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Stable Trace Formula, Automorphic Forms and Galois Representations
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Examples in endoscopy for real groups

Part A : 08/12/08

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We start with a review of regular elliptic Langlands parameters and the characters of discrete series representations of unitary groups. The cardinality of an L-packet of discrete series representations of the unitary group $U(p, n-p)$ is the binomial coefficient $\binom{n}{p}$. We would like the answer to be a power of 2 and because the relevant parameters are strongly regular it should be exactly 2^{n-1}. Modifying what we mean by *unitary group* resolves this. We then attach to each discrete series representation in an L-packet a binary word of length n. For n odd there is just one group and we use all 2^{n-1} even words, while for n even there are two groups and we use even words for the quasi-split one and odd words for the other. The construction determines a simple pairing of an L-packet with a finite group $\mathbb{S}^{sc}$ on the dual side. This group $\mathbb{S}^{sc}$ comes from stabilization of the spectral side of the trace formula, and the pairing plays a central role when we discuss transfer factors and endoscopy in Part B. We review what we have done with a short look at unitary similitude groups (the special unitary case is no different from the unitary case), and end with a very brief remark on base change.

References in this preliminary version are incomplete. Most calculations are well-known, and many are examples for a more general discussion, with references, in *Tempered endoscopy for real groups* I (Contemp. Math, 2008), II (*Automorphic Forms and the Langlands Program*, Intl. Press, 2008), and III (preprint). The binary words come from an ongoing project with a student.

1. The L-group for real unitary groups of $n \times n$ matrices.

There is one L-group for all real unitary groups of $n \times n$ matrices. We set $G^\vee = GL(n, \mathbb{C})$ and $\sigma^\vee : g \to J_n\ {}^t g^{-1} J_n^{-1}$, where

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \cdots & & & \\ (-1)^{n+1} & & & & \end{pmatrix}.$$

Notice that $\sigma^\vee$ preserves the *standard splitting*

$$spl_{G^\vee} = (B^\vee, T^\vee, \{X_i\})$$

of $GL(n, \mathbb{C})$. Here $B^\vee$ consists of the upper triangular matrices and $T^\vee$ of the diagonal matrices in $GL(n, \mathbb{C})$; X_i abbreviates the transvection $X_{i,i+1}$ in $\mathfrak{gl}(n, \mathbb{C})$, $1 \leq i \leq n-1$.

We reserve the notation σ for the nontrivial Galois element:

$$\Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$$

throughout. We use the Weil form of the L-group. Thus

$$W = W_{\mathbb{R}} = \{z \times \tau : z \in \mathbb{C}^\times, \tau \in \Gamma\},$$

a nonsplit extension of Γ by $\mathbb{C}^\times$, with multiplication defined using the fundamental 2-cocycle $a_{\sigma,\sigma} = -1$ of Γ in $\mathbb{C}^\times$. Then W acts on $G^\vee$ through $W \to \Gamma$, and σ acts by $\sigma^\vee$.

Now the L-group is the semidirect product

$$^LG = G^\vee \rtimes W,$$

and we write a typical element as $g \times z \times \tau$.

2. Langlands parameters

A *Langlands parameter* is a $G^\vee$-conjugacy class of homomorphisms $\varphi : W \to {}^LG$ of the form

$$\varphi(w) = \varphi_0(w) \times w, \; w \in W,$$

where φ_0 is a continuous map (which must be a 1-cocycle) of W into the semisimple elements of $G^\vee$. A parameter $\boldsymbol{\varphi}$ represented by homomorphism φ is *tempered* if the image of φ_0 is bounded, and will have discrete series representations attached to it *if and only if* the image of φ lies outside every proper parabolic subgroup of ${}^LG$ (this condition forces $\boldsymbol{\varphi}$ to be tempered). Rather than discuss parabolic subgroups of ${}^LG$ at this point, we introduce the equivalent notion† of *regular elliptic* parameter.

Notice that the centralizer of $\varphi(\mathbb{C}^\times \times 1)$ in $GL(n, \mathbb{C})$ is always a connected reductive subgroup of maximal rank, *i.e.* containing a maximal torus in $GL(n, \mathbb{C})$. We will call $\boldsymbol{\varphi}$ *regular* if this centralizer is abelian and thus a maximal torus in $GL(n, \mathbb{C})$: equivalently, $\boldsymbol{\varphi}$ has a representative φ for which

(i) the centralizer of $\varphi(\mathbb{C}^\times \times 1)$ in $GL(n, \mathbb{C})$ is the diagonal group $T^\vee$.

Then $\varphi(1 \times \sigma)$ normalizes $T^\vee$, and we call $\boldsymbol{\varphi}$ *regular elliptic* if we have further that

(ii) $\varphi(1 \times \sigma)$ acts on $T^\vee$ as $t \to t^{-1}$.

†For a group with noncompact center, such as a group of unitary similitudes, we weaken (ii) to require only that $\varphi(1 \times \sigma)$ acts as $t \to t^{-1}$ on the intersection of the centralizer of $\varphi(\mathbb{C}^\times \times 1)$ with the commutator subgroup of $G^\vee$. A (real algebraic) group G has regular elliptic parameters if and only if G is *cuspidal*, *i.e.* G is a maximal cuspidal Levi subgroup in G or, equivalently, $G(\mathbb{R})$ has Cartan subgroups that are compact modulo the center of $G(\mathbb{R})$. Thus G has regular elliptic parameters if and only $G(\mathbb{R})$ has regular elliptic points.

3. Regular elliptic parameters

Suppose $\boldsymbol{\varphi}$ is a regular elliptic parameter. We now attach to $spl_{G^\vee}$ a representative φ for $\boldsymbol{\varphi}$ which satisfies (i) and (ii) above, and is determined uniquely up to $T^\vee$-conjugacy. In particular, the restriction of φ to $\mathbb{C}^\times \times 1$ will be determined uniquely by the splitting.

First, if $z = re^{i\theta}$ we write $(z/\overline{z})^{m/2}$ for $e^{im\theta}$, $m \in \mathbb{Z}$. Define $\varphi_0(z \times 1)$ to be

$$
\begin{pmatrix}
(z/\overline{z})^{m_1/2} & & & \\
& (z/\overline{z})^{m_2/2} & & \\
& & \ldots & \\
& & & (z/\overline{z})^{m_n/2}
\end{pmatrix},
$$

where $m_1, m_2, ..., m_n$ are given distinct integers. Extending φ_0 to a regular elliptic parameter requires the choice of an element $g = \varphi_0(1 \times \sigma)$ of $GL(n, \mathbb{C})$ which

(i) normalizes $T^\vee$,
(ii) takes the Borel subgroup $B^\vee$ *to* its opposite relative to $T^\vee$, *i.e.* to lower triangular matrices, and
(iii) satisfies

$$
g\sigma^\vee(g) = \begin{pmatrix}
(-1)^{m_1} & & & \\
& (-1)^{m_2} & & \\
& & \ldots & \\
& & & (-1)^{m_n}
\end{pmatrix}.
$$

We can take for g any product of a diagonal matrix with J_n. We will take J_n itself. Since

$$
J_n \sigma^\vee(J_n) = (-1)^{n+1} I,
$$

each m_j must be of parity opposite to that of n, whatever the choice for g. This is the only restriction needed to obtain a well-defined regular elliptic parameter φ. If we replace φ by $Int(h) \circ \varphi$, for any element h in $GL(n, \mathbb{C})$ normalizing $T^\vee$, then (i) - (iii) remain true. To prescribe φ up to $T^\vee$-conjugacy we insist that $m_1 > m_2 > ... > m_n$. Thus:

Proposition

The regular elliptic parameters $\boldsymbol{\varphi}$ are in 1-1 correspondence with tuples $(m_1, m_2, ..., m_n)$ of integers, where $m_1 > m_2 > ... > m_n$ and each m_j is of parity opposite to that of n. Given such a tuple, the corresponding parameter

$$
\boldsymbol{\varphi} = \boldsymbol{\varphi}(m_1, m_2, ..., m_n)
$$

has representative φ given by

$$\varphi(z\times 1)=\begin{pmatrix} (z/\overline{z})^{m_1/2} & & & \\ & (z/\overline{z})^{m_2/2} & & \\ & & ... & \\ & & & (z/\overline{z})^{m_n/2} \end{pmatrix}\times(z\times 1),\quad z\in\mathbb{C}^\times$$

and

$$\varphi(1\times\sigma)=J_n\times(1\times\sigma).$$

4. $\mathbb{S}$-groups for regular elliptic parameters

We attach to a regular elliptic parameter $\boldsymbol{\varphi}$ two finite abelian groups $\mathbb{S}=\mathbb{S}_{\boldsymbol{\varphi}}$ and $\mathbb{S}^{sc}=\mathbb{S}^{sc}_{\boldsymbol{\varphi}}$.

Fix a representative φ and suppose S_φ is the centralizer in $G^\vee=GL(n,\mathbb{C})$ of the image of φ. Then S_φ consists of the elements of order two in the maximal torus $Cent(\varphi(\mathbb{C}^\times\times 1),GL(n,\mathbb{C}))$, and so is a finite abelian group. Suppose

(i) $\mathbb{S}_{\boldsymbol{\varphi}}$ is the image of S_φ in $G^\vee_{ad}=PGL(n,\mathbb{C})$

and

(ii) $\mathbb{S}^{sc}_{\boldsymbol{\varphi}}$ is the preimage of $\mathbb{S}_{\boldsymbol{\varphi}}$ in $G^\vee_{sc}=SL(n,\mathbb{C})$.

Once again our definitions exploit the fact that the center of a unitary group is compact.

Clearly $\mathbb{S}_{\boldsymbol{\varphi}}$ is abelian, a finite subgroup of the image $T^\vee_{ad}$ of $T^\vee$ in $PGL(n,\mathbb{C})$. So also is $\mathbb{S}^{sc}_{\boldsymbol{\varphi}}$ abelian because it is contained in a maximal torus of $SL(n,\mathbb{C})$.

Notice that if we replace representative φ by a conjugate φ' we get unique isomorphisms $\mathbb{S}_{\boldsymbol{\varphi}}\to\mathbb{S}_{\boldsymbol{\varphi}'}$, $\mathbb{S}^{sc}_{\boldsymbol{\varphi}}\to\mathbb{S}^{sc}_{\boldsymbol{\varphi}'}$. We will therefore work with our familiar representative from the proposition, and drop φ from notation. Then S is the group of diagonal matrices of the form

$$\begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & ... & \\ & & & \pm 1 \end{pmatrix}.$$

The group $\mathbb{S}$ may be identified as the quotient of S by $\{\pm I\}$ and so

$$\mathbb{S} \backsimeq (C_2)^{n-1},$$

where we use C_r to denote the cyclic group of order r. Consider the extension

$$1 \to Z_{sc}^{\vee} \to \mathbb{S}^{sc} \to \mathbb{S} \to 1,$$

where $Z_{sc}^{\vee} = C_n$ denotes the center of $G_{sc}^{\vee} = SL(n, \mathbb{C})$.

Assume n is odd. Then each element of $\mathbb{S}$ has a unique matrix in its preimage under $\mathbb{S}^{sc} \to \mathbb{S}$ with diagonal entries all ± 1, and so we have a natural splitting of the extension.

Assume n is even. Then the extension does not split. For example, if $n = 2$ then $\mathbb{S}^{sc}$ is generated by

$$\begin{pmatrix} i & \\ & -i \end{pmatrix},$$

and so $\mathbb{S}^{sc} = C_4$. In general, $\mathbb{S}^{sc}$ is the subgroup of $SL(n, \mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} \pm\epsilon & & & \\ & \pm\epsilon & & \\ & & \ldots & \\ & & & \pm\epsilon \end{pmatrix},$$

where $\epsilon^{2n} = 1$, with the choice of ϵ and signs restricted to yield determinant one. This subgroup is generated by matrices with all entries ± 1 and an even number of -1's, along with a single element

$$\begin{pmatrix} \epsilon & & & & \\ & \epsilon & & & \\ & & \ldots & & \\ & & & \epsilon & \\ & & & & -\epsilon \end{pmatrix},$$

where ϵ is a primitive $2n^{th}$root of unity.

5. Characters on $\mathbb{S}$

Recall that $\mathbb{S}$ is isomorphic to the quotient of S by $\{\pm I\}$. We may then identify the character group of $\mathbb{S}$ as the additive group consisting of the binary words (strings) of length n that are *even* in the sense that the sum of their bits is 0 : the word $\delta_1\delta_2....\delta_n$ is identified with the character given on $diag(\epsilon_1 , \epsilon_2 , ..., \epsilon_n) \in S$, where $\epsilon_1^2 = \epsilon_2^2 = ... = \epsilon_n^2 = 1$, as the product $\epsilon_1^{\delta_1}\epsilon_2^{\delta_2}...\epsilon_n^{\delta_n}$.

Now suppose we regard the characters of $\mathbb{S}$ as the characters on $\mathbb{S}^{sc}$ which are trivial on $Z_{sc}^{\vee}$. Notice that

$$Z_{sc}^{\vee} = (\mathbb{S}^{sc})^2.$$

Thus we have identified the characters on $\mathbb{S}$, or even binary words of length n, with the quadratic characters on $\mathbb{S}^{sc}$.

Those are all the codes we will need for n odd.

Assume from now that n is even. We will also use a second family of characters on $\mathbb{S}^{sc}$, to which we attach the odd binary words of length n. Fix *one* character ζ on $\mathbb{S}^{sc}$ such that $\zeta(-I) = -1$ as follows. We write the value of ζ on an element $diag(\epsilon_1 , \epsilon_2 , ..., \epsilon_n)$ of $\mathbb{S}^{sc}$ as the product $\epsilon_1^{\delta_1}\epsilon_2^{\delta_2}...\epsilon_n^{\delta_n}$, where each δ_j is either 0 or 1 and the sum of all δ_j is odd. We then attach the odd binary word $\delta_1\delta_2....\delta_n$ to ζ. For example, the character 10 maps

$$\begin{pmatrix} i & \\ & -i \end{pmatrix}$$

to i, while 01 maps it to $-i$. Notice that in general ζ has order at least 4 and that since

$$(\epsilon_1^{\delta_1}\epsilon_2^{\delta_2}...\epsilon_n^{\delta_n})^{-1} = \epsilon_1^{1-\delta_1}\epsilon_2^{1-\delta_2}...\epsilon_n^{1-\delta_n}$$

the code for ζ^{-1} is obtained by reversing each bit.

The second family of characters consists of those ζ' which coincide with ζ on $Z_{sc}^{\vee}$. Then $\zeta'\zeta^{-1}$ is trivial on $Z_{sc}^{\vee}$ and so has an even word attached as before. Then the code for ζ' is defined by adding the words for ζ and $\zeta'\zeta^{-1}$ bit by bit. In particular, when we use $\zeta = 100...0$ in calculations we have just to reverse the first bit in the word for $\zeta'\zeta^{-1}$ to obtain the code for ζ'.

6. Unitary groups

For our analysis of discrete series representations we work with the standard representatives† for the isomorphism classes of real unitary groups of $n \times n$ matrices. These are the groups $\mathbb{U}(p,q)$, $p+q=n$. Thus $\mathbb{U}(p,q)(\mathbb{C}) = GL(n,\mathbb{C})$, and we will use the usual Lie group notation $U(p,q)$ for $\mathbb{U}(p,q)(\mathbb{R})$. The Galois action for $\mathbb{U}(p,q)$ is

$$g \to I_{p,q}\ {}^t\overline{g}^{-1} I_{p,q},$$

where the bar denotes complex conjugation of entries, and

$$I_{p,q} = \begin{pmatrix} 1 & & & & & \\ & \dots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \dots & \\ & & & & & -1 \end{pmatrix},$$

with p entries 1 and q entries -1. This action preserves the diagonal subgroup of $GL(n,\mathbb{C})$. We write T for the diagonal subgroup when we use this action which coincides on T with $t \to \overline{t}^{-1}$. The real points of T form a compact Cartan subgroup

$$T(\mathbb{R}) = \{diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n})\}$$

of $U(p,q)$.

The complex Weyl group Ω of T is the permutation group $\mathcal{S}_n$ on the diagonal entries. The real Weyl group $\Omega_{\mathbb{R}}^q$ for $\mathbb{U}(p,q)$ consists all permutations which can realized as conjugation by a matrix in $U(p,q)$. It coincides with the

Weyl group of T in the maximal compact subgroup $U(p) \times U(q)$ of $U(p,q)$, and so is $\mathcal{S}_p \times \mathcal{S}_q$.

† If $I_{p,q}$ is replaced by a diagonal matrix with first p entries positive and the final q entries negative then we modify the frame of inner twists for the K-group of Section 9 to obtain the same final results.

7. L-packets and infinitesimal character

Consider now the unitary group $U(p,q)$, where $p+q=n$. A short description of Langlands' correspondence for discrete series is this: it attaches to regular elliptic $\boldsymbol{\varphi} = \boldsymbol{\varphi}(m_1, m_2, ..., m_n)$ from Section 3 the L-packet consisting of all (finitely many) discrete series representations of $U(p,q)$ with *infinitesimal character*† determined by the linear form

$$\mu : diag(t_1, t_2, ..., t_n) \to \tfrac{m_1}{2}t_1 + \tfrac{m_2}{2}t_2 + ... + \tfrac{m_n}{2}t_n$$

on the space of complex diagonal matrices. We have assumed that $m_1 > m_2 > ... > m_n$; each integer m_j is of parity opposite to that of n.

Recall finite-dimensional irreducible (rational or holomorphic) representations of $GL(n,\mathbb{C})$. The classification by *highest weight* is realized by tuples $(m'_1, m'_2, ..., m'_n)$ of integers such that $m'_1 \geq m'_2 \geq ... \geq m'_n$: let π be an irreducible finite-dimensional representation of $GL(n,\mathbb{C})$ then $d\pi$ is an irreducible finite-dimensional representation of $\mathfrak{gl}(n,\mathbb{C})$. The highest weight for the diagonal subalgebra $\mathfrak{t}_\mathbb{C}$ is of the form

$$\mu' : diag(t_1, t_2, ..., t_n) \to m'_1 t_1 + m'_2 t_2 + ... + m'_n t_n$$

[see Fulton and Harris or Goodman and Wallach]. On the other hand, $d\pi$ determines also an irreducible finite-dimensional representation of the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{gl}(n,\mathbb{C})$. Elements of the center $\mathcal{Z}$ of $\mathcal{U}$ act as scalars and so determine a character χ of $\mathcal{Z}$. We apply the Harish-Chandra isomorphism of $\mathcal{Z}$ with $\mathcal{S}_n$-invariants in the symmetric algebra on $\mathfrak{t}_\mathbb{C}$ to calculate χ as that character of $\mathcal{Z}$ determined by the linear form

$$diag(t_1, t_2, ..., t_n) \to (m'_1 + \tfrac{n-1}{2})t_1 + (m'_2 + \tfrac{n-3}{2})t_2 + ... + (m'_n + \tfrac{-(n-1)}{2})t_n.$$

Setting

$$m_1 = 2m'_1 + n - 1,\ m_2 = 2m'_2 + n - 3,\ ...,\ m_n = 2m'_n - n + 1,$$

we retrieve a tuple $(m_1, m_2, ..., m_n)$ of integers where, as in the last paragraph, $m_1 > m_2 > ... > m_n$ and each m_i is of parity opposite to that of n. Conversely, each such $(m_1, m_2, ..., m_n)$ evidently determines a tuple $(m'_1, m'_2, ..., m'_n)$ of integers such that $m'_1 \geq m'_2 \geq ... \geq m'_n$. We write $\chi = \chi(m_1, m_2, ..., m_n)$.

By Weyl's unitary trick, we may replace $GL(n, \mathbb{C})$ by the compact form $U(n) = U(n, 0)$, along with its complexified Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, to classify irreducible finite dimensional (equivalently, irreducible unitary) representations of $U(n)$ either by highest weight data $(m'_1, m'_2, ..., m'_n)$ or by infinitesimal character $\chi(m_1, m_2, ..., m_n)$.

†This determines the central character of these representations since $T(\mathbb{R})$ is connected.

8. Harish Chandra theorems and stable characters

Assume $\boldsymbol{\varphi} = \boldsymbol{\varphi}(m_1, m_2, ..., m_n)$ is a regular elliptic parameter, with attached L-packet Π of discrete series representations.

We start with $U(n)$. Here Π consists of a single finite-dimensional representation. Using the Weyl formula, we write the character of this representation on matrices in $T(\mathbb{R})$ with distinct entries as

$$\Theta^*(diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n})),$$

given formally by

$$\frac{\sum_{\omega \in \mathcal{S}_n} sign(\omega)\ e^{i(m_{\omega_1}\theta_1\ +\ m_{\omega_2}\theta_2\ +\ ...\ +m_{\omega_n}\theta_n)/2}}{\prod_{j<k}(e^{i(\theta_j\ -\ \theta_k)/2}\ -\ e^{i(\theta_k\ -\ \theta_j)/2})},$$

where $\omega_j = \omega^{-1}(j)$, $1 \leq j \leq n$. To write this in a manner well-defined on the entire regular set in $T(\mathbb{R})$ we extract $\prod_{j<k} e^{i(\theta_j\ -\ \theta_k)/2}$ from both numerator and denominator. The denominator then becomes

$$\prod_{j<k}(1\ -\ e^{-i(\theta_j\ -\ \theta_k)}).$$

The parity conditions on the integers $m_1, m_2, ..., m_n$ ensure that the new numerator is also well-defined: in the term involving $\omega \in \mathcal{S}_n$ the tuple $(m_{\omega_1}, m_{\omega_2}, ..., m_{\omega_n})$ is replaced by the tuple

$$(m_{\omega_1} - (n-1), m_{\omega_2} - (n-3), ..., m_{\omega_n} + (n-1))$$

of even integers.

Passing now to a noncompact form $U(p,q)$, we follow Harish Chandra to regard the character of a discrete series representation π as an invariant tempered distribution $Tr\ \pi : f \to Trace\ \pi(f)$ on $U(p,q)$. Again $\mathcal{Z}$ acts by scalars, defining an infinitesimal character for $Tr\ \pi$ of the form $\chi(m_1, m_2, ..., m_n)$. Until Section 10, however, we limit our attention to *stable* analysis.

According to Harish Chandra, there is a unique tempered invariant eigendistribution (with infinitesimal character $\chi(m_1, m_2, ..., m_n)$) represented on the matrices in $T(\mathbb{R})$ with distinct entries by the function Θ^* above. We write Θ^* also for this distribution. Then $(-1)^{pq}\Theta^*$ is the sum of the characters of all discrete series representations with infinitesimal character $\chi(m_1, m_2, ..., m_n)$, *i.e.* of all members of the packet Π. It is, moreover, a stable distribution in the usual sense (although that is stated slightly differently in Harish Chandra's original theorem). For any π in Π, we write $St\text{-}Tr\ \pi$ for $(-1)^{pq}\Theta^*$.

9. Real unitary groups and K-groups

We follow Vogan's idea of considering several real forms at once, but use Kottwitz's setting in terms of Galois cohomology, following [Arthur, 99duke]. We include various details that will be helpful later. The construction is based on the nontriviality of the cohomology set $H^1(\Gamma, G_{sc})$.

It is convenient to fix the *basic endoscopic group* G^* (attached to trivial endoscopic data) as the unitary group U_n of the hermitian form ϵJ_n, where $\epsilon^2 = (-1)^{n+1}$, along with $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X\})$, where B^* denotes upper triangular matrices, T^* diagonal matrices, and $\{X\}$ the standard simple root vectors. The pairing of character lattices $X^*(T^*)$ and $X^*(T^\vee)$ attached to the pairs (B^*, T^*) and $(B^\vee, T^\vee)$ is the natural one: if $t_j : diag(t_1, t_2, ..., t_n) \to t_j$ and $z_k : diag(z_1, z_2, ..., z_n) \to z_k$ then $< t_j, z_k > = \delta_{jk}$.

We set $G^j = \mathbb{U}(n-j, j)$, so that G^0 is the compact form while G^m is the quasi-split form, where m is the greatest integer less than or equal to $n/2$. We fix some $\mathbb{R}$-isomorphism $\psi : G^m \to G^*$. There is no harm in assuming that ψ is given as conjugation by an element of $GL(n, \mathbb{C})$. For example, we may fix an $\mathbb{R}$-splitting of G^m and choose a conjugation that takes it to spl^*.

We form the groups G^j into a single K-group if n is odd, and into two (disjoint) K-groups if n is even. Whatever the parity of n, there is exactly one K-group $\mathbf{G}$ with the quasi-split form G^m as component. We call this the K-group of quasi-split type, and describe it first.

A real K-group is an algebraic variety over $\mathbb{R}$ whose components are algebraic groups over $\mathbb{R}$ which are all inner forms of each other. We fix a *frame* of inner twists among the components satisfying conditions that are natural for Galois cohomology. We limit our discussion to the results of some simple explicit calculations which exploit the surjectivity of the maps $H^1(\Gamma, T_{sc}) \to H^1(\Gamma, G_{sc})$ and $H^1(\Gamma, T) \to H^1(\Gamma, G)$ determined by inclusion.

Suppose n is odd. Then $\mathbf{G}$ is simply

$$G^0 \sqcup G^1 \sqcup G^2 \sqcup ... \sqcup G^m,$$

with twists $\psi_{i,j} : G^i \to G^j$ all equal to the identity map, and $\psi_j : G^j \to G^*$ equal to the composition of $\psi_{j,m}$ with $\psi : G^m \to G^*$. Because of the different Galois actions on the groups G^j these maps provide us with various cocycles which we need to track. We specify a 1-cocycle of Γ in a group X by an element x_σ of X such that $x_\sigma \sigma(x_\sigma) = 1$. Notice that $\psi_j\, \sigma(\psi_j)^{-1}$ is conjugation by $\psi(x^j_\sigma)$, where $x^m_\sigma = I$ and for $0 \leq j < m$, x^j_σ is the cocycle

$$\begin{pmatrix}
1 & & & & & & & & \\
& ... & & & & & & & \\
& & 1 & (m+1)^{st} & & & & & \\
& & & -1 & & & & & \\
& & & & ... & & & & \\
& & & & & -1 & (n-j)^{th} & & \\
& & & & & & 1 & & \\
& & & & & & & ... & \\
& & & & & & & & 1
\end{pmatrix}$$

in G^m. First we check that the map $H^1(\Gamma, G^*_{sc}) \to H^1(\Gamma, G^*)$ is injective, and then that the cocycles $\psi(x^j_\sigma)$, $0 \leq j \leq m$, provide a complete set of representatives without redundancy for the image of $H^1(\Gamma, G^*_{sc})$ in $H^1(\Gamma, G^*)$: if $\psi(x^j_\sigma)$ has negative determinant we see that it is in the image of the class of $-\psi(x^j_\sigma)$ in $H^1(\Gamma, G^*_{sc})$. Further, this image maps bijectively to $H^1(\Gamma, G^*_{ad})$ under $H^1(\Gamma, G^*) \to H^1(\Gamma, G^*_{ad})$, and that justifies our description of the K-group $\mathbf{G}$.

Suppose n is even. We again have the $m+1$ cocycles $\psi(x^j_\sigma)$ in G^*. Negative entries in x^j_σ now start at the $(m+1)^{st}$ position. However, the class of $\psi(x^j_\sigma)$ lies in the image of $H^1(\Gamma, G^*_{sc}) \to H^1(\Gamma, G^*)$ if and only if $m-j$ is even. Moreover, if $j < m$ as well, there are two distinct classes, represented by $\pm\psi(x^j_\sigma)$, in $H^1(\Gamma, G^*_{sc})$ mapping to distinct classes in $H^1(\Gamma, G^*)$ and then to the same class in $H^1(\Gamma, G^*_{ad})$. If $j = m$ these two classes are represented by $\pm I$ and are both trivial in $H^1(\Gamma, G^*_{sc})$. Finally the classes we have described form the entire image of $H^1(\Gamma, G^*_{sc})$ in $H^1(\Gamma, G^*)$. The K-group of quasi-split type is thus

$$\mathbf{G} = G^m \sqcup G^{m-2} \sqcup G^{m-2} \sqcup G^{m-4} \sqcup G^{m-4} \sqcup \ldots .$$

To describe the inner twists between components, we label the repeated components as $G^{m-2,1}$, $G^{m-2,2}$, etc. We take $\psi^1_{j,m} : G^{j,1} \to G^m$ to be the identity and compose this with ψ to get $\psi^1_j : G^{j,1} \to G^*$. We also take $\psi^2_{j,m} : G^{j,2} \to G^m$ to be the identity, and compose this with ψ to get $\psi^2_j : G^{j,2} \to G^*$. Each remaining twist is defined by the appropriate composition: for example, to get from $G^{j,2}$ to $G^{k,1}$ we use $(\psi^1_k)^{-1} \circ \psi^2_j$.

Thus we have used only some of the inner forms to describe the K-group of quasi-split type. For example, for $n = 4$ we have used only the quasi-split form itself and two copies of the compact form; for $n = 6$ we use two copies of a noncompact form along with the quasi-split form.

The remaining inner forms also constitute a K-group $\mathbf{G}'$, but now every form appears twice. Thus

$$\mathbf{G}' = G^{m-1} \sqcup G^{m-1} \sqcup G^{m-3} \sqcup G^{m-3} \sqcup G^{m-5} \sqcup \ldots .$$

We use a copy of G^{m-1} as our *identity component* $G^{m-1,1}$, and define twists $\psi^1_j : G^{j,1} \to G^*$ and $\psi^2_j : G^{j,2} \to G^*$ as before, for $j = m-1, m-3, m-$

$5, \ldots$. Once again the attached cocycles $\pm\psi(x^j_\sigma)$ provide a complete set of representatives without redundancy for the image of $H^1(\Gamma, G^*_{sc})$ in $H^1(\Gamma, G^*)$, and we are done.

Examples

$\mathbf{n = 2}$

$\mathbf{G} = \mathbb{U}(1,1)$
$\mathbf{G}' = \mathbb{U}(2,0) \sqcup \mathbb{U}(2,0)$

$\mathbf{n = 3}$

$\mathbf{G} = \mathbb{U}(2,1) \sqcup \mathbb{U}(3,0)$

$\mathbf{n = 4}$

$\mathbf{G} = \mathbb{U}(2,2) \sqcup \mathbb{U}(4,0) \sqcup \mathbb{U}(4,0)$
$\mathbf{G}' = \mathbb{U}(3,1) \sqcup \mathbb{U}(3,1)$

$\mathbf{n = 5}$

$\mathbf{G} = \mathbb{U}(3,2) \sqcup \mathbb{U}(4,1) \sqcup \mathbb{U}(5,0)$

$\mathbf{n = 6}$

$\mathbf{G} = \mathbb{U}(3,3) \sqcup \mathbb{U}(5,1) \sqcup \mathbb{U}(5,1)$
$\mathbf{G}' = \mathbb{U}(4,2) \sqcup \mathbb{U}(4,2) \sqcup \mathbb{U}(6,0) \sqcup \mathbb{U}(6,0)$.

10. Harish Chandra theorems and counting L-packets

We return to a regular elliptic parameter $\boldsymbol{\varphi} = \boldsymbol{\varphi}(m_1, m_2, \ldots, m_n)$ and the discrete series representations in the L-packets attached to $\boldsymbol{\varphi}$. Suppose $\mathbf{G} = \sqcup_j G^j$ is a K-group. We will use this notation even when multiple copies of an isomorphism class appear. We write $\Pi = \sqcup_j \Pi^j$, for the disjoint union of the L-packets Π^j for the components G^j. Again our notation will not reflect that we may include two copies of the same representation. There are evident notions of stable conjugacy in $\mathbf{G}(\mathbb{R})$ and stable character $St\text{-}Tr\ \pi$ on $\mathbf{G}(\mathbb{R})$. Notice that the sign $(-1)^{pq}$ is the same for all components of $\mathbf{G}$.

First we consider packet Π^j for a single component G^j. We rewrite the function representing the stable character $St\text{-}Tr\ \pi$ on the regular set in $T(\mathbb{R})$ as

$$\frac{(-1)^{pq}\sum_{\omega\in\Omega}\det\omega\ \ \Lambda_\omega}{\prod_{\alpha>0}\ (1-\alpha^{-1})}.$$

Here $\alpha > 0$ means a root α that is positive in the standard ordering (*i.e.* $\alpha(diag(t_1, ..., t_n) = t_j t_k^{-1}$ for some $j < k$), and ι is one half the sum of the positive roots in this ordering. Also Λ_ω is the (rational) character $\exp(\omega^{-1}\mu - \iota)$. Recall that μ is the linear form from Section 3 defining the infinitesimal character $\chi(m_1, m_2, ..., m_n)$ for the packet, and that we have assumed $m_1 > m_2 > ... > m_n$, *i.e.* μ is regular dominant for the standard ordering on roots.

By a theorem of Harish Chandra there is a unique discrete series representation of $G^j(\mathbb{R})$ whose character $Tr\ \pi$ is given on the regular set in $T(\mathbb{R})$ by the same formula except that the summation is now taken only over the real Weyl group $\Omega^j_{\mathbb{R}}$. More generally, we can sum over any fixed coset $\omega\Omega^j_{\mathbb{R}}$ of $\Omega^j_{\mathbb{R}}$ in Ω to obtain a unique discrete series representation of $G^j(\mathbb{R})$ with same infinitesimal character† and given on the regular set in $T(\mathbb{R})$ by the class function

$$\frac{(-1)^{pq}\det\omega\sum_{\omega_0\in\Omega^j_{\mathbb{R}}}\det\omega_0\ \ \Lambda_{\omega\omega_0}}{\prod_{\alpha>0}(1-\alpha^{-1})}.$$

The term $\det\omega$ disappears if we rewrite the formula using the (unique) positive system for which $\omega^{-1}\mu$ is dominant and replacing ι by $\omega^{-1}\iota$.‡ We then prefer to label the representation as $\pi(\mathcal{C})$, where $\mathcal{C}$ is the Weyl chamber containing the chosen element $\omega^{-1}\mu$ of the complex Weyl group orbit of μ. Here we can take the (open) Weyl chambers as connected components in $\mathbb{R}^n$ of the set of vectors with all entries distinct. So this particular representation is given by the inequality $m_{\omega_1} > m_{\omega_2} > ... > m_{\omega_n}$. We may write the representation $\pi(\mathcal{C})$ also as $\pi(\mathcal{C}')$ if and only if $\mathcal{C}'$ lies in the real ($\Omega^j_{\mathbb{R}}$-) orbit of $\mathcal{C}$.

Since these representations $\pi(\mathcal{C})$ account for all discrete series with infinitesimal character $\chi(m_1, m_2, ..., m_n)$, there are exactly $\left|\Omega/\Omega^j_{\mathbb{R}}\right| = \binom{n}{j}$ discrete series representations in the packet Π^j. Then the cardinality of a discrete series L-packet for $\mathbf{G}$ is $\sum_j \left|\Omega/\Omega^j_{\mathbb{R}}\right|$ which elementary calculation shows to coincide with 2^{n-1} in each of the three cases n odd, n even and quasisplit type, and n even not of quasi-split type. This reflects a property of real Galois cohomology sets that we review next.

†Only the orbit of $(m_1, m_2, ..., m_n)$ under $\Omega = \mathcal{S}_n$ matters for $\chi(m_1, m_2, ..., m_n)$.

‡We may remove all determinants and recover the formula used in [K], [GKM], [Spallone] as follows. Let B be the Borel subgroup defined by $\omega_0^{-1}(\mathcal{C})$ and write Λ_B in place of the (rational) character $\exp(\omega_0^{-1}\omega^{-1}(\mu - \iota))$. Then the character of $\pi(\mathcal{C})$ is given on the regular set of $T(\mathbb{R})$ by

$$(-1)^{pq} \ \sum_B \ \frac{\Lambda_B}{\prod_{\alpha,B} \ (1-\alpha^{-1})},$$

where $\prod_{\alpha,B}$ indicates a product over roots α of T in B.

11. K-groups and $\mathcal{D}_G(T)$

We look more closely at the purpose of the K-group construction, starting with familiar motivation from the geometric side, namely stable conjugacy. Fix a group G^j. Given a regular elliptic stable conjugacy class $\mathcal{O}$ in $G^j(\mathbb{R})$, we choose an element δ of $T(\mathbb{R})$ in $\mathcal{O}$. For $g \in GL(n, \mathbb{C})$, $g^{-1}\delta g$ also lies in $\mathcal{O}$ if and only if $g\sigma(g)^{-1}$ lies in $T(\mathbb{C})$. Then the $G^j(\mathbb{R})$-conjugacy classes in $\mathcal{O}$ are parametrized by the set $\mathcal{D}_j(T)$ of elements in $H^1(\Gamma, T)$ which become trivial in $H^1(\Gamma, G^j)$ under the map determined by inclusion. This set depends on j. For example, it is trivial when $j = 0$, *i.e.* $G^j(\mathbb{R})$ is compact, and is largest† when $j = m$, *i.e.* $G^j(\mathbb{R})$ is quasi-split. On the other hand, each $\mathcal{D}_j(T)$ is contained in the group

$$\mathcal{E}(T) = \mathrm{Im}(H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T))$$

which is the same for all j.

The K-group construction provides a useful partition of $\mathcal{E}(T)$ into subsets $\mathcal{D}'_j(T)$, and also $\mathcal{D}''_j(T)$ when the component G^j is repeated, in bijection with $\mathcal{D}_j(T)$. For convenience we write this partition as $\mathcal{E}(T) = \sqcup_j \mathcal{D}_j(T)$.

For the K-group $\mathbf{G}$ of quasi-split type, the partition is as follows.‡ First identify $\mathcal{D}_j(T)$ with its image in $\mathcal{E}(T)$ under the twist from G^j to G^m. Then $\mathcal{D}'_j(T)$ is the translate of $\mathcal{D}_j(T)$ by the class of the twisting cocycle x_σ^j or $x_\sigma^{j,1}$, while $\mathcal{D}''_j(T)$ is defined relative to the second twist and cocycle. For the K-group $\mathbf{G}'$, we replace G^m by $G^{m-1,1}$ in the definitions.

We can also view $\mathcal{D}_j(T)$ as a quotient of Weyl groups. Each member ω of the complex Weyl group $\Omega \backsimeq \mathcal{S}_n$ acts as a permutation of T preserving $T(\mathbb{R})$. Recall that $\Omega^j_{\mathbb{R}}$ consists of those permutations realized in $G^j(\mathbb{R})$. Because every conjugacy class in a stable regular elliptic conjugacy class meets $T(\mathbb{R})$, we may assume that the element g of the first paragraph normalizes $T(\mathbb{C})$, and so identify $\mathcal{D}_j(T)$ with the quotient set $\Omega \,/\, \Omega^j_{\mathbb{R}}$. In particular, $|\mathcal{D}_j(T)| = \binom{n}{j}$.

Examples

$\mathbf{n=2}$

$$\begin{aligned}\mathcal{E}(T) &= \mathcal{D}_1(T)\\ &= \mathcal{D}_0(T) \sqcup \mathcal{D}_0(T)\\ &\qquad (2 = 2 = 1+1)\end{aligned}$$

$\mathbf{n=3}$

$$\begin{aligned}\mathcal{E}(T) &= \mathcal{D}_1(T) \sqcup \mathcal{D}_0(T)\\ &\qquad (4 = 3+1)\end{aligned}$$

$\mathbf{n=4}$

$$\begin{aligned}\mathcal{E}(T) &= \mathcal{D}_2(T) \sqcup \mathcal{D}_0(T) \sqcup \mathcal{D}_0(T)\\ &= \mathcal{D}_1(T) \sqcup \mathcal{D}_1(T)\\ &\qquad (8 = 6+1+1 = 4+4)\end{aligned}$$

$\mathbf{n=5}$

$$\begin{aligned}\mathcal{E}(T) &= \mathcal{D}_2(T) \sqcup \mathcal{D}_1(T) \sqcup \mathcal{D}_0(T)\\ &\qquad (16 = 10+5+1)\end{aligned}$$

$\mathbf{n=6}$

$$\begin{aligned}\mathcal{E}(T) &= \mathcal{D}_3(T) \sqcup \mathcal{D}_1(T) \sqcup \mathcal{D}_1(T)\\ &= \mathcal{D}_2(T) \sqcup \mathcal{D}_2(T) \sqcup \mathcal{D}_0(T) \sqcup \mathcal{D}_0(T)\\ &\qquad (32 = 20+6+6 = 15+15+1+1).\end{aligned}$$

† This may be false for a unitary similitude group. See Section 16.

‡ Alternatively, we may use the twisted action of Ω from [Borovoi] in the description. For n odd, the partition is exactly into orbits for σ^m-twisted action of Ω on $\mathcal{E}(T)$.

12. Partitions and Tate-Nakayama duality

We use Tate-Nakayama duality to identify $\mathcal{E}(T)$ with

$$Image\ (H^{-1}(\Gamma, X_*(T_{sc})) \to H^{-1}(\Gamma, X_*(T))),$$

or

$$H^{-1}(\Gamma, X_*(T_{sc}))\ /\ Kernel(H^{-1}(\Gamma, X_*(T_{sc})) \to H^{-1}(\Gamma, X_*(T))).$$

In the present example this is simply

$$X_*(T_{sc})\ /\ 2X_*(T_{sc}).$$

Write the roots of T as

$$t_k - t_l : diag(t_1, ..., t_n) \to t_k/t_l,$$

with corresponding coroot

$$e_k - e_l : t \to diag(t_1, ..., t_n),$$

where $t_k = t = t_l^{-1}$ and all other entries are 1. Then each element of $\mathcal{E}(T)$ can be written as a sum of coroots

$$(e_{k_1} - e_{l_1}) + (e_{k_2} - e_{l_2}) + \ ...\ + (e_{k_r} - e_{l_r})$$

modulo $2X_*(T_{sc})$.

We review next how to recognize elements of $\mathcal{D}_j(T)$ among the elements of $\mathcal{E}(T)$. The root $t_k - t_l$ is *compact* in G^j if the k^{th} and l^{th} diagonal entries are the same in the matrix $I_{n-j,j}$, and *noncompact* otherwise. If $t_k - t_l$ is noncompact in G^j then the Weyl reflection $\omega_{k,l}$ relative to $t_k - t_l$ determines the nontrivial element $e_k - e_l$ of $\mathcal{D}_j(T)$. If $t_k - t_l$ is compact then $\omega_{k,l}$ defines the trivial element of $\mathcal{D}_j(T)$. For the contribution of a product $\omega\omega'$ of Weyl reflections (or any product in the Weyl group) we have the rule that if ω, ω' determine the sums χ, χ' respectively, then $\omega\omega'$ determines $\chi + \omega\chi'$.

Examples

n = 2

$\mathcal{D}_0(T) = \{0\}$
$\mathcal{D}_1(T) = \{0, e_1 - e_2\}$

n = 3

$\mathcal{D}_0(T) = \{0\}$
$\mathcal{D}_1(T) = \{0, e_2 - e_3, e_1 - e_3\}$

n = 4

$\mathcal{D}_0(T) = \{0\}$
$\mathcal{D}_1(T) = \{0, e_3 - e_4, e_2 - e_4, e_1 - e_4\}$
$\mathcal{D}_2(T) = \{0, e_2 - e_3, e_2 - e_4, e_1 - e_3, e_1 - e_4, e_1 - e_3 + e_2 - e_4\}$

n = 5

$\mathcal{D}_0(T) = \{0\}$
$\mathcal{D}_1(T) = \{0, e_4 - e_5, e_3 - e_5, e_2 - e_5, e_1 - e_5\}$
$\mathcal{D}_2(T) = \{0, e_3 - e_4, e_3 - e_5, e_2 - e_4, e_2 - e_5, e_1 - e_4, e_1 - e_5, e_3 - e_4 + e_2 - e_5, e_3 - e_4 + e_1 - e_5, e_2 - e_4 + e_1 - e_5\}$

Partitions for G

n = 2

$\mathcal{E}(T) = \mathcal{D}_1(T)$

n = 3

$\mathcal{E}(T) = \mathcal{D}'_0(T) \sqcup \mathcal{D}_1(T),$
where
$\mathcal{D}'_0(T) = e_1 - e_2 + \mathcal{D}_0(T).$

n = 4

$\mathcal{E}(T) = \mathcal{D}'_0(T) \sqcup \mathcal{D}''_0(T) \sqcup \mathcal{D}_2(T),$
where
$\mathcal{D}'_0(T) = e_3 - e_4 + \mathcal{D}_0(T)$
and
$\mathcal{D}''_0(T) = e_1 - e_2 + \mathcal{D}_0(T).$

n = 5

$\mathcal{E}(T) = \mathcal{D}_0'(T) \sqcup \mathcal{D}_1'(T) \sqcup \mathcal{D}_2(T)$,
where
$\mathcal{D}_0'(T) = e_4 - e_5 + \mathcal{D}_0(T)$
and
$\mathcal{D}_1'(T) = e_1 - e_5 + e_2 - e_3 + \mathcal{D}_1(T)$.

Partitions for $\mathbf{G}'$

n = 2

$\mathcal{E}(T) = \mathcal{D}_0(T) \sqcup \mathcal{D}_0'(T)$
where
$\mathcal{D}_0'(T) = e_1 - e_2 + \mathcal{D}_0(T) = \{e_1 - e_2\}$

n = 4

$\mathcal{E}(T) = \mathcal{D}_1(T) \sqcup \mathcal{D}_1'(T)$
where
$\mathcal{D}_1(T) = \{0, e_3 - e_4, e_2 - e_4, e_1 - e_4\}$
and
$\mathcal{D}_1'(T) = e_2 - e_3 + \{0, e_1 - e_2, e_1 - e_3, e_1 - e_4\}$.

In the last example, we identify $\{0, e_1 - e_2, e_1 - e_3, e_1 - e_4\}$ with $\mathcal{D}_1(T)$ for the second component of $\mathbf{G}' = \mathbb{U}(3,1) \sqcup \mathbb{U}(3,1)$.

13. Weyl chambers and Whittaker data

We work with $\mathbf{G}$ of quasi-split type. In analogy with the geometric side, there are two choices that we will need to consider at various points:

(i) a base point π_* for a discrete series L-packet $\Pi = \sqcup_j \Pi^j$ for $\mathbf{G}(\mathbb{R})$,
(ii) an identification of the coroots of T with the roots of the maximal torus $T^\vee$ in $GL(n, \mathbb{C})$.

There will be a natural choice for each once we fix a conjugacy class of *Whittaker data*.

We have fixed (respectively $\mathbb{R}$-, Γ-)splittings $spl_{G^*}, spl_{G^\vee}$. These provide us with the standard identification of the coroots of the diagonal group T^*

in the basic endoscopic group G^* with roots of the diagonal subgroup $T^\vee$ of the dual $GL(n, \mathbb{C}) : e_j - e_k$ is identified with $z_j - z_k$. We *do not* use the same identification for T.

Consider now the diagonal subgroup T of G^m. If B is a Borel subgroup containing T then, using the Galois action of G^m, we have that $\sigma(B)$ is the Borel subgroup opposite to B relative to T. If each B-simple root α of T is noncompact then we may define an $\mathbb{R}$-*opp splitting* $spl = (B, T, \{Y_\alpha\})$. These splittings are characterized by the property that $\sigma(Y_\alpha) = Y_{-\alpha}$ for all B-simple α. Here we make the usual conventions about the choice of $Y_{-\alpha}$, and in particular require that the Killing form takes the value 1 on the pair $(Y_\alpha, Y_{-\alpha})$. Such splittings exist only for quasi-split cuspidal groups.

Suppose now that the simple roots defining the Weyl chamber C_* are all noncompact. Let *spl* be an $\mathbb{R}$-opp splitting for this simple system. Recall that ψ is a *fixed* $\mathbb{R}$-isomorphism of G^m with G^*. We can find a unique isomorphism of the form $Int(g^*) \circ \psi$, where $g^* \in G^*$, carrying *spl* to spl_{G^*}, and thus the chamber $\mathcal{C}_*$ to the chamber defined by the B^*-simple roots of T^*. We may use this to identify coroots of T with coroots of T^* and thence with roots of $T^\vee$.

We are interested only in the $\Omega_{\mathbb{R}}^m$-orbit of chambers $\mathcal{C}_*$ for which the simple roots are noncompact. There is one such orbit if n is odd, and two if n is even. The chamber $\mathcal{C}_{**}$ will always represent the $\Omega_{\mathbb{R}}^m$-orbit not containing $\mathcal{C}_*$.

Whittaker data for G^* consist of a pair (B, λ), where B is a Borel subgroup defined over $\mathbb{R}$, and λ is a generic character on the the real points of the unipotent radical of B. The group $G_{ad}^*(\mathbb{R})$ acts transitively on these pairs by conjugacy. So then does $G^*(\mathbb{R})$ if n is odd. There are two $G^*(\mathbb{R})$-conjugacy classes if n is even.

We have fixed the $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X\})$. Given a $G^*(\mathbb{R})$-conjugacy class of Whittaker data (B, λ), we may assume $B = B^*$. Define the additive characters $\psi_{\mathbb{R}}^{\pm}$ on $\mathbb{R}$ by $\psi_{\mathbb{R}}^{\pm}(x) = \exp(\pm 2\pi i x)$. If n is odd then we may assume λ is defined in terms of the simple root vectors $\{X\}$ from spl^* and either character $\psi_{\mathbb{R}}^{\pm}$. In the case n is even, $\{X\}$ and $\psi_{\mathbb{R}}^{\pm}$ yield two characters $\lambda^{\pm}$ such that exactly one of the pairs (B^*, λ^+) and (B^*, λ^-) lies in the conjugacy class of the given (B, λ). Accordingly, when we apply Vogan's theorem on the Langlands parameters of generic representations to our L-packet of discrete series representations, we determine exactly one of the two chambers $\mathcal{C}_*$, $\mathcal{C}_{**}$ as that for which the attached discrete series representation is (B, λ)-generic.

Thus if we start with Whittaker data (B, λ) we choose between $\mathcal{C}_*$ and $\mathcal{C}_{**}$. We have then both a base point for the packet Π and a preferred identification of the coroots of T with roots in $T^\vee$.

14a. An invariant for representations of $\mathbf{G}(\mathbb{R})$

Consider again the discrete series L-packet Π for $\mathbf{G}(\mathbb{R})$. Fix *any* representation π_* in Π^m ; using the $\mathbb{R}$-isomorphism ψ we can also regard π_* as a representation of $G^*(\mathbb{R})$. Now take any $\pi \in \Pi = \sqcup_j \Pi^j$. Recall that

$$\mathcal{E}(T) = \mathrm{Im}(H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T)).$$

We define

$$inv(\pi, \pi_*) \in \mathcal{E}(T)$$

as follows. Suppose $\pi \in \Pi^j$. Write π_* as $\pi(\mathcal{C}_*)$ and π as $\pi(\mathcal{C})$. Then the Weyl chamber $\mathcal{C}_*$ is uniquely determined up to the action of $\Omega_{\mathbb{R}}^m$, and $\mathcal{C}$ up to the action of $\Omega_{\mathbb{R}}^j$. We may choose $y \in SL(n, \mathbb{C})$ such that $Int(y) \circ \psi_{j,m}$ carries $\mathcal{C}$ to $\mathcal{C}_*$. Then

$$(Int(y) \circ \psi_{j,m}) \circ \sigma(Int(y) \circ \psi_{j,m})^{-1} = Int(v_\sigma),$$

where

$$v_\sigma = y u_\sigma \sigma(y)^{-1}$$

and u_σ is a 1-cocycle in $SL(n, \mathbb{C})$ such that

$$\psi_{j,m} \circ \sigma(\psi_{j,m})^{-1} = Int(u_\sigma).$$

The cocycle u_σ can be read from the calculations in Section 9. Then v_σ is also a 1-cocycle and it evidently takes values in T_{sc}. Moreover its class in $H^1(\Gamma, T_{sc})$ is independent of the choices for $\mathcal{C}$, $\mathcal{C}_*$. We write $inv(\pi, \pi_*)$ for (the inverse of) the image of this class in $H^1(\Gamma, T)$. Then a calculation shows that $inv(\pi, \pi_*)$ lies in the appropriate subset $\mathcal{D}'_j(T)$ or $\mathcal{D}''_j(T)$ of $\mathcal{E}(T)$.

As we have seen in Section 12, Tate-Nakayama duality allows us to regard $inv(\pi, \pi_*)$ as a sum of coroots for T. We fix a conjugacy class of Whittaker data to determine a Weyl chamber $\mathcal{C}_*$ for which the simple roots are noncompact, and then use the identification of Section 13 to regard $inv(\pi, \pi_*)$ as a sum of roots of $T^\vee$.

14b. A relative invariant for representations of $\mathbf{G}'(\mathbb{R})$

There is an analogue within $\mathbf{G}'$ of this last construction, where π_* is replaced by a representation of $G^{m-1,1}(\mathbb{R})$. Instead of using this directly, we define another invariant that is useful for spectral transfer factors and provides a simple formula relating factors for $\mathbf{G}'$ to those for $\mathbf{G}$ (a *local hypothesis*).

Let $\pi_* = \pi(\mathcal{C}_*)$, where initially we allow $\mathcal{C}_*$ to be arbitrary. The representation π_* lies in the packet Π^m for G^m (or the packet for G^*) attached to φ. Now take $\pi \in \Pi$, the packet for $\mathbf{G}'$ attached to φ. Recall that we have two copies of each group appearing in $\mathbf{G}'$; as usual, we will not distinguish between them in notation. Suppose $\pi = \pi(\mathcal{C})$ is a representation of $G^j(\mathbb{R})$. Again we may choose $y \in SL(n, \mathbb{C})$ such that $Int(y) \circ \psi_{j,m}$ carries $\mathcal{C}$ to $\mathcal{C}_*$, and

$$(Int(y) \circ \psi_{j,m}) \circ \sigma(Int(y) \circ \psi_{j,m})^{-1} = Int(v_\sigma),$$

where $v_\sigma = y u_\sigma \sigma(y)^{-1}$ and $u_\sigma \in SL(n, \mathbb{C})$ is such that $\psi_{j,m} \circ \sigma(\psi_{j,m})^{-1} = Int(u_\sigma)$. Now however u_σ, and so also v_σ, is not a cocycle since we have moved outside the K-group of G^m (see the calculations of Section 9). On the other hand, the coboundary of v_σ is the same as that of u_σ, and so is independent of π or Π, and moreover this coboundary lies in the center Z_{sc} of $SL(n, \mathbb{C})$. This allows us to define a *relative* invariant.

Here we describe the relative invariant for representations in two discrete series L-packets Π, Π' for $\mathbf{G}$ with parameters φ, φ' respectively. Suppose that

$\pi' = \pi(\mathcal{C}')$ is a representation of component $G^{j'}$ with cochains u'_σ, v'_σ. Then these have same coboundary as u_σ, v_σ (that is built into the definition of K-group), and so $(v_\sigma^{-1}, v'_\sigma)$ defines a 1-cocycle in the torus $U = T_{sc} \times T_{sc} \ / \ \{(z^{-1}, z) : z \in Z_{sc}\}$. We write $inv_{\mathcal{C}_*}(\pi, \pi')$ for its class in $H^1(\Gamma, U)$.

Using Tate-Nakayama duality for U is not much more effort than for T. The cochains v_σ^{-1}, v'_σ defining $inv_{\mathcal{C}_*}(\pi, \pi')$ project onto cocycles in T_{ad}, and so we may regard them as coweights (rather than the previous integral sums of coroots). Because $(v_\sigma^{-1}, v'_\sigma)$ defines an element of U, the sum of these two coweights must be an integral sum of coroots. We then use $\mathcal{C}_*$ to identify this as a sum of roots of $T^\vee$ which we evaluate at s, writing the result as $< inv_{\mathcal{C}_*}(\pi, \pi'),\ s_U >$.

Now assume also that $\mathcal{C}_*$ has the property that the simple roots for $\mathcal{C}_*$ are noncompact. Then we see that $< inv_{\mathcal{C}_*}(\pi, \pi'),\ s_U >$ is independent of the choice for $\mathcal{C}_*$, and write $inv_{\mathcal{C}_*}(\pi, \pi')$ simply as $inv_*(\pi, \pi')$.

15a. Endoscopic codes for G

For the K-group $\mathbf{G}$ of quasi-split type we will attach to each discrete series representation π a unique even binary word of length n. If n is even this requires an additional choice, one of the two conjugacy classes of Whittaker data. The second set of codes is then obtained from the first by reversing each bit.

Assume first n is odd. We use the unique generic basepoint π_* of the L-packet Π of π to define $inv(\pi, \pi_*)$, and then identify $inv(\pi, \pi_*)$ as a sum of roots of $T^\vee$, as above. Recall that we are using additive notation for the root lattice, but that we regard roots as (rational) characters on $T^\vee$. Then $inv(\pi, \pi_*)$ is trivial on the matrix $-I$, and so may be evaluated on

$$\mathbb{S} = S \ / \ \{\pm I\}.$$

This evaluation defines a character on $\mathbb{S}$ and hence an even binary word of length n (Section 4). Recall also that the map $\mathbb{S}^{sc} \to \mathbb{S}$ identifies characters on $\mathbb{S}$ as the group of quadratic characters on $\mathbb{S}^{sc}$.

Assume now that n is even. We can proceed as for n odd once we choose a conjugacy class of Whittaker data and require π_* to be generic for this class. The assertion about reversing the bits follows from examining the steps of the construction and noting that the cocycle $\sigma \to -I$ determines the character 111...1 for n even.

Examples

$\mathbf{n = 2}$

$U(1,1)$
$m_1 > m_2 : \pi_{00}$
$m_2 > m_1 : \pi_{11}$

$\mathbf{n = 3}$

$U(2,1)$
$m_2 > m_3 > m_1 : \pi_{000}$
$m_3 > m_2 > m_1 : \pi_{110}$
$m_2 > m_1 > m_3 : \pi_{011}$

$U(3,0)$
π_{101}

$\mathbf{n = 4}$

$U(2,2)$

$m_1 > m_3 > m_2 > m_4 : \pi_{0000}$
$m_1 > m_2 > m_3 > m_4 : \pi_{0110}$
$m_3 > m_1 > m_2 > m_4 : \pi_{1100}$
$m_1 > m_3 > m_4 > m_2 : \pi_{0011}$
$m_4 > m_3 > m_2 > m_1 : \pi_{1001}$
$m_4 > m_2 > m_3 > m_1 : \pi_{1111}$

$U(4,0)$, $first$ $copy$

π_{0101}

$U(4,0)$, *second copy*

π_{1010}

$\mathbf{n = 5}$

$U(3,2)$

$m_1 > m_5 > m_3 > m_4 > m_2 : \pi_{00000}$
$m_1 > m_5 > m_4 > m_3 > m_2 : \pi_{00110}$
$m_1 > m_3 > m_5 > m_4 > m_2 : \pi_{01100}$
$m_1 > m_5 > m_3 > m_2 > m_4 : \pi_{00011}$
$m_1 > m_2 > m_3 > m_4 > m_5 : \pi_{01001}$
$m_4 > m_5 > m_3 > m_1 > m_2 : \pi_{10010}$
$m_5 > m_1 > m_3 > m_4 > m_2 : \pi_{11000}$
$m_1 > m_2 > m_4 > m_3 > m_5 : \pi_{01111}$
$m_5 > m_1 > m_4 > m_3 > m_2 : \pi_{11110}$
$m_5 > m_1 > m_3 > m_2 > m_4 : \pi_{11011}$

$U(4,1)$	$U(5,0)$
$m_1 > m_5 > m_3 > m_4 > m_2 : \pi_{11101}$	π_{01010}
$m_1 > m_4 > m_3 > m_5 > m_2 : \pi_{10111}$	
$m_1 > m_3 > m_5 > m_4 > m_2 : \pi_{10001}$	
$m_1 > m_2 > m_3 > m_4 > m_5 : \pi_{10100}$	
$m_5 > m_1 > m_3 > m_4 > m_2 : \pi_{00101}$	

15b. Endoscopic codes for $\mathbf{G}'$

We attach to each discrete series representation π of $\mathbf{G}'$ a unique odd binary word of length n. Now, however, we will be arbitrary about our choice of base point π'. We follow the requirements of [Arthur, L-packets] and attach to π' a character ζ on $\mathbb{S}^{sc}$ such that $\zeta(-I) = -1$, *i.e.* a particular odd binary word of length n as in Section 5.

Recall $inv_*(\pi, \pi')$ is a pair of coweights with sum equal to a sum of coroots of T. To identify this as a sum of roots of $T^\vee$ we fix one of $\mathcal{C}_*$ and $\mathcal{C}_{**}$ (for example, by the choice of a conjugacy class of Whittaker data for the quasi-split type $\mathbf{G}$).

To define the code for π we evaluate the sum of roots of $T^\vee$ attached to $inv_*(\pi, \pi')$ on $\mathbb{S}^{sc}$, so obtaining a quadratic character on $\mathbb{S}^{sc}$. We multiply this character by ζ to determine a character to which we have assigned an odd binary word of length n.

If we change our choice between $\mathcal{C}_*$ and $\mathcal{C}_{**}$ then we again simply reverse the bits provided we also replace ζ by ζ^{-1}.

Examples

$\mathbf{n = 2}$

$U(2,0)$ *first copy*

π_{10} (*specified*)

$U(2,0)$ *second copy*

π_{01}

$\mathbf{n = 4}$

$U(3,1)$ *first copy*

$m_4 > m_2 > m_3 > m_1 : \pi_{1000}$ (*specified*)
$m_3 > m_2 > m_4 > m_1 : \pi_{0010}$
$m_2 > m_4 > m_3 > m_1 : \pi_{0100}$
$m_1 > m_2 > m_3 > m_4 : \pi_{0001}$

$U(3,1)$ *second copy*

$m_4 > m_2 > m_3 > m_1 : \pi_{0111}$
$m_3 > m_2 > m_4 > m_1 : \pi_{1101}$
$m_2 > m_4 > m_3 > m_1 : \pi_{1011}$
$m_1 > m_2 > m_3 > m_4 : \pi_{1110}$

15c. Summary: pairing

Evaluation on $\mathbb{S}^{sc}$ of the endoscopic code characters can be expressed as a pairing. Fix a $G^*(\mathbb{R})$-conjugacy class of Whittaker data (B, λ).

(i) For $\mathbf{G}$ of quasi-split type we have defined a pairing

$$\mathbb{S}^{sc} \times \Pi \to \{\pm 1\}$$

which identifies Π as the group of quadratic characters on $\mathbb{S}^{sc}$, *i.e.* characters trivial on $Z_{sc}^{\vee}$. The unique (B, λ)-generic representation in Π is identified with the trivial character. The pairing is determined uniquely by the conjugacy class of (B, λ).

(ii) For $\mathbf{G}'$ not of quasi-split type we have defined a pairing

$$\mathbb{S}^{sc} \times \Pi \to \mathbb{C}^{\times},$$

with image in $2n^{th}$ roots of unity. This pairing depends also on the choice of base point for Π and character ζ on $\mathbb{S}^{sc}$ with which to identify this base point. The character ζ is required to satisfy $\zeta(-I) = -1$. The packet Π is then identified as the set of all characters on $\mathbb{S}^{sc}$ which agree with ζ modulo squares in $\mathbb{S}^{sc}$, *i.e.* modulo $Z_{sc}^{\vee}$.

We denote the pairing in both cases by

$$(s_{sc}, \pi) \to \prec s_{sc}, \pi \succ .$$

For $\mathbf{G}$ we have then

$$\prec s'_{sc}, \pi \succ = \prec s_{sc}, \pi \succ$$

if s'_{sc} has same image as s_{sc} in $\mathbb{S}$, while for $\mathbf{G}'$ we have

$$\prec s'_{sc}, \pi \succ = \zeta(z_{sc}) \prec s_{sc}, \pi \succ,$$

where $s'_{sc} = z_{sc}s_{sc}$, $z_{sc} \in Z_{sc}^{\vee}$. The factor $\zeta(z_{sc})$ is an n^{th} root of unity.

On the other hand, for both $\mathbf{G}$ and $\mathbf{G}'$ we always have

$$\prec s_{sc}, \pi' \succ = \pm \prec s_{sc}, \pi \succ$$

for any two representations π, π' in Π.

In Part B we will describe how these pairings apply to transfer [and stabilization.]

16. Groups of unitary similitudes

As a further exercise we replace unitary groups by groups of unitary similitudes.

In place of the unitary group $\mathbb{U}(n-j, j)$ we consider the corresponding group of unitary similitudes $\mathbb{GU}(n-j, j)$. Recall

$$\mathbb{GU}(n-j, j)(\mathbb{C}) = GL(n, \mathbb{C}) \times GL(1, \mathbb{C}),$$

and the Galois action is given by

$$\sigma(g,z)=(\overline{z}\sigma^j(g),\overline{z})$$

for $g\in GL(n,\mathbb{C})$, $z\in GL(1,\mathbb{C})$, where σ^j denotes the Galois automorphism for $\mathbb{U}(n-j,j)$. We write $GU(n-j,j)$ for $\mathbb{GU}(n-j,j)(\mathbb{R})$ which consists of those pairs (g,z) for which $z\in\mathbb{R}^\times$ and $\sigma(g)=z^{-1}g$; we will occasionally drop the multiplier z from notation. Notice that for $n=2m$ and $j=m$ the element $(J_n,-1)$ lies in the real points $GU(m,m)$ of the quasi-split form.

Consider first the case n is odd. Then the group $GU(n-j,j)$ of real points is connected because it is the product of

$$U(n-j,j)=U(n-j,j)\times\{1\}$$

and the nonzero complex scalar matrices embedded as

$$\{(zI,\ z\overline{z}):z\in\mathbb{C}^\times\}.$$

If $n=2m$ this is again true unless $j=m$. If $j=m$ the product forms the identity component of $GU(m,m)$. There is then one more component, that of $(J_n,-1)$.

We write $T_{(sim)}$ for the thickened diagonal (and maximal) torus $T_{(sim)}=T\times GL(1)$ in $\mathbb{GU}(n-j,j)$. Notice that for all n, the Cartan subgroup $T_{(sim)}(\mathbb{R})$ consists of pairs

$$diag_{(sim)}(re^{i\theta_1},re^{i\theta_2},...,re^{i\theta_n})=(diag(re^{i\theta_1},re^{i\theta_2},...,re^{i\theta_n}),\ r^2),$$

where $r\in\mathbb{R}^\times$ is positive, and $T_{(sim)}(\mathbb{R})$ is therefore connected. We may then continue to parametrize packets of discrete series representations by infinitesimal character. First, notice that both the complex and the real Weyl groups of $T_{(sim)}$ are unchanged when we pass from $\mathbb{U}(n-j,j)$ to $\mathbb{GU}(n-j,j)$ unless $n=2m$ and $j=m$. Then the real Weyl group $\Omega^j_{\mathbb{R}}(T_{(sim)})$ also contains the action of $(J_n,-1)$ on $T_{(sim)}$.

By a discrete series representation of $GU(n-j,j)$ we will mean an irreducible admissible representation π which is square-integrable modulo the center $Z(\mathbb{R})=\{diag_{(sim)}(re^{i\theta},re^{i\theta},...,re^{i\theta})\}\simeq\mathbb{C}^\times$ of $GU(n-j,j)$. Suppose $\pi_{(un)}$ is a discrete series representation of $U(n-j,j)$ and ξ is a quasicharacter on $Z(\mathbb{R})$ which extends the central character of $\pi_{(un)}$. Excluding the case $n = 2m$ and $j = m$ we attach to $(\pi_{(un)}, \xi)$ a unique discrete series representation of $GU(n-j, j)$ simply by extension. If $n = 2m$ and $j = m$ extension produces a representation of the identity component of $GU(m, m)$ which we then induce to an irreducible representation π of $GU(m, m)$. All discrete series representations of $GU(m, m)$ are obtained this way, and $(\pi_{(un)}, \xi)$ and $(\pi'_{(un)}, \xi')$ yield the same (equivalent) representation if and only if $\xi = \xi'$ and either $\pi_{(un)} = \pi'_{(un)}$ or $\pi_{(un)}, \pi'_{(un)}$ are conjugate by the element $(J_n, -1)$ of $GU(m, m)$.

The modification to K-groups is simple. For n odd we again get a single K-group with all isomophism classes occurring with multiplicity one:

$$\mathbf{G} = \mathbb{GU}(m+1, m) \sqcup \mathbb{GU}(m+2, m-1) \sqcup \ ... \ \mathbb{GU}(n-1, 1) \sqcup \mathbb{GU}(n, 0).$$

Suppose $n = 2m$. Then for each $j = 0, 1, ..., m-1$, the two cocycles $\pm\psi(x_\sigma^j)$ of Section 9 now represent the same class, and we again get two K-groups

$$\mathbf{G} = \mathbb{GU}(m, m) \sqcup \mathbb{GU}(m+2, m-2) \sqcup \mathbb{GU}(m+4, m-4) \sqcup \ ...$$

and

$$\mathbf{G}' = \mathbb{GU}(m+1, m-1) \sqcup \mathbb{GU}(m+3, m-3) \sqcup \mathbb{GU}(m+5, m-5) \sqcup \ ... \ ,$$

but now all isomorphism classes occur with multiplicity one.

The corresponding partition of $\mathcal{E}(T_{(sim)}) = \mathcal{E}(T)$ is the same for n odd. For $n = 2m$ we remark that

$$\left|\mathcal{D}_m(T_{(sim)})\right| = \tfrac{1}{2} \left|\mathcal{D}_m(T)\right| = \tfrac{1}{2}\tbinom{2m}{m},$$

while for $j < m$ we have

$$\mathcal{D}_j(T_{(sim)}) = \mathcal{D}_j(T).$$

Thus $\left|\mathcal{E}(T_{(sim)})\right| = \frac{1}{2} \left|\mathcal{E}(T)\right|$. For the partitions of $\mathcal{E}(T_{(sim)})$, remove one half of the elements from $\mathcal{D}_m(T)$, using

$$e_1 - e_n + e_2 - e_{n-1} + \ ... \ + e_m - e_{m+1} \equiv 0.$$

Also set

$$\mathcal{D}'_j(T_{(sim)}) = \mathcal{D}'_j(T)$$

for $j < m$, and ignore the sets $\mathcal{D}''_j(T)$ attached to the second copies of components.

Let $G^\vee_{(sim)} = GL(n,\mathbb{C}) \times GL(1,\mathbb{C})$. Then σ acts on $G^\vee_{(sim)}$ via

$$\sigma^\vee_{(sim)} : (g,z) \to (\sigma^\vee(g), z\det(g)).$$

We use $W_\mathbb{R} \to \Gamma$ to then make an action of $W_\mathbb{R}$ on $G^\vee_{(sim)}$, and set

$${}^LG_{(sim)} = G^\vee_{(sim)} \rtimes W_\mathbb{R}.$$

There is a variant of the standard splitting with $T^\vee_{(sim)} = T^\vee \times GL(1,\mathbb{C})$ and $B^\vee_{(sim)} = B^\vee \times GL(1,\mathbb{C})$.

Recall that in the setting of unitary groups each regular elliptic parameter has a representative $\varphi = \varphi(m_1,...,m_n)$, where $m_1 > m_2 > ... > m_n$ and each m_j is of parity opposite to that of n. We wrote $\varphi(w)$ as $\varphi_0(w) \times w$. The central character ξ for a discrete series L-packet satisfies

$$\xi((e^{i\theta}I,1)) = e^{im\theta/2},$$

where

$$m = m_1 + m_2 + \, ... \, + m_n \; = 2(m'_1 + m'_2 + \, ... \, + m'_n)$$

in the notation of Section 7. Define $m' = m/2$ and α by

$$\xi(rI, r^2) = r^\alpha.$$

We will typically assume α integral. Define a map $\eta : W_\mathbb{R} \to \mathbb{C}^\times$ by

$$\eta(z \times 1) = z^{\alpha/2}\overline{z}^{\alpha/2+m'} = (z\overline{z})^{(\alpha+m')/2}(z/\overline{z})^{-m'/2}$$

and

$$(\eta(1 \times \sigma))^2 = (-1)^{m'}.$$

Now set

$$\varphi_{(sim)}(w) = (\varphi_0(w), \eta(w)) \times w$$

for w in $W_{\mathbb{R}}$. Then $\varphi_{(sim)}$ is a regular elliptic parameter, and the attached L-packet consists of the representations we have described above for a pair $(\pi_{(un)}, \xi)$, where $\pi_{(un)}$ is in the packet attached to φ.

We return to the constructions of Section 4. The centralizer $S_{(sim)}$ in $G^{\vee}_{(sim)}$ of $\varphi_{(sim)}(W_{\mathbb{R}})$ consists of all pairs $(diag(\pm 1, \pm 1, ..., \pm 1), z)$ with an even number of negative signs and $z \in \mathbb{C}^{\times}$. We are more interested in the component group $\pi_0(S_{(sim)})$ which we may embed in $\mathbb{S}^{sc}_{(sim)}$. The group $\mathbb{S}^{sc}_{(sim)}$ is generated by $\pi_0(S_{(sim)})$ and $Z^{\vee}_{sc}$. Notice that if n is even then $\mathbb{S}^{sc}_{(sim)}$ is a subgroup of index two in $\mathbb{S}^{sc}$ (we replace the earlier generator of order $2n$ with its square, *i.e.* with a generator of $Z^{\vee}_{sc}$), while if n is odd then the two groups coincide.

For n odd, the construction of endoscopic codes is the same. For n even, notice that the character on $\mathbb{S}^{sc}$ with binary code 111....1 is trivial on $\mathbb{S}^{sc}_{(sim)}$. Given a binary word δ write $\overline{\delta}$ for the word obtained by reversing each bit. Then the earlier discussion modifies easily to show that the pairs $\{\delta, \overline{\delta}\}$ of even words of length n parametrize each discrete series packet for $\mathbf{G}$, while the pairs $\{\delta, \overline{\delta}\}$ of odd words parametrize each discrete series packet for $\mathbf{G}'$. The pairing for $\mathbf{G}'$ now takes values in n^{th} roots of unity.

17. Base change example

We return to a unitary group $\mathbf{G}$ or $\mathbf{G}'$. Consider the discrete series packet Π with (regular elliptic) parameter $\varphi = \varphi(m_1, ..., m_n)$, where $m_1 > m_2 > ... > m_n$ and each m_j is of parity opposite to that of n. The *base change* of Π is the (tempered, irreducible) representation $\Pi^{\mathbb{C}}$ of $GL(n, \mathbb{C})$ with complex Langlands parameter obtained by restriction $\varphi^{\mathbb{C}}$ of φ to $\mathbb{C}^{\times} \times 1$. Recall that

$$\varphi(z \times 1) = \begin{pmatrix} (z/\overline{z})^{m_1/2} & & & \\ & (z/\overline{z})^{m_2/2} & & \\ & & ... & \\ & & & (z/\overline{z})^{m_n/2} \end{pmatrix} \times (z \times 1), \quad z \in \mathbb{C}^{\times}.$$

By definition of the Langlands correspondence, $\Pi^{\mathbb{C}}$ is the principal series representation determined by the unitary character

$$diag(t_1, t_2, ..., t_n) \to (t_1/\overline{t_1})^{m_1/2}\ (t_2/\overline{t_2})^{m_2/2} ... (t_n/\overline{t_n})^{m_n/2}$$

on the diagonal subgroup of $GL(n, \mathbb{C})$. This representation has same infinitesimal character as the finite dimensional representation of $GL(n, \mathbb{C})$ attached to $(m_1, ..., m_n)$ as in Section 7 (or its complexification). The infinitesimal character is given by the linear form (μ, μ) on $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$, where $\mathfrak{t}_{\mathbb{C}}$ is the algebra of complex diagonal matrices and μ is the form

$$diag(t_1, t_2, ..., t_n) \to \tfrac{m_1}{2} t_1 + \tfrac{m_2}{2} t_2 + ... + \tfrac{m_n}{2} t_n$$

on $\mathfrak{t}_{\mathbb{C}}$ from Section 7.

The principal series representation $\Pi^{\mathbb{C}}$ is both well-defined and invariant under the Galois automorphisms for $\mathbf{G}$ or $\mathbf{G}'$ independently of the parity conditions on the integers m_j. However if some parities are incorrect for described base change from $\mathbf{G}$ or $\mathbf{G}'$ then they determine instead an endoscopic group $\mathbf{H}$ ... .

Version: August 2, 2008, with minor correction in Section 9, as follows.

Replace $(*)$:

We take $\psi^2_{j,m} : G^{j,2} \to G^m$ to be $\psi^1_{j,m}$ followed by $Int(J_n)$,

by $(**)$:

We also take $\psi^2_{j,m} : G^{j,2} \to G^m$ to be the identity,

****

This does not affect any of the later calculations as they were done using $(**)$. I forgot to go back and replace $(*)$.

Examples in endoscopy for real groups

Addendum

D. Shelstad

Let $G = \mathbb{U}(n,1)$ where $n = 2m$. For this group, standard and Whittaker normalizations are defined for both geometric and spectral factors (since G is a component of the K-group of quasi-split type) and the two normalizations coincide and are canonical. We use this normalization for Δ_{geom} and Δ_{spec}.

We want to calculate $\Delta(\pi_H, \pi)$ for π in the discrete series and π_H in a packet with well-aligned parameter. See examples in banff-notes-B. Suppose π has parameter $\varphi = \varphi(m_1, ..., m_{n+1})$, where $m_1, ..., m_{n+1}$ are even integers (banff-A).

Lemma

$\Delta(\pi_H, \pi) = \Delta(\pi^s, \pi)$ *for some* $s \in S_\varphi$.

Proof: First observe that any elliptic endoscopic data is of the form $\mathfrak{e}(s)$ for some $s \in S_\varphi$ (up to a modification of s which does not matter) ... as constructed in Section 6 of banff-B . Then check definitions to see that all terms in the spectral factor Δ coincide: *well-aligned* is essential for Δ_{II} (see paper II) ■

Recall that $\Delta(\pi^s, \pi) = < s, \pi >$ is trivial to calculate (for this G, in particular) ... banff-A. We have thus to find the correct s. We then write the endoscopic data for H as $\mathfrak{e}(s)$ ...

Before doing this, notice that $\mathbb{U}(2m, 1)$ has a special property:

Each $s = diag(\pm 1, ...) \in S_\varphi$ *with one negative sign (or* n *negative signs) separates off exactly one representation of the packet for* G.

Proof: See pp. 25 - 26 of banff-A for $G = \mathbb{U}(2,1)$, $\mathbb{U}(4,1)$. The ingredients for a general argument along the same lines are set up in the notes: examine $\mathcal{D}_1'(T)$. ■

To separate $\pi^* = \pi(m_{n+1} > m_n > ... > m_1)$, use $s = diag(-1, 1, 1, ..., 1)$.

Lemma:

$$< s, \pi^* > = (-1)^m$$

$$< s, \pi > = (-1)^{m-1} \text{ for } \pi \neq \pi^*.$$

Proof: Another exercise with the setup in banff-A. ■

To separate $\pi^{**} = \pi(m_1 > m_2 > ... > m_{n+1})$ instead, use $s = diag(1, 1, ..., 1, -1)$. The result is the same (with π^{**} in place of π^*).

Version: August 18, 2008

Examples in endoscopy for real groups

Part B: 08/14/08

D. Shelstad

We start by discussing spectral transfer factors along the same lines as the geometric factors of [LS1] and [KS], noticing a simpler structure for geometric factors that is useful for real groups. Although we are interested primarily in examples, it is easier to outline a general transfer theorem in this setting. This brings us to further analysis and simplification of the spectral transfer factors, particularly in the case of Whittaker normalization. Various calculations of spectral transfer factors are included for a pair of examples. We then use the codes from Part A to generate quickly all endoscopic identities involving discrete series packets for unitary groups and unitary similitude groups when $n = 4$. For a general unitary or unitary similitude K-group of quasi-split type we find a canonical basis of endoscopic characters for a discrete series L-packet, with one exception that seems quite transparent. We also discuss the K-group $\mathbf{G}'$ not of quasi-split type along the lines suggested by Arthur [L-packets 2006].

1. A priori definition of transfer factors

In the setting of real groups, we may make a priori definitions of spectral transfer factors Δ_{spec} analogous to those for geometric transfer factors Δ_{geom}. There is a related notion of geometric-spectral compatibility which allows for use of the pair $(\Delta_{geom}, \Delta_{spec})$ in the transfer theorem.

We start with a single (unitary) group G and an (elliptic) endoscopic group H_1. There is an induced central torus Z_1 in H_1 such that the maximal tori over $\mathbb{R}$ in H_1/Z_1 embed over $\mathbb{R}$ as maximal tori in G. In our present example we may take Z_1 trivial, but even here we cannot always ignore an additional datum that comes with H_1: an embedding ξ_1 of the endoscopic datum $\mathcal{H}$ in ${}^L H_1$; this is discussed in some detail in Section 2. On the other hand, all choices of the z-pair (H_1, ξ_1) will determine the same spectral factors.

To recall geometric factors for a single group G we consider first very regular pairs (γ_1, δ) of points: γ_1 is (strongly) G-regular in $H_1(\mathbb{R})$ and δ is (strongly) regular in $G(\mathbb{R})$. Call (γ_1, δ) a *related pair* if γ_1 is an image (or norm) of δ. For two very regular related pairs (γ_1, δ) and (γ_1', δ'), we define first a canonical relative factor $\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$ as in [LS1]. Then a function Δ on very regular related pairs is a geometric transfer factor if

$$\Delta(\gamma_1, \delta) \ / \ \Delta(\gamma_1', \delta') = \Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$$

for all very regular related pairs (γ_1, δ) and (γ_1', δ'). In particular, the functions Δ_0, Δ_λ defined in [LS1] and [KS] for quasi-split G are transfer factors; we will discuss these in detail shortly. In general, the relative factor has the properties

$$\Delta_{geom}(\gamma_1, \delta; \gamma_1, \delta) = 1,$$

$$\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta') \Delta_{geom}(\gamma_1', \delta'; \gamma_1'', \delta'') = \Delta_{geom}(\gamma_1, \delta; \gamma_1'', \delta'').$$

These ensure that geometric transfer factors exist and are determined uniquely up to a complex constant: we may recover each normalization by fixing any one pair (γ_1', δ') and specifying $\Delta(\gamma_1', \delta')$, for then we have

$$\Delta(\gamma_1, \delta) = \Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta') \ \Delta(\gamma_1', \delta')$$

for all very regular related pairs (γ_1, δ).

We plan to make parallel definitions for spectral transfer factors.† The very regular related pairs we start with are now pairs of tempered irreducible admissible representations (π_1, π) with sufficently regular Langlands parameters. We are interested here only in the case that π belongs to the discrete series for $G(\mathbb{R})$, i.e. that its parameter is elliptic as well as regular.

For transfer we consider only representations π_1 of $H_1(\mathbb{R})$ whose restriction to $Z_1(\mathbb{R})$ acts as a character λ_1 prescribed by the z-pair (H_1, ξ_1). There is also an attached mapping on parameters

$$\Phi_{temp}(H_1, \lambda_1) \to \Phi_{temp}(G^*).$$

If (H_1, ξ_1) is replaced by another z-pair (H_2, ξ_2) with attached character λ_2 then we have a bijective map

$$\Phi_{temp}(H_2, \lambda_2) \to \Phi_{temp}(H_1, \lambda_1),$$

with the obvious commutative diagram. We regard $\Phi_{temp}(G)$ as a subset of $\Phi_{temp}(G^*)$. Then (π_1, π) is a *related pair* if the parameter of π_1 maps to that of π.

We start with two very regular related pairs (π_1, π), (π_1', π') and define a canonical relative factor $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$. It has the same general form as Δ_{geom} in that it is the product of three terms which we label $\Delta_I, \Delta_{II}, \Delta_{III}$.[‡] It has properties analogous to those written above for Δ_{geom}. We find moreover that $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ is always simply a sign.

A function Δ on very regular related pairs is a spectral transfer factor if

$$\Delta(\pi_1, \pi) \ / \ \Delta(\pi_1', \pi') = \Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$$

for all very regular related pairs (π_1, π) and (π_1', π'). Examples include spectral analogues of Δ_0, Δ_λ for G quasi-split. In general, we fix (π_1', π'), prescribe $\Delta(\pi_1', \pi')$, and then set

$$\Delta(\pi_1, \pi) = \Delta_{spec}(\pi_1, \pi; \pi_1', \pi') \ \Delta(\pi_1', \pi')$$

for all very regular related pairs (π_1, π). Thus we may always normalize so that each $\Delta(\pi_1, \pi)$ is a sign.

Now write $\Delta_{geom}, \Delta_{spec}$ for some choice (normalization) of geometric, spectral transfer factors. We may define a canonical *compatibility* factor $\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$, again as a product of three terms $\Delta_I, \Delta_{II}, \Delta_{III}$. There are transitivity properties relative to spectral and geometric factors. Then Δ_{geom}, Δ_{spec} are compatible if

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta) \ \Delta_{geom}(\gamma_1, \delta)$$

for one, and hence every, choice of very regular related pairs $(\pi_1, \pi), (\gamma_1, \delta)$. If we define Δ_{geom} by prescribing $\Delta_{geom}(\gamma_1', \delta')$ for given (γ_1', δ'), we obtain compatible Δ_{spec} by fixing (π_1', π') and prescribing $\Delta_{spec}(\pi_1', \pi')$ as

$$\Delta_{comp}(\pi_1', \pi'; \gamma_1', \delta')\ \Delta_{geom}(\gamma_1', \delta').$$

Conversely, we may prescribe Δ_{spec} first.

Before going on to the transfer theorem itself, we describe the transfer factors explicitly in some examples. As in Section A14 where we attached invariants to representations, we also see that K-groups provide a natural setting for normalization.

† This will help in identifying the transfer of a stable character in terms of irreducible characters. In the case of discrete series characters the identification becomes an exercise in cancelling matching geometric and spectral contributions [see II, pp. 273 - 277].

‡ The corresponding arrangement for Δ_{geom} is Δ_I, $\Delta_{II+} = \Delta_{II}\Delta_{III_2}$, and Δ_{III_1}. We ignore Δ_{IV} and use it instead in the definition of orbital integrals.

2. Endoscopic data for unitary groups

Recall that we have organized unitary groups into the following K-groups. For $n = 2m+1$ we have simply

$$\mathbf{G} = \mathbb{U}(m+1, m) \sqcup \mathbb{U}(m+2, m-1) \sqcup \ldots \sqcup \mathbb{U}(2m+1, 0).$$

For $n = 2m$ we have

$$\begin{aligned}\mathbf{G} = \ & \mathbb{U}(m, m) \sqcup \\ & \mathbb{U}(m+2, m-2) \sqcup \mathbb{U}(m+2, m-2) \sqcup \ldots\end{aligned}$$

of quasi-split type, and

$$\begin{aligned}\mathbf{G}' = \ & \mathbb{U}(m+1, m-1) \sqcup \mathbb{U}(m+1, m-1) \sqcup \\ & \mathbb{U}(m+3, m-3) \sqcup \mathbb{U}(m+3, m-3) \sqcup \ldots.\end{aligned}$$

Endoscopic data are the same for all components.

Fix a set of elliptic endoscopic data $(H, \mathcal{H}, s)$. We assume that s lies in the group S of matrices $diag(\pm 1, \pm 1, ..., \pm 1)$ in $G^\vee = GL(n, \mathbb{C})$. If s has a entries $+1$ and b entries -1 (in any order) then H is the product $\mathbb{U}_a \times \mathbb{U}_b$ of the quasi-split unitary groups of Section A9, and for $\mathcal{H}$ we take the subgroup of ${}^L G$ generated by the element $J_n \times (1 \times \sigma)$ and the subgroup $H^\vee \times (\mathbb{C}^\times \times 1)$, where $H^\vee$ is the centralizer of s in $G^\vee$. Alternatively, $\mathcal{H}$ is generated by $H^\vee$ and the image of any regular elliptic parameter $\varphi = \varphi(m_1, m_2, ..., m_n)$.

We attach the K-group $\mathbf{H}$ to H using products of the unitary groups of Section A9. We identify each component of $\mathbf{H}$ with the subgroup of $GL(n)$ consisting of matrices of the same shape as those in $H^\vee$ and endowed with the appropriate Galois action. The diagonal subgroup of each component of $\mathbf{H}$ is again T, and we do have the convenient property that $t_j - t_k$ is a root of T in (a component of) $\mathbf{H}$ if and only if $z_j - z_k$ is a root of $T^\vee$ in $H^\vee$. To simplify calculations, we will pick examples where inclusion of the quasi-split component of $\mathbf{H}$ in the quasi-split component of $\mathbf{G}$ is defined over $\mathbb{R}$.

Example

Let $n = 3$ so that $\mathbf{G} = \mathbb{U}(2,1) \sqcup \mathbb{U}(3,0)$. We choose

$$s = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

and then $\mathbf{H} = \mathbb{U}(1,1) \times \mathbb{U}(1,0)$ which we identify as the group of matrices of the form

$$\begin{pmatrix} a & 0 & b \\ 0 & f & 0 \\ c & 0 & d \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{U}(1,1)$ and $f \in \mathbb{U}(1,0)$.

For the z-pair (H_1, ξ_1) we may take $H_1 = H = \mathbb{U}_a \times \mathbb{U}_b$, but some care is needed with the isomorphism $\xi_1 : \mathcal{H} \to {}^L H$ determining the map $\Phi_{temp}(\mathbf{H}) \to \Phi_{temp}(\mathbf{G})$ on parameters. If $\mathbf{G} = \mathbb{U}(2,1) \sqcup \mathbb{U}(3,0)$ and $\mathbf{H} = \mathbb{U}(1,1) \times \mathbb{U}(1,0)$, as above, then our parameters (m_1, m_2, m_3) for regular elliptic parameters shift parity from $(odd, even, odd)$ for $\mathbf{H}$ to $(even, even, even)$ for $\mathbf{G}$. We prefer to write the details for the following two examples for $n = 4$ which we will continue to follow.

Examples

(i) Let

$$s = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

In this example, the parameter φ_1 for a discrete series L-packet for

$$\mathbf{H} = \mathbb{U}(2,1) \times \mathbb{U}(1) \sqcup \mathbb{U}(3,0) \times \mathbb{U}(1)$$

determines a tuple of integers that are all even, whereas the tuples for

$$\mathbf{G} = \mathbb{U}(2,2) \sqcup \mathbb{U}(4,0) \sqcup \mathbb{U}(4,0)$$

consist of odd integers. The group $\mathcal{H}$ is generated by the subgroup $H^\vee$ of $GL(n, \mathbb{C})$ of all matrices

$$\begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & 0 \\ * & * & 0 & * \end{pmatrix},$$

and the image of $\varphi = \varphi(m_1, m_2, m_3, m_4)$. Taking the isomorphism $\xi_1 : \mathcal{H} \to {}^L H$ to be the identity on $H^\vee$, we have just to specify ξ_1 on the image of φ. We set

$$\xi_1(\varphi(1 \times \sigma)) = h \times (1 \times \sigma),$$

where

$$h = (J_3, J_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, as we noted in Section A4, calculation in ${}^LH$ gives $h\sigma^\vee(h) = I$, so that

$$(\xi_1(\varphi(1\times\sigma)))^2 = I \times ((-1\times 1).$$

But

$$(\varphi(1\times\sigma))^2 = J_4\sigma^\vee(J_4) \times (-1\times 1) = -I \times (-1\times 1) = \varphi(-1\times 1)$$

in ${}^LG$. Thus we cannot take ξ_1 to be the identity on $\varphi(\mathbb{C}^\times \times 1)$. We may choose $\xi_1(\varphi(z\times 1))$ to be

$$\begin{pmatrix} (z/\overline{z})^{(m_1-1)/2} & & & \\ & (z/\overline{z})^{(m_2-1)/2} & & \\ & & (z/\overline{z})^{(m_3-1)/2} & \\ & & & (z/\overline{z})^{(m_4-1)/2} \end{pmatrix} \times (z\times 1),$$

for $z \in \mathbb{C}^\times$. Notice that

$$\varphi_1 = \varphi_1(m_1-1, m_2-1, m_3-1, m_4-1)$$

is a regular elliptic parameter for $\mathbf{H}$ which maps to $\varphi = \varphi(m_1, m_2, m_3, m_4)$ under the map $\Phi_{temp}(\mathbf{H}) \to \Phi_{temp}(\mathbf{G})$, determined by ξ_1, and that the parities shift correctly. We could of course adjust ξ_1 to subtract any given odd integer from each of m_1, m_2, m_3, m_4.

(ii) Let

$$s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

so that $\mathbf{H} = \mathbb{U}(1,1) \times \mathbb{U}(1,1)$, and there is no parity change for parameters. At the same time we see that we may take ξ_1 to be the identity on $H^\vee \times \mathbb{C}^\times \times 1$. We could also adjust ξ_1 to subtract any even integer from each of m_1, m_2, m_3, m_4.

Proposition

For general $s \in S$, $H = \mathbb{U}_a \times \mathbb{U}_b$ we may choose ξ_1 so that $\xi_1(I \times (z\times 1))$, $z \in \mathbb{C}^\times$, is given by

$$\begin{pmatrix} (z/\overline{z})^{-\epsilon_1/2} & & & \\ & (z/\overline{z})^{-\epsilon_2/2} & & \\ & & ... & \\ & & & (z/\overline{z})^{-\epsilon_n/2} \end{pmatrix} \times (z \times 1),$$

where $\epsilon_j = 0$ except if the j^{th} entry of s is $+1$ and a is of opposite parity to $n = a + b$ or if the j^{th} entry of s is -1 and b is of opposite parity to n. For the exceptions we may take $\epsilon_j = 1$.

Returning to the setting of Section A7, we define the linear form

$$\mu^* : diag(t_1, t_2, ..., t_n) \to \tfrac{\varepsilon_1}{2} t_1 + \tfrac{\varepsilon_2}{2} t_2 + ... + \tfrac{\varepsilon_n}{2} t_n$$

on the Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of $T(\mathbb{C})$. Notice that $< \mu^*, \alpha^\vee > = 0$ for each root $\alpha^\vee$ of $T^\vee$ in $H^\vee$. The significance of μ^* for geometric transfer factors is that, in a setting like the present one where $T(\mathbb{R})$ is compact, the linear form $\iota - \iota_H + \mu^*$ exponentiates to a rational character on T.† Here ι is one half the sum of the positive roots of T in a component of $\mathbf{G}$ for some ordering, and ι_H is defined similarly for $\mathbf{H}$. The significance of μ^* for spectral transfer factors is, as we have seen in examples, in the shift of infinitesimal character in related pairs.

We will need another remark. The split rank, or dimension of a maximal $\mathbb{R}$-split torus, of the quasi-split unitary groups $\mathbb{U}(m+1, m)$, $\mathbb{U}(m, m)$ is m. By the split rank of $\mathbf{G}$ we will mean the split rank of its quasi-split component. Notice that the split rank of $H = \mathbb{U}_a \times \mathbb{U}_b$, or the associated K-group $\mathbf{H}$, is the same as that of $\mathbf{G}$ unless $n = a + b$ is even and a, b are odd. In the exceptional case, the split rank of $\mathbf{H}$ is one less than that of $\mathbf{G}$. Example (i) above is of this type, as is the case $\mathbf{G} = \mathbb{U}(1, 1)$, $\mathbf{H} = \mathbb{U}(1) \times \mathbb{U}(1)$ for $n = 2$. This property is significant for the term Δ_{II} in spectral transfer factors.

†These characters appear in the term Δ_{II+}. The term e^{μ^*} also persists in local formulas and in the Lie algebra version of geometric transfer factors.

3. Spectral transfer factors for G

Let π_1 be a discrete series representation of $\mathbf{H}$ and $\varphi_1 = \varphi_1(m'_1, ..., m'_n)$ be its parameter. The parameter $\varphi = \xi_1^{-1} \circ \varphi_1$ for $\mathbf{G}$ is also a tempered elliptic parameter and may written as $\varphi = \varphi(m_1, ..., m_n)$, where $m_j = m'_j +$

ϵ_j, $1 \leq j \leq n$. However, various of the m_j may coincide in which case φ is not regular. Then an attached representation π does not belong to the discrete series. We still define a transfer factor $\Delta_{spec}(\pi_1, \pi)$ for use in the transfer theorem, but not initially. These singular parameters also determine Arthur parameters for nontempered cohomological representations. For now we exclude this case and require instead that φ_1 *is* $G^\vee$*-regular*, *i.e.* that φ is regular.

Thus let (π_1, π) be a related pair of discrete series representations for $\mathbf{H}$ and $\mathbf{G}$, with $G^\vee$-regular elliptic Langlands parameters φ_1, φ. Recall that the geometric transfer factor Δ_0 was defined intially only for quasi-split components where we have an absolute version of the relative term Δ_{III}. The K-group formalism allows us to extend it to all components of $\mathbf{G}$, extension to all components of $\mathbf{H}$ being trivial. We now describe briefly the spectral analogue $\Delta_0(\pi_1, \pi)$.

We start with the main term Δ_{III}. Following Section A14 (see Part b) we fix a Weyl chamber C_* for T and thus a representation π_* in the packet of π. Then we use C_* to identify the coroots of T as roots of $T^\vee$. We again regard $inv(\pi, \pi_*)$ as a sum of roots of $T^\vee$ and evaluate this sum on the given endoscopic datum s to obtain $\Delta_{III}(\pi_1, \pi)$. Thus $\Delta_{III}(\pi_1, \pi)$ is a sign. It depends on the choice of C_* which amounts to a choice of *toral data.*

The sign $\Delta_I(\pi_1, \pi)$, which is the same as the geometric term Δ_I for regular elliptic elements, has cancelling dependence on toral data but introduces additional dependence on a-data. Finally we introduce $\Delta_{II}(\pi_1, \pi)$ which has cancelling dependence on a-data, to obtain

$$\Delta_0(\pi_1, \pi) = \Delta_I(\pi_1, \pi)\ \Delta_{II}(\pi_1, \pi)\ \Delta_{III}(\pi_1, \pi)$$

independent of the choice for toral data and a-data. For n even, $\Delta_I(\pi_1, \pi)$ also depends on the $\mathbb{R}$-splitting spl_* for G^* chosen initially.

There are three contributions to Δ_{II} in general. Two are signs, one from comparing signs $(-1)^q$ for $\mathbf{H}$ and $\mathbf{G}$, and one from the relative positions among Weyl chambers of the tuples $(m'_1, ..., m'_n)$ for φ_1 and $(m_1, ..., m_n)$ for φ. The second sign is also simple to calculate and will not concern us directly here. The third contribution comes from rewriting the stable character formulas on $T(\mathbb{R})$ using a Weyl denominator that depends on a-data rather than on a positive system for the roots of T.†

Examples

We continue with the examples (i) and (ii) from the last section. The first contribution to Δ_{II} is trivial: the number of positive noncompact roots for the quasi-split components of **H**, **G** differs by 2 in each case, and so each sign is $(-1)^2$.

In each example suppose that $\varphi_1 = \varphi_1(m_1', ..., m_4')$ is a $G^\vee$-regular parameter which is well-aligned in the sense that $m_1' > m_2' > m_3' > m_4'$. Then $\varphi = \varphi(m_1, ..., m_4)$ has $m_1 > m_2 > m_3 > m_4$, independently of our choices for ξ_1. Thus both φ_1 and φ are the canonical representatives from Section A3. This makes the second sign in $\Delta_{II}(\pi_1, \pi)$ trivial.

We will use the chamber $\mathcal{C}_* : m_1 > m_3 > m_2 > m_4$ of Section A14 for identifying coroots of T with roots of $T^\vee$, and by a *positive* root of T we will mean a root positive relative to $\mathcal{C}_*$.We use particular a-data in our formulas. Namely, we set $a_\alpha = i$ for each positive root α. Then the third contribution to $\Delta_{II}(\pi_1, \pi)$ is $(\frac{-1}{i})^N = i^N$, where N is the number of positive roots that are not roots for H. In the examples, $N = 3$ and $N = 4$, respectively. Thus:

Example (i): $\quad \Delta_{II}(\pi_1, \pi) = -i$

Example (ii): $\quad \Delta_{II}(\pi_1, \pi) = 1$

Notice that, in general, N is odd exactly when a, b are both odd (recall $H = \mathbb{U}_a \times \mathbb{U}_b$). Because a change in a-data changes $\Delta_{II}(\pi_1, \pi)$ by a sign only, we see then that $\Delta_{II}(\pi_1, \pi)$ is always a sign except in the case n even and both a, b are odd, and in that case it is $\pm i$.

We return now to Δ_I for the two examples. This is a calculation where we use matrices in the (simply-connected) commutator group $SL(4)$. Here is a summary. We assume that the $\mathbb{R}$-isomorphism $\psi : \mathbb{U}(2,2) \to G^*$ is $Int(h_1)$, where h_1 is the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ 0 & -1 & -i & 0 \\ -1 & 0 & 0 & -i \end{pmatrix}.$$

Let B be the Borel subgroup of $GL(4)$ containing the diagonal subgroup $T = T^*$ and with roots determined by $\mathcal{C}_*$. Then $spl = (B, T, \{Y\})$, where $\{Y\}$ consists of the standard root vectors (transvections), is an $\mathbb{R}$-opp splitting for the Galois action of $\mathbb{U}(2,2)$. The same is true of $\psi(spl)$ relative to the Galois action of G^*. Let $h_2 = uh_1^{-1}$, where u is the matrix

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix},$$

with $\epsilon^4 = -1$. Then $Int(h_2)$ carries $\psi(spl)$ to our fixed $\mathbb{R}$-splitting spl_* for G^*. We now calculate the splitting invariant for $\psi(T)$ relative to spl_*, toral data $Int(h_2)$, and given a-data. We find it is the 1-cocycle $t_\sigma \in \psi(T)$ given by

$$h_2^{-1} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \sigma(h_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This cocycle then determines the character 0110 on S by our chosen identifications. We evaluate this character on the endoscopic datum s to obtain $\Delta_I(\pi_1, \pi) = -1$ in (i), while $\Delta_I(\pi_1, \pi) = 1$ in (ii).

In conclusion, we have (for the given choice of toral data or chamber $\mathcal{C}_*$):

Example (i): $\Delta_0(\pi_1, \pi) = i\ \Delta_{III}(\pi_1, \pi)$
In this example, π_1 is a well-aligned G-regular discrete series representation (*i.e.* its parameter is $G^\vee$-regular and well-aligned) of $U(2,1) \times U(1)$ or $U(3,0) \times U(1)$ and π is a related discrete series representation of $U(2,2)$ or of one of two copies of $U(4,0)$.

Example (ii): $\Delta_0(\pi_1, \pi) = \Delta_{III}(\pi_1, \pi)$
In this example, π_1 is a well-aligned G-regular discrete series representation of $U(1,1) \times U(1,1)$ and π is a related discrete series representation of $U(2,2)$ or of one of two copies of $U(4,0)$.

† This denominator cancels conveniently with the term Δ_{II+} of the geometric transfer factor in calculation of the transfer to $\mathbf{G}$ or to $\mathbf{G}'$ of a stable character for $\mathbf{H}$.

4. Relative transfer factors

We will define spectral transfer factors for $\mathbf{G}'$ following the strategy of Section 1, and so focus first on the relative factor. This relative factor is defined for $\mathbf{G}$ also, and we include it for $\mathbf{G}$ to confirm that Δ_0 is a spectral transfer factor.

Let (π_1, π), (π_1', π') be related pairs of discrete series representations for $\mathbf{H}$ and $\mathbf{G}'$, with $G^\vee$-regular elliptic Langlands parameters φ_1, φ and φ_1', φ' as before. The absolute term $\Delta_I(\pi_1, \pi)$, defined above when π is a representation of $\mathbf{G}(\mathbb{R})$, depends only on parameters, and so we may write it as $\Delta_I(\pi_1, \pi)$ in the present setting also. We then set

$$\Delta_I(\pi_1, \pi; \pi_1', \pi') = \Delta_I(\pi_1, \pi) \;/\; \Delta_I(\pi_1', \pi').$$

This depends on toral data and a-data, but is independent of the choice for spl_*. Similarly, the absolute term $\Delta_{II}(\pi_1, \pi)$ is adjusted by a constant sign when we pass to $\mathbf{G}'$, and so the relative term

$$\Delta_{II}(\pi_1, \pi; \pi_1', \pi') = \Delta_{II}(\pi_1, \pi) \;/\; \Delta_{II}(\pi_1', \pi')$$

depends only on parameters, rather than the representations themselves.

The only genuinely relative term $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ is obtained by pairing the relative invariant $inv_{\mathcal{C}_*}(\pi, \pi')$ with the element s_U, as in Section A17.

This finishes the definition of $\Delta(\pi_1, \pi; \pi_1', \pi')$, which is then seen to be independent of the choice of toral data (essentially, the chamber $\mathcal{C}_*$) and a-data. Moreover, we show that $\Delta_{II}(\pi_1, \pi) \;/\; \Delta_{II}(\pi_1', \pi')$ is a sign, so that $\Delta(\pi_1, \pi; \pi_1', \pi')$ is also a sign. The definition of $\Delta(\pi_1, \pi; \pi_1', \pi')$ works for $\mathbf{G}$ as well as $\mathbf{G}'$, and

$$\Delta_{III}(\pi_1, \pi; \pi_1', \pi') = \Delta_{III}(\pi_1, \pi) \;/\; \Delta_{III}(\pi_1', \pi')$$

in the case of $\mathbf{G}$. This implies that Δ_0 is a spectral transfer factor: fix any pair (π_1', π') then Δ_0 is the unique transfer factor Δ for $\mathbf{G}$ such that $\Delta(\pi_1', \pi') = \Delta_0(\pi_1', \pi')$.

Definition of $\Delta(\pi_1, \pi; \pi_1', \pi')$ extends readily to all G-regular tempered related pairs (π_1, π), (π_1', π'). One particular value $\Delta(\pi_1', \pi')$ thus determines the spectral transfer factor $\Delta(\pi_1, \pi)$ for all such pairs.

The extension of $\Delta(\pi_1, \pi)$ to all tempered related pairs (π_1, π) requires more work. It can be done by applying nondegenerate coherent continuation (translation principle) to G-regular transfer identities. The extension is necessary for the converse statement (iii) in the transfer theorem, *i.e.* to deduce geometric matching from spectral matching.

We return to writing $\Delta_{geom}, \Delta_{spec}$ for some choice (normalization) of geometric, spectral transfer factors. It remains to describe the compatibility factor $\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$. This is written as

$$\Delta_I(\pi_1, \pi)/\Delta_I(\gamma_1, \delta).\ \Delta_{II}(\pi_1, \pi)/\Delta_{II}(\gamma_1, \delta).\ \Delta_{III}(\pi_1, \pi; \gamma_1, \delta),$$

where only the last term is yet to be defined and that is now routine. Recall Δ_{geom} and Δ_{spec} are called compatible if

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)\ \Delta_{geom}(\gamma_1, \delta)$$

for one, and hence every, choice of very regular related pairs $(\pi_1, \pi), (\gamma_1, \delta)$. In the case of quasi-split type the standard geometric and spectral factors Δ_0 are compatible. If we multiply each factor in a compatible pair by the same constant, as in Whittaker normalization, we still have a compatible pair of factors.

5. Transfer theorem

We write down a result for general K-group $\mathbf{G}$ and endoscopic data $(\mathbf{H}, \mathcal{H}, s)$, but avoid some minor technicalities by assuming that in the z-pair $(\mathbf{H}_1, \xi_1)$ we have $\mathbf{H}_1 = \mathbf{H}$. Fix Haar measures on $\mathbf{G}(\mathbb{R})$ and $\mathbf{H}(\mathbb{R})$, and assume Haar measures on shared Cartan subgroups are the same. For the initial test functions we use the space $\mathcal{C}(\bullet)$ of Harish Chandra's Schwartz functions.

Theorem

Let Δ_{geom} *and* Δ_{spec} *be transfer factors with compatible normalization. Then:*

(i) *for each* $f \in \mathcal{C}(\mathbf{G}(\mathbb{R}))$ *there exists* $f_1 \in \mathcal{C}(\mathbf{H}(\mathbb{R}))$ *such that*

$$SO_{\boldsymbol{\gamma}_1}(f_1) = \sum_{\boldsymbol{\delta},conj} \Delta_{geom}(\gamma_1, \delta) O_{\boldsymbol{\delta}}(f)$$

for all $\mathbf{G}$*-regular* γ_1 *in* $\mathbf{H}(\mathbb{R})$,

(ii) *there is a dual transfer of stable tempered characters given by*

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\boldsymbol{\pi},temp} \Delta_{spec}(\pi_1, \pi)\ Tr\ \pi(f)$$

for all tempered irreducible representations π *of* $\mathbf{H}(\mathbb{R})$, *and*

(iii) *conversely, if* $f \in \mathcal{C}(\mathbf{G}(\mathbb{R}))$ *and* $f_1 \in \mathcal{C}(\mathbf{H}(\mathbb{R}))$ *satisfy*

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\boldsymbol{\pi},temp} \Delta_{spec}(\pi_1, \pi)\ Tr\ \pi(f)$$

for all tempered irreducible representations π_1 *of* $\mathbf{H}(\mathbb{R})$ *then*

$$SO_{\boldsymbol{\gamma}_1}(f_1) = \sum_{\boldsymbol{\delta},conj} \Delta_{geom}(\gamma_1, \delta) O_{\boldsymbol{\delta}}(f)$$

for all $\mathbf{G}$*-regular* γ_1 *in* $\mathbf{H}(\mathbb{R})$.

Remarks

A theorem of Bouaziz shows that a condition on support characterizes the stable orbital integrals of functions in $C_c^\infty(\mathbf{H}(\mathbb{R}))$ among the stable orbital integrals of Schwartz functions. Thus in (i), if $f \in C_c^\infty(\mathbf{G}(\mathbb{R}))$ we may assume that $f_1 \in C_c^\infty(\mathbf{H}(\mathbb{R}))$.

Dual transfer is then defined for nontempered stable characters on $\mathbf{H}(\mathbb{R})$, but it remains to describe $\Delta_{spec}(\pi_1, \pi)$ explicitly (*i.e.* compatibly with Δ_{geom}). The existence of the factors follows from the Adams-Barbasch-Vogan proof of the Arthur conjectures. See Section 8 of [Arthur: Problems].

It is a straightforward exercise (following constructions for the geometric side) to write down $\Delta_{spec}(\pi_1, \pi)$ for $\mathbf{G}$*-regular* cohomological representations

π_1 of $\mathbf{H}(\mathbb{R})$, and to check that this is correct for the Adams-Johnson proof of dual transfer for these representations. [Describe later].

A theorem of Clozel-Delorme allows us to insert K-finiteness conditions on the test functions.

[references for extending transfer on the geometric side]

6. Factoring parameters and adjoint relations

We apply the spectral ($\mathbb{S}$-) construction of endoscopic data to the regular elliptic parameter $\varphi = \varphi(m_1, ..., m_n)$ from Section A3. Let $s_{sc} \in \mathbb{S}^{sc} = \mathbb{S}^{sc}_{\varphi}$.

(i) Take s in S with same image as s_{sc} in $PGL(n, \mathbb{C})$.

(ii) Let $H^\vee$ be the centralizer in $G^\vee$ of the image of s_{sc}, or of s, in $G^\vee$.

(iii) Let $\mathcal{H}$ be the subgroup of ${}^L G$ generated by $H^\vee$ and the image of φ. This determines dual K-group $\mathbf{H}^s$ of quasi-split type.

(iv) Specify an isomorphism $\xi_s : \mathcal{H} \to {}^L H$ so that $(\mathbf{H}^s, \xi_s)$ is a z-pair.

(v) Set $\varphi^s = \xi_s^{-1} \circ \varphi$.
Denote by π^s any representation of $\mathbf{H}^s(\mathbb{R})$ with parameter φ^s.

We return to the Examples (i) and (ii) from Section 2. In Example (i) we take s_{sc} as

$$\begin{pmatrix} \epsilon & & & \\ & \epsilon & & \\ & & -\epsilon & \\ & & & \epsilon \end{pmatrix},$$

where $\epsilon^4 = -1$. In (ii) we may take s_{sc} as the given s. Then write φ_1, π_1 in each case as φ^s, π^s, and rewrite $\Delta_0(\pi_1, \pi)$ as

$$\Delta_0(\pi^s, \pi) = i \prec s^{sc}, \pi \succ \qquad (Example\ (i))$$

or

$$\Delta_0(\pi^s, \pi) = \prec s^{sc}, \pi \succ \qquad\qquad (\textit{Example (ii)}),$$

for each π in the packet Π attached to φ. Here $\prec *, * \succ$ is the pairing of $\mathbb{S}^{sc}$ with Π from Part A.

There are adjoint relations for general tempered spectral factors $\Delta(\pi^s, \pi)$ parallel to those of Arthur for the geometric factors:

$$\tfrac{1}{\|\Delta\|^2} \ \textstyle\sum_{s \in \mathbb{S}} \ \overline{\Delta(\pi^s, \pi)} \ \Delta(\pi^s, \pi') = |\mathbb{S}| \ \delta_{\pi, \pi'}.$$

Here $\|\Delta\|$ is the absolute value of Δ which is constant since the the relative factor is a sign. These relations provide, for example, a simple inversion formula for the tempered character identities in the transfer theorem.

7. Simplified factors for G

First we review the Whittaker normalization for spectral transfer factors. Fix Whittaker data (B, λ) for G^*. We may assume B is the Borel subgroup B^* from spl^* and that λ is the generic character attached to spl^* and an additive character $\psi_{\mathbb{R}}$ on $\mathbb{R}$ in the usual manner. At the same time, spl^* specifies the spectral transfer factor $\Delta_0(\pi_1, \pi)$. For Whittaker normalization we multiply Δ_0 by the fixed epsilon factor $\varepsilon(V, \psi_{\mathbb{R}})$ described below, to obtain the factor $\Delta_\lambda(\pi_1, \pi)$. This shifts the dependence from the $G^*(\mathbb{R})$-conjugacy class of spl^* to that of (B, λ). Moreover we obtain

$$\Delta_\lambda(\pi_1, \pi) = \pm 1$$

for all tempered (or G-regular cohomological) related pairs (π_1, π).

In the factor $\varepsilon(V, \psi_{\mathbb{R}})$, defined with Langlands' normalization, V denotes the following virtual representation $V_G - V_H$ of degree zero of the Galois group Γ. The space V_H is $X^*(T_1) \otimes \mathbb{C}$ which we identify with $V_G = X^*(T^*) \otimes \mathbb{C}$ by any choice of tora data for the maximally split maximal torus T_1 in $\mathbf{H}$. The action of σ on V_H is by $\sigma_H = \sigma_{T_1}$ while its action on V_G is by $\sigma_{G^*} = \sigma_{T^*}$.

If the split rank of $\mathbf{H}$ is the same as that of $\mathbf{G}$ then V is trivial and $\varepsilon(V, \psi_{\mathbb{R}}) = 1$. Thus if n is odd we always have $\Delta_\lambda = \Delta_0$ is canonical. If n is even and $\mathbf{H} = \mathbb{U}_a \times \mathbb{U}_b$, where each of a, b are even, then again $\Delta_\lambda = \Delta_0$ and we see also that they are canonical, *i.e.* independent of the $G^*(\mathbb{R})$-conjugacy class of (B,λ) or spl^*. This applies to our Example (ii).

On the other hand, Example (i) demonstrates the case (n even and a,b odd) where $\varepsilon(V,\psi_{\mathbb{R}})$ is nontrivial and its product with $\Delta_0(\pi^s,\pi)=\pm i$ is a sign. Here σ_H acts on the standard basis of $X^*(T^*)\otimes\mathbb{C}$ as

$$\begin{pmatrix} & & & -1\\ & -1 & & \\ & & -1 & \\ -1 & & & \end{pmatrix},$$

while σ_{G^*} acts as

$$\begin{pmatrix} & & & -1\\ & & -1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Then we calculate $\varepsilon(V,\psi_+)=-i$ for $\psi_+(x)=\exp(2\pi ix)$, and so

$$\Delta_\lambda(\pi^s,\pi)=(-i)(i)\prec s^{sc},\pi\succ\,=\,\prec s^{sc},\pi\succ,$$

where the pairing is computed relative to the base point $\pi_*=\pi(\mathcal{C}_*)$. We use the fixed $\mathbb{R}$-isomorphism ψ of this example (Section 3) to regard π_* as a representation of $G^*(\mathbb{R})$. Then a calculation shows that π_* is generic for the data attached to spl^* and ψ_+.

The same is true in general. As long as we use Whittaker normalization, the transfer factor for a tempered pair (π^s,π) is the sign determined by the pairing of Part A:

$$\Delta_\lambda(\pi^s,\pi)=\,\prec s^{sc},\pi\succ\,.$$

Here the pairing is computed relative to a chamber determined by the unique λ-generic representation in Π. The main content of this assertion is that

$$\Delta_\lambda(\pi^s,\pi)=1$$

if π is λ-generic.

The transfer theorem thus yields the character identities

$$St\text{-}Tr\ \pi^s(f^s) = \sum_{\pi\in\Pi} \prec s_{sc}, \pi \succ Tr\ \pi(f)$$

when we use Whittaker normalization of transfer factors. These identities generate all endoscopic characters for Π in the following (strong) sense. Suppose we are given an arbitrary set of endoscopic data, a z-pair and a compatible normalization of transfer factors. The transfer theorem then defines a correspondence $(f,\ f_1)$ on test functions. Suppose Π_1 is any tempered L-packet matching Π on the level of parameters. Then for $\pi_1 \in \Pi_1$ the endoscopic character

$$f \to St\text{-}Tr\ \pi_1(f_1)$$

coincides, up to a complex constant, with

$$f \to \sum_{\pi\in\Pi} \prec s_{sc}, \pi \succ Tr\ \pi(f),$$

for some $s_{sc} \in \mathbb{S}^{sc}$.

8. Endoscopic bases for G

We may index the endoscopic characters

$$Tr(\Pi, s_{sc}) : f \to \sum_{\pi\in\Pi} \prec s_{sc}, \pi \succ Tr\ \pi(f)$$

by $\mathbb{S}$ since $\prec s_{sc}, \pi \succ$ depends just on the image of s_{sc} in $\mathbb{S}$. Thus we have a set $\mathcal{B}$ of $|\mathbb{S}|$ linearly independent virtual characters composed from the representations in Π. Moreover the adjoint relations yield an expansion

$$Tr\ \pi = \frac{1}{|\mathbb{S}|} \sum_{\mathcal{B}} \prec s_{sc}, \pi \succ Tr\ (\Pi, s_{sc})$$

for each $\pi \in \Pi$. We call $\mathcal{B}$ an *endoscopic basis for* Π. It is determined uniquely by the given conjugacy class of Whittaker data.

Example

Let $n = 4$ so that $\mathbf{G} = \mathbb{U}(2,2) \sqcup \mathbb{U}(4,0) \sqcup \mathbb{U}(4,0)$.

A discrete series packet Π for $\mathbf{G}$ contains six representations

$$\pi_{0000},\ \pi_{0110},\ \pi_{1100},\ \pi_{0011},\ \pi_{1001},\ \pi_{1111}$$

of $U(2,2)$ and two copies of the one representation of the compact form $U(4,0)$, labelled π_{0101}, π_{1010}. We write

$$f = f_1 + f_2 + f_3$$

where the summands are supported on $U(2,2), U(4,0), U(4,0)$ respectively. Whittaker data was specified in Example (i).

The corresponding endoscopic basis for Π consists of eight combinations.

Example (i) yields one:

$$Tr\ \pi_{0000}\ (f_1) - Tr\ \pi_{0110}\ (f_1) + Tr\ \pi_{1100}\ (f_1) - Tr\ \pi_{0011}\ (f_1)$$

$$+\ Tr\ \pi_{1001}\ (f_1) - Tr\ \pi_{1111}\ (f_1) + Tr\ \pi_{0101}\ (f_2) - Tr\ \pi_{1010}\ (f_3).$$

This is one of four combinations which arise from transfer of stable discrete series characters of

$$U(2,1) \times U(1,0) \sqcup U(3,0) \times U(1,0).$$

Example (ii) yields another type of combination:

$$Tr\ \pi_{0000}\ (f_1) + Tr\ \pi_{0110}\ (f_1) - Tr\ \pi_{1100}\ (f_1) - Tr\ \pi_{0011}\ (f_1)$$

$$+\ Tr\ \pi_{1001}\ (f_1) + Tr\ \pi_{1111}\ (f_1) - Tr\ \pi_{0101}\ (f_2) - Tr\ \pi_{1010}\ (f_3),$$

which is one of three basis elements arising from transfer of stable discrete series characters of

$$U(1,1) \times U(1,1).$$

The remaining combination is the stable sum $\sum_{\pi \in \Pi} Tr\ \pi(f)$.

A change in the conjugacy class of Whittaker data replaces the first four combinations by their negatives but does not affect the others.

Example

Consider the corresponding similitude group $\mathbf{G} = \mathbb{GU}(2,2) \sqcup \mathbb{GU}(4,0)$. There are now only four representations in a discrete series packet. Label them π_{0000}, π_{0110}, π_{1100} and π_{0101}. We write $f = f_1 + f_2$. The first four combinations disappear since there is no endoscopic group for s with an odd number of negative entries.† Example (ii) now corresponds to the combination

$$Tr\ \pi_{0000}\ (f_1) + Tr\ \pi_{0110}\ (f_1) - Tr\ \pi_{1100}\ (f_1) - Tr\ \pi_{0101}\ (f_2).$$

There are also

$$Tr\ \pi_{0000}\ (f_1) - Tr\ \pi_{0110}\ (f_1) + Tr\ \pi_{1100}\ (f_1) - Tr\ \pi_{0101}\ (f_2),$$

and

$$Tr\ \pi_{0000}\ (f_1) - Tr\ \pi_{0110}\ (f_1) + Tr\ \pi_{1100}\ (f_1) + Tr\ \pi_{0101}\ (f_2),$$

and the stable sum.

9. Factors and generators for $\mathbf{G}'$

We work with arbitrary factors $\Delta(\pi_s, \pi)$ for $\mathbf{G}'$. Suppose that π' belongs to the same packet as π and is chosen as base point. We attach the character ζ of $\mathbb{S}^{sc}$ to π' as in Section A15. It is clear from the definitions we have outlined that

$$\Delta(\pi_s, \pi) = \Delta(\pi_s, \pi') < inv_*(\pi, \pi'),\ s_U >$$

where the pairing is the Tate-Nakayama pairing for the torus U from Section A14.

Since

$$\prec s_{sc}, \pi \succ = \zeta(s_{sc}) < inv_*(\pi, \pi'),\ s_U >,$$

we obtain

$$\Delta(\pi_s, \pi) = \rho(\Delta, s_{sc}) \prec s_{sc},\ \pi \succ,$$

where

$$\rho(\Delta, s_{sc}) = \zeta(s_{sc})^{-1}\Delta(\pi_s, \pi').$$

Here we have followed [Arthur: L-packets]. The function ρ has the property

$$\rho(t\Delta, z_{sc}s_{sc}) = t\ \rho(\Delta, s_{sc})\ \zeta(z_{sc})^{-1}$$

for t in $\mathbb{C}^\times$ and z_{sc} in $Z_{sc}^\vee$. This is clear once we observe that $\Delta(\pi_{z_{sc}s}, \pi') = \Delta(\pi_s, \pi')$.

Notice that the combination

$$Tr(\Pi, s_{sc}) = \textstyle\sum_{\pi\in\Pi} \prec s_{sc}, \pi \succ Tr\ \pi$$

does not depend on the normalization of Δ. Now, however, it is multiplied by the n^{th} root of unity $\zeta(z_{sc})$ if s_{sc} is replaced by $s'_{sc} = z_{sc}s_{sc}$ with same image in $\mathbb{S}$. In this setting we will call $Tr(\Pi, s_{sc})$ a generator for Π.

Example

Let $n = 4$, so that $\mathbf{G}' = \mathbb{U}(3,1) \sqcup \mathbb{U}(3,1)$.

A discrete series packet Π for $\mathbf{G}'$ contains four representations

$$\pi_{1000},\ \pi_{0010},\ \pi_{0100},\ \pi_{0001}$$

of the first component $U(3,1)$, along with duplicates (in same order) labelled

$$\pi_{0111},\ \pi_{1101},\ \pi_{1011},\ \pi_{1110}$$

for the second component $U(3,1)$.

We return to Example (i) in Sections 2 and 6. Here

$$s_{sc} = \begin{pmatrix} \epsilon & & & \\ & \epsilon & & \\ & & -\epsilon & \\ & & & \epsilon \end{pmatrix},$$

where $\epsilon^2 = i$. Thus the associated generator is

$$\epsilon\ [Tr\ \pi_{1000}\ (f_1) - Tr\ \pi_{0010}\ (f_1) + Tr\ \pi_{0100}\ (f_1) + Tr\ \pi_{0001}\ (f_1)]$$

$$+\ \ i\epsilon\ [-Tr\ \pi_{0111}\ (f_2) + Tr\ \pi_{1101}\ (f_2) - Tr\ \pi_{1011}\ (f_2) - Tr\ \pi_{1110}\ (f_2)].$$

For Example (ii) we may take $s_{sc} = s$ to obtain

$$Tr\ \pi_{1000}\ (f_1) - Tr\ \pi_{0010}\ (f_1) - Tr\ \pi_{0100}\ (f_1) + Tr\ \pi_{0001}\ (f_1)$$

$$+\ \ Tr\ \pi_{0111}\ (f_2) - Tr\ \pi_{1101}\ (f_2) - Tr\ \pi_{1011}\ (f_2) + Tr\ \pi_{1110}\ (f_2).$$

If we pass to the similitude group $\mathbf{G}' = \mathbb{GU}(3,1)$ then Example (i) no longer applies,[†] while the analogue of Example (ii) removes the f_2 terms. There are two variants of Example (ii) where it is again possible to take $s_{sc} = s$, and finally there is the stable sum attached to $s_{sc} = s = I$.

More generally, for any similitude group $\mathbf{G}'$ we embed $\pi_0(S_{(sim)})$ in $\mathbb{S}^{sc}_{(sim)}$ [see Section A16]. Let $s \in S_{(sim)}$ and denote by s_{sc} the image of the component of s. Then

$$\prec s_{sc}, \pi \succ = \pm 1$$

for any discrete series representation π. These elements s_{sc} determine a complete set of generators involving only signs. There is redundancy: $-s$ determines the negative of the generator for s.

[†] Here is a brief outline of changes, from Section 2 on, for similitude groups in general. For n even only, we now ignore those elements $diag(\pm 1, ..., \pm 1)$ of S with an odd number of negative signs (recall Section A16). Passage to a suitable z-pair allows us to take the endoscopic groups as products of similitude groups $\mathbb{GU}_a \times \mathbb{GU}_b$. For n even, each of a, b must be even. Elliptic endoscopic groups then always have same split rank as $\mathbf{G}$, and transfer factors for $\mathbf{G}$ with Whittaker normalization are just the standard factors Δ_0 (which are canonical).

10. Summary of endoscopic identities

Suppose n is odd and $\mathbf{G}$ is either the unitary or unitary similitude K-group of quasi-split type. A discrete series L-packet Π has cardinality 2^{n-1}. We have described a canonical endoscopic basis for Π.

Suppose n is even and $\mathbf{G}$ is the unitary K-group of quasi-split type, so that again a discrete series packet Π has cardinality 2^{n-1}. To each of the two conjugacy classes of Whittaker data for $\mathbf{G}$ we have attached a unique endoscopic basis for Π. The two bases are related in a simple manner.

Suppose n is even and $\mathbf{G}$ is the unitary similitude K-group of quasi-split type, so that now a discrete series packet Π has cardinality 2^{n-2}. Then we have a canonical endoscopic basis for Π.

Suppose n is even and we consider the unitary K-group $\mathbf{G}'$ not of quasi-split type. Then we have defined generators for Π that involve a $2n^{th}$ root of unity. In the similitude case we may involve only signs in the generators.

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