spin}_{11})$	A_1	A_4
(1.5), n=12	$(\mathrm{Spin}_{12}, \mathrm{HSpin}_{12})$	A_1	A_5
(1.5), n=13	$(\mathrm{Spin}_{13}, \mathrm{Spin}_{13})$	B_2	$A_2\times A_2$
(1.7)	$(SL_6, \wedge^3)$	A_1	$A_2\times A_2$
(1.11)	(E_7, std_{E_7})	A_1	E_6

Table 2. Non-reductive models in Table 1 of [22]

3.1. **The reductive case.** In this subsection we consider the reductive cases, i.e., the ones in Table 1. The nilpotent orbit ι is trivial for all these cases so we will ignore it.

For (1.1) with $p=2m$ (resp. $p=2m+2$), the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2m}, \mathrm{SO}_{2m}, 0) \tag{3.1}$$

$$(\text{resp.} (G,H,\rho_H)=(\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2m+2}, \mathrm{SO}_{2m+1}, 0)) \tag{3.2}$$

which is just the reductive Gross-Prasad model. The unramified computations in [19] prove Theorem 1.7 in these two cases. For the dual side, Theorem 2.5 applied to the theta correspondence between $\mathrm{SO}_{2m}\times \mathrm{Sp}_{2m}$ (resp. $\mathrm{SO}_{2m+2}\times \mathrm{Sp}_{2m}$) implies Conjecture 1.1(2) and this proves Theorem 1.9.

For (1.3) with $m=2$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSp}_4, G(\mathrm{Sp}_4\times \mathrm{Sp}_2), 0)$$

which is the model $(\mathrm{GSp}_6\times \mathrm{GSp}_4, G(\mathrm{Sp}_4\times \mathrm{Sp}_2))$ studied in [36]. The unramified computations in [36] prove Theorem 1.7 in this case.

For (1.3) with $m=3$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSpin}_7, S(\mathrm{GSp}_6\times \mathrm{GSpin}_7), std_{\mathrm{Sp}_6}\otimes \mathrm{Spin}_7).$$

For (1.3) with $m=4$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSpin}_9, S(\mathrm{GSp}_6\times \mathrm{GSpin}_8), std_{\mathrm{Sp}_6}\otimes \mathrm{HSpin}_8). \tag{3.3}$$

Theorem 1.7 and 1.9 for two cases can be established by the same argument as Model (11.11) of [22] (see (5.4) and (5.3) of Section 5.1) together with the triality of D_4.

For (1.6), it is clear that the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of BZSV duality. However, for this specific case, by the work of [16], we expect there is an associated quadruple of the form $(\mathrm{GL}_2, \mathrm{GL}_2, \rho_H, 1)$ where ρ_H is no longer an anomaly free symplectic representation, but rather we understand that ρ_H corresponds to the theta series on $H = \mathrm{GL}_2$ defined via the cubic covering of GL_2 as in [16]. There is a covering group involved in the theta series since the generic stabilizer is not connected. In [16] it is established that the nonvanishing of $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ is equivalent to the nonvanishing of $L(1/2, \Pi, \hat{\rho})$. We expect further that Conjecture 1.1(1) holds in this case.

By the discussion above, the strongly tempered quadruple associated to Table 1 (without the row corresponding to (1.6)) is given as follows. Note that ι is trivial for all these cases.

(G, H, ρ_H)	$\hat{\rho}$
($\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2m}$, SO_{2m}, 0)	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2m}}$
($\mathrm{SO}_{2m+2} \times \mathrm{SO}_{2m+1}$, SO_{2m+1}, 0)	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2m+2}}$
($\mathrm{GSp}_6 \times \mathrm{GSp}_4$, $G(\mathrm{Sp}_4 \times \mathrm{Sp}_2)$,0)	$\mathrm{Spin}_5 \otimes \mathrm{Spin}_7$
($\mathrm{GSp}_6 \times \mathrm{GSpin}_7$, $S(\mathrm{GSp}_6 \times \mathrm{GSpin}_7)$, $std_{\mathrm{Sp}_6} \otimes \mathrm{Spin}_7$)	$std_{\mathrm{Sp}_6} \otimes \mathrm{Spin}_7$
($\mathrm{GSp}_6 \times \mathrm{GSpin}_9, S(\mathrm{GSp}_6 \times \mathrm{GSpin}_8), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8$)	$std_{\mathrm{Sp}_8} \otimes \mathrm{Spin}_7$

Table 3. Dual quadruples of Table 1

3.2. **The non-reductive case.** In this subsection we consider the non-reductive cases, i.e., the ones in Table 2.

For (1.1) with $p = 2n < 2m$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0, (\mathrm{GL}_1)^n \times \mathrm{SO}_{2m-2n+1} \times T_{\mathrm{SO}_{2n}})$$

and it is the Gross-Prasad period for $\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}$. For (1.1) with $p = 2n > 2m+2$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0, T_{\mathrm{SO}_{2m+1}} \times (\mathrm{GL}_1)^m \times \mathrm{SO}_{2n-2m})$$

and it is still the Gross-Prasad period for $\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}$. These two cases are the Whittaker induction of the quadruples (3.1), (3.2). It is clear that Theorem 1.12 holds in these two cases. The unramified computation in [19] proves Theorem 1.7 for these two cases. Theorem 2.5 applied to the theta correspondence between $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2m}$ implies Conjecture 1.1(2) and proves Theorem 1.9 for these two cases.

For (1.3) when $m = 1$, the associated quadruple Δ is

$$(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2, 0, (\mathrm{GL}_3 \times \mathrm{GL}_1) \times T_{\mathrm{GL}_2}) \tag{3.4}$$

and it is the model $(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \ltimes U)$ studied in [36]. This quadruple is the Whittaker induction of the triple product quadruple $((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$ (which a special case of (3.2) with $m = 1$). It is clear that Theorem 1.12 holds in this case and the unramified computation in [36] proves Theorem 1.7 in this case.

For (1.3) when $m > 4$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSp}_6, S(\mathrm{GSpin}_8 \times \mathrm{GSp}_6), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8, L)$$

where L is the Levi subgroup whose projection to GSpin_{2m+1} (resp. GSp_6) is of the form $(\mathrm{GL}_1)^4\times \mathrm{GSpin}_{2m-7}$ (resp. the maximal torus). The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=(\mathrm{GL}_1)^{m-4}\times\mathrm{GSpin}_9\times \mathrm{GSp}_6$ whose stabilizer is $\mathrm{GSpin}_8\times\mathrm{GSp}_6$ and we can naturally embed H into the stabilizer. It is the Whittaker induction of the quadruple (3.3). Theorem 1.7 and 1.9 for this model can be established by the same argument as (5.8) in Section 5.2 together with the triality of D_4.

For (1.4), the associated quadruple Δ is

$$(\mathrm{GSp}_8\times\mathrm{GL}_2, G(\mathrm{SL}_2\times\mathrm{SL}_2), 0, \mathrm{GL}_3\times\mathrm{GL}_1\times\mathrm{GL}_1\times T_{\mathrm{GL}_2}). \tag{3.5}$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_2\times\mathrm{GSp}_4\times\mathrm{GL}_2$ whose stabilizer is $G(\mathrm{SL}_2\times\mathrm{SL}_2)\times\mathrm{GL}_2$. We embeds H into the stabilizer so that the induced embedding from H into M is given by the natural embeddings of H into GSp_4 and into $\mathrm{GL}_2\times\mathrm{GL}_2$. This quadruple is the Whittaker induction of the quadruple $(\mathrm{GSp}_4\times\mathrm{GL}_2\times\mathrm{GL}_2, G(\mathrm{SL}_2\times\mathrm{SL}_2), 0, 1)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_5\times\mathrm{SO}_4$. If we replace the cusp form on GL_2 by an Eisenstein series, we recover the Rankin-Selberg integrals in [5]. It is clear that Theorem 1.12 holds in this case and the unramfied computation in [5] proves Theorem 1.7 in this case.

For (1.5) when $n=11$, the associated quadruple Δ is

$$(\mathrm{GSp}_{10}, \mathrm{GL}_2, 0, \mathrm{GL}_5\times\mathrm{GL}_1) \tag{3.6}$$

and it is the model $(\mathrm{GSp}_{10}, \mathrm{GL}_2\ltimes U)$ studied in [36]. This quadruple is the Whittaker induction of the triple product quadruple $((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [36] proves Theorem 1.7 in this case.

For (1.5) when $n=12$, the associated quadruple Δ is

$$(\mathrm{GSO}_{12}, \mathrm{GL}_2, 0, \mathrm{GL}_6\times\mathrm{GL}_1)$$

and it is the model $(\mathrm{GSO}_{12}, \mathrm{GL}_2\ltimes U)$ studied in [36]. This quadruple is the Whittaker induction of the triple product quadruple $((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [36] proves Theorem 1.7 in this case.

For (1.5) when $n=13$, the associated quadruple Δ is

$$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0, \mathrm{GL}_3\times\mathrm{GL}_3\times\mathrm{GL}_1).$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_4\times\mathrm{GSp}_4$ whose stabilizer is $H=\mathrm{GSp}_4$. The quadruple is the Whittaker induction of the quadruple $(\mathrm{GSp}_4\times\mathrm{GL}_4, \mathrm{GSp}_4, 0, 1)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_6\times\mathrm{SO}_5$. It is clear that Theorem 1.12 holds in this case. In this case the unramified computation can be done in a similar way as [36], which will give Theorem 1.7.

For (1.7), the associated quadruple Δ is

$$(\mathrm{GL}_6, \mathrm{GL}_2, 0, \mathrm{GL}_3\times\mathrm{GL}_3)$$

and it is the Ginzburg-Rallis model $(\mathrm{GL}_6, \mathrm{GL}_2\ltimes U)$ studied in [36]. This quadruple is the Whittaker induction of the triple product quadruple $((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [36] proves Theorem 1.7 in this case.

For (1.11), the associated quadruple Δ is

$$(E_7, \mathrm{PGL}_2, 0, GE_6)$$

and it is the model $(E_7, \mathrm{PGL}_2 \ltimes U)$ studied in [36]. This quadruple is the Whittaker induction of the triple product quadruple $((\mathrm{PGL}_2)^3, \mathrm{PGL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [36] proves Theorem 1.7 in this case.

By the discussion above, the strongly tempered quadruple associated to Table 2 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L.

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0)$	B_{m-n}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0)$	D_{n-m}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
$(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2, 0)$	A_2	$std_{\mathrm{GL}_2} \otimes \mathrm{Spin}_7$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSp}_6, S(\mathrm{GSpin}_8 \times \mathrm{GSp}_6), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8)$	B_{m-4}	$std_{\mathrm{Sp}_{2m}} \otimes \mathrm{Spin}_7$
$(\mathrm{GSp}_8 \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), 0)$	A_2	$std_{\mathrm{GL}_2} \otimes \mathrm{Spin}_9$
$(\mathrm{GSp}_{10}, \mathrm{GL}_2, 0)$	A_4	Spin_{11}
$(\mathrm{GSO}_{12}, \mathrm{GL}_2, 0)$	A_5	HSpin_{12}
$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0)$	$A_2 \times A_2$	Spin_{13}
$(\mathrm{GL}_6, \mathrm{GL}_2, 0)$	$A_2 \times A_2$	$\wedge^3$
$(E_7, \mathrm{PGL}_2, 0)$	E_6	std_{E_7}

TABLE 4. Dual quadruples of Table 2

4. Models in Table 2

In this section we will consider Table 2 of [22], this is for the case when $\hat{\rho} = T(\hat{\tau})$ is the direct sum of two irreducible representations of $\hat{G}$ that are dual to each other. All the representations in Table 2 of [22] are anomaly free, so we need to consider all of them. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [22]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(2.1), m=n	$(\mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}))$	A_{n-1}	0
(2.1), m=n+1 and (2.4), n=2	$(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, T(std_{\mathrm{GL}_{n+1}} \otimes std_{\mathrm{GL}_n}))$	A_{n-1}	0
(2.3)	$(\mathrm{GL}_n, T(Sym^2))$	A_{n-1}	0
(2.6), m=n=2	$(\mathrm{Sp}_4 \times \mathrm{GL}_2, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_2}))$	$A_1 \times A_1$	0
(2.6), m=2, n=3	$(\mathrm{Sp}_4 \times \mathrm{GL}_3, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_3}))$	$C_2 \times A_2$	0
(2.6), m=2, n=4	$(\mathrm{Sp}_4 \times \mathrm{GL}_4, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_4}))$	$C_2 \times A_3$	0
(2.6), m=2, n=5	$(\mathrm{Sp}_4 \times \mathrm{GL}_5, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{SL}_5}))$	$C_2 \times A_3$	0
(2.6), m=n=3	$(\mathrm{Sp}_6 \times \mathrm{GL}_3), T(Std_{\mathrm{Sp}_6} \otimes Std_{\mathrm{GL}_3}))$	$A_3 \times A_2$	0

TABLE 5. Reductive models in Table 2 of [22]

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(2.1), $m>n+1$, and (2.4), $n>2$	$(\mathrm{GL}_m\times\mathrm{GL}_n, T(std_{\mathrm{GL}_m}\otimes std_{\mathrm{GL}_n}))$	A_{n-1}	A_{m-n-1}
(2.2), n=2m	$(\mathrm{GL}_{2m}, T(\wedge^2))$	A_{m-1}	$(A_1)^m$
(2.2),n=2m+1	$(\mathrm{GL}_{2m+1}, T(\wedge^2))$	A_{m-1}	$(A_1)^m$
(2.5)	$(\mathrm{Sp}_{2n}, T(std_{\mathrm{Sp}_{2n}})$	0	C_{m-1}
(2.6), $m>2$, n=2	$(\mathrm{Sp}_{2m}\times\mathrm{SL}_2, T(Std_{\mathrm{Sp}_{2m}}\otimes Std_{\mathrm{SL}_2}))$	$A_1\times A_1$	C_{m-2}
(2.6), m=2, $n>5$	$(\mathrm{Sp}_4\times\mathrm{SL}_m, T(Std_{\mathrm{Sp}_4}\otimes Std_{\mathrm{SL}_m}))$	$C_2\times A_3$	A_{m-5}
(2.6), $m>3$, n=3	$(\mathrm{Sp}_{2m}\times\mathrm{SL}_3, T(Std_{\mathrm{Sp}_{2m}}\otimes Std_{\mathrm{SL}_3}))$	$A_3\times A_2$	C_{m-3}
(2.7), m=2k	$(\mathrm{SO}_{2k}, T(std_{\mathrm{SO}_{2k}}))$	A_1	D_{k-1}
(2.7), m=2k+1	$(\mathrm{SO}_{2k+1}, T(std_{\mathrm{SO}_{2k+1}}))$	A_1	B_{k-1}
(2.8), n=7	$(\mathrm{Spin}_7, T(\mathrm{Spin}_7))$	A_1	A_2
(2.8), n=9	$(\mathrm{Spin}_9, T(\mathrm{Spin}_9))$	$A_1\times A_1$	A_2
(2.8), n=10	$(\mathrm{Spin}_{10}, T(\mathrm{HSpin}_{10}))$	A_1	A_3
(2.9)	$(G_2, T(std_{G_2}))$	A_1	A_1
(2.10)	$(E_6, T(std_{E_6}))$	A_2	D_4

TABLE 6. Non-reductive models in Table 2 of [22]

4.1. **The reductive case.** In this subsection we consider the reductive cases, i.e., the ones in Table 5.

For (2.1) with $m=n$, the associated quadruple Δ is given by

$$(G,H,\rho_H,\iota)=(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n,T(std_{\mathrm{GL}_n}),1). \tag{4.1}$$

For (2.1) with $m=n+1$ and (2.4) with $n=2$, the associated quadruple Δ is given by

$$(G,H,\rho_H,\iota)=(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n,0,1). \tag{4.2}$$

The period integrals in these two cases are exactly the Rankin-Selberg integral for $\mathrm{GL}_n\times\mathrm{GL}_n$ and $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ in [20]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7. For the dual side, Theorem 2.2 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_{n+1}$ and $\mathrm{GL}_n\times\mathrm{GL}_n$ imply Conjecture 1.1(2) and this proves Theorem 1.9.

For (2.3), if $n=2$, the associated quadruple Δ is

$$(\mathrm{GL}_2,\mathrm{SL}_2,T(std_{\mathrm{SL}_2}),1). \tag{4.3}$$

The period integral associated to it is just the Rankin-Selberg integral for symmetric square L-function in [15]. The result in [15] proves Conjecture 1.1(1) and Theorem 1.7.

For (2.3) when $n>2$, the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of the BZSV duality. However, for this specific case, by the Rankin-Selberg integral in [6, 28, 33], we know that the dual quadruple (G,H,ρ_H,ι) should be given by $(\mathrm{GL}_n,\mathrm{GL}_n,\rho_H,1)$ where ρ_H is chosen so that the theta series on $H=\mathrm{GL}_n$ is defined via the double covering of GL_n. As the generic stabilizer is not connected, there are covering groups involved in the theta series.

For (2.6) with $m=n=2$, the associated quadruple Δ is given by

$$(G,H,\rho_H,\iota)=(\mathrm{GSp}_4\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_{2,2},}),1) \tag{4.4}$$

where the embedding of H into G is given by the canonical embedding from $\mathrm{GSpin}_4=G(\mathrm{SL}_2\times\mathrm{SL}_2)$ into $\mathrm{GSpin}_5=\mathrm{GSp}_4$ and the projection of $G(\mathrm{SL}_2\times\mathrm{SL}_2)$ into GL_2 via the first GL_2-copy. The representation ρ_H is the standard representation of the second GL_2-copy of

H. This integral is essentially the Gross-Prasad model for $\mathrm{SO}_5 \times \mathrm{SO}_4$ except we replace the cusp form on one GL_2-copy by the theta series. The unramified computation in [19] proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_2 \times \mathrm{GL}_4$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [8]) for non-tempered Arthur packet for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$. This proves Theorem 1.9.

For (2.6) with $m = 2, n = 3$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GSp}_4 \times \mathrm{GL}_3, \mathrm{GSp}_4 \times \mathrm{GL}_3, T(std_{\mathrm{GSp}_4} \otimes std_{\mathrm{GL}_3}), 1).$$

By the theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ (note that the theta function constructed from $T(std_{\mathrm{GSp}_4} \otimes std_{\mathrm{GL}_3})$ is the restriction of the theta function from $T(std_{\mathrm{GL}_4} \otimes std_{\mathrm{GL}_3})$), the integral over GL_3 of a cusp form on GL_3 with the theta series associated to ρ_H produces an Eisenstein series of GL_4 induced from the cusp form on GL_3 and the trivial character of GL_1. Then the integral over GSp_4 is just the period integral for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$. The unramified computation in [19] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_3$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [7]. This proves Theorem 1.9.

For (2.6) with $m = 2, n = 4$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_4, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4})). \tag{4.5}$$

By the theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$, the integral over Sp_4 of a cusp form on GSp_4 with the theta series associated to ρ_H produces an automorphic form of GL_4. Then the integral over GL_4 is just the Rankin-Selberg integral of $\mathrm{GL}_4 \times \mathrm{GL}_4$ as in [20]. The Rankin-Selberg integral in [20] and Theorems 2.2 and 2.4 applied to theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_4$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [7]. This proves Theorem 1.9. This is a very interesting case because both Δ and $\hat{\Delta}$ are strongly tempered and they are not equal to each other.

For (2.6) with $m = 2, n = 5$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_5, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2). \tag{4.6}$$

By the theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$, the integral over Sp_4 of a cusp form on GSp_4 with the theta series associated to ρ_H produces an automorphic form of GL_4. Then the integral over GL_4 is just the Rankin-Selberg integral of $\mathrm{GL}_5 \times \mathrm{GL}_4$. The Rankin-Selberg integral in [20] and Theorems 2.5 and 2.6 applied to theta correspondence $\mathrm{GSp}_4 \times \mathrm{GSO}_6$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_5$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [7]. This proves Theorem 1.9.

For (2.6) with $m = n = 3$, the associated quadruple Δ is given by

$$(\mathrm{GSpin}_7 \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3})). \tag{4.7}$$

By the theta correspondence for $\mathrm{GL}_3\times\mathrm{GL}_4$ (note that GSpin_6 is essentially GL_4 up to some central isogeny which won't affect the unramified computation) the integral over GL_3 of a cusp form on GL_3 with the theta series associated to ρ_H produces an Eisenstein series of GSpin_6 induced from the cusp form on GL_3 and the trivial character of GL_1. Then the integral over GSpin_6 is just the period integral for the Gross-Prasad model of $\mathrm{GSpin}_7\times\mathrm{GSpin}_6$. The unramified computation in [19] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_3\times\mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_6\times\mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4\times\mathrm{GL}_3$. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 5 is given as follows. Note that ι is trivial for all these cases.

(G,H,ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n,T(std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n,0)$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})$
$(\mathrm{GL}_2,\mathrm{SL}_2,T(std_{\mathrm{SL}_2}))$	$T(Sym^2)$
$(\mathrm{GSp}_4\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_{2,2}}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_2})$
$(\mathrm{GSp}_4\times\mathrm{GL}_3,H=G,T(std_{\mathrm{GSp}_4}\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_3})$
$(\mathrm{GSp}_4\times\mathrm{GL}_4,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{GL}_4}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_4})$
$(\mathrm{GSp}_4\times\mathrm{GL}_5,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2)$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_5})$
$(\mathrm{GSpin}_7\times\mathrm{GL}_3,\mathrm{GSpin}_6\times\mathrm{GL}_3,T(\mathrm{HSpin}_6\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_6}\otimes Std_{\mathrm{GL}_3})$

Table 7. Dual quadruples of Table 5

4.2. **The non-reductive case.** For (2.1) with $m>n+1$ and (2.4) with $n>2$, the associated quadruple Δ is given by

$$(G,H,\rho_H,\iota)=(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_n,0,(\mathrm{GL}_1^n\times\mathrm{GL}_{m-n}\times T_{\mathrm{GL}_n}).$$

When $m-n$ is odd (resp. even), the nilpotent orbit induces a Bessel period (resp. Fourier-Jacobi period) for the unipotent radical of the parabolic subgroup $P=MU$ with $M=(\mathrm{GL}_1)^{m-n-1}\times\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ (resp. $M=(\mathrm{GL}_1)^{m-n}\times\mathrm{GL}_n\times\mathrm{GL}_n$) whose stabilizer in M is $\mathrm{GL}_n\times\mathrm{GL}_n$. We can diagonally embed H into the stabilizer. This model is the Whittaker induction of the quadruple (4.2) (resp. (4.1)). It is clear that Theorem 1.12 holds in this case. The period integral in this case is closely related to the Rankin-Selberg integral in [20]. However the difference is not negligible and we do not claim Theorem 1.7 for this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_m$. This proves Theorem 1.9.

For (2.2) with $n=2m$, the associated quadruple Δ is given by

$$(\mathrm{GL}_{2m},\mathrm{GL}_m,T(std_{\mathrm{GL}_m}),(\mathrm{GL}_2)^m).$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_m\times\mathrm{GL}_m$ whose stabilizer in M is $H=\mathrm{GL}_m$. It is the Whittaker induction of (4.1). It is clear that Theorem 1.12 holds in this case. The period integral in this case is exactly the Rankin-Selberg integral in [21]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7.

For (2.2) with $n=2m+1$, the associated quadruple Δ is given by

$$(\mathrm{GL}_{2m+1},\mathrm{GL}_m,0,(\mathrm{GL}_2)^m\times\mathrm{GL}_1).$$

The nilpotent orbit induces a Fourier-Jacobi period for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_m \times \mathrm{GL}_1 \times \mathrm{GL}_m$ whose stabilizer in M is $\mathrm{GL}_n \times \mathrm{GL}_1$. We can naturally embed H into the stabilizer. It is the Whittaker induction of (4.1). It is clear that Theorem 1.12 holds in this case. The period integral in this case is exactly the Rankin-Selberg integral in [21]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7.

For (2.5), the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0, \mathrm{SO}_{2m-1} \times \mathrm{GL}_1).$$

It is the Gross-Prasad model of $\mathrm{SO}_{2m+1} \times \mathrm{SO}_2$ and it is Whittaker induction of the quadruple (3.1) when $m = 1$. It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{Sp}_{2m} \times \mathrm{SO}_2$ and this proves Theorem 1.9.

For (2.6) with $m > 2, n = 2$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}), (\mathrm{GL}_1)^2 \times \mathrm{GSpin}_{2m-3} \times T_{\mathrm{GL}_2,2}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_5 \times (\mathrm{GL}_1)^{m-2} \times \mathrm{GL}_2$ whose stabilizer in M is $\mathrm{GSpin}_4 \times \mathrm{GL}_2$. We then embeds $H = G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into $\mathrm{GSpin}_4 \times \mathrm{GL}_2$ via the identity map on GSpin_4 and the projection of $G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into GL_2 via the first GL_2-copy. The representation ρ_H is the standard representation of the second GL_2-copy of H. This integral is essentially the Gross-Prasad model for $\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_4$ except we replace the cusp form on one GL_2-copy by theta series. The quadruple is the Whittaker induction of the quadruple (4.4). It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{GSp}_{2n} \times \mathrm{GSO}_4$ and the Rankin-Selberg integral of $\mathrm{GL}_2 \times \mathrm{GL}_1$. This proves Theorem 1.9.

For (2.6) with $m = 2, n > 5$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_n, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2, T_{\mathrm{GSp}_4} \times (\mathrm{GL}_1)^4 \times \mathrm{GL}_{n-4}).$$

When n is odd (resp. even), the nilpotent orbit induces a Bessel period (resp. Fourier-Jacobi period) for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSp}_4 \times \mathrm{GL}_5 \times (\mathrm{GL}_1)^5$ (resp. $M = \mathrm{GSp}_4 \times \mathrm{GL}_4 \times (\mathrm{GL}_1)^4$) whose stabilizer in M is $\mathrm{GSp}_4 \times \mathrm{GL}_4$. We can naturally embed H into the stabilizer. This model is the Whittaker induction of the quadruple (4.6) (resp. (4.5)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_4$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [7]. This proves Theorem 1.9.

For (2.6) with $m > 3, n = 3$, the associated quadruple Δ is given by

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}), (\mathrm{GL}_1)^3 \times \mathrm{GSpin}_{2m-5} \times T_{\mathrm{GL}_3}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_7 \times (\mathrm{GL}_1)^{m-3} \times \mathrm{GL}_3$ whose stabilizer in M is $H = \mathrm{GSpin}_6 \times \mathrm{GL}_3$. This is the Whittaker induction of the quadruple (4.7). It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] and Theorem 2.4

applied to theta correspondence for $\mathrm{GL}_4 \times \mathrm{GL}_3$ proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_{2n} \times \mathrm{GSO}_6$ and the Rankin-Selberg period for $\mathrm{GL}_4 \times \mathrm{GL}_3$. This proves Theorem 1.9.

For (2.7) with $m = 2k$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2k}, \mathrm{GSpin}_3, T(\mathrm{Spin}_3), \mathrm{GL}_1 \times \mathrm{GSpin}_{2k-2}).$$

This is essentially the Gross-Prasad model for $\mathrm{GSpin}_{2k} \times \mathrm{GSpin}_3$ except we replace the cusp form on GSpin_3 by a theta series. It is the Whittaker induction of the quadruple (4.1) when $n = 2$. It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] proves Theorem 1.7.

For (2.7) with $m = 2k + 1$, the associated quadruple Δ is

$$(\mathrm{GSp}_{2k}, \mathrm{SL}_2 \times \mathrm{GL}_1, std_{\mathrm{SL}_2}, \mathrm{GL}_1 \times \mathrm{GSp}_{2n-2}).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSp}_2 \times (\mathrm{GL}_1)^{k-1}$ whose stabilizer in M is $H = \mathrm{SL}_2 \times \mathrm{GL}_1$. It is the Whittaker induction of the quadruple (4.3). It is clear that Theorem 1.12 holds in this case.

For (2.8) with $n = 7$, the associated quadruple Δ is given by

$$(\mathrm{GSp}_6, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}), \mathrm{GL}_3 \times \mathrm{GL}_1).$$

This is essentially the same as the quadruple (3.4) except we replace the cusp form on GL_2 by theta series. The period integral in this case is exactly the Rankin-Selberg integral in [4] and the quadruple is the Whittaker induction of (4.1) when $m = 2$. It is clear that Theorem 1.12 holds in this case. The unramfied computation in [4] and [36] proves Theorem 1.7.

For (2.8) with $n = 9$, the associated quadruple Δ is

$$(\mathrm{GSp}_8, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2,2}), \mathrm{GL}_3 \times \mathrm{GL}_1 \times \mathrm{GL}_1). \tag{4.8}$$

where $std_{\mathrm{GL}_2,2}$ is the standard representation of the second GL_2-copy. This is essentially the same as the quadruple (3.5) except we replace the cusp form on GL_2 by theta series and the period integral in this case is exactly the Rankin-Selberg integral in [5]. This is the Whittaker induction of (4.4). It is clear that Theorem 1.12 holds in this case. The unramfied computation in [5] proves Theorem 1.7.

For (2.8) with $n = 10$, the associated quadruple Δ is

$$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0, \mathrm{GL}_4 \times \mathrm{GL}_1).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{SO}_2$ whose stabilizer in M is $H = \mathrm{GL}_2$ (here the embedding is given by $h \mapsto (h, h, diag(\det(h), 1)))$. It is the Whittaker induction of the quadruple (4.1) when $n = 2$. It is clear that Theorem 1.12 holds in this case. This integral is very close to the Rankin-Selberg integral in [11], though we again do not claim Theorem 1.7 in this case.

For (2.9), the associated quadruple Δ is

$$(G_2, \mathrm{SL}_2, std_{\mathrm{SL}_2}, \mathrm{GL}_2).$$

It is the Whittaker induction of the quadruple (4.3). The period integral associated to it is exactly the Rankin-Selberg integral in [10]. It is clear that Theorem 1.12 holds in this case. The unramified compuation in [10] proves Theorem 1.7.

For (2.10), the associated quadruple Δ is

$$(GE_6, \mathrm{GL}_3, T(std_{\mathrm{GL}_3}), D_4).$$

It is the Whittaker induction of the quadruple (4.1) when $n = 3$. The period integral associated to it is exactly the Rankin-Selberg integral in [9]. It is clear that Theorem 1.12 holds in this case. The unramified compuation in [9] proves Theorem 1.7.

By the discussion above, the strongly tempered quadruple associated to Table 6 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L.

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n, 0)$	A_{m-n-1}	$T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{2m}, \mathrm{GL}_m, T(std_{\mathrm{GL}_m}))$	$(A_1)^m$	$T(\wedge^2)$
$(\mathrm{GL}_{2m+1}, \mathrm{GL}_m, 0)$	$(A_1)^m$	$T(\wedge^2)$
$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0)$	B_{m-1}	$T(std_{\mathrm{Sp}_{2n}})$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	B_{m-2}	$T(Std_{\mathrm{GSp}_{2m}} \otimes Std_{\mathrm{GL}_2})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_n, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2, (\mathrm{GL}_1)^5)$	A_{n-5}	$T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{SL}_n})$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}))$	B_{m-3}	$T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_3})$
$(\mathrm{GSpin}_{2k}, \mathrm{GSpin}_3, T(\mathrm{Spin}_3))$	D_{k-1}	$T(std_{\mathrm{SO}_{2k}})$
$(\mathrm{GSp}_{2k}, \mathrm{SL}_2 \times \mathrm{GL}_1, std_{\mathrm{SL}_2})$	C_{k-1}	$T(std_{\mathrm{SO}_{2k+1}})$
$(\mathrm{GSp}_6, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_7)$
$(\mathrm{GSp}_8, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_9)$
$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0)$	A_3	$T(\mathrm{HSpin}_{10})$
$(G_2, \mathrm{SL}_2, std_{\mathrm{SL}_2})$	A_1	$T(std_{G_2})$
$(GE_6, \mathrm{GL}_3, T(std_{\mathrm{GL}_3}))$	D_4	$T(std_{E_6})$

TABLE 8. Dual quadruples of Table 6

5. Models in Table 11

In this section we will consider Table 11 of [22], this is for the case when $\hat{\rho}$ is the direct sum of two distinct irreducible symplectic representations of $\hat{G}$. It is easy to check that the representations in (11.5), (11.8), (11.13), (11.14), (11.15) of [22] are not anomaly free and the representation in (11.1) (resp. (11.11)) of [22] is only anomaly free when n is even (resp. p odd). Hence it remains to consider the following cases. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [22]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(11.7)	$(\mathrm{Sp}_4 \times \mathrm{Spin}_8 \times \mathrm{SL}_2, std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2})$	$C_2 \times D_4 \times A_1$	0
(11.9)	$(\mathrm{SL}_2 \times \mathrm{Spin}_7 \times SL_2, std_{\mathrm{SL}_2} \otimes \mathrm{Spin}_7 \oplus \mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2})$	$(A_1)^3 \times B_2$	0
(11.10)	$(\mathrm{SL}_2 \times \mathrm{SO}_6 \times \mathrm{SL}_2, std_{\mathrm{SL}_2} \otimes std_{\mathrm{SO}_6} \oplus std_{\mathrm{SO}_6} \otimes std_{\mathrm{SL}_2})$	$A_1 \times A_1 \times B_2$	0
(11.11), p=2m+1	$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}})$	$B_m \times C_m$	0
(11.11), p=2m-1	$(\mathrm{SO}_{2m-1} \times \mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m-1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}})$	$B_{m-1} \times D_m$	0

TABLE 9. Reductive models in Table 11 of [22]

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(11.1), n=2k	$(\mathrm{SL}_2\times\mathrm{SO}_{2k}\times\mathrm{SL}_2, std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_{2k}}\oplus std_{\mathrm{SO}_{2k}}\otimes std_{\mathrm{SL}_2})$	$A_1\times A_1\times B_2$	D_{k-2}
(11.2)	$(\mathrm{Spin}_{12}, \mathrm{HSpin}_{12}^+\oplus\mathrm{HSpin}_{12}^-)$	$(A_1)^2\times B_2$	$A_1\times A_1$
(11.3)	$(\mathrm{SL}_2\times\mathrm{Spin}_{12}, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12})$	$(A_1)^3$	A_3
(11.4)	$(\mathrm{Sp}_4\times\mathrm{Spin}_{12}, std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12})$	$C_2\times A_1\times D_4$	A_1
(11.6)	$(\mathrm{SL}_2\times\mathrm{Spin}_8\times SL_2, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2})$	$(A_1)^3$	A_1
(11.11), $p=2k+1>2m+1$	$(\mathrm{SO}_{2k+1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2k+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_m\times C_m$	B_{k-m}
(11.11), $p=2n-1<2m-1$	$(\mathrm{SO}_{2n-1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2n-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_{n-1}\times D_n$	C_{m-n}
(11.12)	$(\mathrm{Sp}_6, \wedge_0^3\oplus std_{\mathrm{Sp}_6})$	$A_1\times A_1$	A_1

TABLE 10. Non-reductive models in Table 11 of [22]

5.1. **The reductive case.** For (11.7), the associated quadruple Δ is

$$(\mathrm{GSp}_4\times\mathrm{GSpin}_8\times\mathrm{GL}_2, S(\mathrm{GSpin}_8\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}). \tag{5.1}$$

Note that when we take principal series on GSp_4 and GL_2, this period integral recovers the Rankin-Selberg integral in [12]. The unramified computation in loc. cit. proves Theorem 1.7 in this case. This quadruple is self-dual.

For (11.9), the associated quadruple Δ is given by

$$(\mathrm{GSp}_6\times\mathrm{GSO}_4, S(\mathrm{GSO}_4\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4}).$$

By the theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$, the integral over SO_4 of a cusp form on GSO_4 with the theta series associated to ρ_H produces an automorphic form on GSp_4. Then the integral over $G(\mathrm{Sp}_4\times\mathrm{SL}_2)$ is just the period integral for the pair $(\mathrm{GSp}_6\times\mathrm{GSp}_4, G(\mathrm{Sp}_4\times\mathrm{Sp}_2))$ in [36]. The unramified computation in [36] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$ proves Theorem 1.7 in this case.

For (11.10), the associated quadruple Δ is given by

$$(\mathrm{GL}_4\times\mathrm{GSO}_4, S(\mathrm{GSp}_4\times\mathrm{GSO}_4), std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4}). \tag{5.2}$$

By the theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$, the integral over SO_4 of a cusp form on GSO_4 with the theta series associated to ρ_H produces an automorphic form on GSp_4. Then the integral over GSp_4 is just the period integral for the pair $(\mathrm{GL}_4\times\mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_6\times\mathrm{SO}_5$. The unramified computation in [19] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_6\times\mathrm{Sp}_4$ (here we view $\mathrm{SL}_2\times\mathrm{SL}_2$ as a subgroup of Sp_4) and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_5\times\mathrm{SO}_4$ in [7]. This proves Theorem 1.9.

For (11.11) when $p=2m+1$, the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m}, H=G, std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}). \tag{5.3}$$

By the theta correspondence for $\mathrm{SO}_{2m+2}\times\mathrm{Sp}_{2m}$, the integral over Sp_{2m} of a cusp form on Sp_{2m} with the theta series associated to ρ_H produces an automorphic form on SO_{2m+2}. Then the integral over SO_{2m+1} is just the period integral for the Gross-Prasad period for $\mathrm{SO}_{2m+2}\times\mathrm{SO}_{2m+1}$. The unramified computation in [19] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2m+2}\times\mathrm{Sp}_{2m}$ proves Theorem 1.7 in this case. This quadruple is self-dual and it is clear that Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2m+2}\times$

Sp_{2m} and the global period integral conjecture for the Gross-Prasad model of $\mathrm{SO}_{2m+2}\times\mathrm{SO}_{2m+1}$ in [7]. This proves Theorem 1.9.

For (11.11) when $p=2m-1$, the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m-2},\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2},std_{\mathrm{SO}_{2m}}\otimes std_{\mathrm{Sp}_{2m-2}}). \tag{5.4}$$

By the theta correspondence for $\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2}$, the integral over Sp_{2m-2} of a cusp form on Sp_{2m} with the theta series associated to ρ_H produces an automorphic form on SO_{2m}. Then the integral over SO_{2m} is just the Gross-Prasad period for $\mathrm{SO}_{2m+1}\times\mathrm{SO}_{2m}$. The unramified computation in [19] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2}$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2}$ and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_{2m}\times\mathrm{SO}_{2m+1}$ in [7]. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 9 is given as follows (note that ι is trivial for all these cases) where

$$*=(\mathrm{GSp}_4\times\mathrm{GSpin}_8\times\mathrm{GL}_2,S(\mathrm{GSpin}_8\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)),std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2})$$

(G,H,ρ_H)	$\hat{\rho}$
$*$	$std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}$
$(\mathrm{GSp}_6\times\mathrm{GSO}_4,S(\mathrm{GSO}_4\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)),std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2}\otimes\mathrm{Spin}_7\oplus\mathrm{Spin}_7\otimes std_{\mathrm{SL}_2}$
$(\mathrm{GL}_4\times\mathrm{GSO}_4,S(\mathrm{GSp}_4\times\mathrm{GSO}_4),std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_6}\oplus std_{\mathrm{SO}_6}\otimes std_{\mathrm{SL}_2}$
$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m},H=G,std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m-2},\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2},std_{\mathrm{SO}_{2m}}\otimes std_{\mathrm{Sp}_{2m-2}})$	$std_{\mathrm{SO}_{2m-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$

TABLE 11. Dual quadruples of Table 9

5.2. **The non-reductive case.** For (11.1) when $n=2k$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2k}\times\mathrm{GSO}_4,S(\mathrm{GSp}_4\times\mathrm{GSO}_4),std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4},\mathrm{GSpin}_{2k-4}\times(\mathrm{GL}_1)^2\times T_{\mathrm{GSO}_4}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GSpin}_6\times(\mathrm{GL}_1)^{k-3}\times\mathrm{GSO}_4$ whose stabilizer in M is $\mathrm{GSpin}_5\times\mathrm{GSO}_4$. We can embed H into the stabilizer as in (5.2) and this quadruple is the Whittaker induction of (5.2). It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2k}\times\mathrm{Sp}_4$ (here we view $\mathrm{SL}_2\times\mathrm{SL}_2$ as a subgroup of Sp_4) and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_5\times\mathrm{SO}_4$ in [7]. This proves Theorem 1.9.

For (11.2), the associated quadruple Δ is

$$(\mathrm{GSO}_{12},S(\mathrm{GSp}_4\times\mathrm{GSO}_4),0,\mathrm{GL}_2\times\mathrm{GL}_2\times(\mathrm{GL}_1)^3).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_4\times\mathrm{GSO}_4$ whose stabilizer in M is H. It is the Whittaker induction of (5.2). It is clear that Theorem 1.12 holds in this case.

For (11.3), we first introduce a reductive quadruple which belongs to Table S of [22]. Let $G=(\mathrm{GL}_2)^5$ and $H=S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2)$ where the embedding $H\to G$ is given by mapping the first GL_2-copy into the first GL_2-copy, and mapping the second (resp. third) GL_2-copy diagonally into the second and third (resp. fourth and fifth) GL_2-copy. Let

$\rho_H = std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}$ be the triple product representation and ι be trivial. The quadruple

$$\Delta_0 = (G, H, \rho_H, \iota) = ((\mathrm{GL}_2)^5, S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}, 1) \tag{5.5}$$

will be used to explain several models in this paper. This quadruple comes from Table S of [22], it is obtained by combining two copies of Model (S.3) with $n = 4$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2} \otimes std_{\mathrm{GL}_2,3} \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,4} \otimes std_{\mathrm{GL}_2,5}$$

where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy. To justify the duality, we will prove Theorem 1.7 and Theorem 1.9 for this case.

We start with Theorem 1.7. By the theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$, the integral of a cusp form on the first GL_2-copy with the theta series produces cusp forms on the other two GL_2-copies of H. Then the period integral over the remaining two copies of GL_2 are just the period for two trilinear GL_2-models (i.e., the first, second, third GL_2-copies and the first, fourth, fifth GL_2-copies). Then Theorem 1.7 follows from the unramified computation in [19]. In fact, in this case, Conjecture 1.1(1) follows from the result in [18] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$. For the dual side, Conjecture 1.1(2) in this case is also a direct consequence of the result in [18] and Theorem 2.5 applied to theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$. This proves Theorem 1.9.

For (11.3) the associated quadruple Δ is

$$(\mathrm{GSO}_{12} \times \mathrm{PGL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0, \mathrm{GL}_4 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{PGL}_2}). \tag{5.6}$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GSO}_4 \times \mathrm{PGL}_2$ whose stabilizer in M is $S(\mathrm{GL}_2 \times \mathrm{GSO}_4) \times \mathrm{GL}_2$. We can embed H into the stabilizer by mapping the GL_2-copy of H into the GL_2-copy of the stabilizer and by mapping $\mathrm{GSO}_4 = \mathrm{GL}_2 \times \mathrm{GL}_2/\mathrm{GL}_1^{diag}$ into $\mathrm{GSO}_4 \times \mathrm{PGL}_2$ via the idenity map on GSO_4 and the projection map $\mathrm{GSO}_4 = \mathrm{GL}_2 \times \mathrm{GL}_2/\mathrm{GL}_1^{diag} \to \mathrm{PGL}_2$ via the firts GL_2-copy of GSO_4. It is clear that the induced embedding from H into M is the same as (5.5). This quadruple is the Whittaker induction of (5.5). It is clear that Theorem 1.12 holds in this case.

For (11.4), the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8}, T_{\mathrm{GSp}_4} \times \mathrm{GL}_2 \times (\mathrm{GL}_1)^5).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GSpin}_8$ whose stabilizer in M is $\mathrm{GSpin}_4 \times S(\mathrm{GL}_2 \times \mathrm{GSpin}_8)$ and we can naturally embed H into the stabilizer. This quadruple is the Whittaker induction of (5.1). It is clear that Theorem 1.12 holds in this case.

For (11.6), the associated quadruple Δ is

$$(\mathrm{GSO}_8 \times \mathrm{GSO}_4, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0, \mathrm{GL}_2 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{GSO}_4}). \tag{5.7}$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSO}_4 \times \mathrm{GL}_2 \times \mathrm{GSO}_4$ whose stabilizer in M is $S(\mathrm{GSO}_4 \times \mathrm{GL}_2) \times \mathrm{GSO}_4$. We can embed H into the stabilizer by making the GL_2-copy of H into the GL_2-copy of the stabilizer and by mapping the GSO_4-copy of H diagonally into the two GSO_4-copies of the stabilizer. It is clear that the induced embedding from H into M is the same as (5.5). This quadruple is the Whittaker induction of (5.5). It is clear that Theorem 1.12 holds in this case.

For (11.11) when $p = 2k+1 > 2m+1$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1}\times \mathrm{Sp}_{2k}, \mathrm{SO}_{2m+1}\times \mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}, T_{\mathrm{SO}_{2m+1}}\times \mathrm{Sp}_{2k-2m}\times (\mathrm{GL}_1)^m).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{Sp}_{2m}\times (\mathrm{GL}_1)^{k-m}\times \mathrm{SO}_{2m+1}$ whose stabilizer in M is H. This is the Whittaker induction of (5.3). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{Sp}_{2m}\times \mathrm{SO}_{2k+2}$ and the Gan-Gross-Prasad conjecture (Conjecture 9.11 of [8]) for non-tempered Arthur packet of the Gross-Prasad model of $\mathrm{SO}_{2k+2}\times\mathrm{SO}_{2k+1}$. This proves Theorem 1.9.

For (11.11) when $p = 2n-1 < 2m-1$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1}\times \mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n}\times \mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}}\otimes std_{\mathrm{Sp}_{2n-2}}, \mathrm{SO}_{2m-2n+1}\times (\mathrm{GL}_1)^n\times T_{\mathrm{Sp}_{2n-2}}). \tag{5.8}$$

This is the Whittaker induction of (5.4). It is clear that Theorem 1.12 holds in this case. By the theta correspondence for $\mathrm{SO}_{2n}\times\mathrm{Sp}_{2n-2}$, the integral over Sp_{2n-2} of a cusp form on Sp_{2n} with the theta series associated to ρ_H produces an automorphic form on SO_{2n}. Then the integral over SO_{2n} is just the Gross-Prasad period for $\mathrm{SO}_{2m+1}\times\mathrm{SO}_{2n}$. The unramified computation in [19] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2n}\times\mathrm{Sp}_{2n-2}$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{Sp}_{2m}\times\mathrm{SO}_{2n}$ and the global period integral conjecture for the Gross-Prasad period of $\mathrm{SO}_{2n}\times\mathrm{SO}_{2n-1}$ in [7]. This proves Theorem 1.9.

For (11.12), the associated quadruple Δ is

$$(\mathrm{GSpin}_7, \mathrm{GL}_2, S(\mathrm{GL}_2\times\mathrm{GL}_2), std_{\mathrm{GL}_2}, \mathrm{GL}_2\times(\mathrm{GL}_1)^2).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_3\times\mathrm{GL}_2$ whose stabilizer in M is H. The representation ρ_H is the standard representation on the first GL_2-copy. This quadruple is the Whittaker induction of (5.3) when $m = 1$. It is clear that Theorem 1.12 holds in this case.

By the discussion above, the strongly tempered quadruple associated to Table 10 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$* = (\mathrm{GSpin}_4\times\mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}).$$

(G,H,ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSpin}_{2k}\times\mathrm{GSO}_4, S(\mathrm{GSp}_4\times\mathrm{GSO}_4), std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4})$	D_{k-2}	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_{2k}}\oplus std_{\mathrm{SO}_{2k}}\otimes std_{\mathrm{SL}_2}$
$(\mathrm{GSO}_{12}, S(\mathrm{GSp}_4\times\mathrm{GSO}_4), 0)$	$A_1\times A_1$	$\mathrm{HSpin}_{12}^{+}\oplus\mathrm{HSpin}_{12}^{-}$
$(\mathrm{GSO}_{12}\times\mathrm{PGL}_2, S(\mathrm{GL}_2\times\mathrm{GSO}_4), 0)$	A_3	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12}$
$*$	A_1	$std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12}$
$(\mathrm{GSO}_8\times\mathrm{GSO}_4, S(\mathrm{GL}_2\times\mathrm{GSO}_4), 0)$	A_1	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}$
$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2k}, \mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}})$	C_{k-m}	$std_{\mathrm{SO}_{2k+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n}\times\mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}}\otimes std_{\mathrm{Sp}_{2n-2}})$	B_{m-n}	$std_{\mathrm{SO}_{2n-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{GSpin}_7, S(\mathrm{GL}_2\times\mathrm{GL}_2), std_{\mathrm{GL}_2})$	A_1	$\wedge^3\oplus std_{\mathrm{Sp}_6}$

Table 12. Dual quadruples of Table 10

6. Models in Table 12

In this section we will consider Table 12 of [22], this is for the case when $\hat{\rho}$ is the direct sum of three irreducible representations of $\hat{G}$ with two of them dual to each other (i.e. $\hat{\rho}=\hat{\rho}_0\oplus T(\hat{\tau})$). It is easy to check that the representations in (12.4), (12.9), (12.10), (12.11), (12.11) of [22] are not anomaly free. Hence it remains to consider the following cases. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(12.5)	$(\mathrm{SL}_6\times\mathrm{SL}_2,\wedge^3\oplus T(std_{\mathrm{SL}_6}\otimes std_{\mathrm{SL}_2}))$	$A_1\times A_1\times A_3$	0
(12.7), m=1	$(\mathrm{SL}_2\times\mathrm{SL}_4, std_{\mathrm{SL}_2}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4}))$	$A_1\times A_1$	0
(12.7), m=2	$(\mathrm{Sp}_4\times\mathrm{SL}_4, std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4}))$	$C_2\times A_3$	0
(12.7), m=3	$(\mathrm{Sp}_6\times\mathrm{Spin}_6, std_{\mathrm{Sp}_6}\otimes std_{\mathrm{Spin}_6}\oplus T(\mathrm{HSpin}_6))$	$A_3\times A_3$	0
(12.8)	$(\mathrm{SL}_2\times\mathrm{SL}_4\times\mathrm{SL}_2, std_{\mathrm{SL}_2}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4}\otimes std_{\mathrm{SL}_2}))$	$A_1\times A_1\times A_3$	0

Table 13. Reductive models in Table 12 of [22]

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(12.1)	$(\mathrm{Spin}_{12},\mathrm{HSpin}_{12}\oplus T(std_{\mathrm{Spin}_{12}}))$	$A_1\times A_1\times A_1$	A_3
(12.2)	$(\mathrm{SL}_2\times\mathrm{Spin}_{10}, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_{10}}\oplus T(std_{\mathrm{Spin}_{10}}))$	$A_1\times A_1\times A_3$	A_1
(12.3)	$(\mathrm{SL}_2\times\mathrm{Spin}_8, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_8}\oplus T(std_{\mathrm{Spin}_8}))$	$A_1\times A_1\times A_1$	A_1
(12.6)	$(\mathrm{SL}_6,\wedge^3\oplus T(std_{\mathrm{SL}_6}))$	$A_1\times A_1$	$A_1\times A_1$
(12.7), $m>3$	$(\mathrm{Sp}_{2m}\times\mathrm{SO}_6, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_6}\oplus T(\mathrm{HSpin}_6))$	$A_3\times A_3$	C_{m-3}

Table 14. Non-reductive models in Table 12 of [22]

6.1. **The reductive case.**

For (12.5), the associated quadruple Δ is

$$(\mathrm{GL}_6\times\mathrm{GL}_2,\mathrm{GL}_2\times S(\mathrm{GL}_4\times\mathrm{GL}_2),\wedge^2\otimes std_{\mathrm{GL}_2}).$$

At this moment we do not have much evidence that the above is the dual quadruple other than the fact that $\wedge^2\otimes std_{\mathrm{GL}_2}$ is the only feasible choice of symplectic representation. We believe an unramified computation similar to [19] and [36] can confirm the duality in this case.

For (12.7) with $m=1$, the associated quadruple Δ is

$$(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2,0).$$

This is the model $(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2)$ studied in [36] and the unramified computation in [36] proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_2\times\mathrm{GSO}_6$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [8]) for non-tempered Arthur packet of the Rankin-Selberg integral of $\mathrm{GL}_4\times\mathrm{GL}_4$. This proves Theorem 1.9.

For (12.7) with $m=2$, the associated quadruple Δ is

$$(\mathrm{GL}_4\times\mathrm{GSp}_4,\mathrm{GL}_4\times\mathrm{GSp}_4,T(std_{\mathrm{GL}_4}\otimes std_{\mathrm{GSp}_4})).$$

Observe that this is the dual to the quadruple in (4.5), thus both Theorems 1.7 and 1.9 have been proved there.

For (12.7) with $m = 3$, the associated quadruple Δ is

$$(\mathrm{GSpin}_7 \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6)). \tag{6.1}$$

By the theta correspondence for $\mathrm{GL}_4 \times \mathrm{GL}_4$, the integral over the second GSpin_6-copy of a cusp form on GSpin_6 with the theta series associated to ρ_H produces the same cusp form with an extra central value of the Spin L-function. Then the integral over the other copy of GSpin_6 is just the period integral for the Gross-Prasad model $\mathrm{GSpin}_7 \times \mathrm{GSpin}_6$. The unramified computation in [19] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_4 \times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_6 \times \mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4 \times \mathrm{GL}_4$. This proves Theorem 1.9.

For (12.8), the associated quadruple Δ is

$$(\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GL}_4) \times \mathrm{GL}_2, std_{\mathrm{GL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4} \times std_{\mathrm{GL}_2})). \tag{6.2}$$

Note that when we put principal series on both GL_2 copies, this period integral recovers the Rankin-Selberg integral in [29]. The unramified computation in [29] proves Theorem 1.7 in this case. This quadruple is self-dual.

By the discussion above, the strongly tempered quadruple associated to Table 13 is given as follows (ι is trivial for all these cases) where

$$* = (\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GL}_4) \times \mathrm{GL}_2, std_{\mathrm{GL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4} \times std_{\mathrm{GL}_2})).$$

(G, H, ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6} \otimes std_{\mathrm{SL}_2})$
$(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2, 0)$	$std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4})$
$(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GL}_4 \times \mathrm{GSp}_4, T(std_{\mathrm{GL}_4} \otimes std_{\mathrm{GSp}_4}))$	$std_{\mathrm{Sp}_4} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4})$
$(\mathrm{GSpin}_7 \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6))$	$std_{\mathrm{Sp}_6} \otimes std_{\mathrm{Spin}_6} \oplus T(\mathrm{HSpin}_6)$
$*$	$std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4} \otimes std_{\mathrm{SL}_2})$

TABLE 15. Dual quadruples of Table 13

6.2. **The non-reductive case.** For (12.1), we first introduce a reductive quadruple which belongs to Table S of [22]. Let $G = (\mathrm{GL}_2)^4$ and $H = S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2)$ where the embedding $H \to G$ is given by mapping the first two GL_2-copies into the first two GL_2-copy, and mapping the last GL_2-copy diagonally into the third and fourth GL_2-copy. Let $\rho_H = std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \oplus T(std_{\mathrm{GL}_2,2})$ where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy and ι be trivial. This quadruple

$$\Delta_0 = (G, H, \rho_H, \iota) = ((\mathrm{GL}_2)^4, S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \oplus T(std_{\mathrm{GL}_2,2}), 1) \tag{6.3}$$

is almost the same as (5.5) except we replace the cusp form on one GL_2-copy by theta series. It is obtained by combining Model (S.3) and (S.11) in Table S of [22] with $n = 4$ and $m = 2$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1), \ \hat{\rho} = T(std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2}) \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,3} \otimes std_{\mathrm{GL}_2,4}.$$

We can use the same argument as in (5.5) to prove Theorem 1.7 and Theorem 1.9 for this case.

For (12.1), the associated quadruple Δ is

$$(\mathrm{GSO}_{12}, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}), \mathrm{GL}_4 \times (\mathrm{GL}_1)^3).$$

The attached period integral is the same as model in (5.6) except we replace the cusp form on GL_2 by theta series. This is the Whittaker induction of (6.3) and it is clear that Theorem 1.12 holds in this case.

For (12.2), the associated quadruple Δ is

$$(\mathrm{GSpin}_{10} \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSpin}_6) \times \mathrm{GL}_2, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^4 \times T_{\mathrm{GL}_2})$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GSpin}_6 \times \mathrm{GL}_2$ whose stabilizer in M is H. This quadruple is the Whittaker induction of (6.2). It is clear that Theorem 1.12 holds in this case.

For (12.3), the associated quadruple Δ is

$$(\mathrm{GSO}_8 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{GL}_2}).$$

The attached period integral is the same as the model (5.7) except we replace the cusp form on one GL_2-copy by theta series. This is the Whittaker induction of (6.3) and it is clear that Theorem 1.12 holds in this case.

For (12.6), we first introduce a reductive quadruple from Table S of [22] (it is obtained by combining Model (S.10) and Model (S.3) with $n = 4$)

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2, T(std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}), 1) \tag{6.4}$$

where H embeds into G by mapping the first GL_2-copy into the first GL_2-copy and mapping the second GL_2-copy diagonally into the second and third GL_2-copy. We claim the dual quadruple is given by

$$(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2,1}) \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2} \otimes std_{\mathrm{GL}_2,3}$$

where $std_{\mathrm{GL}_2,i}$ is the standard representation of the i-th GL_2-copy. To justify the duality, we will prove Theorem 1.7 and Theorem 1.9 for this case.

We start with Theorem 1.7. By the theta correspondence for $\mathrm{GL}_2 \times \mathrm{GL}_2$, the integral over the first GL_2-copy of a cusp form in π with the theta series gives a cusp form on GL_2 (in the same space π, note though Theorem 2.2 applied to the correspondence does introduce the central value of the standard L-function). Then the integral over the other GL_2-copy is just the period integral for the trilinear GL_2-model. As a result, Conjecture 1.1(1) and Theorem 1.7 follow from the theta correspondence for $\mathrm{GL}_2 \times \mathrm{GL}_2$ and the result in [18]. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$ and the Rankin-Selberg integral of $\mathrm{GL}_2 \times \mathrm{GL}_2$. This proves Theorem 1.9 in this case.

Now we can write down the associated quadruple Δ of (12.6). It is given by

$$(\mathrm{GL}_6, \mathrm{GL}_2 \times \mathrm{GL}_2, 0, \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ whose stabilizer in M is H. This quadruple is the Whittaker induction of (6.4). It is clear that Theorem 1.12 holds in this case.

For (12.7) when $m > 3$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6), \mathrm{GSpin}_{2m-5} \times (\mathrm{GL}_1)^3 \times (\mathrm{GL}_1)^4).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{m-3} \times \mathrm{GSpin}_7 \times \mathrm{GSpin}_6$ whose stabilizer in M is H. It is the Whittaker induction of (6.1). It is clear that Theorem 1.12 holds in this case. The unramified computation in [19] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_4 \times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_{2m} \times \mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4 \times \mathrm{GL}_4$. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 14 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$* = (\mathrm{GSpin}_{10} \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSpin}_6) \times \mathrm{GL}_2, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_2})).$$

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSO}_{12}, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_3	$\mathrm{HSpin}_{12} \oplus T(std_{\mathrm{Spin}_{12}})$
$*$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{10}} \oplus T(std_{\mathrm{Spin}_{10}})$
$(\mathrm{GSO}_8 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus T(std_{\mathrm{Spin}_8})$
$(\mathrm{GL}_6, \mathrm{GL}_2 \times \mathrm{GL}_2, 0)$	$A_1 \times A_1$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6})$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6))$	B_{m-3}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_6} \oplus T(\mathrm{HSpin}_6)$

TABLE 16. Dual quadruples of Table 14

7. MODELS IN TABLE 22

In this section we will consider Table 22 of [22], this is for the case when $\hat{\rho}$ is the direct sum of four irreducible representations of $\hat{G}$ of the form $T(\rho_1) \oplus T(\rho_2)$. All the representations in Table 22 of [22] are anomaly free, so we need to consider all of them. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [22]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(22.2), n=2m	$(\mathrm{SL}_n, T(\wedge^2) \oplus T(std_{\mathrm{SL}_n}))$	$A_{m-1} \times A_{m-1}$	0
(22.2), n=2m+1	$(\mathrm{SL}_n, T(\wedge^2) \oplus T(std_{\mathrm{SL}_n}))$	$A_m \times A_{m-1}$	0
(22.3), m=n	$(\mathrm{SL}_m \times \mathrm{SL}_n, T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1} \times A_{n-1}$	0
(22.3), m=n+1	$(\mathrm{SL}_m \times \mathrm{SL}_n, T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1} \times A_{n-1}$	0
(22.3), m=n-1	$(\mathrm{SL}_m \times \mathrm{SL}_n, T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n}))$	$A_m \times A_{m-1}$	0
(22.3), m=n-2	$(\mathrm{SL}_m \times \mathrm{SL}_n, T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n}))$	$A_m \times A_{m-1}$	0
(22.4), n=3	$(\mathrm{SL}_3, T(std_{\mathrm{SL}_3}) \oplus T(std_{\mathrm{SL}_3}))$	A_1	0
(22.5), m=2	$(\mathrm{Sp}_4, T(std_{\mathrm{Sp}_4}) \oplus T(std_{\mathrm{Sp}_4}))$	$A_1 \times A_1$	0

TABLE 17. Reductive models in Table 22 of [22]

Number in [22]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(22.1)	$(\mathrm{Spin}_8, T(std_{\mathrm{Spin}_8})\oplus T(\mathrm{HSpin}_8))$	$A_1\times A_1\times A_1$	A_1
(22.3), $m>n+1$	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1}\times A_{n-1}$	A_{m-n+1}
(22.3), $m<n-2$	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_m\times A_{m-1}$	A_{n-m-2}
(22.4), $n>3$	$(\mathrm{SL}_n, T(std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	A_1	A_{n-3}
(22.5), $m>2$	$(\mathrm{Sp}_{2m}, T(std_{\mathrm{Sp}_{2m}})\oplus T(std_{\mathrm{Sp}_{2m}}))$	$A_1\times A_1$	C_{m-2}

TABLE 18. Non-reductive models in Table 22 of [22]

7.1. **The reductive case.** For (22.2) with $n=2m$, the associated quadruple Δ is

$$(\mathrm{GL}_{2m}, \mathrm{GL}_m\times \mathrm{GL}_m, T(std_{\mathrm{GL}_m})).$$

The period integral in this case is exactly the Rankin-Selberg integral in [3]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.2) with $n=2m+1$, the associated quadruple Δ is

$$(\mathrm{GL}_{2m+1}, \mathrm{GL}_{m+1}\times \mathrm{GL}_m, T(std_{\mathrm{GL}_{m+1}})).$$

The period integral in this case is exactly the Rankin-Selberg integral in [3]. The unramified computation in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.3) with $m=n$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times \mathrm{GL}_n, \mathrm{GL}_n\times \mathrm{GL}_n, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}\oplus std_{\mathrm{GL}_n})). \tag{7.1}$$

By the theta correspondence for $\mathrm{GL}_n\times \mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to $T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})$ produces a cusp form on GL_n. Then the integral over the other GL_n-copy is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_n$. This quadruple is self-dual. The Rankin-Selberg integral of $\mathrm{GL}_n\times \mathrm{GL}_n$ and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times \mathrm{GL}_n$ proves Conjecture 1.1, Theorem 1.7 and Theorem 1.9. Notice that the theta correspondence introduces an extra central value of the standard L-function in this case.

For (22.3) with $m=n+1$, the associated quadruple Δ is

$$(\mathrm{GL}_{n+1}\times \mathrm{GL}_n, \mathrm{GL}_n\times \mathrm{GL}_n, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})). \tag{7.2}$$

By the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to ρ_H produces another cusp form on GL_n. Then the integral over the other GL_n-copy is just the Rankin-Selberg integral of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$. The Rankin-Selberg integral of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ in [20] and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times \mathrm{GL}_n$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. Again notice that the theta correspondence introduces an extra central value of the standard L-function. For the dual side, Conjecture 1.1(2) follows from the theta correspondence of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ with the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_n$. This proves Theorem 1.9.

For (22.3) with $m=n-1$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times \mathrm{GL}_{n-1}, \mathrm{GL}_n\times \mathrm{GL}_{n-1}, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}}\oplus std_{\mathrm{GL}_n})). \tag{7.3}$$

By the theta correspondence for $\mathrm{GL}_n\times \mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to ρ_H produces another cusp form on GL_n. Then the integral over GL_{n-1} is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$. This quadruple is self-dual. The Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$ and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$ proves Conjecture 1.1, Theorem 1.7 and Theorem 1.9 in this case. As before, the theta correspondence introduces an extra central value of the standard L-function.

For (22.3) with $m=n-2$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times\mathrm{GL}_{n-2},\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2},T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_{n-2}})). \tag{7.4}$$

By the theta correspondence for $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$, the integral over GL_{n-2} of a cusp form on GL_{n-2} with the theta series associated to ρ_H produces an Eisenstein series on GL_{n-1} which is induced from the cuspidal automorphic representation on GL_{n-2} and the trivial character. Then the integral over GL_{n-1} is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$. The Rankin-Selberg integral of $\mathrm{GL}_{n-1}\times\mathrm{GL}_n$ in [20] and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence of $\mathrm{GL}_{n-1}\times\mathrm{GL}_n$ with the Rankin-Selberg integral of $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$. This proves Theorem 1.9.

For (22.4) with $n=3$, the associated quadruple Δ is

$$(\mathrm{GL}_3,\mathrm{GL}_2\times\mathrm{GL}_1,T(std_{\mathrm{GL}_2})). \tag{7.5}$$

The period integral is essentially the Rankin-Selberg integral of $\mathrm{GL}_3\times\mathrm{GL}_2$ except that we replace the cusp form on GL_2 by theta series. The result in [20] proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.5) with $m=2$, the associated quadruple Δ is

$$(\mathrm{GSpin}_5\times\mathrm{GL}_1,\mathrm{GSpin}_4\times\mathrm{GL}_1,T(\mathrm{HSpin}_4^+\oplus\mathrm{HSpin}_4^-\otimes std_{\mathrm{GL}_1})). \tag{7.6}$$

The period integral is essentially the Gross-Prasad period for $\mathrm{GSpin}_5\times\mathrm{GSpin}_4$ except that we replace the cusp form on GSpin_4 by theta series. The unramified computation in [19] proves Theorem 1.7 in this case.

By the discussion above, the strongly tempered quadruple associated to Table 13 is given as follows (ι is trivial for all these cases).

(G,H,ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_{2m},\mathrm{GL}_m\times\mathrm{GL}_m,T(std_{\mathrm{GL}_m}))$	$T(\wedge^2)\oplus T(std_{\mathrm{GL}_{2m}})$
$(\mathrm{GL}_{2m+1},\mathrm{GL}_{m+1}\times\mathrm{GL}_m,T(std_{\mathrm{GL}_{m+1}}))$	$T(\wedge^2)\oplus T(std_{\mathrm{GL}_{2m+1}})$
$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_n\times\mathrm{GL}_{n-1},\mathrm{GL}_n\times\mathrm{GL}_{n-1},T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}})\oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_n\times\mathrm{GL}_{n-2},\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2},T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_{n-2}}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-2}})\oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_3,\mathrm{GL}_2\times\mathrm{GL}_1,T(std_{\mathrm{GL}_2}))$	$T(std_{\mathrm{SL}_3})\oplus T(std_{\mathrm{SL}_3})$
$(\mathrm{GSpin}_5\times\mathrm{GL}_1,\mathrm{GSpin}_4\times\mathrm{GL}_1,T(\mathrm{HSpin}_4^+\oplus\mathrm{HSpin}_4^-\otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{Sp}_4})\oplus T(std_{\mathrm{Sp}_4})$

TABLE 19. Dual quadruples of Table 17

7.2. **The non-reductive case.** For (22.1), we first introduce a reductive quadruple which belongs to Table S of [22]. Let $G=(\mathrm{GL}_2)^3$, $H=S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2)$ and $\rho_H=std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_2,2})\oplus T(std_{\mathrm{GL}_2,3})$ where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy and ι be trivial. This quadruple

(7.7)

$$\Delta_0=(G,H,\rho_H,\iota)=((\mathrm{GL}_2)^3,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_2,2}\oplus T(std_{\mathrm{GL}_2,3}),1)$$

is almost the same as (5.5) except we replace the cusp form on two GL_2-copies by theta series. It is obtained by combining two copies of Model (S.11) in Table S of [22] with $m = 2$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2}) \oplus T(std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,3}).$$

We can use the same argument as in (5.5) to prove Theorem 1.7 and Theorem 1.9 for this case.

For (22.1), the associated quadruple Δ is

$$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^3).$$

The period integral is the same as (5.7) except we replace the cusp form on both GL_2-copies by theta series. This is the Whittaker induction of (7.7) and it is clear that Theorem 1.12 holds in this case.

For (22.3) when $m > n+1$, the associated quadruple Δ is

$$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}), (\mathrm{GL}_1)^n \times \mathrm{GL}_{m-n} \times T_{\mathrm{GL}_n}).$$

When $n-m$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{m-n-1} \times \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ (resp. $M = \mathrm{GL}_1^{m-n} \times \mathrm{GL}_n \times \mathrm{GL}_n$) whose stabilizer in M is H. It is the Whittaker induction of (7.2) (resp. (7.1)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_{m+1}$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [8]) for non-tempered Arthur packet of the Rankin-Selberg integral of $\mathrm{GL}_{m+1} \times \mathrm{GL}_m$. This proves Theorem 1.9.

For (22.3) when $m < n-2$, the associated quadruple Δ is

$$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_m \times \mathrm{GL}_{m+1}, T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_{m+1}}), T_{\mathrm{GL}_m} \times (\mathrm{GL}_1)^{m-1} \times \mathrm{GL}_{n-m-1}).$$

When $n-m-1$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{n-m-2} \times \mathrm{GL}_{m+2} \times \mathrm{GL}_m$ (resp. $M = \mathrm{GL}_1^{n-m-1} \times \mathrm{GL}_{m+1} \times \mathrm{GL}_m$) whose stabilizer in M is H. It is the Whittaker induction of (7.4) (resp. (7.3)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_{m+1}$ and the Rankin-Selberg integral of $\mathrm{GL}_{m+1} \times \mathrm{GL}_m$. This proves Theorem 1.9.

For (22.4) when $n > 3$, we need to introduce another reductive quadruple from Table S of [22] (it is obtained by combining two copies of Model (S.10))

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_2 \times \mathrm{GL}_1, \mathrm{GL}_2 \times \mathrm{GL}_1, T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_1}), 1). \tag{7.8}$$

We claim that the dual quadruple is given by

$$(\hat{G}, \hat{G}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_1}),$$

i.e., it is self-dual. It is easy to see that Conjecture 1.1, Theorem 1.7 and Theorem 1.9 hold in this case.

The associated quadruple Δ for (22.4) with $n > 3$ is given by

$$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}), \mathrm{GL}_{n-2} \times \mathrm{GL}_1 \times \mathrm{GL}_1).$$

When $n-2$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M =$

$\mathrm{GL}_1^{n-3} \times \mathrm{GL}_3$ (resp. $M = \mathrm{GL}_1^{n-2} \times \mathrm{GL}_2$) whose stabilizer in M is H. It is the Whittaker induction of (7.5) (resp. (7.8)). It is clear that Theorem 1.12 holds in this case.

For (22.5) when $m > 2$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1}), \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GSpin}_{2m-3}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{m-2} \times \mathrm{GSpin}_5$ whose stabilizer in M is H. It is the Whittaker induction of (7.6). It is clear that Theorem 1.12 holds in this case. The period integral is essentially the Gross-Prasad period for $\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_4$ except that we replace the cusp form on GSpin_4 by theta series. The unramified computation in [19] proves Theorem 1.7.

By the discussion above, the strongly tempered quadruple associated to Table 18 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$* = (\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1})).$$

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}))$	A_1	$T(std_{\mathrm{Spin}_8}) \oplus T(\mathrm{HSpin}_8)$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}))$	A_{m-n+1}	$T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_m \times \mathrm{GL}_{m+1}, T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_{m+1}}))$	A_{n-m-2}	$T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2})$	A_{n-3}	$T(std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$*$	B_{m-2}	$T(std_{\mathrm{Sp}_{2m}}) \oplus T(std_{\mathrm{Sp}_{2m}})$

TABLE 20. Dual quadruples of Table 18

8. SUMMARY

We summarize our findings in this paper into the following 6 tables.

- Table 21 contains reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) and (2) (i.e., Theorem 1.7 and 1.9).
- Table 22 contains the remaining reductive strongly tempered quadruples. For all of them except $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$, we have provided evidence for Conjecture 1.1(1) (i.e. Theorem 1.7).
- Table 23 contains non-reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) and (2) (i.e., Theorem 1.7, 1.9 and 1.12).
- Table 24 contains non-reductive strongly tempered quadruples for which we have provided evidence only for Conjecture 1.1(1) (i.e., Theorem 1.7 and 1.12).
- Table 25 contains non-reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) by assuming Conjecture 2.10 and we have provided evidence for Conjecture 1.1 (2) (i.e. Theorem 1.9 and 1.12).
- Table 26 contains the remaining non-reductive strongly tempered quadruples. For each of them, we have only provided evidence for Conjecture 1.1(1) by assuming Conjecture 2.10 (i.e., Theorem 1.12).

For quadruples (G, H, ρ_H, ι) in Table 21 and 22, the nilpotent orbit ι is trivial. For all the quadruples $\Delta = (G, H, \rho_H, , \iota)$ in Table 21–26, the dual quadruple is given by $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ where $\hat{\rho}$ is given in the tables and $Z_\Delta = Z_G \cap ker(\rho_H)$.

№	(G, H, ρ_H)	$\hat{\rho}$
1	$(\mathrm{SO}_{2m+1}\times\mathrm{SO}_{2m},\ \mathrm{SO}_{2m},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m}}$
2	$(\mathrm{SO}_{2m+2}\times\mathrm{SO}_{2m+1},\ \mathrm{SO}_{2m+1},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m+2}}$
3	$(\mathrm{GSp}_6\times\mathrm{GSpin}_7,\ S(\mathrm{GSp}_6\times\mathrm{GSpin}_7),\ std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7)$	$std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7$
4	$(\mathrm{GSp}_6\times\mathrm{GSpin}_9, S(\mathrm{GSp}_6\times\mathrm{GSpin}_8), std_{\mathrm{Sp}_6}\otimes\mathrm{HSpin}_8)$	$std_{\mathrm{Sp}_8}\otimes\mathrm{Spin}_7$
5	$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n,T(std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})$
6	$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n,0)$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})$
7	$(\mathrm{GSp}_4\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_{2,2}}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_2})$
8	$(\mathrm{GSp}_4\times\mathrm{GL}_3,H=G,T(std_{\mathrm{GSp}_4}\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_3})$
9	$(\mathrm{GSp}_4\times\mathrm{GL}_4,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{GL}_4}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_4})$
10	$(\mathrm{GSp}_4\times\mathrm{GL}_5,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2)$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_5})$
11	$(\mathrm{GSpin}_7\times\mathrm{GL}_3,\mathrm{GSpin}_6\times\mathrm{GL}_3,T(\mathrm{HSpin}_6\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_6}\otimes Std_{\mathrm{GL}_3})$
12	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m},H=G,std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
13	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m-2},\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2},std_{\mathrm{SO}_{2m}}\otimes std_{\mathrm{Sp}_{2m-2}})$	$std_{\mathrm{SO}_{2m-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
14	$(\mathrm{GL}_4\times\mathrm{GSO}_4,S(\mathrm{GSp}_4\times\mathrm{GSO}_4),std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_6}\oplus std_{\mathrm{SO}_6}\otimes std_{\mathrm{SL}_2}$
15	$(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2,0)$	$std_{\mathrm{SL}_2}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4})$
16	$(\mathrm{GL}_4\times\mathrm{GSp}_4,\mathrm{GL}_4\times\mathrm{GSp}_4,T(std_{\mathrm{GL}_4}\otimes std_{\mathrm{GSp}_4}))$	$std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4})$
17	$(\mathrm{GSpin}_7\times\mathrm{GSpin}_6,\mathrm{GSpin}_6\times\mathrm{GSpin}_6,T(\mathrm{HSpin}_6\otimes\mathrm{HSpin}_6))$	$std_{\mathrm{Sp}_6}\otimes std_{\mathrm{Spin}_6}\oplus T(\mathrm{HSpin}_6)$
18	$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
19	$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
20	$(\mathrm{GL}_n\times\mathrm{GL}_{n-1},\mathrm{GL}_n\times\mathrm{GL}_{n-1},T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
21	$(\mathrm{GL}_n\times\mathrm{GL}_{n-2},\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2},T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_{n-2}}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-2}})\oplus T(std_{\mathrm{GL}_n})$
22	$((\mathrm{GL}_2)^5,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2})$	$*$
23	$\sharp$	$**$
24	$(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2,T(std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}))$	$***$
25	$\sharp\sharp$	$****$
26	$(\mathrm{GL}_2\times\mathrm{GL}_1,\mathrm{GL}_2\times\mathrm{GL}_1,T(std_{\mathrm{GL}_2}\oplus std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{GL}_2}\oplus std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_1})$

TABLE 21. Reductive strongly tempered quadruples 1

$$\sharp=((\mathrm{GL}_2)^4,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_{2,2}})).$$

$$\sharp\sharp=((\mathrm{GL}_2)^3,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_{2,2}}\oplus T(std_{\mathrm{GL}_{2,3}})).$$

$$*=std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}}\otimes std_{\mathrm{GL}_{2,3}}\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,4}}\otimes std_{\mathrm{GL}_{2,5}}.$$

$$**=T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}})\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,3}}\otimes std_{\mathrm{GL}_{2,4}}.$$

$$***=T(std_{\mathrm{GL}_{2,1}})\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}}\otimes std_{\mathrm{GL}_{2,3}}.$$

$$****=T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}})\oplus T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,3}}).$$

№	(G, H, ρ_H)	$\hat{\rho}$
1	$(\mathrm{GSp}_6 \times \mathrm{GSp}_4,\ G(\mathrm{Sp}_4 \times \mathrm{Sp}_2),0)$	$\mathrm{Spin}_5 \otimes \mathrm{Spin}_7$
2	$(\mathrm{GL}_2, \mathrm{SL}_2, T(std_{\mathrm{GL}_2}))$	$T(Sym^2)$
3	$(\mathrm{GSp}_6 \times \mathrm{GSO}_4, S(\mathrm{GSO}_4 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2} \otimes \mathrm{Spin}_7 \oplus \mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2}$
4	$*$	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
5	$(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6} \otimes std_{\mathrm{SL}_2})$
6	$**$	$std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4} \otimes std_{\mathrm{SL}_2})$
7	$(\mathrm{GL}_{2m}, \mathrm{GL}_m \times \mathrm{GL}_m, T(std_{\mathrm{GL}_m}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m}})$
8	$(\mathrm{GL}_{2m+1}, \mathrm{GL}_{m+1} \times \mathrm{GL}_m, T(std_{\mathrm{GL}_{m+1}}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m+1}})$
9	$(\mathrm{GL}_3, \mathrm{GL}_2 \times \mathrm{GL}_1, T(std_{\mathrm{GL}_2}))$	$T(std_{\mathrm{SL}_3}) \oplus T(std_{\mathrm{SL}_3})$
10	$(\mathrm{GSpin}_5 \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{Sp}_4}) \oplus T(std_{\mathrm{Sp}_4})$

Table 22. Reductive strongly tempered quadruples 2

$$* = (\mathrm{GSp}_4 \times \mathrm{GSpin}_8 \times \mathrm{GL}_2, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}).$$

$$** = (\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GL}_4) \times \mathrm{GL}_2, std_{\mathrm{GL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4} \times std_{\mathrm{GL}_2})).$$

№	(G, H, ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0)$	B_{m-n}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
2	$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0)$	D_{n-m}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
3	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSp}_6, S(\mathrm{GSpin}_8 \times \mathrm{GSp}_6), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8)$	B_{m-4}	$std_{\mathrm{Sp}_{2m}} \otimes \mathrm{Spin}_7$
4	$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0)$	B_{m-1}	$T(std_{\mathrm{Sp}_{2n}})$
5	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	B_{m-2}	$T(Std_{\mathrm{GSp}_{2m}} \otimes Std_{\mathrm{GL}_2})$
6	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}))$	B_{m-3}	$T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_3})$
7	$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}} \otimes std_{\mathrm{Sp}_{2n-2}})$	B_{m-n}	$std_{\mathrm{SO}_{2n-1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$
8	$(\mathrm{GSpin}_{2k} \times \mathrm{GSO}_4, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	D_{k-2}	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{SO}_{2k}} \oplus std_{\mathrm{SO}_{2k}} \otimes std_{\mathrm{SL}_2}$
9	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6))$	B_{m-3}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_6} \oplus T(\mathrm{HSpin}_6)$

Table 23. Non-reductive strongly tempered quadruples 1

№	(G,H,ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GSp}_6\times\mathrm{GL}_2,\mathrm{GL}_2,0)$	A_2	$std_{\mathrm{GL}_2}\otimes\mathrm{Spin}_7$
2	$(\mathrm{GSp}_8\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),0)$	A_2	$std_{\mathrm{GL}_2}\otimes\mathrm{Spin}_9$
3	$(\mathrm{GSp}_{10},\mathrm{GL}_2,0)$	A_4	Spin_{11}
4	$(\mathrm{GSO}_{12},\mathrm{GL}_2,0)$	A_5	HSpin_{12}
5	$(\mathrm{GL}_6,\mathrm{GL}_2,0)$	$A_2\times A_2$	$\wedge^3$
6	$(E_7,\mathrm{PGL}_2,0)$	E_6	std_{E_7}
7	$(\mathrm{GL}_{2m},\mathrm{GL}_m,T(std_{\mathrm{GL}_m}))$	$(A_1)^m$	$T(\wedge^2)$
8	$(\mathrm{GL}_{2m+1},\mathrm{GL}_m,0)$	$(A_1)^m$	$T(\wedge^2)$
9	$(\mathrm{GSpin}_{2k},\mathrm{GSpin}_3,T(\mathrm{Spin}_3))$	D_{k-1}	$T(std_{\mathrm{SO}_{2k}})$
10	$(\mathrm{GSp}_6,\mathrm{GL}_2,T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_7)$
11	$(\mathrm{GSp}_8,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_9)$
12	$(G_2,\mathrm{SL}_2,std_{\mathrm{SL}_2})$	A_1	$T(std_{G_2})$
13	$(GE_6,\mathrm{GL}_3,T(std_{\mathrm{GL}_3}))$	D_4	$T(std_{E_6})$
14	$*$	B_{m-2}	$T(std_{\mathrm{Sp}_{2m}})\oplus T(std_{\mathrm{Sp}_{2m}})$

Table 24. Non-reductive strongly tempered quadruples 2

$$*=(\mathrm{GSpin}_{2m+1}\times\mathrm{GL}_1,\mathrm{GSpin}_4\times\mathrm{GL}_1,T(\mathrm{HSpin}_4^+\oplus\mathrm{HSpin}_4^-\otimes std_{\mathrm{GL}_1})).$$

№	(G,H,ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_n,0)$	A_{m-n-1}	$T(std_{\mathrm{GL}_m}\otimes std_{\mathrm{GL}_n})$
2	$(\mathrm{GSp}_4\times\mathrm{GL}_n,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2)$	A_{n-5}	$T(Std_{\mathrm{Sp}_4}\otimes Std_{\mathrm{SL}_m})$
3	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2k},\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m},std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}})$	C_{k-m}	$std_{\mathrm{SO}_{2k+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
4	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}))$	A_{m-n+1}	$T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n})$
5	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_m\times\mathrm{GL}_{m+1},T(std_{\mathrm{GL}_m}\otimes std_{\mathrm{GL}_{m+1}}))$	A_{n-m-2}	$T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n})$

Table 25. Non-reductive strongly tempered quadruples 3

№	(G, H, ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0)$	$A_2 \times A_2$	Spin_{13}
2	$(\mathrm{GSp}_{2k}, \mathrm{SL}_2 \times \mathrm{GL}_1, std_{\mathrm{SL}_2})$	C_{k-1}	$T(std_{\mathrm{SO}_{2k+1}})$
3	$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0)$	A_3	$T(\mathrm{HSpin}_{10})$
4	$(\mathrm{GSO}_{12}, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), 0)$	$A_1 \times A_1$	$\mathrm{HSpin}_{12}^+ \oplus \mathrm{HSpin}_{12}^-$
5	$(\mathrm{GSO}_{12} \times \mathrm{PGL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_3	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
6	$*$	A_1	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
7	$(\mathrm{GSO}_8 \times \mathrm{GSO}_4, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
8	$(\mathrm{GSpin}_7, S(\mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2})$	A_1	$\wedge^3 \oplus std_{\mathrm{Sp}_6}$
9	$(\mathrm{GSO}_{12}, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_3	$\mathrm{HSpin}_{12} \oplus T(std_{\mathrm{Spin}_{12}})$
10	$**$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{10}} \oplus T(std_{\mathrm{Spin}_{10}})$
11	$(\mathrm{GSO}_8 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus T(std_{\mathrm{Spin}_8})$
12	$(\mathrm{GL}_6, \mathrm{GL}_2 \times \mathrm{GL}_2, 0)$	$A_1 \times A_1$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6})$
13	$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}))$	A_1	$T(std_{\mathrm{Spin}_8}) \oplus T(\mathrm{HSpin}_8)$
14	$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2})$	A_{n-3}	$T(std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$

TABLE 26. Non-reductive strongly tempered quadruples 4

$$* = (\mathrm{GSpin}_4 \times \mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8}).$$

$$** = (\mathrm{GSpin}_{10} \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSpin}_6) \times \mathrm{GL}_2, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_2})).$$

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