A local twisted trace formula for Whittaker induction of coregular symmetric pairs: the geometric side

Raphaël Beuzart-Plessis, Chen Wan

June 13, 2024

Abstract

In this paper, we prove the geometric expansion of a local twisted trace formula for the Whittaker induction of any symmetric pairs that are coregular. This generalizes the local (twisted) trace formula for reductive groups proved by Arthur [2] and Waldspurger [28]. We also prove a formula for the regular germs of quasi-characters associated to strongly cuspidal functions in terms of certain weighted orbital integrals. As a consequence of our trace formula and the formula for regular germs of quasi-characters, we prove a simple local trace formula of those models for strongly cuspidal test functions which implies a multiplicity formula for these models. We also present various applications of our trace formula and multiplicity formula, including a necessary condition for a discrete L-packet to contain a representation with a unitary Shalika model (resp. a Galois model for classical groups) in terms of the associated Langlands parameter, and we also compute the summation of the corresponding multiplicities for certain discrete L-packets.

Contents

1 Introduction

Let F be a local non-Archimedean field of characteristic 0, G be a reductive group defined over F, $H \subset G$ be a unimodular subgroup and $\xi : H(F) \to \mathbb{C}^\times$ be a smooth unitary character. Choosing Haar measures on $G(F)$ and $H(F)$ induces an invariant measure on $H(F)\backslash G(F)$ and we let $L^2(H(F)\backslash G(F), \xi)$ be the space of $\varphi : G(F) \to \mathcal{C}^\times$ that transform by left multiplication by $H(F)$ according to the character ξ (i.e. $\varphi(hg) = \xi(h)\varphi(g)$ for $(h, g) \in H(F) \times G(F)$) and whose norm is square-integrable on $H(F)\backslash G(F)$. Then, the natural action of $G(F)$ on $L^2(H(F)\backslash G(F), \xi)$ by right translation is a unitary representation and for $f \in C_c^\infty(G(F))$, we define by integration an operator $R(f)$ on $L^2(H(F)\backslash G(F), \xi)$. This operator is associated to the following kernel function (for simplicity we assume the center of G is trivial in the introduction)

$$K_f(x, y) = \int_{H(F)} f(x^{-1}hy)\xi(h)dh, \quad x, y \in G(F).$$

Formally, the trace of the operator $R(f)$ should be given by the integral of $K_f(x, x)$ over $x \in H(F)\backslash G(F)$. However, neither of these two expressions are well-defined in general. The goal of this paper is to define some canonical regularizations of the integral of K_f over the diagonal for certain triples (G, H, ξ) (essentially associated to symmetric varieties that we name "coregular") and to express the resulting distribution on $G(F)$ as a sum (or integral) of contributions naturally generalizing the weighted orbital integrals of Arthur [2]. This can be considered as the geometric side of a local trace formula for the corresponding unitary representations $L^2(H(F)\backslash G(F), \xi)$. We plan to develop in a subsequent paper a general spectral expansion for those trace formulas.

In the so-called *group-case*, corresponding to $G = H \times H$ with H embedded diagonally in the product, we recover the geometric side of Arthur local trace formula [2]. We actually also consider an enhancement of the previous setting where we fix an extra automorphism θ of the triple (G, H, ξ) and we formally try to compute the trace of the composition $R(f) \circ \theta$. This can be more naturally formulated using the notion of twisted spaces due to Labesse. In the group-case again, we recover the geometric side of the local twisted trace formula due to Waldspurger [28].

Although not implied by our main results, the work of Waldspurger [34] on the local Gan-Gross-Prasad conjecture, whose main innovation was the development of a certain simple local trace formula, has been a main source of inspiration and motivation for this paper.

We also present few applications of our general trace formula. Namely, specializing our geometric expression to a matrix coefficient of a supercuspidal or square-integrable representation, we obtain explicit integral formulas for multiplicities of certain models which generalize our previous results for Galois models [7] and the Shalika model [9]. This can then be further applied to establish necessary conditions, in terms of the associated Langlands parameters, for the distinction of discrete L-packets with respect to a unitary Shalika model or a Galois model for classical groups and we moreover compute the corresponding multiplicities of such packets under an extra assumption. In the case of Galois models, this confirms some consequences of a general conjecture made by D. Prasad [29].

1.1 Main results

Whittaker induction of coregular symmetric varieties

Let ι be an involutive automorphism of G and assume that $(G^\iota)^0 \subset H \subset G^\iota$ where G^ι stands for the subgroup of fixed points and $(G^\iota)^0$ for its neutral component. In this situation, the quotient variety $H\backslash G$ is sometimes called a symmetric variety. In this paper, we impose an important condition on the variety $H\backslash G$ that we decided to name coregularity:

Definition 1.1. *Let $X = H\backslash G$ be a homogeneous G-variety with H reductive. We say that X is* ***coregular*** *if there exists an non-empty open subset $U \subset X \times X$ such that for every $x \in U$, the stabilizer $G_x \subset G$ of x for the diagonal action contains regular elements.*

In Section 3.1 we give various alternative characterizations of coregular homogeneous G-varieties (including the case where H is not reductive). Technically, the most important for us is the following property (where $G_{rs} \subset G$ denotes the open locus of regular semisimple elements and D^G, D^H stand for the usual Weyl discriminants):

A homogeneous G-variety $X = H\backslash G$ is coregular if and only if $H \cap G_{rs}$ is nonempty and the function $h \in H(F) \cap G_{rs}(F) \mapsto \frac{D^H(h)^2}{D^G(h)}$ is locally bounded on $H(F)$.

Examples of coregular symmetric varieties are the group case (that is $X = H^{diag}\backslash H \times H$), Galois symmetric varieties (i.e. homogeneous varieties of the form $X = H\backslash \mathrm{Res}_{E/F} H_E$ where E/F is a quadratic extension and $\mathrm{Res}_{E/F}$ denotes Weil's restriction of scalars) or $\mathrm{Sp}_{2n}\backslash \mathrm{GL}_{2n}$. However, many other natural examples of homogeneous varieties such as $O_n\backslash \mathrm{GL}_n$, $\mathrm{GL}_n \times \mathrm{GL}_n\backslash \mathrm{GL}_{2n}$ or $\mathrm{SO}_n^{\mathrm{diag}}\backslash(\mathrm{SO}_n \times \mathrm{SO}_{n+1})$ (the so-called Gross-Prasad variety, that is not symmetric but at least spherical) are not coregular.

In this paper, we will actually consider a slightly more general setting, essentially including all triples (G, H, ξ) that are in a suitable sense "Whittaker induction" of a coregular symmetric pair (M, H_0). More precisely, the most general triples (G, H, ξ) that we can consider are constructed as follows. There exists an involution ι of G as well as a parabolic subgroup $P \subset G$ that is ι-split (recall that it means that $\overline{P} := \iota(P)$ is opposite to P) and a semi-direct product decomposition $H = H_0 \ltimes N$ where:

- N is the unipotent radical of P and H_0 is a subgroup of the Levi factor $M := P \cap \overline{P}$;
- We have $H_0 = (M^\iota)^0$ and the symmetric variety $H_0\backslash M$ is coregular;
- The restriction of the character ξ to $N(F)$ is non-degenerate i.e. its orbit under the adjoint action of $M(F)$ is open in the F-vector space of all smooth characters $N(F) \to \mathbb{C}^\times$;
- In the case where $P \neq G$, H_0 is precisely the neutral component of the stabilizer of ξ in M.

Following [30, Sect. 2.6], we say that the pair $(H\backslash G, \xi)$ is the *Whittaker induction* of the symmetric (and coregular) variety $H_0\backslash M$. One example of such Whittaker induction is given by the triple $(\mathrm{GL}_{2n}, \mathrm{GL}_n^{\mathrm{diag}} \ltimes \mathrm{Mat}_n, \psi \circ \mathrm{Tr})$, where $\psi : F \to \mathbb{C}^\times$ is a nontrivial character, which is related to so-called *Shalika models* of representations of $\mathrm{GL}_{2n}(F)$. In this particular case, our result on geometric expansions contains the main results of our previous work [9]. There is also a variant of this example for unitary groups that will be described in more details below, related to what we call unitary Shalika models.

Truncation on symmetric varieties

Fix a triple (G, H, ξ) as in the previous paragraph. Our starting point will be to truncate in a meaningful way the (usually non-convergent) integral

$$I(f) = \int_{H(F)\backslash G(F)} K_f(x, x)dx.$$

For this we introduce a sequence of truncation functions $(\kappa_Y)_Y$ indexed by points Y in a certain affine space [1].

[1] For the definition of our truncation functions, we do not need to assume (G, H) is coregular. It works for all the symmetric varieties.

More precisely, we fix from now on a special maximal compact subgroup $K \subset G(F)$ that is in good position with respect to the Levi subgroup M as well as a minimal ι-split parabolic subgroup $P_0 \subset P$[2]. We assume for simplicity that there is only one $H_0(F)$-conjugacy class of minimal ι-split parabolic subgroups in P. (Otherwise we need to replace P_0 by a set of representatives of those conjugacy classes, there are always finitely many, or, even better, we should replace P_0 in the discussion that follows by the union of finitely many $P_0(F)$-orbits in $(P_0 \cap H \backslash P_0)(F)$). Let $\mathcal{A}_{P_0,\iota}$ be the subspace of the real vector space

$$\mathcal{A}_{P_0} := \mathrm{Hom}(X^*(P_0), \mathbb{R})$$

on which ι acts by $-\mathrm{Id}$. Let $\mathcal{A}_{P_0}^{P,+} \subset \mathcal{A}_{P_0}$ be the usual Weyl chamber associated to the parabolic subgroup $P_0 \cap M$ of M and $\mathcal{A}_{P_0,\iota}^{P,+}$ be its projection to $\mathcal{A}_{P_0,\iota}$. We also let A_0 be the maximal central split torus in $M_0 = P_0 \cap \iota(P_0)$ (a Levi factor of P_0) which is *ι-split* (in the sense that $\iota(a) = a^{-1}$ for every $a \in A_0$) and denote by $H_{P_0,\iota} : P_0(F) \to \mathcal{A}_{P_0,\iota}$ the composition of the usual Harish-Chandra morphism $P_0(F) \to \mathcal{A}_{P_0}$ with the projection $\mathcal{A}_{P_0} \to \mathcal{A}_{P_0,\iota}$. Then, by the weak Cartan decomposition of [16] and [10], we can find a compact subset $\omega_{P_0} \subset P_0(F)$ such that setting

$$\mathcal{S}(\omega_{P_0}) = \{x = pak \mid p \in \omega_{P_0}, k \in K, a \in A_0(F), H_{P_0,\iota}(a) \in \mathcal{A}_{P_0,\iota}^{P,+}\}$$

we have the decomposition

$$G(F) = H(F)\mathcal{S}(\omega_{P_0}).$$

Note the formal resemblance with the existence of Siegel domains in a global setting. Let ${}^-\mathcal{A}_{P_0} \subset \mathcal{A}_{P_0}$ be the cone defined by the negative simple weights with respect to P_0 and ${}^-\mathcal{A}_{P_0,\iota}$ be its image in $\mathcal{A}_{P_0,\iota}$. Then, for any $Y \in \mathcal{A}_{P_0,\iota}^+$ that is "sufficiently positive", we denote by κ_Y the characteristic function of the image in $H(F)\backslash G(F)$ of the set $\mathcal{S}(\omega_{P_0}, Y)$ defined by

$$\mathcal{S}(\omega_{P_0}, Y) := \{x \in \mathcal{S}(\omega_{P_0}) \mid H_{P_0,\iota}(x) \in Y + {}^-\mathcal{A}_{P_0,\iota}\}$$

where we have denoted also by $H_{P_0,\iota}$ the unique extension of $H_{P_0,\iota}$ to $G(F)$ that is K-invariant on the right.

Although the family of truncation functions $(\kappa_Y)_Y$ a priori depends on the auxiliary choice of the compact subset ω_{P_0}, it can be shown that it doesn't asymptotically in the following precise sense. For any pair of compact subsets $\omega_{P_0}, \omega'_{P_0} \subset P_0(F)$ such that $G(F) = H(F)\mathcal{S}(\omega_{P_0}) = H(F)\mathcal{S}(\omega'_{P_0})$, we can define as above two families of truncation functions $(\kappa_Y)_Y$ and $(\kappa'_Y)_Y$. Then, there exists $Y_+ \in \mathcal{A}_{P_0,\iota}$ such that $\kappa_Y = \kappa'_Y$ for every $Y \in Y_+ + \mathcal{A}_{P_0,\iota}^+$. In particular, it makes sense to study the asymptotic behavior of the expression

$$I_Y(f) := \int_{H(F)\backslash G(F)} K_f(x,x)\kappa_Y(x)dx$$

[2]Here, by a *minimal ι-split* parabolic subgroup we mean a parabolic subgroup that is ι-split and minimal for this property.

when $Y \xrightarrow{P_0} \infty$ (where the latter notation means asymptotic along the filter generated by translates $Y_+ + \mathcal{A}_{P_0,\iota}^+$ of the positive Weyl chamber). Moreover, the functions κ_Y are so defined that the integrand in the above expression is compactly supported (see Lemma 6.3).

Finally, we can also suppress the dependence of our truncation process on the choice of P_0 (but not on that of K) as follows: for any other choice of a minimal ι-split parabolic subgroup $P_0' \subset P$, there exists a natural affine isomorphism

$$\iota_{P_0,P_0',K} : \mathcal{A}_{P_0,\iota} \simeq \mathcal{A}_{P_0',\iota}, \tag{1.1.1}$$

such that as $Y \in \mathcal{A}_{P_0,\iota} \xrightarrow{P_0} \infty$ we eventually have $\kappa_Y = \kappa_{Y'}$ where $Y' = \iota_{P_0,P_0',K}(Y)$. We emphasize that $\iota_{P_0,P_0',K}$ is not the most obvious isomorphism $\mathcal{A}_{P_0,\iota} \simeq \mathcal{A}_{P_0',\iota}$, namely the one induced by conjugation by an element $p \in P(F)$ such that $pP_0p^{-1} = P_0'$, which is not only affine but linear. Indeed, in general the map (1.1.1) does not preserve the origins; a fact that is related to the existence of more than one $H(F) \cap K$-conjugacy class of minimal ι-split parabolic subgroups $P_0 \subset P$.

Therefore, we can as well think of Y as living in the inverse limit

$$\mathcal{A}_{X,K} = \varprojlim_{P_0} \mathcal{A}_{P_0,\iota}$$

where P_0 runs over the set of minimal ι-split parabolic subgroups of P and the transition maps are given by the affine isomorphisms (1.1.1). This is the point of view that we will adopt in the body of the paper.

The geometric expansion of a general local trace formula.

Let $\Gamma(H_0)$ (resp. $\Gamma_{\text{ell}}(H_0)$) be the set of regular semisimple (resp. regular elliptic) conjugacy classes in $H_0(F)$. These two sets can be naturally equipped with measures, see Sections 6.3 and 7.1 for details.

For $t \in \Gamma(H_0)$, that we identify with a representative in $H(F)$, we denote by H_t, G_t, M_t, N_t and $B_t = M_tN_t$ the neutral components of the centralizers of t in H, G M, N and P respectively. Then, for t in general position B_t is a Borel subgroup of G_t and $\xi \mid_{N_t(F)}$ is a non-degenerate character on its unipotent radical (see Lemma 6.1, here we need to use the coregular assumption). For $f \in C_c^\infty(G(F))$ and $Y \in \mathcal{A}_{X,K}$, we define the following expression

$$J_Y(f) = \int_{\Gamma(H_0)} D^H(t)\xi(t)J_Y(t,f)dt$$

where $J_Y(t,f)$ denotes some kind of "weighted orbital integral". More precisely, $J_Y(t,.)$ is a distribution of the form

$$J_Y(t,f) = \int_{\mathcal{O}_t} f(g)v_{\xi,\iota,Y}(g)dg$$

where $\mathcal{O}_t$ denotes the union of the (finitely many) *regular* $G(F)$-conjugacy classes with semisimple part t and the function $g \mapsto v_{\xi,\iota,Y}(g)$ is a certain weight function. When $\xi = 1$

(so that $P = G$ and, by the coregular assumption, t is already regular in G at least when it is in general position), this weight is very similar to the one appearing in the definition of Arthur's weighted orbital integrals as $v_{\xi,\iota,Y}(g^{-1}tg)$ is given by the volume of the convex hull of a certain family $(-H_{\overline{Q},\iota}(g) + Y_Q)_Q$ where Q runs over the minimal ι-split parabolic subgroups of G containing t, $H_{\overline{Q},\iota} : G(F) \to \mathcal{A}_{L,\iota}$ denotes the usual Harish-Chandra map for the parabolic subgroup $\overline{Q} = \iota(Q) = LN_{\overline{Q}}$ (chosen to be K-invariant on the right) composed with the projection $\mathcal{A}_{\overline{Q}} \to \mathcal{A}_{L,\iota}$ to the ι-antifixed points in $\mathcal{A}_L$ and $Y \mapsto Y_Q \in \mathcal{A}_{Q,\iota} = \mathcal{A}_{L,\iota}$ is the composition of the canonical isomorphism $\mathcal{A}_{X,K} \simeq \mathcal{A}_{P_0,\iota}$ with the natural projection $\mathcal{A}_{P_0,\iota} \to \mathcal{A}_{Q,\iota}$ for any minimal ι-split parabolic subgroup $P_0 \subset Q$ (it can be shown that the composition doesn't depend on P_0). In general, the precise definition looks like

$$J_Y(t,f) = \int_{B_t(F)\backslash G(F)} \int_{N_t(F)} f(x^{-1}tux) v_{B_t,\xi,\iota,Y}(x,u)\,du\,dx$$

where we refer the reader to Section 6.3 for the definition of the weight $v_{B_t,\xi,\iota,Y}(x,u)$ when $\xi \neq 1$. Another important point is that, after Harish-Chandra, it is known that near singular point the typical order of growth of (weighted) orbital integrals is as the inverse of the square root of the Weyl discriminant. Therefore, our assumption on coregularity of the pair (G,H) is what guarantees the absolute convergence of the expression defining $J_Y(f)$ above. Then, the aforementioned geometric expansion of the local trace formula for (G,H,ξ) is contained in the following theorem.

Theorem 1.2. *Let $0 < \epsilon < 1$ and fix $f \in C_c^\infty(G(F))$. Then, for any $k > 0$, we have*

$$|I_Y(f) - J_Y(f)| \ll N(Y)^{-k}$$

for every $Y \in \mathcal{A}_{X,K}$ with $d(Y) > \epsilon N(Y)$. Moreover, the function $Y \in \mathcal{A}_{X,K} \mapsto J_Y(f)$ is a polynomial-exponential function in a suitable sense (see Section 2.9) and if the variety $X = H\backslash G$ is tempered (see Section 3.2), then the same statement holds for functions f in the Harish-Chandra Schwartz space $\mathcal{C}(G(F))$.

In the above statement, $N(Y)$ stands for any norm on the affine space $\mathcal{A}_{X,K}$ whereas, fixing a minimal ι-split parabolic subgroup $P_0 \subset P$ for convenience, the *depth* $d(Y)$ of Y is defined by

$$d(Y) = \min_{\alpha\in\Delta_0} \langle \alpha, Y - Y_0\rangle$$

where Δ_0 stands for the set of simple roots with respect to P_0. Therefore, in some loose sense, the above theorem describes the asymptotic behavior of $I_Y(f)$ as Y goes to infinity in the direction of the positive Weyl chamber and "sufficiently far from the walls".

As already mentioned, in the main body of the paper we actually prove a more general theorem of the above form for suitable *twisted* triples $(\widetilde{G}, \widetilde{H}, \xi)$. In particular, in the group case (i.e. when H is diagonally embedded in $G = H \times H$) this recovers the geometric side of the twisted local trace formula [28].

The case of strongly cuspidal functions and integral formulas for multiplicities

Most applications of our trace formula comes from a simple version obtained by specializing it to the case of *strongly cuspidal* test functions. More precisely, we recall following [34] that a function $f \in C_c^\infty(G(F))$ is said to be *strongly cuspidal* if for every proper parabolic subgroup $Q = LV \subset G$ we have

$$\int_{V(F)} f(lu)du = 0, \quad \text{for every } l \in L(F).$$

It is then shown in *loc. cit.* that the regular semisimple weighted orbital integrals (in the sense of Arthur) of a strongly cuspidal function f don't depend on any choice (except that of a Haar measure on $G(F)$) and that, correctly normalized by certain signs, they define a function

$$\Theta_f : G_{\text{rs}}(F) \to \mathbb{C}$$

which is $G(F)$-invariant by conjugation and a *quasi-character* in the following sense: for every semisimple element $x \in G(F)$, there exists an expansion

$$\Theta_f(x\exp(X)) = \sum_{\mathcal{O}\in \text{Nil}(\mathfrak{g}_x^*)} c_{f,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, X), \quad X \in \omega \cap \mathfrak{g}_{x,\text{rs}}(F),$$

where:

- $\omega \subset \mathfrak{g}_x(F)$ is a sufficiently small neighborhood of 0 in the Lie algebra of $G_x(F)$;
- $\text{Nil}(\mathfrak{g}_x^*)$ denotes the (finite) set of nilpotent coadjoint orbits in the dual $\mathfrak{g}_x(F)^*$ of $\mathfrak{g}_x(F)$;
- for $\mathcal{O} \in \text{Nil}(\mathfrak{g}_x^*)$, $\widehat{j}(\mathcal{O}, .)$ stands for the unique smooth function on $\mathfrak{g}_{x,\text{rs}}(F)$ that is locally integrable on $\mathfrak{g}_x(F)$ and represents the Fourier transform of the orbital integral over $\mathcal{O}$ i.e. for every $\varphi \in C_c^\infty(\mathfrak{g}_x(F))$ we have

$$\int_{\mathfrak{g}_x(F)} \varphi(X)\widehat{j}(\mathcal{O}, X)dX = \int_{\mathcal{O}} \widehat{\varphi}(Y)dY$$

  where dX is a Haar measure on $\mathfrak{g}_x(F)$, $\widehat{\varphi}(Y) = \int_{\mathfrak{g}_x(F)} \varphi(X)\psi(\langle X, Y\rangle)dX$, $Y \in \mathfrak{g}_x(F)^*$, denotes the Fourier transform of φ (which depend in the auxiliary choice of a non-trivial additive character $\psi : F \to \mathbb{C}^\times$) and dY is the canonical Kirillov-Kostant measure on the coadjoint orbit $\mathcal{O}$.

For $t \in H_{0,\text{rs}}(F)$ in general position, the restriction $\xi_t := \xi \mid_{N_t(F)}$ is a generic character of $N_t(F)$. We let $\mathcal{O}_t \in \text{Nil}(\mathfrak{g}_t^*)$ be the orbit associated to ξ_t[3]. Then, for every strongly cuspidal test function $f \in C_c^\infty(G(F))$ we set

$$I_{\text{geom}}(f) := \int_{\Gamma_{\text{ell}}(H_0)} D^H(t)c_{f,\mathcal{O}_t}(t)\xi(t)dt.$$

[3]More precisely, $\mathcal{O}_t$ is the unique nilpotent coadjoint orbit in $\mathfrak{g}_t(F)^*$ containing an element Y such that $\xi(\exp(X)) = \psi(\langle Y, X\rangle)$ for every $X \in \mathfrak{n}_t(F)$, the Lie algebra of $N_t(F)$.

Theorem 1.3. *Let $f \in C_c^\infty(G(F))$ be a strongly cuspidal function. Then,*

1. *We have*
$$\lim_{Y \xrightarrow{P_0} \infty} I_Y(f) = I_{\text{geom}}(f),$$
in particular the limit exists.

2. *If moreover f is a matrix coefficient of a supercuspidal representation π of $G(F)$ and the dimension $m_{H,\xi}(\pi^\vee)$ of the space $\mathrm{Hom}_H(\pi^\vee, \xi)$ of $(H(F), \xi)$-equivariant linear forms on (the space of) the contragredient representation $\pi^\vee$ is finite, then the integral defining $I(f)$ is already convergent and we have*
$$I(f) = \frac{f(1)}{d(\pi)} m_{H,\xi}(\pi^\vee)$$
where $d(\pi)$ stands for the formal degree of π.

Furthermore, if the pair (G, H) is tempered then the same holds for strongly cuspidal test functions $f \in \mathcal{C}(G(F))$ and matrix coefficients of square-integrable representations π respectively.

As a corollary of the above theorem we can also obtain general integral formulas for the multiplicities $m_{H,\xi}(\pi)$. More precisely, for π an irreducible representation of $G(F)$, it is known that the Harish-Chandra character Θ_π is also a quasi-character in the above sense. Therefore, we can define an expression $m_{\text{geom},H,\xi}(\pi)$ similar to $I_{\text{geom}}(f)$ by formally replacing Θ_f by Θ_π. Then, we have the following. (see Theorem 7.4).

Corollary 1.4. 1. *Assume that π is supercuspidal and the multiplicity $m_{H,\xi^{-1}}(\pi)$ is finite. Then, we have*

$$m_{H,\xi^{-1}}(\pi) = m_{\text{geom},H,\xi}(\pi). \tag{1.1.2}$$

2. *If the pair (G, H) is tempered, π is square-integrable and the multiplicity $m_{H,\xi}(\pi)$ is finite, then the equality* (1.1.2) *also holds.*

In the case of Galois models or the Shalika model, the above corollary recovers one of the main result in [7] and [9] respectively. Actually for Galois models associated to classical groups, we can also deduce new results from the analog of the above corollary in certain twisted situations as explained in more details below.

An integral formula for regular germs of quasi-characters

One important technical result that we prove along the way to Theorem 1.2, and that may be of independent interest, is a certain explicit formula for some singular weighted orbital integrals of strongly cuspidal functions. More precisely, we are able to write down a set of measures on regular (but not necessarily semi-simple) conjugacy classes representing the distributions $f \mapsto c_{f,\mathcal{O}}(x)$ for $x \in G(F)$ semisimple and $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$ a regular nilpotent coadjoint orbit.

More precisely, let us fix a Borel subgroup B_x of G_x with a Levi decomposition $B_x = T_xN_x$ as well as a generic character ξ_x of $N_x(F)$, and we let $\mathcal{O}_x \in \mathrm{Nil}(\mathfrak{g}_x^*)$ be the corresponding regular nilpotent coadjoint orbit (every regular nilpotent coadjoint orbit arises in this way). In Section 4 we will define a weighted function $v_{B_x,\xi_x}(g,u)$ for $g \in G(F)$ and $u \in N_x(F)$ *regular*. The next theorem expresses the regular germ of the quasi-character Θ_f in terms of certain weighted orbital integral (we refer the reader to Section 2-4 for various notation). It will be proved in Section 4.

Theorem 1.5. *For every strongly cuspidal function $f \in \mathcal{C}(G(F))$, we have*

$$c_{f,-\mathcal{O}_x}(x) = \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} f(g^{-1}xug)v_{B_x,\xi_x}(u,g)dudg.$$

The Galois model for classical groups

Let E/F be a quadratic extension, H be a reductive group defined over F, χ be a character of $H(F)$ and $G = Res_{E/F}H_E$. The model (G,H,χ) is the so-called Galois model. In [29], Prasad made a general conjectural regarding the multiplicity of Galois model. In this paper, we will study the case when H is a classical group.

Let H be a quasi-split special orthogonal group or a symplectic group and $G = Res_{E/F}H_E$. If H is the even special orthogonal group, let H_0 be a quasi-split special orthogonal group that is not a pure inner form of H and such that $G = Res_{E/F}H_E = Res_{E/F}H_{0,E}$ (i.e. the determinants of the quadratic forms defining H and H_0 belong to the same square class in $E^\times/(E^\times)^2$ but belong to different square classes in $F^\times/(F^\times)^2$). If $H = \mathrm{Sp}_{2n}$ or SO_{2n}, let χ be the trivial character on H (and H_0 if $H = \mathrm{SO}_{2n}$). If $H = \mathrm{SO}_{2n+1}$, let $\chi \in \{1, \eta_n\}$ where η_n is the composition of the Spin norm character of SO_{2n+1} with the quadratic character $\eta_{E/F}$.

Our first result is a necessary condition for a discrete L-packet to be distinguished.

Theorem 1.6. *Let $H = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$ or SO_{2n+1}, $G = Res_{E/F}H$, $\chi = 1$ if $H = \mathrm{Sp}_{2n}$ or SO_{2n}, and $\chi \in \{1, \eta_n\}$ if $H = \mathrm{SO}_{2n+1}$. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ and $\Pi_\phi(G')$ be the endoscopic transfer of the L-packet to the general linear group $G' = \mathrm{GL}_a(E)$ (here $a = 2n$ if $H = \mathrm{SO}_{2n}$ or SO_{2n+1} and $a = 2n+1$ if $H = \mathrm{Sp}_{2n}$). Then the packet $\Pi_\phi(G)$ is distinguished (i.e. $m(\pi,\chi) \neq 0$ for some $\pi \in \Pi_\phi(G)$) only if $\Pi_\phi(G')$ is $(\mathrm{GL}_a(F),\chi')$-distinguished. Here $\chi' = 1$ if $\chi = 1$ and $\chi' = \eta_n' := \eta_{E/F} \circ \det$ if $\chi = \eta_n$.*

Our second result is to compute the summation of the multiplicities over certain discrete L-packets. Assume that $\Pi_\phi(G')$ is $(\mathrm{GL}_a(F),\chi')$-distinguished. By Theorem 4.2 of [26], $\Pi_\phi(G')$ is of the form

$$\Pi_\phi(G') = (\tau_1 \times \cdots \times \tau_l) \times (\sigma_1 \times \bar{\sigma}_1) \times \cdots \times (\sigma_m \times \bar{\sigma}_m)$$

where

- τ_i is a discrete series of $\mathrm{GL}_{a_i}(E)$ that is conjugate self-dual. Moreover, if $(H,\chi) = (\mathrm{SO}_{2n+1},\eta_n)$, τ_i is self-dual of symplectic type; otherwise, τ_i is self-dual of orthogonal type.
- σ_j is a discrete series of $\mathrm{GL}_{b_i}(E)$ that is NOT conjugate self-dual. Moreover, if $(H,\chi) = (\mathrm{SO}_{2n+1},\eta_n)$, σ_j is self-dual of symplectic type; otherwise, σ_j is self-dual of orthogonal type.
- τ_i, σ_j are all distinct.
- $\sum_{i=1}^{l} a_i + 2\sum_{j=1}^{m} b_j = a$.

We will consider the special case when $m = 0$. The general case will be consider in our future paper. When $m = 0$, $\Pi_\phi(G')$ appears discretely in the L^2 space of the Galois model $(\mathrm{GL}_a(E), \mathrm{GL}_a(F), \chi')$.

Theorem 1.7. *With the notation above, if H is the symplectic group or the odd special orthogonal group, we have*

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi) = 2^{l-1}.$$

If H is the even special orthogonal group, we let H_0 be another even special orthogonal group as above. We use $m_0(\pi,\chi)$ to denote the multiplicity for the model (G,H_0,χ). Then we have

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi) + m_0(\pi,\chi) = 2^{l-1}.$$

Remark 1.8. *By Theorem 1 of [7], the above two theorems also hold if we replace H (and H_0 if we are in the even orthogonal group case) by the non quasi-split classical group.*

The unitary Shalika model

Let Z be a E-vector space of finite dimension $n \geqslant 1$. Let $Z^{*,c}$ be the conjugate-dual of Z that is the space of c-linear forms on Z (a similar notation will be applied later to other vector spaces). Set $V = Z \oplus Z^{*,c}$ and we equip with the nondegenerate Hermitian form

$$h(v+v^*, w+w^*) = \langle v, w^*\rangle + \langle w, w^*\rangle^c, \quad (v,v^*),(w,w^*) \in Z \oplus Z^{*,c}.$$

Here $\langle .,.\rangle$ stands for the canonical pairing between Z and $Z^{*,c}$. Let $G = U(V,h)$ be the unitary group associated to this Hermitian form. We define two maximal parabolic subgroups Q and $\overline{Q}$ of G as the stabilizers of the maximal isotropic subspaces Z and $Z^{*,c}$ respectively. Then, $L = Q \cap \overline{Q}$ is a Levi component of Q and restriction to Z induces an isomorphism

$$L \simeq \mathrm{Res}_{E/F} GL(Z). \tag{1.1.3}$$

Let N be the unipotent radical of Q. Thus $Q = LN$ and restriction to $Z^{*,c}$ induces an isomorphism

$$N \simeq \left\{X \in \mathrm{Hom}(Z^{c,*}, Z) \mid {}^T X^c = -X\right\} \tag{1.1.4}$$

where ${}^T X^c$ denotes the transpose conjugate of X (seen as a linear endomorphism $Z \to Z^{*,c}$ through the canonical identification $(Z^{*,c})^{*,c} = Z$). We will actually identify the right hand side above with the Lie algebra $\mathfrak{n}$ of N in a way such that the above isomorphism becomes the exponential map.

We henceforth choose two isomorphisms $W_+, W_- : Z \to Z^{*,c}$ satisfying ${}^T W_{\pm}^c = -W_{\pm}$ and such that the corresponding antihermitian forms on Z are not equivalent (there are actually only two equivalence classes of antihermitian forms on Z). For $\epsilon \in \{\pm\}$, we let $H_{0,\epsilon} \subset L \simeq \mathrm{Res}_{E/F} GL(Z)$ be the unitary group associated to W_ϵ, that is the stabilizer of W_ϵ for the obvious action. Then, $H_{0,\epsilon}(F)$ coincides with the stabilizer in $L(F)$ of the character

$$\xi_\epsilon : N(F) \to \mathbb{C}^\times,$$

$$\exp(X) \mapsto \psi(\mathrm{Tr}(W_\epsilon X)) \ \ (X \in \mathfrak{n}(F)).$$

We will henceforth assume, as we may, that $W_\pm$ have been chosen so that $H_{0,+}$ is quasi-split.

Set $H_\epsilon = H_{0,\epsilon} \ltimes N$. We extend ξ_ϵ to a character of $H_\epsilon(F)$ trivial on $H_{0,\epsilon}(F)$. We also fix a character χ of $E^1 = \ker(N_{E/F})$ that we will consider as a character of $H_{0,\epsilon}(F)$ through composition with the determinant $\det : H_{0,\epsilon}(F) \to E^1$. The model $(G, H_\epsilon, \chi \otimes \xi_\epsilon)$ is an analogue of the Shalika model for unitary groups and we will call it the unitary Shalika model. For a smooth irreducible representation π of $G(F)$, we define the multiplicity

$$m_\epsilon(\pi, \chi) := \dim(\mathrm{Hom}_{H_\epsilon(F)}(\pi, \chi \otimes \xi_\epsilon)).$$

Our first result for the unitary Shalika model is that the multiplicity for the two models are equal to each other for all stable discrete series.

Theorem 1.9. *1. Let π be a finite length discrete series of $G(F)$ with central character χ^n. If Θ_π is a stable distribution, then $m_+(\pi, \chi) = m_-(\pi, \chi)$.*

2. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ with central character χ^n. Then we have

$$\sum_{\pi \in \Pi_\phi(G)} m_+(\pi, \chi) = \sum_{\pi \in \Pi_\phi(G)} m_-(\pi, \chi).$$

Our second result for the unitary Shalika model is a necessary condition for a discrete L-packet to be distinguished. The character χ of E^1 induces a character χ' of $E^\times$ by $\chi'(x) = \chi(x/x^c)$. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ and let $\Pi_\phi(G')$ be its base change to $G'(F) = \mathrm{GL}_{2n}(E)$. Then $\Pi_\phi(G')$ is an irreducible tempered representation. Let $H'(F) = \{\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \mid h \in \mathrm{GL}_n(E), X \in Mat_{n\times n}(E)\}$ be the Shalika subgroup and we define the character $\chi' \otimes \xi'$ on it to be

$$\chi' \otimes \xi'(\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}) = \chi'(\det(h))\psi(\mathrm{tr}_{E/F}(\mathrm{tr}(X))).$$

Theorem 1.10. *With the notation above, the packet $\Pi_\phi(G)$ is $(H_+, \chi \otimes \xi_+)$-distinguished (i.e. $m_+(\pi, \chi) \neq 0$ for some $\pi \in \Pi_\phi(G)$) only if $\Pi_\phi(G')$ is distinguished by the Shalika model $(H', \chi' \otimes \xi')$.*

Remark 1.11. *By Theorem 1.9, we know that the packet $\Pi_\phi(G)$ is $(H_+, \chi\otimes\xi_+)$-distinguished if and only if it is $(H_-, \chi \otimes \xi_-)$-distinguished.*

Our next result for the unitary Shalika model is to compute the summation of the multiplicities over certain discrete L-packets. Assume that $\Pi_\phi(G')$ is distinguished by the Shalika model $(H', \chi' \otimes \xi')$. By Corollary 1.1 of [25], $\Pi_\phi(G')$ is of the form (χ'' is a character of $E^\times$ with $\chi' = (\chi'')^2$)

$$\Pi_\phi(G') \otimes (\chi'' \circ \det)^{-1} = (\tau_1 \times \cdots \times \tau_l) \times (\sigma_1 \times \sigma_1^\vee) \times \cdots \times (\sigma_m \times \sigma_m^\vee)$$

where

- τ_i is a discrete series of $\mathrm{GL}_{2a_i}(E)$ that is conjugate self-dual, self-dual and of symplectic type. In particular, a_i is even.
- σ_j is a discrete series of $\mathrm{GL}_{b_i}(E)$ that is conjugate self-dual, but NOT self-dual.
- τ_i, σ_j are all distinct.
- $\sum_{i=1}^l a_i + 2\sum_{j=1}^m b_j = 2n$.

We will consider the special case when $m = 0$. The general case will be consider in our future paper. When $m = 0$, $\Pi_\phi(G')$ appears discretely in the L^2-space of the Shalika model.

Theorem 1.12. *With the notation above, we have*

$$\sum_{\pi\in\Pi_\phi(G)} m_+(\pi, \chi) = \sum_{\pi\in\Pi_\phi(G)} m_-(\pi, \chi) = 2^{l-1}.$$

The idea to prove our main theorems for the unitary Shalika model (resp. the Galois model for classical groups) is by comparing the simple trace formula of the unitary Shalika model (resp. the Galois model for classical groups) with the twisted simple trace formula for the Shalika model (resp. Galois model for general linear groups), we refer the reader to Section 8 (resp. Section 9) for details.

In our next paper, we will prove the spectral side of the trace formula in the general case and we will use it to compute the multiplicity of all the discrete series for the Galois model for classical groups and the unitary Shalika model.

1.2 Organization of the paper

In Section 2, we introduce basic notations and conventions of this paper. This include some extended discussions of twisted weighted orbital integrals, germ expansions and the twisted local trace formula for strongly cuspidal functions.

In Section 3, we introduce the notion of coregular varieties and we will have an extended discussion of symmetric varieties.

In Section 4, we will introduce certain (ι-)weighted functions associated to singular semisimple elements and we will prove an integral formula of the regular germs of quasi-characters. We will also prove a descent formula for the ι-weighted functions which will be used in later section.

We prove a special case of the spectral expansion of the trace formula in Section 5 and in Section 6 we will prove the geometric expansion.

In Section 7 we will discuss our first two applications of the trace formula, namely a simple trace formula for strongly cuspidal functions and a multiplicity formula.

In Section 8 and 9 we will discuss another two applications of the trace formula. In Section 8 we will prove our theorems for the unitary Shalika models and in Section 9 we will prove our theorems for the Galois models of classical groups.

In Appendix A we will prove some results regarding finitely generated convex sets and in Appendix B we will prove the Howe's conjecture for twisted weighted orbital integrals. The results in the two appendices will be used in Section 4 in our proof of the integral formula of the regular germs of quasi-characters.

1.3 Acknowledgement

We thank Rui Chen for pointing out that the base change of a character is always a square of another character. We thank Spencer Leslie and Yiannis Sakellaridis for a remark on the notion of coregular spherical varieties (Remark 3.5). Finally, the first author is grateful to Yiannis Sakellaridis for many very inspiring discussions over the years on local trace formulas for spherical varieties that have partly inspired this work.

The work of first author was funded by the European Union ERC Consolidator Grant, RELANTRA, project number 101044930. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. The work of the second author is partially supported by the NSF grant DMS-2000192 and DMS-2103720.

2 Groups and twisted spaces

Throughout the paper, F will be a non-Archimedean local field of characteristic zero with normalized absolute value $|.|$. Unless otherwise specified, all the groups and varieties that we will consider are implicitely supposed defined over F. We fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$ and, whenever convenient, we will also fix an algebraic closure $\overline{F}$ of F.

For V a real vector space, we write V^* for its dual and we denote by $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Moreover, $iV \subset V_{\mathbb{C}}$ will stand for the real subspace of purely imaginary vectors.

For two complex valued functions f and g on a set X with g taking values in $\mathbb{R}_{\geq 0}$, we write that

$$f(x) \ll g(x), \quad x \in X,$$

and say that f is essentially bounded by g, if there exists a constant $c > 0$ such that for all $x \in X$, we have

$$|f(x)| \leq cg(x).$$

We say f and g are equivalent, which is denoted by

$$f(x) \sim g(x)$$

if f is essentially bounded by g and g is essentially bounded by f.

2.1 Groups

In this section, we fix some notation relative to the datum of a linear algebraic group G defined over F. First, we write $rk(G)$ for the (absolute) rank of G, that is $\dim(T)$ for any maximal torus $T \subset G$, U_G for the unipotent radical of G and we will denote Lie algebras by the corresponding gothic letter such as $\mathfrak{g}$ for G. The adjoint representations $G \to \mathrm{GL}(\mathfrak{g})$ and $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ will be denoted by $g \mapsto \mathrm{Ad}_g$ and $X \mapsto \mathrm{ad}_X$ respectively. We also write $\mathfrak{g}^*$ for the dual of $\mathfrak{g}$. The exponential map, which is well-defined on some neighborhood of 0 in $\mathfrak{g}(F)$ to $G(F)$, will be denoted by exp. For every $g \in G$, we write Ad_g both for the adjoint action of g on G and $\mathfrak{g}$. We also denote by $X^*(G)$ the group of algebraic characters $G \to \mathbb{G}_m$ defined over F and by A_G the maximal central split torus in G/U_G. We set

$$\mathcal{A}_G = \mathrm{Hom}(X^*(G), \mathbb{R}) = \mathrm{Hom}(X^*(A_G), \mathbb{R})$$

and we let as usual $H_G : G(F) \to \mathcal{A}_G$ be the homomorphism defined by $\langle H_G(g), \chi \rangle = \log|\chi(g)|$ for every $(g, \chi) \in G(F) \times X^*(G)$. We denote by G_{rs} and $\mathfrak{g}_{\mathrm{rs}}$ (resp. G_{reg} and $\mathfrak{g}_{\mathrm{reg}}$) the open subsets of regular semi-simple elements (resp. regular elements) in G and $\mathfrak{g}$ respectively. The notation δ_G will stand for the modular character of $G(F)$ that is the character $\delta_G : G(F) \to \mathbb{R}_+^*$ defined by $\delta_G(g) = |\det \mathrm{Ad}_g\,|_{\mathfrak{g}}|$. For every semi-simple element $X \in \mathfrak{g}(F)$, we let

$$D^G(X) = |\det \mathrm{ad}_X\,|_{\mathfrak{g}/\mathfrak{g}_X}|$$

be its Weyl discriminant, where $\mathfrak{g}_X$ stands for the centralizer of X. The Weyl discriminant $D^G(g)$ for $g \in G$ semi-simple is defined in a similar way. We also define algebraic variants of the Weyl discriminant by

$$D^G_{alg}(X) = \det \mathrm{ad}_X\,|_{\mathfrak{g}/\mathfrak{g}_X}, \;\; D^G_{alg}(x) = \det 1 - \mathrm{Ad}_x\,|_{\mathfrak{g}/\mathfrak{g}_x}$$

for $X \in \mathfrak{g}_{\mathrm{rs}}$ and $x \in G_{\mathrm{rs}}$ regular semi-simple elements. Note that these extend to regular functions on $\mathfrak{g}$ and G respectively.

When G is connected and $P \subset G$ is a parabolic subgroup, there is a natural splitting $\mathcal{A}_P = \mathcal{A}_G \oplus \mathcal{A}_P^G$ and we can define as usual subsets $\Delta_P^G \subset (\mathcal{A}_P^G)^*$, $\Delta_P^{G,\vee} \subset \mathcal{A}_P^G$, that we call by abuse of terminology the sets of simple roots and coroots associated to the pair $P \subset G$, see e.g. [23, §1.2]. When G is moreover reductive and clear from the context, we will sometimes drop the superscript.

2.2 Twisted spaces

In this paper, we will freely use the notion of twisted space due to Labesse as well as corresponding terminology. The main references are [23] and [28] but for the reader's convenience we recall most of the definitions here.

A *twisted space* is a pair $(G, \widetilde{G})$ where G is a group and $\widetilde{G}$ is a set equipped with two commuting left and right actions

(2.2.1) $$G \times \widetilde{G} \times G \to \widetilde{G}, \quad (g, \gamma, g') \mapsto g\gamma g'$$

each making $\widetilde{G}$ into a principal G-homogeneous space and such that $\widetilde{G} \neq \emptyset$. Similarly, a *twisted space over* F is a pair $(G, \widetilde{G})$ where G is an algebraic group over F and $\widetilde{G}$ is an algebraic variety over F equipped with two commuting regular actions as in (2.2.1) both making $\widetilde{G}$ into a principal G-torsor and such that $\widetilde{G}(F) \neq \emptyset$. We moreover say that $(G, \widetilde{G})$ is *reductive* (resp. *connected*) if G is so.

Let $(G, \widetilde{G})$ be a twisted space over F with G linear. For every $\gamma \in \widetilde{G}$, we denote by Ad_γ the unique automorphism of G such that

$$\gamma g = \mathrm{Ad}_\gamma(g)\gamma, \text{ for every } g \in G.$$

We will denote by $\theta_{\widetilde{G}}$ the outer automorphism of G over F (i.e. an element of $\mathrm{Aut}_F(G)/\mathrm{Ad}(G(F))$) associated to Ad_γ for any $\gamma \in \widetilde{G}(F)$ (it is independent of γ). We will also write θ for $\theta_{\widetilde{G}}$ when the twisted space $\widetilde{G}$ is clear from the context. When $\theta_{\widetilde{G}}$, or Ad_γ, induces natural automorphisms on related objects these will invariantly be denoted by the same symbol. For example, $\theta_{\widetilde{G}}$ induces an automorphism of A_G and $\mathcal{A}_G$. We write $A_{\widetilde{G}}$ for the connected component of the subgroup of fixed points A_G^θ. The following condition on $\theta_{\widetilde{G}}$ will be assumed throughout:

(2.2.2) the outer automorphism $\theta_{\widetilde{G}}$ is of finite order.

Note that if G is reductive, this is equivalent to the restriction $\theta \mid_{Z(G)}$ to the center of G being of finite order. We set

$$\mathcal{A}_{\widetilde{G}} = \mathrm{Hom}(X^*(A_{\widetilde{G}}), \mathbb{R}).$$

Then, $\mathcal{A}_{\widetilde{G}}$ can naturally be identified with the subspace of $\theta_{\widetilde{G}}$-invariants in $\mathcal{A}_G$ and due to condition (2.2.2) it admits a unique $\theta_{\widetilde{G}}$-stable complement so that we have a canonical projection $\mathcal{A}_G \to \mathcal{A}_{\widetilde{G}}$. We denote by

$$H_{\widetilde{G}} : G(F) \to \mathcal{A}_{\widetilde{G}}$$

the composite of H_G with that projection and by $\mathcal{A}_{\widetilde{G},F}$ the lattice $H_{\widetilde{G}}(G(F))$. We also set $i\mathcal{A}^\vee_{\widetilde{G},F} = \mathrm{Hom}(\mathcal{A}_{\widetilde{G},F}, 2i\pi\mathbb{Z})$, a subgroup of $i\mathcal{A}^*_{\widetilde{G}}$, and $i\mathcal{A}^*_{\widetilde{G},F} = i\mathcal{A}^*_{\widetilde{G}}/i\mathcal{A}^\vee_{\widetilde{G},F}$. We also denote by $\delta_{\widetilde{G}}$ the "modular character" of $\widetilde{G}(F)$ that is the function $\delta_{\widetilde{G}} : \widetilde{G}(F) \to \mathbb{R}^+$ defined by

$$\delta_{\widetilde{G}}(\gamma) = |\det \mathrm{Ad}_\gamma\ |_{\mathfrak{g}}|.$$

For a subset $X \subset G$, the *normalizer* of X in $\widetilde{G}$ is the subset of $\gamma \in \widetilde{G}$ such that $\mathrm{Ad}_\gamma(X) = X$. We denote by $\mathrm{Norm}_{\widetilde{G}}(X)$ the normalizer of X in $\widetilde{G}$ and by $\mathrm{Norm}_{\widetilde{G}(F)}(X) = \mathrm{Norm}_{\widetilde{G}}(X) \cap \widetilde{G}(F)$. Similarly, for a subset $X \subset \widetilde{G}$ we write $\mathrm{Norm}_G(X)$ (resp. $Z_G(X)$) for the normalizer (resp. the centralizer) of X in G that is the subset of $x \in G$ such that $x^{-1}Xx = X$ (resp. $x^{-1}\gamma x = \gamma$ for every $\gamma \in X$) and we set $\mathrm{Norm}_{G(F)}(X) = \mathrm{Norm}_G(X) \cap G(F)$ (resp. $Z_{G(F)}(X) = Z_G(X) \cap G(F)$). When $\gamma \in \widetilde{G}$, we simply write $Z_G(\gamma)$ for $Z_G(\{\gamma\})$ and we denote by G_γ the neutral component of $Z_G(\gamma)$.

We henceforth assume that G is connected and reductive. A *twisted parabolic subspace* of $\widetilde{G}$ is the normalizer $\widetilde{P} = \mathrm{Norm}_{\widetilde{G}}(P)$ of a parabolic subgroup $P \subset G$ satisfying $\widetilde{P}(F) \neq \emptyset$ (or equivalently $\widetilde{P}(\overline{F}) \neq \emptyset$). Note that the parabolic subgroup P is entirely determined by $\widetilde{P}$ and that $(P, \widetilde{P})$ is a twisted space over F. If $\widetilde{P}$ is a twisted parabolic subspace of $\widetilde{G}$, a *Levi component* of $\widetilde{P}$ is the normalizer $\widetilde{M} = \mathrm{Norm}_{\widetilde{P}}(M)$ in $\widetilde{P}$ of a Levi component M of P. Note that the condition $\widetilde{P}(F) \neq \emptyset$ implies $\widetilde{M}(F) \neq \emptyset$. A *twisted Levi subspace* of $\widetilde{G}$ is a Levi component $\widetilde{M}$ of some twisted parabolic subspace $\widetilde{P}$ of $\widetilde{G}$. Note that if $\widetilde{P} \subset \widetilde{G}$ is a parabolic subspace and $\widetilde{M} \subset \widetilde{P}$ is a Levi component of it, we have (canonically) $A_{\widetilde{M}} = A_{\widetilde{P}}$ and $\mathcal{A}_{\widetilde{M}} = \mathcal{A}_{\widetilde{P}}$.

Let $\widetilde{M} \subset \widetilde{G}$ be a twisted Levi subspace. We denote by $\mathcal{P}(\widetilde{M})$ (resp. $\mathcal{F}(\widetilde{M})$) the set of twisted parabolic subspaces with Levi component $\widetilde{M}$ (resp. containing $\widetilde{M}$). For $\widetilde{Q} \in \mathcal{F}(\widetilde{M})$, we have a natural decomposition

$$\mathcal{A}_{\widetilde{M}} = \mathcal{A}_{\widetilde{Q}} \oplus \mathcal{A}^{\widetilde{Q}}_{\widetilde{M}}.$$

We will also write $\mathcal{A}^{\widetilde{Q}}_{\widetilde{P}}$ for $\mathcal{A}^{\widetilde{Q}}_{\widetilde{M}}$ for every $\widetilde{P} \in \mathcal{P}(\widetilde{M})$.

For two parabolic subspaces $\widetilde{P} \subset \widetilde{Q}$, we denote by $\Delta^{\widetilde{Q},\vee}_{\widetilde{P}}$ and $\Delta^{\widetilde{Q}}_{\widetilde{P}}$ the respective images of $\Delta^{Q,\vee}_P$ and Δ^Q_P by the natural projections $\mathcal{A}^Q_P \to \mathcal{A}^{\widetilde{Q}}_{\widetilde{P}}$ and $(\mathcal{A}^Q_P)^* \to (\mathcal{A}^{\widetilde{Q}}_{\widetilde{P}})^*$.

If $M \subset G$ is a Levi subgroup (not necessarily corresponding to any twisted Levi subspace of $\widetilde{G}$), we set

$$W^G(M) = \mathrm{Norm}_{G(F)}(M)/M(F) \text{ and } W^{\widetilde{G}}(M) = \mathrm{Norm}_{\widetilde{G}(F)}(M)/M(F).$$

Note that if $W^{\widetilde{G}}(M) \neq \emptyset$ then $(W^G(M), W^{\widetilde{G}}(M))$ is a twisted group.

Two twisted parabolic subspaces $\widetilde{P}$ and $\widetilde{Q}$ of $\widetilde{G}$ are called *opposite* if the corresponding parabolic subgroups P and Q of G are so or, equivalently, if the intersection $\widetilde{P} \cap \widetilde{Q}$ is a common Levi component of $\widetilde{P}$ and $\widetilde{Q}$. If $\widetilde{M} \subset \widetilde{G}$ is a twisted Levi subspace and $\widetilde{P} \in \mathcal{P}(\widetilde{M})$, there exists a unique $\widetilde{Q} \in \mathcal{P}(\widetilde{M})$ which is opposite to $\widetilde{P}$.

There is also a notion of *twisted maximal torus*: it is a subvariety $\widetilde{T} \subset \widetilde{G}$ defined over F for which there exists a Borel pair (B, T), not necessarily defined over F, such that $\widetilde{T} = \mathrm{Norm}_{\widetilde{G}}(B) \cap \mathrm{Norm}_{\widetilde{G}}(T)$ and $\widetilde{T}(F) \neq \emptyset$. If $\widetilde{T} \subset \widetilde{G}$ is a twisted maximal torus, the torus $T \subset G$ is uniquely determined by $\widetilde{T}$ and is defined over F. Moreover, the pair $(T, \widetilde{T})$ is a twisted space over F. We say that a twisted maximal torus $\widetilde{T} \subset \widetilde{G}$ is *elliptic* if $A_{\widetilde{T}} = A_{\widetilde{G}}$. For any twisted maximal torus $\widetilde{T} \subset \widetilde{G}$ we set

$$W(G, \widetilde{T}) = \mathrm{Norm}_{G(F)}(\widetilde{T})/T(F).$$

An element $\gamma \in \widetilde{G}$ is *semisimple* if Ad_γ normalizes a Borel pair (B, T) (not necessarily defined over F). The subset of semisimple elements is denoted $\widetilde{G}_{ss}$. Also, we say that $\gamma \in \widetilde{G}$ (resp $\gamma \in \widetilde{G}(F)$) is *regular semisimple* (resp. *regular elliptic*) if the neutral component G_γ of its centralizer $Z_G(\gamma)$ is a torus (resp. a torus anisotropic modulo $A_{\widetilde{G}}$). We denote by $\widetilde{G}_{\mathrm{rs}}$ (resp. $\widetilde{G}(F)_{\mathrm{ell}}$) the open subset of regular semi-simple elements in $\widetilde{G}$ (resp. of regular elliptic elements in $\widetilde{G}(F)$). We write $\Gamma_{\mathrm{ell}}(\widetilde{G})$ for the set of $G(F)$-conjugacy classes in $\widetilde{G}(F)_{\mathrm{ell}}$ and for $\gamma \in \widetilde{G}_{ss}(F)$ we define its *Weyl discriminant* by

$$D^{\widetilde{G}}(\gamma) = |\det(1 - \mathrm{Ad}_\gamma)\ |_{\mathfrak{g}/\mathfrak{g}_\gamma}|$$

where $\mathfrak{g}_\gamma$ stands for the Lie algebra of G_γ.

We henceforth fix a minimal parabolic subgroup $P_{\min}$ of G with a Levi decomposition $P_{\min} = M_{\min} U_{\min}$ and we let $\widetilde{P}_{\min} = \mathrm{Norm}_{\widetilde{G}}(P_{\min})$, $\widetilde{M}_{\min} = \mathrm{Norm}_{\widetilde{P}_{\min}}(M_{\min})$. Then, $\widetilde{P}_{\min}$ is a minimal parabolic subspace of $\widetilde{G}$ and $\widetilde{M}_{\min}$ a Levi component of it. We denote by $\mathcal{L}(\widetilde{M}_{\min})$ the set of twisted Levi subspaces of $\widetilde{G}$ containing $\widetilde{M}_{\min}$ and for every $\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})$ we set

$$\widetilde{W}^M = \mathrm{Norm}_{M(F)}(\widetilde{M}_{\min}(F))/M_{\min}(F).$$

We also fix a special maximal compact subgroup K of $G(F)$ in good position relative to $M_{\min}$.

2.3 Log-norms and Harish-Chandra Ξ function

In this paper we shall freely use the notion of log-norms on algebraic varieties over F as defined in [5, §1.2], which are simple variants of the norms introduced by Kottwitz in [22, §18]. For every algebraic variety X over F, we will fix a log-norm σ_X on it and, for $C > 0$, we denote by $\mathbf{1}_{\sigma_X \leq C}$ the characteristic function of the set

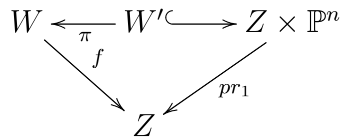
$$\{x \in X(F) |\ \sigma_X(x) \leq C\}.$$

In particular, we have log-norms σ_G and $\sigma_{\widetilde{G}}$ on G and $\widetilde{G}$ respectively that for simplicity we will both denote by σ. For any given base-point $\gamma_0 \in \widetilde{G}(F)$ we have $\sigma(g\gamma_0) \sim \sigma(g)$ for $g \in G(F)$. Moreover, it will be convenient to assume, as we may, that σ is left and right K-invariant for some chosen special maximal compact subgroup $K \subset G(F)$.

Lemma 2.1. *Let W, Z be two algebraic varieties over F and $f : W \to Z$ be a proper morphism. Then, we have*

$$(2.3.1) \qquad \sigma_Z(f(x)) \sim \sigma_W(x), \ \textit{for } x \in W(\overline{F}).$$

Proof. By Chow's lemma [32, Tag 02O2], there exists for some $n \geqslant 0$ a closed subscheme $W' \hookrightarrow Z \times \mathbb{P}^n$ with a surjective regular morphism $\pi : W' \to W$ such that the following diagram commutes



where $pr_1 : Z \times \mathbb{P}^n \to Z$ stands for the first projection. It is readily seen that

$$\sigma_{\mathbb{P}^n}(y) \sim 1, \ \text{for } y \in \mathbb{P}^n(\overline{F})$$

and therefore

$$\sigma_{Z\times\mathbb{P}^n}(z, y) \sim \sigma_Z(z) + \sigma_{\mathbb{P}^n}(y) \sim \sigma_Z(z), \ \text{for } (z, y) \in Z(\overline{F}) \times \mathbb{P}^n(\overline{F}).$$

As $W' \hookrightarrow Z \times \mathbb{P}^n$ is a closed immersion, it follows that

$$\sigma_Z(f(\pi(x'))) \ll \sigma_W(\pi(x')) \ll \sigma_{W'}(x') \sim \sigma_{Z\times\mathbb{P}^n}(x') \sim \sigma_Z(pr_1(x')) = \sigma_Z(f(\pi(x'))), \ x' \in W'(\overline{F}).$$

Hence, $\sigma_Z(f(\pi(x'))) \sim \sigma_W(\pi(x'))$ for $x' \in W'(\overline{F})$. As π is surjective, this implies (2.3.1). □

We also denote by Ξ^G, or simply by Ξ, the basic spherical function of Harish-Chandra i.e. the normalized spherical matrix coefficient (with respect to some choice of special compact subgroup $K \subset G(F)$) of the unramified representation with trivial Satake parameter. Fixing a base-point $\gamma_0 \in \widetilde{G}(F)$, we also define a function $\Xi^{\widetilde{G}}$ on $\widetilde{G}(F)$ by

$$\Xi^{\widetilde{G}}(g\gamma_0) = \Xi^G(g), \ \text{for } g \in G(F).$$

Standard properties of Ξ^G have obvious analogs for $\Xi^{\widetilde{G}}$ e.g. we have (see [33, proposition II.4.5]):

(2.3.2) Let $\widetilde{P} = \widetilde{M}U_P$ be a parabolic subspace of $\widetilde{G}$. Then, for every $d > 0$, there exists $d' > 0$ such that

$$\delta_{\widetilde{P}}(\widetilde{m})^{1/2} \int_{U_P(F)} \Xi^{\widetilde{G}}(\widetilde{m}u)\sigma(\widetilde{m}u)^{-d'} du \ll \Xi^{\widetilde{M}}(\widetilde{m})\sigma(\widetilde{m})^{-d}, \ \text{for } \widetilde{m} \in \widetilde{M}(F).$$

From the 'doubling principle' [33, lemme II.1.3] we also deduce:

(2.3.3) For every compact-open subset $\omega_{\widetilde{G}}$ of $\widetilde{G}(F)$ we have

$$\int_{\omega_{\widetilde{G}}} \Xi^{\widetilde{G}}(x\gamma y)d\gamma \ll \Xi^G(x)\Xi^G(y), \ \text{for } x, y \in G(F).$$

We let $\mathcal{C}(\widetilde{G}(F))$ be the *Harish-Chandra Schwartz space* of $\widetilde{G}(F)$ i.e. the space of functions $f : \widetilde{G}(F) \to \mathbb{C}$ that are left and right invariant by some compact-open subgroup of $G(F)$ and such that, for every $d > 0$, we have

$$\sup_{\gamma \in \widetilde{G}(F)} |f(\gamma)| \Xi^{\widetilde{G}}(\gamma)^{-1} \sigma(\gamma)^d < \infty.$$

For every compact-open subgroup $J \subset G(F)$, the subspace $\mathcal{C}(J\backslash\widetilde{G}(F)/J)$ of J-biinvariant functions is naturally a Fréchet space, the topology being associated to the seminorms defined by the above supremum for every $d > 0$, and $\mathcal{C}(\widetilde{G}(F)) = \bigcup_J \mathcal{C}(J\backslash\widetilde{G}(F)/J)$ is a strict LF space. Moreover, the subspace $C_c^\infty(\widetilde{G}(F))$ of locally constant compactly supported functions is dense in $\mathcal{C}(\widetilde{G}(F))$. The Harish-Chandra Schwartz space $\mathcal{C}(G(F))$ of $G(F)$ is defined similarly (it suffices to replace $\Xi^{\widetilde{G}}$ and $\sigma_{\widetilde{G}}$ by Ξ^G and σ_G in the definition). We denote by ${}^0\mathcal{C}(G(F))$ the subspace of *cusp forms* i.e. of functions $f \in \mathcal{C}(G(F))$ such that for every proper parabolic subgroup $P = MU_P \subsetneq G$,

$$\int_{U_P(F)} f(xu)du = 0, \quad \text{for every } x \in G(F).$$

2.4 Measures

Let T be a torus (over F). We equip $T(F)$ with a Haar measure as follows: if T is split we choose the unique Haar measure giving to the maximal compact subgroup $T(F)_c$ measure one, in general we endow $T(F)$ with the measure such that its quotient by the measure just defined on $A_T(F)$ gives $T(F)/A_T(F) = (T/A_T)(F)$ a total mass of one.

Let $\widetilde{T} \subset \widetilde{G}$ be a twisted maximal torus. The neutral connected component $T^{\theta,0}$ of the subgroup T^θ of $\theta_{\widetilde{T}}$-fixed points is a torus and therefore, $T^{\theta,0}(F)$ is already equipped with a measure as in the above discussion. Let $\widetilde{T}(F)/(1-\theta)(T(F))$ be the quotient of $\widetilde{T}(F)$ by the adjoint action of $T(F)$. We endow $\widetilde{T}(F)/(1-\theta)(T(F))$ with the unique left and right $T(F)$-invariant measure such that, for every $\gamma \in \widetilde{T}(F)$, the application

$$T^{\theta,0}(F) \to \widetilde{T}(F)/(1-\theta)(T(F))$$

$$t \mapsto \gamma t$$

is locally measure preserving. Set $\widetilde{T}_{\text{reg}}(F) := \widetilde{T}(F) \cap \widetilde{G}_{\text{rs}}(F)$. Then, $\widetilde{T}_{\text{reg}}(F)/(1-\theta)(T(F))$ is an open subset of $\widetilde{T}(F)/(1-\theta)(T(F))$ (for the quotient topology). To simplify notation, we will write $\widetilde{T}(F)_{/\theta}$ (resp. $\widetilde{T}_{reg}(F)_{/\theta}$) for $\widetilde{T}(F)/(1-\theta)(T(F))$ (resp. $\widetilde{T}_{\text{reg}}(F)/(1-\theta)(T(F))$). Similarly, we write $\widetilde{T}_{/\theta}$ for the GIT quotient of $\widetilde{T}$ by the adjoint action of T.

We endow the real vector spaces $\mathcal{A}_G$ and $\mathcal{A}_{\widetilde{G}}$ with the unique Haar measures for which the lattices $H_G(A_G(F))$ and $H_{\widetilde{G}}(A_{\widetilde{G}}(F))$ are of covolume one. Through the exponential map, $i\mathcal{A}_G^*$ and $i\mathcal{A}_{\widetilde{G}}^*$ can be identified with the Pontryagin duals of $\mathcal{A}_G$ and $\mathcal{A}_{\widetilde{G}}$. We equip them with the dual measures.

Let $\mathcal{T}_{\rm ell}(\widetilde{G})$ (resp. $\mathcal{T}(\widetilde{G})$) be a set of representatives of the $G(F)$-conjugacy classes of elliptic twisted maximal tori (resp. twisted maximal tori) in $\widetilde{G}$. We equip the set $\Gamma_{\rm ell}(\widetilde{G})$ (resp. $\Gamma(\widetilde{G})$) of regular elliptic conjugacy classes in $\widetilde{G}(F)$ (resp. regular semisimple conjugacy classes in $\widetilde{G}(F)$) with a measure which is characterized by:

$$\int_{\Gamma_{\rm ell}(\widetilde{G})} \varphi(\gamma)d\gamma = \sum_{\widetilde{T}\in\mathcal{T}_{\rm ell}(\widetilde{G})} |W(G,\widetilde{T})|^{-1}[T^\theta(F):T^{\theta,0}(F)]^{-1}\int_{\widetilde{T}_{\rm reg}(F)_{/\theta}}\varphi(t)dt$$

$$\left(\text{resp. } \int_{\Gamma(\widetilde{G})} \varphi(\gamma)d\gamma = \sum_{\widetilde{T}\in\mathcal{T}(\widetilde{G})} |W(G,\widetilde{T})|^{-1}[T^\theta(F):T^{\theta,0}(F)]^{-1}\int_{\widetilde{T}_{\rm reg}(F)_{/\theta}}\varphi(t)dt\right).$$

for every "reasonable" function φ on $\Gamma_{\rm ell}(\widetilde{G})$ (resp. on $\Gamma(\widetilde{G})$).

These conventions also apply to the parabolic and Levi subgroups (resp. parabolic and Levi subspaces) of G (resp. of $\widetilde{G}$). In particular, the definition of the measures on $\Gamma_{\rm ell}(\widetilde{G})$ and $\Gamma(\widetilde{G})$ was chosen so that Weyl's integration formula for $\widetilde{G}(F)$ (see [28, §4.1]) takes the following forms:

$$(2.4.1)\qquad \int_{\widetilde{G}(F)} f(\gamma)d\gamma = \int_{\Gamma(\widetilde{G})} D^{\widetilde{G}}(\gamma)\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dgd\gamma$$
$$= \sum_{\widetilde{M}\in\mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M||\widetilde{W}^G|^{-1}\int_{\Gamma_{\rm ell}(\widetilde{M})} D^{\widetilde{G}}(\gamma)\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dgd\gamma$$

for every $f\in L^1(\widetilde{G}(F))$ where in the above formula, we have chosen a Haar measure on $G(F)$ from which we deduce a measure on $\widetilde{G}(F)$ by translation by any element $\gamma\in\widetilde{G}(F)$ and we put on the F-points of the torus G_γ the canonical measure defined above.

There is yet another description of the measure on $\Gamma(\widetilde{G})$ that will be useful to us. More precisely, for every $\gamma\in\widetilde{G}_{\rm rs}(F)$ set $\widetilde{G}_\gamma=\gamma G_\gamma$. Then, the pair $(G_\gamma,\widetilde{G}_\gamma)$ is a twisted torus of a very special form, namely every element of G_γ commutes with every element of $\widetilde{G}_\gamma$, which is however rarely a maximal twisted torus of $\widetilde{G}$ (unless the outer automorphism $\theta_{\widetilde{G}}$ is trivial). Let $\mathcal{S}(\widetilde{G})$ be a set of representatives of twisted torus of the form $(G_\gamma,\widetilde{G}_\gamma)$, $\gamma\in\widetilde{G}_{\rm rs}(F)$, for the natural action by conjugation of $G(F)$. Then, there exist constants $(c_{\widetilde{S}})_{\widetilde{S}\in\mathcal{S}(\widetilde{G})}$ such that the measure on $\Gamma(\widetilde{G})$ takes the form:

$$(2.4.2)\qquad \int_{\Gamma(\widetilde{G})}\varphi(\gamma)d\gamma = \sum_{\widetilde{S}\in\mathcal{S}(\widetilde{G})} c_{\widetilde{S}}\int_{\widetilde{S}_{\rm reg}(F)}\varphi(s)ds$$

where, as before, we have set $\widetilde{S}_{\rm reg}=\widetilde{S}\cap\widetilde{G}_{\rm rs}$ and the measure on $\widetilde{S}_{\rm reg}(F)$ is the restriction of the translation of the natural Haar measure on $S(F)$ introduced above to $\widetilde{S}(F)$. More precisely, we can take for $\mathcal{S}(\widetilde{G})$ the set of twisted tori $(T^{\theta,0},T^{\theta,0}t)$ where $\widetilde{T}$ runs over the set of representatives $\mathcal{T}(\widetilde{G})$ fixed above and t describes a set of representatives of the $T^{\theta,0}(F)$

orbits in $\widetilde{T}(F)/\theta$. Then, the above integration formula follows readily from the definition of the measure on $\Gamma(\widetilde{G})$ with constants

$$c_{\widetilde{S}} = |W(G,\widetilde{T})|^{-1}[T^{\theta}(F) : T^{\theta,0}(F)]^{-1}|T^{\theta,0}(F) \cap (1-\theta)(T(F))|^{-1}.$$

For $P \subset Q$ (resp. $\widetilde{P} \subset \widetilde{Q}$) two parabolic subgroups of G (resp. two parabolic subspaces of $\widetilde{G}$) we equip

$$\mathcal{A}_P^Q = \mathcal{A}_P/\mathcal{A}_Q \text{ (resp. } \mathcal{A}_{\widetilde{P}}^{\widetilde{Q}} = \mathcal{A}_{\widetilde{P}}/\mathcal{A}_{\widetilde{Q}})$$

with the quotient of the two Haar measures just defined.

All other groups considered will be equipped with Haar measures whose normalization does not really matter. However, for some intermediate steps, it will be convenient to assume that for $P = MU_P$ a parabolic subgroup of G, the Haar measures are chosen so that we have the following integration formula:

$$\int_{G(F)} f(g)dg = \int_{M(F)} \int_{U_P(F)} \int_K f(muk)dkdudm.$$

Finally, for a Levi subspace $\widetilde{M}$ of $\widetilde{G}$, we endow $\widetilde{M}(F)$ with the unique (biinvariant) measure such that for every $\gamma \in \widetilde{M}(F)$ the bijection $m \in M(F) \mapsto \gamma m \in \widetilde{M}(F)$ is measure-preserving.

2.5 Estimates

Let $\widetilde{T} \subset \widetilde{G}$ be a twisted maximal torus. In this section, we denote by $\theta = \theta_{\widetilde{T}}$ the restriction of Ad_t to T for any $t \in \widetilde{T}$ (it does not depend on t). As in the previous section, we write $\widetilde{T}_{\mathrm{reg}}(F)_{/\theta} = \widetilde{T}_{\mathrm{reg}}(F)/(1-\theta)(T(F))$ for the quotient of $\widetilde{T}_{\mathrm{reg}}(F)$ by the adjoint action of $T(F)$ and we denote by $\widetilde{T}_{/\theta} = \widetilde{T}/(1-\theta)T$ the categorical quotient of $\widetilde{T}$ by the adjoint action of T.

Lemma 2.2. *We have*

$$\inf_{t \in T(F)} \sigma_G(tg) \ll \sigma_{\widetilde{G}}(g^{-1}\gamma g) + |\log D^{\widetilde{G}}(\gamma)|$$

for $(g,\gamma) \in G(F) \times \widetilde{T}_{\mathrm{reg}}(F)$.

Proof. Let $Y = \widetilde{T}_{\mathrm{reg}} \times^T G$ be the quotient of $\widetilde{T}_{\mathrm{reg}} \times G$ by the free action of T given by $t \cdot (\gamma, g) = (t\gamma t^{-1}, tg)$. Then, the regular map

$$Y \to \widetilde{G}_{\mathrm{rs}},\ [\gamma, g] \mapsto g^{-1}\gamma g$$

is finite. Thus, by [22, Proposition 18.1(1)], we have

$$\sigma_Y(\gamma, g) \sim \sigma_{\widetilde{G}_{\mathrm{rs}}}(g^{-1}\gamma g) \sim \sigma_{\widetilde{G}}(g^{-1}\gamma g) + |\log D^{\widetilde{G}}(\gamma)|, \quad \text{for } [\gamma, g] \in Y(F). \tag{2.5.1}$$

On the other hand, the regular map $Y \to T\backslash G$, $[\gamma, g] \mapsto Tg$, implies that

$$\sigma_{T\backslash G}(g) \ll \sigma_Y(\gamma, g), \quad \text{for } [\gamma, g] \in Y(F). \tag{2.5.2}$$

Finally, by [22, Proposition 18.3], we have

$$\sigma_{T\backslash G}(g) \sim \inf_{t\in T(F)} \sigma(tg), \quad \text{for } g \in G(F). \tag{2.5.3}$$

The lemma readily follows from the combination of (2.5.1), (2.5.2) and (2.5.3). □

For every positive function f on $\widetilde{G}(F)$ and $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)$ we set

$$J_{\widetilde{G}}(\gamma, f) = D^{\widetilde{G}}(t)^{1/2} \int_{A_{\widetilde{T}}(F)\backslash G(F)} f(g^{-1}\gamma g)dg$$

whether the integral is convergent or not. Note that this expression only depends on the image of γ in $\widetilde{T}_{\mathrm{reg}}(F)_{/\theta}$.

Proposition 2.3. *For every $d > 0$ there exists $d' > 0$ such that the orbital integral $J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d'})$ is convergent for all $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)$ and we have*

$$\sup_{\gamma\in\widetilde{T}_{\mathrm{reg}}(F)_{/\theta}} \sigma_{\widetilde{T}_{/\theta}}(\gamma)^d J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d'}) < \infty.$$

Proof. Let $\widetilde{M} \subset \widetilde{G}$ be the centralizer of $A_{\widetilde{T}}$. Then, $\widetilde{M}$ is a twisted Levi subspace. Choose a parabolic subspace $\widetilde{P} = \widetilde{M}U_P \in \mathcal{P}(\widetilde{M})$. By the Iwasawa decomposition $G(F) = P(F)K$ and a standard Jacobian computation, up to a constant depending on measures, for every positive function f on $\widetilde{G}(F)$ we have

$$J_{\widetilde{G}}(\gamma, f) = J_{\widetilde{M}}(\gamma, f_{\widetilde{P}})$$

where $f_{\widetilde{P}}$ is the function on $\widetilde{M}(F)$ defined by

$$f_{\widetilde{P}}(\widetilde{m}) = \delta_{\widetilde{P}}(\widetilde{m})^{1/2} \int_K \int_{U_P(F)} f(k^{-1}\widetilde{m}uk)dudk, \quad \widetilde{m} \in \widetilde{M}(F).$$

Therefore, by (2.3.2), up to replacing $\widetilde{G}$ by $\widetilde{M}$ we may assume that $A_{\widetilde{T}} = A_{\widetilde{G}}$ i.e. that $\widetilde{T}$ is elliptic in $\widetilde{G}$. The statement of the proposition can also be readily reduced to the case where $A_{\widetilde{G}} = 1$ which we assume from now on. Then, as $\widetilde{T}$ is elliptic the quotient $\widetilde{T}(F)_{/\theta}$ is compact and we just need to show the existence of $d_0 > 0$ such that

$$\sup_{\gamma\in\widetilde{T}_{\mathrm{reg}}(F)_{/\theta}} J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0}) < \infty.$$

Assume for one moment the following claim:

(2.5.4) There exists $d_0 > 0$ such that for almost all $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)_{/\theta}$, the integral defining $J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0})$ converges.

Then, we can conclude as in [12, Corollary 2] using Howe's conjecture for twisted groups [28, Chap. 2, théorème 2.1]. Indeed, let $(\Omega_n)_{n\geqslant 1}$ be an increasing and exhaustive[4] sequence of K-biinvariants compact subsets of $\widetilde{G}(F)$ and set $f_n = \mathbf{1}_{\Omega_n}\Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0}$. Then, (f_n) is an increasing sequence of functions in $C_c(K\backslash\widetilde{G}(F)/K)$ converging pointwise to $\Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0}$ hence $J_{\widetilde{G}}(\gamma, f_n)$ converges to $J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0})$ for all $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)_{/\theta}$ (whether the last integral is finite or infinite). However, by [28, Chap. 1, 4.2 (1)], the functions $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)_{/\theta} \mapsto J_{\widetilde{G}}(\gamma, f_n)$ are locally constant and bounded whereas by [28, Chap. 2, théorème 2.1] ("Howe's conjecture" for the twisted group $\widetilde{G}$) the vector space they span is finite dimensional. It follows that the function $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)_{/\theta} \mapsto J_{\widetilde{G}}(\gamma, \Xi^{\widetilde{G}}\sigma_{\widetilde{G}}^{-d_0})$ has the same properties (i.e. it is locally constant and bounded) and this proves the proposition.

It remains to show (2.5.4). Set T^θ for the subgroup of θ-fixed points in T (recall that $\theta = \theta_{\widetilde{T}}$). Let $\gamma \in \widetilde{T}_{\mathrm{reg}}(F)$ and let $(T^\theta)'$ be the inverse image of $\widetilde{G}_{\mathrm{rs}}$ by the morphism $t \in T^\theta \mapsto \gamma t$. Then, the map $t \in (T^\theta)'(F) \mapsto \gamma t((1-\theta)(T(F))) \in \widetilde{T}_{\mathrm{reg}}(F)_{/\theta}$ is a local homeomorphism. Therefore, by Fubini, we just need to check that for every compact-open subset $\omega \subset (T^\theta)'(F)$, the integral

$$\int_\omega \int_{G(F)} \Xi^{\widetilde{G}}(g^{-1}\gamma t g)\sigma_{\widetilde{G}}(g^{-1}\gamma t g)^{-d_0} dg dt \tag{2.5.5}$$

converges. First, we show that

$$\sigma_G(g) \ll \sigma_{\widetilde{G}}(g^{-1}\gamma t g), \text{ for } (g,t) \in G(F) \times \omega. \tag{2.5.6}$$

The morphism

$$(T^\theta)' \times T^\theta\backslash G \to \widetilde{G}_{\mathrm{rs}}, \quad (t,g) \mapsto g^{-1}\gamma t g \tag{2.5.7}$$

is finite étale. Therefore, we have

$$\sigma_{T^\theta\backslash G}(g) + \sigma_{(T^\theta)'}(t) \ll \sigma_{\widetilde{G}_{\mathrm{rs}}}(g^{-1}\gamma t g) \sim \sigma_{\widetilde{G}}(g^{-1}\gamma t g) + |\log D^{\widetilde{G}}(\gamma t)|$$

for $(g,t) \in G(F) \times (T^\theta)'(F)$. On the other hand, since ω is compact, we have $\sigma_{(T^\theta)'}(t) \sim 1$ and $|\log D^{\widetilde{G}}(\gamma t)| \sim 1$ for $t \in \omega$. Combining this with the previous inequality, gives

$$\sigma_{T^\theta\backslash G}(g) \ll \sigma_{\widetilde{G}}(g^{-1}\gamma t g), \text{ for } (g,t) \in G(F) \times \omega.$$

Moreover, by [22, Proposition 18.3] we have

$$\sigma_{T^\theta\backslash G}(g) \sim \inf_{t \in T^\theta(F)} \sigma_G(tg) \sim \sigma_G(g), \text{ for } g \in G(F),$$

[4]Meaning that $\widetilde{G}(F) = \bigcup_n \Omega_n$.

(Recall that the twisted torus $\widetilde{T}$ is elliptic and $A_{\widetilde{G}} = 1$ hence $T^\theta(F)$ is compact.) and this implies (2.5.6).

We now consider the integral (2.5.5). By (2.5.6) it is essentially bounded by

$$\int_\omega \int_{G(F)} \Xi^{\widetilde{G}}(g^{-1}\gamma tg)\sigma_G(g)^{-d_0} dgdkdt$$

which can be rewritten as

$$\int_\omega \int_K \int_{G(F)} \Xi^{\widetilde{G}}(g^{-1}k^{-1}\gamma tkg)\sigma_{\widetilde{G}}(g)^{-d_0} dgdkdt.$$

Since the map $(T^\theta)'(F) \times T^\theta(F)\backslash G(F) \ni (t,g) \mapsto g^{-1}\gamma tg \in \widetilde{G}(F)$ is a local F-analytic isomorphism, the last expression above is bounded up to a (multiplicative) constant by

$$\int_{\omega_{\widetilde{G}}} \int_{G(F)} \Xi^{\widetilde{G}}(g^{-1}\widetilde{g}g)\sigma_{\widetilde{G}}(g)^{-d_0} dgd\widetilde{g}$$

for some compact-open subset $\omega_{\widetilde{G}}$ of $\widetilde{G}(F)$. Furthermore, by (2.3.3), we have $\int_{\omega_{\widetilde{G}}} \Xi^{\widetilde{G}}(g^{-1}\widetilde{g}g)d\widetilde{g} \ll \Xi^G(g)^2$ for $g \in G(F)$ and the integral (2.5.5) is therefore bounded up to a constant by

$$\int_{G(F)} \Xi^G(g)^2 \sigma_G(g)^{-d_0} dg$$

which is well-known to converge for d_0 sufficiently large, see [33, lemme II.1.5]. □

2.6 Quasi-characters

Following [36, §1.6], by a *quasi-character* on $\widetilde{G}(F)$ we mean a function $\Theta : \widetilde{G}_{\mathrm{rs}}(F) \to \mathbb{C}$ such that for every semisimple element $x \in \widetilde{G}_{\mathrm{ss}}(F)$, there is a local expansion

(2.6.1) $$\Theta(x\exp(X)) = \sum_{\mathcal{O}\in \mathrm{Nil}(\mathfrak{g}_x^*)} c_{\Theta,\mathcal{O}}(x)\widehat{j}_\psi(\mathcal{O},X)$$

valid for $X \in \mathfrak{g}_{x,\mathrm{rs}}(F)$ sufficiently close to 0 and where

- $\mathrm{Nil}(\mathfrak{g}_x^*)$ stands for the set of nilpotent $G_x(F)$-coadjoint orbits in $\mathfrak{g}_x^*(F)$;
- $c_{\Theta,\mathcal{O}}(x) \in \mathbb{C}$ for every $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$;
- For $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$, $\widehat{j}_\psi(\mathcal{O},.)$ is the unique locally integrable function on $\mathfrak{g}_x(F)$ which is locally constant on $\mathfrak{g}_{x,\mathrm{rs}}(F)$ and such that

$$\int_{\mathfrak{g}_x(F)} \varphi(X)\widehat{j}_\psi(\mathcal{O},X)dX = \int_{\mathcal{O}} \widehat{\varphi}(Z)dZ$$

for every $\varphi \in C_c^\infty(\mathfrak{g}_x(F))$, where $\widehat{\varphi} \in C_c^\infty(\mathfrak{g}_x^*(F))$ denotes the Fourier transform

$$Y \in \mathfrak{g}_x^*(F) \mapsto \widehat{\varphi}(Y) = \int_{\mathfrak{g}_x(F)} \varphi(X)\psi(\langle X, Y\rangle)dX$$

and dZ is the Kirillov-Kostant $G_x(F)$-invariant measure on $\mathcal{O}$ deduced from the canonical symplectic form on $\mathcal{O}$ and the self-dual measure on F associated to ψ (see [17]).

For $x \in \widetilde{G}_{\mathrm{ss}}(F)$, we denote by $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x^*) \subset \mathrm{Nil}(\mathfrak{g}_x^*)$ the subset of regular nilpotent coadjoint orbits.

Lemma 2.4. *Let Θ be a quasi-character on $\widetilde{G}(F)$. The function*

$$x \in \widetilde{G}_{ss}(F) \mapsto D^{\widetilde{G}}(x)^{1/2} \max_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x^*)} |c_{\Theta,\mathcal{O}}(x)|$$

is locally bounded.

Proof. Let $x \in \widetilde{G}_{ss}(F)$ be a semisimple element. By [17, Lemma 3.2], we have

$$\widehat{j}_\psi(\mathcal{O}, tX) = |t|^{-\dim(\mathcal{O})/2}\widehat{j}_\psi(\mathcal{O}, X)$$

for every $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$, $X \in \mathfrak{g}_{x,\mathrm{rs}}(F)$ and $t \in F^{\times,2}$. Moreover, for $X \in \mathfrak{g}_{x,\mathrm{rs}}(F)$ sufficiently close to 0, we have

$$D^{\widetilde{G}}(x\exp(X)) = D^{\widetilde{G}}(x)D^{G_x}(X) \text{ and } D^{G_x}(tX) = |t|^{\delta(G_x)}D^{G_x}(X)$$

for every $t \in F^\times$ where we have set $\delta(G_x) = \dim(G_x) - rk(G_x)$. As for every $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$ we have $\dim(\mathcal{O}) \leqslant \delta(G_x)$ with equality if and only if $\mathcal{O}$ is regular, we deduce from the expansion (2.6.1) that for every $X \in \mathfrak{g}_{x,\mathrm{rs}}(F)$ we have

$$\lim_{t \in F^{\times,2}, t\to 0} D^{\widetilde{G}}(x\exp(tX))^{1/2}\Theta(x\exp(tX)) = D^{\widetilde{G}}(x)^{1/2}D^{G_x}(X)^{1/2} \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x^*)} c_{\Theta,\mathcal{O}}(x)\widehat{j}_\psi(\mathcal{O}, X).$$

Since the functions $\widehat{j}_\psi(\mathcal{O}, .)$, $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x^*)$, are linearly independent this implies

$$D^{\widetilde{G}}(x)^{1/2} \max_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x^*)} |c_{\Theta,\mathcal{O}}(x)| \leqslant c(G_x) \limsup_{y \in \widetilde{G}_{\mathrm{rs}}(F), y\to x} D^{\widetilde{G}}(y)^{1/2}|\Theta(y)|$$

where $c(G_x) > 0$ is a constant that depends only on the isomorphism class of G_x. By [17, Corollary 6.3], the function $(D^{\widetilde{G}})^{1/2}\Theta$ is locally bounded and the lemma follows as there are only finitely many isomorphism classes of centralizers G_x for $x \in \widetilde{G}_{ss}(F)$. □

2.7 Representations

In this paper, by a *representation* of $G(F)$ we mean a pair (π, V_π) where V_π is a complex vector space and $\pi : G(F) \to GL(V_\pi)$ is a smooth representation of $G(F)$ on V_π. Most of the time we will omit the space V_π and just write π for a representation of $G(F)$. For $\lambda \in \mathcal{A}^*_{G,\mathbb{C}}$, we denote by $\pi \mapsto \pi_\lambda$, where $\pi_\lambda(g) = e^{\langle \lambda, H_G(g)\rangle}\pi(g)$, the twisting operation by λ on representations of $G(F)$.

Let π be a representation of $G(F)$. We denote by $\pi^\vee$ the smooth contregredient of π realized in the usual way on the space $V_\pi^\vee$ of smooth functionals on V_π. We denote by $\langle ., .\rangle$ the canonical pairing on $V_\pi \times V_\pi^\vee$. We say that π is *tempered* if it is of finite length and for every $(v, v^\vee) \in V_\pi \times V_\pi^\vee$ there exists a constant $C > 0$ such that

$$|\langle \pi(g)v, v^\vee\rangle| \leqslant C\Xi^G(g), \quad \text{for every } g \in G(F).$$

We write $\Pi_2(G)$ (resp. $\mathrm{Temp}(G)$) for the set of isomorphism classes of unitary square-integrable (resp. tempered) irreducible representations of $G(F)$. If $P = MU$ is a parabolic subgroup of G and σ is a representation of $M(F)$, we let $I_P^G(\sigma)$ be the smooth normalized parabolic induction of σ to $G(F)$. When $\sigma \in \mathrm{Temp}(M)$, we write $I_M^G(\sigma)$ for the isomorphism class of $I_P^G(\sigma)$ where $P \in \mathcal{P}(M)$ (it does not depend on this choice). Define $\mathrm{Temp}_{\mathrm{ind}}(G)$ as the set of isomorphism classes of representations of $G(F)$ of the form $I_M^G(\sigma)$ where M is a Levi subgroup of G and $\sigma \in \Pi_2(M)$. According to Harish-Chandra, every $\pi \in \mathrm{Temp}(G)$ can be embedded in $I_M^G(\sigma)$ for such a pair (M, σ) which is moreover unique up to conjugacy by $G(F)$. Thus, we get a map

$$pr_G : \mathrm{Temp}(G) \to \mathrm{Temp}_{\mathrm{ind}}(G).$$

We equip $\mathrm{Temp}_{\mathrm{ind}}(G)$ with a topology that can be described as follows. Let M be a Levi subgroup of G and $\sigma \in \Pi_2(M)$. Then, the set

$$\mathcal{O} = \{I_M^G(\sigma_\lambda) \mid \lambda \in i\mathcal{A}_M^*\}$$

is a connected component of $\mathrm{Temp}_{\mathrm{ind}}(G)$ and the topology on $\mathcal{O}$ is the quotient topology inherited from $i\mathcal{A}_M^*$ via the surjection

$$\lambda \in i\mathcal{A}_M^* \mapsto I_M^G(\sigma_\lambda) \in \mathcal{O}. \tag{2.7.1}$$

We say that a function $z : \mathrm{Temp}_{\mathrm{ind}}(G) \to \mathbb{C}$ is *smooth* if for every pair (M, σ) as before, the composition of z with the map (2.7.1) gives a C^∞ function on $i\mathcal{A}_M^*$ in the usual sense. We denote by $C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))$ the vector space of smooth functions on $\mathrm{Temp}_{\mathrm{ind}}(G)$. It is an algebra for pointwise multiplication. Moreover, by the description of the image by Fourier transform of the Harish-Chandra Schwartz space $\mathcal{C}(G(F))$ [33], there exists an action

$$C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G)) \times \mathcal{C}(G(F)) \to \mathcal{C}(G(F)), \quad (z, f) \mapsto z \star f \tag{2.7.2}$$

of $C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))$ on $\mathcal{C}(G(F))$ which is characterized by

$$\pi(z \star f) = z(\pi)\pi(f) \tag{2.7.3}$$

for every $(\pi, z, f) \in \mathrm{Temp}_{\mathrm{ind}}(G) \times C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G)) \times \mathcal{C}(G(F))$. See also [31] for a different approach where $C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))$ is shown to coincide with the so-called *tempered Bernstein center* of $G(F)$.

The outer automorphism θ of $G(F)$ induces a bijection $\theta : \mathrm{Temp}_{\mathrm{ind}}(G) \to \mathrm{Temp}_{\mathrm{ind}}(G)$. We denote by $\mathrm{Temp}_{\mathrm{ind}}(G)^\theta$ the subset of fixed points.

2.8 Twisted representations

A (smooth) *representation* of the twisted space $\widetilde{G}(F)$ is a pair $(\pi, \widetilde{\pi})$ where π is a representation of $G(F)$ and $\widetilde{\pi}$ is a map $\widetilde{G}(F) \to GL(V_\pi)$ satisfying

$$\widetilde{\pi}(g\gamma g') = \pi(g)\widetilde{\pi}(\gamma)\pi(g'), \text{ for every } (g, \gamma, g') \in G(F) \times \widetilde{G}(F) \times G(F).$$

Most of the time, we will simply refer to a representation of $\widetilde{G}(F)$ by the map $\widetilde{\pi}$, the underlying representation (π, V_π) of $G(F)$ being understood. Note that if $\widetilde{\pi}$ is a representation of $\widetilde{G}(F)$ then so is $c\widetilde{\pi}$ for every $c \in \mathbb{C}^\times$. Moreover, a representation π of $G(F)$ extends to a representation $(\pi, \widetilde{\pi})$ of $\widetilde{G}(F)$ (although not uniquely) if and only if its isomorphism class is fixed by the outer automorphism θ.

Let $\widetilde{\pi}$ be a representation of $\widetilde{G}(F)$. We say that $\widetilde{\pi}$ is *G-irreducible* if π is irreducible in the usual sense i.e. if there is no nontrivial $G(F)$-invariant subspace of V_π. We also say that $\widetilde{\pi}$ is *admissible* (resp. *tempered*) if π is so. We denote by $\widetilde{\pi}^\vee$ the *smooth contragredient* of $\widetilde{\pi}$ that is the representation of $\widetilde{G}(F)$ on the space $V_\pi^\vee$ of smooth functionals on V_π characterized by

$$\langle \widetilde{\pi}(\gamma)v, \widetilde{\pi}^\vee(\gamma)v^\vee \rangle = \langle v, v^\vee \rangle, \text{ for } (\gamma, v, v^\vee) \in \widetilde{G}(F) \times V_\pi \times V_\pi^\vee.$$

Assume that π is of finite length. For every $f \in C_c^\infty(\widetilde{G}(F))$, we define as usual an operator $\widetilde{\pi}(f)$ on V_π characterized by

$$\langle \widetilde{\pi}(f)v, v^\vee \rangle = \int_{\widetilde{G}(F)} f(\gamma) \langle \widetilde{\pi}(\gamma)v, v^\vee \rangle d\gamma, \text{ for } (v, v^\vee) \in V_\pi \times V_\pi^\vee. \tag{2.8.1}$$

These operators are of finite rank and, according to [11, Theorem 3], there exists a quasi-character $\Theta_{\widetilde{\pi}}$ on $\widetilde{G}(F)$ (in the sense of Section 2.6), called the *Harish-Chandra character of* $\widetilde{\pi}$, such that

$$\mathrm{Tr}\, \widetilde{\pi}(f) = \int_{\widetilde{G}(F)} f(g)\Theta_{\widetilde{\pi}}(g) dg, \text{ for every } f \in C_c^\infty(\widetilde{G}(F)). \tag{2.8.2}$$

For ease of notation, we will denote by

$$c_{\widetilde{\pi}, \mathcal{O}}(x) = c_{\Theta_{\widetilde{\pi}}, \mathcal{O}}(x), \text{ for every } x \in \widetilde{G}_{\mathrm{ss}}(F) \text{ and } \mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x),$$

the various coefficients of the germs expansions of $\Theta_{\widetilde{\pi}}$. If moreover $\widetilde{\pi}$ is tempered, the definition (2.8.1) of the operator $\widetilde{\pi}(f)$ still makes sense and the formula (2.8.2) is still valid for $f \in \mathcal{C}(\widetilde{G}(F))$ (the integral being absolutely convergent).

Let $\widetilde{P} = \widetilde{M}U$ be a parabolic subspace of $\widetilde{G}$ and $\widetilde{\sigma}$ be a representation of $\widetilde{M}(F)$. We denote by $I_{\widetilde{P}}^{\widetilde{G}}(\widetilde{\sigma})$ the *normalized parabolic induction of* $\widetilde{\sigma}$ i.e. the representation of $\widetilde{G}(F)$ on the space of smooth functions $e : G(F) \to V_\sigma$ satisfying

$$e(mug) = \delta_P(m)^{1/2}\sigma(m)e(g)$$

for every $(m, u, g) \in M(F) \times U(F) \times G(F)$ the action of $\widetilde{G}(F)$ being given by

$$(I_{\widetilde{P}}^{\widetilde{G}}(\gamma, \widetilde{\sigma})e)(g) = \delta_{\widetilde{P}}(\gamma')^{1/2}\widetilde{\sigma}(\gamma')e(g')$$

for $(\gamma, g) \in \widetilde{G}(F) \times G(F)$ where $g\gamma = \gamma' g'$ is any decomposition with $(\gamma', g') \in \widetilde{M}(F) \times G(F)$ (the right hand side is readily seen to be independent of this decomposition). Note that the underlying representation $I_P^G(\sigma)$ of $G(F)$ is the usual normalized parabolic induction of σ.

Let $M \subset G$ be a Levi subgroup and $\sigma \in \Pi_2(M)$. We set

$$\mathrm{Norm}_{\widetilde{G}(F)}(\sigma) = \{\widetilde{n} \in \mathrm{Norm}_{\widetilde{G}(F)}(M) \mid \sigma \circ \mathrm{Ad}_{\widetilde{n}} \simeq \sigma\},$$

$$\mathrm{Norm}_{G(F)}(\sigma) = \{n \in \mathrm{Norm}_{G(F)}(M) \mid \sigma \circ \mathrm{Ad}_n \simeq \sigma\}$$

and

$$W^{\widetilde{G}}(\sigma) = \mathrm{Norm}_{\widetilde{G}(F)}(\sigma)/M(F), \;\; W^G(\sigma) = \mathrm{Norm}_{G(F)}(\sigma)/M(F).$$

Assume that $W^{\widetilde{G}}(\sigma) \neq \emptyset$. Then, $W^{\widetilde{G}}(\sigma)$ is a torsor under $W^G(\sigma)$ both for left and right multiplication i.e. the pair $(W^G(\sigma), W^{\widetilde{G}}(\sigma))$ is a twisted space. Let $P \in \mathcal{P}(M)$. As in [28, Chap 1, §2.8], and making auxilliary choices (including a regularization of standard intertwining operators), we can define for each $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ an extension of $I_P^G(\sigma)$ to a representation $\widetilde{I_P^G(\sigma)}(\widetilde{w}, .)$ of $\widetilde{G}(F)$. This extension depends on the auxilliary choices only up to multiplication by a nonzero scalar. Let $W_0^G(\sigma)$ be the distinguished subgroup of elements $w \in W_0^G(\sigma)$ such that for each $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ the representations $\widetilde{I_P^G(\sigma)}(\widetilde{w}, .)$ and $\widetilde{I_P^G(\sigma)}(\widetilde{w}w, .)$ are the same up to a scalar. The *twisted R-group of* (M, σ) is the quotient $R^{\widetilde{G}}(\sigma) = W^{\widetilde{G}}(\sigma)/W_0^G(\sigma)$. We also denote by $R^G(\sigma) = W^G(\sigma)/W_0^G(\sigma)$ the corresponding R-group so that $(R^G(\sigma), R^{\widetilde{G}}(\sigma))$ is again a twisted space. To every $r \in R^{\widetilde{G}}(\sigma)$, we associate the twisted representation $\widetilde{I_P^G(\sigma)}(r, .) = \widetilde{I_P^G(\sigma)}(\widetilde{w}_r, .)$ where $\widetilde{w}_r \in W^{\widetilde{G}}(\sigma)$ is some chosen lift of r. This representation still depends, up to a scalar, on various choices but, henceforth, we will always assume that all such choices have been made and we will denote its isomorphism class by $\widetilde{I_M^G(\sigma)}(r, .)$ (which, as the notation suggests, does not depend on P, at least up to a scalar). Note, however, that the isomorphism class of the twisted representation $\widetilde{I_M^G(\sigma)}(r, .) \otimes \widetilde{I_M^G(\sigma)}(r, .)^\vee$ of $\widetilde{G}(F) \times \widetilde{G}(F)$ is completely canonical and independent of all the choices involved.

Let $E(\widetilde{G})$ be the set of $G(F)$-conjugacy classes of triples (M, σ, r) where (M, σ) is as above and $r \in R^{\widetilde{G}}(\sigma)$ is such that the character of $\widetilde{I_M^G(\sigma)}(r, .)$ (which, again, is only well-defined up to a scalar) is nonzero. For $\tau \in E(\widetilde{G})$ represented by a triple (M, σ, r), we will write $\widetilde{\pi}_\tau$ for the twisted representation $\widetilde{I_M^G(\sigma)}(r, .)$. Actually, for $\tau, \tau' \in E(\widetilde{G})$ the representations $\widetilde{\pi}_\tau$ and $\widetilde{\pi}_{\tau'}$ are isomorphic if and only if $\tau = \tau'$ (this follows e.g. from [28, Chap. 1, proposition 2.9]) and we will also sometimes identify $E(\widetilde{G})$ with the set of isomorphism classes $\{\widetilde{\pi}_\tau \mid \tau \in E(\widetilde{G})\}$. Note that for every $\widetilde{\pi} \in E(\widetilde{G})$ the isomorphism class of the underlying representation π belongs to $\mathrm{Temp}_{\mathrm{ind}}(G)^\theta$.

Each $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ induces an automorphism of $\mathcal{A}_M$ (induced from $\mathrm{Ad}_{\widetilde{n}}$ for any lifting $\widetilde{n} \in \widetilde{G}(F)$ of $\widetilde{w}$). Let $W^{\widetilde{G}}_{\mathrm{reg}}(\sigma)$ be the subset of $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ such that $\mathcal{A}_M^{\widetilde{w}} = \mathcal{A}_{\widetilde{G}}$. Following [28, §2.11], we define $E_{\mathrm{disc}}(\widetilde{G})$ (resp. $E_{\mathrm{ell}}(\widetilde{G})$) to be the subset of triples $\tau = [M, \sigma, r] \in E(\widetilde{G})$ such that $W_0^G(\sigma) r \cap W^{\widetilde{G}}_{\mathrm{reg}}(\sigma) \neq \emptyset$ (resp. $W_0^G(\sigma) = \{1\}$ and $r \in W^{\widetilde{G}}_{\mathrm{reg}}(\sigma)$). We also introduce the further subset $E_2(\widetilde{G})$ of triples $\tau = [M, \sigma, r] \in E(\widetilde{G})$ such that $W^{\widetilde{G}}(\sigma) = W^{\widetilde{G}}_{\mathrm{reg}}(\sigma)$. By [28, lemme 2.11], we have $E_2(\widetilde{G}) \subset E_{\mathrm{ell}}(\widetilde{G}) \subset E_{\mathrm{disc}}(\widetilde{G})$.

Remark 2.5. *The set $E(\widetilde{G})$, $E_{\mathrm{disc}}(\widetilde{G})$ and $E_{\mathrm{ell}}(\widetilde{G})$ do not exactly coincide with the ones defined in [28, Chap. 1, §2.9] but correspond rather to the sets denoted by $E(\widetilde{G})/conj$, $E_{\mathrm{disc}}(\widetilde{G})/conj$ and $E_{\mathrm{ell}}(\widetilde{G})/conj$ in* loc. cit.

There is a natural action of $i\mathcal{A}^*_{\widetilde{G}}$ on $E(\widetilde{G})$ given by $\lambda \cdot [M, \sigma, r] = [M, \sigma_\lambda, r]$ [5]. This action factors through $i\mathcal{A}^*_{\widetilde{G},F}$ and preserves the subsets $E_{\mathrm{disc}}(\widetilde{G})$, $E_{\mathrm{ell}}(\widetilde{G})$ and $E_2(\widetilde{G})$. Let $J \subset G(F)$ be a compact-open subgroup. Then, we have:

(2.8.3) the subset $E_{\mathrm{disc}}(\widetilde{G})^J$ of triples $\tau \in E_{\mathrm{disc}}(\widetilde{G})$ such that the representation π_τ admits nonzero J-invariant vectors is finite modulo the action of $i\mathcal{A}^*_{\widetilde{G}}$;

(see [28, Chap. 2, Proposition 2.2] for the case of $E_{\mathrm{ell}}(\widetilde{G})$ the proof being entirely similar for $E_{\mathrm{disc}}(\widetilde{G})$).

We equip $E_{\mathrm{disc}}(\widetilde{G})$ with the unique measure such that for every $\tau \in E_{\mathrm{disc}}(\widetilde{G})$, the twisting map $\lambda \in i\mathcal{A}^*_{\widetilde{G}} \mapsto \lambda \cdot \tau$ is locally measure preserving. Thus, denoting by $E_{\mathrm{disc}}(\widetilde{G})/i\mathcal{A}^*_{\widetilde{G}}$ the set of orbits in $E_{\mathrm{disc}}(\widetilde{G})$ under the action of $i\mathcal{A}^*_{\widetilde{G}}$, for every sufficiently nice function $\varphi : E_{\mathrm{disc}}(\widetilde{G}) \to \mathbb{C}$[6] we have

$$\int_{E_{\mathrm{disc}}(\widetilde{G})} \varphi(\tau) d\tau = \sum_{\tau \in E_{\mathrm{disc}}(\widetilde{G})/i\mathcal{A}^*_{\widetilde{G}}} |\mathrm{Stab}(i\mathcal{A}^*_{\widetilde{G},F}, \tau)|^{-1} \int_{i\mathcal{A}^*_{\widetilde{G},F}} \varphi(\lambda \cdot \tau) d\lambda$$

where we have denoted by $\mathrm{Stab}(i\mathcal{A}^*_{\widetilde{G},F}, \tau)$ the stabilizer of τ in $i\mathcal{A}^*_{\widetilde{G},F}$.

[5] Identifying $E(\widetilde{G})$ with a set of isomorphism classes of tempered representations of $\widetilde{G}(F)$ as before, this action is also sending $\widetilde{\pi}$ to its "twist" by λ but this twist is only well-defined up to a scalar (it requires the choice of an extension to $\widetilde{G}(F)$ of the unramified character associated to λ e.g. through the choice of a base-point).

[6] In practice, we will only consider functions φ that are supported in a finite number of $i\mathcal{A}^*_{\widetilde{G}}$-orbits and such that for every $\tau \in E_{\mathrm{disc}}(\widetilde{G})$, $\lambda \in i\mathcal{A}^*_{\widetilde{G}} \mapsto \varphi(\lambda \cdot \tau)$ is continuous (even C^∞).

For $\tau = [M, \sigma, r] \in E_{\text{disc}}(\widetilde{G})$, we set (following [28, Sect. 2.11])

$$\iota(\tau) = |R^G(\sigma)_r|^{-1} |W_0^G(\sigma)|^{-1} \sum_{\widetilde{w} \in W_0^G(\sigma) r \cap W_{\text{reg}}^{\widetilde{G}}(\sigma)} \epsilon_\sigma(\widetilde{w}) |\det(1 - \widetilde{w}) \mid_{\mathcal{A}_M^{\widetilde{G}}}|^{-1} \tag{2.8.4}$$

where $\mathcal{A}_M^{\widetilde{G}} = \mathcal{A}_M / \mathcal{A}_{\widetilde{G}}$, $R^G(\sigma)_r$ denotes the centralizer of r in $R^G(\sigma)$ and the $\epsilon_\sigma(\widetilde{w})$ are certain signs defined in *loc. cit.* In the particular case where $\tau \in E_{\text{ell}}(\widetilde{G})$ this simplifies to

$$\iota(\tau) = D(\tau) := |R^G(\sigma)_r|^{-1} |\det(1 - r) \mid_{\mathcal{A}_M^{\widetilde{G}}}|^{-1}.$$

Lemma 2.6. *Let $\widetilde{\pi} \in E(\widetilde{G})$. Then, $\widetilde{\pi} \in E_2(\widetilde{G})$ if and only if $\{\pi_\lambda \mid \lambda \in i\mathcal{A}_{\widetilde{G}}^*\}$ is a connected component of* $\text{Temp}_{\text{ind}}(G)^\theta$.

Proof. Let $M \subset G$ be a Levi subgroup and $\sigma \in \Pi_2(M)$. Then, $\pi = I_M^G(\sigma) \in \text{Temp}_{\text{ind}}(G)^\theta$ if and only if $W^{\widetilde{G}}(\sigma) \neq \emptyset$. Assume this is the case and set $\pi_\lambda = I_M^G(\sigma_\lambda)$ for every $\lambda \in i\mathcal{A}_M^*$. Then, it suffices to show that

$$\pi \otimes i\mathcal{A}_{\widetilde{G}}^* := \{\pi_\lambda \mid \lambda \in i\mathcal{A}_{\widetilde{G}}^*\}$$

is a connected component of $\text{Temp}_{\text{ind}}(G)^\theta$ if and only if $W^{\widetilde{G}}(\sigma) = W_{\text{reg}}^{\widetilde{G}}(\sigma)$. This, in turn, is an easy consequence of the following claim:

(2.8.5) There exists a neighborhood $U \subset i\mathcal{A}_M^*$ of 0 such that for every $\lambda \in U$, $\pi_\lambda \in \text{Temp}_{\text{ind}}(G)^\theta$ if and only if there exists $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ such that $\widetilde{w}\lambda = \lambda$.

To prove the claim, we first observe that, for $\lambda \in i\mathcal{A}_M^*$, $\pi_\lambda \in \text{Temp}_{\text{ind}}(G)^\theta$ if and only if there exists $\widetilde{w} \in W^{\widetilde{G}}(M)$ such that $\sigma_\lambda \circ \text{Ad}_{\widetilde{w}} \simeq \sigma_\lambda$. Moreover, we can find a sufficiently small $W^G(\sigma)$-invariant neighborhood $U \subset i\mathcal{A}_M^*$ of 0 such that:

- For every $\lambda, \mu \in U$, we have $\sigma_\lambda \simeq \sigma_\mu$ if and only if $\lambda = \mu$;
- for every $\widetilde{w} \in W^{\widetilde{G}}(M) \setminus W^{\widetilde{G}}(\sigma)$ and $\lambda \in U$, we have $\sigma_\lambda \circ \text{Ad}_{\widetilde{w}} \notin \sigma \otimes U$.

It follows that, for $\lambda \in U$, we have $\pi_\lambda \in \text{Temp}_{\text{ind}}(G)^\theta$ if and only if there exists $\widetilde{w} \in W^{\widetilde{G}}(\sigma)$ such that $\sigma_\lambda \circ \text{Ad}_{\widetilde{w}} \simeq \sigma_\lambda$ or equivalently, since $\sigma_\lambda \circ \text{Ad}_{\widetilde{w}} \simeq \sigma_{\widetilde{w}^{-1}\lambda}$, $\widetilde{w}\lambda = \lambda$.

□

We can extend (2.7.2) to an action of $C^\infty(\text{Temp}_{\text{ind}}(G))$ on $\mathcal{C}(\widetilde{G}(F))$ as follows. Choose $\gamma \in \widetilde{G}(F)$ and set, for every $f \in \mathcal{C}(\widetilde{G}(F))$, $f_\gamma(g) = f(g\gamma)$ ($g \in G(F)$). This function belongs to $\mathcal{C}(G(F))$ and for $(z, f) \in C^\infty(\text{Temp}_{\text{ind}}(G)) \times \mathcal{C}(\widetilde{G}(F))$, we define $z \star f \in \mathcal{C}(\widetilde{G}(F))$ by

$$(z \star f)(g\gamma) := (z \star f_\gamma)(g), \quad \text{for } g \in G(F).$$

As the endomorphism $z\star$ commutes with right translations, this definition is easily seen to be independent on the choice of γ. Moreover, we have

$$(zz') \star f = z \star (z' \star f) \tag{2.8.6}$$

and

$$\widetilde{\pi}(z \star f) = z(\pi)\widetilde{\pi}(f) \tag{2.8.7}$$

for every $(z, z') \in C^\infty(\text{Temp}_{\text{ind}}(G)) \times C^\infty(\text{Temp}_{\text{ind}}(G))$, $f \in \mathcal{C}(\widetilde{G}(F))$ and $\widetilde{\pi} \in E(\widetilde{G})$.

2.9 Orthogonal sets

Let $(G, \widetilde{G})$ be a twisted space. We briefly recall the notion of $(\widetilde{G}, \widetilde{M})$-families from [23].

Let $\widetilde{M}$ be a Levi subspace of $\widetilde{G}$. Two parabolic subspaces $\widetilde{P}, \widetilde{Q} \in \mathcal{P}(\widetilde{M})$ are said to be *adjacent* if the intersection $\Delta_{\widetilde{P}}^{\vee} \cap -\Delta_{\widetilde{Q}}^{\vee}$ is a singleton $\{\alpha_{\widetilde{P},\widetilde{Q}}^{\vee}\}$. If this is the case, the hyperplane $\{X \in i\mathcal{A}_{\widetilde{M}}^{*} \mid \langle \alpha_{\widetilde{P},\widetilde{Q}}^{\vee}, X \rangle = 0\}$ is called *the wall separating* $\widetilde{P}$ and $\widetilde{Q}$.

By definition $(\widetilde{G}, \widetilde{M})$-*orthogonal set* is a family $\mathcal{Y} = (Y_{\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{M})}$ of points in $\mathcal{A}_{\widetilde{M}}$ such that for every adjacent parabolic subspaces $\widetilde{P}, \widetilde{Q} \in \mathcal{P}(\widetilde{M})$, we have

$$Y_{\widetilde{P}} - Y_{\widetilde{Q}} \in \mathbb{R}\alpha_{\widetilde{P},\widetilde{Q}}^{\vee}$$

where $\Delta_{\widetilde{P}}^{\vee} \cap -\Delta_{\widetilde{Q}}^{\vee} = \{\alpha_{\widetilde{P},\widetilde{Q}}^{\vee}\}$. We further say that $\mathcal{Y}$ is *positive* if

$$Y_{\widetilde{P}} - Y_{\widetilde{Q}} \in \mathbb{R}_{>0}\alpha_{\widetilde{P},\widetilde{Q}}^{\vee}$$

for every pair of adjacent parabolic subspaces $\widetilde{P}, \widetilde{Q} \in \mathcal{P}(\widetilde{M})$.

For any $(\widetilde{G}, \widetilde{M})$-orthogonal set $\mathcal{X} = (X_{\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{M})}$, we set

$$d(\mathcal{X}) = \min_{\widetilde{P} \in \mathcal{P}(\widetilde{M})} \min_{\alpha \in \Delta_{\widetilde{P}}} \alpha(X_{\widetilde{P}}), \quad N(\mathcal{X}) = \max_{\widetilde{P} \in \mathcal{P}(\widetilde{M})} \max_{\alpha \in \Delta_{\widetilde{P}}} |\alpha(X_{\widetilde{P}})|$$

that we shall call the *depth* and the *norm* of $\mathcal{X}$ respectively.

Let $\mathcal{Y} = (Y_{\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{M})}$ be a $(\widetilde{G}, \widetilde{M})$-orthogonal set. For $\widetilde{Q} = \widetilde{L}U_Q \in \mathcal{F}(\widetilde{M})$, we denote by $Y_{\widetilde{Q}}$ the projection to $\mathcal{A}_{\widetilde{L}}$ of $Y_{\widetilde{P}}$ for any $\widetilde{P} \in \mathcal{P}(\widetilde{M})$ such that $\widetilde{P} \subset \widetilde{Q}$ (this projection does not depend on the choice of $\widetilde{P}$). To $\mathcal{Y}$ we associate functions $\Gamma_{\widetilde{L}}^{\widetilde{Q}}(., \mathcal{Y})$ on $\mathcal{A}_{\widetilde{L}}^{\widetilde{Q}}$ and complex numbers $v_{\widetilde{L}}^{\widetilde{Q}}(\mathcal{Y}) \in \mathbb{C}$ for every $\widetilde{L} \in \mathcal{L}(\widetilde{M})$ and $\widetilde{Q} \in \mathcal{F}(\widetilde{L})$ as follows:

$$\Gamma_{\widetilde{L}}^{\widetilde{Q}}(H, \mathcal{Y}) = \sum_{\widetilde{P} \in \mathcal{F}(\widetilde{L}), \widetilde{P} \subset \widetilde{Q}} (-1)^{a_{\widetilde{P}}^{\widetilde{Q}}} \widehat{\tau}_{\widetilde{P}}^{\widetilde{Q}}(H - Y_{\widetilde{P}}), \quad H \in \mathcal{A}_{\widetilde{L}}^{\widetilde{Q}},$$

and

$$v_{\widetilde{L}}^{\widetilde{Q}}(\mathcal{Y}) = \int_{\mathcal{A}_{\widetilde{L}}^{\widetilde{Q}}} \Gamma_{\widetilde{L}}^{\widetilde{Q}}(H, \mathcal{Y}) dH.$$

Here $\widehat{\tau}_{\widetilde{P}}^{\widetilde{Q}}$ denotes the characteristic function of the cone in $\mathcal{A}$ characterized by (where ϖ_α is the fundamental weight associated to α)

$$\widehat{\tau}_{\widetilde{P}}^{\widetilde{Q}}(H) = 1 \Leftrightarrow \varpi_\alpha(H) > 0, \ \forall \alpha \in \Delta_{\widetilde{P}}^{\widetilde{Q}}.$$

When $\widetilde{Q} = \widetilde{G}$, we will sometimes drop the superscript $\widetilde{Q}$. If $\mathcal{Y}$ is positive, $v_{\widetilde{L}}^{\widetilde{Q}}(\mathcal{Y})$ is simply the volume of the convex hull of $(Y_{\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{L}), \widetilde{P} \subset \widetilde{Q}}$. Once again, we will sometimes drop the superscript when $\widetilde{Q}=\widetilde{G}$. We will also use $\tau_{\widetilde{P}}^{\widetilde{Q}}$ to denote the characteristic function of the cone in $\mathcal{A}$ characterized by

$$\tau_{\widetilde{P}}^{\widetilde{Q}}(H)=1 \Leftrightarrow \alpha(H)>0,\ \forall\alpha\in\Delta_{\widetilde{P}}^{\widetilde{Q}}.$$

Let K be a special compact subgroup of $G(F)$. Using the Iwasawa decomposition $G(F)=P(F)K$, for every parabolic subspace $\widetilde{P}\subset\widetilde{G}$, we can extend the homomorphism $H_{\widetilde{P}}$ to a map $G(F)\to\mathcal{A}_{\widetilde{P}}$. Then, for every Levi subspace $\widetilde{M}\subset\widetilde{G}$ and $g\in G(F)$, the family $\mathcal{H}_{\widetilde{M}}(g)=(-H_{\widetilde{P}}(g))_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}$ is a positive $(\widetilde{G},\widetilde{M})$-orthogonal set and we define

$$v_{\widetilde{M}}^{\widetilde{Q}}(g)=v_{\widetilde{M}}^{\widetilde{Q}}(\mathcal{H}_{\widetilde{M}}(g)),\quad \text{for } \widetilde{Q}\in\mathcal{F}(\widetilde{M}).$$

Let $\Lambda\subset\mathcal{A}_{\widetilde{M},\mathbb{Q}}:=X_*(A_{\widetilde{M}})\otimes_{\mathbb{Z}}\mathbb{Q}$ be a $\mathbb{Z}$-lattice. We say that a $(\widetilde{G},\widetilde{M})$-orthogonal set $\mathcal{Y}=(Y_{\widetilde{P}})_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}$ is Λ-*rational* if for every $\widetilde{P}\in\mathcal{P}(\widetilde{M})$, we have $Y_{\widetilde{P}}\in\Lambda$ and we say that it is *rational* if it is Λ-rational for some lattice Λ. We denote by $\mathcal{C}_\Lambda(\widetilde{G},\widetilde{M})$ (resp. $\mathcal{C}_{\mathbb{Q}}(\widetilde{G},\widetilde{M})$) the set of all Λ-rational (resp. rational) $(\widetilde{G},\widetilde{M})$-orthogonal sets. Then, a function $\mathcal{Y}\in\mathcal{C}_{\mathbb{Q}}(\widetilde{G},\widetilde{M})\mapsto f(\mathcal{Y})\in\mathbb{C}$ is said to be a *unitary polynomial-exponential* if for every lattice $\Lambda\subset\mathcal{A}_{\widetilde{M},\mathbb{Q}}$ we can find a family of polynomial functions $Q_{\mu,\Lambda,\widetilde{P}}\in\mathbb{C}[\mathcal{A}_{\widetilde{M}}]$ for $\widetilde{P}\in\mathcal{P}(\widetilde{M})$ and $\mu\in\widehat{\Lambda}:=\mathrm{Hom}(\Lambda,\mathbb{S}^1)$ that are almost all equal to 0 and such that

$$f(\mathcal{Y})=\sum_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}\sum_{\mu\in\widehat{\Lambda}}Q_{\mu,\Lambda,\widetilde{P}}(Y_{\widetilde{P}})\mu(Y_{\widetilde{P}})$$

for every $\mathcal{Y}=(Y_{\widetilde{P}})_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}\in\mathcal{C}_\Lambda(\widetilde{G},\widetilde{M})$. Moreover, we say that a unitary polynomial-exponential function f is *of degree at most* r if the polynomials $Q_{\mu,\Lambda,\widetilde{P}}$ are of degree at most r for every lattice $\Lambda\subset\mathcal{A}_{\widetilde{M},\mathbb{Q}}$, $\widetilde{P}\in\mathcal{P}(\widetilde{M})$ and $\mu\in\widehat{\Lambda}$.

2.10 Weighted orbital integrals

Let $\widetilde{M}$ be a Levi subspace of $\widetilde{G}$, $\gamma\in\widetilde{M}(F)\cap\widetilde{G}_{rs}(F)$ and $\widetilde{Q}\in\mathcal{F}(\widetilde{M})$. For $f\in\mathcal{C}(\widetilde{G}(F))$, we define the *twisted weighted orbital integral*

$$\Phi_{\widetilde{M}}^{\widetilde{Q}}(\gamma,f)=\int_{G_\gamma(F)\backslash G(F)}f(g^{-1}\gamma g)v_{\widetilde{M}}^{\widetilde{Q}}(g)dg$$

as well as its normalized version

$$J_{\widetilde{M}}^{\widetilde{Q}}(\gamma,f)=D^{\widetilde{G}}(\gamma)^{1/2}\Phi_{\widetilde{M}}^{\widetilde{Q}}(\gamma,f).$$

The above integral is absolutely convergent. More precisely, for $\widetilde{T}\subset\widetilde{M}$ a maximal twisted torus, we claim:

(2.10.1) There exist $p > 0$ and, for every $d > 0$, a continuous semi-norm ν_d on $\mathcal{C}(\widetilde{G}(F))$ such that
$$\left| J^{\widetilde{Q}}_{\widetilde{M}}(\gamma, f) \right| \leqslant \nu_d(f)(1 + |\log D^{\widetilde{G}}(\gamma)|)^p \sigma_{\widetilde{T}_{/\theta}}(\gamma)^{-d}$$
for every $\gamma \in \widetilde{T}_{\text{reg}}(F)$ and $f \in \mathcal{C}(\widetilde{G}(F))$.

Indeed, there exists $p > 0$ such that $v^{\widetilde{Q}}_{\widetilde{M}}(g) \ll \sigma_G(g)^p$ for $g \in G(F)$. As $v^{\widetilde{Q}}_{\widetilde{M}}$ is left invariant by $T(F)$, by Lemma 2.2 this implies $v^{\widetilde{Q}}_{\widetilde{M}}(g) \ll (1 + |\log D^{\widetilde{G}}(\gamma)|)^p \sigma_{\widetilde{G}}(g^{-1}\gamma g)^p$ for $(g, \gamma) \in G(F) \times \widetilde{T}_{\text{reg}}(F)$. The claim is now a straightforward consequence of Proposition 2.3.

Now consider the case where $\widetilde{G} = \widetilde{H} \times \widetilde{H}$ where $\widetilde{H}$ is a connected twisted reductive space over F (with underlying reductive group H). Let $\widetilde{M}_H$ be a Levi subspace of $\widetilde{H}$. Then, $\widetilde{M} = \widetilde{M}_H \times \widetilde{M}_H$ is a Levi subspace of $\widetilde{G}$. Let $\gamma \in \widetilde{M}_H(F) \cap \widetilde{H}_{\text{rs}}(F)$ and $f_1, f_2 \in \mathcal{C}(\widetilde{H}(F))$. We set (following [28, Chap. 1, §4.8])

$$J^{\widetilde{H}}_{\widetilde{M}_H}(\gamma, f_1, f_2) = \int_{H_\gamma(F)\backslash H(F) \times H_\gamma(F)\backslash H(F)} f_1(x^{-1}\gamma x) f_2(y^{-1}\gamma y) v^{\widetilde{H}}_{\widetilde{M}_H}(x, y) dx dy$$

where the "weight" $v^{\widetilde{H}}_{\widetilde{M}_H}(x, y)$ is the volume associated to the positive $(\widetilde{H}, \widetilde{M}_H)$-orthogonal set
$$\widetilde{P}_H \in \mathcal{P}(\widetilde{M}_H) \mapsto H_{\overline{\widetilde{P}}_H}(y) - H_{\widetilde{P}_H}(x).$$

(Here $\overline{\widetilde{P}}_H$ denotes the unique parabolic subspace opposite to $\widetilde{P}_H$ such that $\overline{\widetilde{P}}_H \cap \widetilde{P}_H = \widetilde{M}$.)

By the descent formulas of [28, Chap. 1, lemme 5.4], we have

$$J^{\widetilde{H}}_{\widetilde{M}_H}(\gamma, f_1, f_2) = \sum_{\widetilde{L}_1, \widetilde{L}_2 \in \mathcal{L}(\widetilde{M}_H)} d^{\widetilde{H}}_{\widetilde{M}_H}(\widetilde{L}_1, \widetilde{L}_2) J^{\widetilde{Q}_1}_{\widetilde{M}_H}(\gamma, f_1) J^{\widetilde{Q}_2}_{\widetilde{M}_H}(\gamma, f_2) \tag{2.10.2}$$

where $\widetilde{Q}_1, \widetilde{Q}_2$ are certain parabolic subspaces in $\mathcal{P}(\widetilde{L}_1), \mathcal{P}(\widetilde{L}_2)$ respectively and $d^{\widetilde{H}}_{\widetilde{M}_H}(\widetilde{L}_1, \widetilde{L}_2)$ is a certain real numbers which is zero unless $\mathcal{A}^{\widetilde{H}}_{\widetilde{M}_H} = \mathcal{A}^{\widetilde{H}}_{\widetilde{L}_1} \oplus \mathcal{A}^{\widetilde{H}}_{\widetilde{L}_2}$ and moreover $d^{\widetilde{H}}_{\widetilde{M}_H}(\widetilde{H}, \widetilde{M}_H) = 1$.

2.11 Twisted weighted characters

Let $\widetilde{M}$ be a Levi subspace of $\widetilde{G}$, $\widetilde{R} \in \mathcal{F}(\widetilde{M})$ and $\widetilde{\pi}$ be a tempered representation of $\widetilde{M}(F)$. First assume that $\widetilde{\pi}$ is in "general position" (more precisely, this means that π is in some open-dense subset of the family $\{\pi_\lambda \mid \lambda \in i\mathcal{A}^*_{\widetilde{M}}\}$). Then, we define as in [28, Chap.1, §2.7] a *weighted character*

$$f \in \mathcal{C}(\widetilde{G}(F)) \mapsto J^{\widetilde{R}}_{\widetilde{M}}(\widetilde{\pi}, f) := \mathrm{Tr}(\mathcal{M}^{\widetilde{R}}_{\widetilde{M}}(\pi) I^{\widetilde{G}}_{\widetilde{P}}(\widetilde{\pi}, f))$$

where $\widetilde{P}$ is any chosen parabolic subspace in $\mathcal{P}(\widetilde{M})$ (the distribution $J^{\widetilde{R}}_{\widetilde{M}}(\widetilde{\pi}, .)$ does not depend on this choice) and $\mathcal{M}^{\widetilde{R}}_{\widetilde{M}}(\pi)$ is the operator on (the space of) $I^G_P(\pi)$ associated to the $(\widetilde{G}, \widetilde{M})$-family of operators $(\mathcal{M}(\pi; \Lambda, \widetilde{Q}))_{\widetilde{Q} \in \mathcal{P}(\widetilde{M})}$ defined as in *loc. cit.*. Similarly, for $f_1, f_2 \in \mathcal{C}(G(F))$

we set

$$J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}, f_1, f_2) = \mathrm{Tr}\left(\mathcal{M}^{\widetilde{G}}_{\widetilde{M}}(\pi^\vee \otimes \pi) I^{\widetilde{G}\times\widetilde{G}}_{\widetilde{P}\times\widetilde{P}}(\widetilde{\pi}^\vee \otimes \widetilde{\pi}, f_1 \otimes f_2)\right),$$

where this time the operator $\mathcal{M}^{\widetilde{G}}_{\widetilde{M}}(\pi^\vee \otimes \pi)$ is associated to the $(\widetilde{G}, \widetilde{M})$-family

$$\widetilde{Q} \in \mathcal{P}(\widetilde{M}) \mapsto \mathcal{M}(\pi^\vee \otimes \pi; \Lambda, \widetilde{Q}) = \mathcal{M}(\pi^\vee; \Lambda, \overline{\widetilde{Q}}) \otimes \mathcal{M}(\pi; \Lambda, \widetilde{Q})$$

of operators on $I^{G\times G}_{P\times P}(\pi^\vee \otimes \pi)$. The genericity assumption on $\widetilde{\pi}$ is necessary for the above $(\widetilde{G}, \widetilde{M})$-families to be well-defined. However, the definitions of $J^{\widetilde{R}}_{\widetilde{M}}(\widetilde{\pi}, f)$ and $J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}, f_1, f_2)$ extend to every tempered representation $\widetilde{\pi}$ thanks to the following property (see [28, Chap. 1, proposition 2.7]):

(2.11.1) The operator valued functions $\lambda \mapsto \mathcal{M}^{\widetilde{R}}_{\widetilde{M}}(\pi_\lambda)$ and $\lambda \mapsto \mathcal{M}^{\widetilde{G}}_{\widetilde{M}}(\pi^\vee_\lambda \otimes \pi_\lambda)$, a priori only well-defined on an dense open subset of $i\mathcal{A}^*_{\widetilde{M}}$, extend to smooth functions on all of $i\mathcal{A}^*_{\widetilde{M}}$.

Finally, by the descent formula of [28, Chap 1, lemme 5.4], for every $f_1, f_2 \in \mathcal{C}(\widetilde{G}(F))$ we have

$$J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}, f_1, f_2) = \sum_{\widetilde{L}_1, \widetilde{L}_2 \in \mathcal{L}(\widetilde{M})} d^{\widetilde{G}}_{\widetilde{M}}(\widetilde{L}_1, \widetilde{L}_2) J^{\overline{\widetilde{Q}}_1}_{\widetilde{M}}(\widetilde{\pi}, f_1) J^{\widetilde{Q}_2}_{\widetilde{M}}(\widetilde{\pi}, f_2) \tag{2.11.2}$$

where $\widetilde{Q}_1$, $\widetilde{Q}_2$ and $d^{\widetilde{G}}_{\widetilde{M}}(\widetilde{L}_1, \widetilde{L}_2)$ are as in (2.10.2).

2.12 Twisted strongly cuspidal functions

We say that a function $f \in \mathcal{C}(\widetilde{G}(F))$ is *strongly cuspidal*, if for every parabolic subspace $\widetilde{P} = \widetilde{M}U_P$ of $\widetilde{G}$ and $x \in G(F)$, the function defined by

$${}^x f_{(\widetilde{P})}(\widetilde{m}) := \delta_{\widetilde{P}}(\widetilde{m})^{1/2} \int_{U_P(F)} f(x^{-1}\widetilde{m}ux)du, \quad \text{for } \widetilde{m} \in \widetilde{M}(F), \tag{2.12.1}$$

is identically zero. By a change of variable, this last condition is equivalent to

$$\int_{U_P(F)} f(x^{-1}u^{-1}\widetilde{m}ux)du = 0, \quad \text{for every } \widetilde{m} \in \widetilde{M}(F) \cap \widetilde{G}_{\mathrm{rs}}(F) \text{ and } x \in G(F). \tag{2.12.2}$$

We denote by $\mathcal{C}_{\mathrm{scusp}}(\widetilde{G}(F)) \subseteq \mathcal{C}(\widetilde{G}(F))$ the subspace of strongly cuspidal functions.

Let $f \in \mathcal{C}_{\mathrm{scusp}}(\widetilde{G}(F))$. Let $\widetilde{M}$ be a Levi subspace and $\gamma \in \widetilde{M}(F) \cap \widetilde{G}_{rs}(F)$. For $\widetilde{Q} = \widetilde{L}U_Q \in \mathcal{F}(\widetilde{M})$, the weight $v^{\widetilde{Q}}_{\widetilde{M}}$ is left invariant by $U_Q(F)$. Hence, by (2.12.2), we have

$$J^{\widetilde{Q}}_{\widetilde{M}}(\gamma, f) = 0 \text{ unless } \widetilde{Q} = \widetilde{G}. \tag{2.12.3}$$

Then, by the same argument as for [34, lemme 5.2], it follows that the weighted orbital integral $\Phi_{\widetilde{M}}(\gamma, f)$ does not depend on the choice of K.

We define a function Θ_f on $\widetilde{G}_{rs}(F)$ by

$$\Theta_f(\gamma) = (-1)^{a_{G_\gamma} - a_{\widetilde{G}}} \Phi^{\widetilde{G}}_{\widetilde{M(\gamma)}}(\gamma, f), \quad \gamma \in \widetilde{G}_{\rm rs}(F),$$

where $\widetilde{M(\gamma)}$ stands for the centralizer of A_{G_γ} in $\widetilde{G}$ (it is the minimal Levi subspace containing γ), $a_{G_\gamma} = \dim(A_{G_\gamma})$ and $a_{\widetilde{G}} = \dim(A_{\widetilde{G}})$. It is proved in [36, proposition 1.7] that if f is compactly supported then Θ_f is a *quasi-character* on $\widetilde{G}(F)$ in the sense of Section 2.6. We extend this result to every strongly cuspidal function $f \in \mathcal{C}_{\rm scusp}(\widetilde{G}(F))$ in Section 2.13 (see Corollary 2.11). For ease of notation, for every $x \in \widetilde{G}_{\rm ss}(F)$, we set

$$c_{f,\mathcal{O}}(x) = c_{\Theta_f,\mathcal{O}}(x), \quad \mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x),$$

for the coefficients of the germ expansion of Θ_f near x.

Let again $f \in \mathcal{C}_{\rm scusp}(\widetilde{G}(F))$. Let $\widetilde{M} \subset \widetilde{G}$ be a twisted Levi subspace, $\widetilde{\pi}$ be a tempered representation of $\widetilde{M}(F)$ and $\widetilde{Q} \in \mathcal{F}(\widetilde{M})$. By [36, lemme 1.13], we have

(2.12.4) $$J^{\widetilde{Q}}_{\widetilde{M}}(\widetilde{\pi}, f) = 0 \text{ unless } \widetilde{Q} = \widetilde{G}.$$

Still by [36, lemme 1.13], we also have

(2.12.5) $J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}, f) = 0$ if $\widetilde{\pi}$ is properly parabolically induced (e.g. if $\widetilde{\pi} \in E(\widetilde{M}) \setminus E_{\rm ell}(\widetilde{M})$).

On the other hand, for $\widetilde{\pi} \in E_{\rm ell}(\widetilde{M})$, we set

$$\widehat{\Theta}_f(\widetilde{\pi}) = (-1)^{a_{\widetilde{M}} - a_{\widetilde{G}}} J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}, f), \text{ for } \widetilde{\pi} \in E_{\rm ell}(\widetilde{M}).$$

Recall that in Section 2.8, we have defined an action of $C^\infty(\mathrm{Temp}_{\rm ind}(G))$ on $\mathcal{C}(\widetilde{G}(F))$. We denote by $C^\infty(\mathrm{Temp}_{\rm ind}(G))^\theta$ the subspace of θ-invariant functions in $C^\infty(\mathrm{Temp}_{\rm ind}(G))$.

Lemma 2.7. *Let $f \in \mathcal{C}(\widetilde{G}(F))$ and $z \in C^\infty(\mathrm{Temp}_{\rm ind}(\widetilde{G}))^\theta$. Then, if f is strongly cuspidal so is $z \star f$.*

Proof. Let $\widetilde{P} = \widetilde{M}U_P$ be a proper parabolic subspace of $\widetilde{G}$. By (2.3.2), for every $x \in G(F)$ the function ${}^x f_{(\widetilde{P})}$ defined by the integral (2.12.1) belongs to $\mathcal{C}(\widetilde{M}(F))$. For $\widetilde{\sigma}$ a tempered representation of $\widetilde{M}(F)$, we set

$$I^{\widetilde{G}}_{\widetilde{P}}(\widetilde{\sigma}, f)(x,y) = \int_{\widetilde{M}(F)} \delta_{\widetilde{P}}(\widetilde{m})^{1/2} \int_{U_P(F)} f(x^{-1}\widetilde{m}uy)du\, \widetilde{\sigma}(\widetilde{m})d\widetilde{m}, \text{ for } (x,y) \in G(F) \times G(F).$$

This operator-valued function is the kernel of the operator $I^{\widetilde{G}}_{\widetilde{P}}(\widetilde{\sigma}, f)$ in the sense that

$$(I^{\widetilde{G}}_{\widetilde{P}}(\widetilde{\sigma}, f)e)(x) = \int_{P(F)\backslash G(F)} I^{\widetilde{G}}_{\widetilde{P}}(\widetilde{\sigma}, f)(x,y)e(y)dy$$

for every $e \in I_P^G(\sigma)$ and $x \in G(F)$. Note that

$$I_{\widetilde{P}}^{\widetilde{G}}(\widetilde{\sigma}, f)(x,x) = \widetilde{\sigma}({}^x f_{(\widetilde{P})}), \text{ for } x \in G(F). \tag{2.12.6}$$

Let $d \geqslant 1$ be the order of the outer automorphism $\theta = \theta_{\widetilde{M}}$ of $M(F)$ and set, for $\sigma \in \mathrm{Temp}_{\mathrm{ind}}(M)$,

$$\sigma\langle\theta\rangle := \sigma \oplus \sigma^\theta \oplus \ldots \oplus \sigma^{\theta^{d-1}}.$$

It is clear that $\sigma\langle\theta\rangle$ extends to a twisted representation of $\widetilde{M}(F)$ and we will denote by $\widetilde{\sigma\langle\theta\rangle}$ one such extension. It is well-known, and this follows e.g. from the Harish-Chandra-Plancherel formula [33], that a function $f' \in \mathcal{C}(M(F))$ is zero if and only if $\sigma(f') = 0$ for every $\sigma \in \mathrm{Temp}_{\mathrm{ind}}(M)$. This implies a similar equivalence for $\widetilde{M}$: a function $f' \in \mathcal{C}(\widetilde{M}(F))$ is zero if and only if $\widetilde{\sigma\langle\theta\rangle}(f') = 0$ for every $\sigma \in \mathrm{Temp}_{\mathrm{ind}}(M)$. Therefore, from (2.12.6) and the definition of a strongly cuspidal function, we see that f is strongly cuspidal if and only if $I_{\widetilde{P}}^{\widetilde{G}}(\widetilde{\sigma\langle\theta\rangle}, f)(x,x) = 0$ for every proper parabolic subspace $\widetilde{P} = \widetilde{M}U_P$, every $\sigma \in \mathrm{Temp}_{\mathrm{ind}}(M)$ and every $x \in G(F)$. The lemma is a direct consequence of this characterization since for $z \in C^\infty(\mathrm{Temp}_{\mathrm{ind}}(\widetilde{G}))^\theta$ and every $\widetilde{P}$ and σ as before, since $z(I_M^G(\sigma^{\theta^i})) = z(I_M^G(\sigma))$ for all i (by θ-invariance of z), we have

$$I_{\widetilde{P}}^{\widetilde{G}}(\widetilde{\sigma\langle\theta\rangle}, z \star f) = z(I_M^G(\sigma)) I_{\widetilde{P}}^{\widetilde{G}}(\widetilde{\sigma\langle\theta\rangle}, f).$$

□

2.13 Twisted local trace formula for strongly cuspidal functions

The twisted local trace formula of [28, Chap. 1, théorème 5.1] is an equality of distributions

$$J_{\mathrm{spec}}^{\widetilde{G}}(f_1, f_2) = J_{\mathrm{geom}}^{\widetilde{G}}(f_1, f_2) \tag{2.13.1}$$

where $f_1, f_2 \in C_c^\infty(\widetilde{G}(F))$ and

$$J_{\mathrm{spec}}^{\widetilde{G}}(f_1, f_2) = \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M||\widetilde{W}^G|^{-1}(-1)^{a_{\widetilde{M}} - a_{\widetilde{G}}} \int_{E_{\mathrm{disc}}(\widetilde{M})} \iota(\tau) J_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\pi}_\tau, f_1, f_2) d\tau, \tag{2.13.2}$$

$$J_{geom}^{\widetilde{G}}(f_1, f_2) = \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M||\widetilde{W}^G|^{-1}(-1)^{a_{\widetilde{M}} - a_{\widetilde{G}}} \int_{\Gamma_{\mathrm{ell}}(\widetilde{M})} J_{\widetilde{M}}^{\widetilde{G}}(\gamma, f_1, f_2) d\gamma. \tag{2.13.3}$$

We refer the reader to Section 2.8 for the definition of $\iota(\tau)$ as well as of the measure on $E_{\mathrm{disc}}(\widetilde{M})$ and to Sections 2.10 and 2.11 for the definitions of $J_{\widetilde{M}}^{\widetilde{G}}(\gamma, f_1, f_2)$ and $J_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\pi}_\tau, f_1, f_2)$ respectively.

Remark 2.8. *Despite the notation, the distributions $J_{\mathrm{spec}}^{\widetilde{G}}$ and $J_{\mathrm{geom}}^{\widetilde{G}}$ depend on the choice of the pair $(M_{\min}, K)$ (at least up to conjugacy). They also depend, incidentally, on the choice of the Haar measure on $G(F)$.*

Proposition 2.9. *The expressions* (2.13.2) *and* (2.13.3) *are both absolutely convergent for* $(f_1, f_2) \in \mathcal{C}(\widetilde{G}(F))^2$ *and they define continuous bilinear forms on* $\mathcal{C}(\widetilde{G}(F)) \times \mathcal{C}(\widetilde{G}(F))$*. In particular, the identity* (2.13.1) *extends by continuity to all* $f_1, f_2 \in \mathcal{C}(\widetilde{G}(F))$*.*

Proof. The same argument as in the non-twisted case [3, p.189] applies here noticing that (2.10.1) gives the required twisted analog of the estimates (5.7) of *loc. cit.*. □

Let $f_1, f_2 \in \mathcal{C}(\widetilde{G}(F))$ and assume that f_1 is strongly cuspidal. By the descent formulas (2.10.2), (2.11.2) as well as the vanishing (2.12.3), (2.12.4), (2.12.5) we then have

$$J^{\widetilde{G}}_{\widetilde{M}}(\gamma, f_1, f_2) = (-1)^{a_{\widetilde{M}} - a_{\widetilde{G}}} D^{\widetilde{G}}(\gamma)^{1/2} \Theta_{f_1}(\gamma) J_{\widetilde{G}}(\gamma, f_2)$$

and

$$J^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\pi}_\tau, f_1, f_2) = \begin{cases} (-1)^{a_{\widetilde{M}} - a_{\widetilde{G}}} \widehat{\Theta}_{f_1}(\widetilde{\pi}_\tau^\vee) J_{\widetilde{G}}(\widetilde{\pi}_\tau, f_2) \text{ if } \tau \in E_{\mathrm{ell}}(\widetilde{M}); \\ \\ 0 \text{ otherwise,} \end{cases}$$

for every $\gamma \in \Gamma_{\mathrm{ell}}(\widetilde{M})$ and $\tau \in E_{\mathrm{disc}}(\widetilde{M})$. Thus, in this case the distributions $J^{\widetilde{G}}_{\mathrm{spec}}$ and $J^{\widetilde{G}}_{\mathrm{geom}}$ can be rewritten as

$$J^{\widetilde{G}}_{\mathrm{spec}}(f_1, f_2) = \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M| |\widetilde{W}^G|^{-1} \int_{E_{\mathrm{ell}}(\widetilde{M})} D(\tau) \widehat{\Theta}_{f_1}(\widetilde{\pi}_\tau^\vee) J_{\widetilde{G}}(\widetilde{\pi}_\tau, f_2) d\tau$$

and

$$\begin{aligned} J^{\widetilde{G}}_{geom}(f_1, f_2) &= \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M| |\widetilde{W}^G|^{-1} \int_{\Gamma_{\mathrm{ell}}(\widetilde{M})} D^{\widetilde{G}}(\gamma)^{1/2} \Theta_{f_1}(\gamma) J_{\widetilde{G}}(\gamma, f_2) d\gamma \\ &= \int_{\widetilde{G}(F)} \Theta_{f_1}(\gamma) f_2(\gamma) d\gamma \end{aligned}$$

where the last equality follows from the Weyl integration formula (2.4.1). Moreover, by definition of the Harish-Chandra characters $\Theta_{\widetilde{\pi}_\tau}$, the spectral side $J^{\widetilde{G}}_{\mathrm{spec}}(f_1, f_2)$ can be further rewritten as

$$J^{\widetilde{G}}_{\mathrm{spec}}(f_1, f_2) = \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M| |\widetilde{W}^G|^{-1} \int_{E_{\mathrm{ell}}(\widetilde{M})} \int_{\widetilde{G}(F)} D(\tau) \widehat{\Theta}_{f_1}(\widetilde{\pi}_\tau^\vee) \Theta_{\widetilde{\pi}_\tau}(\gamma) f_2(\gamma) d\gamma d\tau.$$

The above expression being absolutely convergent (note that, by (2.8.3), the support of $\tau \mapsto \widehat{\Theta}_{f_1}(\widetilde{\pi}_\tau^\vee)$ in $E_{\mathrm{ell}}(\widetilde{M})$ is contained in a finite union of orbits under the action of $i\mathcal{A}^*_{\widetilde{M}}$), from (2.13.1) we get the identity

$$\int_{\widetilde{G}(F)} \Theta_{f_1}(\gamma) f_2(\gamma) d\gamma = \int_{\widetilde{G}(F)} \sum_{\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M| |\widetilde{W}^G|^{-1} \int_{E_{\mathrm{ell}}(\widetilde{M})} D(\tau) \widehat{\Theta}_{f_1}(\widetilde{\pi}_\tau^\vee) \Theta_{\widetilde{\pi}_\tau}(\gamma) d\tau f_2(\gamma) d\gamma$$

for every $f_1, f_2 \in \mathcal{C}(\widetilde{G}(F))$ with f_1 strongly cuspidal. Fixing f_1 and varying f_2, we deduce:

Proposition 2.10. *Let $f \in \mathcal{C}(\widetilde{G}(F))$ be a strongly cuspidal function. Then, for every $\gamma \in \widetilde{G}_{\rm rs}(F)$, we have*

$$\Theta_f(\gamma) = \sum_{\widetilde{M}\in\mathcal{L}(\widetilde{M}_{\min})} |\widetilde{W}^M||\widetilde{W}^G|^{-1} \int_{E_{\rm ell}(\widetilde{M})} D(\tau)\widehat{\Theta}_f(\widetilde{\pi}_\tau^\vee)\Theta_{\widetilde{\pi}_\tau}(\gamma)d\tau \tag{2.13.4}$$

where the right hand side is absolutely convergent.

As a corollary, we can now show the following extension of [36, proposition 1.7].

Corollary 2.11. *For $f \in \mathcal{C}(\widetilde{G}(F))$ strongly cuspidal, the function Θ_f is a quasi-character.*

Proof. This is a direct consequence of Proposition 2.10 combined with the following facts (valid for every $\widetilde{M} \in \mathcal{L}(\widetilde{M}_{\min})$):

- For every $\tau \in E_{\rm ell}(\widetilde{M})$, $\Theta_{\widetilde{\pi}_\tau}$ is a quasi-character [11, Theorem 3];
- The function $\tau \in E_{\rm ell}(\widetilde{M}) \mapsto \widehat{\Theta}_f(\widetilde{\pi}_\tau^\vee)$ is supported on a finite number of orbits under the action of $i\mathcal{A}^*_{\widetilde{M}}$ (see (2.8.3));
- For every $i\mathcal{A}^*_{\widetilde{M}}$-orbit $\Omega \subset E_{\rm ell}(\widetilde{M})$ and compact subset $\mathcal{K} \subset \widetilde{G}(F)$, the vector space spanned by the restrictions
$$\{\Theta_{\widetilde{\pi}_\tau}|_{\mathcal{K}_{\rm rs}} |\; \tau \in \Omega\},$$
where $\mathcal{K}_{\rm rs} := \mathcal{K} \cap \widetilde{G}_{\rm rs}(F)$, is of finite dimension (this follows e.g. from the induction formula [36, lemme 1.12]).

□

2.14 Spectral localization of strongly cuspidal functions

Let $f \in \mathcal{C}(\widetilde{G}(F))$ and choose a base-point $\gamma_0 \in \widetilde{G}(F)$. Put $f_{\gamma_0}(g) = f(g\gamma_0)$ for every $g \in G(F)$. Note that $f_{\gamma_0} \in \mathcal{C}(G(F))$. We define the *spectral support* of f, henceforth denoted by $\text{Supp}_{\text{spec}}(f)$ to be the support of the operator-valued function

$$\text{Temp}_{\text{ind}}(G) \ni \pi \mapsto \pi(f_{\gamma_0}) \in \text{End}(V_\pi).$$

Note that $\text{Supp}_{\text{spec}}(f)$ does not depend on the choice of γ_0: changing the base-point replaces f_{γ_0} by one of its right translates which acts non trivially on the same tempered representations as f_{γ_0}.

Proposition 2.12. *Let $\tau \in E_2(\widetilde{G})$ (see Section 2.8 for the definition of $E_2(\widetilde{G})$) and ω be a compact neighborhood of π_τ in* $\text{Temp}_{\text{ind}}(G)$ *(see Section 2.7 for the topology on* $\text{Temp}_{\text{ind}}(G)$*). Then, there exists a strongly cuspidal function $f \in \mathcal{C}(\widetilde{G}(F))$ such that*

$$\text{Supp}_{\text{spec}}(f) \subset \omega \tag{2.14.1}$$

and for every $\tau' \in E(\widetilde{G})$ we have

$$\text{(2.14.2)} \qquad \operatorname{Tr} \widetilde{\pi}_{\tau'}(f) = \begin{cases} 0 \text{ if } \tau' \neq \lambda \cdot \tau \text{ for every } \lambda \in i\mathcal{A}^*_{\widetilde{G}}, \\ \\ 1 \text{ if } \tau' = \tau. \end{cases}$$

Moreover, if $f \in \mathcal{C}(\widetilde{G}(F))$ is such a strongly cuspidal function, we have

$$\text{(2.14.3)} \qquad \Theta_f(\gamma) = |\mathrm{Stab}(i\mathcal{A}^*_{\widetilde{G},F}, \tau)|^{-1} D(\tau) \int_{i\mathcal{A}^*_{\widetilde{G},F}} \operatorname{Tr} \widetilde{\pi}_{\lambda \cdot \tau}(f) \Theta_{\widetilde{\pi}^\vee_{\lambda\cdot\tau}}(\gamma) d\lambda$$

for every $\gamma \in \widetilde{G}_{\mathrm{rs}}(F)$ where $\mathrm{Stab}(i\mathcal{A}^*_{\widetilde{G},F}, \tau)$ stands for the stabilizer of τ in $i\mathcal{A}^*_{\widetilde{G},F}$ (for the action by twisting).

Proof. For simplicity of notation, let us set $\widetilde{\pi} = \widetilde{\pi}_\tau$, $\pi = \pi_\tau$ as well as $\widetilde{\pi}_\lambda = \widetilde{\pi}_{\lambda\cdot\tau}$ and $\pi_\lambda = \pi_{\lambda\cdot\tau} = (\pi_\tau)_\lambda$ for every $\lambda \in i\mathcal{A}^*_{\widetilde{G}}$. By Lemma 2.6, up to shrinking ω we may assume that it is θ-stable and that

$$\text{(2.14.4)} \qquad \omega \cap \mathrm{Temp}_{\mathrm{ind}}(G)^\theta \subset \{\pi_\lambda \mid \lambda \in i\mathcal{A}^*_{\widetilde{G}}\}.$$

Let S be the finite set of $\tau' \in E(\widetilde{G})$ such that $\pi_{\tau'} = \pi_\tau$. By definition of $E_2(\widetilde{G})$, we have $S \subset E_2(\widetilde{G})$. Let $S_0 \subset S$ be a subset such that for every $\tau' \in S$ there exists an unique $\tau'_0 \in S_0$ as well as $\lambda \in i\mathcal{A}^*_{\widetilde{G}}$ (not necessarily unique) such that $\tau' = \lambda \cdot \tau'_0$. We may and will assume that $\tau \in S_0$. Moreover, by (2.14.4), we have:

(2.14.5) for $\tau' \in E(\widetilde{G})$ if $\pi_{\tau'} \in \omega$ then there exist $\lambda \in i\mathcal{A}^*_{\widetilde{G}}$ and $\tau'_0 \in S_0$ such that $\tau' = \lambda \cdot \tau'_0$.

Let $G(F)^1$ be the kernel of the homomorphism $H_G : G(F) \to \mathcal{A}_G$. By the orthogonality relations [28, théorème 7.3] between elliptic twisted characters, the restrictions of the twisted characters $\Theta_{\widetilde{\pi}_{\tau'}}$, for $\tau' \in S$, to the elliptic locus $\widetilde{G}(F)_{\mathrm{ell}}$ are linearly independent. More precisely, fixing $\gamma \in \widetilde{G}(F)_{\mathrm{ell}}$ and since elements of S_0 all have different orbits under $i\mathcal{A}^*_{\widetilde{G}}$, the restrictions of the twisted characters $\Theta_{\widetilde{\pi}_{\tau'}}$, for $\tau' \in S_0$, to $\widetilde{G}(F)_{\mathrm{ell}} \cap G(F)^1\gamma$ are linearly independent. Thus, we can find a function $f_0 \in C_c^\infty(\widetilde{G}(F))$ supported in $\widetilde{G}(F)_{\mathrm{ell}} \cap G(F)^1\gamma$ such that

$$\text{(2.14.6)} \qquad \operatorname{Tr} \widetilde{\pi}_{\tau'}(f_0) = \begin{cases} 0 \text{ if } \tau' \neq \tau, \\ \\ 1 \text{ if } \tau' = \tau \end{cases}$$

for every $\tau' \in S_0$. Note that, since f_0 is supported in $\widetilde{G}(F)_{\mathrm{ell}}$, it is a strongly cuspidal function. Moreover, since f_0 is supported in a unique coset modulo $G(F)^1$, for every $\tau' \in E(\widetilde{G})$ and $\lambda \in i\mathcal{A}^*_{\widetilde{G}}$, $\operatorname{Tr} \widetilde{\pi}_{\lambda\cdot\tau'}(f_0)$ is equal (up to a non-zero multiplicative constant which depends on how we normalized $\widetilde{\pi}_{\lambda\cdot\tau'}$) to $\operatorname{Tr} \widetilde{\pi}_{\tau'}(f_0)$. In particular, by (2.14.6), we also have

$$\text{(2.14.7)} \qquad \operatorname{Tr} \widetilde{\pi}_{\lambda\cdot\tau'}(f_0) = 0$$

for every $\tau' \in S_0 \setminus \{\tau\}$ and $\lambda \in i\mathcal{A}^*_{\widetilde{G}}$.

Let now $z \in C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))^\theta$ be a θ-invariant C^∞ function on $\mathrm{Temp}_{\mathrm{ind}}(G)$ which is supported in ω and such that $z(\pi) = 1$ (such a function certainly exists). Using the action of $C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))^\theta$ on $\mathcal{C}(\widetilde{G}(F))$ defined in Section 2.8, we set $f = z \star f_0 \in \mathcal{C}(\widetilde{G}(F))$. By Lemma 2.7, f is strongly cuspidal. On the other hand, by the spectral characterization of the action of $C^\infty(\mathrm{Temp}_{\mathrm{ind}}(G))^\theta$ on $\mathcal{C}(\widetilde{G}(F))$, f clearly satisfies condition (2.14.1). Similarly, (2.14.2) follows from the combination of (2.14.5), (2.14.6) and (2.14.7). Finally, the equality (2.14.3) is an immediate consequence of Proposition 2.10, remembering that the restriction of the measure on $E_{\mathrm{ell}}(\widetilde{G})$ to the orbit $\{\lambda \cdot \tau \mid \lambda \in i\mathcal{A}^*_{\widetilde{G}}\}$ is equal to $|\mathrm{Stab}(i\mathcal{A}^*_{\widetilde{G},F}, \tau)|^{-1}$ times the pushforward of the measure on $i\mathcal{A}^*_{\widetilde{G},F}$ by the map

$$\lambda \in i\mathcal{A}^*_{\widetilde{G},F} \mapsto \lambda \cdot \tau.$$

□

3 Spherical spaces

3.1 Coregular varieties

Let G be a connected reductive group over F and $H \subset G$ be a closed subgroup. We let $X = H\backslash G$ be the corresponding homogenous variety. We let TX, T^*X be the tangent and cotangent bundles of X respectively. Both are naturally equipped with a right action of G.

Let $\mathcal{B}$ be the flag variety of G. Recall that the variety X is called *spherical* if H has an open orbit in $\mathcal{B}$ or, equivalently, if G has an open orbit in $X \times \mathcal{B}$ for the diagonal action.

In the proposition below, by the *generic stabilizer* of a G-variety Y we mean a conjugacy class of closed subgroups $S \subset G$ such that for some dense open subset $U \subset Y$, the stabilizer of every $y \in U$ is conjugated to S. Generic stabilizers do not always exist but they do in the cases considered in the proposition below by the references cited in the proof, namely [20] and [21].

Proposition 3.1. *Assume that $X = H\backslash G$ is quasi-affine and that H is connected. Then, the following assertions are equivalent:*

(i) The generic stabilizer of T^*X contains regular elements;

(i') The generic stabilizer of T^*X contains regular semisimple elements;

(ii) The generic stabilizer of $X \times \mathcal{B}$ contains regular elements;

(ii') The generic stabilizer of $X \times \mathcal{B}$ contains regular semisimple elements;

(iii) We have $H \cap G_{\mathrm{rs}} \neq \emptyset$ and the function $h \in H \cap G_{\mathrm{rs}} \mapsto \frac{D^H_{alg}(h)^2}{D^G_{alg}(h)}$ extends to a regular function on H.

(iii') We have $H \cap G_{\mathrm{rs}} \neq \emptyset$ and the function $h \in H(F) \cap G_{\mathrm{rs}}(F) \mapsto \frac{D^H(h)^2}{D^G(h)}$ is locally bounded on $H(F)$ (i.e. it is bounded on the intersection of $G_{\mathrm{rs}}(F)$ with any compact subset of $H(F)$).

Moreover, the above assertions imply that X is spherical. If furthermore H is reductive, then the above conditions are also equivalent to:

(iv) The generic stabilizer of TX in G contains regular elements;

(v) The generic stabilizer of $X \times X$ in G for the diagonal action contains regular elements.

Remark 3.2. *The above proposition does not hold without the assumption that H is connected as the example of $X = O(2)\backslash \mathrm{GL}_2$ shows. Indeed, for $X = O(2)\backslash \mathrm{GL}_2$, conditions (i), (i'), (ii), (ii') are satisfied but not (iii) and (iii'). We also believe that (i), (ii), (ii'), (iii) and (iii') are still equivalent when X is not necessarily quasi-affine (but still assuming that H is connected) but that (i') is strictly stronger (e.g. take $X = \mathcal{B}$).*

We will say that the variety X is *coregular*, or that the pair (G, H) is *coregular*, if the equivalent conditions (i)-(iii) (or (i)-(v) if H is reductive) of the above proposition are satisfied.

Proof. Pick a Borel subgroup $B \subset G_{\overline{F}}$ with unipotent radical N and let $P(X)$, $U(X)$ be the respective stabilizers of the generic B and N orbits in X. In other words, there exists an open dense subset $\mathcal{U} \subset X_{\overline{F}}$ such that $xp \in xB$ (resp. $xu \in xN$) for every $(x, p) \in \mathcal{U} \times P(X)$ (resp. $(x, u) \in \mathcal{U} \times U(X)$) and the subgroups $P(X)$, $U(X)$ are maximal for these properties.

Let $L(X) \subset P(X)$ be a Levi factor and set $S(X) = L(X) \cap U(X)$. By [20, Korollar 2.9], we know that $U(X)$ is a normal subgroup of $P(X)$ and the quotient $A_X := P(X)/U(X) = L(X)/S(X)$ is a torus. Moreover, by the local structure theorem of [21, Theorem 2.3, Proposition 2.4], there exists a locally closed subvariety $\Sigma \subset X_{\overline{F}}$ which is $L(X)$-stable, on which the $L(X)$-action factors through the quotient $L(X) \to A_X$ and on which the resulting A_X-action is free, such that the $P(X)$-action induces an open embedding:

$$\Sigma \times^{L(X)} P(X) \hookrightarrow X. \tag{3.1.1}$$

Since $P(X) = L(X)B$, it follows that the generic stabilizer of $X \times \mathcal{B}$ exists and is given by the conjugacy class of $S(X) \cap B$. On the other hand, by the construction of [21, §3] there exists a $L(X)$-equivariant embedding

$$\Sigma \times \mathfrak{a}_X^* \hookrightarrow T^*X$$

whose image intersects every generic G-orbit in T^*X [21, Theorem 3.2, Lemma 3.1] and whose composition with the moment map $T^*X \to \mathfrak{g}^*$ is the second projection $\Sigma \times \mathfrak{a}_X^* \to \mathfrak{a}_X^*$ (followed by the natural inclusions $\mathfrak{a}_X^* \subset \mathfrak{l}(X)^* \subset \mathfrak{g}^*$). As the centralizer of a generic element in $\mathfrak{a}_X^*$ is $L(X)$ [21, Lemma 2.1], this shows that the generic stabilizer of T^*X exists and is the same as that of Σ in $L(X)$, i.e. $S(X)$.

As every conjugacy class, over $\overline{F}$, in $L(X)$ meets $L(X)\cap B$ and $S(X)$ is normal in $L(X)$, every element of $S(X)$ is G-conjugated to an element of $S(X)\cap B$ and this shows $(i)\Leftrightarrow(ii)$, $(i')\Leftrightarrow(ii')$. Moreover, if $S(X)$ contains a regular element of G then it contains a regular semisimple one. Indeed, if $g\in S(X)$ is G-regular and $g=su$ is its Jordan decomposition, then u is a regular unipotent element of the connected centralizer $Z_G(s)^0$. However, u belongs to the Levi subgroup $L(X)\cap Z_G(s)^0$ of $Z_G(s)^0$ and therefore $Z_G(s)^0\subset L(X)$. If this is so, an element of the form st for $t\in Z_G(s)^0_{der}$ in general position will be regular semisimple in G and this proves the claim as $L(X)_{der}\subset S(X)$ implies $Z_G(s)^0_{der}\subset S(X)$.

Thus, if (i) and (ii) are satisfied, $S(X)$ contains a regular semisimple element and so does $S(X)\cap B$. This proves the equivalence between (i), (i'), (ii) and (ii').

Assume now that (i') is satisfied i.e. that there exists $h\in S(X)$ which is regular semisimple in G. Then, $T_G=Z_G(h)^0\subset L(X)$ acts transitively on all the connected components of the subvariety of fixed points X^h. As Σ is a connected subvariety of X^h this shows that Σ is actually homogeneous under $L(X)$ and it follows, by the open embedding (3.1.1), that X is spherical. Up to conjugacy, we may assume that the canonical base point $x_0=H1$ of X belongs to Σ i.e. that HB is open in G. Then by (3.1.1), choosing a splitting of the surjection $\mathfrak{a}_{L(X)}\twoheadrightarrow\mathfrak{a}_X$, we have a direct sum decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{a}_X\oplus\mathfrak{n}(X)$$

where $\mathfrak{n}(X)$ denotes the nilradical of $\mathfrak{p}(X)$. Note that this decomposition is stable under the adjoint action of $T_H=Z_H(h)^0$ as the latter is a maximal torus of H contained in $L(X)$. Therefore, for $t\in T_H\cap G_{rs}$ we have

$$D^G_{alg}(t)=D^H_{alg}(t)\det(1-\mathrm{Ad}_t\mid_{\mathfrak{n}(X)})$$

from which it follows that

$$\frac{D^H_{alg}(t)^2}{D^G_{alg}(t)}=\frac{D^G_{alg}(t)}{\det(1-\mathrm{Ad}_t\mid_{\mathfrak{n}(X)})^2}=\delta^{alg}_{P(X)}(t)^{-1}D^{L(X)}_{alg}(t)$$

which implies (iii).

It is clear that (iii) implies (iii').

Assume now that (G,H) satisfies (iii') and let us show that (i') is also satisfied. First, we make a reduction to the case where the generic stabilizer $S(X)$ of T^*X is a torus. Indeed, since $(T^*X)(F)$ is Zariski dense in T^*X, up to conjugating we may assume that $S(X)$ is the stabilizer of a point $p=(x,\xi)\in(T^*X)(F)$, hence is defined over F, and even that $x=x_0$ is the canonical base-point of $X=H\backslash G$ (so that, in particular, $S(X)\subset H$). Let $T_S\subset S(X)$ be a maximal torus and let $H'=Z_H(T_S)$, $G'=Z_G(T_S)$ be the centralizers of T_S in H and G respectively. We need to show that G' is a torus (i.e. that T_S contains regular semi-simple elements of G). Let $X'=H'\backslash G'$ be the connected component of the subvariety of fixed points X^{T_S} containing x. Then, X' is a homogeneous G'-variety and $T^*(X')$ is a connected component of the subvariety of fixed points $(T^*X)^{T_S}$ in the cotangent bundle. We claim that:

(3.1.2) The pair (G', H') also satisfies condition (iii') i.e. $H' \cap G'_{rs} \neq \emptyset$ and the function $h \in H'(F) \cap G'_{\mathrm{rs}}(F) \mapsto \frac{D^{H'}(h)^2}{D^{G'}(h)}$ is locally bounded on $H'(F)$.

Indeed, H' contains a maximal torus of H hence regular semisimple elements of G by assumption but such elements are a fortiori also regular semisimple in G'. Moreover, for $h \in H'(F)$ and $t \in T_S(F)$ we have

$$D^H(ht) = D^{H'}(h)|\det(1 - \mathrm{Ad}_{ht}\mid_{\mathfrak{h}/\mathfrak{h}'})|,\ D^G(ht) = D^{G'}(h)|\det(1 - \mathrm{Ad}_{ht}\mid_{\mathfrak{g}/\mathfrak{g}'})|$$

and for each $h_0 \in H'(F)$ we can find $t \in T_S(F)$ as well as an open neighborhood $U \subset H'(F)$ of h_0 such that $h \mapsto |\det(1 - \mathrm{Ad}_{ht}\mid_{\mathfrak{h}/\mathfrak{h}'})|$ is bounded from below and $h \mapsto |\det(1 - \mathrm{Ad}_{ht}\mid_{\mathfrak{g}/\mathfrak{g}'})|$ is bounded from above on U. By the assumption that (G, H) satisfies condition (iii'), this shows that the function $h \mapsto \frac{D^{H'}(h)^2}{D^{G'}(h)}$ is bounded on U hence the function is locally bounded everywhere.

Let $B \subset G$ be a Borel subgroup containing T_S (not necessarily defined over F) and set $B' = Z_B(T_S)$, a Borel subgroup of G' containing a maximal torus T of G, and let $T_H \subset H'$ be a maximal torus. Taking T_S-invariants of the embedding (3.1.1), we see that X' contains an open subset B'-equivariantly isomorphic to $\Sigma \times^T B'$. Furthermore, in a neighborhood of $0 \in \mathfrak{t}_H(F)$, the functions $X \mapsto D^{H'}(e^X)$ and $X \mapsto D^{G'}(e^X)$ are products of $\dim(H') - \dim(T_H)$ and $\dim(G') - \dim(T)$ absolute values of linear forms respectively. Thus, (3.1.2) implies that

$$\dim(G') - \dim(T) \leqslant 2(\dim(H') - \dim(T_H))$$

or equivalently

$$\dim(X') \leqslant \frac{\dim(G') + \dim(T)}{2} - \dim(T_H) = \dim(B') - \dim(T_H). \tag{3.1.3}$$

However, as $\Sigma \times^T B'$ is open in X', we also have

$$\dim(X') = \dim(\Sigma) + \dim(B') - \dim(T). \tag{3.1.4}$$

Combining (3.1.3) with (3.1.4), we obtain that

$$\dim(\Sigma) \leqslant \dim(T) - \dim(T_H).$$

However, as T/T_S acts freely on Σ and T_H contains T_S this last inequality is only possible if $T_S = T_H$. But then, by the assumption that $H \cap G_{rs} \neq \emptyset$ and since H is connected, this implies that T_S and hence also $S(X)$ contains a regular semisimple element i.e. (i') is verified. This proves that (iii')$\Rightarrow$ (i') and therefore that the conditions (i), (i'), (ii), (ii'), (iii) and (iii') are all equivalent.

It remains to show that these are also equivalent to (iv) and (v) when H is reductive. The equivalence (iv)$\Leftrightarrow$ (v) follows from Luna's slice theorem [24] applied to the diagonal

G-orbit in $X \times X$ and noting that the normal bundle to the diagonal in $X \times X$ is isomorphic to TX. On the other hand, we have

$$TX = \mathfrak{g}/\mathfrak{h} \times^H G, \;\; T^*X = \mathfrak{h}^\perp \times^H G$$

where $\mathfrak{h}^\perp$ stands for the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}^*$. As both H and G are reductive, the adjoint representation of H on $\mathfrak{h}$ is isomorphic to the coadjoint action of H on $\mathfrak{h}^\perp$ and this shows that (v)$\Leftrightarrow$(i). $\square$

It is clear from the above discussion that if (G, H) is coregular then $H_{rs} \subset G_{rs}$. However, the opposite direction is not true in general. For example, when $(G, H) = (\mathrm{GL}_3, \mathrm{SL}_2)$, we have $H_{rs} \subset G_{rs}$ but the pair is not coregular. The next lemma shows that in the case of symmetric pairs, the coregular condition is equivalent to $H_{rs} \subset G_{rs}$.

Lemma 3.3. *A symmetric pair (G, H) is coregular if and only if $H_{rs} \subset G_{rs}$.*

Proof. The "only if" direction is obvious, we will only prove the other direction. So assume that $H_{rs} \subset G_{rs}$. Let $T_H \subset H$ be a maximal torus and let $T = Z_G(T_H)$ be its centralizer in G (a maximal torus of G by the assumption that $H_{\mathrm{rs}} \subset G_{\mathrm{rs}}$). Pick a cocharacter $\lambda \in X_*(T_H)$ that is G-regular and let $B \subset G$ be the Borel subgroup consisting of elements b_inG such that $\lambda(t)b\lambda(t)^{-1}$ has a limit when $t \to 0$. Let ι be the involution of G such that $H = (G^\iota)^\circ$. Obviously, B is ι-stable and contains T. Moreover, $B_H := B \cap H$ is a Borel subgroup of H containing the maximal torus T_H.

Let Σ^+ to denote the set of positive roots of T with respect to $B = TN$ and for each $\alpha \in \Sigma$, let $X_\alpha \in \mathfrak{n}(F)$ denote a nonzero element in the root subspace corresponding to α. Then we can decompose Σ into a union of three subsets $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where

$$\Sigma_1 = \{\alpha \in \Sigma |\ \iota(\alpha) \neq \alpha\}, \ \Sigma_2 = \{\alpha \in \Sigma |\ \iota(\alpha) = \alpha, \ \iota(X_\alpha) = X_\alpha\},$$

$$\Sigma_3 = \{\alpha \in \Sigma |\ \iota(\alpha) = \alpha, \ \iota(X_\alpha) = -X_\alpha\}.$$

Note that ι induces an involution without fixed points of Σ_1 and that, denoting by Σ_1/ι the set of ι-orbits in Σ_1, the set of positive roots Σ_H^+ of T_H with respect to B_H is in bijection with the set $\Sigma_1/\iota \cup \Sigma_2$ by the map

$$\alpha \in \Sigma_1/\iota \cup \Sigma_2 \mapsto \alpha \in \Sigma_H^+.$$

In particular, we have

$$D^G_{alg}(t) = \prod_{\alpha\in\Sigma^+} (1-\alpha(t))(1-\alpha(t)^{-1}), \;\; D^H_{alg}(t) = \prod_{\alpha\in\Sigma_1/\iota\cup\Sigma_2} (1-\alpha(t))(1-\alpha(t)^{-1}) \text{ for } t \in T_H.$$

From these identities, we see that it suffices to show that $\Sigma_3 = \emptyset$.

We will prove this by contradiction. Assume Σ_3 is non-empty and let $\alpha \in \Sigma_3$. The complements of $T_H \cap H_{\mathrm{rs}}$ in T_H can be described as a union of divisors as follows

$$T_H \setminus (T_H \cap G_{\mathrm{rs}}) = \bigcup_{\beta\in\Sigma_1/\iota\cup\Sigma_2^+} D_\beta$$

where $D_\beta := \{t \in T_H \mid \beta(t) = 1\}$. Note that each of these divisors D_β is actually a finite disjoint union of translates by the subtorus $T_\beta = \mathrm{Ker}(\beta \mid_{T_H})^0$ and that the subtori $(T_\beta)_\beta$ are two by two distinct (as Σ_H^+ is reduced). Since $T_H \setminus (T_H \cap G_{\mathrm{rs}})$ also contains the divisor $D_\alpha = \{t \in T_H \mid \alpha(t) = 1\}$ and $T_H \cap G_{\mathrm{rs}} = T_H \cap H_{\mathrm{rs}}$ by assumption, there must exist a $\beta \in \Sigma_1 \cup \Sigma_2$ such that $T_\beta = \mathrm{Ker}(\alpha \mid_{T_H})^0$ and so $D_\alpha \subset D_\beta$ i.e. $\beta \mid_{T_H} = k\alpha \mid_{T_H}$ for some integer $k \in \mathbb{Z}$. Note that this last equality is equivalent to

$$\beta + \iota(\beta) = k(\alpha + \iota(\alpha)) = 2k\alpha.$$

We now distinguish two cases. First, if $\beta \in \Sigma_2$ then $\iota(\beta) = \beta$ and the above identity becomes $\beta = k\alpha$ hence $\alpha = \beta \in \Sigma_2$ (as the system of positive roots Σ^+ is reduced) which is a contradiction. On the other hand, if $\beta \in \Sigma_1$, denoting by $\alpha^\vee$ the coroot associated to α, we have

$$2\langle \alpha^\vee, \beta \rangle = \langle \alpha^\vee, \beta + \iota(\beta) \rangle = 2k\langle \alpha^\vee, \alpha \rangle = 4k.$$

Hence, $\langle \alpha^\vee, \beta \rangle = 2k$ and therefore $k = 1$. Thus, we have $2\alpha = \beta + \iota(\beta)$ and $\langle \alpha^\vee, \beta \rangle = 2$. Since $\beta \neq \iota(\beta)$ this also implies that the length of β is twice the length of α and thus $\langle \beta^\vee, \alpha \rangle = \langle \alpha^\vee, \beta \rangle / 2 = 1$. Finally, denoting by s_β the simple reflection corresponding to β, we deduce that $s_\beta \alpha = \alpha - \beta = (\iota(\beta) - \beta)/2$ is a root of T in G. But that is a contradiction since this root would be ι-antiinvariant and there is no such root. □

Definition 3.4. *Assume now given twisted spaces $(G, \widetilde{G})$ and $(H, \widetilde{H})$ with an embedding $\widetilde{H} \hookrightarrow \widetilde{G}$ that is compatible with the inclusion $H \subset G$. Following the above discussion, we will say that the pair $(\widetilde{G}, \widetilde{H})$ is coregular if the following condition is satisfied:*

(3.1.5) We have $\widetilde{H} \cap \widetilde{G}_{rs} \neq \emptyset$ and the function $h \in \widetilde{H}(F) \cap \widetilde{G}_{rs}(F) \mapsto \frac{D^{\widetilde{H}}(h)^2}{D^{\widetilde{G}}(h)}$ is locally bounded on $\widetilde{H}(F)$.

Remark 3.5.

- *There is yet another characterization of the coregular spherical varieties which reads as follows: if $X = H\backslash G$ is a quasi-affine homogeneous spherical variety then X is coregular if and only if $\mathrm{rk}(X) = \mathrm{rk}(G) - \mathrm{rk}(H)$ where $\mathrm{rk}(G)$, $\mathrm{rk}(H)$, $\mathrm{rk}(X)$ denote the (absolute) ranks of G, H and X respectively. (We recall that the rank of a G-variety is, by definition, the rank of the torus $T_X = B/B_xU$ where $B = TU \subset G$ is a Borel subgroup and $x \in X$ a point in general position e.g., in the case of a spherical variety, a point in the open B-orbit.)*

  Indeed, if $\mathrm{rk}(X) = \mathrm{rk}(G) - \mathrm{rk}(H)$ and $B \subset G$ is a Borel subgroup with HB open, $B \cap H$ contains a maximal torus T_H of H and we have an isomorphism of T_H-representations $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{h}/\mathfrak{b} \cap \mathfrak{h}$. However, as a T_H-representation, $\mathfrak{g}/\mathfrak{b}$ is isomorphic to the dual of $\mathfrak{u}$ (the nilradical of $\mathfrak{b}$) and $\mathfrak{h}/\mathfrak{t}_H$ does not contain the trivial representation of T_H. Therefore, all restrictions of the roots of G to T_H are non-trivial which implies that T_H contains regular semi-simple element. As T_H is contained in the generic stabilizer of $X \times \mathcal{B}$ this proves point (ii') of the above proposition.

  Conversely, assume that X is coregular. Let again B be a Borel subgroup with HB open, then $H \cap B$ contains a G-regular semi-simple element h (again by characterization *(ii')), hence the maximal torus $T = G_h$ of G is included in B and therefore also the maximal torus $T_H = H_h$ of H and this shows that the universal Cartan T_X of X is a finite quotient of T/T_H hence* $\mathrm{rk}(X) = \mathrm{rk}(G) - \mathrm{rk}(H)$.

- *It is not true that $(\widetilde{G}, \widetilde{H})$ is coregular if and only if (G, H) is so. For example, let H be connected reductive and take $\widetilde{G} = (H \times H)\iota$ where $\iota(h_1, h_2) = (h_2, h_1)$ and $\widetilde{H} = H$ with the embedding $\widetilde{H} \hookrightarrow \widetilde{G}$ given by $h \mapsto (h, h)\iota$. Then, the pair (G, H) is always coregular whereas the pair $(\widetilde{G}, \widetilde{H})$ is coregular if and only if for every $h \in H(F)$,* $\det(1+\mathrm{Ad}_h) \neq 0$.

3.2 Tempered varieties

We continue to consider the setting at the end of the last section: $(G, \widetilde{G})$ is a connected reductive twisted space over F and $(H, \widetilde{H})$ is a closed connected twisted subspace of it. We also assume that H is unimodular (this implies that $X = H\backslash G$ is quasi-affine).

Following [7, §2.7], we say that the pair $(\widetilde{G}, \widetilde{H})$ is *tempered* if it satisfies the following condition:

(3.2.1) There exists $d > 0$ such that the integral

$$\int_{\widetilde{H}(F)} \Xi^{\widetilde{G}}(h)\sigma_{\widetilde{G}}(h)^{-d}dh$$

is convergent.

Note that the pair $(\widetilde{G}, \widetilde{H})$ is tempered if and only if (G, H) is so. Moreover, by *loc. cit.*, a pair (G, H) is tempered in the above sense if and only if $L^2(H(F)\backslash G(F))$ is tempered as a unitary representation of $G(F)$. (But we will not need this fact.)

Lemma 3.6. *Assume that the pair $(\widetilde{G}, \widetilde{H})$ is coregular and tempered. Then, the function*

$$h \in \widetilde{H}(F) \cap \widetilde{G}_{rs}(F) \mapsto \frac{D^{\widetilde{H}}(h)}{D^{\widetilde{G}}(h)^{1/2}}$$

is globally bounded.

Proof. Let $\widetilde{T} \subset \widetilde{H}$ be a maximal twisted torus. It is enough to show that the function

$$t \in \widetilde{T}(F) \cap \widetilde{G}_{rs}(F) \mapsto \frac{D^{\widetilde{H}}(t)}{D^{\widetilde{G}}(t)^{1/2}}$$

is globally bounded.

Set $M = Z_G(A_{\widetilde{T}})$ and $\widetilde{M} = Z_{\widetilde{G}}(A_{\widetilde{T}}) = M\widetilde{T}$. Then, $\widetilde{M}$ is the minimal Levi subspace of $\widetilde{G}$ containing $\widetilde{T}$. For each $\widetilde{P} \in \mathcal{F}(\widetilde{M})$, we let $\widetilde{T}_P^+$ be the subset of those $t \in \widetilde{T}(F)$ such that all the eigenvalues of the restriction of Ad_t to $\mathfrak{p}(\overline{F})$ are of absolute value ≥ 1. Then, we have a partition

$$\widetilde{T}(F) = \bigsqcup_{\widetilde{P} \in \mathcal{F}(\widetilde{M})} \widetilde{T}_P^+$$

and it is enough to show that, for any fixed $\widetilde{P} \in \mathcal{F}(\widetilde{M}_T)$, the function

$$t \in \widetilde{T}_P^+ \mapsto \frac{D^{\widetilde{H}}(t)}{D^{\widetilde{G}}(t)^{1/2}}$$

is bounded.

Let $\widetilde{P} \in \mathcal{F}(\widetilde{M})$ and assume that $\widetilde{T}_P^+ \neq \emptyset$. Let $\widetilde{L}$ be the unique Levi factor of $\widetilde{P}$ containing $\widetilde{M}$ and set

$$P_H = P \cap H,\ L_H = L \cap H,\ A_P^+ = \{a \in A_L(F) \mid |\alpha(a)| \geqslant 1\ \forall \alpha \in \Delta_P\} \text{ and } A_{P,H}^+ = A_P^+ \cap H(F).$$

Then, $\widetilde{T}_P^+$ is right invariant by the monoid $A_{P,H}^+$ and the quotient $\widetilde{T}_P^+/A_{P,H}^+$ is compact. Moreover, for $t \in \widetilde{T}_P^+$, the Lie algebras $\mathfrak{p}(\overline{F})$ and $\mathfrak{p}_H(\overline{F})$ (resp. $\mathfrak{l}(\overline{F})$ and $\mathfrak{l}_H(\overline{F})$) are the maximal subspaces of $\mathfrak{g}(\overline{F})$ and $\mathfrak{h}(\overline{F})$ where all the eigenvalues of Ad_t are of absolute value ≥ 1 (resp. $= 1$). Setting $\widetilde{P}_H = \widetilde{P} \cap \widetilde{H}$ and $\widetilde{L}_H = \widetilde{L} \cap \widetilde{H}$, it follows that

$$D^{\widetilde{G}}(t) = D^{\widetilde{L}}(t)\delta_{\widetilde{P}}(t),\ D^{\widetilde{H}}(t) = D^{\widetilde{L}_H}(t)\delta_{\widetilde{P}_H}(t)$$

for every $t \in \widetilde{T}_P^+$. Therefore,

$$\text{(3.2.2)} \qquad \frac{D^{\widetilde{H}}(t)}{D^{\widetilde{G}}(t)^{1/2}} = \frac{D^{\widetilde{L}_H}(t)}{D^{\widetilde{L}}(t)^{1/2}} \cdot \frac{\delta_{\widetilde{P}_H}(t)}{\delta_{\widetilde{P}}(t)^{1/2}}, \text{ for } t \in \widetilde{T}_P^+.$$

Since $(\widetilde{G}, \widetilde{H})$ is coregular, this in particular entails that the function

$$t \in \widetilde{T}_P^+ \mapsto D^{\widetilde{L}_H}(t) D^{\widetilde{L}}(t)^{-1/2}$$

is locally bounded. Because $\widetilde{T}_P^+/A_{P,H}^+$ is compact and $D^{\widetilde{L}_H}$, $D^{\widetilde{L}}$ are both $A_L(F)$-invariant, we deduce that the function $t \in \widetilde{T}_P^+ \mapsto D^{\widetilde{L}_H}(t) D^{\widetilde{L}}(t)^{-1/2}$ is globally bounded. Therefore, by (3.2.2) it only remains to check that the function

$$t \in \widetilde{T}_P^+ \mapsto \frac{\delta_{\widetilde{P}_H}(t)}{\delta_{\widetilde{P}}(t)^{1/2}}$$

is bounded. Again because $\widetilde{T}_P^+/A_{P,H}^+$ is compact, it is equivalent to work with the function

$$a \in A_{P,H}^+ \mapsto \frac{\delta_{P_H}(a)}{\delta_P(a)^{1/2}}.$$

Let $J_H \subset H(F)$ be a compact-open subgroup. Then, we have

$$\text{vol}(J_H a J_H) \sim \delta_{P_H}(a), \text{ for } a \in A^+_{P,H}. \tag{3.2.3}$$

Moreover, we can assume that $J_H \cap A_{L_H}(F) = A^c_{L_H}$ is the maximal compact subgroup of $A_{L_H}(F)$ and that the cosets $J_H a J_H$, $a \in A^+_{P,H}/A^c_{L_H}$ are disjoint. As (see [33, lemme II.1.1])

$$\delta_P(a)^{-1/2} \ll \Xi^G(a), \text{ for } a \in A^+_P,$$

and (G,H) is tempered, we can find $d > 0$ such that

$$\begin{aligned} \sum_{a \in A^+_{P,H}/A^c_{L_H}} \frac{\delta_{P_H}(a)}{\delta_P(a)^{1/2}} \sigma(a)^{-d} &\ll \sum_{a \in A^+_{P,H}/A^c_{L_H}} \Xi^G(a)\sigma(a)^{-d} \text{vol}(J_H a J_H) \\ &\leqslant \int_{H(F)} \Xi^G(h)\sigma(h)^{-d} dh < \infty. \end{aligned}$$

Since $A^+_{P,H}/A^c_{L_H}$ is a finitely generated monoid and δ_{P_H}, δ_P are characters on it, the above estimate implies that $a \in A^+_{P,H} \mapsto \frac{\delta_{P_H}(a)}{\delta_P(a)^{1/2}}$ is bounded and the lemma is proved.

□

3.3 Symmetric pairs

In this paper, by a *symmetric pair* (over F) we will mean a pair (G, ι) where G is a connected linear group over F and ι an involutive automorphism of G defined over F. Let (G,ι) be a symmetric pair. We denote by G^ι the closed subgroup of ι-fixed points, by $X^*(G)_\iota$ the subgroup of characters $\chi \in X^*(G)$ that are trivial on G^ι and by $A_{G,\iota}$ the neutral component of the subgroup $\{a \in A_G \mid \iota(a) = a^{-1}\}$. We also set

$$\mathcal{A}^*_{G,\iota} = X^*(A_{G,\iota}) \otimes \mathbb{R} = X^*(G)_\iota \otimes \mathbb{R}, \quad \mathcal{A}_{G,\iota} = X_*(A_{G,\iota}) \otimes \mathbb{R} = \text{Hom}(X^*(G)_\iota, \mathbb{R}).$$

Then, $\mathcal{A}^*_{G,\iota}$ (resp. $\mathcal{A}_{G,\iota}$) can be identified with the subspace of ι-antiinvariant vectors in $\mathcal{A}^*_G$ (resp. in $\mathcal{A}_G$). We also denote by $H_{G,\iota} : G(F) \to \mathcal{A}_{G,\iota}$ the composition of H_G and of the natural projection $\mathcal{A}_G \to \mathcal{A}_{G,\iota}$.

Assume from now on that G is reductive and connected. Recall that a parabolic subgroup $P \subset G$ is said to be *ι-split* if $\iota(P)$ is opposite to P and that a Levi subgroup $M \subset G$ is said to be *ι-split* if there exists a ι-split parabolic subgroup P such that $M = P \cap \iota(P)$.

We will denote by $\mathcal{P}_\iota$ and $\mathcal{L}_\iota$ the sets of all ι-split parabolic subgroups and ι-split Levi subgroups of G respectively. We will also write $[\mathcal{P}_\iota] = \mathcal{P}_\iota/\sim$ where, for $P, P' \in \mathcal{P}_\iota$, $P \sim P'$ if P and P' are $G(F)$-conjugate (or, equivalently, $G(\overline{F})$-conjugate). For $P \in \mathcal{P}_\iota$, we set $\overline{P} = \iota(P)$ for the unique ι-split parabolic subgroup opposite to P, $M_P = P \cap \iota(P) \in \mathcal{L}_\iota$ for its unique ι-split Levi factor and we denote by $[P]$ its image in $[\mathcal{P}_\iota]$.

By a *minimal ι-split parabolic subgroup* of G, we mean a parabolic subgroup that is ι-split, defined over F and minimal for these properties. We denote by $\mathcal{P}^{\min}_\iota \subseteq \mathcal{P}_\iota$ the subset of minimal ι-split parabolic subgroups. By [18, Proposition 4.9], all minimal ι-split parabolic subgroups are conjugated under $G(F)$ i.e. the image $[\mathcal{P}_\iota^{\min}]$ of $\mathcal{P}_\iota^{\min}$ in $[\mathcal{P}_\iota]$ is a singleton. For $M \in \mathcal{L}_\iota$, we set $\mathcal{P}_\iota(M) = \mathcal{P}(M) \cap \mathcal{P}_\iota$, $\mathcal{F}_\iota(M) = \mathcal{F}(M) \cap \mathcal{P}_\iota$ and $\mathcal{L}_\iota(M) = \mathcal{L}(M) \cap \mathcal{L}_\iota$.

For every $P \in \mathcal{P}_\iota$, we set $A_{P,\iota} = A_{M_P,\iota}$, $\mathcal{A}_{P,\iota} = \mathcal{A}_{M_P,\iota}$ and we denote by $H_{P,\iota} : P(F) \to \mathcal{A}_{P,\iota}$ the composition of the projection $P(F) \to M_P(F)$ with $H_{M_P,\iota}$. Then, for $P, Q \in \mathcal{P}_\iota$ with $P \subset Q$ we have the decomposition $\mathcal{A}_{P,\iota} = \mathcal{A}_{P,\iota}^Q \oplus \mathcal{A}_{Q,\iota}$ where $\mathcal{A}_{P,\iota}^Q = \mathcal{A}_{P,\iota} \cap \mathcal{A}_P^Q$ and we denote by $\Delta_{P,\iota}^Q$, $\Delta_{P,\iota}^{Q,\vee}$ the respective projections of Δ_P^Q and $\Delta_P^{Q,\vee}$ to $\mathcal{A}_{P,\iota}^{Q,*}$ and $\mathcal{A}_{P,\iota}^Q$. When $Q = G$, we will sometimes drop the superscript and when $M = M_P$ we will sometimes write $\mathcal{A}_{M,\iota}^Q$ for $\mathcal{A}_{P,\iota}^Q$. For $P \in \mathcal{P}_\iota$, we also set

$$A_{P,\iota}^+ = \{a \in A_P(F) \mid \langle \alpha, H_{P,\iota}(a) \rangle \geq 0 \ \forall \alpha \in \Delta_{P,\iota}\}.$$

Let $P_0 \in \mathcal{P}_\iota^{\min}$ and set $M_0 = M_{P_0} = P_0 \cap \iota(P_0)$, $A_{0,\iota} = A_{M_0,\iota}$ and $\mathcal{A}_{0,\iota}^G = \mathcal{A}_{M_0,\iota}^G$. It is known that the set $\Sigma_{0,\iota} \subseteq \mathcal{A}_{0,\iota}^{G,*}$ of nonzero weights for the adjoint action of $A_{0,\iota}$ on $\mathfrak{g}$ is a root system and that the subset of weights $\Sigma_{0,\iota}^+ \subset \Sigma_{0,\iota}$ appearing in $\mathfrak{p}_0$ forms a system of positive roots with associated set of simple roots $\Delta_{0,\iota} = \Delta_{P_0,\iota}$ see [18, §5]. We will denote by $W_{0,\iota}$ the Weyl group of this root system. We also set

$$\mathcal{A}_{P_0,\iota}^+ = \{X \in \mathcal{A}_{P_0,\iota} \mid \langle \alpha, X \rangle \geq 0 \ \forall \alpha \in \Delta_{0,\iota}\}, \ {}^-\mathcal{A}_{P_0,\iota} = \{X \in \mathcal{A}_{P_0,\iota} \mid \langle \varpi_\alpha, X \rangle \leq 0 \ \forall \alpha \in \Delta_{0,\iota}\}.$$

There is a bijection between $\mathcal{F}_\iota(M_0)$ and the collection of parabolic subsets of $\Sigma_{0,\iota}$ obtained by sending $Q \in \mathcal{F}_\iota(M_0)$ to the set $\Sigma_{0,Q,\iota}$ of nonzero weights of $A_{0,\iota}$ in $\mathfrak{q}$. Furthermore, for every ι-split parabolic subgroup $P \supset P_0$, the subset $\Delta_{0,\iota}^P := \Delta_{P_0,\iota}^P$ coincides with the set of simple roots $\Sigma_{0,P,\iota} \cap -\Sigma_{0,P,\iota} \cap \Delta_{0,\iota}$ and elements of $\Delta_{0,\iota}^{P,\vee}$ are *positively proportional* to the coroots associated to $\Delta_{0,\iota}^P$.

Let $P, Q \in \mathcal{P}_\iota$ with $[P] = [Q]$ and choose $\gamma \in G(F)$ such that $\gamma P \gamma^{-1} = Q$ and $\gamma M_P \gamma^{-1} = M_Q$. Then, Ad_γ induces isomorphisms

$$I_{P,Q} : \mathcal{A}_{P,\iota} \simeq \mathcal{A}_{Q,\iota}, \quad I_{P,Q} : A_{P,\iota} \simeq A_{Q,\iota}$$

sending respectively $\Delta_{P,\iota}$ to $\Delta_{Q,\iota}$, $A_{P,\iota}^+$ to $A_{Q,\iota}^+$ and which is independent of the choice of γ. Moreover, it is readily seen that the element $\iota(\gamma)$ still conjugates the pair (P, M_P) to (Q, M_Q) from which it follows that $\gamma\iota(\gamma)^{-1} \in M_Q(F)$.

Let us further fix a special maximal compact subgroup $K \subset G(F)$ that we use to extend $H_{P,\iota}$ to a right K-invariant map $G(F) \to \mathcal{A}_{P,\iota}$ for every $P \in \mathcal{P}_\iota$ by mean of the Iwasawa decomposition $G(F) = P(F)K$. Then, to every $P, Q \in \mathcal{P}_\iota$ with $[P] = [Q]$ we associate a point $Y_{Q,P}^K \in \mathcal{A}_{Q,\iota}$ as follows. Pick $\gamma \in G(F)$ such that $\gamma P \gamma^{-1} = Q$ and $\gamma M_P \gamma^{-1} = M_Q$. Then, recalling that $\gamma\iota(\gamma)^{-1} \in M_Q(F)$, we set

$$Y_{Q,P}^K := H_{\overline{Q},\iota}(\gamma) - \frac{1}{2} H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1})$$

where $\overline{Q} = \iota(Q)$ is the ι-split parabolic subgroup opposite to Q.

Lemma 3.7. *The element $Y_{Q,P}^K \in \mathcal{A}_{Q,\iota}$ so constructed doesn't depend on the choice of γ (i.e. it only depends on P, Q and K).*

Proof. Because the normalizer of the pair (Q, M_Q) in G is equal to M_Q, for any other element $\gamma' \in G(F)$ satisfying $\gamma' P(\gamma')^{-1} = Q$, $\gamma' M_P(\gamma')^{-1} = M_Q$, there exists $m \in M_Q(F)$ such that $\gamma' = m\gamma$. Then, it follows that

$$H_{\overline{Q},\iota}(\gamma') = H_{M_Q,\iota}(m) + H_{\overline{Q},\iota}(\gamma) \text{ and } H_{M_Q,\iota}(\gamma'\iota(\gamma')^{-1}) = 2H_{M_Q,\iota}(m) + H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1}),$$

hence

$$H_{\overline{Q},\iota}(\gamma') - \frac{1}{2}H_{M_Q,\iota}(\gamma'\iota(\gamma')^{-1}) = H_{\overline{Q},\iota}(\gamma) - \frac{1}{2}H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1}).$$

□

3.4 Symmetric varieties

Let (G, ι) be a symmetric pair with G reductive and connected as in the previous section. We set $H = G^\iota$ and let $X = H\backslash G$ be the corresponding *symmetric variety*.

For every $M \in \mathcal{L}_\iota$, we set $H_M = H \cap M$ and $X_M = H_M\backslash M$. Note that X_M is the symmetric variety associated to the symmetric pair $(M, \iota\mid_M)$ and that it is naturally a closed subvariety of X. Any character $\chi \in X^*(M)_\iota$ is by definition trivial on H and therefore descends to a regular map $X_M \to \mathbb{G}_m$ that we shall denote by the same letter. We define a *Harish-Chandra map*

$$H_{M,\iota} : X_M(F) \to \mathcal{A}_{M,\iota} = \mathrm{Hom}(X^*(M)_\iota, \mathbb{R})$$

by $\langle \chi, H_{M,\iota}(x)\rangle = \log|\chi(x)|$ for every $x \in X_M(F)$ and $\chi \in X^*(M)_\iota$. We also have a left action of A_M on X_M given by $a \cdot x = xa$ for $(a, x) \in A_M \times X_M$ which commutes with the (right) M-action and satisfies $H_{M,\iota}(a \cdot x) = H_{M,\iota}(a) + H_{M,\iota}(x)$ for every $(a, x) \in A_M(F) \times X_M(F)$.

For $P_0 \in \mathcal{P}_\iota^{\min}$ with $M_0 = M_{P_0}$, we set

$$X_{P_0}^+ = \{x \in X_{M_0}(F) \mid \langle \alpha, H_{M_0,\iota}(x)\rangle \geqslant 0 \ \forall \alpha \in \Delta_{P_0,\iota}\}.$$

If moreover $P \in \mathcal{P}_\iota$ is such that $P_0 \subset P$ and $C \geq 0$ we define

$$X_{P_0}^+(\geq C, P) = \{x \in X_{P_0}^+ \mid \langle \alpha, H_{M_0,\iota}(x)\rangle \geqslant C \ \forall \alpha \in \Delta_{P_0,\iota} \setminus \Delta_{P_0,\iota}^P\}.$$

Set

$$\mathcal{A}_X := \varprojlim_{P_0 \in \mathcal{P}_\iota^{\min}} \mathcal{A}_{P_0,\iota}$$

where the transition maps are given by the isomorphisms $I_{P_0,P_0'} : \mathcal{A}_{P_0,\iota} \simeq \mathcal{A}_{P_0',\iota}$ for $P_0, P_0' \in \mathcal{P}_\iota^{\min}$. Then, $\mathcal{A}_X$ is a real vector space equipped with canonical isomorphisms $\mathcal{A}_X \simeq \mathcal{A}_{P_0,\iota}$ for every $P_0 \in \mathcal{P}_\iota^{\min}$. The images by this isomorphism of the cones $\mathcal{A}_{P_0,\iota}^+$ and ${}^-\mathcal{A}_{P_0,\iota}$ don't depend on the choice of P_0, we will denote them by $\mathcal{A}_X^+$, ${}^-\mathcal{A}_X$ respectively. Moreover, for every $P \in \mathcal{P}_\iota$ we can choose $P_0 \in \mathcal{P}_\iota^{\min}$ such that $P_0 \subset P$ and we get an embedding $\mathcal{A}_{P,\iota} \hookrightarrow \mathcal{A}_X$ given by the composition of the natural inclusion $\mathcal{A}_{P,\iota} \subset \mathcal{A}_{P_0,\iota}$ with the isomorphism $\mathcal{A}_{P_0,\iota} \simeq \mathcal{A}_X$. This embedding actually does not depend on the choice of P_0 as can readily be checked.

Let $P, Q \in \mathcal{P}_\iota$ with $[P] = [Q]$ and choose $\gamma \in G(F)$ such that $\gamma P \gamma^{-1} = Q$, $\gamma M_P \gamma^{-1} = M_Q$. Then, we have $\gamma \in HM_P$: indeed, as already argued $\iota(\gamma)^{-1}\gamma \in M_P$ but this element is also in the neutral component of the subvariety of ι-antiinvariant elements in M_P hence there exists $m \in M_P$ such that $\iota(\gamma)^{-1}\gamma = \iota(m)^{-1}m$ or equivalently $\gamma \in Hm$. It now readily follows that $X_{M_Q} = X_{M_P}\gamma^{-1}$ and
(3.4.1)

$$H_{M_Q,\iota}(x\gamma^{-1}) = \mathrm{Ad}_\gamma(H_{M_P,\iota}(x)) - \frac{1}{2}H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1}) = I_{P,Q}(H_{M_P,\iota}(x)) + \frac{1}{2}H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1})$$

for every $x \in X_{M_P}(F)$. (Recall that $\gamma\iota(\gamma)^{-1} \in M_Q(F)$.)

3.5 Neighborhoods of infinity

Recall the following *weak Cartan decomposition* from [16] and [10]: for every $P_0 \in \mathcal{P}_\iota^{\min}$ we can find a compact subset $\mathcal{K} \subset G(F)$ such that

$$X(F) = X_{P_0}^+ \mathcal{K}. \tag{3.5.1}$$

Let $P \in \mathcal{P}_\iota$ and set $M = M_P$. Choose $P_0 \in \mathcal{P}_\iota^{\min}$ with $P_0 \subset P$ and a compact subset $\mathcal{K}$ satisfying the equality (3.5.1). Then, following [15], we define a *neighborhood of* ∞_P in $X(F)$ to be a subset of the latter containing

$$X_{P_0}^+(\geq C, P)\mathcal{K}$$

for some large enough constant $C > 0$. This notion actually only depends on the class $[P]$ in $[\mathcal{P}_\iota]$, and in particular not on the auxilliary choices of P_0 and $\mathcal{K}$. Indeed, using (3.4.1) this readily reduces to showing the following: if $\mathcal{K}' \supset \mathcal{K}$ is a bigger compact subset then for every $C \geq 0$ we can find $C' \geq 0$ with $X_{P_0}^+(\geq C, P)\mathcal{K} \supset X_{P_0}^+(\geq C', P)\mathcal{K}'$. This, in turn, is a consequence of the following lemma.

Lemma 3.8. *Let $P_0 \in \mathcal{P}_\iota^{\min}$, $\mathcal{K} \subset G(F)$ be a compact subset and set $M_0 = M_{P_0}$. Then, there exists $d > 0$ such that for every $x, y \in X_{P_0}^+$, $x\mathcal{K} \cap y\mathcal{K} \neq \emptyset$ implies $\|H_{M_0,\iota}(x) - H_{M_0,\iota}(y)\| \leq d$.*

Proof. Set $A_{0,\iota} = A_{M_0,\iota}$ and recall from [18, Proposition 4.7 (iii)] that $X_{M_0}(F)/A_{0,\iota}(F)$ is compact. It follows that we can find a compact subset $\Omega_0 \subset X_{M_0}(F)$ such that

$$X_{P_0}^+ \subset \Omega_0 A_{P_0,\iota}^+. \tag{3.5.2}$$

Let $X^*(M_0)_\iota^+$ be the subset of *dominant* weights $\chi \in X^*(M_0)_\iota$ i.e. such that $\langle \alpha^\vee, \chi\rangle \geq 0$ for every $\alpha^\vee \in \Delta_{P_0}^\vee$. Then, for every $\chi \in X^*(M_0)_\iota^+$ there exists a nonzero regular function $f_{2\chi} \in F[X]$ such that $f_{2\chi}(xp_0) = f(x)\chi(p_0)^2$ for every $(x, p_0) \in X \times P_0$[7]. Moreover, up to scaling $f_{2\chi}$ we may assume that $f_{2\chi}(x) = \chi(x)^2$ for every $x \in X_{M_0}$. Let $V_\chi \subset F[X]$ be the G-submodule generated by $f_{2\chi}$ for the action by right translation R. Then the weights of

[7]Indeed, since $\overline{P}_0 = \iota(P_0) = M_0\overline{N}_0$ is opposite to P_0, there exists a nonzero regular function $\varphi_\chi \in F[G]$ such that $\varphi_\chi(\overline{u}p) = \chi(p)$ for $(\overline{u}, p) \in \overline{N}_0 \times P_0$ and it suffices to take $f_{2\chi}(x) = \varphi_\chi(\iota(x)^{-1}x)$.

A_{M_0} in V are of the form $2\chi - \sum_{\alpha\in\Delta_{P_0}} n_\alpha \alpha$ where $n_\alpha \in \mathbb{N}$. From this and (3.5.2) it follows that for every compact subset $L \subset V$ we can find $c_\chi^L > 0$ such that

$$|f'(x)| \leq c_\chi^L |\chi(x)|^2 \text{ for every } (f', x) \in L \times X_{P_0}^+. \tag{3.5.3}$$

We will apply this to $L = R(\mathcal{K}')f_{2\chi}$ where $\mathcal{K}' = \mathcal{K}\mathcal{K}^{-1}$, setting $c_\chi = c_\chi^L$ for simplicity. Indeed, for $x, y \in X_{P_0}^+$ such that $x\mathcal{K} \cap y\mathcal{K} \neq \emptyset$ we can find $f' \in L$ such that $f_{2\chi}(x) = f'(y)$. Thus, applying (3.5.3) we get

$$|\chi(x)|^2 = |f_{2\chi}(x)| = |f'(y)| \leq c_\chi |\chi(y)|^2$$

i.e. $\langle \chi, H_{M_0,\iota}(x)\rangle \leq \langle \chi, H_{M_0,\iota}(y)\rangle + \frac{1}{2}\log(c_\chi)$. By symmetry, we also have the inequality with x, y permuted. As this holds for every $\chi \in X^*(M_0)_\iota^+$ and $X^*(M_0)_\iota^+$ generates $\mathcal{A}_{M_0,\iota}^*$ this gives the desired result. □

We shall denote by $\mathcal{N}(\infty_P)$ the collection of all neighborhoods of ∞_P in $X(F)$. The set $\mathcal{N}(\infty_P)$ is stable by finite intersections and translations by elements of $G(F)$. By a *basis* of $\mathcal{N}(\infty_P)$ we mean a subset $\mathcal{N}' \subset \mathcal{N}(\infty_P)$ such that every element $\Omega \in \mathcal{N}(\infty_P)$ contains at least one $\Omega' \in \mathcal{N}'$.

We define similarly the notion of *neighborhood of* ∞_P^M in $X_M(F)$ as follows. By the weak Cartan decomposition (3.5.1) applied to the symmetric variety X_M, we can find a compact subset $\mathcal{K}_M \subseteq M(F)$ such that $X_M(F) = X_{P_0\cap M}^+ \mathcal{K}_M$. Then, by definition, a neighborhood of ∞_P^M in $X_M(F)$ is a subset of the latter containing $X_{P_0}^+(\geq C, P)\mathcal{K}_M$ for a suitable $C > 0$. Once again, using the above lemma, we can show that this notion is independent on the choices of P_0 and $\mathcal{K}_M$. We will denote by $\mathcal{N}(\infty_P^M)$ the collection of all neighborhoods of ∞_P^M in $X_M(F)$. Note that $\mathcal{N}(\infty_P^M)$ admits a basis consisting of (left) $A_{P,\iota}^+$-invariant subsets (e.g. the family of subsets $X_{P_0}^+(\geq C, P)\mathcal{K}_M$ would do).

Let us now fixed a special maximal compact subgroup $K \subseteq G(F)$. Let $\overline{P} = MU_{\overline{P}} \in \mathcal{P}^\iota$ be the parabolic subgroup opposite to P with respect to M. Then, every $\gamma \in G(F)$ admits an Iwasawa decomposition $\gamma = m_{\overline{P}}(\gamma)u_{\overline{P}}(\gamma)k_{\overline{P}}(\gamma)$ with $m_{\overline{P}}(\gamma) \in M(F)$, $u_{\overline{P}}(\gamma) \in U_{\overline{P}}(F)$ and $k_{\overline{P}}(\gamma) \in K$.

Lemma 3.9. *Let $\mathcal{K} \subset G(F)$ be a compact subset. Then, we can find $\Omega_P^M \in \mathcal{N}(\infty_P^M)$ such that*

$$x\gamma K = x m_{\overline{P}}(\gamma) K$$

for every $(x, \gamma) \in \Omega_P^M \times \mathcal{K}$.

Proof. For every neighborhood $\Omega \in \mathcal{N}(\infty_P^M)$, we can find another one $\Omega' \in \mathcal{N}(\infty_P^M)$ such that $\Omega' m_{\bar{P}}(\gamma) \subset \Omega$ for every $\gamma \in \mathcal{K}$. It follows that we may assume that $\mathcal{K} \subset U_{\bar{P}}(F)$ and $m_{\overline{P}}(\gamma) = 1$ for every $\gamma \in \mathcal{K}$.

The lemma is then a variant of the wavefront Lemma [30, Corollary 5.3.2]. Indeed, let us fix $P_0 = M_0U_0 \in \mathcal{P}_\iota^{\min}$ with $P_0 \subset P$ as well as representatives $x_1, \ldots, x_n$ for the $M_0(F)$-orbits in $X_{M_0}(F)$. Set, for $C \geq 0$,

$$M_0^+(\geq C, P) = \{m_0 \in M_0(F) \mid \langle \alpha, H_{M_0}(m_0)\rangle \geq 0 \; \forall \alpha \in \Delta_{P_0},\ \langle \alpha, H_{M_0}(m_0)\rangle \geq C \; \forall \alpha \in \Delta_{P_0} \backslash \Delta_{P_0}^P \}.$$

Then, there exists a compact $\mathcal{K}_M \subset M(F)$ such that the subsets

$$\bigsqcup_i x_i M_0^+(\geq C, P)\mathcal{K}_M, \quad C > 0,$$

form a basis of neighborhoods of ∞_P^M in $X_M(F)$. Fix $1 \leq i \leq n$ and set

$$\mathcal{K}_{\overline{U}} := \bigcap_{k \in \mathcal{K}_M} k\mathcal{K}k^{-1} \cap U_{\overline{P}}(F), \quad \mathcal{K}' := \bigcup_{k \in \mathcal{K}_M} k^{-1}Kk,$$

two compact subsets of $U_{\bar{P}}(F)$ and $G(F)$ respectively. It suffices to show that

$$x_i M_0^+(\geq C, P)\mathcal{K}_{\overline{U}} \subset x_i M_0^+(\geq C, P)\mathcal{K}'$$

for C large enough. Let $J_{P_0} \subset P_0(F)$ be a compact-open subgroup small enough so that $m_0^{-1} J_{P_0} m_0 \subset \mathcal{K}'$ for every $m_0 \in M_0^+$. For every compact-open subgroup $J_{\overline{U}} \subset U_{\bar{P}}(F)$, provided C is large enough we have $m_0 \mathcal{K}_{\overline{U}} m_0^{-1} \subset J_{\overline{U}}$ for every $m_0 \in M_0^+(\geq C, P)$. Therefore, it only remains to check that $J_{\overline{U}}$ can be chosen such that $x_i J_{\overline{U}} \subset x_i J_{P_0}$ but this follows from the fact that $x_i P_0$, which is the image of HP_0 by the natural projection $G \to X$, is open in X (so that $x_i J_{P_0}$ contains a neighborhood of x_i in $X(F)$) since P_0 is ι-split. □

A consequence of the previous lemma is that for every $\Omega_P^M \in \mathcal{N}(\infty_P^M)$ we have $\Omega_P := \Omega_P^M K \in \mathcal{N}(\infty_P)$ and moreover that, if Ω_P^M is sufficiently small, the natural surjection $\Omega_P^M \to \Omega_P/K$ descends to a map $\Omega_P^M/K_M \to \Omega_P/K$ where K_M denotes the image of $K \cap P(F)$ by the natural surjection $P(F) \to M(F)$. We recall the following result from [15, Theorem 2]:

(3.5.4) If Ω_P^M is sufficiently small, the map $\Omega_P^M/K_M \to \Omega_P/K$ is a bijection.

In particular, if Ω_P^M is sufficiently small and $A_{P,\iota}^+$-invariant, there exists a map $H_{\overline{P},\iota} : \Omega_P/K \to \mathcal{A}_{M,\iota}$ and a left action of $A_{P,\iota}^+$ on Ω_P/K characterized by

$$H_{\overline{P},\iota}(xK) = H_{M,\iota}(x) \text{ and } a \cdot (xK) = (a \cdot x)K$$

for every $x \in \Omega_P^M$ and $a \in A_{P,\iota}^+$. For simplicity, we will henceforth assume that such a choice of Ω_P^M has been made for every parabolic $P \in \mathcal{P}_\iota$ so that if $\Omega_P \in \mathcal{N}(\infty_P)$ is sufficently small, $H_{\overline{P},\iota}(x)$ and $a \cdot x$ are well-defined for every $x \in \Omega_P/K$ and $a \in A_{P,\iota}^+$. These satisfy

$$H_{\overline{P},\iota}(a \cdot x) = H_{M,\iota}(a) + H_{\overline{P},\iota}(x) \tag{3.5.5}$$

for every $(a, x) \in A_{P,\iota}^+ \times \Omega_P/K$.

Lemma 3.10. *Let $P, Q \in \mathcal{P}_\iota$ with $[P] = [Q]$. Then, for $\Omega \in \mathcal{N}(\infty_P) = \mathcal{N}(\infty_Q)$ sufficiently small we have*

$$H_{\overline{Q},\iota}(x) = I_{P,Q}(H_{\overline{P},\iota}(x)) + Y_{P,Q}^K \text{ and } I_{P,Q}(a) \cdot x = a \cdot x$$

for every $(a, x) \in A_{P,\iota}^+ \times \Omega/K$.

Proof. Let us choose $\gamma \in G(F)$ such that $\gamma P \gamma^{-1} = Q$, $\gamma M_P \gamma^{-1} = M_Q$. Then, the map $\Omega_P^{M_P} \mapsto \Omega_P^{M_P}\gamma^{-1}$ induces a bijection $\mathcal{N}(\infty_P^{M_P}) \simeq \mathcal{N}(\infty_Q^{M_Q})$ and thus we may assume that there exists a small enough $\Omega_P^{M_P} \in \mathcal{N}(\infty_P^{M_P})$ such that $\Omega \subset \Omega_P^{M_P}K$ and $\Omega \subset \Omega_Q^{M_Q}K$ where we have set $\Omega_Q^{M_Q} := \Omega_P^{M_P}\gamma^{-1}$. Thus, an element $x \in \Omega/K$ can both be written as $x = x_P K$ and $x = x_Q K$ for $x_P \in \Omega_P^{M_P}$, $x_Q \in \Omega_Q^{M_Q}$. Then, since $x_P\gamma^{-1} \in \Omega_Q^{M_Q}$, provided $\Omega_P^{M_P}$ has been chosen sufficiently small, by Lemma 3.9 we have

$$x = (x_P\gamma^{-1})\gamma K = (x_P\gamma^{-1})m_{\overline{Q}}(\gamma)K.$$

Together with (3.5.4) we get $x_Q K_{M_Q} = (x_P\gamma^{-1})m_{\overline{Q}}(\gamma)K_{M_Q}$. By (3.4.1) and Lemma 3.7 this implies that

$$H_{\overline{Q},\iota}(x) = H_{M_Q,\iota}(x_P\gamma^{-1}m_{\overline{Q}}(\gamma)) = I_{P,Q}(H_{M_P,\iota}(x_P)) + \frac{1}{2}H_{M_Q,\iota}(\gamma\iota(\gamma)^{-1}) + H_{\overline{Q},\iota}(\gamma) = I_{P,Q}(H_{P,\iota}(x)) + Y_{P,Q}^K$$

and this shows the first equality of the lemma. For the second one, we notice that if $\Omega_P^{M_P}$ has been chosen sufficiently small and $A_{P,\iota}^+$-invariant, we have

$$\begin{aligned} a \cdot x &= x_P a K = x_P a\gamma^{-1}\gamma K = x_P a \gamma^{-1} m_{\overline{Q}}(\gamma)K \\ &= x_P\gamma^{-1}m_{\overline{Q}}(\gamma)\gamma a \gamma^{-1}K = I_{P,Q}(a)\cdot(x_P\gamma^{-1}m_{\overline{Q}}(\gamma)K) = I_{P,Q}(a)\cdot x \end{aligned}$$

for every $a \in A_{P,\iota}^+$. □

3.6 The map H_X

In this subsection, we continue to fix a special maximal compact subgroup $K \subset G(F)$. Recall that for every $P, Q \in \mathcal{P}_\iota$ with $[P] = [Q]$, we have introduced an element $Y_{P,Q}^K \in \mathcal{A}_{Q,\iota}$. For two minimal ι-split parabolic subgroups P_0, P_0', we introduce the following affine isomorphism

$$I_{P_0,P_0'}^K : \mathcal{A}_{P_0,\iota} \simeq \mathcal{A}_{P_0',\iota}, \;\; I_{P_0,P_0'}^K(H) = I_{P_0,P_0'}(H) + Y_{P_0,P_0'}^K.$$

These isomorphisms compose well (i.e. for any third $P_0'' \in \mathcal{P}_\iota^{\min}$ we have $I_{P_0,P_0''}^K = I_{P_0',P_0''}^K I_{P_0,P_0'}^K$) and we can introduce the real affine space

$$\mathcal{A}_{X,K} := \varprojlim_{P_0} \mathcal{A}_{P_0,\iota}$$

where the transition maps are this time given by the (affine) isomorphisms $I_{P_0,P_0'}^K$. Note that the space of translations of $\mathcal{A}_{X,K}$ is $\mathcal{A}_X$ and for every $P_0 \in \mathcal{P}_\iota^{\min}$ there is an affine isomorphism $\mathcal{A}_{X,K} \simeq \mathcal{A}_{P_0,\iota}$ compatible with the identification $\mathcal{A}_X \simeq \mathcal{A}_{P_0,\iota}$.

Let $P \in \mathcal{P}_\iota$ be a ι-split parabolic subgroup (not necessarily minimal) and choose $P_0 \in \mathcal{P}_\iota^{\min}$ with $P_0 \subset P$. Then, the composition of the isomorphism $\mathcal{A}_{X,K} \simeq \mathcal{A}_{P_0,\iota}$ with the projection $\mathcal{A}_{P_0,\iota} \twoheadrightarrow \mathcal{A}_{P,\iota}$ is independent on the choice of P_0 and will be denoted

$$proj_P : \mathcal{A}_{X,K} \to \mathcal{A}_{P,\iota}$$

or simply $Y \mapsto Y_P$.

In the following, we fix a norm $\|.\|$ on the real vector space $\mathcal{A}_X$ that we transfer to $\mathcal{A}_{X,K}$ through the choice of (an arbitrary) base-point.

Proposition 3.11. *There exists a K-invariant map $H_X : X(F) \to \mathcal{A}_{X,K}$ satisfying the following conditions: for every $P \in \mathcal{P}_\iota$, there exists a small enough neighborhood Ω_P of ∞_P in $X(F)$ such that:*

1. *For every $x \in \Omega_P$, we have $proj_P(H_X(x)) = H_{\overline{P},\iota}(x)$;*
2. *For every $(a, x) \in A^+_{P,\iota} \times \Omega_P/K$, $H_X(a \cdot x) = H_{M_P,\iota}(a) + H_X(x)$.*
3. *$1 + \|H_X(x)\| \sim \sigma_X(x)$ for $x \in X(F)$.*
4. *For every $P_0 \in \mathcal{P}^{\min}_\iota$, we can find $Y^-_{P_0,\iota} \in \mathcal{A}_{P_0,\iota}$ such that $H_X(x)_{P_0} \in Y^-_{P_0,\iota} + \mathcal{A}^+_{P_0,\iota}$ for every $x \in X(F)$.*

Proof. Let $P, Q \in \mathcal{P}_\iota$ be such that $[P] = [Q]$. Then, it readily follows from Lemma 3.10 that a K-invariant map $H_X : X(F) \to \mathcal{A}_{X,K}$ satisfies conditions 1 and 2 for P if and only if it satisfies the same conditions for Q. Similarly, for $P_0, P_0' \in \mathcal{P}^{\min}_\iota$, condition 4 holds for P_0 if and only if it holds for P_0'. Therefore, fixing $P_0 \in \mathcal{P}^{\min}_\iota$, it suffices to show the existence of a K-invariant map $H_X : X(F) \to \mathcal{A}_{X,K}$ satisfying conditions 1.-4. for every parabolic subgroup $P \in \mathcal{P}_\iota$ with $P_0 \subset P$. We will call such parabolics standard and we will denote by $\mathcal{P}^{std}_\iota$ the subset of them. We will also use the identification $\mathcal{A}_{X,K} = \mathcal{A}_{P_0,\iota}$.

For each $P \in \mathcal{P}^{std}_\iota$ we can find a neighborhood $\Omega_P \in \mathcal{N}(\infty_P)$ such that:

- For each $P \in \mathcal{P}^{std}_\iota$, Ω_P is $A^+_{P,\iota} \times K$-stable and is small enough that the map $H_{\overline{P},\iota} : \Omega_P \to \mathcal{A}_{P,\iota}$ as well as the action of $A^+_{P,\iota}$ on Ω_P/K are well-defined;
- For each $P, Q \in \mathcal{P}^{std}_\iota$, $\Omega_P \cap \Omega_Q \subseteq \Omega_{P\cap Q}$.

Then, for each $P \in \mathcal{P}^{std}_\iota$, we set

$$\omega_P := \Omega_P \setminus \bigcup_{Q \subsetneq P} \Omega_Q.$$

From the second bullet point above, it follows that we have a partition in K-invariant subsets

$$X(F) = \bigsqcup_{P \in \mathcal{P}_\iota P_0 \subset P} \omega_P. \tag{3.6.1}$$

We define a map $H_X : X(F) \to \mathcal{A}_{X,K}$ by $H_X(x) = H_{\overline{P},\iota}(x)$ for $x \in \omega_P$. Clearly H_X is K-invariant.

Let $P \in \mathcal{P}_\iota$ be standard and let us check that H_X satisfies conditions 1. and 2. Let $x \in \Omega_P$. By definition of the partition (3.6.1) there exists a standard $Q \in \mathcal{P}^{std}_\iota$ with $Q \subset P$ such that $x \in \omega_Q$. Since, by definition of $H_{\overline{P},\iota}$ and $H_{\overline{Q},\iota}$, we have $proj_P H_{\overline{Q},\iota}(x) = H_{\overline{P},\iota}(x)$, condition 1. is immediate. Also, since, by our choice of neighborhoods $(\Omega_P)_P$, ω_Q/K is invariant by $A^+_{Q,\iota}$, hence also by $A^+_{P,\iota}$, from (3.5.5) we deduce that for every $a \in A^+_{P,\iota}$ we have

$$H_X(a \cdot x) = H_{\overline{Q},\iota}(a \cdot x) = H_{M_Q,\iota}(a) + H_{\overline{Q},\iota}(x) = H_{M_P,\iota}(a) + H_X(x)$$

and this proves condition 2.

Let us now check condition 3. First, since $\|H_{\overline{P},\iota}(x)\| \ll \sigma_X(x)$ for every $P \in \mathcal{P}_\iota^{std}$ and $x \in \Omega_P$, it follows from the above definition of H_X that we have

$$\|H_X(x)\| \ll \sigma_X(x), \quad \text{for } x \in X(F).$$

Thus, we just need to prove the converse inequality. By the weak Cartan decomposition (3.5.1), it suffices to check it for $x \in X^+_{P_0}$. Let $C > 0$ that will be assumed large enough in what follows. Let $x \in X^+_{P_0}$ and let $P \in \mathcal{P}_\iota^{std}$ be such that

$$\Delta_{P_0,\iota} \setminus \Delta^P_{P_0,\iota} = \{\alpha \in \Delta_{P_0,\iota} \mid \langle \alpha, H_{M_0,\iota}(x)\rangle \geq C\}.$$

Then, provided C is large enough, we have $x \in \Omega_P$. Hence, by property 1. we have $proj_P H_X(x) = H_{\overline{P},\iota}(x)$. On the other hand, it is easy to see that $\sigma_X(x) \ll 1 + \|H_{\overline{P},\iota}(x)\|$. Hence, $\sigma_X(x) \ll 1 + \|H_X(x)\|$ and we are done.

It only remains to prove that H_X satisfies condition 4. Let us fix a weak Cartan decomposition like (3.5.1). Then, by definition of neighborhoods of ∞_P, there exists $C > 0$ such that $X^+_{P_0}(\geq C, P)\mathcal{K} \subset \Omega_P$ for every $P \in \mathcal{P}_\iota^{std}$. Then, for $P \subset Q$ we have $X^+_{P_0}(\geq C, P) \subseteq X^+_{P_0}(\geq C, Q)$ and the subsets

$$X^+_{P_0}(\geq C, P) \setminus \bigcup_{Q \subsetneq P} X^+_{P_0}(\geq C, Q)$$

are relatively compact modulo $A^+_{P,\iota}$. It follows that we can find compact subsets $\omega'_P \subset \Omega_P$ such that

$$X(F) = \bigcup_{P \in \mathcal{P}_\iota^{std}} A^+_{P,\iota}\omega'_P.$$

By property 2., we have $H_X(A^+_{P,\iota}\omega'_P) = H_{P,\iota}(A^+_{P,\iota}) + H_X(\omega'_P)$ for each $P \in \mathcal{P}_\iota^{std}$. Since, by property 3., $H_X(\omega'_P) \subset \mathcal{A}_{X,K}$ is relatively compact and $H_{P,\iota}(A^+_{P,\iota}) \subset \mathcal{A}^+_{P,\iota} \subset \mathcal{A}^+_{P_0,\iota}$, property 4. follows. □

Proposition 3.12. *Let $H_X : X(F) \to \mathcal{A}_{X,K}$ be the map as in the previous proposition. Let $P \in \mathcal{P}_\iota$ and set $M = M_P$. Then, there exists $c > 0$ such that for every $(a, x) \in A^+_{P,\iota} \times X_M(F)$, we can find $Q \in \mathcal{F}_\iota(P)$ such that*

(3.6.2) $$proj_Q H_X(ax) = H_{M_Q,\iota}(ax),$$

(3.6.3) $$\|H_X(ax) - proj_Q H_X(ax)\| \leqslant c\sigma_X(x),$$

and

(3.6.4) $$\|H_{M_Q,\iota}(ax) - H_{M_P,\iota}(ax)\| \leqslant c\sigma_X(x).$$

Proof. We prove this by induction on $\dim(A_P)$. So we assume that the statement holds for P replaced by any parabolic $R \in \mathcal{P}_\iota$ with $P \subsetneq R$. Let $\Omega_P^M \in \mathcal{N}(\infty_P^M)$ be a small enough $A_{P,\iota}^+$-stable neighborhood of ∞_P^M in $X_M(F)$ such that the first and second points of the previous proposition are satisfied for $\Omega_P = \Omega_P^M K$ and $H_{\overline{P},\iota}(x) = H_{M,\iota}(x)$ for every $x \in \Omega_P^M$. Then, there exists a constant $c_1 > 0$ such that for every $(a, x) \in A_{P,\iota}^+ \times X_M(F)$ satisfying

$$\langle \alpha, H_{M,\iota}(a) \rangle \geq c_1 \sigma_X(x), \quad \text{for every } \alpha \in \Delta_{P,\iota},$$

we have $ax \in \Omega_P^M$. (This follows e.g. from using a weak Cartan decomposition for $X_M(F)$.) Moreover, there also exists a constant $c' > c_1$ such that for any such a and x we can find $a' \in A_{P,\iota}^+$ with $\sigma(a') \leq c'\sigma_X(x)$, $a'x \in \Omega_P^M$ and $a \in a' A_{P,\iota}^+$. From this and Proposition 3.11, we get

$$proj_P H_X(ax) = H_{\overline{P},\iota}(ax) = H_{M,\iota}(ax) = H_{M,\iota}(a) + H_{M,\iota}(x)$$

and

$$H_X(ax) = H_X(a(a')^{-1}a'x) = H_{M,\iota}(a(a')^{-1}) + H_X(a'x) = H_{M,\iota}(a) - H_{M,\iota}(a') + H_X(a'x).$$

Hence,

$$\|H_X(ax) - proj_P H_X(ax)\| = \|H_X(a'x) - H_{M,\iota}(a') - H_{M,\iota}(x)\|.$$

Since $\|H_{M,\iota}(a')\| \ll \sigma_X(x)$, $\|H_X(a'x)\| \ll \sigma_X(x)$ and $\|H_{M,\iota}(x)\| \ll \sigma_X(x)$, we get that in this case both (3.6.2) and (3.6.3) holds for $Q = P$ and a suitable constant $c > 0$ (note that (3.6.4) is trivial when $P = Q$).

It remains to treat the case where there exists $\alpha \in \Delta_{P,\iota}$ such that $\langle \alpha, H_{M,\iota}(a) \rangle < c_1 \sigma_X(x)$. Let $R \in \mathcal{P}_\iota$ be the unique ι-split parabolic subgroup such that $R \supset P$ and $\Delta_{P,\iota}^R = \{\alpha\}$. Then, for every $(a, x) \in A_{P,\iota}^+ \times X_M(F)$ with $\langle \alpha, H_{M,\iota}(a) \rangle < c_1 \sigma_X(x)$ we can find $a_R \in A_{R,\iota}^+$ such that $\sigma(a a_R^{-1}) \ll \sigma_X(x)$. Then, writing $ax = a_R(a_R^{-1}ax)$ where $a_R^{-1}ax \in X_M(F) \subset X_{M_R}(F)$, by the induction hypothesis there exists a constant $c_R > 0$ as well as $Q \in \mathcal{P}_\iota$, $Q \supset R$ such that $proj_Q H_X(ax) = H_{M_Q,\iota}(ax)$ and

$$\|H_X(ax) - proj_Q H_X(ax)\| \leq c_R \sigma_X(a_R^{-1}ax), \; \|H_{M_Q,\iota}(ax) - H_{M_R,\iota}(ax)\| \leq c_R \sigma_X(a_R^{-1}ax).$$

Since $\sigma_X(a_R^{-1}ax) \ll \sigma_X(x)$ and $\|H_{M_R,\iota}(ax) - H_{M_P,\iota}(ax)\|| \ll \sigma_X(x)$, this again gives (3.6.2), (3.6.3) and (3.6.4) for a suitable constant $c > 0$ and the proposition is proved.

□

3.7 Twisted symmetric pairs

We define a *twisted symmetric pair* (over F) to be a triple $(G, \widetilde{G}, \iota)$ where $(G, \widetilde{G})$ is a linear twisted space, (G, ι) is a symmetric pair both defined over F and we have extended ι to an involutive automorphism of $\iota : \widetilde{G} \to \widetilde{G}$ (still defined over F) with $\iota(g_1 \gamma g_2) = \iota(g_1)\iota(\gamma)\iota(g_2)$ for every $(\gamma, g_1, g_2) \in \widetilde{G} \times G \times G$. We will usually refer to twisted symmetric pairs by $(\widetilde{G}, \iota)$, the underlying group G being implicit.

Let $(\widetilde{G},\iota)$ be a twisted symmetric pair. We denote by $A_{\widetilde{G},\iota}$ the neutral component of the subgroup $\{a\in A_{\widetilde{G}}\mid \iota(a)=a^{-1}\}$ and we set

$$\mathcal{A}^*_{\widetilde{G},\iota}=X^*(A_{\widetilde{G},\iota})\otimes\mathbb{R},\quad \mathcal{A}_{\widetilde{G},\iota}=X_*(A_{\widetilde{G},\iota})\otimes\mathbb{R}.$$

Then, $\mathcal{A}^*_{\widetilde{G},\iota}$ (resp. $\mathcal{A}_{\widetilde{G},\iota}$) can be identified with the subspace of ι-antiinvariant vectors in $\mathcal{A}^*_{\widetilde{G}}$ (resp. in $\mathcal{A}_{\widetilde{G}}$). We also denote by $H_{\widetilde{G},\iota}:G(F)\to\mathcal{A}_{\widetilde{G},\iota}$ the composition of $H_{\widetilde{G}}$ with the natural projection $\mathcal{A}_{\widetilde{G}}\to\mathcal{A}_{\widetilde{G},\iota}$.

We assume from now on that G is connected and reductive. Let $H=G^\iota$, $\widetilde{H}=\widetilde{G}^\iota$ be the subvarieties of ι-fixed points in G and $\widetilde{G}$ respectively. Then, $(H,\widetilde{H})$ is a reductive twisted space over F and we will always assume that $\widetilde{H}(F)\neq\emptyset$.

A parabolic subspace $\widetilde{P}\subset\widetilde{G}$ is called *ι-split* if the underlying parabolic subgroup $P\subset G$ is ι-split or equivalently if $\iota(\widetilde{P})$ is a parabolic subspace opposite to $\widetilde{P}$. Similarly, a Levi subspace $\widetilde{M}\subset\widetilde{G}$ is said to be *ι-split* if there exists a ι-split parabolic subspace $\widetilde{P}$ such that $\widetilde{M}=\widetilde{P}\cap\iota(\widetilde{P})$. For $\widetilde{M}$ a ι-split Levi subspace, we denote by $\mathcal{P}_\iota(\widetilde{M})$ (resp. $\mathcal{F}_\iota(\widetilde{M})$, resp. $\mathcal{L}_\iota(\widetilde{M})$) the set of ι-split parabolic subspaces having $\widetilde{M}$ as a Levi component (resp. containing $\widetilde{M}$, resp. the set of ι-split Levi subspaces containing $\widetilde{M}$).

We equip $\mathcal{A}_{\widetilde{M},\iota}$ with the unique Haar measure for which the lattice $H_{\widetilde{M},\iota}(A_{\widetilde{M}}(F))$ is of covolume one. We will also write $\mathcal{A}_{\widetilde{P},\iota}$ for $\mathcal{A}_{\widetilde{M},\iota}$ for every $\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})$ and we denote by $H_{\widetilde{P},\iota}:P(F)\to\mathcal{A}_{\widetilde{P},\iota}$ the composition of the projection $P(F)\to M(F)$ with $H_{\widetilde{M},\iota}$.

We will denote by $\widetilde{\mathcal{P}}_\iota$ and $\widetilde{\mathcal{L}}_\iota$ the sets of all ι-split parabolic subspaces and ι-split Levi subspaces of $\widetilde{G}$ respectively.

Let $\widetilde{M}$ be a ι-split Levi subspace of $\widetilde{G}$, $\widetilde{Q}\in\mathcal{F}_\iota(\widetilde{M})$ and set $\mathcal{A}^{\widetilde{Q}}_{\widetilde{M},\iota}=\mathcal{A}_{\widetilde{M},\iota}/\mathcal{A}_{\widetilde{Q},\iota}$. We equip this space with the quotient of the Haar measures on $\mathcal{A}_{\widetilde{M},\iota}$ and $\mathcal{A}_{\widetilde{Q},\iota}$. For $\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})$ with $\widetilde{P}\subset\widetilde{Q}$, we denote by

$$\Delta^{\widetilde{Q},\vee}_{\widetilde{P},\iota}\subset\mathcal{A}^{\widetilde{Q}}_{\widetilde{M},\iota}\text{ and }\Delta^{\widetilde{Q}}_{\widetilde{P},\iota}\subset\mathcal{A}^{\widetilde{Q},*}_{\widetilde{M},\iota}$$

the images of $\Delta^{Q,\vee}_{P,\iota}$ and $\Delta^{Q}_{P,\iota}$ by the natural projections

$$\mathcal{A}^{Q}_{M,\iota}\twoheadrightarrow\mathcal{A}^{\widetilde{Q}}_{\widetilde{M},\iota}\text{ and }\mathcal{A}^{Q,*}_{M,\iota}\twoheadrightarrow\mathcal{A}^{\widetilde{Q},*}_{\widetilde{M},\iota}$$

respectively. These form basis of $\mathcal{A}^{\widetilde{Q}}_{\widetilde{M},\iota}$ and $\mathcal{A}^{\widetilde{Q},*}_{\widetilde{M},\iota}$ respectively and we write $\widehat{\Delta}^{\widetilde{Q}}_{\widetilde{P},\iota}\subseteq\mathcal{A}^{\widetilde{Q},*}_{\widetilde{M},\iota}$ for the basis dual to $\Delta^{\widetilde{Q},\vee}_{\widetilde{P},\iota}$. We denote by $\tau^{\widetilde{Q}}_{\widetilde{P},\iota}$, $\hat{\tau}^{\widetilde{Q}}_{\widetilde{P},\iota}$ the characteristic functions of the cone in $\mathcal{A}$ characterized by

$$\tau^{\widetilde{Q}}_{\widetilde{P},\iota}(H)=1\iff\langle\alpha,H\rangle>0,\ \forall\alpha\in\Delta^{\widetilde{Q}}_{\widetilde{P},\iota},\ \hat{\tau}^{\widetilde{Q}}_{\widetilde{P},\iota}(H)=1\iff\langle\varpi,H\rangle>0,\ \forall\varpi\in\widehat{\Delta}^{\widetilde{Q}}_{\widetilde{P},\iota}$$

respectively. When $\widetilde{Q}=\widetilde{G}$ we will sometimes drop the superscript $\widetilde{Q}$.

Recall that $\mathcal{P}^{\min}_\iota$ stands for the set of all minimal ι-split parabolic subgroups of G. We have

(3.7.1) For every $P_0 \in \mathcal{P}_\iota^{\min}$, $\widetilde{P}_0 := Norm_{\widetilde{G}}(P_0)$ is a ι-split parabolic subspace of $\widetilde{G}$ i.e. $\widetilde{P}_0(F) \neq \emptyset$.

Indeed, we just need to check the existence of an element $\gamma \in \widetilde{G}(F)$ such that the parabolic subgroups P_0 and $\mathrm{Ad}_\gamma(P_0)$ are in the same conjugacy class. However, for $\gamma \in \widetilde{H}(F)$ the parabolic subgroup $\mathrm{Ad}_\gamma(P_0)$ is also ι-split minimal and by [18, Proposition 4.9] all minimal ι-split parabolic subgroups are in the same conjugacy class.

Let $\widetilde{\mathcal{P}}_\iota^{\min} \subset \widetilde{\mathcal{P}}_\iota$ be the subset of minimal elements of $\widetilde{\mathcal{P}}_\iota$ (for the inclusion relation). Then, by (3.7.1), the map $P_0 \mapsto \widetilde{P}_0$ gives a bijection $\mathcal{P}_\iota^{\min} \simeq \widetilde{\mathcal{P}}_\iota^{\min}$.

Set $\widetilde{M}_0 = \widetilde{P}_0 \cap \sigma(\widetilde{P}_0)$. It is easy to see that the automorphism θ of $\mathcal{A}_{M_{0,\iota}}^{G,*}$ preserves the root system $\Sigma_{0,\iota}$ as well as its subset of simple roots $\Delta_{0,\iota}$.

Let $X = H\backslash G$ be the homogeneous symmetric variety associated to (G,ι). Then, there exists a unique regular map $X \times \widetilde{G} \to X$, $(x,\gamma) \mapsto x\gamma$ such that $(Hg)\gamma = H\mathrm{Ad}_\gamma^{-1}(g)$ for every $(g,\gamma) \in G \times \widetilde{H}$. Note that we have

$$((xg_1)\gamma)g_2 = x(g_1\gamma g_2), \text{ for every } (x,g_1,g_2,\gamma) \in X \times G \times G \times \widetilde{G}.$$

We will usually write $\widetilde{X}$ to mean X equipped with this "twisted action" of $\widetilde{G}$. This twisted action naturally induces an automorphism θ of the real vector space $\mathcal{A}_X$ and we set

$$\mathcal{A}_{\widetilde{X}} := \mathcal{A}_X^\theta.$$

Then, for any $P_0 \in \mathcal{P}_\iota^{\min}$, the canonical isomorphism $\mathcal{A}_X \simeq \mathcal{A}_{P_0,\iota}$ induces an isomorphism $\mathcal{A}_{\widetilde{X}} \simeq \mathcal{A}_{\widetilde{P}_0,\iota}$. We will also write $\mathcal{A}_{\widetilde{X}}^+$, ${}^-\mathcal{A}_{\widetilde{X}}$ for the respective images of $\mathcal{A}_X^+$, ${}^-\mathcal{A}_X$ by the natural projection $\mathcal{A}_X \to \mathcal{A}_{\widetilde{X}}$ and $\phi_{\widetilde{X}}$ for the characteristic function of the cone ${}^-\mathcal{A}_{\widetilde{X}}$.

Let $K \subset G(F)$ be a special maximal compact subgroup and let $\mathcal{A}_X(1-\theta)$ be the kernel of the natural projection $\mathcal{A}_X \to \mathcal{A}_{\widetilde{X}}$. We set

$$\mathcal{A}_{\widetilde{X},K} := \mathcal{A}_{X,K}/\mathcal{A}_X(1-\theta).$$

It is an affine space with direction $\mathcal{A}_{\widetilde{X}}$ and for every $P_0 \in \mathcal{P}_\iota^{\min}$ the (affine) isomorphism $\mathcal{A}_{X,K} \simeq \mathcal{A}_{P_0,\iota}$ induces an isomorphism $\mathcal{A}_{\widetilde{X},K} \simeq \mathcal{A}_{\widetilde{P}_0,\iota}$. Moreover, for every $\widetilde{P} \in \widetilde{\mathcal{P}}_\iota$ there is a natural affine projection

$$\mathcal{A}_{\widetilde{X},K} \to \mathcal{A}_{\widetilde{P},\iota} \tag{3.7.2}$$

that can be described as the composition of $\mathcal{A}_{\widetilde{X},K} \simeq \mathcal{A}_{\widetilde{P}_0,\iota}$ with the projection $\mathcal{A}_{\widetilde{P}_0,\iota} \to \mathcal{A}_{\widetilde{P},\iota}$ for any $\widetilde{P}_0 \in \widetilde{\mathcal{P}}_\iota$ with $\widetilde{P}_0 \subset \widetilde{P}$. For every $Y \in \mathcal{A}_{\widetilde{X},K}$, we will denote by $Y_{\widetilde{P}}$ its image by the projection (3.7.2).

If $H_X : X(F) \to \mathcal{A}_{X,K}$ is a map as in Proposition 3.11, we will usually write $H_{\widetilde{X}} : \widetilde{X}(F) \to \mathcal{A}_{\widetilde{X},K}$ for the composition of H_X with the natural projection $\mathcal{A}_{X,K} \to \mathcal{A}_{\widetilde{X},K}$.

3.8 Orthogonal sets

Let $(\widetilde{G},\iota)$ be a twisted symmetric pair. In [7, §2.8.2], we have introduced notions of (G,M,ι)-families extending in an obvious way Arthur's definition of (G,M)-families in the context of symmetric pairs. This actually exactly corresponds to Arthur's theory applied to the root system $\Sigma_{0,\iota}$. There is a similar combinatorics for twisted groups as developed in [23] which can be applied to any automorphism of a root system preserving a positive system. In particular, starting from the pair $(\Sigma_{0,\iota},\theta)$ there is a corresponding notion of $(\widetilde{G},\widetilde{M},\iota)$-orthogonal sets that we now briefly describe.

Let $\widetilde{M}$ be a ι-split Levi subspace of $\widetilde{G}$. Two parabolic subspaces $\widetilde{P},\widetilde{Q}\in\mathcal{P}_\iota(\widetilde{M})$ are said to be *ι-adjacent* if the intersection $\Delta^\vee_{\widetilde{P},\iota}\cap-\Delta^\vee_{\widetilde{Q},\iota}$ is a singleton $\{\alpha^\vee_{\widetilde{P},\widetilde{Q}}\}$. If this is the case, the hyperplane $\{X\in i\mathcal{A}^*_{\widetilde{M},\iota}\mid\langle\alpha^\vee_{\widetilde{P},\widetilde{Q}},X\rangle=0\}$ is called *the wall separating* $\widetilde{P}$ and $\widetilde{Q}$.

By definition $(\widetilde{G},\widetilde{M},\iota)$-*orthogonal set* is a family $\mathcal{X}=(X_{\widetilde{P},\iota})_{\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})}$ of points in $\mathcal{A}_{\widetilde{M},\iota}$ such that for every ι-adjacent parabolic subspaces $\widetilde{P},\widetilde{Q}\in\mathcal{P}_\iota(\widetilde{M})$, we have

$$X_{\widetilde{P},\iota}-X_{\widetilde{Q},\iota}\in\mathbb{R}\alpha^\vee_{\widetilde{P},\widetilde{Q}}$$

where $\Delta^\vee_{\widetilde{P},\iota}\cap-\Delta^\vee_{\widetilde{Q},\iota}=\{\alpha^\vee_{\widetilde{P},\widetilde{Q}}\}$. We further say that $\mathcal{X}$ is *positive* if

$$X_{\widetilde{P},\iota}-X_{\widetilde{Q},\iota}\in\mathbb{R}_{\geqslant 0}\alpha^\vee_{\widetilde{P},\widetilde{Q}}$$

for every pair of ι-adjacent parabolic subspaces $\widetilde{P},\widetilde{Q}\in\mathcal{P}_\iota(\widetilde{M})$.

As in Subsection 2.9, we define the *depth* and the *norm* of a $(\widetilde{G},\widetilde{M},\iota)$-orthogonal set $\mathcal{X}=(X_{\widetilde{P}})_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}$ by

$$d(\mathcal{X})=\min_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}\min_{\alpha\in\Delta_{\widetilde{P}}}\alpha(X_{\widetilde{P}})\text{ and }N(\mathcal{X})=\max_{\widetilde{P}\in\mathcal{P}(\widetilde{M})}\max_{\alpha\in\Delta_{\widetilde{P}}}|\alpha(X_{\widetilde{P}})|$$

respectively. Note that $\mathcal{X}$ is positive if and only if $d(\mathcal{X})\geq 0$.

Let $\mathcal{X}=(X_{\widetilde{P},\iota})_{\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})}$ be a $(\widetilde{G},\widetilde{M},\iota)$-orthogonal set. For $\widetilde{Q}=\widetilde{L}U_Q\in\mathcal{F}_\iota(\widetilde{M})$, we denote by $X_{\widetilde{Q},\iota}$ the projection to $\mathcal{A}_{\widetilde{L},\iota}$ of $X_{\widetilde{P},\iota}$ for any $\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})$ such that $\widetilde{P}\subset\widetilde{Q}$ (this projection does not depend on the choice of $\widetilde{P}$). To $\mathcal{X}$ we associate functions $\Gamma^{\widetilde{Q}}_{\widetilde{L},\iota}(.,\mathcal{X})$ on $\mathcal{A}^{\widetilde{Q}}_{\widetilde{L},\iota}$ and complex numbers $v^{\widetilde{Q}}_{\widetilde{L},\iota}(\mathcal{X})\in\mathbb{C}$ for every $\widetilde{L}\in\mathcal{L}_\iota(\widetilde{M})$ and $\widetilde{Q}\in\mathcal{F}_\iota(\widetilde{L})$ as follows:

$$\Gamma^{\widetilde{Q}}_{\widetilde{L},\iota}(H,\mathcal{X})=\sum_{\widetilde{P}\in\mathcal{F}_\iota(\widetilde{L}),\widetilde{P}\subset\widetilde{Q}}(-1)^{a^{\widetilde{Q}}_{\widetilde{P},\iota}}\widehat{\tau}^{\widetilde{Q}}_{\widetilde{P},\iota}(H-X_{\widetilde{P},\iota}),\quad H\in\mathcal{A}^{\widetilde{Q}}_{\widetilde{L},\iota}$$

and

$$v^{\widetilde{Q}}_{\widetilde{L},\iota}(\mathcal{X})=\int_{\mathcal{A}^{\widetilde{Q}}_{\widetilde{L},\iota}}\Gamma^{\widetilde{Q}}_{\widetilde{L},\iota}(H,\mathcal{X})dH.$$

If $\mathcal{X}$ is positive, $v^{\widetilde{Q}}_{\widetilde{L},\iota}(\mathcal{X})$ is simply the volume of the convex hull of the family $(X_{\widetilde{P},\iota})_{\widetilde{P}\in\mathcal{P}_\iota(\widetilde{L}),\widetilde{P}\subset\widetilde{Q}}$. Once again, we will sometimes drop the superscript when $\widetilde{Q}=\widetilde{G}$.

Let K be a special compact subgroup of $G(F)$. Using the Iwasawa decomposition $G(F) = P(F)K$, for every ι-split parabolic subspace $\widetilde{P} \subset \widetilde{G}$, we can extend the homomorphism $H_{\widetilde{P},\iota}$ to a map $G(F) \to \mathcal{A}_{\widetilde{P},\iota}$. Then, for every ι-split Levi subspace $\widetilde{M} \subset \widetilde{G}$ and $g \in G(F)$, the family $\mathcal{H}_{\widetilde{M},\iota}(g) = (-H_{\widetilde{P},\iota}(g))_{\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})}$ is a positive $(\widetilde{G}, \widetilde{M}, \iota)$-orthogonal set and we define

$$v^{\widetilde{Q}}_{\widetilde{M},\iota}(g) = v^{\widetilde{Q}}_{\widetilde{M},\iota}(\mathcal{H}_{\widetilde{M},\iota}(g)), \quad \text{for } \widetilde{Q} \in \mathcal{F}_\iota(\widetilde{M}).$$

Let $Y \in \mathcal{A}_{\widetilde{X},K}$ and $\widetilde{M} \in \widetilde{\mathcal{L}}_\iota$. Then, the family $\mathcal{Y}_{\widetilde{M}} := (Y_{\widetilde{P}})_{\widetilde{P}\in\mathcal{P}_\iota(\widetilde{M})}$ is $(\widetilde{G}, \widetilde{M}, \iota)$-orthogonal set. Indeed, this follows from the fact that if $\widetilde{P}, \widetilde{Q} \in \mathcal{P}_\iota(\widetilde{M})$ are ι-adjacent then $Y^K_{Q,P} \in \mathbb{R}\alpha^\vee_{\widetilde{P},\widetilde{Q}}$ as can be directly checked on the definition.

Whenever convenient, we will also fix a minimal ι-split parabolic subspace $\widetilde{P}_0 \subset \widetilde{G}$ to define the *depth* and *norm* of an element $Y \in \mathcal{A}_{\widetilde{X},K}$ by

$$d(Y) = \min_{\alpha\in\Delta_{\widetilde{P}_0,\iota}} \alpha(Y_{\widetilde{P}_0,\iota}) \text{ and } N(Y) = \max_{\alpha\in\Delta_{\widetilde{P}_0,\iota}} |\alpha(Y_{\widetilde{P}_0,\iota})|$$

respectively. We note that for every ι-split Levi subspace $\widetilde{M} \subset \widetilde{G}$, there exist constants $c_1, c_2 > 0$ such that

$$d(Y) - c_1 \leq d\left(\mathcal{Y}_{\widetilde{M}}\right) \text{ and } N\left(\mathcal{Y}_{\widetilde{M}}\right) \leq N(Y) + c_2$$

for every $Y \in \mathcal{A}_{\widetilde{X},K}$.

3.9 ι-weighted orbital integrals

Let $\widetilde{M}$ be a ι-split Levi subspace of $\widetilde{G}$, $\gamma \in \widetilde{M}(F)\cap\widetilde{G}_{rs}(F)$ and $\widetilde{Q} \in \mathcal{F}_\iota(\widetilde{M})$. For $f \in \mathcal{C}(\widetilde{G}(F))$, we define the *$\iota$-twisted weighted orbital integral*

$$\Phi^{\widetilde{Q}}_{\widetilde{M},\iota}(\gamma, f) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g) v^{\widetilde{Q}}_{\widetilde{M},\iota}(g) dg$$

as well as its normalized version

$$J^{\widetilde{Q}}_{\widetilde{M},\iota}(\gamma, f) = D^{\widetilde{G}}(\gamma)^{1/2}\Phi^{\widetilde{Q}}_{\widetilde{M},\iota}(\gamma, f).$$

By the same argument as in (2.10.1), the above integral is absolutely convergent, and for $\widetilde{T} \subset \widetilde{M}$ a maximal twisted torus, we have:

(3.9.1) There exist $p > 0$ and, for every $d > 0$, a continuous semi-norm ν_d on $\mathcal{C}(\widetilde{G}(F))$ such that

$$\left|J^{\widetilde{Q}}_{\widetilde{M},\iota}(\gamma, f)\right| \leqslant \nu_d(f)(1 + |\log D^{\widetilde{G}}(\gamma)|)^p \sigma_{\widetilde{T}_{/\theta}}(\gamma)^{-d}$$

for every $\gamma \in \widetilde{T}_{\text{reg}}(F)$ and $f \in \mathcal{C}(\widetilde{G}(F))$.

4 Harmonic analysis for certain singular conjugacy classes

In this section, we fix a connected reductive twisted space $\widetilde{G}$ as well as a semisimple element $x \in \widetilde{G}_{ss}(F)$ and a regular nilpotent coadjoint orbit $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x^*)$. Our goal is to establish an alternative description of the distribution

$$f \in \mathcal{C}_{\mathrm{scusp}}(\widetilde{G}(F)) \mapsto c_{f,\mathcal{O}}(x)$$

that was defined in Subsection 2.12.

Note that since G_x admits a regular coadjoint orbit, it is quasi-split. We fix once and for all a Borel subgroup B_x of G_x with a Levi decomposition $B_x = T_x N_x$. Let $Y \in \mathcal{O}$ whose stabilizer in N_x is trivial. It defines a generic character ξ of $N_x(F)$ by the formula

$$\xi(\exp(X)) = \psi(\langle X, Y\rangle), \quad X \in \mathfrak{n}_x(F). \tag{4.0.1}$$

We then say that the generic character ξ is *associated* to the nilpotent coadjoint orbit $\mathcal{O}$. The formula we will give for the distribution $f \mapsto c_{f,\mathcal{O}}(x)$ will be in term of this character ξ. Note that there are more than one generic characters of $N_x(F)$ associated to $\mathcal{O}$ but they form a unique $T_x(F)$-orbit and the resulting description of $c_{f,\mathcal{O}}(x)$ will be easily seen to be independent of the choice of ξ. Conversely, given a generic character $\xi : N_x(F) \to \mathbb{C}^\times$, we can always find a regular nilpotent element $Y \in \mathfrak{g}_x^*(F)$ such that ξ is given by formula (4.0.1) and moreover such Y is unique up to $N_x(F)$-conjugacy. We will then say that the (nilpotent) coadjoint orbit of Y is *associated* to the generic character ξ and we will dentoe it by $\mathcal{O}_\xi$.

4.1 The function $\Gamma_{B_x}(., \mathcal{X})$

Let $A_x \subset T_x$ be the maximal split subtorus, $\mathcal{A}_x = X_*(A_x) \otimes \mathrm{R}$, $M(x) = Z_G(A_x)$ and $\widetilde{M}(x) = M(x)x$, a Levi subspace of $\widetilde{G}$. We let $\mathcal{P}_{B_x}(\widetilde{M}(x))$ (resp. $\mathcal{F}_{B_x}(\widetilde{M}(x))$) be the set of parabolic subspaces $\widetilde{P} \in \mathcal{P}(\widetilde{M}(x))$ (resp. $\widetilde{Q} \in \mathcal{F}(\widetilde{M}(x))$) such that $P \cap G_x = B_x$ (resp. $Q \supset B_x$). Let also

$$W_x = Norm_{G_x(F)}(T_x)/T_x$$

be the Weyl group of T_x in G_x.

Lemma 4.1. *(i) We have* $A_{\widetilde{M}(x)} = A_x$.

(ii) There is a natural embedding of W_x *into the Weyl group* $W(G, \widetilde{M}(x)) = Norm_{G(F)}(\widetilde{M}(x))/M(x)(F)$ *and we have a partition*

$$\mathcal{P}(\widetilde{M}(x)) = \bigsqcup_{w \in W_x} w\mathcal{P}_{B_x}(\widetilde{M}(x)). \tag{4.1.1}$$

Proof. (i) Since A_x centralizes with $M(x)$ and x, it centralizes $\widetilde{M}(x)$ and therefore $A_x \subset A_{\widetilde{M}(x)}$. On the other hand, every element of $A_{\widetilde{M}(x)}$ centralizes x (so that $A_{\widetilde{M}(x)} \subset G_x$) and T_x. It follows that $A_{\widetilde{M}(x)} \subset Z_{G_x}(T_x) = T_x$ and finally $A_{\widetilde{M}(x)} \subset A_x$.

(ii) Since every element of $Norm_{G_x}(T_x)$ centralizes x and normalizes A_x, $Norm_{G_x}(T_x)$ is contained in the normalizer of $Z_G(A_x)x = \widetilde{M}(x)$ i.e. $Norm_{G_x}(T_x) \subset Norm_G(\widetilde{M}(x))$. Moreover, $Norm_{G_x}(T_x) \cap M(x)$ is equal to T_x because the centralizer of A_x in G_x is T_x. This explains the "natural" embedding $W_x \hookrightarrow W(G, \widetilde{M}(x))$.

We have

$$w\mathcal{P}_{B_x}(\widetilde{M}(x)) = \{\widetilde{P} \in \mathcal{P}(\widetilde{M}(x)) \mid P \cap G_x = wB_x\}$$

so that (4.1.1) is just the partition corresponding to the fibers of the map $\mathcal{P}(\widetilde{M}(x)) \to \mathcal{P}^{G_x}(T_x)$, $\widetilde{P} \mapsto P \cap G_x$.

□

By the above lemma, we have a containment of set of roots

$$\Sigma(A_x, G_x) \subset \Sigma(A_x, G) = \Sigma(A_{\widetilde{M}(x)}, G).$$

Thus, for every $\alpha \in \Sigma(A_x, G_x)$ there are a priori two associated coroots $\alpha_1^\vee, \alpha_2^\vee \in \mathcal{A}_x$. Namely, we can either see α as a root of A_x in G_x and consider the corresponding coroot $\alpha_1^\vee \in X_*(A_x)$ or we can view α as a root of $A_{\widetilde{M}(x)}$ in G and consider the corresponding coroot $\alpha_2^\vee \in \mathcal{A}_x$. It turns out that $\alpha_1^\vee$ and $\alpha_2^\vee$ are always positively proportional. This can be seen as follows. Take a maximal split torus $A_x \subset A_{\min} \subset G$ and fix an inner product on $\mathcal{A}_{\min} := \mathcal{A}_{A_{\min}}$ which is invariant under the action of the Weyl group $W_{\min} = \mathrm{Norm}_{G(F)}(A_{\min}(F))/\mathrm{Cent}_{G(F)}(A_{\min})$. By restriction to $\mathcal{A}_x \subset \mathcal{A}_{\min}$, this gives an inner product on $\mathcal{A}_x$, hence an identification $\mathcal{A}_x \simeq \mathcal{A}_x^*$ such that for every $\alpha \in \Sigma(A_{\widetilde{M}(x)}, G)$, $\alpha_2^\vee$ is positively proportional to α. This inner product is still $W(G, \widetilde{M}(x))$-invariant, hence W_x-invariant by the second point of the above lemma. It follows that, for every $\alpha \in \Sigma(A_x, G_x)$, the identification $\mathcal{A}_x \simeq \mathcal{A}_x^*$ also sends $\alpha_1^\vee$ to a positive multiple of α. Since in what follows, the coroots will only matter up to a positive scalar, we will not really have to distinguish between $\alpha_1^\vee$ and $\alpha_2^\vee$. However, to fix ideas, when there is an ambiguity we will always use $\alpha_2^\vee$ instead of $\alpha_1^\vee$.

Let $\mathcal{X} = (X_{\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{M}(x))}$ be a $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set in $\mathcal{A}_x = \mathcal{A}_{\widetilde{M}(x)}$. For $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ and $H \in \mathcal{A}_x$, we set (the function $\widehat{\tau}_{\widetilde{P}}^{\widetilde{Q}}$ is defined in Section 2.9)

$$\Gamma_{B_x}^{\widetilde{Q}}(H, \mathcal{X}) = \sum_{\widetilde{P} \in \mathcal{F}_{B_x}(\widetilde{M}(x)), \widetilde{P} \subset \widetilde{Q}} (-1)^{a_{\widetilde{P}}^{\widetilde{Q}}} \widehat{\tau}_{\widetilde{P}}^{\widetilde{Q}}(H - X_{\widetilde{P}}).$$

For the next lemma, we recall, for two parabolic subspaces $\widetilde{Q}, \widetilde{R} \in \mathcal{F}(\widetilde{M}(x))$ with $\widetilde{Q} \subset \widetilde{R}$, the function

$$\Gamma_{\widetilde{Q}}^{\widetilde{R}} : \mathcal{A}_{\widetilde{Q}} \times \mathcal{A}_{\widetilde{Q}} \to \mathbb{R}$$

that can be defined by

$$\Gamma_{\widetilde{Q}}^{\widetilde{R}}(H, X) = \sum_{\widetilde{Q} \subset \widetilde{P} \subset \widetilde{R}} (-1)^{a_{\widetilde{P}}^{\widetilde{R}}} \tau_{\widetilde{Q}}^{\widetilde{P}}(H) \widehat{\tau}_{\widetilde{P}}^{\widetilde{R}}(H - X).$$

Then, provided $X \in \mathcal{A}^{+}_{\widetilde{Q}}$, we have [23, lemme 1.8.3]

$$\Gamma^{\widetilde{R}}_{\widetilde{Q}}(H,X) = \tau^{\widetilde{R}}_{\widetilde{Q}}(H)\phi^{\widetilde{R}}_{\widetilde{Q}}(H-X) \tag{4.1.2}$$

where $\phi^{\widetilde{R}}_{\widetilde{Q}}$ is the characteristic function of those $Y \in \mathcal{A}_{\widetilde{Q}}$ such that $\langle \varpi, Y\rangle \leq 0$ for every $\varpi \in \widehat{\Delta}^{\widetilde{R}}_{\widetilde{Q}}$ (in other words it is the characteristic function with support the closure of the support of the function $X \mapsto \widehat{\tau}^{\widetilde{R}}_{\widetilde{Q}}(-X)$).

Lemma 4.2. *For two $(\widetilde{G}, \widetilde{M}(x))$-orthogonal sets $\mathcal{X}$ and $\mathcal{Y}$, we have*

$$\Gamma^{\widetilde{R}}_{B_x}(H, \mathcal{X}+\mathcal{Y}) = \sum_{\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x)),\widetilde{Q}\subset\widetilde{R}} \Gamma^{\widetilde{Q}}_{B_x}(H,\mathcal{X})\Gamma^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}}, Y_{\widetilde{Q}}).$$

Proof. The proof follows from the same argument as Lemma 1.8.6 of [23]. □

Proposition 4.3. *For every $H \in \mathcal{A}_x$ and $\widetilde{R} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$, we have (the function $\tau^{\widetilde{R}}_{\widetilde{Q}}$ is defined in Section 2.9)*

$$\sum_{\widetilde{R}\supset\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))} \Gamma^{\widetilde{Q}}_{B_x}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}}) = 1. \tag{4.1.3}$$

Moreover, if $\mathcal{X}$ is positive *the function $H \mapsto \Gamma_{B_x}(H,\mathcal{X})$ is the characteristic function of the set of $H \in \mathcal{A}_x$ such that*

$$\varpi_\alpha(H - X_{\widetilde{P}}) \leqslant 0$$

for every $\widetilde{P} \in \mathcal{P}_{B_x}(\widetilde{M}(x))$ and $\alpha \in \Delta_{\widetilde{P}}$.

Proof. Let $\widetilde{R} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$. By definition of $\Gamma^{\widetilde{Q}}_{B_x}(., \mathcal{X})$, for $H \in \mathcal{A}_x$, we have

$$\begin{aligned}
&\sum_{\widetilde{R}\supset\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))} \Gamma^{\widetilde{Q}}_{B_x}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}})\\
&= \sum_{\widetilde{R}\supset\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))} \sum_{\mathcal{F}_{B_x}(\widetilde{M}(x))\ni\widetilde{P}\subset\widetilde{Q}} (-1)^{a^{\widetilde{Q}}_{\widetilde{P}}}\widehat{\tau}^{\widetilde{Q}}_{\widetilde{P}}(H-X_{\widetilde{P}})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}})\\
&= \sum_{\widetilde{P}\in\mathcal{F}_{B_x}(\widetilde{M}(x))} \sum_{\substack{\mathcal{F}(\widetilde{M}(x))\ni\widetilde{Q}\\ \widetilde{P}\subset\widetilde{Q}\subset\widetilde{R}}} (-1)^{a^{\widetilde{Q}}_{\widetilde{P}}}\widehat{\tau}^{\widetilde{Q}}_{\widetilde{P}}(H-X_{\widetilde{P}})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}}).
\end{aligned}$$

Moreover, by [23, proposition 1.7.1, lemme 2.9.2] the inner sum

$$\sum_{\substack{\mathcal{F}(\widetilde{M}(x))\ni\widetilde{Q}\\ \widetilde{P}\subset\widetilde{Q}\subset\widetilde{R}}} (-1)^{a^{\widetilde{Q}}_{\widetilde{P}}}\widehat{\tau}^{\widetilde{Q}}_{\widetilde{P}}(H-X_{\widetilde{P}})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}})$$

equals 1 if $\widetilde{P}=\widetilde{R}$ and 0 otherwise. The identity (4.1.3) follows.

Assume now that $\mathcal{X}$ is positive. Fix $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$. For $\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))$, we denote by $\phi_{\widetilde{P}',\widetilde{P}}$ the characteristic function of the set of $H\in\mathcal{A}_x$ such that for every $\alpha\in\Delta_{\widetilde{P}'}$ we have

$$\varpi_\alpha(H)\leqslant 0 \text{ if } \alpha\in\Sigma^+_{\widetilde{P}}$$

and

$$\varpi_\alpha(H)>0 \text{ if } \alpha\in\Sigma^-_{\widetilde{P}}.$$

Then, we have

$$\begin{aligned}
\Gamma_{B_x}(.,\mathcal{X}) &= \sum_{\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))}(-1)^{a^{\widetilde{G}}_{\widetilde{Q}}}\widehat{\tau}_{\widetilde{Q}}(.-X_{\widetilde{Q}}) \\
&= \sum_{\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))}\sum_{\substack{\widetilde{P}'\subset\widetilde{Q}\\ \widetilde{P}\cap\widetilde{Q}=\widetilde{P}\cap\widetilde{P}'}}(-1)^{a^{\widetilde{G}}_{\widetilde{Q}}}\widehat{\tau}_{\widetilde{Q}}(.-X_{\widetilde{Q}}) \\
&= \sum_{\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))}(-1)^{|\Delta_{\widetilde{P}'}\cap\Sigma^-_{\widetilde{P}}|}\phi_{\widetilde{P}',\widetilde{P}}(.-X_{\widetilde{P}'})
\end{aligned} \tag{4.1.4}$$

where the last identity follows from [23, lemme 1.7.4, lemme 2.9.2]. From the above we deduce that for every $H\in\mathcal{A}_x$ satisfying $\Gamma_{B_x}(H,\mathcal{X})\neq 0$ there exists $\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))$ such that $\phi_{\widetilde{P}',\widetilde{P}}(H-X_{\widetilde{P}'})=1$ which, since $\mathcal{X}$ is positive, further implies

$$\varpi_\alpha(H)\leqslant\varpi_\alpha(X_{\widetilde{P}'})\leqslant\varpi_\alpha(X_{\widetilde{P}})$$

for every $\alpha\in\Delta_{\widetilde{P}}$. Thus, as $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$ was arbitrary, we see that $\mathrm{Supp}(\Gamma_{B_x}(.,\mathcal{X}))$ is included in the subset of those $H\in\mathcal{A}_x$ such that

$$\varpi_\alpha(H-X_{\widetilde{P}})\leqslant 0,\ \forall\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x)),\ \forall\alpha\in\Delta_{\widetilde{P}}. \tag{4.1.5}$$

Conversely, assume that $H\in\mathcal{A}_x$ satisfies the inequalities (4.1.5). Then, for a chosen $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$, we have $\phi_{\widetilde{P},\widetilde{P}}(H-X_{\widetilde{P}})=1$ whereas for $\widetilde{P}\neq\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))$, as $\Delta_{\widetilde{P}'}\cap\Sigma^-_{\widetilde{P}}\neq\emptyset$, we have $\phi_{\widetilde{P}',\widetilde{P}}(H-X_{\widetilde{P}'})=0$. From identity (4.1.4) this readily implies that $\Gamma_{B_x}(H,\mathcal{X})=1$. This gives the last part of the proposition.

□

Corollary 4.4. *Let $\mathcal{X},\mathcal{Y}$ be two positive $(\widetilde{G},\widetilde{M}(x))$-orthogonal sets. Then, for every $\widetilde{Q},\widetilde{R}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$ and $Y\in\mathcal{A}_x$ we have*

$$\Gamma^{\widetilde{Q}}_{B_x}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}})\Gamma^{\widetilde{R}}_{B_x}(H,\mathcal{X}+\mathcal{Y})=\Gamma^{\widetilde{Q}}_{B_x}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}})\phi^{\widetilde{R}}_{\widetilde{Q}}(H-X_{\widetilde{Q}}-Y_{\widetilde{Q}}).$$

In other words, on the support of the function $\Gamma^{\widetilde{Q}}_{B_x}(.,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(.-X_{\widetilde{Q}})$, $\Gamma^{\widetilde{R}}_{B_x}(.,\mathcal{X}+\mathcal{Y})$ is equal to $\phi^{\widetilde{R}}_{\widetilde{Q}}(.-X_{\widetilde{Q}}-Y_{\widetilde{Q}})$.

Proof. This follows from the combination of Lemma 4.2 with the identities (4.1.3) and (4.1.2), noting that each of the functions $\Gamma^{\widetilde{Q}}_{B_x}(.,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q}}(.-X_{\widetilde{Q}})$, $\widetilde{Q},\widetilde{R}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$, are characteristic functions by Proposition 4.3. $\square$

To simplify the notation, we will use Δ_x (resp. $\Delta_x^\vee$) to denote the set of roots $\Delta_{B_x}\subset X^*(A_x)$ (resp. of coroots $\Delta^\vee_{B_x}\subset X_*(A_x)$).

Proposition 4.5. *Assume that $\mathcal{X}$ is positive. Then, $\Gamma_{B_x}(.,\mathcal{X})$ is the characteristic function of*

$$Conv\{X_{\widetilde{P}}\mid \widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))\}+{}^-\mathcal{A}_{B_x}+\mathcal{A}_{\widetilde{G}}$$

where $Conv\{X_{\widetilde{P}}\mid \widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))\}$ denotes the convex hull of the finite set $\{X_{\widetilde{P}}\mid \widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))\}$ whereas ${}^-\mathcal{A}_{B_x}$ stands for the closed cone generated by $-\Delta_x^\vee$.

Proof. Set

$$\mathcal{C}_{B_x}(\mathcal{X}):=Conv\{X_{\widetilde{P}}\mid \widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))\}+{}^-\mathcal{A}_{B_x}+\mathcal{A}_{\widetilde{G}}.$$

This is obviously a closed convex subset of $\mathcal{A}_x$. Moreover, its set of extreme points is contained in $\{X_{\widetilde{P}}\mid \widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))\}$. Recall that for every closed convex subset $\mathcal{C}\subset\mathcal{A}_x$, denoting by $Ext(\mathcal{C})$ its set of extreme points and, for $X\in Ext(\mathcal{C})$, by $\mathcal{C}_X$ the cone centered at X generated by $\mathcal{C}$, that is $\mathcal{C}_X=\{X+t(Y-X)\mid Y\in\mathcal{C},t\geq 0\}$, we have

$$\mathcal{C}=\bigcap_{X\in Ext(\mathcal{C})}\mathcal{C}_X.$$

According to the previous proposition $\Gamma_{B_x}(.,\mathcal{X})$ is the characteristic function of

$$\{H\in\mathcal{A}_x\mid \varpi_\alpha(H-X_{\widetilde{P}})\leqslant 0\ \forall\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x)),\ \forall\alpha\in\Delta_{\widetilde{P}}\}.$$

Therefore, it suffices to show that for every $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$ we have

$$\mathcal{C}_{B_x}(\mathcal{X})_{X_{\widetilde{P}}}=\{H\in\mathcal{A}_x\mid \varpi_\alpha(H-X_{\widetilde{P}})\leqslant 0\ \forall\alpha\in\Delta_{\widetilde{P}}\}(=X_{\widetilde{P}}+{}^-\mathcal{A}_{\widetilde{P}}+\mathcal{A}_{\widetilde{G}}).$$

As $\mathcal{X}$ is positive, and ${}^-\mathcal{A}_{B_x}\subset{}^-\mathcal{A}_{\widetilde{P}}$, the inclusion $\mathcal{C}_{B_x}(\mathcal{X})_{X_{\widetilde{P}}}\subseteq X_{\widetilde{P}}+{}^-\mathcal{A}_{\widetilde{P}}+\mathcal{A}_{\widetilde{G}}$ is clear. On the other hand, for every $\alpha\in\Delta_{\widetilde{P}}$ we either have:

- $\alpha^\vee$ is positively proportional to an element of $\Delta_x^\vee$;
- there exists $\widetilde{P}'\in\mathcal{P}_{B_x}(\widetilde{M}(x))$ such that $\Sigma^+_{\widetilde{P}}\cap\Sigma^-_{\widetilde{P}'}=\{\alpha\}$ in which case $X_{\widetilde{P}'}-X_{\widetilde{P}}\in\mathbb{R}_{>0}\alpha^\vee$.

This implies that, in both cases, $\mathcal{C}_{B_x}(\mathcal{X})_{X_{\widetilde{P}}}$ is invariant by translation by $\mathbb{R}_{\leq 0}\alpha^\vee$. As this holds for all $\alpha\in\Delta_{\widetilde{P}}$, this gives the reverse inclusion $X_{\widetilde{P}}+{}^-\mathcal{A}_{\widetilde{P}}+\mathcal{A}_{\widetilde{G}}\subseteq\mathcal{C}_{B_x}(\mathcal{X})_{X_{\widetilde{P}}}$ and therefore $\mathcal{C}_{B_x}(\mathcal{X})_{X_{\widetilde{P}}}=X_{\widetilde{P}}+{}^-\mathcal{A}_{\widetilde{P}}+\mathcal{A}_{\widetilde{G}}$. $\square$

4.2 The weight $v_{B_x,\xi}(u,g)$

Let $N_{x,\mathrm{reg}} = N_x \cap G_{x,\mathrm{reg}}$ be the open subset of regular elements in N_x and $T_{x,c} \subset T_x(F)$ be the maximal compact subgroup. We equip $T_{x,c}$ with the Haar measure of total mass 1 and we also fix a log-norm $\sigma_{x,\mathrm{reg}} : G_{x,reg}(F) \to \mathbb{R}_{\geqslant 1}$ on $G_{x,reg}(F)$ (see Section 2.3). Set $r = \dim(\mathcal{A}_x) - a_{\widetilde{G}}$.

Lemma 4.6. *For any $u \in N_{x,\mathrm{reg}}(F)$ and any positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, the iterated integral*

(4.2.1)
$$\int_{T_x(F)/A_{\widetilde{G}}(F)} \int_{T_{x,c}} \xi(a^{-1}t^{-1}uta)\, dt\, \Gamma_{B_x}(H_{T_x}(a), \mathcal{X})\, da$$

is absolutely convergent in that order and will be denoted by

$$\tilde{v}_{B_x,\xi}(u,\mathcal{X}) := \int^*_{T_x(F)/A_{\widetilde{G}}(F)} \xi(a^{-1}ua)\Gamma_{B_x}(H(a),\mathcal{X})\, da.$$

Moreover, there exists a constant $C>0$ such that for every $u \in N_{x,reg}(F)$ and every positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, we have (where $N(\mathcal{X})$ denotes the norm of $\mathcal{X}$ defined in Section 2.9)

$$|\tilde{v}_{B_x,\xi}(u,\mathcal{X})| \leqslant C(\sigma_{x,\mathrm{reg}}(u) + N(\mathcal{X}))^r.$$

Proof. The inner integral over $T_{x,c}$ in (4.2.1) is obviously convergent. Let $N_{x,der}$ be the derived subgroup of N_x and let $N_x/N_{x,der} = \bigoplus_{\alpha\in\Delta_x}(N_x/N_{x,der})_\alpha$ be the isotypic decomposition with respect to the adjoint action of A_x. We fix a norm $\|.\|$ on the F-vector space $N_x(F)/N_{x,der}(F)$ and for every $u \in N_x$ and $\alpha \in \Delta_x$, let us we denote by u_α the projection of u to $(N_x/N_{x,der})_\alpha$. Then, since ξ is a generic character, there exists $C_1 > 0$ such that for all $a \in T_x(F)$ and $u \in N_{x,reg}(F)$ we have

$$\int_{T_{x,c}} \xi(a^{-1}t^{-1}uta)dt \neq 0 \Rightarrow \langle \alpha, H_{T_x}(a)\rangle \geqslant \log\|u_\alpha\| - C_1 \text{ for all } \alpha \in \Delta_x.$$

On the other hand, there exists $C_2 > 0$ such that $\log\|u_\alpha\| - C_1 \geqslant -C_2\sigma_{x,\mathrm{reg}}(u)$ for all $(u,\alpha) \in N_{x,reg}(F) \times \Delta_x$. Combining this with Proposition 4.5, we see that, for $u \in N_{x,reg}(F)$ and $\mathcal{X}$ a positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set, the function

(4.2.2)
$$a \in T_x(F)/A_{\widetilde{G}}(F)T_{x,c} \mapsto \Gamma_{B_x}(H_{T_x}(a),\mathcal{X}) \int_{T_{x,c}} \xi(a^{-1}t^{-1}uta)\, dt$$

is supported in the compact subset (where we identify $T_x(F)/T_{x,c}$ with a subset of $\mathcal{A}_x$ via the map H_{T_x})

(4.2.3)
$$\Big(Conv\{X_{\widetilde{P}} \mid \widetilde{P} \in \mathcal{P}_{B_x}(\widetilde{M}(x))\} + {}^-\mathcal{A}_{B_x} + \mathcal{A}_{\widetilde{G}}\Big) \cap \{H \in \mathcal{A}_x \mid \langle \alpha, H\rangle \geqslant -C_2\sigma_{x,\mathrm{reg}}(u),\ \forall \alpha \in \Delta_x\}$$

of $\mathcal{A}_x/\mathcal{A}_{\widetilde{G}}$. Since the function (4.2.2) is also obviously bounded by 1, the lemma follows up to noticing the existence of $C_3 > 0$ such that the subset (4.2.3) is contained in $\mathbb{B}(C_3(\sigma_{x,\mathrm{reg}}(u) + N(\mathcal{X}))) + \mathcal{A}_{\widetilde{G}}$ for any $u \in N_{x,reg}(F)$ and for any positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$. Here for $R > 0$, we use $\mathbb{B}(R)$ to denote the ball of radius R centered at 0 in $\mathcal{A}_x$ for a given norm. □

Lemma 4.7. *There exists $C > 0$ and, for every $u \in N_{x,reg}(F)$, a unique unitary polynomial-exponential function $v_{B_x,\xi}(u,.)$ on $\mathcal{C}_{\mathbb{Q}}(\widetilde{G}, \widetilde{M}(x))$ such that for every rational $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X} \in \mathcal{C}_{\mathbb{Q}}(\widetilde{G}, \widetilde{M}(x))$ with $d(\mathcal{X}) \geqslant C\sigma(u)$ (we refer the reader to Section 2.9 for various notation), we have*

$$v_{B_x,\xi}(u, \mathcal{X}) = \widetilde{v}_{B_x,\xi}(u, \mathcal{X}).$$

Moreover, as u varies, the set of those unitary polynomial-exponential functions $\{v_{B_x,\xi}(u, .)|\ u \in N_{x,reg}(F)\}$ spans a finite dimensional vector space and there exists $C' > 0$ such that for every $u \in N_{x,reg}(F)$ and $\mathcal{X} \in \mathcal{C}_{\mathbb{Q}}(\widetilde{G}, \widetilde{M}(x))$ we have

$$|v_{B_x,\xi}(u, \mathcal{X})| \leqslant C'(\sigma_{x,reg}(u) + N(\mathcal{X}))^r.$$

Proof. Before proving the lemma, we need some preparation. For every $u \in N_{x,reg}(F)$ and $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ with Levi decomposition $Q = L_Q U_Q$ (where $M(x) \subset L_Q$), there is a unique decomposition $u = u^Q u_Q$ where $u^Q \in L_Q(F)$, $u_Q \in U_Q(F)$ and we set

$$\xi_x^{c,u,\widetilde{Q}}(t) := \int_{T_{x,c}} \xi(t^{-1}t_c^{-1}u^Q t_c t)dt_c, \quad \text{for } t \in T_x(F).$$

Then, these functions satisfy:

- For every $u \in N_{x,reg}(F)$ and $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$, $\xi^{c,u,\widetilde{Q}}$ is invariant by translation by $A_{\widetilde{Q}}(F)$;
- There exists $C_1 > 0$ such that for every $u \in N_{x,reg}(F)$, $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$, $t \in T_x(F)$ and $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ satisfying $d(\mathcal{X}) \geqslant C_1\sigma(u)$ the condition

  $$\Gamma_{B_x}^{\widetilde{Q}}(H(t), \mathcal{X})\tau_{\widetilde{Q}}(H(t) - X_{\widetilde{Q}}) \neq 0$$

  implies $\xi^{c,u,\widetilde{G}}(t) = \xi^{c,u,\widetilde{Q}}(t)$.

The first bullet point is obvious. Let's prove the second bullet point. Pick $C_1 > 0$ and let $\widetilde{Q}$, u, t, $\mathcal{X}$ be as above satisfying

$$d(\mathcal{X}) \geqslant C_1\sigma(u), \tag{4.2.4}$$

$$\Gamma_{B_x}^{\widetilde{Q}}(H(t), \mathcal{X})\tau_{\widetilde{Q}}(H(t) - X_{\widetilde{Q}}) \neq 0. \tag{4.2.5}$$

Then, we will show that provided C_1 is large enough, we have

$$\xi^{c,u,\widetilde{G}}(t) = \xi^{c,u,\widetilde{Q}}(t). \tag{4.2.6}$$

By Proposition 4.5 (applied to $\widetilde{L}_Q$ instead of $\widetilde{G}$), the condition (4.2.5) is equivalent to

$$H^{\widetilde{Q}}(t) \in Conv\{X^{\widetilde{Q}}_{\widetilde{P}} \mid \widetilde{B}_x \subset \widetilde{P} \subset \widetilde{Q}\} + {}^{-}\mathcal{A}^{\widetilde{Q}}_{B_x} \text{ and } H_{\widetilde{Q}}(t) \in X_{\widetilde{Q}} + \mathcal{A}^+_{\widetilde{Q}}$$

where $H^{\widetilde{Q}}(t)$, $H_{\widetilde{Q}}(t)$ denote the respective projections of $H(t)$ onto $\mathcal{A}^{\widetilde{Q}}_{\widetilde{M}(x)}$, $\mathcal{A}_{\widetilde{Q}}$ and we have set $\widetilde{B}_x = xB_x$. Hence, it implies that

$$H(t) = H^{\widetilde{Q}}(t) + H_{\widetilde{Q}}(t) \in Conv(X_{\widetilde{P}} \mid \widetilde{B}_x \subset \widetilde{P} \subset \widetilde{Q}) + \mathcal{A}^+_{\widetilde{Q}} + {}^{-}\mathcal{A}^{\widetilde{Q}}_{B_x}. \tag{4.2.7}$$

On the other hand, we have

$$\xi^{c,u,\widetilde{G}}(t) = \int_{T_{x,c}} \xi(t^{-1}t_c^{-1}u^Q t_c t)\xi(t^{-1}t_c^{-1}u_Q t_c t) dt_c.$$

Thus, it suffices to show that, when C_1 is sufficiently large, we have

$$\xi(t^{-1}t_c^{-1}u_Q t_c t) = 1, \quad \forall t_c \in T_{x,c}.$$

There exists $C_2 > 0$ such that this last condition is implied by the inequalities

$$\langle \alpha, H(t)\rangle \geqslant C_2\sigma(u), \quad \text{for every } \alpha \in \Delta_x \setminus \Delta_x^{\widetilde{Q}}.$$

where $\Delta_x^{\widetilde{Q}} = \Delta_{B_x}^{Q\cap B_x}$. However, as every $\alpha \in \Delta_x \setminus \Delta_x^{\widetilde{Q}}$ takes non-negative values on ${}^{-}\mathcal{A}^{\widetilde{Q}}_{B_x}$ and on $\mathcal{A}^+_{\widetilde{Q}}$, (4.2.7) implies

$$\langle \alpha, H(t)\rangle \geqslant \min_{\widetilde{B}_x \subset \widetilde{P} \subset \widetilde{Q}} \langle \alpha, X_{\widetilde{P}}\rangle \geqslant d(\mathcal{X}), \quad \text{for every } \alpha \in \Delta_x \setminus \Delta_x^{\widetilde{Q}}. \tag{4.2.8}$$

Therefore, taking $C_1 \geqslant C_2$ gives the required identity.

Let us now prove the lemma. Fix a lattice $\Lambda \subset \mathcal{A}_{x,\mathbb{Q}}$. Then, we can find a constant $C_2 > 0$ and for every $u \in N_x(F)$, a Λ-rational orthogonal set $\mathcal{X}_u = (X_{u,\widetilde{P}})_{\widetilde{P} \in \mathcal{P}(\widetilde{M}(x))} \in \mathcal{C}_\Lambda(\widetilde{G}, \widetilde{M}(x))$ such that $d(\mathcal{X}_u) \geq C_1\sigma(u)$ and $N(\mathcal{X}_u) \leq C_2\sigma(u)$. Obviously, it suffices to show that for every $u \in N_{x,reg}(F)$, the function

$$\mathcal{Y} \in \mathcal{C}_{\mathbb{Q}}(\widetilde{G}, \widetilde{M}(x)) \mapsto \tilde{v}_{B_x,\xi}(u, \mathcal{X}_u + \mathcal{Y})$$

coincides, for $\mathcal{Y}$ positive, with a unitary polynomial-exponential function. Applying the splitting formula of Lemma 4.2 as well as the two bullet points above, we obtain

$$\tilde{v}_{B_x,\xi}(u, \mathcal{X}_u + \mathcal{Y}) = \sum_{\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))} \int_{T_x(F)/A_{\widetilde{G}}(F)} \xi^{c,u,\widetilde{G}}(t) \Gamma^{\widetilde{Q}}_{B_x}(H(t), \mathcal{X}_u) \Gamma_{\widetilde{Q}}(H(t) - X_{u,\widetilde{Q}}, Y_{\widetilde{Q}}) dt$$

$$= \sum_{\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))} \int_{T_x(F)/A_{\widetilde{G}}(F)} \xi^{c,u,\widetilde{Q}}(t) \Gamma^{\widetilde{Q}}_{B_x}(H(t), \mathcal{X}_u) \Gamma_{\widetilde{Q}}(H(t) - X_{u,\widetilde{Q}}, Y_{\widetilde{Q}}) dt$$

$$= \sum_{\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))} \int_{T_x(F)/A_{\widetilde{Q}}(F)} \xi^{c,u,\widetilde{Q}}(t) \Gamma^{\widetilde{Q}}_{B_x}(H(t), \mathcal{X}_u) \int_{A_{\widetilde{Q}}(F)/A_{\widetilde{G}}(F)} \Gamma_{\widetilde{Q}}(H_{\widetilde{Q}}(at) - X_{u,\widetilde{Q}}, Y_{\widetilde{Q}}) da dt$$

for every positive $(\widetilde{G},\widetilde{M}(x))$-orthogonal set $\mathcal{Y}$ and where the second equality is based on the fact that since $Y_{\widetilde{Q}}\in\mathcal{A}^+_{\widetilde{Q}}$, $\Gamma_{\widetilde{Q}}(H-X_{u,\widetilde{Q}},Y_{\widetilde{Q}})\neq 0$ implies $\tau_{\widetilde{Q}}(H-X_{u,\widetilde{Q}})\neq 0$. For $\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$, the function

$$t\in T_x(F)/A_{\widetilde{Q}}(F)\mapsto \xi^{c,u,\widetilde{Q}}(t)\Gamma^{\widetilde{Q}}_{B_x}(H(t),\mathcal{X}_u)$$

is compactly supported so that in the above expression the integral over $T_x(F)/A_{\widetilde{Q}}(F)$ can be written as a finite sum. On the other hand for every fixed $t\in T_x(F)$, the function

$$Y_{\widetilde{Q}}\in\mathcal{A}_{\widetilde{Q},\mathbb{Q}}\mapsto\int_{A_{\widetilde{Q}}(F)/A_{\widetilde{G}}(F)}\Gamma_{\widetilde{Q}}(H_{\widetilde{Q}}(at)-X_{u,\widetilde{Q}},Y_{\widetilde{Q}})da$$

is a unitary polynomial-exponential function and the set of these functions, as $t\in T_x(F)$ and $X_{u,\widetilde{Q}}\in\Lambda$ vary, spans a finite dimensional vector space. This shows the lemma except for the last estimate.

By the above computation, we have

$$v_{B_x,\xi}(u,\mathcal{X}_u+\mathcal{Y})=\sum_{\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))}\int_{T_x(F)/A_{\widetilde{Q}}(F)}\xi^{c,u,\widetilde{Q}}(t)\Gamma^{\widetilde{Q}}_{B_x}(H(t),\mathcal{X}_u)\int_{A_{\widetilde{Q}}(F)/A_{\widetilde{G}}(F)}\Gamma_{\widetilde{Q}}(H_{\widetilde{Q}}(at)-X_{u,\widetilde{Q}},Y_{\widetilde{Q}})dadt$$

for every $u\in N_{x,\mathrm{reg}}(F)$ and $\mathcal{Y}\in\mathcal{C}_{\mathbb{Q}}(\widetilde{G},\widetilde{M}(x))$. However, the integral

$$\int_{A_{\widetilde{Q}}(F)/A_{\widetilde{G}}(F)}|\Gamma_{\widetilde{Q}}(H_{\widetilde{Q}}(at)-X_{u,\widetilde{Q}},Y_{\widetilde{Q}})|da$$

is essentially bounded by $(1+N(\mathcal{X}_u)+N(\mathcal{Y}))^{a^{\widetilde{G}}_{\widetilde{Q}}}$ whereas by a similar reasoning as in the proof of Lemma 4.6, the integral

$$\int_{T_x(F)/A_{\widetilde{Q}}(F)}\left|\xi^{c,u,\widetilde{Q}}(t)\Gamma^{\widetilde{Q}}_{B_x}(H(t),\mathcal{X}_u)\right|dt$$

is essentially bounded by $(N(\mathcal{X}_u)+\sigma_{x,\mathrm{reg}}(u))^{r-a^{\widetilde{G}}_{\widetilde{Q}}}$. Since $N(\mathcal{X}_u)\ll\sigma(u)$ this shows that $|v_{B_x,\xi}(u,\mathcal{X}_u+\mathcal{Y})|\ll(\sigma_{x,\mathrm{reg}}(u)+N(\mathcal{X}_u+\mathcal{Y}))^r$ and the lemma is proved. □

For $g\in G(F)$, applying the above definition to the $(\widetilde{G},\widetilde{M}(x))$-orthogonal set $\mathcal{Y}(g)=(H_{\widetilde{P}}(g))_{\widetilde{P}\in\mathcal{P}(\widetilde{M}(x))}$, we define the weight

$$v_{B_x,\xi}(u,g)=v_{B_x,\xi}(u,\mathcal{Y}(g)).$$

It satisfies the relation

(4.2.9) $\quad v_{B_x,\xi}(u,bg)=v_{B_x,\xi}(b^{-1}ub,g)$ for every $(u,b,g)\in N_{x,\mathrm{reg}}(F)\times B_x(F)\times G(F)$.

4.3 A formula of regular germs for quasi-characters

Theorem 4.8. *For every strongly cuspidal function $f \in \mathcal{C}(\widetilde{G}(F))$, we have* [8]

$$c_{f,-\mathcal{O}_\xi}(x) = \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} f(g^{-1}xug) v_{B_x,\xi}(u,g) du dg.$$

Let us remark that thanks to (4.2.9), the expression in the right-hand side of the above theorem makes sense formally. We will check its absolute convergent in the next subsection.

4.4 Some estimates

In this subsection we prove some estimates that in particular imply the convergence of the right-hand side of Theorem 4.8.

Let S be the connected center of G_x (a torus) and set $\widetilde{S} = Sx$. Let $\widetilde{S}'$ the open subset of those $s \in \widetilde{S}$ such that $G_s = G_x$.

Recall that $r = \dim(A_x) - \dim(A_{\widetilde{G}})$ and $\sigma_{x,\mathrm{reg}}$ denotes a log-norm on $N_{x,\mathrm{reg}} = N_x \cap G_{x,\mathrm{reg}}$. We fix log-norms σ_{reg} and $\sigma_{\widetilde{S}'}$ on $\widetilde{G}_{\mathrm{reg}}(F)$ and $\widetilde{S}'(F)$ respectively.

Lemma 4.9. *We have inequalities*

$$\inf_{b \in B_x(F)} (\sigma_{x,\mathrm{reg}}(bub^{-1}) + \sigma(bg)) \ll \sigma_{\mathrm{reg}}(g^{-1}sug) + \sigma_{\widetilde{S}'}(s), \tag{4.4.1}$$

and

$$\inf_{b \in B_x(F)} (\sigma(bub^{-1}) + \sigma(bg)) \ll \sigma(g^{-1}sug) + \sigma_{\widetilde{S}'}(s) \tag{4.4.2}$$

for $(s,u,g) \in \widetilde{S}'(F) \times N_{x,\mathrm{reg}}(F) \times G(F)$.

Proof. Let $\mathcal{N}_x \subset G_x$ be the unipotent cone and $\mathcal{N}_x \times^{G_x} G$ be the quotient of $\mathcal{N}_x \times G$ by the free action of G_x given by $g_x \cdot (u,g) = (g_x u g_x^{-1}, g_x g)$. Then, the regular map

$$\mathcal{N}_x \times^{G_x} G \times \widetilde{S}' \to \widetilde{G} \times \widetilde{S}', \;\; (u,g,s) \mapsto (g^{-1}sug, s)$$

is a closed embedding with image the subset of those $(\gamma, s) \in \widetilde{G} \times \widetilde{S}'$ such that the semisimple part of γ is in the same geometric conjugacy class as s. Let $\mathcal{N}_{x,\mathrm{reg}} = \mathcal{N}_x \cap G_{x,\mathrm{reg}}$ be the open subset of regular unipotent elements. Then, the previous map restricts to a closed embedding $\mathcal{N}_{x,\mathrm{reg}} \times^{G_x} G \times \widetilde{S}' \to \widetilde{G}_{\mathrm{reg}} \times \widetilde{S}'$. Furthermore, the natural map $\mathcal{N}_x \times^{B_x} G \to \mathcal{N}_x \times^{G_x} G$ is proper and N_x (resp. $N_{x,\mathrm{reg}}$) is a closed subset of $\mathcal{N}_x$ (resp. $\mathcal{N}_{x,\mathrm{reg}}$). It follows that the two regular maps

$$N_x \times^{B_x} G \times \widetilde{S}' \to \widetilde{G} \times \widetilde{S}' \text{ and } N_{x,\mathrm{reg}} \times^{B_x} G \times \widetilde{S}' \to \widetilde{G}_{\mathrm{reg}} \times \widetilde{S}'$$

[8] $-\mathcal{O}_\xi$ is the same as $\mathcal{O}_{\xi^{-1}}$

are proper (the second one being actually a closed embedding). By Lemma 2.1, we have

$$\sigma_{N_x\times^{B_x}G}(u,g)\ll\sigma(g^{-1}sug)+\sigma_{\widetilde{S}'}(s),\quad \sigma_{N_{x,\mathrm{reg}}\times^{B_x}G}(u,g)\ll\sigma_{\mathrm{reg}}(g^{-1}sug)+\sigma_{\widetilde{S}'}(s)$$

for $(u,s,g)\in N_{x,\mathrm{reg}}(F)\times\widetilde{S}'(F)\times G(F)$. It remains to check that

$$\sigma_{N_x\times^{B_x}G}(u,g)\sim\inf_{b\in B_x(F)}(\sigma(bub^{-1})+\sigma(bg))\text{ and }\sigma_{N_{x,\mathrm{reg}}\times^{B_x}G}(u,g)\sim\inf_{b\in B_x(F)}(\sigma_{x,\mathrm{reg}}(bub^{-1})+\sigma(bg))$$

for $(u,g)\in N_{x,\mathrm{reg}}(F)\times G(F)$ i.e. that the two natural projections $N_x\times G\to N_x\times^{B_x}G$ and $N_{x,\mathrm{reg}}\times G\to N_{x,\mathrm{reg}}\times^{B_x}G$ have the norm descent property. Since both are pullbacks of the projection $G\to B_x\backslash G$, it suffices to check that the latter has the norm descent property.

Choose $P=M(x)U\in\mathcal{P}_{B_x}(M(x))$ a parabolic subgroup with Levi $M(x)$ such that $P\cap G_x=B_x$. As $P\backslash G$ is proper, we just need to check that $P\to B_x\backslash P$ has the norm descent property. Let $\pi:B_x\backslash P\to T_x\backslash M(x)$ be the natural map. According to Kottwitz, $M(x)\to T_x\backslash M(x)$ already has the norm descent property. Thus, for every $p\in P(F)$, we can find $m\in M(x)(F)$ such that $p\in T_x(F)U(F)m$ and

$$\sigma(m)\sim\sigma_{T_x\backslash M(x)}(\pi(B_xp))\ll\sigma_{B_x\backslash P}(B_xp).$$

Choose $C>0$ large enough such that for all $p\in P(F)$, there exists $p'=u'm'\in B_x(F)p$ with $u'\in U(F)$ and $m'\in M(x)(F)$ such that

$$\sigma(m')\leq C\sigma_{B_x\backslash P}(B_xp).$$

Fix another constant $C_1>0$ large enough (with respect to C). If $\sigma_{N_x\backslash U}(u')\leq C_1\sigma_{B_x\backslash P}(B_xp)$, then since $U\to N_x\backslash U$ has the norm descent property (this is because this quotient map admits a regular section), there exists $n\in N_x(F)$ such that

$$\sigma(nu'm')\leq 2C_1\sigma_{B_x\backslash P}(B_xp).$$

This implies that

$$\inf_{b\in B_x(F)}\sigma(bp)\leq 2C_1\sigma_{B_x\backslash P}(B_xp).$$

If $\sigma_{N_x\backslash U}(u')>C_1\sigma_{B_x\backslash P}(B_xp)$, since $N_x\backslash U\to B_x\backslash P$ is a closed embedding and since $\sigma(m')\leq C\sigma_{B_x\backslash P}(B_xp)$, we have

$$\sigma_{B_x\backslash P}(p)=\sigma_{B_x\backslash P}(u'm')\geq\frac{1}{2}\sigma_{B_x\backslash P}(u')=\frac{1}{2}\sigma_{N_x\backslash U}(u')$$

and

$$\inf_{b\in B_x(F)}\sigma(bp)\leq\inf_{n\in N_x(F)}\sigma(nu'm')\leq 2\sigma_{N_x\backslash U}(u').$$

This implies that

$$\inf_{b\in B_x(F)}\sigma(bp)\leq 4\sigma_{B_x\backslash P}(B_xp).$$

As a result we have proved that the map $P\to B_x\backslash P$ has the norm descent property and this finishes the proof of the lemma. $\square$

Corollary 4.10. *We have*

$$|v_{B_x,\xi}(u,g)| \ll (\sigma_{\mathrm{reg}}(g^{-1}sug) + \sigma_{\widetilde{S}'}(s))^r$$

for every $u \in N_{x,reg}(F)$, $g \in G(F)$ *and* $s \in \widetilde{S}'(F)$.

Proof. According to Lemma 4.7, we have $|v_{B_x,\xi}(u,g)| \ll (\sigma_{x,\mathrm{reg}}(u) + \sigma(g))^r$. Combining this with the equation (4.2.9) and the previous lemma, we obtain

$$|v_{B_x,\xi}(u,g)| \ll \inf_{b\in B_x(F)} (\sigma_{x,\mathrm{reg}}(bub^{-1}) + \sigma(bg))^r \ll (\sigma_{\mathrm{reg}}(g^{-1}sug) + \sigma_{\widetilde{S}'}(s))^r$$

for $u \in N_{x,reg}(F)$, $g \in G(F)$ and $s \in \widetilde{S}'(F)$. □

Proposition 4.11. *Let* $r_0 > 0$. *Then, for every* $f \in \mathcal{C}(\widetilde{G}(F))$ *and every* $d > 0$, *we have*

$$D^{\widetilde{G}}(s)^{1/2} \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} |f(g^{-1}sug)|\sigma_{\mathrm{reg}}(g^{-1}sug)^{r_0} du dg \ll_{f,d} \sigma(s)^{-d}\sigma_{\widetilde{S}'}(s)^{r_0} \quad (4.4.3)$$

for $s \in \widetilde{S}'(F)$. *In particular, the integral in Theorem 4.8 is absolutely convergent.*

Proof. Let $K \subset G(F)$ be a compact-open subgroup. First, we show that

$$\mathrm{vol}(gKg^{-1} \cap N_x(F))^{-1} \int_{gKg^{-1}\cap N_x(F)} \sigma_{\mathrm{reg}}(g^{-1}sukg)^{r_0} dk \ll (\sigma(g^{-1}sug) + \sigma_{\widetilde{S}'}(s))^{r_0} \quad (4.4.4)$$

for every $(g,s,u) \in G(F) \times \widetilde{S}'(F) \times N_x(F)$. Since the left-hand side of the inequality is invariant by the transformation $(g,s,u) \mapsto (bg,s,bub^{-1})$, by Lemma 4.9 it suffices to establish that

$$\mathrm{vol}(gKg^{-1} \cap N_x(F))^{-1} \int_{gKg^{-1}\cap N_x(F)} \sigma_{\mathrm{reg}}(g^{-1}sukg)^{r_0} dk \ll (\sigma(g) + \sigma(u) + \sigma_{\widetilde{S}'}(s))^{r_0}$$

for $(g,s,u) \in G(F) \times \widetilde{S}'(F) \times N_x(F)$. Note that $\sigma_{\mathrm{reg}}(g^{-1}sug) \ll \sigma(g) + \sigma_{\mathrm{reg}}(su) \ll \sigma(g) + \sigma_{\widetilde{S}'}(s) + \sigma_{x,\mathrm{reg}}(u)$. Therefore, we are reduced to show

$$\mathrm{vol}(gKg^{-1}\cap N_x(F))^{-1} \int_{gKg^{-1}\cap N_x(F)} \sigma_{x,\mathrm{reg}}(uk)^{r_0} dk \ll (\sigma(g)+\sigma(u))^{r_0} \text{ for } u \in N_x(F). \quad (4.4.5)$$

Let Δ_x be the set of simple roots of A_x in B_x and for $\alpha \in \Delta_x$, let $\mathfrak{n}_{x,\alpha} \subset \mathfrak{n}_x$ be the corresponding root subspace. Then, we have a natural projection $\mathfrak{n}_x \to \mathfrak{n}_{x,\alpha}$ and for $u \in N_x$, we denote by $\log(u)_\alpha$ the image of $\log(u)$ in $\mathfrak{n}_{x,\alpha}$ where $\log : N_x \to \mathfrak{n}_x$ denotes the logarithmic map (a regular morphism). Fix an ultrametric norm $\|.\|$ on $\mathfrak{n}_x(F)$ and set $v(.) = -\log\|.\|$. Then, we have

$$\sigma_{x,reg}(u)^{r_0} \sim \sigma(u)^{r_0} + \sum_{\alpha\in\Delta_x} \max(1, v(\log(u)_\alpha))^{r_0} \text{ for } u \in N_{x,\mathrm{reg}}(F).$$

Thus, to show (4.4.5) it suffices to bound the integral

$$\text{vol}(gKg^{-1} \cap N_x(F))^{-1} \int_{gKg^{-1}\cap N_x(F)} \max(1, v(\log(u)_\alpha + \log(k)_\alpha))^{r_0} dk \tag{4.4.6}$$

by a constant times $(\sigma(u) + \sigma(g))^{r_0}$. For this, we remark that there exists $C > 0$ such that the image of $gKg^{-1} \cap N_x(F)$ in $\mathfrak{n}_{x,\alpha}(F)$ contains the ball

$$\mathbb{B}(C\sigma(g)) := \{X \in \mathfrak{n}_{x,\alpha}(F) \mid v(X) \geq C\sigma(g)\}$$

for every $g \in G(F)$. Therefore, since $\sigma(uk) \ll \sigma(u) + \sigma(g)$ for every $g \in G(F)$, $u \in N_x(F)$ and $k \in gKg^{-1}$, the desired estimate for (4.4.6) follows from the elementary inequality

$$\text{vol}(\mathbb{B}(R))^{-1} \int_{\mathbb{B}(R)} \max(1, v(X+Y))^{r_0} dY \ll R^{r_0} \text{ for every } R \geq 1, X \in \mathfrak{n}_{x,\alpha}(F),$$

and this ends the proof of (4.4.4).

From (4.4.4) applied to some compact-open subgroup $K \subset G(F)$ leaving f invariant in the right, we get that the left-hand side of (4.4.3) is essentially bounded by

$$\sigma_{\widetilde{S}'}(s)^{r_0} D^{\widetilde{G}}(s)^{1/2} \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} |f(g^{-1}sug)|\sigma(g^{-1}sug)^{r_0} dudg.$$

Note that the function $\sigma^{r_0}|f|$ belongs to the Harish-Chandra Schwartz space $\mathcal{C}(G(F))$. Therefore, up to replacing f by this function, it suffices to show that for every $d > 0$ we have

$$D^{\widetilde{G}}(s)^{1/2} \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} |f(g^{-1}sug)| dudg \ll_d \sigma(s)^{-d} \tag{4.4.7}$$

for $s \in \widetilde{S}'(F)$.

Pick a parabolic subspace $\widetilde{P} = \widetilde{M}(x)U \in \mathcal{P}_{B_x}(\widetilde{M}(x))$ with $P \cap G_x = B_x$ as well as a compact subgroup $K \subset G(F)$ such that $G(F) = P(F)K$. Then, by the usual change of variable the last integral above is equal to

$$D^{\widetilde{M}(x)}(s)^{1/2} \delta_{\widetilde{P}}(s)^{1/2} \int_{T_x(F)\backslash M(x)(F)} \int_K \int_{U(F)} |f(k^{-1}m^{-1}smuk)| dudkdm.$$

Thus, since the function $m \in \widetilde{M}(x)(F) \mapsto \delta_{\widetilde{P}}(m)^{1/2} \int_K \int_{U(F)} |f(k^{-1}muk)| du$ is Harish-Chandra Schwartz [33, Proposition II.4.5], the estimate (4.4.7) is now a consequence of [7, Lemma 2.9.2]. □

4.5 Definition of a sequence of test functions

As a preparation for the proof of Theorem 4.8 we introduce a sequence of functions $\phi_n \in C_c^\infty(G_x(F))$ as follows, the construction being inspired from [27].

First, let Ξ be the unique element of $\overline{\mathfrak{b}}_x^{\perp}(F) \subset \mathfrak{g}^*(F)$ (where $\overline{B}_x = T_x\overline{N}_x$ denotes the Borel opposite to B_x) such that for every $X \in \mathfrak{n}_x(F)$ we have $\xi(\exp(X)) = \psi(\langle \Xi, X\rangle)$. Then, $\Xi \in \mathcal{O}$ (by definition of the generic character ξ) and, denoting by $\mathfrak{g}_{x,\Xi}$ the centralizer of Ξ in $\mathfrak{g}_x$, we have $\mathfrak{n}_x \cap \mathfrak{g}_{x,\Xi} = 0$. Moreover, the image of $\mathfrak{n}_x(F)$ in the quotient $\mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F)$ is maximal isotropic with respect to the bicharacter

$$(X,Y) \in \mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F) \mapsto \psi(\langle \Xi, [X,Y]\rangle). \tag{4.5.1}$$

Let $L \subset \mathfrak{g}_x(F)$ be a lattice such that:

- The image L^ξ of L in $\mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F)$ is self-dual with respect to the bicharacter (4.5.1) i.e. $L^\xi = \{X \in \mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F) \mid \psi(\langle \Xi, [X,Y]\rangle) = 1\ \forall Y \in L^\xi\}$;
- The preimage of L^ξ in $\mathfrak{n}_x(F)$ is $\mathfrak{n}_x(F) \cap L$.

We then choose an integer $n_0 > 0$ large enough such that:

(4.5.2)
- The exponential map $\exp : \mathfrak{g}_x(F) \to G_x(F)$ is well-defined on $\varpi^{n_0}L$;
- For every $n \geq n_0$, $m \geq n_0$, $X \in \varpi^n L$ and $Y \in \varpi^m L$, we have $e^X e^Y \in \exp(X + Y + \varpi^{m+n-n_0}L)$;
- $[L, L] \subset \varpi^{-n_0}L$;
- The restriction of ξ to $\varpi^{n_0}L$ is trivial;
- For every $n \geq n_0$, $m \in \mathbb{Z}$, $Y \in \varpi^n L$, $X \in \varpi^m L$, we have

$$\mathrm{Ad}_{e^Y}(X) - X - [Y,X] \in \varpi^{2n+m-n_0}L.$$

That the last point above is satisfied for n_0 large enough is a consequence of the series expansion $\mathrm{Ad}_{e^Y} X = \sum_{k\geq 0} \frac{\mathrm{ad}_Y^k(X)}{k!}$ (valid for Y small enough).

For every integer $n \geqslant n_0$, we set

- $a_n = (2\rho_x^\vee)(\varpi)^n$ where $\rho_x^\vee$ denotes half the sum of the positive coroots of A_x with respect to B_x,;
- $L_n = (Ad_{a_n})^{-1}\varpi^n L$, $K_n = \exp(L_n)$ and $K_n' = \exp(\varpi^n L)$;
- $\xi_n : K_n \to \mathbb{C}^\times$, $\xi_n' : K_n' \to \mathbb{C}^\times$ the locally constant functions defined by $\xi_n(\exp(X)) = \psi(\langle \Xi, X\rangle)$ and $\xi_n'(\exp(Y)) = \psi(\langle \Xi, \varpi^{-2n}Y\rangle)$ for every $X \in L_n$ and $Y \in \varpi^n L$ respectively.

Note that, by the second condition on n_0, K_n and K_n' are compact-open subgroups of $G_x(F)$. Moreover, we have $K_n' = a_n K_n a_n^{-1}$ and $\xi_n'(a_n k a_n^{-1}) = \xi_n(k)$ for every $k \in K_n$ which follows from the fact that $\mathrm{Ad}^*_{a_n}\Xi = \varpi^{-2n}\Xi$. From the last condition on n_0, we also deduce that the function ξ_n (resp. ξ_n') is K_n-invariant (resp. K_n'-invariant) by conjugation[9]

[9]Actually, for n large enough and if the residue characteristic is different from 2, it can be shown that ξ_n and ξ_n' are characters of K_n and K_n' respectively. But we will not need this fact in the sequel.

We fix Haar measures on $\mathfrak{g}_x(F)$ and $\mathfrak{n}_x(F)$ compatibly with the measures on $G_x(F)$ and $N_x(F)$ i.e. such that exponential maps are locally measure preserving. Using the additive character ψ, we can identify $\mathfrak{n}_x^\perp(F)$ with the Pontryagin dual of $\mathfrak{g}_x(F)/\mathfrak{n}_x(F)$, and we endow $\mathfrak{n}_x^\perp(F)$ with the dual of the quotient measure on the latter. This is the only invariant measure on $\mathfrak{n}_x^\perp(F)$ such that for every lattice $\Lambda \subset \mathfrak{g}_x(F)$ we have $\mathrm{vol}(\Lambda^\perp \cap \mathfrak{n}_x^\perp(F)) = \mathrm{vol}(\Lambda \cap \mathfrak{n}_x(F))\,\mathrm{vol}(\Lambda)^{-1}$.

For $n \geq n_0$, we let $\varphi_n \in C_c^\infty(\mathfrak{g}_x(F))$ be the function that is equal to $X \mapsto \mathrm{vol}(L_n)^{-1}\psi(\langle \Xi, X\rangle)$ on L_n and is equal to zero outside. We define similarly $\phi_n \in C_c^\infty(G_x(F))$ as the function that coincides with $\mathrm{vol}(K_n)^{-1}\xi_n$ on K_n and is equal to zero outside. Note that, $\phi_n(\exp(X)) = \varphi_n(X)$ for every $X \in L_n$ and that the Fourier transform $\widehat{\varphi}_n$ is the characteristic function of the coset $-\xi + L_n^\perp$. Here the Fourier transform is normalized as in Subsection 2.6 and for any $\mathcal{O}_F$-lattice $\mathcal{L} \subset \mathfrak{g}(F)$, we denote by $\mathcal{L}^\perp$ the dual lattice in $\mathfrak{g}^*(F)$ with respect to ψ, that is

$$\mathcal{L}^\perp = \{Y \in \mathfrak{g}^*(F) \mid \psi(\langle Y, X\rangle) = 1,\ \forall X \in \mathcal{L}\}.$$

Lemma 4.12. *For n large enough, the following hold.*

(i) $\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F))$ *is invariant under the conjugation of* $K_n \cap N_x(F)$.

(ii) For $u \in N_x(F)$, *if* $u^{-1}(\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)))u \cap \Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)) \neq \emptyset$, *then* $u \in K_n \cap N_x(F)$.

(iii) For every $f \in C_c^\infty(\mathfrak{g}_x(F))$, *we have*

$$\int_{\mathfrak{g}_x(F)} \varphi_n(X) f(X) dX = \mathrm{vol}(K_n)^{-1}\,\mathrm{vol}(K_n \cap N_x(F))^{-1} \int_{K_n} \int_{\Xi + \mathfrak{n}_x^\perp(F)} \mathbf{1}_{\Xi + L_n^\perp}(Y) \widehat{f}(kYk^{-1}) dY dk.$$

(iv) For every $\mathcal{O}' \in \mathrm{Nil}(\widehat{\mathfrak{g}}_x)$, *the coadjoint orbital integral of* $\widehat{\varphi}_n$ *on* $\mathcal{O}'$ *(normalized using the Kirillov-Kostant measure as in Subsection 2.6) is given by*

$$\int_{\mathcal{O}'} \widehat{\varphi}_n(Y) dY = \begin{cases} 1 & \text{if } \mathcal{O}' = -\mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) We have

$$\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)) = \left(\Xi + \mathfrak{n}_x^\perp(F)\right) \cap \left(\Xi + L_n^\perp\right).$$

Furthermore, since Ξ restricts to a $N_x(F)$-invariant character on $\mathfrak{n}_x(F)$, $\Xi + \mathfrak{n}_x^\perp(F)$ is $N_x(F)$-invariant whereas, since the function ξ_n is K_n-invariant, $\Xi + L_n^\perp$ is K_n-invariant. The claim follows.

(ii) Let $u \in N_x(F)$ and set $X = \log(u) \in \mathfrak{n}_x(F)$. After conjugating everything by a_n, the statement is equivalent to

$$u^{-1}(\Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp))u \cap \Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp) \neq \emptyset \Rightarrow X \in \varpi^n L.$$

By the theory of Kostant section, the map

$$N_x \times \Xi + \mathfrak{n}_x^\perp \to \left(\Xi + \mathfrak{n}_x^\perp\right) \times \left(\Xi + \mathfrak{n}_x^\perp\right),\ (n, X) \mapsto (X, n^{-1}Xn)$$

is a closed embedding. Hence, for any $m > 0$ we can choose n large enough such that if

$$u^{-1}(\Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp))u \cap \Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp) \neq \emptyset,$$

then $X \in \varpi^m L$. Now let k be the largest integer such that $X \in \varpi^k L$. We know that $k \geq m$ and we need to show $k \geq n$. There exists some absolute constant $C \in \mathbb{N}$ such that

$$u^{-1}\Xi u \in \Xi - [X, \Xi] + \varpi^{2k-C}L^\perp,\ u^{-1}(\varpi^n L^\perp)u \subset \varpi^n L^\perp + \varpi^{n+k-C}L^\perp.$$

If $n > k$, once we choose m to be large enough (with respect to C), the above relations imply that

$$u^{-1}(\Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp(F)))u \subset \Xi + [X, \Xi] + (\varpi^{k+1}L^\perp \cap \mathfrak{n}_x^\perp(F)).$$

Since the image of L in $\mathfrak{g}_x/\mathfrak{g}_{x,\Xi}$ is self-dual, we know that $[X, \Xi] \in \varpi^k L^\perp \cap \mathfrak{n}_x^\perp(F)$ and $[X, \Xi] \notin \varpi^{k+1}L^\perp \cap \mathfrak{n}_x^\perp(F)$. This implies that

$$u^{-1}(\Xi + (\varpi^n L^\perp \cap \mathfrak{n}_x^\perp(F)))u \cap \Xi + (\varpi^{k+1}L^\perp \cap \mathfrak{n}_x^\perp(F)) = \emptyset$$

which is a contradiction. Hence we must have $k \geq n$ and this proves the lemma.

(iii) Let D be the distribution on $\mathfrak{g}_x(F)$ defined by

$$D(f) = \int_{K_n} \int_{\Xi + \mathfrak{n}_x^\perp(F)} \mathbf{1}_{\Xi + L_n^\perp}(Y)\widehat{f}(kYk^{-1})dYdk,\quad f \in C_c^\infty(\mathfrak{g}_x(F)).$$

Then, it has the following properties:

(a) It is K_n-invariant: $D({}^k f) = D(f)$ for every $(k, f) \in K_n \times C_c^\infty(\mathfrak{g}_x(F))$;

(b) It is supported in $\mathrm{Ad}(K_n)(L_n + \mathfrak{n}_x(F))$: this follows from the Fourier inversion formula

$$\int_{\Xi + \mathfrak{n}_x^\perp(F)} \mathbf{1}_{\Xi + L_n^\perp}(Y)\widehat{f}(kYk^{-1})dY = \int_{L_n + \mathfrak{n}_x(F)} f(k^{-1}Xk)\psi(\langle \Xi, X\rangle)dX;$$

(c) It is (L_n, ξ_n)-equivariant: $D(L(X)f) = \psi(\langle \Xi, X\rangle)D(f)$ for every $(X, f) \in L_n \times C_c^\infty(\mathfrak{g}_x(F))$ (this is a consequence of the same Fourier inversion formula and the fact that the restriction of $\psi(\langle \Xi, .\rangle)$ to L_n is K_n-invariant).

We claim:

(4.5.3) For n large enough, every distribution on $\mathfrak{g}_x(F)$ satisfying the properties (a), (b) and (c) above is proportional to the distribution

$$f \in C_c^\infty(\mathfrak{g}_x(F)) \mapsto \int_{\mathfrak{g}_x(F)} \varphi_n(X) f(X) dX.$$

Indeed, every distribution D verifying (a), (b), (c) is represented by a function $F \in C^\infty(\mathfrak{g}_x(F))$ which is K_n-invariant, satisfies $F(Y+X) = \psi(\langle \Xi, X\rangle)F(Y)$ for $(X,Y) \in L_n \times \mathfrak{g}_x(F)$ and is supported in $\mathrm{Ad}(K_n)(L_n + \mathfrak{n}_x(F))$. It then suffices to show that such a function is necessarily supported in L_n which would be a consequence of the following property: for every $X \in \mathfrak{n}_x(F) \setminus (L_n \cap \mathfrak{n}_x(F))$, we can find $k \in K_n$ such that $\mathrm{Ad}_k X - X \in L_n$ and $\psi(\langle \Xi, \mathrm{Ad}(k)X - X\rangle) \neq 1$. Conjugating everything by a_n, this property can be restated as:

(4.5.4) Provided n is large enough, for every $X \in \mathfrak{n}_x(F) \setminus (L \cap \mathfrak{n}_x(F))$, we can find $k \in K_n'$ such that $\mathrm{Ad}(k)X - X \in L$ and $\psi(\varpi^{-n}\langle \Xi, \mathrm{Ad}_k X - X\rangle) \neq 1$.

Indeed, let $X \in \mathfrak{n}_x(F)\setminus(L\cap\mathfrak{n}_x(F))$ and set $-k = val_L(X) < 0$. Set $m = \max(n, k+2n_0)$. Then, for every $Y \in \varpi^m L$, by the last and third points of (4.5.2) respectively, we have

$$\mathrm{Ad}_{e^Y}(X) - X \in [Y,X] + \varpi^{2m-k-n_0}L \subset [Y,X] + \varpi^{n+n_0}L$$

and

$$[Y,X] \in \varpi^{m-k-n_0}L \subset \varpi^{n_0}L.$$

Thus, $\mathrm{Ad}_{e^Y}(X) - X \in L$ and, by the fourth point of (4.5.2),

$$\psi(\varpi^{-n}\langle \Xi, \mathrm{Ad}_{e^Y}(X) - X\rangle) = \psi(\varpi^{-n}\langle \Xi, [Y,X]\rangle)$$

for every $Y \in \varpi^m L$. Since $\exp(\varpi^m L) \subset K_n'$ it therefore suffices to find $Y \in \varpi^m L$ such that $\psi(\varpi^{-n}\langle \Xi, [Y,X]\rangle) \neq 1$ (provided n is large enough). However, if $\psi(\varpi^{-n}\langle \Xi, [Y,X]\rangle) = 1$ for all $Y \in \varpi^m L$ then, since the lattice $L^\xi \subset \mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F)$ is self-dual, the image of X in $\mathfrak{g}_x(F)/\mathfrak{g}_{x,\Xi}(F)$ belongs to $\varpi^{n-m}L^\xi$. As the preimage of L^ξ in $\mathfrak{n}_x(F)$ is $\mathfrak{n}_x(F) \cap L$, this would imply $Y \in \varpi^{n-m}L$ hence $n - m \leq -k$ or equivalently $n \leq m - k = \max(n-k, 2n_0)$. This last inequality is obviously false for $n > 2n_0$ so that the claim (4.5.4) is satisfied for such a n.

This shows (4.5.3). As a consequence, we can find a constant c such that

$$\int_{\mathfrak{g}_x(F)} \varphi_n(X) f(X) dX = c \int_{K_n} \int_{\Xi + \mathfrak{n}_x^\perp(F)} \mathbf{1}_{\Xi + L_n^\perp}(Y) \widehat{f}(kYk^{-1}) dY dk$$

for every $f \in C_c^\infty(\mathfrak{g}_x(F))$. Plugging in $f = \overline{\varphi_n}$, we have $\widehat{f} = \mathbf{1}_{\Xi + L_n^\perp}$ and we obtain

$$\mathrm{vol}(L_n)^{-1} = c \,\mathrm{vol}(K_n)\, \mathrm{vol}(L_n^\perp \cap \mathfrak{n}_x(F)^\perp).$$

By our choice of measures, we have

$$\mathrm{vol}(L_n)\,\mathrm{vol}(L_n^\perp \cap \mathfrak{n}_x(F)^\perp) = \mathrm{vol}(L_n \cap \mathfrak{n}_x(F)) = \mathrm{vol}(K_n \cap N_x(F))$$

and therefore $c = \mathrm{vol}(K_n)^{-1}\,\mathrm{vol}(K_n \cap N_x(F))^{-1}$ as claimed.

(iv) This follows from the computation in the middle of p.437 of [27].

□

4.6 Application of the local trace formula

We now start the proof of Theorem 4.8 which will be finished in the next subsection. The proof for general reductive twisted spaces is basically the same as in the untwisted case, i.e. when $\widetilde{G} = G$, the only difference is to replace the local trace formula in [2] (resp. Howe's conjecture for weighted orbital integrals in [3, Lemma 8.2] [4]) by the local twisted trace formula in [28] (resp. Howe's conjecture for twisted weighted orbital integrals in Appendix B). Hence to simplify notation, we will only write the proof when $\widetilde{G} = G$. For further simplification of notation, we will also assume that the split center A_G is trivial.

We need to recall some material on the local trace formula from [2]. Fix a minimal Levi subgroup M_{min} of G as well as $P_{min} \in \mathcal{P}(M_{min})$ and a special maximal compact subgroup $K \subset G(F)$ in good position relative to M_{min}. We set $W = Norm_{G(F)}(M_{min})/M_{min}(F)$. Let $T \in \mathcal{A}_{\min,\mathbb{Q}}$. For $P \in \mathcal{P}(M_{min})$, we set $T_P = w_P T$ where $w_P \in W$ is the unique element such that $w_P P_{min} w_P^{-1} = P$ and, for $g_1, g_2 \in G(F)$, we define a (G, M_{min})-orthogonal set by

$$\mathcal{Y}_P(g_1, g_2, T) = T_P + H_P(g_1) - H_{\overline{P}}(g_2), \quad P \in \mathcal{P}(M_{min}),$$

where $\overline{P}$ denotes the parabolic subgroup opposite to P (with respect to $M_{\min}$). For $M \in \mathcal{L}(M_{\min})$, we set

$$v_M(g_1, g_2, T) = \int_{A_M(F)} \Gamma_M(H(a), \mathcal{Y}(g_1, g_2, T))da.$$

We also denote by $M_{min,\leqslant T}$ the set those $m \in M_{min}(F)$ such that

$$0 \leqslant \langle \alpha, H_{M_{min}}(m)\rangle \text{ and } \langle \varpi_\alpha, H_{M_{min}}(m)\rangle \leqslant \langle \varpi_\alpha, T\rangle, \text{ for every } \alpha \in \Delta_{min},$$

and we let $u(., T)$ be the characteristic function of the subset $KM^+_{min,\leqslant T}K$ of $G(F)$.

For $f \in \mathcal{C}(G(F))$ and $\varphi \in C_c^\infty(G(F))$, we define

$$\widetilde{J}^T(f, \varphi) = \int_{G(F)} u(g, T) \int_{G(F)} f(g^{-1}xg)\varphi(x)dxdg.$$

and

$$J^T(f, \varphi) := \sum_{M \in \mathcal{L}(M_{min})} \frac{|W^M|}{|W|} \int_{\Gamma_{ell}(M)} J^T_M(\gamma, f, \varphi)d\gamma$$

where

$$J^T_M(\gamma, f, \varphi) = D^G(\gamma) \int_{(A_M(F)\backslash G(F))^2} f(g_1^{-1}\gamma g_1)\varphi(g_2^{-1}\gamma g_2)v_M(g_1, g_2, T)dg_1dg_2.$$

Proposition 4.13. *(i) For every $f \in \mathcal{C}(G(F))$ and $\varphi \in C_c^\infty(G(F))$, the function*

$$T \in \mathcal{A}_{min,\mathbb{Q}} \mapsto J^T(f,\varphi)$$

is a unitary polynomial-exponential. Moreover, for any compact-open subgroup $J \subset G(F)$ and subset $\Omega \subset G(F)$ that is bounded modulo conjugation, the subspace of $\mathcal{C}(J\backslash G(F)/J)^*$ spanned by

$$\left\{ f \in \mathcal{C}(J\backslash G(F)/J) \mapsto J^T(f,\varphi) \mid \varphi \in C_c^\infty(\Omega), T \in \mathcal{A}_{\min,\mathbb{Q}} \right\}$$

is finite dimensional.

(ii) Let $\epsilon > 0$. Then, for every $r > 0$ there exists a constant $C_r > 0$ such that

$$\left| J^T(f,\varphi) - \widetilde{J}^T(f,\varphi) \right| \leqslant C_r \|T\|^{-r}$$

for every $T \in \mathcal{A}^+_{P_{min},\mathbb{Q}}$ satisfying $\langle \alpha, T\rangle \geqslant \epsilon\|T\|$ for every $\alpha \in \Delta_{min}$.

(iii) For every $T \in \mathcal{A}_{\min,\mathbb{Q}}$, $\varphi \in C_c^\infty(G(F))$ and strongly cuspidal function $f \in \mathcal{C}_{scusp}(G(F))$ we have

$$J^T(f,\varphi) = \int_{G(F)} \Theta_f(g)\varphi(g)dg.$$

Proof. (i) is a consequence of the splitting formulas [3, Equation (5.5)] and of Howe's conjecture for weighted orbital integrals [3, Lemma 8.2]. (see Appendix B for the twisted case.)

(ii) follows directly from the proof of the geometric side of the local trace formula in [2].

(iii) follows from the splitting formula [3, Equation (5.5)] and the same argument as in Section 2.13.

□

Let $\omega_x \subset \mathfrak{g}_x(F)$ be a sufficiently small invariant neighborhood of 1 and define, for n sufficiently large, $\phi_n^G \in C_c^\infty(G(F))$ by (here $K_x = K \cap G_x$)

$$\phi_n^G(g) = \begin{cases} \int_{K_x} \phi_n(a_n^{-1}k_x^{-1}yk_xa_n)dk_x & \text{if } g = kxyk^{-1} \text{ for some } (y,k) \in \exp(\omega_x) \times K \\ 0 & \text{otherwise.} \end{cases}$$

For every $f \in \mathcal{C}(G(F))$, we set

$$\widetilde{J}^T_{x,\xi,n}(f) = \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} f(g^{-1}xug)\widetilde{v}_{B_x,\xi}(u, \mathcal{Y}(g,T) + H(a_n))dudg \tag{4.6.1}$$

where $\mathcal{Y}(g,T)$ denotes the $(G, M(x))$-family defined by

$$\mathcal{Y}_P(g,T) = H_P(g) + T_P, \quad P \in \mathcal{P}(M(x)),$$

and where we recall that for every $(G, M(x))$-family $\mathcal{X}$ and $u \in N_x(F)$, we have set

$$\widetilde{v}_{B_x,\xi}(u,\mathcal{X}) := \int^{reg}_{T_x(F)} \xi(t^{-1}ut)\Gamma_{B_x}(H(t),\mathcal{X})dt.$$

Proposition 4.14. *(i) Let $f \in \mathcal{C}_{scusp}(G(F))$. Then, for n large enough we have*

$$J^T(f, \phi_n^G) = c_{f,-\mathcal{O}}(x)$$

for every $T \in \mathcal{A}_{P_{min},\mathbb{Q}}$.

(ii) Let $f \in C_c^\infty(G(F))$. Then, there exists $n_f > 0$ and $C = C_f > 0$ (both depend on the support and the level of f) such that

$$\widetilde{J}^T(f, \phi_n^G) = \widetilde{J}^T_{x,\xi,n}(f)$$

for every $n \geqslant n_f$ and $T \in \mathcal{A}^+_{P_{min}}$ satisfying $\alpha(T) \geqslant C$ for every $\alpha \in \Delta_{min}$.

Proof. (i) Applying Proposition 4.13 (iii) to $\varphi = \phi_n^G$ and by usual arguments of semisimple descent and descent to the Lie algebra, together with the germ expansion of Θ_f, for n large enough we get

$$\begin{aligned} J^T(f, \varphi) &= \int_{G(F)} \Theta_f(g)\phi_n^G(g)dg = \int_{G_x(F)} \Theta_f(xy)\phi_n(y)dy \\ &= \sum_{\mathcal{O}' \in \mathrm{Nil}(\widehat{\mathfrak{g}}_x)} c_{f,\mathcal{O}'}(x) \int_{\mathcal{O}'} \widehat{\varphi}_n(Y)dY. \end{aligned}$$

The result then follows immediately from Lemma 4.12 (iv).

(ii) We may assume that f is invariant under K-conjugation. For every $g \in G(F)$, we define a function ${}^g f_{x,\omega_x} \in C_c^\infty(\mathfrak{g}_x(F))$ by

$${}^g f_{x,\omega_x}(X) = \begin{cases} f(g^{-1}x\exp(X)g) & \text{if } X \in \omega_x, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by a standard descent argument to the Lie algebra and Lemma 4.12 (iii), we have

$$\begin{aligned} \widetilde{J}^T(f, \phi_n^G) &= \int_{G(F)} u(g,T) \int_{\mathfrak{g}_x} {}^g f_{x,\omega_x}(X)\varphi_n(a_n^{-1}Xa_n)dXdg \\ &= \int_{G(F)} u(a_n g,T) \int_{\mathfrak{g}_x} {}^g f_{x,\omega_x}(X)\varphi_n(X)dXdg \\ &= \mathrm{vol}(K_n)^{-1}\,\mathrm{vol}(K_n \cap N_x(F))^{-1} \int_{G(F)} u(a_n g,T) \int_{K_n} \int_{\Xi+\mathfrak{n}_x^\perp} \widehat{{}^{kg} f}_{x,\omega_x}(Y)\mathbf{1}_{\Xi+L_n^\perp}(Y)dY\,dk \end{aligned}$$

where the first equality follows from the change of variables $g \mapsto a_n g$ and $X \mapsto a_n X a_n^{-1}$. Since the function $g \mapsto u(a_n g,T)$ is left-invariant by K_n for n large enough, this gives

$$\begin{aligned} J^T(f, \varphi_n) &= \mathrm{vol}(K_n \cap N_x)^{-1} \int_{G(F)} u(a_n g,T) \int_{\Xi+(L_n^\perp \cap \mathfrak{n}_x^\perp)} \widehat{{}^g f}_{x,\omega_x}(Y)dY\,dg \\ &= \mathrm{vol}(K_n \cap N_x)^{-1} \int_{B_x(F)\backslash G(F)} \int_{B_x(F)} u(a_n bg,T) \int_{\Xi+(L_n^\perp \cap \mathfrak{n}_x^\perp)} \widehat{{}^g f}_{x,\omega_x}(b^{-1}Yb)dY\,d_Lb\,dg. \end{aligned}$$

Thus, it suffices to show that, for every $g \in G(F)$, we have

$$\text{vol}(K_n \cap N_x)^{-1} \int_{B_x(F)} u(a_n bg, T) \int_{\Xi+(L_n^\perp \cap \mathfrak{n}_x^\perp)} \widehat{{}^g f}_{x,\omega_x}(b^{-1}Yb) dY d_L b$$
$$= \int_{N_x(F)} f(g^{-1}xug) \widetilde{v}_{B_x,\xi}(u, \mathcal{Y}(g,T) + H(a_n)) du. \tag{4.6.2}$$

Note that both sides of the above equation are (B_x, δ_{B_x})-equivariant on the left. Moreover, since f is compactly supported and x is semisimple, for g outside of a set that is compact modulo $G_x(F)$, the function ${}^g f$ is zero. As $B_x(F)$ is cocompact in $G_x(F)$, we may restrict to establish the above identity for g in some fixed compact subset $\mathcal{K} \subset G(F)$. We need a lemma.

Lemma 4.15. *Let us set $T_x^+[\geq -C] := \{t \in T_x(F) \mid \langle \alpha, H_{T_x}(t) \rangle \leq -C\}$ for every $C > 0$. Then, there exist two large enough constants $C > 0$ and $n_1 > 0$ such that:*

(i) For every $g \in \mathcal{K}$, the function

$$T_x(F) \times (\Xi + \mathfrak{n}_x^\perp(F)) \ni (t, Y) \mapsto \widehat{{}^g f}_{x,\omega_x}(tYt^{-1})$$

is supported on $T_x^+[\geq -C] \times (\Xi + \mathfrak{n}_x^\perp(F))$.

(ii) For every $g \in \mathcal{K}$ and $n \geq n_1$, the function

$$B_x(F) \times (\Xi + L_n^\perp \cap \mathfrak{n}_x^\perp(F)) \ni (b, Y) \mapsto \widehat{{}^g f}_{x,\omega_x}(bYb^{-1})$$

is supported on $T_x^+[\geq -C](N_x(F) \cap K_n) \times (\Xi + \mathfrak{n}_x^\perp(F))$.

Proof. (i) There exists a compact $\mathcal{K}^L \subset \mathfrak{g}_x(F)$ such that $\text{Supp}(\widehat{{}^g f}_{x,\omega_x}) \subset \mathcal{K}^L$ for every $g \in \mathcal{K}$. Thus, it suffices to see that $(t\Xi t^{-1} + \mathfrak{n}_x^\perp(F)) \cap \mathcal{K}^L \neq \emptyset$ implies $t \in T_x^+[\geq -C]$ for $C > 0$ large enough but this is clear as decomposing Ξ along eigenspaces for $A_x(F)$, it has nonzero components in all the root subspaces corresponding the simple negative roots with respect to B_x.

(ii) Let $b \in B_x(F)$ and $Y \in \Xi + L_n^\perp \cap \mathfrak{n}_x^\perp(F)$ be such that $\widehat{{}^g f}_{x,\omega_x}(bYb^{-1}) \neq 0$ and write b as $b = tu$ where $t \in T_x(F)$ and $u \in N_x(F)$. Since $uYu^{-1} \in \Xi + \mathfrak{n}_x^\perp(F)$, by point (i) we have $t \in T_x^+[\geq -C]$ for some $C > 0$ large enough. Moreover, we have $u(\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)))u^{-1} \cap t^{-1}\mathcal{K}^L t \neq \emptyset$. As

$$u(\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)))u^{-1} \subset \Xi + \mathfrak{n}_x^\perp(F),$$

and there exists a compact subset $\mathcal{K}' \subset \Xi + \mathfrak{n}_x^\perp(F)$ such that $t^{-1}\mathcal{K}^L t \cap (\Xi + \mathfrak{n}_x^\perp(F)) \subset \mathcal{K}'$ for every $t \in T_x^+[\geq -C]$, this implies

$$u(\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F)))u^{-1} \cap \mathcal{K}' \neq \emptyset.$$

Using that $(L_n^\perp \cap \mathfrak{n}_x^\perp(F))_n$ is an increasing and exhausting family of compact subsets of $\mathfrak{n}_x^\perp(F)$, for n large enough it follows that

$$u(\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp))u^{-1} \cap \Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp) \neq \emptyset$$

which implies, by Lemma 4.12, that $u \in N_x(F) \cap K_n$. □

Let $g \in \mathcal{K}$. By the lemma above and since $\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp)$ is invariant under the conjugation of $N_x(F) \cap K_n$ (Lemma 4.12), we have

(4.6.3)
$$\begin{aligned} &\mathrm{vol}(K_n \cap N_x)^{-1} \int_{B_x(F)} u(a_n bg, T) \int_{\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp)} \widehat{{}^g f}_{x,\omega_x}(b^{-1}Yb) dY d_L b \\ &= \int_{T_x(F)} u(a_n t^{-1} g, T) \int_{\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F))} \widehat{{}^g f}_{x,\omega_x}(tYt^{-1}) dY \delta_{B_x}(t)^{-1} dt. \end{aligned}$$

and moreover the integrand is supported in $T_x^+[\geq -C]$ for some large enough $C > 0$ (independent of n). For n large enough, as seen in the proof of the last lemma, for every $t \in T_x^+[\geq -C]$ the function

$$Y \in \Xi + \mathfrak{n}_x^\perp(F) \mapsto \widehat{{}^g f}_{x,\omega_x}(tYt^{-1})$$

is supported in $\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F))$. Hence we can replace the above integral over $\Xi + (L_n^\perp \cap \mathfrak{n}_x^\perp(F))$ by an integral over $\Xi + \mathfrak{n}_x^\perp(F)$ (since the integrand stays supported in $T_x^+[\geq -C]$ by the lemma). Then by a simple change of variable, we obtain that the above expression is equal to

(4.6.4)
$$\int_{T_x(F)} u(a_n t^{-1} g, T) \int_{t\Xi t^{-1} + \mathfrak{n}_x^\perp(F)} \widehat{{}^g f}_{x,\omega_x}(Y) dY dt.$$

Next we show that

(4.6.5) For n sufficiently large and T sufficiently regular, we have

$$u(a_n t^{-1} g, T) = \Gamma_{B_x}(H(t), \mathcal{Y}(g,T) + H(a_n))$$

for every $g \in \mathcal{K}$ and $t \in T_x^+[\geq -C]$.

Let t be as above. First, we note that

$$\Gamma_{B_x}(H(t), \mathcal{Y}(g,T) + H(a_n)) = \Gamma_{B_x}(H(a_n^{-1}t), \mathcal{Y}(g,T))$$

and

$$\langle \alpha, H_{T_x}(a_n^{-1}t) \rangle = 2n \log(q) + \langle \alpha, H_{T_x}(t) \rangle \geqslant 2n \log(q) - C$$

for all $\alpha \in \Delta_x$. From this, we see that (4.6.5) actually reduces to the following statement:

(4.6.6) There exist $C' > 0$ such that

$$u(t^{-1}g, T) = \Gamma_{B_x}(H(t), \mathcal{Y}(g, T))$$

for every $g \in \mathcal{K}$, $t \in T_x(F)$ and $T \in \mathcal{A}^+_{P_{min}}$ satisfying

$$\langle \alpha, H(t) \rangle \geqslant 0, \ \ \forall \alpha \in \Delta_x,$$

and

$$\langle \alpha, T \rangle \geqslant C', \ \ \forall \alpha \in \Delta_{min}.$$

Let $t \in T_x(F)$ and $T \in \mathcal{A}^+_{P_{min}}$ be elements satisfying the above inequalities. Then, provided C' is large enough, the $(G, M(x))$-orthogonal set $\mathcal{Y}(g, T)$ is positive for every $g \in \mathcal{K}$ and therefore by Proposition 4.3 and the assumption on t we have

$$\Gamma_{B_x}(H(t), \mathcal{Y}(g, T)) = \Gamma_{M(x)}(H(t), \mathcal{Y}(g, T)).$$

Furthermore, by the identity at the bottom of p.38 of [2], provided again that C' is large enough, we also have

$$\Gamma_{M(x)}(H(t), \mathcal{Y}(g, T)) = u(t^{-1}g, T)$$

for every $g \in \mathcal{K}$. This shows (4.6.6) and ends the proof of (4.6.5).

Now, from (4.6.5) and (4.6.4), we deduce that, for n sufficiently large and T sufficiently regular, we have

$$\begin{aligned}
&\mathrm{vol}(K_n \cap N_x)^{-1} \int_{B_x(F)} u(a_n b g, T) \int_{\Xi + (L_n^\perp \cap \mathfrak{n}_x)} \widehat{{}^g f}_{x,\omega_x}(b^{-1} Y b) dY d_L b \\
&= \int_{T_x(F)} \Gamma_{B_x}(H(t), \mathcal{Y}(g, T) + H(a_n)) \int_{t \Xi t^{-1} + \mathfrak{n}_x^\perp} \widehat{{}^g f}_{x,\omega_x}(Y) dY dt \\
&= \int_{T_x(F)} \Gamma_{B_x}(H(t), \mathcal{Y}(g, T) + H(a_n)) \int_{N_x(F)} f(g^{-1} x u g) \xi(t^{-1} u t) du dt \\
&= \int_{N_x(F)} f(g^{-1} x u g) \int^{reg}_{T_x(F)} \xi(t^{-1} u t) \Gamma_{B_x}(H(t), \mathcal{Y}(g, T) + H(a_n)) dt du \\
&= \int_{N_x(F)} f(g^{-1} x u g) \tilde{v}_{B_x, \xi}(u, \mathcal{Y}(g, T) + H(a_n)) du
\end{aligned}$$

for every $g \in \mathcal{K}$. This gives (4.6.2) and therefore closes the proof of the proposition. □

4.7 End of the proof of Theorem 4.8

In this subsection we will prove the formula of regular germs in Theorem 4.8. Fix a strongly cuspidal function $f \in \mathcal{C}(G(F))$, we need to show that

$$c_{f,-\mathcal{O}_\xi}(x) = \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} f(g^{-1}xug) v_{B_x,\xi}(u,g) dudg.$$

By Proposition 4.14(i), there exists $n_f > 0$ such that for $n > n_f$ we have

$$J^T(f, \phi_n^G) = c_{f,-\mathcal{O}_\xi}(x)$$

for every $T \in \mathcal{A}_{P_{min},\mathbb{Q}}$. Let J be an open compact subgroup of $G(F)$ by which f is biinvariant. By Proposition 4.13(i), once we choose n_f large enough, we can find a sequence of functions $f_N \in C_c^\infty(G(F))^{J\times J}$ such that

- $f_N \to f$ (in $\mathcal{C}(G(F))$) as $N \to \infty$;
- $J^T(f, \phi_n^G) = J^T(f_N, \phi_n^G)$ for all $N > 0$, T and $n > n_f$.

Indeed, by Proposition 4.13(i), we know that the span of the linear forms

$$f \in \mathcal{C}(G(F))^{J\times J} \mapsto J^T(f, \phi_n^G)$$

for all T and $n > n_f$ is finite dimensional. Let $J_1, \cdots, J_k$ be a basis of this span. Since these linear forms are continuous, by density we know that $J_1, \cdots, J_k$ are also linearly independent when restricted to $C_c^\infty(G(F))^{J\times J}$. Thus, we can find $g_i \in C_c^\infty(G(F))^{J\times J}$ $(i = 1, \ldots, k)$ such that $J_i(g_j) = \delta_{i,j}$. Choose now an arbitrary sequence $f'_N \in C_c^\infty(G(F))^{J\times J}$ converging to $f \in \mathcal{C}(G(F))^{J\times J}$. Then the modified sequence $f_N = f'_N + \sum_{i=1}^k (J_i(f) - J_i(f'_N))g_i$ satisfies the required conditions.

Let $N > 0$ be fixed for the moment. By Proposition 4.14(ii), we have

(4.7.1)
$$\tilde{J}^T(f_N, \phi_n^G) = \tilde{J}^T_{x,\xi,n}(f_N) = \int_{B_x(F)\backslash G(F)} \int_{N_x(F)} f_N(g^{-1}xug) \widetilde{v}_{B_x,\xi}(u, \mathcal{Y}(g,T) + H(a_n)) dudg$$

for n sufficiently large and T sufficiently regular (both with respect to f_N). Let $\epsilon > 0$ be sufficiently small. Then, by the local trace formula (Proposition 4.13 (ii)), as $T \in \mathcal{A}^+_{P_{\min},\mathbb{Q}}$ goes to ∞ in the cone

$$\mathcal{C} = \{\langle\alpha, T\rangle \geq \epsilon\|T\|,\ \forall\ \alpha \in \Delta_{min}\},$$

$\tilde{J}^T(f_N, \phi_n^G)$ is asymptotic to the polynomial-exponential (with unitary exponents) $T \mapsto J^T(f_N, \phi_n^G) = J^T(f, \phi_n^G)$. On the other hand, by Lemma 4.7, there exists a constant $C > 0$ such that the equality of weights

$$\widetilde{v}_{B_x,\xi}(u, \mathcal{Y}(g,T) + H(a_n)) = v_{B_x,\xi}(u, \mathcal{Y}(g,T) + H(a_n))$$

holds whenever the depth of the $(G, M(x))$-orthogonal set $\mathcal{Y}(g,T) + H(a_n)$ is bigger than $C\sigma(u)$. Thus, as f_N is compactly supported, this holds for T large enough in the cone $\mathcal{C}$ and for every $(g,u)\in\mathcal{K}\times N_x(F)$ such that $f_N(g^{-1}xug)\neq 0$. Thus, as $T\to\infty$ in $\mathcal{C}$, the right hand side of (4.7.1) is also asymptotic to the polynomial-exponential

$$T\mapsto\int_{B_x(F)\backslash G(F)}\int_{N_x(F)}f_N(g^{-1}xug)v_{B_x,\xi}(u,\mathcal{Y}(g,T)+H(a_n))dudg.$$

From this, we deduce the equality of polynomial-exponentials

$$J^T(f,\phi_n^G)=\int_{B_x(F)\backslash G(F)}\int_{N_x(F)}f_N(g^{-1}xug)v_{B_x,\xi}(u,\mathcal{Y}(g,T)+H(a_n))dudg$$

for every $T\in\mathcal{A}_{P_{min},\mathbb{Q}}$ and n large enough. However, we know that the left hand side is identically equal to $c_{f,-\mathcal{O}_\xi}(x)$ whereas the right-hand side is a polynomial-exponential in both T and $H(a_n)$. This polynomial-exponential is therefore constant and the same identity holds for $T=0$ and $H(a_n)=0$ which gives the equality of Theorem 4.8 except with f_N instead of f in the right-hand side. Thus, letting $N\to\infty$, we obtain the desired identity.

4.8 A descent formula

In this subsection, we will prove a descent formula that will be used in later section. We keep the notation as in the previous subsections. Moreover, we assume that $\iota(x)=x$, B_x is ι-split and $T_x^\iota\subset Z_{G_x}$.

The action of ι naturally descends to $\mathcal{A}_x$ and this induces a decomposition $\mathcal{A}_x=\mathcal{A}_x^\iota\oplus\mathcal{A}_{x,\iota}$ where $\mathcal{A}_x^\iota$ (resp. $\mathcal{A}_{x,\iota}$) denotes the subspace consisting of elements $H\in\mathcal{A}_x$ satisfying $\iota(H)=H$ (resp. $\iota(H)=-H$). For $H\in\mathcal{A}_x$ we will denote without further comment by H^ι, H_ι the respective projections of X with respect to this decomposition. Similarly, if C is a subset of $\mathcal{A}_x$ (typically the positive cone associated to a parabolic subspace) we will denote by C^ι the image of its projection to $\mathcal{A}_x^\iota$.

Let $\mathcal{X}=(X_{\tilde{P}})_{\widetilde{P}\in\mathcal{P}(\widetilde{M}(x))}$ be a $(\widetilde{G},\widetilde{M}(x))$-orthogonal set. For every $\widetilde{Q}\in\mathcal{F}_{B_x,\iota}(\widetilde{M}(x))$ (resp. $\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$ such that $\mathcal{A}_x^\iota\cap\mathcal{A}^{\widetilde{Q}}=0$), we define a function $\Gamma^{\widetilde{Q}}_{B_x,\iota}(.,\mathcal{X})$ (resp. $\Gamma^{\widetilde{Q},\iota}_{B_x}(.,\mathcal{X})$) on $\mathcal{A}_x$ by

$$\Gamma^{\widetilde{Q}}_{B_x,\iota}(H,\mathcal{X})=\sum_{\widetilde{P}\in\mathcal{F}_{B_x,\iota}(\widetilde{M}(x)),\widetilde{P}\subset\widetilde{Q}}(-1)^{a^{\widetilde{Q}}_{\widetilde{P},\iota}}\widehat{\tau}^{\widetilde{Q}}_{\widetilde{P},\iota}(H-X_{\widetilde{P},\iota}),\quad H\in\mathfrak{a}_x,$$

$$\text{(resp. }\Gamma^{\widetilde{Q},\iota}_{B_x}(H,\mathcal{X})=\begin{cases}\Gamma^{\widetilde{Q}}_{B_x}(Y^{\widetilde{Q}},\mathcal{X}) & \text{if } H\in X_{\widetilde{Q}}+Y^{\widetilde{Q}}+\mathcal{A}_x^\iota+\mathcal{A}_{\widetilde{G}}\text{ for some }Y^{\widetilde{Q}}\in\mathcal{A}^{\widetilde{Q}};\\ 0 & \text{otherwise.}\end{cases},\quad H\in\mathcal{A}_x.)$$

Proposition 4.16. *For every $\widetilde{R}\in\mathcal{F}_{B_x,\iota}(\widetilde{M}(x))$, we have the following identity of functions on $\mathcal{A}_x$:*

$$\sum_{\widetilde{R}\supset\widetilde{Q}\in\mathcal{F}_{B_x,\iota}(\widetilde{M}(x))}\Gamma^{\widetilde{Q}}_{B_x,\iota}(.,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q},\iota}(.-X_{\widetilde{Q}})=1.\tag{4.8.1}$$

If $\mathcal{X}$ is positive, $\Gamma^{\widetilde{G}}_{B_x,\iota}(.,\mathcal{X})$ is the characteristic function of either of the two following subsets

$$\left\{H \in \mathcal{A}_x \mid \varpi_\alpha(H - X_{\widetilde{P},\iota}) \leqslant 0,\ \forall \widetilde{P} \in \mathcal{P}_{B_x,\iota}(\widetilde{M}(x)), \forall \alpha \in \Delta_{\widetilde{P},\iota}\right\}, \tag{4.8.2}$$

$$Conv\left\{X_{\widetilde{P},\iota} \mid \widetilde{P} \in \mathcal{P}_{B_x,\iota}(\widetilde{M}(x))\right\} + {}^-\mathcal{A}_{B_x,\iota} + \mathcal{A}_x^\iota + \mathcal{A}_{\widetilde{G}}. \tag{4.8.3}$$

Moreover, if $\mathcal{X}$ is positive and $\mathcal{Y} = (Y_{\tilde{P}})_{\widetilde{P}\in\mathcal{P}(\widetilde{M}(x))}$ is another positive $(\widetilde{G},\widetilde{M}(x))$-orthogonal sets, then, for every $\widetilde{Q},\widetilde{R} \in \mathcal{F}_{B_x,\iota}(\widetilde{M}(x))$ and $Y \in \mathcal{A}_x$ we have

(4.8.4)

$$\Gamma^{\widetilde{Q}}_{B_x,\iota}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q},\iota}(H - X_{\widetilde{Q}})\Gamma^{\widetilde{R}}_{B_x,\iota}(H,\mathcal{X}+\mathcal{Y}) = \Gamma^{\widetilde{Q}}_{B_x,\iota}(H,\mathcal{X})\tau^{\widetilde{R}}_{\widetilde{Q},\iota}(H - X_{\widetilde{Q}})\phi^{\widetilde{R}}_{\widetilde{Q},\iota}(H - X_{\widetilde{Q}} - Y_{\widetilde{Q}})$$

where $\phi^{\widetilde{R}}_{\widetilde{Q},\iota}$ denotes the characteristic function of the set of those $Z \in \mathcal{A}_{\widetilde{Q},\iota}$ such that $\langle\varpi, Z\rangle \leq 0$ for every $\varpi \in \widehat{\Delta}^{\widetilde{R}}_{\widetilde{Q},\iota}$.

Proof. The proof is basically the same as for Proposition 4.3, Proposition 4.5 and Corollary 4.4 adding some ι's in indices along the way. We skip the details. □

Proposition 4.17. *Assume that $\mathcal{X}$ is positive and let $\varepsilon \in (\mathcal{A}^+_{B_x})^\iota$ that is in general position. For every $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ such that $\varepsilon \in (\mathcal{A}^+_{\widetilde{Q}})^\iota$ we define a number $d_\varepsilon(\widetilde{Q})$ inductively by the relation*

$$\sum_{\substack{\widetilde{Q}\subset\widetilde{R}\in\mathcal{F}_{B_x}(\widetilde{M}(x))\\ \varepsilon\in(\mathcal{A}^+_{\widetilde{R}})^\iota}} d_\varepsilon(\widetilde{R}) = 1. \tag{4.8.5}$$

Then, we have the following equality of functions on $\mathcal{A}_x$

$$\Gamma^{\widetilde{G}}_{B_x,\iota}(.,\mathcal{X}) = \sum_{\substack{\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))\\ \varepsilon\in(\mathcal{A}^+_{\widetilde{Q}})^\iota}} d_\varepsilon(\widetilde{Q})\Gamma^{\widetilde{Q},\iota}_{B_x}(.,\mathcal{X}). \tag{4.8.6}$$

Proof. For $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$, we set

$$\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X}) := Conv\left\{X_{\widetilde{P}} \mid \widetilde{P} \in \mathcal{P}_{B_x}(\widetilde{M}(x)), \widetilde{P} \subset \widetilde{Q}\right\} + {}^-\mathcal{A}^{\widetilde{Q}}_{B_x}.$$

Then $\Gamma^{\widetilde{Q}}_{B_x}(.,\mathcal{X})$ is the characteristic function of $\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X}) + \mathcal{A}_{\widetilde{Q}}$ (by Proposition 4.5) and:

(4.8.7) If $\mathcal{A}^\iota_x \cap \mathcal{A}^{\widetilde{Q}} = 0$, $\Gamma^{\widetilde{Q},\iota}_{B_x}(.,\mathcal{X})$ is the characteristic function of $\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X}) \oplus (\mathcal{A}^\iota_x + \mathcal{A}_{\widetilde{G}})$.

(This follows from the definition of $\Gamma^{\widetilde{Q},\iota}_{B_x}(.,\mathcal{X})$ and the previous point.) Furthermore, we claim that:

(4.8.8) $\Gamma^{\widetilde{G}}_{B_x,\iota}(.,\mathcal{X})$ is the characteristic function of $\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})+\mathcal{A}^{\iota}_x+\mathcal{A}_{\widetilde{G}}$.

Indeed, by the previous proposition it suffices to check that for every $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$, $\widetilde{P}'\in\mathcal{P}_{B_x,\iota}(\widetilde{M}(x))$ and $\alpha\in\Delta_{\widetilde{P}',\iota}$ we have

$$\varpi_\alpha(X_{\widetilde{P},\iota}-X_{\widetilde{P}',\iota})\leqslant 0.$$

But this follows, after projection onto $\mathcal{A}_{x,\iota}$, from the fact that $X_{\widetilde{P},\iota}-X_{\widetilde{P}',\iota}$ is a linear combination with negative coefficients of elements of $\Delta_{\widetilde{P}'}$ (by definition of a positive $(\widetilde{G},\widetilde{M}(x))$-family).

With the terminology and notation from Appendix A, we also have:

(4.8.9) $\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})$ is a finitely generated convex set with faces $F^{\widetilde{Q}}:=\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X})$, $\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$, and corresponding (open) cones $\mathcal{A}^+_{F^{\widetilde{Q}}}=\mathcal{A}^+_{\widetilde{Q}}$.

Indeed, that $\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})$ is a finitely generated convex set is clear from its definition. Let $\lambda\in\mathcal{A}_x$ and $c\in\mathbb{R}$ be such that $\langle\lambda,H\rangle\leqslant c$ for every $H\in\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})$. Applying this inequality to $H\in{}^-\mathcal{A}_{B_x}$, we see that $\lambda\in\overline{\mathcal{A}^+_{B_x}}$. As

$$\overline{\mathcal{A}^+_{B_x}}=\bigcup_{\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))}\mathcal{A}^+_{\widetilde{Q}},$$

we have $\lambda\in\mathcal{A}^+_{\widetilde{Q}}$ for some $\widetilde{Q}\in\mathcal{F}_{B_x}(\widetilde{M}(x))$. For $H\in{}^-\mathcal{A}_{B_x}$, we have

$$\langle\lambda,H\rangle\leqslant 0$$

with equality if and only if $H\in{}^-\mathcal{A}^{\widetilde{Q}}_{B_x}$. Furthermore, as $\mathcal{X}$ is positive, for every $\widetilde{P}\in\mathcal{P}_{B_x}(\widetilde{M}(x))$ we have

$$\langle\lambda,X_{\widetilde{P}}\rangle\leqslant\langle\lambda,X_{\widetilde{Q}}\rangle$$

with equality if and only if $\widetilde{P}\subset\widetilde{Q}$. Therefore,

$$\langle\lambda,H\rangle\leqslant\langle\lambda,X_{\widetilde{Q}}\rangle$$

for $H\in\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})$ with equality if and only if $H\in\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X})$ and it follows that the intersection

$$\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})\cap\{H\in\mathcal{A}_x\mid\langle\lambda,H\rangle=c\}$$

is either empty or equal to $\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X})$. The claim (4.8.9) follows.

From (4.8.9) and Proposition A.3(i) (applied to $\mathfrak{b}=\mathcal{A}_{x,\iota}$), we deduce that

$$\mathcal{C}^{\widetilde{G}}_{B_x}(\mathcal{X})+\mathcal{A}^{\iota}_x+\mathcal{A}_{\widetilde{G}}=\bigcup_{\widetilde{Q}\in\mathcal{F}_{B_x,\varepsilon}(\widetilde{M}(x))}\mathcal{C}^{\widetilde{Q}}_{B_x}(\mathcal{X})+\mathcal{A}^{\iota}_x+\mathcal{A}_{\widetilde{G}}\tag{4.8.10}$$

where we have denoted by $\mathcal{F}_{B_x,\varepsilon}(\widetilde{M}(x))$ the subset of $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ such that $\varepsilon \in (\mathcal{A}_{\widetilde{Q}}^+)^\iota$.

Thus, by (4.8.7) and (4.8.8), to get the identity (4.8.6) it only remains to check that

$$\sum_{\substack{\widetilde{Q} \in \mathcal{F}_{B_x,\varepsilon}(\widetilde{M}(x)) \\ H \in \mathcal{C}_{B_x}^{\widetilde{Q}}(\mathcal{X}) + \mathcal{A}_x^\iota + \mathcal{A}_{\widetilde{G}}}} d_\varepsilon(\widetilde{Q}) = 1 \tag{4.8.11}$$

for every $H \in \mathcal{C}_{B_x}^{\widetilde{G}}(\mathcal{X}) + \mathcal{A}_x^\iota + \mathcal{A}_{\widetilde{G}}$. By Proposition A.3(ii), there exists a minimal $\widetilde{Q} \in \mathcal{F}_{B_x,\varepsilon}(\widetilde{M}(x))$ such that $H \in \mathcal{C}_{B_x}^{\widetilde{Q}}(\mathcal{X}) + \mathcal{A}_x^\iota + \mathcal{A}_{\widetilde{G}}$ and, by the relation (4.8.5), it suffices to show that, for $\widetilde{R} \in \mathcal{F}_{B_x,\varepsilon}(\widetilde{M}(x))$, we have $\mathcal{C}_{B_x}^{\widetilde{Q}}(\mathcal{X}) \subset \mathcal{C}_{B_x}^{\widetilde{R}}(\mathcal{X})$ if and only if $\widetilde{Q} \subset \widetilde{R}$ but this follows from (4.8.9) (as this shows that both inclusions are equivalent to $\mathcal{A}_{\widetilde{R}}^+ \subset \overline{\mathcal{A}_{\widetilde{Q}}^+}$). □

As in the previous subsection, let $N_{x,reg} \subset N_x$ be the subset of regular elements in N_x and $T_{x,c} \subset T_x(F)$ be the maximal compact subgroup. We equip $T_{x,c}$ with the Haar measure of total mass 1 and we also fix a log-norm $\sigma_{reg} : N_{x,reg}(F) \to \mathbb{R}_{\geqslant 1}$ on $N_{x,reg}(F)$. Set $r = \dim(\mathfrak{a}_x)$. The next two lemmas can be proved by the same argument as in Lemma 4.6. We will skip the proofs here.

Lemma 4.18. *For any $u \in N_{x,reg}(F)$ and any positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, the iterated integral*

$$\int_{T_x(F)/A_{\widetilde{G}}(F)T_x^\iota(F)} \int_{T_{x,c}} \xi(a^{-1}t^{-1}uta)dt\Gamma_{B_x,\iota}(H_{T_x}(a), \mathcal{X})da \tag{4.8.12}$$

is absolutely convergent in that order and will be denoted by

$$\tilde{v}_{B_x,\xi,\iota}(u, \mathcal{X}) := \int^*_{T_x(F)/A_{\widetilde{G}}(F)T_x^\iota(F)} \xi(a^{-1}ua)\Gamma_{B_x,\iota}(H(a), \mathcal{X})da.$$

Moreover, there exists a constant $C > 0$ such that for every $u \in N_{x,reg}(F)$ and every positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, we have

$$|\tilde{v}_{B_x,\xi,\iota}(u, \mathcal{X})| \leqslant C(\sigma_{reg}(u) + N(\mathcal{X}))^r.$$

Lemma 4.19. *For any $u \in N_{x,reg}(F)$, $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ such that $\mathcal{A}_x^\iota \cap \mathcal{A}^{\widetilde{Q}} = 0$ and any positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, the iterated integral*

$$\int_{T_x(F)/A_{\widetilde{G}}(F)T_x^\iota(F)} \int_{T_{x,c}} \xi(a^{-1}t^{-1}uta)dt\Gamma_{B_x}^{\widetilde{Q},\iota}(H_{T_x}(a), \mathcal{X})da \tag{4.8.13}$$

is absolutely convergent in that order and will be denoted by

$$\tilde{v}_{B_x,\xi}^{\widetilde{Q},\iota}(u, \mathcal{X}) := \int^*_{T_x(F)/A_{\widetilde{G}}(F)T_x^\iota(F)} \xi(a^{-1}ua)\Gamma_{B_x}^{\widetilde{Q},\iota}(H(a), \mathcal{X})da.$$

Moreover, there exists a constant $C > 0$ such that for every $u \in N_{x,reg}(F)$ and every positive $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$, we have

$$\left|\tilde{v}^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{X})\right| \leqslant C(\sigma_{reg}(u) + N(\mathcal{X}))^r.$$

Lemma 4.20. *There exists $C > 0$ such that for every $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ satisfying $d(\mathcal{X}) \geqslant C\sigma(u)$, we have*

$$\tilde{v}^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{X}) = \tilde{v}^{\widetilde{Q},\iota}_{B_x,\xi}(u^Q,\mathcal{X})$$

where $u = u^Q u_Q$ is the unique decomposition with $u^Q \in L_{\widetilde{Q}}(F)$ and $u_Q \in U_{\widetilde{Q}}(F)$.

Proof. The proof is the same as the proof of the second bullet point in the proof of Lemma 4.7 (we just need to use our assumption that $T_x^\iota \subset Z_{G_x}$). □

The next two lemmas can be proved by the same argument as in Lemma 4.7. We will skip the proof here.

Lemma 4.21. *There exists $C > 0, r > 0$ and, for every $u \in N_{x,reg}(F)$, a unique exponential polynomial $v_{B_x,\xi,\iota}(u,.) \in Pol_{\leqslant r}$ whose exponents belong to a finite set independent of u such that for every $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ satisfying $d(\mathcal{X}) \geqslant C\sigma(u)$, we have*

$$v_{B_x,\xi,\iota}(u,\mathcal{X}) = \widetilde{v}_{B_x,\xi,\iota}(u,\mathcal{X}).$$

Moreover, there exists $C' > 0$ and $R > 0$ such that for every $u \in N_{x,reg}(F)$ and every $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ we have

$$|v_{B_x,\xi,\iota}(u,\mathcal{X})| \leqslant C'(\sigma_{reg}(u) + N(\mathcal{X}))^R.$$

Lemma 4.22. *For $\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x))$ such that $\mathcal{A}_x^\iota \cap \mathcal{A}^{\widetilde{Q}} = 0$, there exists $C > 0, r > 0$ and, for every $u \in N_{x,reg}(F)$, a unique exponential polynomial $v^{\widetilde{Q},\iota}_{B_x,\xi}(u,.) \in Pol_{\leqslant r}$ whose exponents belong to a finite set independent of u such that for every $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ satisfying $d(\mathcal{X}) \geqslant C\sigma(u)$, we have*

$$v^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{X}) = \widetilde{v}^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{X}).$$

Moreover, there exists $C' > 0$ and $R > 0$ such that for every $u \in N_{x,reg}(F)$ and every $(\widetilde{G}, \widetilde{M}(x))$-orthogonal set $\mathcal{X}$ we have

$$\left|v^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{X})\right| \leqslant C'(\sigma_{reg}(u) + N(\mathcal{X}))^R.$$

Following the above two lemmas, we define

$$v_{B_x,\xi,\iota}(u,g) = v_{B_x,\xi,\iota}(u,\mathcal{Y}(g)),\ v^{\widetilde{Q},\iota}_{B_x,\xi}(u,g) = v^{\widetilde{Q},\iota}_{B_x,\xi}(u,\mathcal{Y}(g))$$

The following corollary is a direct consequence of the two lemmas above.

Corollary 4.23. *There exists $d > 0$ such that*

$$v_{B_x,\xi,\iota}(u,g) \ll \sigma_G(g)^d \sigma_{N_{x,reg}}(u)^d,\ v^{\widetilde{Q},\iota}_{B_x,\xi}(u,g) \ll \sigma_G(g)^d \sigma_{N_{x,reg}}(u)^d$$

for all $u \in N_{x,reg}(F)$ and $g \in G(F)$.

Corollary 4.24. *The function $v^{\widetilde{Q},\iota}_{B_x,\xi}(u,g)$ is left $N_x \cap U_{\widetilde{Q}}(F)$ on u and left $U_{\widetilde{Q}}(F)$-invariant on g.*

Proof. The left $U_{\widetilde{Q}}(F)$-invariant on g is clear from the definition. The left $N_x \cap U_{\widetilde{Q}}(F)$ on u follows from Lemma 4.20. □

Corollary 4.25. *We have the decent formula*

$$v_{B_x,\xi,\iota}(u,g) = \sum_{\substack{\widetilde{Q} \in \mathcal{F}_{B_x}(\widetilde{M}(x)) \\ \varepsilon \in (\mathcal{A}^+_{\widetilde{Q}})^\iota}} d_\varepsilon(\widetilde{Q}) v^{\widetilde{Q},\iota}_{B_x,\xi}(u,g).$$

Proof. This is a direct consequence of Proposition 4.17. □

5 On the spectral expansion

Let $(G, \widetilde{G})$ be a connected reductive twisted space over F. Let H be a closed unimodular subgroup of G defined over F and $(H, \widetilde{H})$ be a twisted space over F equipped with an embedding $\widetilde{H} \subset \widetilde{G}$ which is $H \times H$-equivariant. Let $(\chi, \tilde{\chi})$ be a one-dimensional unitary representation of $\widetilde{H}(F)$ i.e. $\chi : H(F) \to \mathbb{C}^\times$ is a (smooth) unitary character and $\tilde{\chi} : \widetilde{H}(F) \to \mathbb{C}^\times$ is a map satisfying $\tilde{\chi}(h_1 \widetilde{h} h_2) = \chi(h_1 h_2)\tilde{\chi}(\widetilde{h})$ for $(\widetilde{h}, h_1, h_2) \in \widetilde{H}(F) \times H(F) \times H(F)$. Let ω be a character of $A_{\widetilde{G}}(F)$ which coincides with χ on the intersection $A_{\widetilde{G}}(F) \cap H(F)$.

Denote by $L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$ the Hilbert space of functions $\varphi : G(F) \to \mathbb{C}$ satisfying $\varphi(hag) = \chi(h)\omega(a)\varphi(g)$ for $(h,a,g) \in H(F) \times A_{\widetilde{G}}(F) \times G(F)$ and such that $g \mapsto |\varphi(g)|^2$ is integrable on $H(F)A_{\widetilde{G}}(F)\backslash G(F)$. The representation by right translation of $G(F)$ on that space will be denoted by R. This extends to a twisted representation $\tilde{R}$ of $\widetilde{G}(F)$ defined by

$$(\tilde{R}(\widetilde{h}g)\varphi)(x) = \tilde{\chi}(\widetilde{h})\varphi(\mathrm{Ad}^{-1}_{\widetilde{h}}(x)g)$$

for every $(\widetilde{h}, g, x) \in \widetilde{H}(F) \times G(F) \times G(F)$ and $\varphi \in L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$. For $f \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$, the operator $\tilde{R}(f)$ is given by

$$(\tilde{R}(f)\varphi)(x) = \int_{\widetilde{G}(F)/A_{\widetilde{G}}(F)} f(\widetilde{g})(\tilde{R}(\widetilde{g})\varphi)(x) d\widetilde{g},\ \ \varphi \in L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega).$$

This operator is associated with the kernel function $\nu(\widetilde{H})^{-1} K_f(x,y)$ where

$$K_f(x,y) = \int_{\widetilde{H}(F)/A^H_{\widetilde{G}}(F)} f(x^{-1}\widetilde{h}y)\tilde{\chi}(\widetilde{h}) d\widetilde{h},\ \ x,y \in G(F) \tag{5.0.1}$$

and $\nu(\widetilde{H}) = |H(F) \cap A_{\widetilde{G}}(F) : A_{\widetilde{G}}^H(F)|$. Here $A_{\widetilde{G}}^H$ is the maximal split torus of $A_{\widetilde{G}} \cap H$. We define

$$I(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} K_f(x,x)dx, \text{ for } f \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$$

provided the integral is absolutely convergent.

If the pair (G,H) is tempered (see Section 3.2 for the definition of tempered), we can define in a similar way operators $\tilde{R}(f)$ for $f \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega)$ and these operators are associated to kernel functions given by the same expression (5.0.1) (which is absolutely convergent) and we also define $I(f)$ by the same formula provided the integral is abstolutely convergent.

Let now f be in $C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ or, if X is tempered, in $\mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ and assume that it satisfies the following very strong condition:

(5.0.2) The operator $\tilde{R}(f)$ is of finite rank.

This implies that the integral defining $I(f)$ is convergent and equals $\nu(\widetilde{H}) \operatorname{Tr} \tilde{R}(f)$:

$$I(f) = \nu(\widetilde{H}) \operatorname{Tr} \tilde{R}(f).$$

Let $L^2_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$ be the sum of all the irreducible unitary subrepresentations of $L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$ and $L^2_{\text{cont}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$ be its orthogonal complement. The assumption (5.0.2) also implies that $\tilde{R}(f)$ acts by zero on $L^2_{\text{cont}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$, therefore

$$I(f) = \nu(\widetilde{H}) \operatorname{Tr} \tilde{R}_{\text{disc}}(f)$$

where $\tilde{R}_{\text{disc}}(f)$ stands for the restriction of $\tilde{R}(f)$ to $L^2_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$.

Let $\Pi_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$ be the set of isomorphism classes of irreducible subrepresentations of $L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)$. Then, we have the isotypic decomposition

$$L^2_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega) = \bigoplus_{\pi \in \Pi_{disc}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)} \pi \otimes M_{L^2}(\pi)$$

where $M_{L^2}(\pi) := \operatorname{Hom}_G(\pi, L^2(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega))$ are multiplicity spaces. Let $\Pi_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)^\theta$ be the subset of isomorphism classes fixed by θ and choose for every $\pi \in \Pi_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)^\theta$ an extension $\widetilde{\pi}$ of π to a representation of the twisted space $\widetilde{G}(F)$. Then, there is an unique endomorphism $\theta\langle\widetilde{\pi}\rangle$ of $M_{L^2}(\pi)$ such that the restriction of $\tilde{R}(\widetilde{g})$ to the isotypic component $\pi \otimes M_{L^2}(\pi)$ is equal to $\widetilde{\pi}(\widetilde{g}) \otimes \theta\langle\widetilde{\pi}\rangle$ for $\widetilde{g} \in \widetilde{G}(F)$. Using these notations, and under the assumption (5.0.2), we have

$$\operatorname{Tr}(\tilde{R}_{\text{disc}}(f)) = \sum_{\pi \in \Pi_{\text{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi \otimes \omega)} \operatorname{Tr}(\widetilde{\pi}(f)) \times \operatorname{Tr}(\theta\langle\widetilde{\pi}\rangle \mid M_{L^2}(\pi))$$

for $f \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ (or $f \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ if (G,H) is tempered). Note that a priori we didn't assume the multiplicity spaces $M(\pi)$ to be of finite dimension but, by the assumption (5.0.2), this is automatic whenever $\mathrm{Tr}(\widetilde{\pi}(f)) \neq 0$, so that the above expression makes sense.

Summarizing the discussion so far, we have the following proposition:

Proposition 5.1. *Let f be in $C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ or, if X is tempered, in $\mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ and assume that it satisfies* (5.0.2). *Then, the integral defining $I(f)$ converges and, with the above notation, we have*

$$I(f) = \nu(\widetilde{H}) \sum_{\pi \in \Pi_{\mathrm{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi\otimes\omega)} \mathrm{Tr}(\widetilde{\pi}(f)) \times \mathrm{Tr}(\theta\langle\widetilde{\pi}\rangle \mid M(\pi)).$$

When $X = H\backslash G$ is wavefront spherical and G is split [30, Theorem 9.2.1] or when X is symmetric [15, Theorem 4], we have[10]:

(5.0.3) For every compact-open subgroup $J \subset G(F)$, the subspace

$$L^2_{\mathrm{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi\otimes\omega)^J$$

of J-invariants in $L^2_{\mathrm{disc}}(H(F)A_{\widetilde{G}}(F)\backslash G(F), \chi\otimes\omega)$ is finite dimensional.

This readily implies that for every f in $C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ or, if X is tempered, in $\mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$, the operator $\tilde{R}_{\mathrm{disc}}(f)$ is of finite rank so that, in those cases, we have

(5.0.4) the assumption (5.0.2) is equivalent to $\tilde{R}(f) = \tilde{R}_{\mathrm{disc}}(f)$.

Two other situations where condition (5.0.2) is automatically satisfied are as follows:

(5.0.5) $\bar{f} \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega)$ is a matrix coefficient of a supercuspidal representation $(\pi, \widetilde{\pi})$ of $\widetilde{G}(F)$ with

$$m_{L^2}(\pi) := \dim M_{L^2}(\pi) < \infty.$$

(5.0.6) The pair (G, H) is tempered and $\bar{f} \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega)$ is a matrix coefficient of a discrete series representation $(\pi, \widetilde{\pi})$ of $\widetilde{G}(F)$ with

$$m_{L^2}(\pi) := \dim M_{L^2}(\pi) < \infty.$$

By [14], the finite multiplicity assumption in 5.0.5 and 5.0.6 is automatically satisfied when $H = H_0 \ltimes N$, with N the unipotent radical of some parabolic subgroup $P = MN$ of G, H_0 a symmetric subgroup of a Levi factor M (i.e. there exists an involution ι of M such that $(M^\iota)^0 \subset H_0 \subset M^\iota$), and the restriction of the character χ to $N(F)$ is *generic* (in the sense that its orbit under the adjoint action of $M(F)$ is open in the group of all continuous characters $\mathrm{Hom}_{\mathrm{cont}}(N(F), \mathbb{C}^\times)$).

[10] This property is of course expected to hold for all spherical varieties.

6 The geometric expansion

6.1 The setup

Let $(\widetilde{G}, \iota)$ be a twisted symmetric pair (see §3.7) with G connected and reductive, $\widetilde{P} = \widetilde{M}N$ be a ι-split parabolic subspace with $\widetilde{M} = \widetilde{P} \cap \iota(\widetilde{P})$, and $\xi : N(F) \to \mathbb{C}^\times$ be a generic character of $N(F)$ i.e. a character whose the stabilizer in M under the adjoint action is of minimal dimension. Let $H_0 = (M^\iota)^\circ$ and $\widetilde{H}_0$ be a connected component of the subvariety of ι-fixed points $\widetilde{M}^\iota$ so that $(H_0, \widetilde{H}_0)$ is a twisted reductive space. We make the following two assumptions:

- $\widetilde{H}_0$ stabilizes the character ξ under the adjoint action. Moreover, if ξ is nontrivial (i.e. if $\widetilde{P}$ is a proper parabolic subspace), we assume that H_0 is the neutral component of the stabilizer of the character ξ in M.
- The twisted symmetric pair $(\widetilde{M}, \widetilde{H}_0)$ is coregular in the sense of Subsection 3.1, i.e. $\widetilde{H}_0(F) \cap \widetilde{M}_{\mathrm{rs}}(F) \neq \emptyset$ and the function

$$t \in \widetilde{H}_0(F) \cap \widetilde{M}_{\mathrm{rs}}(F) \mapsto \frac{D^{\widetilde{H}_0}(t)}{D^{\widetilde{M}}(t)^{1/2}}$$

  is locally bounded on $\widetilde{H}_0(F)$.

For every $h \in \widetilde{H}_{0,ss}(F)$, we have

$$D^{\widetilde{H}}(h) = D^{\widetilde{H}_0}(h) D^{\widetilde{G}}(h)^{1/2} D^{\widetilde{M}}(h)^{-1/2} \delta_{\widetilde{P}}(h)^{-1/2}$$

and $\delta_{\widetilde{P}}(h) = \delta_{\widetilde{P}}(\iota(h))^{-1} = \delta_{\widetilde{P}}(h)^{-1}$, since $\widetilde{P}$ is ι-split, hence $\delta_{\widetilde{P}}(h) = 1$. Therefore, the second assumption implies that:

(6.1.1) the function $h \in \widetilde{H}_0(F) \cap \widetilde{M}_{rs}(F) \mapsto \frac{D^{\widetilde{H}}(h)}{D^{\widetilde{G}}(h)^{1/2}}$ is locally bounded on $\widetilde{H}_0(F)$.

We set

$$\widetilde{H} = \widetilde{H}_0 \ltimes N$$

and we denote again by $\xi : \widetilde{H}(F) \to \mathbb{C}^\times$ the twisted character that is trivial on $\widetilde{H}_0(F)$ and coincides with the previous character on $N(F)$. We also fix a unitary twisted character $\chi : \widetilde{H}_0(F) \to \mathbb{C}^\times$ and we denote by $\xi \otimes \chi$ the twisted character of $\widetilde{H}(F)$ given by

$$\xi \otimes \chi : h_0 u \in \widetilde{H}(F) = \widetilde{H}_0(F) \ltimes N(F) \mapsto \chi(h_0)\xi(u).$$

Let $t \in \widetilde{H}_{0,rs}(F)$. By the coregular assumption, t is also regular in $\widetilde{M}$ and this implies that G_t is quasi-split over F with $P_t = M_t N_t$ as a Borel subgroup where G_t (resp. P_t, M_t and N_t) denotes the neutral component of the centralizer of t in G (resp. in P, M and N).

Let ξ_t be the restriction of ξ to $N_t(F)$. Similarly, if we let $S = H_{0,t}$ and $\widetilde{T} = St$, $G_{\widetilde{S}}$ is quasi-split over F, $M_{\widetilde{S}}N_{\widetilde{S}}$ is a Borel subgroup of $G_{\widetilde{S}}$ and we let $\xi_{\widetilde{S}}$ be the restriction of ξ to $N_{\widetilde{S}}(F)$ where $G_{\widetilde{S}}$ (resp. $P_{\widetilde{S}}$, $M_{\widetilde{S}}$ and $N_{\widetilde{S}}$) is the centralizer of $\widetilde{S}$ in G (resp. P, M and N). Note that $G_t = G_{\widetilde{S}}$ for almost all $t \in \widetilde{S}(F)$ and $M_{\widetilde{S}} \cap H_0 = S$ belongs to the center of $G_{\widetilde{S}}$.

Lemma 6.1. *With the notation above, $\xi_{\widetilde{S}}$ is a generic character of $N_{\widetilde{S}}(F)$.*

Proof. We denote by the same letter the pullbacks of ξ and $\xi_{\widetilde{S}}$ to $\mathfrak{n}(F)$ and $\mathfrak{n}_{\widetilde{S}}(F)$ (via the exponential maps). Let $\mathfrak{n}^{\widetilde{S}}$ be the unique $\mathrm{Ad}(\widetilde{S})$-stable complement of $\mathfrak{n}_{\widetilde{S}}$ in $\mathfrak{n}$. Then, since $\widetilde{S}$ stabilizes ξ, ξ is trivial on $\mathfrak{n}^{\widetilde{S}}(F)$ and it follows that an element of $M_{\widetilde{S}}$ stabilizes $\xi_{\widetilde{S}}$ if and only if it stabilizes ξ. However, by our first assumption, $S = M_{\widetilde{S}} \cap H_0$ is the neutral component of the stabilizer of ξ in $M_{\widetilde{S}}$. As S is included in the center of $G_{\widetilde{S}}$, this implies that $\xi_{\widetilde{S}}$ is generic. □

6.2 Truncations

Let $\widetilde{X}$, $\widetilde{X}_M$ be the twisted symmetric spaces associated to $(\widetilde{G}, \iota)$, $(\widetilde{M}, \iota)$ respectively (see Section 3.7). More precisely, the underlying varieties are $X = G^\iota\backslash G$, $X_M = M^\iota\backslash M$ and these are equipped with the natural twisted actions of $\widetilde{G}$, $\widetilde{M}$ respectively. We fix from now on a special compact subgroup $K \subset G(F)$ in good position relative to M and we set $K_M = K \cap M(F)$. In Section 3.7, we have defined real affine spaces $\mathcal{A}_{\widetilde{X},K}$ and $\mathcal{A}_{\widetilde{X}_M,K_M}$. We claim that there is a natural identification $\mathcal{A}_{\widetilde{X},K} \simeq \mathcal{A}_{\widetilde{X}_M,K_M}$. Indeed, for any minimal ι-split parabolic subspace $\widetilde{P}_0 \subset \widetilde{P}$, we have by definition canonical isomorphisms of real affine spaces

$$\mathcal{A}_{\widetilde{X},K} \simeq \mathcal{A}_{\widetilde{P}_0,\iota} \simeq \mathcal{A}_{\widetilde{P}_0\cap\widetilde{M},\iota} \simeq \mathcal{A}_{\widetilde{X}_M,K_M}$$

and the resulting isomorphism $\mathcal{A}_{\widetilde{X},K} \simeq \mathcal{A}_{\widetilde{X}_M,K_M}$ does not depend on the choice of $\widetilde{P}_0$. We fix a map $H_{X_M} : X_M(F)/K_M \to \mathcal{A}_{X_M,K_M}$ satisfying the requirements of Proposition 3.11 and, as in Section 3.7, we let $H_{\widetilde{X}_M} : X_M(F)/K_M \to \mathcal{A}_{\widetilde{X}_M,K_M}$ be the composition of H_{X_M} with the natural projection $\mathcal{A}_{X_M,K_M} \to \mathcal{A}_{\widetilde{X}_M,K_M}$.

Recall also that the vector space associated to $\mathcal{A}_{\widetilde{X},K}$ is the limit $\mathcal{A}_{\widetilde{X}} = \varprojlim_{\widetilde{P}_0} \mathcal{A}_{\widetilde{P}_0,\iota}$ where $\widetilde{P}_0$ runs over all minimal ι-split parabolic subspaces $\widetilde{P}_0 \subset \widetilde{G}$ and the transition maps are given by conjugation by elements of $G(F)$. As explained in Section 3.7, there is a characteristic function $\phi_{\widetilde{X}} : \mathcal{A}_{\widetilde{X}} \to \{0,1\}$ which, upon identifying $\mathcal{A}_{\widetilde{X}}$ with $\mathcal{A}_{\widetilde{P}_0,\iota}$, is given by $\phi_{\widetilde{P}_0,\iota}$ for any minimal ι-split parabolic subspace $\widetilde{P}_0 \subset \widetilde{G}$.

Note that by the Iwasawa decomposition $G(F) = P(F)K$, we have a natural identifications of cosets

$$H(F)\backslash G(F)/K = H_0(F)\backslash M(F)/K_M.$$

Moreover, there is a natural map $H_0(F)\backslash M(F)/K_M \to X_M(F)/K_M$ given by the composition of the surjection $H_0(F)\backslash M(F) \twoheadrightarrow M^\iota(F)\backslash M(F)$ with the natural inclusion $M^\iota(F)\backslash M(F) \subset X_M(F) = (M^\iota\backslash M)(F)$.

For $Y \in \mathcal{A}_{\widetilde{X},K}$, we define a characteristic function $\kappa_Y : H(F)\backslash G(F)/K \to \{0,1\}$ by the following (commutative) diagram:

$$\begin{array}{ccccc} H(F)\backslash G(F)/K & = & H_0(F)\backslash M(F)/K_M & \longrightarrow & X_M(F)/K_M \\ \downarrow{\scriptstyle \kappa_Y} & & & & \downarrow{\scriptstyle H_{\widetilde{X}_M}} \\ \{0,1\} & \xleftarrow[\phi_{\widetilde{X}}(.-Y)]{} & \mathcal{A}_{\widetilde{X},K} & = & \mathcal{A}_{\widetilde{X}_M,K_M}. \end{array}$$

In other words, identifying elements in $M(F)$ with their image in $\widetilde{X}_M(F)$, κ_Y is characterized by the following property: for every $(m,u,k) \in M(F) \times N(F) \times K$ we have $\kappa_Y(muk) = \phi_{\widetilde{X}}(H_{\widetilde{X}_M}(m) - Y)$.

Proposition 6.2. *(1) For every ι-split parabolic subspace $\widetilde{Q} \subset \widetilde{\widetilde{P}}$, there is a constant $\epsilon > 0$ such that, setting $\widetilde{L} = \widetilde{Q} \cap \iota(\widetilde{Q})$, the following holds: for every $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$, $x \in L[\leq \epsilon d(Y)]$ and $a \in A^+_{\widetilde{Q},\iota}$, we have*

$$\kappa_Y(ax) = \phi_{\widetilde{Q},\iota}(H_{\widetilde{L},\iota}(ax) - Y_{\widetilde{Q},\iota}).$$

(2) There exists a constant $C > 0$, such that for every $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$, we have

$$\mathrm{Supp}(\kappa_Y) \cap X_M(F) \subseteq A_{P,\iota}(F) X_M[\leq CN(Y)].$$

(3) For any fixed $x \in G(F)$, there exists $C_2 > 0$ such that for every $a \in A_{P,\iota}(F)$ satisfying $\kappa_Y(ax) = 1$ we have $\langle \varpi, H_{\widetilde{M},\iota}(a)\rangle \leq C_2$ for every $\varpi \in \widehat{\Delta}_{\widetilde{P},\iota}$.

Proof. Let us fix a minimal ι-split parabolic subspace $\widetilde{P}_0 \subset \widetilde{\widetilde{P}}$ and identify both $\mathcal{A}_{\widetilde{X}}$ and $\mathcal{A}_{\widetilde{X},K}$ with $\mathcal{A}_{\widetilde{P}_0,\iota}$ via the natural isomorphisms. Thus, for every $x \in X_M(F)$ we have $\kappa_Y(x) = \phi_{\widetilde{P}_0,\iota}(H_{\widetilde{X}_M}(x) - Y)$ where we recall that

$$\phi_{\widetilde{P}_0,\iota}(H_{\widetilde{X}_M}(x) - Y) = 1 \Leftrightarrow \varpi(H_{\widetilde{X}_M}(x) - Y) \leq 0 \ \forall \varpi \in \widehat{\Delta}_{\widetilde{P}_0,\iota}.$$

Let $\widetilde{Q} \subset \widetilde{\widetilde{P}}$ be a ι-split parabolic subspace $Y \in \mathcal{A}_{\widetilde{X},K}$ and $\epsilon > 0$. We are going to prove that point (1) is satisfied provided ϵ is sufficiently small. Of course, we may and will assume that $\widetilde{P}_0 \subset \widetilde{Q}$. Let $x \in X_L[\leqslant \epsilon d(Y)]$ and $a \in A^+_{\widetilde{Q},\iota}$. By Proposition 3.12, there exists an absolute constant $c > 0$ (depending only on $\widetilde{Q}$) and a ι-split parabolic subspace $\widetilde{Q} \subset \widetilde{Q}' \subset \tilde{\bar{P}}$ satisfying (where we have set $\widetilde{L}' = \widetilde{Q}' \cap \iota(\widetilde{Q}')$)

$$||H_{\widetilde{X}_M}(xa) - H_{\widetilde{L}',\iota}(xa)|| < c\epsilon d(Y), \ proj_{\widetilde{Q}'} H_{\widetilde{X}_M}(xa) = H_{\widetilde{L}',\iota}(xa),$$

and

$$||H_{\widetilde{L}',\iota}(xa) - H_{\widetilde{L},\iota}(xa)|| < c\epsilon d(Y).$$

Furthermore, there exists an (absolute) constant $c_0 > 0$ such that $\varpi(Y) > c_0 d(Y)$ for every $\varpi \in \widehat{\Delta}_{\widetilde{P}_0,\iota}^{\widetilde{Q}'}$. This implies, by the above, that for $\varpi \in \widehat{\Delta}_{\widetilde{P}_0,\iota}^{\widetilde{Q}'}$, provided ϵ is small enough, we have

$$\varpi(H_{\widetilde{X}_M}(xa) - Y) = \varpi(H_{\widetilde{X}_M}(xa) - Y - H_{\widetilde{L}',\iota}(xa)) \leq \varpi(H_{\widetilde{X}_M}(xa) - H_{\widetilde{L}',\iota}(xa)) - c_0 d(Y) \leq 0.$$

Similarly, for $\varpi \in \widehat{\Delta}_{\widetilde{Q},\iota}^{\widetilde{Q}'}$ and provided ϵ is small enough, we have

$$\varpi(H_{\widetilde{L},\iota}(xa) - Y_{\widetilde{Q},\iota}) = \varpi(H_{\widetilde{L},\iota}(xa) - Y - H_{\widetilde{L}',\iota}(xa)) \leq 0.$$

That is we have

$$\phi_{\widetilde{P}_0,\iota}^{\widetilde{Q}'}(H_{\widetilde{X}_M}(xa) - Y) = \phi_{\widetilde{Q},\iota}^{\widetilde{Q}'}(H_{\widetilde{L},\iota}(xa) - Y_{\widetilde{Q},\iota}) = 1.$$

Now, as is well-known, for $H \in \mathcal{A}_{\widetilde{P}_0,\iota}$ with $\phi_{\widetilde{P}_0,\iota}^{\widetilde{Q}'}(H) = 1$, we have $\phi_{\widetilde{P}_0,\iota}(H) = \phi_{\widetilde{Q}',\iota}(H)$. Therefore,

$$\phi_{\widetilde{P}_0,\iota}(H_{\widetilde{X}_M}(xa) - Y) = \phi_{\widetilde{Q}',\iota}(H_{\widetilde{X}_M}(xa) - Y) = \phi_{\widetilde{Q}',\iota}(H_{\widetilde{L}',\iota}(xa) - Y_{\widetilde{Q}',\iota})$$

and

$$\phi_{\widetilde{Q},\iota}(H_{\widetilde{L},\iota}(xa) - Y_{\widetilde{Q},\iota}) = \phi_{\widetilde{Q}',\iota}(H_{\widetilde{L}',\iota}(xa) - Y_{\widetilde{Q}',\iota})$$

where we have used that the projections of $H_{\widetilde{L},\iota}(xa) - Y_{\widetilde{Q},\iota}$ and $H_{\widetilde{X}_M}(xa) - Y$ to $\mathcal{A}_{\widetilde{Q}',\iota}$ are $H_{\widetilde{L}',\iota}(xa) - Y_{\widetilde{Q}',\iota}$ and $H_{\widetilde{L}',\iota}(xa) - Y_{\widetilde{Q}',\iota}$ respectively. This proves point (1).

(2) follows from Proposition 3.11(3) and (4). Indeed, let $Y^- \in \mathcal{A}_{P_0,\iota}$ be such that H_{X_M} has image in $Y^- + \mathcal{A}_{P_0^M,\iota}^+$ where we have set $P_0^M = P_0 \cap M$ (a minimal ι-split parabolic subgroup of M). Then, the restriction of the projection map $\mathcal{A}_{P_0,\iota}^P \to \mathcal{A}_{\widetilde{P}_0,\iota}^{\widetilde{P}}$ to the image of $Y^- + \mathcal{A}_{P_0^M,\iota}^+$ is proper and this implies that the intersection of the support of $\phi_{\widetilde{P}_0,\iota}(. - Y_{\widetilde{P}_0,\iota})$ (seen as a characteristic function on $\mathcal{A}_{P_0,\iota}$ by the previous projection) with $Y^- + \mathcal{A}_{P_0^M,\iota}^+$ is contained in the sum of $\mathcal{A}_{P,\iota}$ with a ball centered at 0 of radius $C'N(Y)$ for a certain $C' > 0$.

For (3), by the Iwasawa decomposition we may assume that $x \in M(F)$. Then $H_{\widetilde{X}_M}(ax) = H_{\widetilde{M},\iota}(a) + H_{\widetilde{X}_M}(x)$. Then (3) just follows from the definition of κ_Y.

□

6.3 The geometric expansion

For $f \in C_c^\infty(\widetilde{G}(F))$ and $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$, define

$$I(f,x) = \int_{\widetilde{H}(F)} f(x^{-1}hx)(\xi \otimes \chi)(h)dh, \ x \in G(F);$$

$$I_Y(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} I(f,x)\kappa_Y(x)dx.$$

If the (G,H) is tempered (see Subsection 3.2), we can also define $I(f,x)$ and $I_Y(f)$ for $f \in \mathcal{C}(\widetilde{G}(F))$. It is then clear that the integral defining $I(f,x)$ is absolutely convergent.

Lemma 6.3. *The integral defining $I_Y(f)$ is absolutely convergent.*

Proof. It is enough to show that the integral

$$\int_{H(F)A_P(F)\backslash G(F)}\int_{(A_P\cap H)(F)A_{\widetilde{G}}(F)\backslash A_P(F)} I(f,ax)\kappa_Y(ax)\delta_P(a)^{-1}dadx$$

is absolutely convergent. By Proposition 6.2(2) and the Iwasawa decomposition, the integrand of the outer integral over $H(F)A_P(F)\backslash G(F)$ is compactly supported, hence it is enough to show that, for each $x\in H(F)A_P(F)\backslash G(F)$, the inner integral is absolutely convergent, which (since $(A_P\cap H_0)(F)A_{P,\iota}(F)$ is of finite index in $A_P(F)$) is equivalent to show that the absolute convergence of the expression

$$\int_{A_{\widetilde{G},\iota}(F)\backslash A_{P,\iota}(F)} I(f,ax)\kappa_Y(ax)\delta_P(a)^{-1}da =$$
$$\int_{A_{G,\iota}(F)\backslash A_{P,\iota}(F)}\int_{A_{\widetilde{G},\iota}(F)\backslash A_{G,\iota}(F)} I(f,a_1a_2x)da_1\kappa_Y(a_2x)\delta_P(a_2)^{-1}da_2.$$

Since ξ is a generic character and the function $\gamma\in\widetilde{G}(F)\mapsto f(x^{-1}\gamma x)$ is right-invariant by a compact-open subgroup, there exists a constant $C_{f,x}>0$ (depending on f and x) such that, for $a\in A_{P,\iota}(F)$,

$$I(f,ax)\neq 0\Rightarrow \langle\alpha, H_{M,\iota}(a)\rangle\geq -C_{f,x} \text{ for every } \alpha\in\Delta_{P,\iota}. \tag{6.3.1}$$

Combining this with Proposition 6.2(3), it follows that the integrand of the outer integral over $A_{G,\iota}(F)\backslash A_{P,\iota}(F)$ is compactly supported. Indeed, the above inequality together with Proposition 6.2(3) imply that when $I(f,ax)\kappa_Y(ax)\neq 0$ the image of $H_{\widetilde{M},\iota}(a)$ in $\mathcal{A}_{\widetilde{P},\iota}^{\widetilde{G}}$ belongs to a fixed compact subset (depending only on x and f). However, since the automorphism θ of $\mathcal{A}_{P,\iota}$ induced from the twisted space $(P,\widetilde{P})$ preserves the set of simple roots $\Delta_{P,\iota}$, the restriction of the natural projection $\mathcal{A}_{P,\iota}^{G}\to\mathcal{A}_{\widetilde{P},\iota}^{\widetilde{G}}$ to any translate of the chamber $\mathcal{A}_{P,\iota}^{G,+}$ is proper which implies (using again the inequality (6.3.1)) that the support of the function $a\in A_{P,\iota}(F)\mapsto I(f,ax)\kappa_Y(ax)$ is compact modulo $A_{G,\iota}(F)$.

Finally, we are reduced to show the convergence of

$$\int_{A_{G,\iota}(F)/A_{\widetilde{G},\iota}(F)} I(f,ax)da \tag{6.3.2}$$

for any given $x\in G(F)$. Up to replacing f by its conjugate by x, we may assume that $x=1$. Then

$$I(f,a)=\int_{\widetilde{H}(F)} f(a^{-1}ha)(\xi\otimes\chi)(h)dh=\int_{\widetilde{H}(F)} f(h\theta(a)^{-1}a)(\xi\otimes\chi)(h)dh.$$

where we have denoted by θ the automorphism of $A_{G,\iota}$ induced from the twisted space $\widetilde{G}$. Consider the regular map

$$H\times A_{G,\iota}/A_{\widetilde{G},\iota}\to G$$

$$(h,a) \mapsto h\theta(a)^{-1}a.$$

It is a morphism of linear groups with finite kernel and image HA_G, in particular it is finite. This implies that

(6.3.3) $$\sigma(h\theta(a)^{-1}a) \gg \sigma(h) + \sigma_{A_{G,\iota}/A_{\widetilde{G},\iota}}(a) \text{ for } (h,a) \in \widetilde{H}(F) \times A_{G,\iota}(F)/A_{\widetilde{G},\iota}(F).$$

In particular, if f is compactly supported, the function $a \in A_{G,\iota}(F)/A_{\widetilde{G},\iota}(F) \mapsto I(f,a)$ is also compactly supported which of course implies the convergence of (6.3.2).

Assume now that the pair (G,H) is tempered and that f is a Harish-Chandra Schwartz function. Then, by (6.3.3) and since the function $\Xi^{\widetilde{G}}$ is invariant by $A_G(F)$, for every $d>0$ the integral (6.3.2) is essentially bounded by

$$\int_{\widetilde{H}(F)} \Xi^{\widetilde{G}}(h)\sigma(h)^{-d}dh \times \int_{A_{G,\iota}(F)/A_{\widetilde{G},\iota}(F)} \sigma_{A_{G,\iota}/A_{\widetilde{G},\iota}}(a)^{-d}da.$$

However, as (G,H) is tempered the first integral converges for d large enough and similarly for the second integral. This proves the lemma. □

For $t \in (\widetilde{H}_0)_{rs}(F)$, let $S = H_{0,t}$, $T = M_t$, $\widetilde{S} = St$ and $\widetilde{T} = Tt$. Let $N_{\widetilde{S}}$ be the centralizer of $\widetilde{S}$ in N, which is a maximal unipotent subgroup of $G_{\widetilde{S}}$, the centralizer of $\widetilde{S}$ in G. By Lemma 6.1, we know that $\xi|_{N_{\widetilde{S}}(F)}$ is generic. Also $B_{\widetilde{S}} = SN_{\widetilde{S}}$ is a Borel subgroup of $G_{\widetilde{S}}$.

Let $M(t)$ be the centralizer of the maximal split torus of T in G and let $\widetilde{M}(t) = M(t)t$. Define $Y(g) = (Y_{\widetilde{Q}}(g))_{\widetilde{Q}\in\mathcal{F}_\iota(\widetilde{M}(t))}$ to be

$$Y_{\widetilde{Q}}(g) = Y_{\widetilde{Q},\iota} - H_{\bar{\widetilde{Q}},\iota}(g).$$

We then define

$$\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S) = \int^*_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \Gamma_{\bar{B}_{\widetilde{S}},\iota}(H_{\widetilde{M}(t),\iota}(t^M), Y(x))\xi(t^M n_S (t^M)^{-1})dt^M$$

for $x \in G(F)$ and $n_S \in N_{\widetilde{S},reg}(F)$. We refer the reader to Lemma 4.18 for the definition of the normalized integral $\int^*_{A_{\widetilde{G}}(F)S(F)\backslash T(F)}$. By Lemma 4.21, there exists $C>0, r>0$ and, for any (x,n_S), a unique exponential polynomial $v_{B_{\widetilde{S}},\xi,\iota,\cdot}(x,n_S) \in Pol_{\leqslant r}$ whose exponents belongs to a finite set independent of (x,n_S), such that

$$\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S) = v_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)$$

for all Y and (x,n_S) such that $1_{<d(Y)/C}(x,n_S) \neq 0$. Here, we recall that $d(Y)$ denotes the depth Y defined in Subsection 3.8 (we also refer the reader to this subsection for the norm $N(Y)$ that appears in the lemma below). Also for $c>0$, $1_{<c}(\cdot,\cdot)$ stands for the characteristic function of the subset

$$\{(x,n_S) \in A_{\widetilde{G}}(F)\backslash G(F) \times N_{\widetilde{S}}(F) |\ \sigma_{A_{\widetilde{G}}\backslash G}(x), \sigma_{N_{\widetilde{S},reg}}(n_S) < c\}.$$

From Lemma 4.18, we have:

Lemma 6.4. *There exists $d>0$ such that*

$$\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)\ll N(Y)^d\cdot\sigma_{G/A_{\widetilde{G}}}(x)^d\sigma_{N_{\widetilde{S},reg}}(n_S)^d$$

and

$$v_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)\ll N(Y)^d\cdot\sigma_{G/A_{\widetilde{G}}}(x)^d\sigma_{N_{\widetilde{S},reg}}(n_S)^d$$

for all $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M}$, $x\in G(F)$ and $n_S\in N_{\widetilde{S},reg}(F)$.

Recall that the set of regular semisimple conjugacy classes $\Gamma(\widetilde{H}_0)$ was equipped with a measure in Subsection 2.4. We define

$$\tilde{J}_Y(f)=\int_{\Gamma(\widetilde{H}_0)}D^{\widetilde{H}}(t)\tilde{\chi}(t)\int_{B_{\widetilde{S}}(F)\backslash G(F)}\int_{N_{\widetilde{S}}(F)}f(x^{-1}tn_Sx)\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)dn_Sdxdt$$

and

$$J_Y(f)=\int_{\Gamma(\widetilde{H}_0)}D^{\widetilde{H}}(t)\tilde{\chi}(t)\int_{B_{\widetilde{S}}(F)\backslash G(F)}\int_{N_{\widetilde{S}}(F)}f(x^{-1}tn_Sx)v_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)dn_Sdxdt.$$

By Lemma 2.9.3 of [7], Proposition 4.11 and the lemma above, we know that the above two integrals are absolutely convergent for all $f\in C_c^\infty(\widetilde{G}(F))$ (resp. for all $f\in\mathcal{C}(\tilde{G}(F))$ if the model (M,H_0) is tempered). Note that since we have assumed that $(\widetilde{M},\widetilde{H}_0)$ is coregular, the function $t\in\widetilde{H}_0(F)\cap\widetilde{M}_{rs}(F)\mapsto\frac{D^{\widetilde{H}_0}(t)}{D^{\widetilde{M}}(t)^{1/2}}=\frac{D^{\widetilde{H}}(t)}{D^{\widetilde{G}}(t)^{1/2}}$ is locally bounded on $\widetilde{H}_0(F)$. If we further assume that $(\widetilde{M},\widetilde{H}_0)$ is tempered, that function is globally bounded. The geometric expansion is the following theorem (see Lemma 3.6).

Theorem 6.5. *Let $0<\epsilon<1$ and fix $f\in C_c^\infty(\widetilde{G}(F))$. For $k>0$, we have*

$$|I_Y(f)-J_Y(f)|\ll N(Y)^{-k}$$

for every $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y)>\epsilon N(Y)$. Moreover, if the model (M,H_0) is tempered, then the estimates hold for all $f\in\mathcal{C}(\tilde{G}(F))$.

Remark 6.6. *It is clear that in order to prove the above theorem, we only need to prove*

$$|I_Y(f)-\tilde{J}_Y(f)|\ll N(Y)^{-k}$$

and

$$|J_Y(f)-\tilde{J}_Y(f)|\ll N(Y)^{-k}.$$

To end this subsection, we will also state an analogue of the trace formula in Theorem 6.5 for functions with a fixed central character. We fix a character $\omega:A_{\widetilde{G}}(F)\to\mathbb{C}^\times$ whose restriction to $A_{\widetilde{G}}^{H_0}(F)$ coincides with χ where $A_{\widetilde{G}}^{H_0}$ stands for the connected component of $A_{\widetilde{G}}\cap H_0$. For $f\in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F),\omega^{-1})$ and $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M}$, define

$$I(f,x)=\int_{N(F)}\int_{\widetilde{H}_0(F)/A_{\widetilde{G}}^{H_0}(F)}f(x^{-1}hnx)\xi(n)\chi(h)dhdn,\ x\in G(F);$$

$$I_Y(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} I(f,x)\kappa_Y(x)dx.$$

If (M, H_0) is tempered, we can also define $I(f,x)$ and $I_Y(f)$ for $f \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$.

For the geometric side, we define

$$J_Y(f) = \int_{\Gamma(\widetilde{\overline{H_0}})} D^{\widetilde{H}}(t)\chi(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x) v_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S) dn_S dx dt.$$

where $\overline{H_0} = H_0/A_{\widetilde{G}}^{H_0}$ and $\Gamma(\widetilde{\overline{H_0}})$ is the set of regular semisimple conjugacy classes of the twisted space $\widetilde{\overline{H_0}}(F)$ equipped with a measure defined in Subsection 2.4. The following theorem is an analogue of Theorem 6.5 and it can be proved by the exactly same argument.

Theorem 6.7. *Let $0 < \epsilon < 1$ and fix $f \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$. For $k > 0$, we have*

$$|I_Y(f) - J_Y(f)| \ll N(Y)^{-k}$$

for every $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y) > \epsilon N(Y)$. Moreover, if the model (M, H_0) is tempered, then the estimates hold for all $f \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$.

The goal of the rest of this section is to prove Theorem 6.5.

6.4 Some reduction

Recall that we have defined

$$I(f,x) = \int_{N(F)} \int_{\widetilde{H}_0(F)} f(x^{-1}hnx)\xi(n)\tilde{\chi}(h)dhdn, \; x \in G(F);$$

$$I_Y(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} I(f,x)\kappa_Y(x)dx.$$

By the Weyl's integration formula (applied to $\widetilde{H}_0$), we have

$$I_Y(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} \kappa_Y(x) \int_{N(F)} \int_{\widetilde{H}_0(F)} f(x^{-1}hnx)\xi(n)\tilde{\chi}(h)dhdndx$$

$$= \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} \kappa_Y(x) \int_{N(F)} \int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}_0}(t) \int_{S(F)\backslash H_0(F)} f(x^{-1}h^{-1}thnx)\xi(n)\tilde{\chi}(t)dhdtdndx$$

$$= \int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}_0}(t)\tilde{\chi}(t) \int_{A_{\widetilde{G}}(F)S(F)N(F)\backslash G(F)} \kappa_Y(x) \int_{N(F)} f(x^{-1}tnx)\xi(n)dndxdt$$

$$= \int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}_0}(t)\tilde{\chi}(t) \int_{T(F)N(F)\backslash G(F)} \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x)$$

$$\int_{N(F)} f(x^{-1}tnx)\xi(t^M n(t^M)^{-1})dndt^M dxdt.$$

Here as in the previous subsection, for $t \in (\widetilde{H}_0)_{rs}(F)$, we let $S = H_{0,t}$, $T = M_t$, $\widetilde{S} = St$ and $\widetilde{T} = Tt$. It is easy to see that the isomorphism

$$N_{\widetilde{S}}(F) \times^{N_{\widetilde{S}}(F)} N(F) \to N(F) : \ (n_S, n) \mapsto (Ad_t)^{-1}(n^{-1})n_S n$$

has Jacobian $D^{\widetilde{H}_0}(t)^{-1}D^{\widetilde{H}}(t)$, hence the above expression is equal to

$$\int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{T(F)N(F)\backslash G(F)} \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x) \int_{N_{\widetilde{S}}(F)\times^{N_{\widetilde{S}}(F)} N(F)}$$
$$f(x^{-1}t(Ad_t)^{-1}(n^{-1})n_S nx)\xi(t^M(Ad_t)^{-1}(n^{-1})n_S n(t^M)^{-1})d(n_S,n)dt^M dxdt$$

$$= \int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{T(F)N(F)\backslash G(F)} \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x) \int_{N_{\widetilde{S}}(F)\times^{N_{\widetilde{S}}(F)} N(F)}$$
$$f(x^{-1}n^{-1}tn_S nx)\xi(t^M n_S(t^M)^{-1})d(n_S,n)dt^M dxdt$$
$$= \int_{\Gamma(\widetilde{H}_0)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{T(F)N_{\widetilde{S}}(F)\backslash G(F)}$$
$$\int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x) \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x)\xi(t^M n_S(t^M)^{-1})dn_S dt^M dxdt.$$

Definition 6.8. *With the notation above, we define*

$$\kappa_{Y,\widetilde{S},\xi}(x,n_S) = \int^*_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x)\xi(t^M n_S(t^M)^{-1})dt^M$$

$$:= \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \int_{T_c} \kappa_Y(t^M tx)\xi(t^M tn_S(t^M t)^{-1})dtdt^M$$

and

$$I_{Y,\widetilde{S}}(f) = \int_{\widetilde{S}(F)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x)\kappa_{Y,\widetilde{S},\xi}(x,n_S)dn_S dxdt.$$

We also define

$$\tilde{J}_{Y,\widetilde{S}}(f) = \int_{\widetilde{S}(F)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x)\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)dn_S dxdt$$

and

$$J_{Y,\widetilde{S}}(f) = \int_{\widetilde{S}(F)} D^{\widetilde{H}}(t)\tilde{\chi}(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x)v_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)dn_S dxdt.$$

Lemma 6.9. *There exists $d > 0$ such that*

$$\kappa_{Y,\widetilde{S},\xi}(x, n_S) \ll N(Y)^d \cdot \sigma_{G/A_{\widetilde{G}}}(x)^d \sigma_{N_{\widetilde{S},reg}}(n_S)^d$$

for all $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$, $x \in G(F)$ and $n_S \in N_{\widetilde{S},reg}(F)$.

Proof. This follows from Proposition 6.2. □

We fix $t, S, T, \widetilde{S}, \widetilde{T}$ as above. By the integration formula (2.4.2), in order to prove Theorem 6.5, we only need to establish the following theorem.

Theorem 6.10. *Let $0 < \epsilon < 1$ and fix $f \in C_c^\infty(\widetilde{G}(F))$. For $k > 0$, we have*

$$|I_{Y,\widetilde{S}}(f) - \tilde{J}_{Y,\widetilde{S}}(f)| \ll N(Y)^{-k}$$

and

$$|J_{Y,\widetilde{S}}(f) - \tilde{J}_{Y,\widetilde{S}}(f)| \ll N(Y)^{-k}$$

for every $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y) > \epsilon N(Y)$. Moreover, if the model (M, H_0) is tempered, then the estimates hold for all $f \in \mathcal{C}(\tilde{G}(F))$.

6.5 Comparison of the weights

Recall that for $c > 0$ we have defined the function $1_{<c}(\cdot,\cdot)$ to be the characteristic function of

$$\{(x, n_S) \in A_{\widetilde{G}}(F)\backslash G(F) \times N_{\widetilde{S}}(F) |\ \sigma_{A_{\widetilde{G}}\backslash G}(x) < c, \sigma_{N_{\widetilde{S},reg}}(n_S) < c\}.$$

The goal of this subsection is to prove the following lemma.

Lemma 6.11. *There exists $C > 0$ such that*

$$\kappa_{Y,\widetilde{S},\xi}(x, n_S) = \tilde{v}_{B_{\widetilde{T}},\xi,\iota,Y}(x, n_S)$$

for all $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ and (x, n_S) such that $1_{<N(Y)^\epsilon}(x, n_S) \neq 0$, $d(Y) > \epsilon N(Y)$ and $d(Y) > C$.

Let $M(\widetilde{S}) = M(t)$ be the centralizer of the maximal split torus of T in G and let $\widetilde{M}(\widetilde{S}) = \widetilde{M}(t) = M(\widetilde{S})\widetilde{S}$. For all $\widetilde{Q} \in \mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))$ (resp. $\widetilde{Q} \in \mathcal{F}_{\bar{B}_{\widetilde{S}},\iota}(\widetilde{M}(\widetilde{S}))$), let $\widetilde{L}$ be the Levi factor containing $\widetilde{M}(\widetilde{S})$ and let $\bar{Q}$ be the opposite parabolic subgroup of Q with respect to L. We first need a lemma.

Lemma 6.12. *For all $\widetilde{Q} \in \mathcal{F}_{\bar{B}_{\widetilde{S}},\iota}(\widetilde{M}(\widetilde{S}))$, we have $U_{\bar{Q}} \subset P$.*

The above lemma is a direct consequence of the next lemma.

Lemma 6.13. *Let $A_{T,\iota}$ be the maximal split ι-split torus in $T(F)$. Every root in $\Delta(A_{T,\iota}, N)$ can be written as a linear combination of roots in $\Delta(A_{T,\iota}, N_{\widetilde{S}})$ with nonnegative coefficients.*

Proof. For $a \in A_{T,\iota}$ and a character ξ' of $N(F)$, we say a shrinks ξ' if

$$\lim_{i\to\infty} \xi'(a^{-i}na^i) = 1$$

for all $n \in N(F)$. To prove the lemma, we only need to show that for $a \in A_{T,\iota}$, if a shrinks ξ, then a shrinks all the characters of $N(F)$. The characters of $N(F)$ can be naturally identified with the vector space $\bar{\mathfrak{n}}(F)/[\bar{\mathfrak{n}}(F), \bar{\mathfrak{n}}(F)]$ and we endow it with the natural topology coming from the vector space. We only need to show that a shrinks an open subset of the characters. There exists a ι-split parabolic subgroup $P_{a,\iota}$ of $M = G_0$ such that $\lim_{i\to\infty} a^{-i}pa^i$ exists for all $p \in P_{a,\iota}(F)$. Then we know that a shrinks all the characters ξ' of $N(F)$ of the form

$$\xi'(n) = \xi(m^{-1}nm),\ m \in P_{a,\iota}(F)H_0(F).$$

Since ξ is a generic character and $P_{a,\iota}H_0$ is Zariski open in $G_0 = M$, we know that a shrinks an open subset of the characters. This proves the lemma. $\square$

Let $C' > 0$ be a constant large enough (with respect to $\frac{1}{\epsilon}$ and the inverse of the constant in Prop 6.2(1)). For every positve $(\widetilde{G}, M(\widetilde{S}), \iota)$-orthogonal set $Y_0 = (Y_{0,\widetilde{Q},\iota})_{\widetilde{Q}\in\mathcal{F}_\iota(\widetilde{M}(\widetilde{S}))}$, by using the parabolic subgroup $P = MN$ it induces a $(\widetilde{M}, M(\widetilde{S}), \iota)$-orthogonal set $Y_{0,M} = (Y_{0,M,\widetilde{Q},\iota})_{\widetilde{Q}\in\mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))}$ where $Y_{0,M,\widetilde{Q},\iota} = Y_{0,\widetilde{Q}\cdot\bar{N},\iota}$. We fix an auxiliary positive $(\widetilde{G}, M(\widetilde{S}), \iota)$-orthogonal set $Y_0 = (Y_{0,\widetilde{Q},\iota})_{\widetilde{Q}\in\mathcal{F}_\iota(\widetilde{M}(\widetilde{S}))}$ satisfies the following conditions.

- We have
  (6.5.1)
  $$\frac{1}{C'^2}N(Y) < N(Y_{0,M}) < \frac{1}{C'}N(Y),\ \frac{1}{C'^2}N(Y) < N(Y_0) < \frac{1}{C'}N(Y),\ d(Y_{0,M}) > \epsilon N(Y_{0,M}).$$
- For any $\widetilde{Q} \in \mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))$ and $\alpha \in \Delta(A_{T,\iota}, N)$, we have

$$\alpha(Y_{0,M,\widetilde{Q},\iota}) > \frac{1}{C'^3}N(Y). \tag{6.5.2}$$

It is clear such Y_0 exists (for example, we can take it to be of the form $(\frac{Y_{\widetilde{Q},\iota}}{C''} + X)_{\widetilde{Q}}$ for some $C'' > 0$ and $X \in \mathcal{A}_{\widetilde{M},\iota}$). For $\widetilde{Q} \in \mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))$, define

$$\kappa_{Y,\widetilde{S},\xi}^{Y_0,\widetilde{Q}}(x, n_S) = \int^*_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \kappa_Y(t^M x)\xi(t^M n_S(t^M)^{-1})$$
$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota})dt^M,$$
$$\tilde{v}^{Y_0,\widetilde{Q}}_{B_{\widetilde{S}},\xi,\iota,Y}(x, n_S) = \int^*_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \Gamma_{\bar{B}_{\widetilde{S}},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y(x))\xi(t^M n_S(t^M)^{-1})$$
$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota})dt^M.$$

By (4.8.1), in order to prove Lemma 6.11, it is enough to compare $\kappa_{Y,\widetilde{S},\xi}^{Y_0,\widetilde{Q}}(x, n_S)$ and $\tilde{v}^{Y_0,\widetilde{Q}}_{B_{\widetilde{S}},\xi,\iota,Y}(x, n_S)$ for all $\widetilde{Q} \in \mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))$. Fix $\widetilde{Q} \in \mathcal{F}_\iota^{\widetilde{M}}(\widetilde{M}(\widetilde{S}))$. We are reduced to prove the following lemma.

Lemma 6.14. *There exists $C > 0$ such that*

$$\kappa^{Y_0,\widetilde{Q}}_{Y,\widetilde{S},\xi}(x,n_S) = \tilde{v}^{Y_0,\widetilde{Q}}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)$$

for all $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ and (x,n_S) such that $1_{<N(Y)^\epsilon}(x,n_S) \neq 0$, $d(Y) > \epsilon N(Y)$ and $d(Y) > C$.

Proof. We can rewrite the weighted functions as (T_c is the maximal compact subgroup of $T(F)$)

$$\kappa^{Y_0,\widetilde{Q}}_{Y,\widetilde{S},\xi}(x,n_S) = \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \int_{T_c} \kappa_Y(tt^M x)\xi(tt^M n_S(t^M)^{-1}t^{-1})$$

$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota})dtdt^M,$$

$$\tilde{v}^{Y_0,\widetilde{Q}}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S) = \int_{A_{\widetilde{G}}(F)S(F)\backslash T(F)} \int_{T_c} \Gamma_{\bar{B}_{\widetilde{S}},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y(x))\xi(tt^M n_S(t^M)^{-1}t^{-1})$$

$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota})dt^M.$$

Hence it is enough to show that the two functions

$$t^M \mapsto \Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M)-Y_{0,M,\widetilde{Q},\iota})\cdot\int_{T_c} \kappa_Y(tt^M x)\xi(tt^M n_T(t^M)^{-1}t^{-1})dt$$

$$t^M \mapsto \Gamma_{\bar{B}_{\widetilde{S}},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y(x))\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{T}),\iota}(t^M),Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota})$$

$$\cdot\int_{T_c} \xi(tt^M n_S(t^M)^{-1}t^{-1})dt$$

on $A_{\widetilde{G}}(F)S(F)\backslash T(F)$ are equal to each other. We denote these two functions by F_{1,x,n_S} and F_{2,x,n_S}.

Let $x = mnk$ be the Iwasawa decomposition of x with respect to $P = MN$. Since κ_Y is left $N(F)$-invariant and right K-invariant, the function F_{1,x,n_S} only depends on m. By Lemma 6.12, we know that $N \subset Q$ for all $\widetilde{Q} \in \mathcal{P}_{\bar{B}_{\widetilde{S}},\iota}(\widetilde{M}(\widetilde{S}))$. This implies that the function F_{2,x,n_S} also only depends on m. So we may assume that $x = m \in M(F)$. Let $x = luk$ be the Iwasawa decomposition of x with respect to $\bar{Q} = LU_{\bar{Q}}$. We first prove the following statement.

(1) With the assumption on Y and (x,n_S), once we choose C large enough the above two functions F_{1,x,n_S} and F_{2,x,n_S} only depends on l.

For the function F_{1,x,n_S}, since κ_Y is right K-invariant, we know that it only depends on lu. It is enough to show that for $t^M \in A_{\widetilde{G}}(F)S(F)\backslash T(F)$ with

$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M),Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota}) \neq 0,$$

we have

$$\kappa_Y(t^M lu) = \kappa_Y(t^M l).$$

For $t^M \in A_{\widetilde{G}}(F)S(F)\backslash T(F)$, we can always choose a representative of t^M in $T(F)$, denoted by t_2, such that the projection of $H_{\widetilde{M}(\widetilde{S})}(t_2) \in \mathcal{A}_{\widetilde{M}(\widetilde{S})}$ to $\mathcal{A}^{\iota}_{\widetilde{M}(\widetilde{S})}$ belongs to a compact subset that only depends on $\widetilde{T}$. It is enough to show that

$$\kappa_Y(t_2 lu) = \kappa_Y(t_2 l).$$

The argument is the same as the proof of (5.3.14) of [7] and we will skip it here. This proves (1) for the function F_{1,x,n_S}.

For the function F_{2,x,n_S}, let $\tilde{Q}' = \tilde{Q}\bar{N} \in \mathcal{F}_{\bar{B}_{\widetilde{S}},\iota}(\widetilde{M}(\widetilde{S}))$ and $\tilde{\bar{Q}}' = \tilde{\bar{Q}}N \in \mathcal{F}_{B_{\widetilde{S}},\iota}(\widetilde{M}(\widetilde{S}))$. By Lemma 6.13, we know that there exists $c_1 > 0$ such that if

$$\int_{T_c} \xi(tt^M n_S (t^M)^{-1} t^{-1}) dt \neq 0,$$

then

$$\alpha(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M)) \leq c_1 N(Y)^{\epsilon},\ \forall \alpha \in \Delta(A_{T,\iota}, N). \tag{6.5.3}$$

Combining with the second condition (6.5.2) of Y_0, we know that once we choose $C, C' > 0$ large enough, we have

$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_{0,M})\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,M,\widetilde{Q},\iota}) \int_{T_c} \xi(tt^M n_S (t^M)^{-1} t^{-1}) dt \neq 0$$

$$\Rightarrow \Gamma^{\widetilde{Q}'}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_0)\tau_{\widetilde{Q}',\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q}',\iota}) \neq 0.$$

Combining with (4.8.4) (applied to the case when $\mathcal{X} = Y_0$ and $\mathcal{Y} = Y(x) - Y_0$, note that by the first condition (6.5.1) of Y_0 we know that $\mathcal{Y}$ is positive with $d(\mathcal{Y}) > \frac{d(Y)}{2}$) we get the following statement.

(2) If

$$\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_0)\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q},\iota}) \int_{T_c} \xi(tt^M n_S (t^M)^{-1} t^{-1}) dt \neq 0,$$

then

$$\Gamma_{\bar{B}_{\widetilde{S}},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y(x)) = \phi_{\widetilde{Q}',\iota}(H_{\widetilde{L},\iota}(t^M) - Y_{\widetilde{Q}',\iota} + H_{\tilde{\bar{Q}}',\iota}(x)).$$

In particular, we have proved statement (1) for the function F_{2,x,n_S}.

From now on assume that $x = l \in L(F)$. We only need to prove the following two statements.

(3) With the assumption on Y and (x, n_S), once we choose $C > 0$ large enough, for $t^M \in A_{\widetilde{G}}(F)S(F)\backslash T(F)$ with $F_{1,x,n_S}(t^M) \neq 0$, the following holds.

– $\phi_{\widetilde{Q}',\iota}(H_{\widetilde{L},\iota}(t^M x) - Y_{\widetilde{Q}',\iota}) = 1$.

– $F_{1,x,n_S}(t^M) = \Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_0)\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q},\iota}) \cdot \int_{T_c} \xi(tt^M n_T(t^M)^{-1}t^{-1})dt$.

(4) With the assumption on Y and (x, n_S), once we choose $C > 0$ large enough, for $t^M \in A_{\widetilde{G}}(F)S(F)\backslash T(F)$ with $F_{2,x,n_S}(t^M) \neq 0$, the following holds.

– $\phi_{\widetilde{Q}',\iota}(H_{\widetilde{L},\iota}(t^M x) - Y_{\widetilde{Q}',\iota}) = 1$.

– $F_{2,x,n_S}(t^M) = \Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_0)\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q},\iota}) \cdot \int_{T_c} \xi(tt^M n_T(t^M)^{-1}t^{-1})dt$.

Statement (4) follows from (2). It remains to prove (3). As in the proof of (2), by Lemma 6.13 and (6.5.2), we know that once we choose $C > 0$ large enough, then

$$\tau^{\widetilde{M}}_{\widetilde{Q},\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q},\iota}) \cdot \int_{T_c} \kappa_Y(tt^M x)\xi(tt^M n_S(t^M)^{-1}t^{-1})dt \neq 0 \tag{6.5.4}$$

$$\Rightarrow \tau_{\widetilde{Q}',\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M) - Y_{0,\widetilde{Q}',\iota}) \neq 0.$$

We can choose a representative of t^M in $T(F)$ of the form $t' t_1 a$ where t' belongs to a compact set, $a \in A_{\widetilde{L},\iota}(F)$ and $t_1 \in T_\iota(F)$ with $H_{\widetilde{L},\iota}(t_1) = 0$. Here T_ι is the maximal ι-split torus of T.

Since $\Gamma^{\widetilde{Q}}_{\widetilde{M}(\widetilde{S}),\iota}(H_{\widetilde{M}(\widetilde{S}),\iota}(t^M), Y_{0,M}) \neq 0$, by Proposition 4.16 and (6.5.1), we know that once we choose $C > 0$ large enough, we have

$$\sigma_{G/A_{\widetilde{G}}}(t' t_1) < \frac{d(Y)}{\sqrt{C'}}. \tag{6.5.5}$$

Combining with (6.5.4) and (6.5.2), we know that once we choose $C > 0$ large enough, we can write a as $a = a_1 a_2$ such that $a_1 \in A^+_{\widetilde{Q}',\iota}$ and

$$\sigma_{G/A_{\widetilde{G}}}(t' t_1 a_2 x) < \frac{d(Y)}{\sqrt{C'}}. \tag{6.5.6}$$

Combining (6.5.6) with Proposition 6.2, we know that

$$\kappa_Y(tt^M x) = \kappa_Y(tt' t_1 a_2 x a_1)$$

is equal to 1 if and only if $\phi_{\widetilde{Q}',\iota}(H_{\widetilde{L},\iota}(t^M x) - Y_{\widetilde{Q}',\iota}) = 1$. This proves (3) and finishes the proof of the proposition.

□

6.6 The proof of Theorem 6.10

We have

$$|I_{Y,\widetilde{S}}(f)-\tilde{J}_{Y,\widetilde{S}}(f)|\leq\int_{\widetilde{S}(F)}D^{\widetilde{H}}(t)\int_{B_{\widetilde{S}}(F)\backslash G(F)}\int_{N_{\widetilde{S}}(F)}$$
$$|f(x^{-1}tn_S x)|\cdot|\kappa_{Y,\widetilde{S},\xi}(x,n_S)-\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)|dn_S dx dt.$$

Let $N>0$. By Lemma 6.4, 6.9 and 6.11, there exists $d_0>0$ such that

$$|\kappa_{Y,\widetilde{S},\xi}(x,n_S)-\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)|\ll N(Y)^{-N}(\sigma_{G/A_{\widetilde{G}}}(x)+\sigma_{N_{\widetilde{S},reg}}(n_S))^{d_0}$$

for all $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M},x\in G(F),n_S\in N_{\widetilde{S},reg}(F)$ with $d(Y)>\epsilon N(Y)$. Since the left hand side is invariant under the transform $(x,n_s)\mapsto(bx,bn_s b^{-1})$ for all $b\in B_{\widetilde{S}}(F)$, by Lemma 4.9 it follows that

$$|\kappa_{Y,\widetilde{S},\xi}(x,n_S)-\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)|\ll N(Y)^{-N}(\sigma_{\widetilde{G}_{reg}}(x^{-1}tn_S x)+\sigma_{\widetilde{S}'}(t))^{d_0}$$

for all $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M},x\in G(F),n_S\in N_{\widetilde{S},reg}(F),t\in\widetilde{S}'(F)$ with $d(Y)>\epsilon N(Y)$. Combining this with Proposition 4.11, we deduce that, for any $d>0$, the integral

$$D^{\widetilde{H}}(t)\int_{B_{\widetilde{S}}(F)\backslash G(F)}\int_{N_{\widetilde{S}}(F)}|f(x^{-1}tn_S x)|\cdot|\kappa_{Y,\widetilde{S},\xi}(x,n_S)-\tilde{v}_{B_{\widetilde{S}},\xi,\iota,Y}(x,n_S)|dn_S dx$$

is essentially bounded by

$$N(Y)^{-N}\cdot\sigma_{\widetilde{S}'}(t)^{d_0}\cdot\sigma_{\widetilde{G}}(t)^{-d}$$

for $t\in\widetilde{S}(F)$ and $Y\in\mathcal{A}_{\widetilde{X},K}=\mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y)>\epsilon N(Y)$. Moreover, we can choose $d>0$ large enough such that the expression

$$\int_{\widetilde{S}(F)}\sigma_{\widetilde{G}}(t)^{-d}\sigma_{\widetilde{S}'}(t)^{d_0}dt$$

is convergent (see Lemma 2.9.3 of [7], note that $\sigma_{\widetilde{S}'}(t)\sim\sigma_{\widetilde{G}}(t)+\log(2+D^{\widetilde{G}}(t)^{-1})$). This proves the first inequality of Theorem 6.10. The second inequality follows from the same argument except that we replace Lemma 6.11 by Lemma 4.21. This finishes the proof of Theorem 6.5 and 6.10.

7 Application of the geometric expansion

In this section, we will discuss the application of the geometric expansion in Theorem 6.10. The first application is a simple local twisted trace formula for strongly cuspidal functions in the coregular case. The second application is a multiplicity formula for Whittaker induction of coregular symmetric pairs. We use the same notation as in the previous section.

7.1 A simple local trace formula

With the same notation as in Theorem 6.7, for $f \in C_c^\infty(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ (or $f \in \mathcal{C}(\widetilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ if the model (M, H_0) is tempered) and $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$, we have defined

$$I(f,x) = \int_{N(F)} \int_{\widetilde{H}_0(F)/A_{\widetilde{G}}^{H_0}(F)} f(x^{-1}hnx)\xi(n)\chi(h)dhdn, \ x \in G(F);$$

$$I_Y(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} I(f,x)\kappa_Y(x)dx.$$

We also define

$$I(f) = \int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)} I(f,x)dx$$

whenever this integral is convergent. The next proposition has been proved in Proposition 5.1.

Proposition 7.1. *The integrals defining $I(f,x)$ and $I(f)$ are absolutely convergent for all $f \in C_{c,scusp}^\infty(\tilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ satisfies* (5.0.2)*. If (M, H_0) is tempered, then both integrals are absolutely convergent for all $f \in \mathcal{C}_{scusp}(\tilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ satisfies* (5.0.2).

Remark 7.2. *In fact we can even prove the convergence without the assumption on $\tilde{R}(f)$. But since we will not use it here, we will postpone the proof of the general convergence to our next paper.*

For $t \in \Gamma_{rs}(\widetilde{H}_0)$, let $S, T, \widetilde{S}, \widetilde{T}$ be the same as in the previous section and we let $\mathcal{O}_t = \mathcal{O}_{\widetilde{S}} \in Nil_{reg}(\mathfrak{g}_{\widetilde{S}}^*)$ be the orbit associated to $\xi_{\widetilde{S}}$ as explained at the beginning of Section 4. For a quasi-character Θ on $\widetilde{G}(F)$ with central character ω^{-1}, we define

$$m_{geom,\widetilde{H}}(\Theta) := \int_{\Gamma_{ell}(\widetilde{\widetilde{H_0}})} D^{\widetilde{H}}(t)c_{\Theta,-\mathcal{O}_t}(t)\chi(t)dt.$$

Here $\Gamma_{ell}(\widetilde{\widetilde{H_0}})$ is the set of regular elliptic semisimple conjugacy classes of the twisted space $\widetilde{\widetilde{H_0}}(F)$ equipped with a measure defined in Subsection 2.4. By Lemma 2.4 and coregular assumption, we know that the integral defining $m_{geom,\widetilde{H}}(\Theta)$ is absolutely convergent.

For $f \in \mathcal{C}_{scusp}(\tilde{G}(F)/A_{\widetilde{G}}(F), \omega)$, we define

$$I_{geom}(f) = \nu(\widetilde{H}_0)m_{geom,\widetilde{H}}(\Theta_f), \ \nu(\widetilde{H}_0) = |H_0(F) \cap A_{\widetilde{G}}(F) : A_{\widetilde{G}}^{H_0}(F)|.$$

The next theorem is the geometric side of a simple local twisted trace formula in the coregular case.

Theorem 7.3. *1. We have $I(f) = I_{geom}(f)$ for all $f \in C_{c,scusp}^\infty(\tilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ such that the integral defining $I(f)$ is absolutely convergent.*

2. *If the symmetric pair (M, H_0) is tempered, then $I(f) = I_{geom}(f)$ for all $f \in \mathcal{C}_{scusp}(\tilde{G}(F)/A_{\widetilde{G}}(F), \omega^{-1})$ such that the integral defining $I(f)$ is absolutely convergent.*

Proof. It is enough to show that the limit of $I_Y(f)$ is equal to $I_{geom}(f)$ as $N(Y)$ goes to infinity where Y runs over all the elements in $\mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y) > \epsilon N(Y)$. In the previous section we have also defined

$$J_Y(f) = \int_{\Gamma(\widetilde{\widetilde{H_0}})} D^{\widetilde{H}}(t)\chi(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x) v_{B_{\widetilde{S}},\xi,\iota,Y}(x, n_S) dn_S dx dt$$

and we proved in Theorem 6.7 that

$$|I_Y(f) - J_Y(f)| \ll N(Y)^{-k} \tag{7.1.1}$$

for every $Y \in \mathcal{A}_{\widetilde{X},K} = \mathcal{A}_{\widetilde{X}_M,K_M}$ with $d(Y) > \epsilon N(Y)$. This implies that the limit of $I_Y(f)$ is equal to the limit of $J_Y(f)$ as $N(Y)$ goes to infinity. Since the function

$$Y \mapsto v_{B_{\widetilde{S}},\xi,\iota,Y}(x, n_S)$$

is an exponential polynomial with bounded degree and with exponents in a fixed finite set (both independent of x and n_S), we know that the limit of $J_Y(f)$ is equal to

$$J(f) = \int_{\Gamma(\widetilde{\widetilde{H_0}})} D^{\widetilde{H}}(t)\chi(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x) v_{B_{\widetilde{S}},\xi,\iota}(x, n_S) dn_S dx dt.$$

Here $v_{B_{\widetilde{S}},\xi,\iota}(x, n_S)$ is defined in Section 4.8. In particular, as Y goes to infinity, $J_Y(f)$ is a constant independent of Y.

Fix $t \in \Gamma_{rs}(\widetilde{H}_0)$ and let $S, T, \widetilde{S}, \widetilde{T}$ be as in the previous section. Define

$$J_{\widetilde{S}}(f) = \int_{\widetilde{S}(F)/A_{\widetilde{G}}^{H_0}} D^{\widetilde{H}}(t)\chi(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x) v_{B_{\widetilde{S}},\xi,\iota}(x, n_S) dn_S dx dt.$$

Fix $\varepsilon \in (\mathcal{A}_{B_{\widetilde{S}}}^{+})^{\iota}$ in general position. By Corollary 4.25, we have the descent formula

$$v_{B_{\widetilde{S}},\xi,\iota}(x, n_S) = \sum_{\substack{\widetilde{Q} \in \mathcal{F}_{B_{\widetilde{S}}}(\widetilde{M}(\widetilde{S})) \\ \varepsilon \in (\mathcal{A}_{\widetilde{Q}}^{+})^{\iota}}} d_\varepsilon(\widetilde{Q}) v_{B_{\widetilde{S}},\xi}^{\widetilde{Q},\iota}(x, n_S).$$

This implies that

$$J_{\widetilde{S}}(f) = \sum_{\substack{\widetilde{Q} \in \mathcal{F}_{B_{\widetilde{S}}}(\widetilde{M}(\widetilde{S})) \\ \varepsilon \in (\mathcal{A}_{\widetilde{Q}}^{+})^{\iota}}} d_\varepsilon(\widetilde{Q}) J_{\widetilde{S}}^{\widetilde{Q}}(f)$$

where

$$J_{\widetilde{S}}^{\widetilde{Q}}(f) = \int_{\widetilde{S}(F)/A_{\widetilde{G}}^{H_0}} D^{\widetilde{H}}(t)\chi(t) \int_{B_{\widetilde{S}}(F)\backslash G(F)} \int_{N_{\widetilde{S}}(F)} f(x^{-1}tn_S x) v_{B_{\widetilde{S}},\xi}^{\widetilde{Q},\iota}(x, n_S) dn_S dx dt.$$

By Corollary 4.24 and the assumption that f is strongly cuspidal, we know that $J_{\widetilde{S}}^{\widetilde{Q}}(f)=0$ if $\widetilde{Q}\neq\widetilde{G}$. If $\widetilde{Q}=\widetilde{G}$, then $\varepsilon\in(\mathcal{A}_{\widetilde{G}})^{\iota}$. Since $\varepsilon\in(\mathcal{A}_{B_{\widetilde{S}}}^{+})^{\iota}$ is in general position, we must have $(\mathcal{A}_{B_{\widetilde{S}}}^{+})^{\iota}=(\mathcal{A}_{G_x})^{\iota}=(\mathcal{A}_{\widetilde{G}})^{\iota}$. This implies that $\widetilde{S}$ is elliptic. If this is the case, the function $\Gamma_{B_{\widetilde{S}}}^{\widetilde{G},\iota}$ in the definition of $v_{B_{\widetilde{S}},\xi}^{\widetilde{G},\iota}$ is just the function $\Gamma_{B_{\widetilde{S}}}$. Hence we have

$$v_{B_{\widetilde{S}},\xi}^{\widetilde{G},\iota}(x,n_S)=\nu(H_0)v_{B_{\widetilde{S}},\xi}(x,n_S).$$

Here $\nu(H_0)=|H_0(F)\cap A_{\widetilde{G}}(F):A_{\widetilde{G}}^{H_0}(F)|$ comes from the volume of $S(F)A_{\widetilde{G}}(F):A_{\widetilde{G}}(F)$ which is equal to the volume of $S(F)/A_{\widetilde{G}}^{H_0}(F)$ (which is equal to 1) times $|S(F)\cap A_{\widetilde{G}}(F):A_{\widetilde{G}}^{H_0}(F)|=|H_0(F)\cap A_{\widetilde{G}}(F):A_{\widetilde{G}}^{H_0}(F)|=\nu(\widetilde{H}_0)$. This implies that $J_{\widetilde{S}}(f)$ is equal to 0 if $\widetilde{S}$ is not elliptic, and is equal to

$$\nu(H_0)\int_{\widetilde{S}(F)/A_{\widetilde{G}}^{H_0}}D^{\widetilde{H}}(t)\chi(t)\int_{B_{\widetilde{S}}(F)\backslash G(F)}\int_{N_{\widetilde{S}}(F)}f(x^{-1}tn_Sx)v_{B_{\widetilde{S}},\xi}(x,n_S)dn_Sdxdt$$

if $\widetilde{S}$ is elliptic. Then the theorem follows from Theorem 4.8. □

7.2 The multiplicity formula

We use the same notation as the previous subsection and we assume that the twisted space $\widetilde{G}$ is just G (i.e. the automorphism θ is the identity map). Let π be an irreducible smooth representation of G with central character ω^{-1}. Define

$$m(\pi)=\mathrm{Hom}_{H(F)}(\pi,\chi^{-1}\otimes\xi^{-1}),\ m_{geom}(\pi):=\int_{\Gamma_{ell}(\overline{H_0}}D^{H}(t)\chi(t)c_{\Theta,-\mathcal{O}_t}(t)dt.$$

By [14], we know that the multiplicity $m(\pi)$ is finite. The goal of this subsection is to prove the following multiplicity formula.

Theorem 7.4. *The multiplicity formula $m(\pi)=m_{geom}(\pi)$ holds for all supercuspidal representations of $G(F)$ with central character ω^{-1}. If (M,H_0) is tempered, then the multiplicity formula holds for all discrete series of $G(F)$ with central character ω^{-1}.*

Proof. For $f\in C_{c,scusp}^{\infty}(G(F)/A_G(F),\omega^{-1})$ (or $f\in\mathcal{C}_{scusp}(G(F)/A_G(F),\omega^{-1})$ if (M,H_0) is tempered), we have defined

$$I(f,x)=\int_{N(F)}\int_{H_0(F)/A_G^{H_0}(F)}f(x^{-1}hnx)\xi(n)\chi(h)dhdn,\ I(f)=\int_{H(F)A_{\widetilde{G}}(F)\backslash G(F)}I(f,x)dx.$$

in the previous subsection. Proposition 5.1, (5.0.5), (5.0.6) and Theorem 7.3 implies that

$$I(f)=\nu(H_0)m_{geom,H}(\Theta_f)$$

for all $f \in {}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1}) \cap C_c^\infty(G(F)/A_G(F),\omega^{-1})$ (or $f \in {}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1})$ if (M,H_0) is tempered). Here ${}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1})$ is the span of matrix coefficients of discrete series of $G(F)$ with central character ω^{-1}. For $f \in {}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1})$, define

$$I_{spec}(f) = \nu(\widetilde{H}_0) \sum_{\pi \in \Pi_{disc}(G,\omega^{-1})} \mathrm{tr}(\pi(\bar{f}))m(\pi).$$

By Theorem 4.1.1 of [7], we have the spectral expansion

$$\nu(H_0)m_{geom,H}(\Theta_f) = I(f) = I_{spec}(f) \tag{7.2.1}$$

for all $f \in {}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1}) \cap C_c^\infty(G(F)/A_G(F),\omega^{-1})$ (or $f \in {}^\circ\mathcal{C}(G(F)/A_G(F),\omega^{-1})$ if (M,H_0) is tempered). Then the multiplicity formula follows from (7.2.1). All we need to do is to let f be the matrix coefficient of a supercuspidal representation (or a discrete series if (M,H_0) is tempered). This finishes the proof of the multiplicity formula. □

Remark 7.5. *Some special cases of the multiplicity formula proved in the above theorem are the multiplicity formulas for the Galois models and the generalized Shalika models proved in [7] and [9] .*

8 The unitary Shalika model

In this section we will prove our main theorems for the unitary Shalika model (i.e. Theorem 1.9, 1.10, and 1.12). In Section 8.1 we will recall the defintion of the models and prove a comparison between the unitary Shalika model and the twisted Shalika model (for general linear groups). Then in Section 8.2 we will prove Theorem 1.9, 1.10, and 1.12.

8.1 Some comparison

Let Z be a E-vector space of finite dimension $n \geqslant 1$. Let $Z^{*,c}$ be the conjugate-dual of Z that is the space of c-linear forms on Z (a similar notation will be applied later to other vector spaces). Set $V = Z \oplus Z^{*,c}$ and we equip with the nondegenerate Hermitian form

$$h(v+v^*, w+w^*) = \langle v, w^*\rangle + \langle w, w^*\rangle^c, \quad (v,v^*),(w,w^*) \in Z \oplus Z^{*,c}.$$

Here $\langle .,. \rangle$ stands for the canonical pairing between Z and $Z^{*,c}$. Let $G = U(V,h)$ be the unitary group associated to this Hermitian form. We define two maximal parabolic subgroups Q and $\overline{Q}$ of G as the stabilizers of the maximal isotropic subspaces Z and $Z^{*,c}$ respectively. Then, $L = Q \cap \overline{Q}$ is a Levi component of Q and restriction to Z induces an isomorphism

$$L \simeq \mathrm{Res}_{E/F} GL(Z). \tag{8.1.1}$$

Let N be the unipotent radical of Q. Thus $Q = LN$ and restriction to $Z^{*,c}$ induces an isomorphism

$$N \simeq \left\{ X \in \mathrm{Hom}(Z^{c,*}, Z) \mid {}^T X^c = -X \right\} \tag{8.1.2}$$

where ${}^T X^c$ denotes the transpose conjugate of X (seen as a linear endomorphism $Z \to Z^{*,c}$ through the canonical identification $(Z^{*,c})^{*,c} = Z$). We will actually identify the right hand side above with the Lie algebra $\mathfrak{n}$ of N in a way such that the above isomorphism becomes the exponential map.

We henceforth choose two isomorphisms $W_+, W_- : Z \to Z^{*,c}$ satisfying ${}^T W_\pm^c = -W_\pm$ and such that the corresponding antihermitian forms on Z are not equivalent (there are actually only two equivalence classes of antihermitian forms on Z). For $\epsilon \in \{\pm\}$, we let $H_{0,\epsilon} \subset L \simeq \mathrm{Res}_{E/F} GL(Z)$ be the unitary group associated to W_ϵ, that is the stabilizer of W_ϵ for the obvious action. Then, $H_{0,\epsilon}(F)$ coincides with the stabilizer in $L(F)$ of the character

$$\xi_\epsilon : N(F) \to \mathbb{C}^\times,$$

$$\exp(X) \mapsto \psi(\mathrm{Tr}(W_\epsilon X)) \;\; (X \in \mathfrak{n}(F)).$$

We will henceforth assume, as we may, that $W_\pm$ have been chosen so that $H_{0,+}$ is quasi-split.

Set $H_\epsilon = H_{0,\epsilon} \ltimes N$. We extend ξ_ϵ to a character of $H_\epsilon(F)$ trivial on $H_{0,\epsilon}(F)$. We also fix a character χ of $E^1 = \ker(N_{E/F})$ that we will consider as a character of $H_{0,\epsilon}(F)$ through composition with the determinant $\det : H_{0,\epsilon}(F) \to E^1$. For a smooth irreducible representation π of $G(F)$, we define the multiplicity

$$m_\epsilon(\pi, \chi) := \dim(\mathrm{Hom}_{H_\epsilon(F)}(\pi, \chi \otimes \xi_\epsilon)).$$

For $x \in H_{0,\epsilon}(F)_{\mathrm{ell}}$, the centralizer $G_x = Z_G(x)$ is quasi-split and the intersection $N_x := G_x \cap N$ is a maximal unipotent subgroup of it. Moreover, by restriction ξ_ϵ induces a non-degenerate character of $N_x(F)$. We let $\mathcal{O}_x$ be the regular coadjoint nilpotent orbit in $\mathfrak{g}_x^*$ associated to it. For any quasi-character Θ on $G(F)$, we set

$$J_{\epsilon,\chi,geom}(\Theta) = \int_{\Gamma_{\mathrm{ell}}(H_{0,\epsilon})} D^G(x)^{1/2} c_{\Theta,\mathcal{O}_x}(x)\chi(x)^{-1} dx, \; J_{\chi,geom}(\Theta) = J_{+,\chi,geom}(\Theta) + J_{-,\chi,geom}(\Theta).$$

By Theorem 7.4, the multiplicity formula

$$m_\epsilon(\pi, \chi) = J_{\epsilon,\chi,geom}(\Theta_\pi)$$

holds for all discrete series.

Recall that two semisimple regular elements $x, y \in G_{\mathrm{rs}}(F)$ are said to be *stably conjugated* if they are conjugated in $G(\overline{F})$ and that a quasi-character Θ on $G(F)$ is called *stable* if it is constant on stable conjugacy classes (that is if $x, y \in G_{\mathrm{rs}}(F)$ are stably conjugated then $\Theta(x) = \Theta(y)$). If Θ is stable it is clear that we have $c_{\Theta,\mathcal{O}_x}(x) = c_\Theta(x)$. The following comparison between the geometric sides will be used in our applications.

Proposition 8.1. *Assume that Θ is a stable quasi-character on $G(F)$. Then*

$$J_{\chi,geom,+}(\Theta) = J_{\chi,geom,-}(\Theta).$$

Proof. This follows from the following two facts

- there is a natural measure-preserving bijection $x_+ \leftrightarrow x_-$ between the regular elliptic stable conjugacy classes of $H_{0,+}(F)$ and the regular elliptic stable conjugacy classes of $H_{0,-}(F)$;
- under the above bijection $x_+ \leftrightarrow x_-$, the number of rational conjugacy classes in a regular elliptic stable conjugation class x_+ of $H_{0,+}(F)$ is equal to the number of rational conjugacy classes in a regular elliptic stable conjugation class x_- of $H_{0,-}(F)$ and we have $c_\Theta(x_+) = c_\Theta(x_-)$.

□

Set $G' = Res_{E/F}\mathrm{GL}(V)$ and let Q', $\overline{Q}'$ be the maximal parabolic subgroups of G' stabilizing the subspaces Z and $Z^{*,c}$ respectively. Then, $L' := Q' \cap \overline{Q}'$ is a Levi component of Q' and we have an isomorphism (given by restriction)

$$L' \simeq \mathrm{Res}_{E/F}(\mathrm{GL}(Z) \times \mathrm{GL}(Z^{*,c})). \tag{8.1.3}$$

We fix an isomorphism $W : Z \simeq Z^{*,c}$ satisfying ${}^TW^c = -W$ and we let $H_0' \subset L'$ be the subgroup $\{(h, WhW^{-1}) \mid h \in \mathrm{Res}_{E/F}\mathrm{GL}(Z)\}$.

Let N' be the unipotent radical of Q' (so that $Q' = L'N'$). We will identify its Lie algebra $\mathfrak{n}'$ with $\mathrm{Res}_{E/F}\mathrm{Hom}(Z^{*,c}, Z)$ and we define a character of $N'(F)$ by

$$\xi' : \exp(X) \in N'(F) \mapsto \psi(\mathrm{tr}_{E/F}(\mathrm{Tr}(WX))), \;\; X \in \mathfrak{n}'(F).$$

We let $H' = H_0' \ltimes N'$ be the *Shalika subgroup*. The character χ of E^1 induces a character χ' of $E^\times$ by $\chi'(x) = \chi(x/x^c)$ and we will identify χ' with the character of $H_0'(F)$ given by $(h, WhW^{-1}) \mapsto \chi'(\det h))$.

For every $g \in G$, let us denote by $g^\star$ the adjoint linear map with respect to the Hermitian form h on V. We define θ to be the automorphism $g \mapsto (g^\star)^{-1}$ of G and we let $\widetilde{G} = G\theta$ be the nonneutral component of the nonconnected group $G \rtimes \{1, \theta\}$. It is a twisted space in the sense of §2.2. We also set $\tilde{Q}' = Q'\theta$, $\tilde{L}' = L'\theta$. These are respectively a twisted parabolic subspace of $\tilde{G}'$ and a Levi component of it. The automorphism θ preserves H_0' and H' and we let $\tilde{H}_0' = H_0'\theta$, $\tilde{H}' = H'\theta$ be the corresponding twisted spaces. The character χ' of $H_0'(F)$ being conjugate self-dual, it can be extended to the twisted space $\widetilde{H}_0'$. We fix such an extension whose value at θ is equal to 1 and we still denote by χ'.

For every quasi-character $\widetilde{\Theta}$ on $\widetilde{G}'(F)$, we define

$$\tilde{J}_{\chi',geom}(\widetilde{\Theta}) = \int_{\Gamma_{\mathrm{ell}}(\widetilde{H}_0')} D^{\widetilde{G}'}(x)^{1/2} c_{\widetilde{\Theta}}(x)\chi'(x)^{-1}dx.$$

Let

$$Nr : \widetilde{G}'(F) \to G'(F), \;\; g\theta \mapsto g\theta(g)$$

be the norm map. Recall that an element $x \in \widetilde{G}'_{\mathrm{rs}}(F)$ is said to be *G-regular* if $Nr(x)$ is regular and that if $x \in \widetilde{G}'_{\mathrm{rs}}(F)$ is G-regular, an element $y \in G_{\mathrm{rs}}(F)$ is called a *norm of* x if it is conjugated to $Nr(x)$ inside $G'(F)$ (note that $G(F) \subset G'(F)$). Remark that if $y \in G_{\mathrm{rs}}(F)$ is a norm of x and $y' \in G_{\mathrm{rs}}(F)$ is stably conjugated to y then y' is also a norm of x (this is because in $G'(F)$ there is no difference between conjugation and stable conjugation). Let Θ be a stable quasi-character on $G(F)$. We also recall that a quasi-character $\widetilde{\Theta}$ on $\widetilde{G}'(F)$ is said to be a *transfer* of Θ if for every G-regular element $x \in \widetilde{G}'_{\mathrm{rs}}(F)$ and every $y \in G_{\mathrm{rs}}(F)$ that is a norm of x, we have

$$D_0^{\widetilde{G}'}(x)^{1/2}\widetilde{\Theta}(x) = D^G(y)^{1/2}\Theta(y).$$

Here $D_0^{\widetilde{G}'}(x) = D^{\widetilde{G}'}(x)d_{\widetilde{G}'}(x)^{-1}$ where $d_{\widetilde{G}'}(x)$ is defined in Section 1.6 of [35] (it is 1 unless the residue characteristic is 2). To end this subsection, we prove a comparison between $J_{\chi,geom}$ and $\tilde{J}_{\chi',geom}$. This will be used in our application.

Proposition 8.2. *Let Θ be a stable quasi-character on $G(F)$ and $\widetilde{\Theta}$ be a quasi-character on $\widetilde{G}'(F)$. If $\widetilde{\Theta}$ is a transfer of Θ, we have*

$$J_{\chi,geom}(\Theta) = J_{\chi,geom,+}(\Theta) + J_{\chi,geom,-}(\Theta) = \tilde{J}_{\chi',geom}(\widetilde{\Theta}).$$

Proof. Recall that (note that since Θ is stable we have $c_{\Theta,\mathcal{O}_x}(x) = c_\Theta(x)$)

$$J_{\chi,geom}(\Theta) = \int_{\Gamma_{\mathrm{ell}}(H_{0,+})\cup\Gamma_{\mathrm{ell}}(H_{0,+})} D^G(x)^{1/2}c_\Theta(x)\chi(x)^{-1}dx,$$

$$\tilde{J}_{\chi',geom}(\tilde{\Theta}) = \int_{\Gamma_{\mathrm{ell}}(\widetilde{H}_0')} D^{\widetilde{G}'}(x)^{1/2}c_{\widetilde{\Theta}}(x)\chi'(x)^{-1}dx.$$

There is a natural bijection (denoted by $t \leftrightarrow t'$) given by the norm map described above between the regular stable elliptic conjugacy classes of $H_{0,+}(F) \cup H_{0,-}(F)$ and the regular stable elliptic twisted conjugacy classes of $\tilde{H}_0'(F)$. For each $t \leftrightarrow t'$, the number of conjugacy classes in t is equal to the number of twisted conjugacy classes in t'. By the definition of the character χ' we know that $\chi(t) = \chi'(t')$ for $t \leftrightarrow t'$. Moreover, by Section 2.2 of [6], we know that under this bijection we have $dt' = d_{\tilde{H}_0'}(t')^{-1}dt$ where $d_{\tilde{H}_0'}(t')$ is defined in Section 1.6 of [35] (it is equal to 1 unless the residue characteristic of F is 2). Hence it is enough to show that for all $t \leftrightarrow t'$, we have

$$D^G(t)^{1/2}c_\Theta(t) = d_{\tilde{H}_0'}(t')^{-1}D^{\tilde{G}'}(t')^{1/2}c_{\tilde{\Theta}}(t').$$

We fix a representative of t (resp. t') and by abusing of language we still denoted it by t (resp. t'). Let a be the natural isomorphism between $E^\times$ and $Z_L(F)$. Also let W be the Weyl group of $G_t(F)$ (which is also equal to the Weyl group of $(G')_{t'}(F)$). By Proposition 4.5.1 of [5], we have

$$D^G(t)^{1/2}c_\Theta(t) = \lim_{\lambda\in F^\times\to 1} \frac{D^G(ta(\lambda))^{1/2}\Theta(ta(\lambda))}{|W|},$$

$$D^{\tilde{G}'}(t')^{1/2}c_{\tilde{\Theta}}(t') = \lim_{\lambda\in F^\times\to 1} \frac{D^{\tilde{G}'}(t'a(\lambda))^{1/2}\tilde{\Theta}(t'a(\lambda))}{|W|}.$$

Hence it is enough to show that

$$D^{G}(ta(\lambda))^{1/2}\Theta(ta(\lambda)) = d_{\tilde{H}_0'}(t')^{-1}D^{\tilde{G}'}(t'a(\lambda))^{1/2}\tilde{\Theta}(t'a(\lambda)) \tag{8.1.4}$$

for $1 \neq \lambda \in F^\times$ that is close to 1. For $\lambda \neq 1$ that is close to 1, we know that $ta(\lambda)$ (resp. $t'a(\lambda)$) is a regular semisimple element of G (resp. $\tilde{G}'$). Since $t \leftrightarrow t'$, we know that the stable conjugacy class of $ta(\lambda)$ corresponds to the stable conjugacy class of $t'a(\lambda)$. Then (8.1.4) follows from the fact that Θ and $\tilde{\Theta}$ are the transfer of each other (note that by the definition of the constant $d(\cdot)$ we have $d_{\tilde{H}_0'}(t')^2 = d_{\tilde{G}'}(t'a(\lambda))$). This proves the proposition. □

8.2 The proof of the main results for the unitary Shalika model

In this section, we will prove Theorem 1.9, 1.10, and 1.12. We start with Theorem 1.9.

Theorem 8.3. *1. Let π be a finite length discrete series of $G(F)$ with central character χ^n. If Θ_π is a stable distribution, then $m_+(\pi,\chi) = m_-(\pi,\chi)$.*

2. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ with central character χ^n. Then we have

$$\sum_{\pi\in\Pi_\phi(G)} m_+(\pi,\chi) = \sum_{\pi\in\Pi_\phi(G)} m_-(\pi,\chi).$$

Proof. The first part is a direct consequence of the multiplicity formula and Proposition 8.1. The second part follows from the first part together with the fact that the distribution character $\Theta_{\Pi_\phi(G)} = \sum_{\pi\in\Pi_\phi(G)}\Theta_\pi$ is stable. □

Next we will prove a necessary condition for a discrete L-packet to be distinguished and compute the summation of the multiplicity for some special cases. Let $(G, H_\epsilon, \chi\otimes\xi_\epsilon)$ be the unitary Shalika model defined in the previous subsection. Let $\Pi_\phi(G)$ be a discrete L-packet of G and let $\Pi_\phi(G')$ be its base change to $G'(F) = \mathrm{GL}_{2n}(E)$. Then $\Pi_\phi(G')$ is an irreducible tempered representation and we can extend it to a unitary twisted representation on $\tilde{G}'(F)$ (denoted by $\widetilde{\Pi_\phi(G')}$) so that $\Theta_{\widetilde{\Pi_\phi(G')}}$ is a transfer of $\Theta_{\Pi_\phi(G)}$. Our goal is to prove the following theorem.

Theorem 8.4. *With the notation above, the packet $\Pi_\phi(G)$ is $(H_+,\chi\otimes\xi_+)$-distinguished (i.e. $m_+(\pi)\neq 0$ for some $\pi\in\Pi_\phi(G)$) only if $\Pi_\phi(G')$ is distinguished by the Shalika model $(H',\chi'\otimes\xi')$.*

Remark 8.5. *By Theorem 8.3, we know that the packet $\Pi_\phi(G)$ is $(H_+,\chi\otimes\xi_+)$-distinguished if and only if it is $(H_-,\chi\otimes\xi_-)$-distinguished.*

Proof. Assume that $\Pi_\phi(G')$ is not distinguished by the Shalika model, we need to show that the packet $\Pi_\phi(G)$ is not $(H_+,\chi\otimes\xi_+)$-distinguished. It is enough to show that

$$J_{\chi,geom}(\Theta_{\Pi_\phi(G)})=0$$

where $\Theta_{\Pi_\phi(G)}=\sum_{\pi\in\Pi_\phi(G)}\Theta_\pi$. By Proposition 8.2, we only need to show that

$$\tilde{J}_{\chi',geom}(\Theta_{\widetilde{\Pi_\phi(G')}})=0.$$

Since $\Pi_\phi(G')$ is not distinguished by the Shalika model, by Corollary 1.1 of [25], we can choose a small neighborhood ω of $\Pi_\phi(G')$ in $Temp_{ind}(\mathrm{GL}_{2n}(E))$ such that every element in ω is not distinguished by the Shalika model. By Proposition 2.12, we can find a strongly cuspidal function $\tilde{f}$ on $\tilde{G}'(F)$ such that $\tilde{f}$ is supported on ω and $\Theta_{\tilde{f}}=\Theta_{\widetilde{\Pi_\phi(G')}}$. Hence it is enough to show that $\tilde{J}_{\chi',geom}(\Theta_{\tilde{f}})=0$.

By our assumption on the support of $\tilde{f}$ and Plancherel formula of Shalika model in [13] we known that $\tilde{R}(\tilde{f})=0$ and hence $\tilde{f}$ satisfies (5.0.2). Applying Proposition 5.1 and Theorem 7.3 to the twisted Shalika model, we have

$$\tilde{J}_{\chi',geom}(\Theta_{\tilde{f}})=\int_{H'(F)\backslash G'(F)}\int_{N'(F)}\int_{\tilde{H}_0'(F)}\tilde{f}(x^{-1}hnx)\xi'(n)^{-1}\chi'(h)^{-1}dhdndx=\nu(\tilde{H}')\mathrm{tr}(\tilde{R}_{disc}(\tilde{f}))=0.$$

This finishes the proof of the theorem. □

Remark 8.6. *The Plancherel decomposition proved in [13] is for the case when $\chi'=1$. However, by our definition of χ' we know that $\chi'(-1)=1$ which implies that the character χ' is a square of another character χ'' of $E^\times$. Then we just need to twist the Plancherel decomposition in [13] by the character $\chi''\circ\det$.*

Now assume that $\Pi_\phi(G')$ is distinguished by the Shalika model $(H',\chi'\otimes\xi')$. By Corollary 1.1 of [25], $\Pi_\phi(G')$ is of the form (note that χ'' is a character of $E^\times$ with $\chi'=(\chi'')^2$)

$$\Pi_\phi(G')\otimes(\chi''\circ\det)^{-1}=(\tau_1\times\cdots\times\tau_l)\times(\sigma_1\times\sigma_1^\vee)\times\cdots\times(\sigma_m\times\sigma_m^\vee)$$

where

- τ_i is a discrete series of $\mathrm{GL}_{2a_i}(E)$ that is conjugate self-dual, self-dual and of symplectic type. In particular, a_i is even.
- σ_j is a discrete series of $\mathrm{GL}_{b_i}(E)$ that is conjugate self-dual, but NOT self-dual.
- τ_i,σ_j are all distinct.
- $\sum_{i=1}^l a_i+2\sum_{j=1}^m b_j=2n$.

We will consider the special case when $m=0$. The general case will be consider in our future paper. When $m=0$, by the Plancherel decomposition proved in [13], $\Pi_\phi(G')$ appears discretely in the L^2-space of the Shalika model. Our goal is to prove the following theorem.

Theorem 8.7. *With the notation above, we have*

$$\sum_{\pi\in\Pi_\phi(G)} m_+(\pi,\chi)=\sum_{\pi\in\Pi_\phi(G)} m_-(\pi,\chi)=2^{l-1}.$$

Proof. It is enough to show that $J_{\chi,geom}(\Theta_{\Pi_\phi(G)})=2^l$ where $\Theta_{\Pi_\phi(G)}=\sum_{\pi\in\Pi_\phi(G)}\Theta_\pi$. Since $J_{\chi,geom}(\Theta_{\Pi_\phi(G)})=\sum_{\pi\in\Pi_\phi(G)} m_+(\pi,\chi)+m_-(\pi,\chi)$ is a non-negative integer, we only need to show that $|J_{\chi,geom}(\Theta_{\Pi_\phi(G)})|=2^l$. By our assumption of $\Pi_\phi(G')$ and the Plancherel formula of Shalika model [13], it appears discretely in the L^2 space of the Shalika model and hence we can choose a small neighborhood ω of $\Pi_\phi(G')$ in $Temp_{ind}(\mathrm{GL}_{2n}(E))$ such that $\Pi_\phi(G')$ is the only element in ω distinguished by the Shalika model. By Proposition 2.12, we can find a strongly cuspidal function $\tilde{f}$ on $\tilde{G}'(F)$ such that $\tilde{f}$ is supported on ω, $\Theta_{\tilde{f}}=\Theta_{\widetilde{\Pi_\phi(G')}}$ and $\mathrm{tr}(\widetilde{\Pi_\phi(G')}(\tilde{f}))=2^l$. Note that the number $|\mathrm{Stab}(i\mathcal{A}^*_{\tilde{G},F},\tau)|^{-1}D(\tau)$ in Proposition 2.12 is equal to 2^{-l} for $\widetilde{\Pi_\phi(G')}$. By Proposition 8.2, we only need to show that $|\tilde{J}_{\chi',geom}(\Theta_{\widetilde{\Pi_\phi(G')}})|=2^l$.

By our assumption on the support of $\tilde{f}$ and Plancherel formula of Shalika model in [13] we known that $\tilde{R}(\tilde{f})$ satisfies (5.0.2). By Proposition 5.1 and Theorem 7.3, it is enough to show that

$$|\mathrm{tr}(\widetilde{\Pi_\phi(G')}(\tilde{f}))\cdot\mathrm{tr}(\theta\langle\widetilde{\Pi_\phi(G')}\rangle|M(\widetilde{\Pi_\phi(G')}))|=2^l. \tag{8.2.1}$$

Since $\widetilde{\Pi_\phi(G')}$ is unitary, so is $\theta\langle\widetilde{\Pi_\phi(G')}\rangle|M(\widetilde{\Pi_\phi(G')})$. As the multiplicity space $M(\widetilde{\Pi_\phi(G')})$ is one dimensional, this implies that $|\mathrm{tr}(\theta\langle\widetilde{\Pi_\phi(G')}\rangle|M(\widetilde{\Pi_\phi(G')}))|=1$. Then (8.2.1) follows from the facts that $\mathrm{tr}(\widetilde{\Pi_\phi(G')}(\tilde{f}))=2^l$. This proves the theorem. □

9 Galois model for classical groups

In this section we will prove our main theorems for the Galois models (i.e. Theorem 1.6 and 1.7). In Section 9.1 we will prove a comparison between the Galois model for classical groups and the twisted Galois model for general linear groups. Then in Section 9.2 we will prove Theorem 1.6 and 1.7.

9.1 Some comparison

Let H be a quasi-split special orthogonal group or a symplectic group defined over F and $G=Res_{E/F}H_E$. Let $G'=Res_{E/F}H'_E$ where $H'=\mathrm{GL}_{2n}$ if $H=\mathrm{SO}_{2n}$ or SO_{2n+1} and $H'=\mathrm{GL}_{2n+1}$ if $H=\mathrm{Sp}_{2n}$. Let θ be the involution of G given by $\theta(g)=w(g^t)^{-1}w^{-1}$ where w is the longest Weyl element. Let $\tilde{G}'$ be the non-neutral component of $G'\rtimes\{1,\theta\}$ and let $\tilde{H}'=H'\theta$. Then $\tilde{G}'$ (resp. $\tilde{H}'$) is a twisted space of G' (resp. H'). Finally, if H is the even special orthogonal group, let H_0 be a quasi-split special orthogonal group that is not a pure inner form of H and such that $G = Res_{E/F}H_E = Res_{E/F}H_{0,E}$ (i.e. the determinanats of the quadratic forms defining H and H_0 belong to the same square class in $E^\times/(E^\times)^2$ but belong to different square classes in $F^\times/(F^\times)^2$). If $H = \mathrm{Sp}_{2n}$ or SO_{2n}, let χ be the trivial character on H (and H_0 if $H = \mathrm{SO}_{2n}$) and let χ' be the trivial character on H'. If $H = \mathrm{SO}_{2n+1}$, let $\chi \in \{1, \eta_n\}$ where η_n is the composition of the Spin norm character of SO_{2n+1} with the quadratic character $\eta_{E/F}$. In this case, we let $\chi' = 1$ if $\chi = 1$ and $\chi' = \eta_n' := \eta_{E/F} \circ \det$ if $\chi = \eta_n$. In both cases, we can extend the character χ' to the twisted space $\tilde{H}'$ by making it equal to 1 on θ.

For a quasi-character Θ (resp. twisted quasi-character $\tilde{\Theta}$) on $G(F)$ (resp. $\tilde{G}'(F)$), define

$$J_{geom}(\Theta) = \int_{\Gamma_{ell}(H)} D^G(t)^{1/2}\Theta(t)\chi(t)^{-1}dt, \text{ if } H = \mathrm{SO}_{2n+1},\ \mathrm{Sp}_{2n},$$

$$J_{geom}(\Theta) = \int_{\Gamma_{ell}(H)\cup\Gamma_{ell}(H_0)} D^G(t)^{1/2}\Theta(t)\chi(t)^{-1}dt, \text{ if } H = \mathrm{SO}_{2n},$$

$$\tilde{J}_{geom}(\tilde{\Theta}) = \int_{\Gamma_{ell}(\tilde{H}')} D^{\tilde{G}'}(t)^{1/2}\tilde{\Theta}(t)\chi'(t)^{-1}dt.$$

Proposition 9.1. *Let Θ be a stable quasi-character on $G(F)$ and $\tilde{\Theta}$ be a twisted quasi-character on $\tilde{G}(F)$. If H is the even orthogonal group, we fix a Whittaker datum in the definition of the transfer factor so that the element η in Section 1.6 of [37] is equal to 1. Assume that Θ and $\tilde{\Theta}$ are the transfer of each other (in the sense of Section 1.6 of [35]). Then we have*

$$2 \cdot J_{geom}(\Theta) = \tilde{J}_{geom}(\tilde{\Theta}).$$

Proof. When H is the odd orthogonal group, the proposition is a direct consequence of the following four facts

- There is a natural bijection (denoted by $t \leftrightarrow \tilde{t}$) between the stable regular elliptic conjugacy classes of $H(F)$ and of $\tilde{H}'(F)$. Under this bijection, we have $d\tilde{t} = d_{\tilde{H}'}(\tilde{t})^{-1}dt$ (Section 1.4 of [35]).
- We have

$$D^G(t)^{1/2}\Theta(t) = D^{\tilde{G}}(\tilde{t})^{1/2}d_{\tilde{G}'}(\tilde{t})^{-1/2}\tilde{\Theta}(\tilde{t}) = D^{\tilde{G}}(\tilde{t})^{1/2}d_{\tilde{H}'}(\tilde{t})^{-1}\tilde{\Theta}(\tilde{t})$$

  for all $t \leftrightarrow \tilde{t}$ (note that the transfer factor between t and any rational twisted conjugacy class in $\tilde{t}$ is trivial by Section 1.10 of [37]).
- For $t \leftrightarrow \tilde{t}$, the number of $H(F)$-conjugacy classes in t is half of the number of $\tilde{H}'(F)$-conjugacy classes in $\tilde{t}$ (the other half belongs to the pure inner form of the odd special orthogonal group).
- For all $t \leftrightarrow \tilde{t}$, we have $\chi(t) = \chi'(\tilde{t})$.

The first three facts are straightforward. For the last one, it is trivial when $\chi = 1$. It remains to consider the case when $\chi = \eta_n$ and $\chi' = \eta_n'$. In this case, the stable conjugacy class t (resp. $\tilde{t}$) corresponds to (see Section 1.3 of [37])

$$(F_i, F_{\pm i}, t_i)_{1\leq i\leq h}$$

where

- $F_{\pm i}/F$ is a finite extension of degree d_i with $\sum_{i=1}^h d_i = n$;
- F_i is a quadratic extension of $F_{\pm i}$;
- $t_i \in ker(N_{F_i/F_{\pm i}})$.

It is easy to see from the definition that

$$\chi(t) = \chi'(\tilde{t}) = \eta_{E/F}(\Pi_{i=1}^h N_{F_i/F}(e_i))$$

where e_i is any element in $F_i^\times$ such that $\frac{e_i}{\bar{e}_i} = t_i$ ($\bar{e}_i$ is the conjugation of e_i under the nontrivial element of $Gal(F_i/F_{\pm i})$). This proves the last fact.

For the rest two cases, the characters χ and χ' are trivial. When H is the symplectic group, the proposition is a direct consequence of the following three facts (all of them are straightforward)

- There is a natural bijection (denoted by $t \leftrightarrow \tilde{t}$) between the stable regular elliptic conjugacy classes of $H(F)$ and of $\tilde{H}'(F)$. Under this bijection, we have $d\tilde{t} = \frac{1}{2}\cdot |2|_F \cdot d_{\tilde{H}'}(\tilde{t})^{-1}dt$ (Section 1.6 of [35], note that in this case $|T(F)^\theta : T_\theta(F)| = 2$ for any maximal elliptic twisted torus $\widetilde{T}$ of $\widetilde{H}'(F)$).
- We have

$$D^G(t)^{1/2}\Theta(t) = D^{\tilde{G}}(\tilde{t})^{1/2} d_{\tilde{G}'}(\tilde{t})^{-1/2}\tilde{\Theta}(\tilde{t}) = D^{\tilde{G}}(\tilde{t})^{1/2} d_{\tilde{H}'}(\tilde{t})^{-1}\tilde{\Theta}(\tilde{t})$$

  for all $t \leftrightarrow \tilde{t}$ (note that the transfer factor between t and any rational twisted conjugacy class in $\tilde{t}$ is trivial by Section 1.10 of [37]).
- For $t \leftrightarrow \tilde{t}$, the number of $H(F)$-conjugacy classes in t is equal to the number of $\tilde{H}'(F)$-conjugacy classes in $\tilde{t}$ divided by $|F^\times/(F^\times)^2|$. Moreover, $|F^\times/(F^\times)^2| = 4\cdot|\frac{1}{2}|_F$.

When H is the even special orthogonal group, the proposition is a direct consequence of the following three facts

- There is a natural map $\Gamma_{st,ell}(H \cup H_0) \to \Gamma_{st,ell}(\tilde{H}')$ (denoted by $t \to \tilde{t}$) from the stable regular elliptic conjugacy classes of $H(F)$ and $H_0(F)$ to the stable regular elliptic conjugacy classes of $\tilde{H}'(F)$. The fiber of each element in the image of this map has exactly two elements (differed by the outer automorphism of the even special orthgonal group). Under this map, we have $d\tilde{t} = d_{\tilde{H}'}(\tilde{t})^{-1}dt$ (Section 1.6 of [35]).

- We have

$$D^G(t)^{1/2}\Theta(t) + D^G(t)^{1/2}\Theta(t') = D^{\tilde{G}}(\tilde{t})^{1/2} d_{\tilde{G}'}(\tilde{t})^{-1/2}\tilde{\Theta}(\tilde{t}) = D^{\tilde{G}}(\tilde{t})^{1/2} d_{\tilde{H}'}(\tilde{t})^{-1}\tilde{\Theta}(\tilde{t})$$

  for all $t \to \tilde{t}$ where t' is another element in the fiber of $\tilde{t}$. On the other hand, if $\tilde{t}$ is a stable regular elliptic conjugacy class of $\tilde{H}'(F)$ that does not belong to the image of $\Gamma_{st,ell}(H \cup H_0) \to \Gamma_{st,ell}(\tilde{H}')$, then $\tilde{\Theta}(\tilde{t}) = 0$.

- For $t \to \tilde{t}$, the number of $H(F)$-conjugacy classes (or $H_0(F)$-conjugacy classes) in t is half of the number of $\tilde{H}'(F)$-conjugacy classes in $\tilde{t}$ (the other half belongs to the pure inner form of the even special orthogonal group).

The first and third facts are straightforward. For the second one, we need to show that the trasnfer factor between t and any rational twisted conjugacy class in $\tilde{t}$ is trivial for all $t \in \Gamma_{st,ell}(H \cup H_0) \to \tilde{t} \in \Gamma_{st,ell}(\tilde{H}')$. We follow the notation in [37]. Under the notation in Section 1.3 of [37], the stable conjugacy class t is of the form

$$(F_i, F_{\pm i}, t_i)_{1 \leq i \leq h}$$

where

- $F_{\pm i}/F$ is a finite extension of degree d_i with $\sum_{i=1}^h d_i = n$;
- F_i is a quadratic extension of $F_{\pm i}$;
- $t_i \in ker(N_{F_i/F_{\pm i}})$.

A rational twisted conjugacy class in $\tilde{t}$ is of the form

$$(F_i, F_{\pm i}, t_i, c_i)_{1 \leq i \leq h}$$

where $c_i \in F_i^\times / Im(N_{E_i/F_i})$. Next we need to describe how does $(F_i, F_{\pm i}, t_i, c_i)_{1 \leq i \leq h}$ behave under base change. There are three types:

Type 1 If E is not contained in E_i, then $(F_i/F_{\pm i}, t_i, c_i)$ becomes $(E_i, E_{\pm i}, t_i, 1)$ where $E_i = F_i \otimes_F E$ and $E_{\pm i} = F_{\pm i} \otimes_F E$. Here we view t_i as an element of $ker(N_{E_i/E_{\pm i}})$ via the canonical embedding from $ker(N_{F_i/F_{\pm i}})$ to $ker(N_{E_i/E_{\pm i}})$.

Type 2 If E is not contained in $F_{\pm i}$ and $E \subset F_i$, then $(F_i/F_{\pm i}, t_i, c_i)$ becomes $(F_i \oplus F_i, F_i, (t_i, t_i^{-1}), 1)$.

Type 3 If $E \subset F_{\pm i}$, let $F_{\pm i} = E[x]/f(x)$ and we define the field $F'_{\pm i}$ to be $F'_{\pm i} = E[x]/\bar{f}(x)$ where $f \mapsto \bar{f}$ is the conjugation map on $E[x]$ induced by the non-trivial element of $Gal(E/F)$. Similarly we can also define the field F'_i which will be a quadratic extension of $F'_{\pm i}$. Moreover, we have a natural isomorphism (denoted by $x \mapsto \bar{x}$) between $ker(N_{F_i/F_{\pm i}})$ and $ker(N_{F'_i/F'_{\pm i}})$. Then $(F_i/F_{\pm i}, t_i, c_i)$ becomes $(F_i/F_{\pm i}, t_i, c_i) \cup (F'_i/F'_{\pm i}, \bar{t}_i, \bar{c}_i)$.

We decompose the set I into $I_1 \cup I_2 \cup I_3$ where I_j consists of those $i \in I$ such that $(F_i, F_{\pm i})$ belongs to Type j above.

Then if we view t as a stable conjugacy class of $G(F)$, it is of the form

$$(E_i, E_{\pm i}, t_i)_{i\in I_1} \cup (F_i \oplus F_i, F_i, (t_i, t_i^{-1}))_{i\in I_2} \cup ((F_i/F_{\pm i}, t_i) \cup (F_i'/F_{\pm i}', \bar{t}_i))_{i\in I_3}.$$

Similarly, if we view a rational twisted conjugacy class in $\tilde{t}$ as a rational twisted conjugacy class of $\tilde{G}'(F)$, it is of the form

$$(E_i, E_{\pm i}, t_i, 1)_{i\in I_1} \cup (F_i \oplus F_i, F_i, (t_i, t_i^{-1}), 1)_{i\in I_2} \cup ((F_i/F_{\pm i}, t_i, c_i) \cup (F_i'/F_{\pm i}', \bar{t}_i, \bar{c}_i))_{i\in I_3}.$$

For $i \in I_1$, the quadratic character $\eta_{E_i/E_{\pm i}}$ is trivial on $F_{\pm i}^{\times}$. For $i \in I_2$, the quadratic character $\eta_{F_i\oplus F_i/F_i}$ is the trivial character. For $i \in I_3$, the natural isomorphism from $F_{\pm i}^{\times}$ to $(F_{\pm i}')^{\times}$ maps the quadratic character $\eta_{F_i/F_{\pm i}}$ to the quadratic character $\eta_{F_i'/F_{\pm i}'}$. Combining these three facts with the definition of transfer factor in Section 1.10 [37] (note that we have choosen the Whittaker datum so that the number η in loc. cit. is equal to 1), we know that the transfer factor between t and any rational twisted conjugacy class in $\tilde{t}$ is trivial. This finishes the proof of the proposition.

□

9.2 The proof of the main theorem for Galois model

We start with a necessary condition for a discrete L-packet to be distinguished (i.e. Theorem 1.6).

Theorem 9.2. *Let $H = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$ or SO_{2n+1}, $G = Res_{E/F}H_E$, $\chi = 1$ if $H = \mathrm{Sp}_{2n}$ or SO_{2n}, and $\chi \in \{1, \eta_n\}$ if $H = \mathrm{SO}_{2n+1}$. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ and $\Pi_\phi(G')$ be the endoscopic transfer of the L-packet to the general linear group $G' = \mathrm{GL}_a(E)$ (here $a = 2n$ if $H = \mathrm{SO}_{2n}$ or SO_{2n+1} and $a = 2n+1$ if $H = \mathrm{Sp}_{2n}$). Then the packet $\Pi_\phi(G)$ is distinguished (i.e. $m(\pi, \chi) \neq 0$ for some $\pi \in \Pi_\phi(G)$) only if $\Pi_\phi(G')$ is $(\mathrm{GL}_a(F), \chi')$-distinguished. Here $\chi' = 1$ if $\chi = 1$ and $\chi' = \eta_n' := \eta_{E/F} \circ \det$ if $\chi = \eta_n$.*

For the summation of the multiplicities (i.e. Theorem 1.7), assume that $\Pi_{\phi(G')}$ is $(\mathrm{GL}_a(F), \chi')$-distinguished. By Theorem 4.2 of [26], $\Pi_\phi(G')$ is of the form

$$\Pi_\phi(G') = (\tau_1 \times \cdots \times \tau_l) \times (\sigma_1 \times \bar{\sigma}_1) \times \cdots \times (\sigma_m \times \bar{\sigma}_m)$$

where

- τ_i is a discrete series of $\mathrm{GL}_{a_i}(E)$ that is conjugate self-dual. Moreover, if $(H, \chi) = (\mathrm{SO}_{2n+1}, \eta_n)$, τ_i is self-dual of symplectic type; otherwise, τ_i is self-dual of orthogonal type.
- σ_j is a discrete series of $\mathrm{GL}_{b_i}(E)$ that is NOT conjugate self-dual. Moreover, if $(H, \chi) = (\mathrm{SO}_{2n+1}, \eta_n)$, σ_j is self-dual of symplectic type; otherwise, σ_j is self-dual of orthogonal type.

- τ_i, σ_j are all distinct.
- $\sum_{i=1}^{l} a_i + 2\sum_{j=1}^{m} b_j = a$.

We will consider the special case when $m = 0$. The general case will be consider in our future paper. When $m = 0$, by the Plancheral formula for the Galois model proved in [8], $\Pi_\phi(G')$ appears discretely in the L^2 space of the Galois model $(\mathrm{GL}_a(E), \mathrm{GL}_a(F), \chi')$.

Theorem 9.3. *With the notation above, if H is the symplectic group or the odd special orthogonal group, we have*

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi) = 2^{l-1}.$$

If H is the even special orthogonal group, we let H_0 be another even special orthogonal group as in the previous subsection. We use $m_0(\pi,\chi)$ to denote the multiplicity for the model (G, H_0, χ). Then we have

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi) + m_0(\pi,\chi) = 2^{l-1}.$$

The proof of the above two theorem is almost the same as the unitary Shalika model case. The only differences is to replace Proposition 8.2 by Proposition 9.1, and to replace the Plancherel formula for Shalika model in [13] by the Plancheral formula for the Galois model in [8]. We will skip the details here.

A Projections of finitely generated convex sets

In this appendix, we state and prove a decomposition result for orthogonal projections of finitely generated convex sets that is directly inspired from [1, Appendix].

Let $\mathfrak{a}$ be a real Euclidean space with scalar product denoted by $(.,.)$. A subset $\mathcal{C} \subset \mathfrak{a}$ is a *finitely generated convex* set if it satisfies one of the following equivalent properties:

- $\mathcal{C}$ is a finite intersection of *half-spaces* (by which we mean subsets of the form $\{X \in \mathfrak{a} \mid (Y,X) \leqslant c\}$ for some $Y \in \mathfrak{a}$ and $c \in \mathbb{R}$);
- There exists finite subsets $\{X_i \mid i \in I\}$ and $\{v_j \mid j \in J\}$ of $\mathfrak{a}$ such that
$$\mathcal{C} = Conv\{X_i \mid \in I\} + \sum_{j\in J} \mathbb{R}_+ v_j;$$
- $\mathcal{C} \times \{1\}$ is the intersection of $\mathfrak{a} \times \{1\}$ with a finitely generated cone in $\mathfrak{a} \oplus \mathbb{R}$.

Remark that any finitely generated convex set is automatically closed.

Let $\mathcal{C} \subset \mathfrak{a}$ be a finitely generated set. We let

$$C := \{X \in \mathfrak{a} \mid \mathcal{C} + \mathbb{R}_+ X = \mathcal{C}\}$$

be its *asymptotic cone* and denote by

$$C^\vee := \{Y \in \mathfrak{a} \mid (Y, X) \leqslant 0 \ \forall X \in C\}$$

be the corresponding *dual cone.*

For $H \in \mathcal{C}$, we define its *tangent cone* and *normal cone* respectively by

$$T_{\mathcal{C}}(H) := \mathbb{R}_+(\mathcal{C} - H), \quad N_{\mathcal{C}}(H) := T_{\mathcal{C}}(H)^\vee = \{Y \in \mathfrak{a} \mid (Y, X) \leqslant 0 \ \forall X \in T_{\mathcal{C}}(H)\}.$$

Note that both are finitely generated cones.

Lemma A.1. *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathfrak{a}$ be two finitely generated convex sets. Then, for $H \in \mathcal{C}_1 \cap \mathcal{C}_2$ we have*

$$T_{\mathcal{C}_1\cap\mathcal{C}_2}(H) = T_{\mathcal{C}_1}(H) \cap T_{\mathcal{C}_2}(H) \text{ and } N_{\mathcal{C}_1\cap\mathcal{C}_2}(H) = N_{\mathcal{C}_1}(H) + N_{\mathcal{C}_2}(H).$$

Proof. The first equality is obvious by definition. The second follows from the first one and the relation $(C_1 \cap C_2)^\vee = C_1^\vee + C_2^\vee$ that holds for every finitely generated cones $C_1, C_2 \subset \mathfrak{a}$. □

A face of $\mathcal{C}$ is its nonempty intersection with a supporting (affine) hyperplane i.e. a subset of $\mathcal{C}$ of the form

$$F = \{H \in \mathcal{C} \mid (\lambda, H) = c\}$$

where $\lambda \in \mathfrak{a}$ and $c \in \mathbb{R}$ are such that $(\lambda, H) \leqslant c$ for every $H \in \mathcal{C}$ with equality for at least one such H. Note that we allow $\lambda = 0$ so that $\mathcal{C}$ is a face of itself. We let $\mathcal{F}(\mathcal{C})$ be the set of faces of $\mathcal{C}$.

To every face $F \in \mathcal{F}(\mathcal{C})$, we associate the subspace $\mathfrak{a}^F$ that is the span of $F - X_F$ for any $X_F \in F$. Moreover, the normal cone $N_{\mathcal{C}}(X_F)$ is independent of X_F when the latter is chosen in the relative interior $\mathring{F}$ of F (that is its interior relative to $F + \mathfrak{a}^F$) and we shall denote by $\mathfrak{a}_F^+$ the relative interior of this normal cone. For any $X_F \in \mathring{F}$ we have

$$\mathfrak{a}_F^+ = \{Y \in \mathfrak{a} \mid (Y, H - X_F) < 0 \ \forall H \in \mathcal{C} - F\}.$$

Lemma A.2. *We have a partition*

$$C^\vee = \bigsqcup_{F\in\mathcal{F}(\mathcal{C})} \mathfrak{a}_F^+.$$

Proof. We lemma reduces to the three following claims:

(A.0.1) the cones $\mathfrak{a}_F^+$, $F \in \mathcal{F}(\mathcal{C})$, are mutually disjoint.

Indeed, let $F, F' \in \mathcal{F}(\mathcal{C})$ be distinct faces and choose $X_F \in \mathring{F}$, $X_{F'} \in \mathring{F}'$. Without loss of generality we may assume that $F' \not\subset F$ so that $X_{F'} \notin F$. Then, for $H \in \mathfrak{a}_F^+ \cap \mathfrak{a}_{F'}^+$ we have

$$(H, X_F - X_{F'}) \leqslant 0 \text{ and } (H, X_{F'} - X_F) < 0.$$

As these two inequalities are incompatible this shows that $\mathfrak{a}_F^+ \cap \mathfrak{a}_{F'}^+ = \emptyset$.

(A.0.2) For every $F \in \mathcal{F}(\mathcal{C})$ we have $\mathfrak{a}_F^+ \subset C^\vee$.

Indeed, for $X_F \in \mathring{F}$ we have $C \subset T_\mathcal{C}(X_F)$ hence $\mathfrak{a}_F^+ \subset N_\mathcal{C}(X_F) \subset C^\vee$.

(A.0.3) For every $X \in C^\vee$, there exists $F \in \mathcal{F}(\mathcal{C})$ such that $X \in \mathfrak{a}_F^+$.

Indeed, the function $Y \mapsto (X, Y)$ attains a maximum on $\mathcal{C}$ (as follows from the fact that $\mathcal{C}$ can be written as the sum of a convex hull of finitely many points and C) say $c \in \mathbb{R}$. Then,

$$F := \{H \in \mathcal{C} \mid (X, H) = c\}$$

is a face of $\mathcal{C}$ and $X \in \mathfrak{a}_F^+$. □

Let $\mathfrak{b}$ be a vector subspace of $\mathfrak{a}$ and $\mathfrak{b}^\perp$ be its orthogonal complement. Denote by $p : \mathfrak{a} \to \mathfrak{b}$ and $p^\perp : \mathfrak{a} \to \mathfrak{b}^\perp$ the two orthogonal projections. For $\xi \in \mathfrak{b}^\perp$ we set

$$\mathcal{F}(\mathcal{C}, \xi) = \{F \in \mathcal{F}(\mathcal{C}) \mid \xi \in p^\perp(\mathfrak{a}_F^+)\}.$$

Proposition A.3. *Assume that* $\dim(C^\vee + \mathfrak{b}) = \dim(\mathfrak{a})$ *and* $\xi \in p(\mathfrak{a}_C^+)$ *is in general position. (More precisely, we require that for every face* $F \in \mathcal{F}(\mathcal{C})$ *with* $\dim p^\perp(\mathfrak{a}_F^+) < \dim \mathfrak{b}^\perp$ *we have* $\xi \notin p^\perp(\mathfrak{a}_F^+)$*). Then, we have:*

(i) p induces a bijection between

$$\bigcup_{F \in \mathcal{F}(\mathcal{C},\xi)} F \text{ and } p(\mathcal{C}).$$

(ii) For $F_1, F_2 \in \mathcal{F}(\mathcal{C}, \xi)$ *we have*

$$p(F_1) \cap p(F_2) = p(F_1 \cap F_2).$$

Proof. Let $H \in \mathcal{C}$ and consider the intersection

$$\mathcal{C}_{H,\mathfrak{b}} := (H + \mathfrak{b}^\perp) \cap \mathcal{C}.$$

It is a finitely generated set with asymptotic cone $C_\mathfrak{b} := \mathfrak{b}^\perp \cap C$ and dual cone $C_\mathfrak{b}^\vee = \mathfrak{b} + C^\vee$. As $\xi \in C_\mathfrak{b}^\vee$, by the decomposition of Lemma A.2, there exists a unique face $F_{H,\mathfrak{b}} \in \mathcal{F}(\mathcal{C}_{H,\mathfrak{b}})$ such that $\xi \in \mathfrak{a}^+_{F_{H,\mathfrak{b}}}$. Take $X \in \mathring{F}_{H,\mathfrak{b}}$. Then, by Lemma A.1, we have

$$\mathfrak{a}^+_{F_{H,\mathfrak{b}}} = N_{\mathcal{C}_{H,\mathfrak{b}}}(X)^\circ = N_\mathcal{C}(X)^\circ + \mathfrak{b} = \mathfrak{a}_F^+ + \mathfrak{b}$$

where $F \in \mathcal{F}(\mathcal{C})$ is the unique face such that $X \in \mathring{F}$. In particular, we see that $F \in \mathcal{F}(\mathcal{C}, \xi)$. This already shows that p induces a surjection

$$\bigcup_{F \in \mathcal{F}(\mathcal{C},\xi)} F \to p(\mathcal{C}).$$

To prove that this map is also injective, it only remains to check that for the face $F_{H,\mathfrak{b}}$ is a singleton. But, by the assumption that ξ is in general position, $\xi \in \mathfrak{b} + \mathfrak{a}_F^+ = \mathfrak{a}^+_{F_{H,\mathfrak{b}}}$ implies that $\dim(\mathfrak{a}^+_{F_{H,\mathfrak{b}}}) = \dim(\mathfrak{a})$ i.e. that $F_{H,\mathfrak{b}}$ is reduced to one extreme point of $\mathcal{C}_{H,\mathfrak{b}}$. This proves (i). Note that (ii) is a direct consequence of (i). □

B Howe's conjecture for twisted weighted orbital integrals

The purpose of this appendix is to establish an analog of Howe's conjecture [19] for weighted orbital integrals on a p-adic twisted space. This result is needed for the proof of Theorem 4.8. A similar extension of Howe's conjecture to weighted orbital integrals was established for honest reductive groups by Arthur [3] based on his local trace formula but to the best of our knowledge Arthur's argument hasn't been extended to twisted spaces. The proof presented here is a direct adaptation of the work of Barbasch and Moy [4] which has the advantage of allowing non-Archimedean local fields of arbitrary characteristics. Actually, the reasoning in [4] extends without much effort to twisted spaces but for the comfort of the reader, as well as for the authors own edification, we reproduce below with some details Barbasch and Moy's beautiful argument.

The first section of this appendix contains the precise statement of the "Howe conjecture for twisted weighted orbital integrals" as well as a reduction to a certain property of twisted Hecke modules (Proposition B.2). The proof of this proposition will be given in Section B.3 following very closely the paper [4]. The intermediate Section B.2 aims to collect necessary material on Bruhat-Tits buldings and the Moy-Prasad filtrations.

B.1 The statement

We will freely use the basic notations introduced in Chapter 2 for twisted spaces and their subgroups. The main objects under consideration will a priori depend on the choices of Haar measures. However, the precise normalization of those are completely irrelevant for the main result of this appendix and we will therefore assume that Haar measures have been fixed every time they appear in a formula.

Let $(G,\widetilde{G})$ be a twisted reductive space defined over F. Let K be a special maximal compact subgroup of $G(F)$ so that for every parabolic subgroup $P\subset G$ we have an Iwasawa decomposition $G(F)=P(F)K$. Let $\widetilde{M}\subset\widetilde{G}$ be a Levi subspace. For every parabolic subspace $\widetilde{P}\in\mathcal{P}(\widetilde{M})$ and $g\in G(F)$ we set $H_{\widetilde{P}}(g):=H_{\widetilde{M}}(m_P(g))$ where $g=m_P(g)u_P(g)k_P(g)$ is an arbitrarily chosen decomposition with $m_P(g)\in M(F)$, $u_P(g)\in N_P(F)$ and $k_P(g)\in K$. Note that for $g\in G(F)$, the convex hull $Conv\{H_{\widetilde{P}}(g)\mid \widetilde{P}\in\mathcal{P}(\widetilde{M})\}$ is contained in a translate of the subspace $\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}}$. We let $v_{\widetilde{M}}(g)$ be the volume of that convex hull with respect to a given Haar measure on $\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}}$.

For $\gamma\in\widetilde{M}(F)\cap\widetilde{G}_{rs}(F)$ and $f\in C_c^\infty(\widetilde{G}(F))$, we can form the *weighted orbital integral*

$$WO_{\widetilde{M}}(\gamma,f)=WO_{\widetilde{M}}^{\widetilde{G}}(\gamma,f)=\int_{G_\gamma(F)\backslash G(F)}f(g^{-1}\gamma g)v_{\widetilde{M}}(g)dg$$

for some choice of invariant measure on $G_\gamma(F)\backslash G(F)$.

For any subset $\Omega\subseteq\widetilde{M}(F)$, we denote by $WO_{\widetilde{M}}(\Omega)$ the span of the linear functionals $f\in C_c^\infty(\widetilde{G}(F))\mapsto WO_{\widetilde{M}}(\gamma,f)$ for $\gamma\in\Omega\cap\widetilde{G}_{rs}(F)$. Also, for $J\subset G(F)$ a compact open subgroup we set

$$\widetilde{\mathcal{H}}_J = \widetilde{\mathcal{H}}_J^{\widetilde{G}} = C_c(\widetilde{G}(F)/J).$$

Howe's conjecture for twisted weighted orbital integrals can now be stated as follows:

Theorem B.1. *Assume that $\Omega \subseteq \widetilde{M}(F)$ is compact modulo conjugation and let $J \subset G(F)$ be a compact-open subgroup. Then, the restriction of $WO_{\widetilde{M}}(\Omega)$ to $\widetilde{\mathcal{H}}_J$ is finite dimensional.*

We will now reduce the above theorem to a statement about twisted Hecke modules. The space $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{\widetilde{G}} = C_c^\infty(\widetilde{G}(F))$ is a bimodule over the Hecke algebra $\mathcal{H} := C_c^\infty(G(F))$ for the action by left and right convolution (after fixing a Haar measure on $G(F)$). For $J \in CO(G)$, we denote by $[\mathcal{H}, \widetilde{\mathcal{H}}_J]$ the span of the commutators $[\phi, f] = \phi * f - f * \phi$ for $(\phi, f) \in \mathcal{H} \times \widetilde{\mathcal{H}}_J$. Equivalently, $[\mathcal{H}, \widetilde{\mathcal{H}}_J]$ is the span of the differences ${}^g f - f$ for $(g, f) \in G(F) \times \widetilde{\mathcal{H}}_J$ where we have set ${}^g f(\gamma) := f(g^{-1}\gamma g)$, $\gamma \in \widetilde{G}(F)$.

For any subset $\Omega \subset \widetilde{G}(F)$, we define let $\widetilde{\mathcal{H}}_J(\Omega)^c$ be the subspace of functions $f \in \widetilde{\mathcal{H}}_J$ that are supported in $\widetilde{G}(F) \setminus \Omega J$. We will establish the theorem through the following more technical statement.

Proposition B.2. *Let $\Omega \subset \widetilde{G}(F)$ be a subset that is compact modulo conjugation and $J \subset G(F)$ be a compact-open subgroup. Then, there exists an open subgroup $J' \subset J$ such that the quotient space*

$$\widetilde{\mathcal{H}}_J/([\mathcal{H}, \widetilde{\mathcal{H}}_{J'}] \cap \widetilde{\mathcal{H}}_J + \widetilde{\mathcal{H}}_J(\Omega)^c)$$

is of finite dimension.

To end this section, we now explain why Proposition B.2 implies Theorem B.1. Let

$$\tau_\Omega = \tau_\Omega^{\widetilde{G}} : \widetilde{\mathcal{H}} \to C^\infty(\Omega \cap \widetilde{G}_{rs}(F))$$

be the linear map sending $f \in \widetilde{\mathcal{H}}$ to the function

$$\gamma \in \Omega \cap \widetilde{G}_{rs}(F) \mapsto WO_{\widetilde{M}}(\gamma, f).$$

We need to show that $\tau_\Omega(\widetilde{\mathcal{H}}_J)$ is finite dimensional. The proof is by induction on the semisimple rank of G and thus we assume that the result already holds for all the proper Levi subspaces of $\widetilde{G}$.

For $g \in G(F)$, $f \in \widetilde{\mathcal{H}}$ and $\gamma \in \widetilde{M}(F) \cap \widetilde{G}_{rs}(F)$ we have the splitting formula [23, Proposition 2.9.4 (4)]

$$\frac{D^{\widetilde{G}}(\gamma)^{1/2}}{D^{\widetilde{M}}(\gamma)^{1/2}} WO_{\widetilde{M}}(\gamma, {}^g f) = \sum_{\widetilde{Q} \in \mathcal{F}(\widetilde{M})} WO_{\widetilde{M}}^{\widetilde{L}_Q}(\gamma, f_{g,\widetilde{Q}}) \tag{B.1.1}$$

where $\widetilde{L}_Q$ stands for the unique Levi factor of $\widetilde{Q}$ containing $\widetilde{Q}$, $f_{g,\widetilde{Q}} \in C_c^\infty(\widetilde{L}_Q(F))$ is the function given by

$$f_{g,\widetilde{Q}}(\widetilde{m}) = \delta_{\widetilde{P}}(\widetilde{m})^{1/2} \int_{K \times N_Q(F)} f(k^{-1}\widetilde{m}uk) u_{\widetilde{Q}}(kg^{-1}) du dk, \;\; \widetilde{m} \in \widetilde{L}_Q(F)$$

and

$$u_{\widetilde{Q}}(h) := \int_{\mathfrak{a}_{\widetilde{Q}}^{\widetilde{G}}} \Gamma_{\widetilde{Q}}(H, -H_{\widetilde{Q}}(h))dH, \text{ for } h \in G(F).$$

Let $\Omega^G \subseteq \widetilde{G}(F)$ be the union of all $G(F)$-conjugates of Ω. As Ω is compact modulo $M(F)$-conjugation, Ω^G is similarly compact modulo $G(F)$-conjugation. Let $J' \subset J$ be as in Proposition B.2 with Ω^G instead of Ω. Then, for every $\widetilde{Q} \in \mathcal{F}(\widetilde{M})$ we can find a compact-open subgroup $J_Q \subset L_Q(F)$ such that

(B.1.2) $$f_{g,\widetilde{Q}} \in \widetilde{\mathcal{H}}_{J_Q}^{\widetilde{L}_Q}, \text{ for every } f \in \widetilde{\mathcal{H}}_{J'} \text{ and } g \in G(F).$$

From (B.1.1) and (B.1.2) we deduce that

$$\tau_\Omega([\mathcal{H}, \widetilde{\mathcal{H}}_{J'}]) \subseteq \sum_{\widetilde{G} \neq \widetilde{Q} \in \mathcal{F}(\widetilde{M})} \tau_\Omega^{\widetilde{L}_Q}(\widetilde{\mathcal{H}}_{J_Q}^{\widetilde{L}_Q}).$$

By the induction hypothesis, this implies that $\tau_\Omega([\mathcal{H}, \widetilde{\mathcal{H}}_{J'}])$ has finite dimension. Furthermore, since the distribution $WO_{\widetilde{M}}(\gamma, .)$ for $\gamma \in \Omega \cap \widetilde{G}_{rs}(F)$ is supported in the $G(F)$-conjugacy class of γ, the image of $\widetilde{\mathcal{H}}_J(\Omega^G)^c$ by τ_Ω is zero. By Proposition B.2, it follows that $\tau_\Omega(\widetilde{\mathcal{H}}_J)$ is also of finite dimension Q.E.D.

B.2 Bruhat-Tits building and the Moy-Prasad filtrations

Let $\mathcal{B}$ be the restricted Bruhat-Tits building of G. It is a polysimplicial complex carrying polysimplicial actions of $G(F)$ and $\widetilde{G}(F)$ that are compatible in the sense that

$$(g\gamma g') \cdot x = g \cdot (\gamma \cdot (g' \cdot x)), \text{ for every } (g, \gamma, g') \in G(F) \times \widetilde{G}(F) \times G(F) \text{ and } x \in \mathcal{B}.$$

Moreover, these actions factor through $G(F)/Z_G(F)$ and $\widetilde{G}(F)/Z_G(F)$ respectively and the resulting actions are proper. Picking, for some minimal Levi subgroup $M_0 \subset G$, a scalar product on $\mathfrak{a}_{M_0}^G$ that is invariant under $Norm_{\widetilde{G}(F)}(M_0)$ yields a distance function

$$dist : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_+$$

on $\mathcal{B}$ that is uniquely geodesic and invariant under $\widetilde{G}(F)$. For $x, y \in \mathcal{B}$ we shall denote by $[x, y]$ the unique geodesic joining x and y and we set $]x, y[= [x, y] \setminus \{x, y\}$.

By a *chamber* of $\mathcal{B}$, we shall mean the closure of a facet of maximal dimension.

For $\gamma \in \widetilde{G}(F)$, the *displacement function* $d_\gamma : \mathcal{B} \to \mathbb{R}_+$ is defined by

$$d_\gamma(x) = dist(x, \gamma \cdot x), \quad x \in \mathcal{B}.$$

For any chamber $C \subset \mathcal{B}$ and $\gamma \in \widetilde{G}(F)$ we set

$$d_C(\gamma) := \inf_{x \in C} d_\gamma(x).$$

We also set

$$d(\gamma) := \inf_{x\in\mathcal{B}} d_\gamma(x), \text{ for } \gamma \in \widetilde{G}(F).$$

Note that, as the set of all chambers cover $\mathcal{B}$, we have

$$d(\gamma) = \inf_{C\subset\mathcal{B}} d_C(\gamma) \tag{B.2.1}$$

where the infimum is taken over the set of all chambers in $\mathcal{B}$.

Lemma B.3. *(i) For any chamber $C \subset \mathcal{B}$, the value set $d_C(\widetilde{G}(F))$ is a closed discrete subset of $\mathbb{R}_+$ and can thus be linearly ordered*

$$d_C(\widetilde{G}(F)) = \{0 = r_0 < r_1 < r_2 < \ldots\}.$$

(ii) Let $x \in \mathcal{B}$ and $\gamma \in \widetilde{G}(F)$ be such that $d_\gamma(x) > d(x)$. Then, for every $y \in]x, \gamma\cdot x[$ we have

$$d_\gamma(y) < d_\gamma(x).$$

(iii) Let $x \in \mathcal{B}$ and $\gamma \in \widetilde{G}(F)$. Then, if d_γ attains a local minimum at x we have $d_\gamma(x) = d(x)$.

(iv) The function $d : \widetilde{G}(F) \to \mathbb{R}_+$ is invariant by $G(F)$-conjugation and locally constant.

Proof. (i) The statement is equivalent to $d_C(\widetilde{G}(F)) \cap [0, R]$ being finite for every $R > 0$. The set

$$B(C, R) := \{x \in \mathcal{B} \mid \inf_{y\in C} dist(x, y) \leqslant R\}$$

is compact. Thus, by the properness of the action of $\widetilde{G}(F)/Z_G(F)$, the set

$$\{\gamma \in \widetilde{G}(F) \mid d_C(\gamma) \leqslant R\}$$

is compact modulo $Z_G(F)$. However, d_C is also right invariant by the pointwise stabilizer G_C of C which is an open subgroup of $G(F)$ containing $Z_G(F)$. The claim follows.

(ii) By the triangular inequality, and since $y \in]x, \gamma x[$, we have

$$\begin{aligned} d_\gamma(y) = dist(y, \gamma y) &\leqslant dist(y, \gamma x) + dist(\gamma x, \gamma y) \\ &= dist(y, \gamma x) + dist(x, y) = dist(x, \gamma x) = d_\gamma(x). \end{aligned}$$

Moreover, as $\mathcal{B}$ is uniquely geodesic, equality holds if and only if $\gamma x \in]y, \gamma y[$ or equivalently $\gamma x \in [x, \gamma^2 x]$. Assume by way of contradiction that $d_\gamma(y) = d_\gamma(x)$. Then, we have $\gamma^n x \in [\gamma^{n-1}x, \gamma^{n+1}x]$ for every $n \geqslant 1$ from which if follows that the geodesics $[x, \gamma x], \ldots, [\gamma^{n-1}x, \gamma^n x]$ piece together to form the geodesic $[x, \gamma^n x]$ and so

$$dist(x, \gamma^n x) = nd_\gamma(x), \text{ for every } n \geqslant 0.$$

On the other hand, as $d_\gamma(x) > d(\gamma)$, we can find $z \in \mathcal{B}$ such that $d_\gamma(z) < d_\gamma(x)$. By the triangular inequality again, we have

$$\begin{aligned} nd_\gamma(x) = dist(x, \gamma^n x) &\leqslant dist(x,z) + dist(z, \gamma^n z) + dist(\gamma^n z, \gamma^n x) \\ &\leqslant 2dist(x,z) + nd_\gamma(z) \end{aligned}$$

for each $n \geqslant 0$. Letting n goes to infinity leads to a contradiction. Therefore, $d_\gamma(y) < d_\gamma(x)$ and we are done.

(iii) This follows from (ii), noting that if $\gamma x \neq x$ every neighborhood of x meets $]x, \gamma x[$.

(iv) It is clear that d is invariant by $G(F)$-conjugation. Let us show that it is also locally constant. Let $\gamma \in \widetilde{G}(F)$. Then, by (i) and (B.2.1) there exists a chamber $C \subset \mathcal{B}$ such that $d(\gamma) = d_C(\gamma)$. As C is compact and $x \in C \mapsto d_\gamma(x)$ is continuous, d_γ attains its infimum on C and therefore $d(\gamma) = d_\gamma(x)$ for some $x \in C$. Let $J \subset G(F)$ be a compact-open subgroup that fixes pointwise some neighborhood of x in $\mathcal{B}$. Then, for each $k \in J$ the function $d_{\gamma k}$ attains a local minimum at x from which we deduce, by (iii), that $d(\gamma k) = d_{\gamma k}(x) = d_\gamma(x) = d(\gamma)$ i.e. d is constant on the coset γJ. □

Let $x \in \mathcal{B}$. For every real number $r \geqslant 0$, Moy and Prasad have defined an open-compact subgroup $K_{x,r} \subset G(F)$ with the following properties:

(B.2.2) For every $s \geqslant r \geqslant 0$ and $x \in \mathcal{B}$, we have $K_{x,s} \subseteq K_{x,r}$;

(B.2.3) For each $x \in \mathcal{B}$, $\bigcap_{r \geqslant 0} K_{x,r} = \{1\}$;

(B.2.4) For any $x \in \mathcal{B}$, $r \geqslant 0$ and $\gamma \in \widetilde{G}(F)$, we have $K_{\gamma \cdot x, r} = \mathrm{Ad}_\gamma(K_{x,r})$;

(B.2.5) There exists $h > 0$ such that for each integer $n \geqslant 0$ and $x \in \mathcal{B}$, $K_{x,nh}$ only depends on the facet F containing x;

(B.2.6) For $r > 0$ and $x, y, z \in \mathcal{B}$ such that $y \in [x, z]$ we have

$$K_{y,r} \subset K_{x,r} \cdot K_{z,r}.$$

By (B.2.5), for any chamber $C \subset \mathcal{B}$ we may define $K_{C,n}$ as $K_{x,nh}$ for any point x in the relative interior of C.

B.3 Proof of Proposition B.2

Fix a chamber $C \subset \mathcal{B}$. It suffices to prove Proposition B.2 for $J = J' = K_{C,n}$ and n large enough. In particular, we will assume that n is sufficiently large that J fixes pointwise all the chambers $C' \subset \mathcal{B}$ with $C \cap C' \neq \emptyset$.

By Lemma B.3(i), we can write

$$d_C(\widetilde{G}(F)) = \{0 = r_0 < r_1 < r_2 < \dots\}$$

and for each $i \geqslant 0$ we let $\widetilde{\mathcal{H}}_{J,\leqslant r_i}$ be the subspace of $f \in \widetilde{\mathcal{H}}_J$ which are supported in the set of $\gamma \in \widetilde{G}(F)$ with $d_C(\gamma) \leqslant r_i$. Then, $i \mapsto \widetilde{\mathcal{H}}_{J,\leqslant r_i}$ is an increasing and exhaustive filtration of $\widetilde{\mathcal{H}}_J$ and since the action of $\widetilde{G}(F)/Z_G(F)$ on the building is proper, the quotients $\widetilde{\mathcal{H}}_{J,\leqslant r_i}/\widetilde{\mathcal{H}}_{J,\leqslant r_i} \cap \widetilde{\mathcal{H}}_J(\Omega)^c$ are finite dimensional. Therefore, it suffices to check that for i sufficiently large we have

$$\widetilde{\mathcal{H}}_{J,\leqslant r_i} \subseteq \widetilde{\mathcal{H}}_{J,\leqslant r_{i-1}} + [\mathcal{H}, \widetilde{\mathcal{H}}_J] + \widetilde{\mathcal{H}}_J(\Omega)^c. \tag{B.3.1}$$

We will actually show that the above inclusion holds as soon as

$$r_i > \sup_{\gamma\in\Omega} d(\gamma). \tag{B.3.2}$$

We thus assume that the above inequality is satisfied. The quotient $\widetilde{\mathcal{H}}_{J,\leqslant r_i}/\widetilde{\mathcal{H}}_{J,\leqslant r_{i-1}}$ is spanned by the images of the functions $\mathbf{1}_{\gamma J}$ for $\gamma \in \widetilde{G}(F)$ with $d_C(\gamma) = r_i$ and it suffices to show that for such γ, $\mathbf{1}_{\gamma J} \in \widetilde{\mathcal{H}}_{J,\leqslant r_{i-1}} + [\mathcal{H}, \widetilde{\mathcal{H}}_J] + \widetilde{\mathcal{H}}_J(\Omega)^c$. For this we distinguish two cases:

First we assume that $d_C(\gamma) = d(\gamma)$. Let $x \in C$ be such that $d_C(\gamma) = d_\gamma(x)$. Then, as J fixes pointwise a neighborhood of x, for every $k \in J$ the displacement function $d_{\gamma J}$ attains a local minimum at x and therefore, by Lemma B.3(iii), we have

$$d(\gamma k) = d_{\gamma k}(x) = d_\gamma(x) = d_C(x) = r_i.$$

By (B.3.2), this implies $\gamma J \subseteq \widetilde{G}(F) \setminus \Omega$ and therefore $\mathbf{1}_{\gamma J} \in \widetilde{\mathcal{H}}_J(\Omega)^c$.

Assume now that $d_C(\gamma) > d(\gamma)$. Let again $x \in C$ be such that $d_C(\gamma) = d_\gamma(x)$. Then by Lemma B.3(ii), we have $[x, \gamma x] \cap C = \{x\}$. Let $y \in]x, \gamma x[$ be sufficiently close to x so that if F denotes the facet containing y we have $x \in \overline{F}$ (where $\overline{F}$ denotes the closure of F). We can find a chamber D containing y and $x' \in \mathring{C}$ (where $\mathring{C}$ denotes the interior of C) such that $[x', \gamma x'] \cap \mathring{D} \neq \emptyset$. By (B.2.4) and (B.2.6), for any $y' \in [x', \gamma x'] \cap \mathring{D}$, we have

$$K_{D,n} = K_{y',nh} \subset K_{x',nh} K_{\gamma x',nh} = J Ad_\gamma(J).$$

Let $k_1, \ldots, k_\ell \in Ad_\gamma(J)$ be such that

$$JK_{D,n} = \bigcup_{i=1}^{\ell} Jk_i.$$

Then, since $k_i^{-1}\gamma J = \gamma J$ for any i, we have

$$\mathbf{1}_{\gamma JK_{D,n}} = \sum_{i=1}^{\ell} \mathbf{1}_{k_i^{-1}\gamma J k_i} = \sum_{i=1}^{\ell} {}^{k_i^{-1}}\mathbf{1}_{\gamma J}.$$

This shows that

$$\ell^{-1}\mathbf{1}_{\gamma JK_{D,n}} - \mathbf{1}_{\gamma J} \in [\mathcal{H}, \widetilde{\mathcal{H}}_J]. \tag{B.3.3}$$

Furthermore, as J fixes D pointwise (since $D\cap C$ contains x and is therefore nonempty) and $y\in D\cap]x,\gamma x[$, by Lemma B.3(ii) we have

$$d_D(\gamma k)=d_D(\gamma)\leqslant d_\gamma(y)<d_\gamma(x)=d_C(x)=r_i$$

for every $k\in J$. Let $g\in G(F)$ be such that $C=gD$. Then, we have $gK_{D,n}g^{-1}=K_{C,n}=J$ and, by the above,

$$d_C(g\gamma kg^{-1})=d_D(\gamma k)<r_i$$

for every $k\in J$. This shows that the function

$${}^g\mathbf{1}_{\gamma JK_{D,n}}=\mathbf{1}_{g\gamma JK_{D,n}g^{-1}}=\mathbf{1}_{g\gamma Jg^{-1}J}$$

belongs to $\widetilde{\mathcal{H}}_{J,\leqslant r_{i-1}}$. Combining this with (B.3.3), we deduce that $\mathbf{1}_{\gamma J}\in\widetilde{\mathcal{H}}_{J,\leqslant r_{i-1}}+[\mathcal{H},\widetilde{\mathcal{H}}_J]$ and the claim follows.

References

[1] J. Arthur, *The invariant trace formula. I. Local theory*, J. Amer. Math. Soc. 1 (1988), no. 2, 323-383.

[2] J. Arthur, *A local trace formula*, Inst. Hautes Études Sci. Publ. Math. No. 73 (1991), 5-96.

[3] J. Arthur: *On the Fourier transforms of weighted orbital integrals*, J. Reine Angew. Math. 452 (1994), 163-217.

[4] D. Barbasch, A. Moy: *A new proof of the Howe conjecture.* J. Amer. Math. Soc. 13 (2000), no. 3, 639–650.

[5] R. Beuzart-Plessis, *A local trace formula for the local Gan-Gross-Prasad conjecture for unitary groups: The archimedean case*, Astérisque No. 418 (2020).

[6] R. Beuzart-Plessis, *Endoscopie et conjecture locale raffinée de Gan–Gross–Prasad pour les groupes unitaires*, Compositio Mathematica , Volume 151 , Issue 7 , July 2015 , pp. 1309-1371

[7] R. Beuzart-Plessis, *On distinguished square-integrable representations for Galois pairs and a conjecture of Prasad*, Invent. math. 214 (2018), 437-521

[8] R. Beuzart-Plessis, *Plancherel formula for* $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(E)$ *and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups.* Inventiones mathematicae volume 225, 159-297 (2021).

[9] R. Beuzart-Plessis, C. Wan, *A local trace formula for the generalized Shalika model*, Duke Mathematical Journal Volume 168, Number 7 (2019), 1303-1385.

[10] Y. Benoist, H. Oh, *Polar decomposition for p-adic symmetric spaces,* Int. Math. Res. Not. IMRN 2007, no. 24, Art. ID rnm121, 20 pp

[11] L. Clozel: *Characters of nonconnected, reductive p-adic groups*, Canad. J. Math. 39 (1987), no. 1, 149-167.

[12] L. Clozel, *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*, in "Harmonic analysis on reductive groups" (Brunswick, ME, 1989), 101-121, Progr. Math., 101, Birkhäuser Boston, Boston, MA, 1991.

[13] N. Duhamel, *Formule de Plancherel sur* $GL_n \times GL_n \backslash GL_{2n}$. ArXiv:1912.08497.

[14] P. Delorme: *Constant term of smooth* H_ψ*-spherical functions on a reductive p-adic group*, Trans. Amer. Math. Soc.362(2010), no.2, 933-955.

[15] P. Delorme: *Neighborhoods at infinity and the Plancherel formula for a reductive p-adic symmetric space*, Math. Ann. 370 (2018), no. 3-4, 1177–1229.

[16] P. Delorme, V. Secherre, *An analogue of the Cartan decomposition for p-adic reductive symmetric spaces of split p-adic reductive groups*, Pacific J. Math. 251 (2011), no. 1, 1-21

[17] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, notes par S. DeBacker et P. Sally, University Lecture series 16, AMS (1999)

[18] A. G. Helminck, S.P. Wang, *On rationality properties of involutions of reductive groups*, Adv. Math. 99 (1993), no. 1, 26-96

[19] R. Howe: *Two conjectures about reductive p-adic groups.* Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pp. 377–380. Amer. Math. Soc., Providence, R.I., 1973.

[20] F. Knop, *Über Bewertungen, welche unter einer reduktiven Gruppe invariant sind.* Math. Ann. 295 (1993), no. 2, 333-363.

[21] F. Knop, *The asymptotic behavior of invariant collective motion*, Invent. Math. 116 (1994), no. 1-3, 309-328.

[22] R.E. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras*, in "Harmonic analysis, the trace formula, and Shimura varieties", 393522, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005

[23] J.-P. Labesse, J.-L. Waldspurger, *La formule des traces tordue d'après le Friday Morning Seminar* CRM Monograph Series, 31. American Mathematical Society, Providence, RI, 2013. xxvi+234 pp

[24] D. Luna, *Slices étales*, Sur les groupes algébriques, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33 Soc. Math. France, Paris, 1973.

[25] N. Matringe, *Shalika periods and parabolic induction for* GL(n) *over a non-archimedean local field.* Bull. Lond. Math. Soc. 49 (2017), no. 3, 417-427.

[26] N. Matringe, *Distinction and Asai L-functions for generic representations of general linear groups over p-adic fields.* IMRN, volume 2011, issue 1, 74-95.

[27] C. Mœglin, J.-L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques*, Math. Z. 196 (1987), no. 3, 427-452

[28] C. Mœglin, J.-L. Waldspurger, *La formule des traces locales tordue.* Mem. Amer. Math. Soc. 251 (2018), no. 1198, v+183 pp.

[29] D. Prasad, *A 'relative' local Langlands correspondence*, arXiv preprint 2015, arXiv:1512.04347

[30] Y. Sakellaridis, A. Venkatesh, *Periods and harmonic analysis on spherical varieties.* Astérisque No. 396 (2017), viii+360 pp.

[31] P. Schneider, E.-W. Zink, *The algebraic theory of tempered representations of p-adic groups. II. Projective generators.* Geom. Funct. Anal. 17 (2008), no. 6, 2018-2065.

[32] The Stacks project authors, , *The Stacks project*, `https://stacks.math.columbia.edu`, 2020.

[33] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu 2 (2003), no. 2, 235-333

[34] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad*, Compos. Math. 146 (2010), no. 5, 1180-1290

[35] J.-L.Waldspurger, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spéciaux orthogonaux*, dans "Sur les conjectures de Gross et Prasad II" Astérisque No. 347 (2012), 103-165

[36] J.-L.Waldspurger, *Calcul d'une valeur d'un facteur ϵ par une formule intégrale*, dans "Sur les conjectures de Gross et Prasad II" Astérisque No. 347 (2012), 1-102.

[37] J.-L.Waldspurger, *Les facteurs de transfert pour les groupes classiques : un formulaire,* Manuscripta Math. 133 (2010), no. 1-2, 41-82

Raphaël Beuzart-Plessis
Aix Marseille Univ,
CNRS, I2M,
Marseille, France
raphael.beuzart-plessis@univ-amu.fr

Chen Wan

Department of Mathematics & Computer Science
Rutgers University – Newark
Smith Hall, 101 Warren St.,
Newark, NJ 07102, USA
chen.wan@rutgers.edu