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Error Exponents for Capacity-Achieving Signaling on Wideband Rayleigh Fading Channels

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Abstract

We have previously presented the exponent for an upper bound on the error probability of a "peaky" signaling scheme that achieves the capacity of the Rayleigh fading channel under an average power constraint in the limit of infinite bandwidth. In the present work, we complement this result with a lower bound. We find that the exponents of the upper and lower bounds coincide in the wideband limit and therefore yield the reliability function of the Rayleigh fading channel using peaky signaling. We illustrate the behavior of the reliability function and the upper and lower error probability bounds with some numerical examples.

1. Introduction

Interest in the study of fading channels with very large bandwidths has been spurred in recent times with the emergence of proposals for ultra-wideband (UWB) radio and wideband CDMA systems. The use of such spread-spectrum signaling schemes in the very large bandwidth regime, however, contradicts established information-theoretic results. Indeed, it is known that direct-sequence spread-spectrum signals perform poorly in this regime in terms of mutual information. More precisely, assuming that the channel exhibits time and frequency decorrelation and has no specular component and that the signal scales inversely with bandwidth in an appropriate manner (which induces a constraint on the fourth moment) the mutual information approaches zero with increasing bandwidth. This result has been shown with several variations in the channel model and the assumptions on the properties of the input signal [1, 2, 3].

Rather, given an infinite band, capacity can be reached using "peaky" signaling, i.e. transmitting with a low-duty-cycle frequency-shift keying scheme. Such signaling is described as peaky because transmission energy is concentrated into narrow regions of time (owing to the low duty cycle) and frequency (owing to the employment of frequency-shift keying). The capacity reached by this scheme, which assumes no channel state information at either the receiver or the sender, is the same as that of the AWGN channel. This result was presented by Kennedy [4] and by Gallager [5, §8.6] for the case of Rayleigh fading, and most recently by Telatar and Tse [3] for general multipath fading.

Therefore, if the bandwidth is large enough, then spreading energy over that band in an even manner that keeps the fourth moment constrained, for example with direct-sequence or related spread spectrum techniques, is not advisable. In addition, peaky signaling should yield good performance. The bandwidth at which spreading begins to become detrimental, however, is not entirely clear (though the issue is partially addressed in [6]) nor is the bandwidth at which peaky signaling begins to become advantageous. We address the latter issue in the case where the fading process is modeled as Rayleigh.

We have previously derived the exponent for an upper bound on the error probability of this scheme that vanishes with increasing bandwidth [7]. This is similar to the random coding exponent obtained by Gallager [5, §5.6, §7] for discrete-time memoryless channels, which is the exponent at which an upper bound on the error probability of random block coding vanishes with increasing block length. And, just as the random coding exponent brings out the relationship among the error probability, data rate, block length, and channel behavior, the exponent that we derive brings out the relationship among the error probability, data rate, bandwidth, "peakiness", and fading parameters such as the coherence time. The relationships drawn are, however, based only upon upper bounds on the error probability. In the case of random block coding over discrete memoryless channels, Gallager [5, §5.8] has found exponents for lower bounds, namely the sphere-packing and straight-line exponents, to better determine the true exponential behavior of the error probability. Likewise, we obtain in the present work, a lower bound in the case of peaky signaling over Rayleigh fading channels and use it to determine the tightness of the upper bound. We shall see that the exponent of the upper bound represents the true exponential dependence of the error probability in the wideband limit and is hence the *reliability function*.

2. Channel model

The channel is modeled as a Rayleigh fading channel with block fading in time. For a given input waveform $x(t)$, the output waveform $y(t)$ is given by

$$y(t) = \sum_{l=1}^{L} a_l(t)x(t - d_l(t)) + z(t), \qquad (1)$$

where L is the number of paths, $a_l(t)$ and $d_l(t)$ are the gain and delay on the lth path at time t respectively, and $z(t)$ is white Gaussian noise with power spectral density $N_0/2$. The total effect of the paths is consistent with Rayleigh fading. Let T_c and T_d be the coherence time and delay spread of the fading channel respectively. We assume that the processes $\{a_l(t)\}$ and $\{d_l(t)\}$ are constant and i.i.d. over time intervals of T_c, and that the channel is underspread (i.e. $T_d \ll T_c$).

3. Capacity-achieving scheme

A full description of the capacity-achieving scheme can be found in [3]. We briefly summarize it below.

Suppose that the average power constraint is P, and let $\theta \in (0, 1]$. Suppose further that we have a code-book of size M. The mth code word is represented at baseband as a complex sinusoid of amplitude $\sqrt{P/\theta}$ at frequency f_m, i.e.

$$x_m(t) = \begin{cases} \sqrt{P/\theta}\exp(j2\pi f_m t) & 0 \le t \le T_s, \\ 0 & \text{otherwise}; \end{cases} \qquad (2)$$

where the time duration of the signal T_s is taken to be the coherence time T_c. The frequency f_m is an integer multiple of $1/T_s'$, where $T_s' = T_s - 2T_d$.

Let us consider the channel output over the interval $[T_d, T_s - T_d]$ (the time axis at the receiver is shifted by the shortest path delay). During this interval, $\{a_l(t)\}$ and $\{d_l(t)\}$ are constant owing to the assumptions of the model, and we denote their values by $\{a_l\}$ and $\{d_l\}$ respectively. So by (1), the received signal when message m is sent is

$$y(t) = G\sqrt{P/\theta}\exp(j2\pi f_m t) + z(t) \qquad (3)$$

where $G = \sum_{l=1}^{L} a_l \exp(-j2\pi f_m d_l)$ is a circularly-symmetric complex Gaussian random variable since the fading process is Rayleigh. We define signal power in the conventional sense as the received signal power, and thus normalize the channel gain so that $\mathbf{E}[|G|^2] = 1$.

At the receiver, we form the correlator outputs

$$R_k = \frac{1}{\sqrt{N_0 T_s'}} \int_{T_d}^{T_s - T_d} \exp(-j2\pi f_k t) y(t) dt \qquad (4)$$

for $1 \le k \le M$.

The message is repeated over N disjoint time intervals to obtain time diversity. Hence for $1 \le k \le M$ and $1 \le n \le N$, we have

$$R_{k,n} = \delta_{km} G_n \sqrt{\frac{PT_s'}{\theta N_0}} + W_{k,n}, \qquad (5)$$

where $\{G_n\}$ and $\{W_{k,n}\}$ are sets of i.i.d. circularly-symmetric complex Gaussian random variables of unit variance. We construct the decision variables

$$S_k = \frac{1}{N} \sum_{n=1}^{N} |R_{k,n}|^2 \qquad (6)$$

and use a threshold decoding rule: Let

$$A = 1 + (1 - \epsilon)\frac{PT_s'}{\theta N_0} \qquad (7)$$

(where $\epsilon \in (0, 1)$ is an arbitrary parameter) be the threshold. If S_k exceeds A for one value of k only, then we estimate $\hat{m} = k$; otherwise we declare an error.

We transmit using the above scheme only for a fraction of time θ. Hence the average power is P. Note that the scheme transmits $\ln M$ nats in NT_s/θ seconds, so the rate is $R = \theta \ln(M)/(NT_s)$.

4. Upper bound on the error probability

Owing to symmetry, we can assume without loss of generality that $m = 1$. An error occurs if $S_1 < A$ or if $S_k \ge A$ for some $2 \le k \le M$. Let B_1 be the event $S_1 < A$ and let B_k be the event $S_k \ge A$ for $2 \le k \le M$. Then, denoting the error probability by p_e, we have

$$p_e = \Pr\left\{\bigcup_{k=1}^{M} B_k\right\} \le \Pr\{B_1\} + M\Pr\{B_2\}. \qquad (8)$$

For notational convenience, we define

$$p_e^{(1)} \triangleq \Pr\{B_1\}, \qquad p_e^{(2)} \triangleq \Pr\{B_2\}. \qquad (9)$$

In [7], we obtain exponential upper bounds to $p_e^{(1)}$ and $p_e^{(2)}$ using the Chernoff bound, viz.

$$p_e^{(1)} \le \exp(-N[A' - 1 - \ln(A')]), \qquad (10)$$

$$p_e^{(2)} \le \exp(-N[A - 1 - \ln(A)]); \qquad (11)$$

where

$$A' = \frac{A}{1 + PT_s'/(\theta N_0)}. \qquad (12)$$

Thus

$$p_e \le \exp(-N[A' - 1 - \ln(A')]) + \exp(-N[A - 1 - \ln(A) - RT_s/\theta]). \qquad (13)$$

Since ϵ is a characteristic of the decision rule with no implication on physical quantities of interest, we minimize over ϵ and get

$$p_e \leq 2\exp(-\ln(M)E(R,\theta)) \tag{14}$$

where

$$E(R,\theta) = \frac{\theta}{RT_s}\left\{\frac{RT_sN_0}{PT_s'} + \frac{\theta N_0}{PT_s'}\ln\left(1+\frac{PT_s'}{\theta N_0}\right) - 1 - \ln\left(\frac{RT_sN_0}{PT_s'} + \frac{\theta N_0}{PT_s'}\ln\left(1+\frac{PT_s'}{\theta N_0}\right)\right)\right\} \tag{15}$$

for

$$0 \leq R < \frac{T_s'P}{T_sN_0} - \frac{\theta}{T_s}\ln\left(1+\frac{PT_s'}{\theta N_0}\right). \tag{16}$$

The error exponent $E(R,\theta)$ is positive over its domain of definition, so recalling that $T_s = T_c$, we see that there exists $\theta \in (0,1]$ such that p_e vanishes as $M \to \infty$ as long as R does not exceed $(1-2T_d/T_c)P/N_0$. Therefore, since the channel is underspread, rates very close to the infinite-bandwidth AWGN capacity of $C = P/N_0$ can be achieved.

5. Lower bound on the error probability

Conditioned upon $m = 1$, we have

$$S_1 = \frac{1}{N}\sum_{n=1}^{N}\left|G_n\sqrt{\frac{PT_s'}{\theta N_0}} + W_1,n\right|^2 \tag{17}$$

and

$$S_2 = \frac{1}{N}\sum_{n=1}^{N}|W_2,n|^2. \tag{18}$$

Recall that $\{G_n\}$ and $\{W_{k,n}\}$ are sets of i.i.d. circularly-symmetric complex Gaussian random variables of unit variance, so S_1 and S_2 conditioned upon $m = 1$ are both χ^2 random variables with $2N$ degrees of freedom, and $p_e^{(1)}$ and $p_e^{(2)}$ can in fact be exactly evaluated using the cdf for χ^2 random variables with an even number of degrees of freedom [8, §2.1.4]:

$$\begin{aligned} p_e^{(1)} &= \Pr\left\{\sum_{n=1}^{N}\left|G_n\sqrt{\frac{PT_s'}{\theta N_0}} + W_{1,n}\right|^2 < NA\right\} \\ &= \exp(-NA')\sum_{k=N}^{\infty}\frac{(NA')^k}{k!} \end{aligned} \tag{19}$$

and similarly,

$$p_e^{(2)} = \exp(-NA)\sum_{k=0}^{N-1}\frac{(NA)^k}{k!}. \tag{20}$$

Since $(NA')^k/k!$ and $(NA)^k/k!$ are both positive for all k, we have the inequalities

$$\sum_{k=N}^{\infty}\frac{(NA')^k}{k!} \geq \frac{(NA')^N}{N!}, \tag{21}$$

$$\sum_{k=0}^{N-1}\frac{(NA)^k}{k!} \geq \frac{(NA)^{(N-1)}}{(N-1)!} \tag{22}$$

by taking only one of the summation terms. Therefore,

$$p_e^{(1)} \geq \exp\left\{-NA' + \ln\left[\frac{(NA')^N}{N!}\right]\right\} \tag{23}$$

$$p_e^{(2)} \geq \exp\left\{-NA + \ln\left[\frac{(NA)^{N-1}}{(N-1)!}\right]\right\}. \tag{24}$$

Applying Stirling's formula, we then obtain

$$p_e^{(1)} > \exp\{-N(A' - 1 - \ln(A') + o_1(N))\}, \tag{25}$$

$$p_e^{(2)} > \exp\{-N(A - 1 - \ln(A) + o_2(N))\}; \tag{26}$$

where $o_1(N)$ and $o_2(N)$ are quantities that go to zero with increasing N that are given by

$$o_1(N) = \frac{1}{2N}\ln(2\pi N) + \frac{1}{12N^2}, \tag{27}$$

$$o_2(N) = \frac{1}{2N}\ln(2\pi NA^2) + \frac{1}{12N^2}. \tag{28}$$

Thus the upper and lower bounds on $p_e^{(1)}$ and $p_e^{(2)}$ are exponentially tight, i.e. the error exponents are arbitrarily close for N, or equivalently $\ln(M)$, sufficiently large.

We now observe that

$$\begin{aligned} p_e &\geq \sum_{k=1}^{M}\Pr\{B_k\} - \sum_{j\neq k}\Pr\{B_j \cap B_k\} \\ &= p_e^{(1)} + (M-1)p_e^{(2)} - (M-1)p_e^{(1)}p_e^{(2)} \\ &\quad - \frac{(M-1)(M-2)}{2}p_e^{(2)2}. \end{aligned} \tag{29}$$

After substituting the upper and lower bounds on $p_e^{(1)}$ and $p_e^{(2)}$ and applying l'Hôpital's rule, we obtain

$$\begin{aligned} &\lim_{N\to\infty}\frac{-\ln(p_e)}{N} \\ &\leq \min(A' - 1 - \ln(A'),\ A - 1 - \ln(A) - RT_s/\theta) \end{aligned} \tag{30}$$

for $RT_s/\theta < A - 1 - \ln(A)$. The reverse inequality follows straightforwardly from (13). Thus (30) holds with equality. For $RT_s/\theta \geq A - 1 - \ln(A)$, we use

$$p_e = 1 - \Pr\left\{\bigcap_{k=1}^{M}B_k^c\right\} \geq 1 - (1 - p_e^{(2)})^{(M-1)}, \tag{31}$$

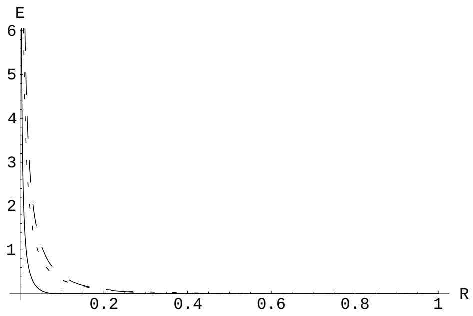

Figure 1: Reliability function $E(R, \theta)$ as a function of R for $\theta = 10^{-2}$ (solid), $\theta = 10^{-3}$ (dashed), and $\theta = 10^{-4}$ (dotted).

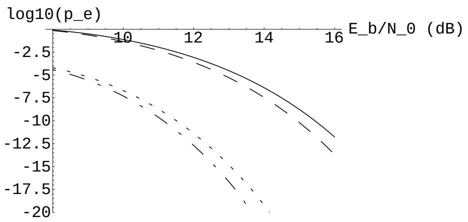

Figure 2: Error probability bounds as functions of E_b/N_0 for bandwidths of 1 GHz (upper bound solid, lower bound dotted) and 10 GHz (upper bound dashed, lower bound dot-dashed). The duty factor is $\theta = 4 \times 10^{-3}$.

and get

$$\lim_{N \to \infty} \frac{-\ln(p_e)}{N} \leq 0. \tag{32}$$

By noting that p_e is a probability and is therefore at most 1, the reverse inequality follows. Hence (32) also holds with equality. After optimization over ϵ, we have

$$\lim_{M \to \infty} \frac{-\ln(p_e)}{\ln(M)} = \lim_{N \to \infty} \frac{-\theta \ln(p_e)}{NRT_s} = E(R, \theta). \tag{33}$$

Therefore $E(R, \theta)$ represents the true exponential dependence of the error probability on $\ln(M)$ for M sufficiently large — it is the reliability function of the Rayleigh fading channel using peaky signaling with duty factor θ.

6. Numerical results

For numerical results, we choose fading parameters that are typical for very-high frequency transmission in an indoor environment (the proposed setting for UWB systems): Let $T_d = 10^{-7}$ s and $T_s = T_c = 2 \times 10^{-3}$ s. Let $P = N_0 = 1$ so $C = 1$. Figure 1 illustrates the behavior of the reliability function for various values of the duty factor θ. It is evident that smaller values of the duty factor are required to achieve higher rates, though the optimal θ for a given rate is not immediately apparent. Suppose we impose a peak power constraint of $P/\theta \leq 250$, which is rather reasonable. It follows that θ must be at least 4×10^{-3}. Given this restriction on θ, we find, using numerical methods, that the optimal reliability function is achieved for all rates under capacity by taking $\theta = 4 \times 10^{-3}$. With this choice of parameters, the behavior of the upper and lower error probability bounds as functions of E_b/N_0 for bandwidths of 1 GHz and 10 GHz is shown in Figure 2.

7. Conclusion

We presented a lower bound on the probability of error of a capacity-achieving scheme for the infinite-bandwidth Rayleigh fading channel. This result complements the upper bound that we previously presented in [7]. We used the lower bound to show that the exponent of the upper bound is the true exponential dependence of the error probability in the wideband limit and is thus the reliability function of the Rayleigh fading channel using peaky signaling.

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