On some early sources for the notion of transfer in Langlands Functoriality

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Part I

An overview with examples

1 Introduction

The purpose of this paper is to explain a quite simple-minded way of looking at some of Langlands' vast and visionary program of conjectures. Part II of our project, more concerned with precise general statements and their proofs, will be presented elsewhere.

What does *transfer* mean? We may just as well ask: what does *Langlands functoriality* mean? The two notions, whatever they are or should be, are intricately intertwined with each other.

To get started, what are the objects we study? And then, what does it mean to transfer them? Where do functoriality principles come into play? After very limited remarks towards answers in some generality, we examine, also briefly, a concrete example where we do have quite simple explicit answers. We also include sources and hints for our approach, including remarks on a short expository gem from Harish-Chandra in 1966.

2 Settings

We limit our attention to fields that have characteristic zero, even when that is not necessary. Then we fix a **field** F that either is **local**, *i.e.* a finite extension of a completion of $\mathbb{Q}$, or is **global**, *i.e.* a finite extension of $\mathbb{Q}$ itself (a number field). We are interested in connected reductive algebraic groups defined over F. There are three types of problems: those in the local setting, those in the global setting, and those concerned with the relationships among objects in the two settings.

In the local setting, our objects of study fall into two types, geometric and spectral. The geometric objects are the so-called **orbital integrals** on $G(F)$ and the spectral objects are **irreducible representations** of $G(F)$. For us, geometric transfer (the transfer of orbital integrals) emphatically comes first.

3 Flavors for transfer

Following Langlands' vision, transfer itself comes in two flavors. First there is endoscopic transfer which involves a severely limited family of groups and must be viewed as a preliminary step for the second transfer which fully embraces the notion of Langlands functoriality and is very different both in overview and in details. We label the second transfer as stable-stable transfer (for reasons that will become apparent). Again we stress that our goal in this paper is to explain how these principles may be built out of elementary considerations.

4 Endoscopic transfer

First some more words about orbital integrals and endoscopic transfer. A deep analysis of orbital integrals played a central role in the monumental work of Harish-Chandra in the 1940s, 50s and 60s on representation theory of real reductive Lie groups. That analysis is our starting point, and we can't stress enough that we plan to do only simple things with it.

Thus let G be a connected reductive linear algebraic group defined over $\mathbb{R}$, the field of real numbers. This forces $G(\mathbb{C})$ to be a connected complex Lie group and $G(\mathbb{R})$ to be a real Lie group with finitely many connected components. What is an orbital integral (on $G(\mathbb{R})$)? By an orbital integral we mean the set of integrals of a nice function f_G on $G(\mathbb{R})$ along the various conjugacy classes in $G(\mathbb{R})$. The measure on each conjugacy class must be specified. For the purposes of actually defining endoscopic transfer we are able to limit our attention to the so-called strongly regular conjugacy classes. We stress that this will be enough to provide us with a transfer for *all* conjugacy classes.

5 Focus on geometric side

The strongly regular classes are the conjugacy classes of the regular semisimple elements γ_G in $G(\mathbb{R})$ for which the centralizer $Cent(\gamma_G, G)$ of γ_G in G is connected as algebraic group and so coincides with the maximal torus T_{γ_G} of G containing γ_G. Here is how we choose a measure on the conjugacy class of such γ_G. Let dg and dt_{γ_G} be Haar measures on $G(\mathbb{R})$ and $T_{\gamma_G}(\mathbb{R})$ respectively (the choices will be of no consequence when we arrive at a careful statement of endoscopic transfer). Then $\frac{dg}{dt_{\gamma_G}}$ will be the quotient measure on the space $T_{\gamma_G}(\mathbb{R})\backslash G(\mathbb{R})$. This quotient is diffeomorphic to the conjugacy class of γ_G, a closed subset of $G(\mathbb{R})$, via $T_{\gamma_G}(\mathbb{R})g \longmapsto g^{-1}\gamma_G g$. We define the orbital integral $O(\gamma_G, f_G)$ at γ_G of f_G to be $\int_{T_{\gamma_G}(\mathbb{R})\backslash G(\mathbb{R})} f_G(g^{-1}\gamma_G g)\frac{dg}{dt_{\gamma_G}}$. We organize the strongly regular classes using Harish-Chandra's F_f-transform, or more generally his $'F_f-$transform, defined for all regular semisimple conjugacy classes, as our inspiration. Harish-Chandra made two different definitions of his transforms and it is crucial to our considerations that we use the second (final) version [6].

6 Stable conjugacy

In fact, what will work much better for our goals, is to work with the notion of stable conjugacy. The stable conjugacy class of a strongly regular element in $G(\mathbb{R})$ consists of all elements in $G(\mathbb{R})$ that are conjugate to that element by an element of $G(\mathbb{C})$. Langlands' general definition of stable conjugacy of two elements requires further conditions on the chosen elements of $G(\mathbb{C})$, but what we may show eventually is that in endoscopic transfer basic results for *all* classes follow a simple pattern heralded by the strongly regular case.

7 Algebraic groups foremost

For all that we do it is crucial that we work in the algebraic group setting. There is much apparently lucky cancellation in otherwise unwieldy formulas. Nevertheless, the resulting simple formulas hold deep information (here we will concern ourselves only with some fairly immediate examples of our evidence for this).

One thing to notice is that while we work with the results of classical theory of real reductive Lie groups, we do not fix a Cartan decomposition up front. That comes only after we have our algebraic setting in place. Again, that we do these things, with algebraic information at the forefront, is key to our agenda.

Another point, quite minor, is that once we can deal with the case $F = \mathbb{R}$, it takes comparatively little effort to talk in terms of the general archimedean setting. We will save that for elsewhere, noting that much of what we need is found in Langlands' paper [8] on real groups.

8 Working with real groups: one algebraic feature that is harder

Thus we start with a connected reductive linear algebraic group G defined over $\mathbb{R}$. And we are looking for a welldefined notion of endoscopic transfer. An immediate stumbling block is that the stable conjugacy classes are not quite big enough ... for example, for a nonanistropic unitary group G in 3 variables, there are 3 conjugacy classes in a stable conjugacy class of regular elliptic elements in $G(\mathbb{R})$, whereas a little work shows we might reasonably expect 4 conjugacy classes. Where do we find the missing class?

The idea for our answer is due essentially to Vogan, although he considered not conjugacy classes but dual objects, namely irreducible representations, and we further capitalize on a refinement due to Kottwitz; see [3]. We define an extended group over $\mathbb{R}$ to be a (necessarily finite) collection of connected reductive linear algebraic groups G_i, each defined over $\mathbb{R}$, together with a family ψ_{ij} of isomorphisms $G_i \longrightarrow G_j$ over $\mathbb{C}$ for which $\sigma(\psi_{ij})\psi_{ij}^{-1}$ is inner, i.e., each ψ_{ij} is an inner twist, subject to constraints we will come to later.

An extended group may include several copies of an individual group, but there can be at most one copy of a group that is quasi-split over $\mathbb{R}$. Every group appears in some extended group. We call an extended group quasi-split over $\mathbb{R}$ if it does include a group that is quasi-split over $\mathbb{R}$. We stress that not every group appears in an extended group quasi-split over $\mathbb{R}$.

9 Endoscopic transfer: difficulty making a well-defined notion

We emphasize again that we seek a welldefined notion of geometric endoscopic transfer, and this is a most delicate issue. The considerations of Adams and Johnson in [1] are not adequate, nor are those of Adams, Barbasch and Vogan [2]. The notion of geometric transfer discussed in the Wikipedia article on the Fundamental Lemma is not welldefined. We simply cannot use an endoscopic group alone as primary datum.

10 Our primary datum

Instead, we look to embeddings of (Weil group versions of) L-groups. Actually we need a technical modification of no serious interest here so we ignore that. Our primary datum for an endoscopic transfer will then be a pair $(s, {}^L H \hookrightarrow {}^L G)$, where ${}^L G = G^\vee \rtimes W_\mathbb{R}$ is the L-group of the extended group $\{(G_i, \psi_{ij})\}$, s is a semisimple element of $G^\vee$, and ${}^L H = Cent(s, G^\vee)^0 \rtimes W_\mathbb{R}$.

Another point to stress is that we want a robust notion of transfer: our "nice functions" must form a large enough space defined independently of our problem, although we keep in mind that the larger the geometric transfer is, the more limited the dual spectral transfer must be. We will find that there is a remarkable balance between our geometric and spectral transfers.

11 More on well-defined notions

What precisely do we mean by a *well-defined notion of transfer*? And what is its significance?

Our first observation is that any geometric transfer uniquely determines a dual transfer of distributions.

Is it clear that this dual transfer is an endoscopic transfer of characters of irreducible representations? The short answer is *no.* However, there is considerable progress.

Could we start with making a well-defined notion for the transfer of (some) characters and then get a uniquely determined transfer of orbital integrals? In principle, the answer may be *yes.* However, a deep understanding of the representation theory of our group would be needed, and so we insist on geometric transfer as the starting point.

We have introduced our *primary datum* for endoscopic transfer, certain L-group information. Does this determine a unique endoscopic transfer? Almost! In fact, we get a family of transfers with a simply transitive $\mathbb{C}^\times$-action on the family. A more technical analysis shows this is exactly what we want, and so it is our definition of a well-defined transfer.

12 Measures in place of functions

It turns out that certain related measures are simpler to work with than nice functions themselves. We deal with that now. By a nice measure on $G(\mathbb{R})$ we will mean a measure of the form $f dg$, where f is a nice function on $G(\mathbb{R})$ and dg is a Haar measure on $G(\mathbb{R})$. This is all expressed more elegantly in terms of tensor products: see, for example,[10]. However, our more concrete approach will serve us well enough here.

13 What is a nice function?

We have two answers. First, we define a nice function to be a smooth function on $G(\mathbb{R})$ that is rapidly decreasing in the sense of Harish-Chandra. The set of all such functions forms a complete topological vector space, the Harish-Chandra Schwartz space $\mathcal{C}(G(\mathbb{R}))$ via the wellknown Harish-Chandra seminorms. We will label the corresponding nice measures as HCS-measures. Our second space of nice functions is $C_c^\infty(G(\mathbb{R}))$, the set of smooth compactly supported functions on $G(\mathbb{R})$ under the topology of uniform convergence on compact sets. We then label the attached measures as C_c^∞-measures. The natural embedding of $C_c^\infty(G(\mathbb{R}))$ in $\mathcal{C}(G(\mathbb{R}))$, being continuous, provides us with a compatibility demand for the two transfers that we define. That demand will be satisfied thanks to work of Bouaziz [4].

14 A different and simpler problem

We pause to look at an evidently much less complicated problem, combinatorial in nature. The results will be critical not only for our work on endoscopic transfer but also for the second, and main, transfer we have not yet addressed. We start with a single connected reductive group G defined over $\mathbb{R}$.

Let T, T' be maximal tori in G, each defined over $\mathbb{R}$. Then we say that T, T' are stably conjugate if there is $g \in G(\mathbb{C})$ such that the restriction of $Int(g)$ to T is defined over $\mathbb{R}$ and carries $T(\mathbb{C})$ to $T'(\mathbb{C})$. In that case, $Int(g)$ is easily seen to carry $T(\mathbb{R})$ to $T'(\mathbb{R})$, and by an old theorem [9], g may be chosen in $G(\mathbb{R})$. Thus for the base field $F = \mathbb{R}$, stable conjugacy for maximal tori over F coincides with $G(F)$-conjugacy.

We write $t_{st}(G)$ for the set of (stable) conjugacy classes of maximal tori in G that are defined over $\mathbb{R}$. This finite set has a partial ordering: let T, T' be maximal tori over $\mathbb{R}$, and write $\{T\}$, $\{T'\}$ for their stable conjugacy classes.

Then we define $\{T\} \preceq \{T'\}$ if the unique maximal $\mathbb{R}$-split torus S_T in T is $G(\mathbb{R})$-conjugate to an $\mathbb{R}$-split torus in T' or, equivalently, there is $g \in G(\mathbb{R})$ such that $Int(g)$ carries S_T into $S_{T'}$. This partial ordering makes $t_{st}(G)$ a lattice with a unique minimal element, namely the class of fundamental maximal tori over $\mathbb{R}$, and a unique maximal element, namely the class of those maximal tori over $\mathbb{R}$ containing a maximal $\mathbb{R}$-split torus in G. In our pictures of these lattices we place the minimal element at the top.

15 Examples

We consider a few low-dimensional cases.

$G = SL(2)$:

$$\begin{array}{c} o \\ \downarrow \\ o \end{array}$$

$G = SU(2,1)$:

$$\begin{array}{c} o \\ \downarrow \\ o \end{array}$$

$G = Sp_4$, the $\mathbb{R}$-split symplectic group in 4 variables :

$$\begin{array}{ccccc} & & o & & \\ & \swarrow & & \searrow & \\ o & & & & o \\ & \searrow & & \swarrow & \\ & & o & & \end{array}$$

$G = Sp(2,2)$, an example of a hyperbolic symplectic group.

$$\begin{array}{c} o \\ \downarrow \\ o \end{array}$$

16 Concrete view in general case

We have a concrete description of the structure of the lattice $t_{st}(G)$ in terms of the structure of G as algebraic group. Define T to be *adjacent to* (or to *immediately precede*) T' and $\{T\}$ to be adjacent to $\{T'\}$ if $\{T\} \preceq \{T'\}$ and $\dim S_{T'} = 1 + \dim S_T$.

Adjacency is key to the structure of $t_{st}(G)$, and the *symmetric orbits* which come next are key to understanding adjacency.

First we describe adjacency in concrete terms using Harish-Chandra's classification of roots, but in purely algebraic terms. Let α be a root of T in G. Then α is a rational character on T, and so is the root $\sigma\alpha$, the image of α under the action of the nontrivial element σ in $Gal(C/R)$. We consider the Galois orbit $\mathcal{O}_\alpha = \{\alpha, \sigma\alpha\}$ in $X^*(T)$, the module of all rational characters on T. If $\mathcal{O}_\alpha = \mathcal{O}_{-\alpha}$, where we write $-\alpha$ for the root $t \mapsto \alpha(t)^{-1}$, then we call $\mathcal{O}_\alpha$ symmetric. An orbit $\mathcal{O}_\alpha$ that is not symmetric must have the property that $\mathcal{O}_\alpha$ and $\mathcal{O}_{-\alpha}$ are disjoint, and we then call $\mathcal{O}_\alpha$ antisymmetric. These orbits are important too, but not yet.

17 More prep for this view

For $\mathcal{O}_\alpha$ symmetric, the roots $\pm\alpha$ are *imaginary* in the sense of Harish-Chandra or they are *real* in his sense. This is according as $\sigma\alpha = -\alpha$ or $\sigma\alpha = \alpha$. In contrast to the usual practice, we now make a purely algebraic definition. Thus we call an imaginary root α compact or nonsingular according as the 3-dimensional simple group G_α over $\mathbb{R}$ determined by α is $\mathbb{R}$-isomorphic to $SU(2)$ or to $SL(2)$.

The imaginary roots of T in G are exactly the roots of T in the connected reductive subgroup $M_T = Cent(S_T, G)$ of G. The group M_T is defined over $\mathbb{R}$. We describe the Weyl group Ω_{M_T} of T in M_T, usually called the imaginary Weyl group of T, concretely as the group $Norm(T(\mathbb{C}), M_T(\mathbb{C}))/T(\mathbb{C})$. This Weyl group acts on the set of imaginary roots. We call an orbit for this action *totally compact* if each root in it is compact. This sets up our concrete algebraic description of adjacency.

18 General picture

Suppose that the imaginary root α of T is not totally compact. Then we find an element s of $M_T(\mathbb{C})$ such that (i) $T' = sTs^{-1}$ is defined over $\mathbb{R}$ and (ii) $\sigma(s)s^{-1}$, which then normalizes T, acts on T as the Weyl reflection ω_α for α. This ensures that T is adjacent to T'. Conversely, given adjacent pair T and T', we can find such an α.

If T does not contain a maximal $\mathbb{R}$-split torus of G then T has imaginary roots, and if at least one of these roots, say α, is not totally compact then there exists T' adjacent to T. Replacing α by a root in its imaginary Weyl group orbit does not change the stable conjugacy class of T'. Passing to a not totally compact imaginary root outside the Weyl orbit of α does change the stable conjugacy class of T'. Finally, suppose that T' is any given maximal torus over $\mathbb{R}$. Then, apart from the case T' is fundamental, there exists T adjacent to T'.

It is instructive to check how this view works in our examples above, but details are not included here.

19 Remark on other fields

How do things change when we replace $\mathbb{R}$ by other fields of interest to us here? Assume for the rest of this paragraph that F is nonarchimedean. Then a fundamental maximal torus over F in G is elliptic. On the other hand, stable conjugacy for maximal tori over F does not coincide with $G(F)$-conjugacy except in certain cases. We no longer have a unique fundamental (elliptic) stable conjugacy class. We do have a unique maximally F-split stable conjugacy class which is then a single conjugacy class.

20 $t_{st}(G)$ and inner forms

A lemma of Langlands [8, Lemma 3.2] shows that an inner twist $\psi : G \to G^*$, where the connected reductive group G^* is quasi-split over $\mathbb{R}$, determines a map $\psi^{(t)} : t_{st}(G) \to t_{st}(G^*)$. We see easily from our analysis above of familiar results on roots that $\psi^{(t)}$ maps $t_{st}(G)$ to an initial segment of $t_{st}(G^*)$. More precisely, $\psi^{(t)}$ is injective and maps the class of fundamental maximal tori over $\mathbb{R}$ in G to the corresponding class in G^*. Further, there is a unique maximal element in the image of $t_{st}(G)$, namely the image of the class of maximal tori containing a maximal $\mathbb{R}$-split torus in G. This image is the class of maximal tori containing a maximal $\mathbb{R}$-split torus in G^* only if G is quasi-split over $\mathbb{R}$ (and then ψ must be an isomorphism over $\mathbb{R}$). The notion of $t_{st}(G)$ as simply an initial segment of $t_{st}(G^*)$ is developed extensively as we go on.

21 Back to endoscopic transfer

How is $t_{st}(G)$ helpful in visualizing endoscopic transfer?

First, we recall the original goal in endoscopic transfer of orbital integrals. For some groups at least, the (finite) set of conjugacy classes in the stable conjugacy classes of a strongly regular element in $G(\mathbb{R})$ has the structure of a finite abelian group, a sum of Z/2's. An immediate difficulty for us is that this group structure is not uniquely determined by the stable conjugacy class. Nevertheless, first attempts at endoscopic transfer involved picking families of structures and showing that certain combinations of orbital integrals associated with these families could be identified with stable orbital integrals on a certain lower dimensional group, an endoscopic group. This point of view prevailed a long time despite the fact, already emphasized, that it was clear that this does not lead to a well-defined notion of endoscopic transfer.

Another difficulty is that for other groups, only parts of the mentioned finite abelian groups appear in the considerations for a single group G. That is resolved by a variant of an already discussed Vogan technique. It is not really significant for our present concerns, so we will just assume that we may work with a single group G.

While it is hardly surprising that a detailed analysis of the structure of $t_{st}(G)$ played an important role in the original point of view, can it really matter in the proof of existence of a well-defined geometric endoscopic transfer?

Our answer is that the only way we know to prove the existence is to explicitly construct it ... in this one case $F = \mathbb{R}$. Moreover, our approach gives us much more: indeed, we see the form the transfer statement must take in the nonarchimedean case in order to satisfy local-global compatibility demands, although of course our methods do not offer a proof of the existence for the nonarchimedean case. We will see that these constructive methods are heavily influenced by the structure of $t_{st}(G)$.

22 Our results for endoscopic transfer for $F = \mathbb{R}$

Our primary datum for endoscopic transfer is (on ignoring an easily-handled technical modification) a pair $(\mathbf{s}, {}^L H \hookrightarrow {}^L G)$, where ${}^L G = G^\vee \rtimes W_\mathbb{R}$ is the L-group of a given extended group $\{(G_i, \psi_{ij})\}$, $\mathbf{s}$ is a semisimple element of $G^\vee$, and ${}^L H = Cent(\mathbf{s}, G^\vee)^0 \rtimes W_\mathbb{R}$.

Consider the product of the set of strongly regular stable conjugacy classes in $H(\mathbb{R})$ with the set of strongly regular conjugacy classes in $G(\mathbb{R})$. We identify a certain subset of this product as the set of *very regular pairs.* For each very regular pair $(\Gamma_H^{st}, \Gamma_G)$ we define a complex number $\Delta(\Gamma_H^{st}, \Gamma_G)$ such that for each nice measure m_G on $G(\mathbb{R})$ there exists a nice measure m_H on $H(\mathbb{R})$ satisfying

$$O^{st}(\Gamma_H^{st}, m_H) = \sum \Delta(\Gamma_H^{st}, \Gamma_G) O(\Gamma_G, m_G)$$

for all Γ_H^{st} contributing to very regular pairs (*i.e.*, for all strongly G-regular stable classes in $H(\mathbb{R})$). The proof, while firmly based on only elementary consequences of the Harish-Chandra theory, is long and quite complicated. The result is sufficient to establish our main goal, a well-defined geometric transfer in the endoscopic setting.

What about the attached dual transfer ... does it behave as desired regarding representations? The answer is *yes* in the *HCS* case, also called the *tempered* case. Our constructive methods for the orbital integral matching greatly simplify the arguments on the spectral side; this will be explained in Part II of the present project. We will also describe some progress we have made for the C_c^∞-case, partially by recasting some results of others. For example, motivated by important work of Waldspurger, we see that the dual transfer builds in a natural way on the *elliptic* representations. We note in passing that Knapp-Zuckerman decomposition of unitary principal series builds from a wider, more complicated family of representations.

23 Algebraic point of view again

As we have indicated already, we have used a simple algebraic method for the normalization of Haar measures. Starting instead with Cartan involutions, usually called the geometric method, we get very different normalizations. This is proved by an elaborate calculation available already in the 1960's in work of Harish-Chandra (see [6]) using different language. The algebraic approach clearly works better for our intended applications, including geometric ones. We will say no more about this in the present paper.

24 Getting started on *stable-stable transfer*

We come now to the main, and entirely different, type of transfer. Before we start, we ask again about what endoscopic transfer has achieved. It tells us that orbital integrals along conjugacy classes in $G(\mathbb{R})$. can be expressed in terms of orbital integrals along *stable* conjugacy classes from a certain related finite collection of groups $H(\mathbb{R})$. These groups include a quasi-split inner form of G, and all other groups in the collection are of lower dimension.For the attached dual transfer, if we consider the HCS-case then we know that all tempered characters on $G(\mathbb{R})$ are nicely expressed in terms of stable characters on the $H(\mathbb{R})$. Consider what we will call the *trivial case of endoscopic transfer*, that where the endoscopic group is a quasi-split inner form, say G^*, of G. It shows that stable orbital integrals on $G(\mathbb{R})$ may be viewed as stable orbital integrals on $G^*(\mathbb{R})$, and stable tempered characters on $G(\mathbb{R})$ as stable tempered characters on $G^*(\mathbb{R})$. This will allow us to reduce our new transfer involving only stable orbital integrals to the case where both groups, say G_1 and G_2, are quasi-split over $\mathbb{R}$. A more elaborate reduction will then bring us to the case that G_1 and G_2 have same split rank over $\mathbb{R}$. That is the only case we will investigate here.

Our primary datum is (up to a technicality we continue to ignore here) an L-homomorphism from ${}^LG_1$ to ${}^LG_2$. What if G_1 is endoscopic for G_2? Have we already solved the second transfer problem by doing endoscopic transfer? Emphatically, no ... unless we are in the trivial case ... where, because both groups are quasi-split over $\mathbb{R}$) the L-homomorphism determines an $\mathbb{R}$-isomorphism from G_1 to G_2.

25 Working concretely

Consider the following example. We take G_1 to be a 1-dimensional torus anisotropic over $\mathbb{R}$ and G_2 to be $SL(2)$. Before starting, we remark that we will apply to the general case a principle of Harish-Chandra that pervades his work on real groups. We have called it the *Semiregular is sufficient* Principle, and, very roughly, it tells us how this little example can be applied over and over, along with various elementary arguments, to generate the general case.

Now, for details in the example, we identify $G_1(\mathbb{R})$ as the group of rotation matrices $r(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \theta \in \mathbb{R}$. We notice that $r(\theta)$ is strongly $SL(2)$-regular if and only if $\theta \neq 0 \bmod \pi$. The endoscopic transfer tells us that, as function of θ, the suitably normalized unstable combination of orbital integrals of nice measure $m_{SL(2)}$ along the stable conjugacy class of $r(\theta)$ extends smoothly across the points $\theta = 0 \bmod \pi$ on the real line. In the language of transfer, the stable orbital integrals, *i.e.*, the point values at $r(\theta)$ for $\theta \neq 0 \bmod \pi$, of the function so defined match the unstable combination of orbital integrals of $m_{SL(2)}$. Here we have been careless in notation when switching back and forth between nice measures and nice functions.

This smoothness is a simpler preliminary version of the second formula in [5, page 40], wellknown even longer than the formula itself. Notice that translation of our language to that of Harish-Chandra's F_f requires a change from our difference of two terms to the sum of his two terms.

Harish-Chandra's first formula in [5, page 40] points us towards the statement of stable-stable transfer. His formula tells us for $\theta \neq 0 \bmod \pi$ how to write a stable orbital integral on $SL(2,\mathbb{R})$ at $r(\theta)$ in terms of stable tempered characters on $SL(2,\mathbb{R})$. To review, we denote by $Ch(\Pi_n,*)$ the stable discrete series character attached to the positive integer n,.and by $Ch(\Pi_{\lambda,+},*), Ch(\Pi_{\lambda,-},*)$ the two unitary prinicipal series characters attached to the positive real number λ. Here we are following, as closely as our different conventions allow, Harish-Chandra's notation in [5]. Then we write the first formula as

$$\begin{gathered}\widehat{O}(\Gamma_\theta, m) = \\ \sum_{n>0} \Delta(\Gamma_\theta, \Pi_n) Ch(\Pi_n, \Gamma_\theta) \\ + \int_0^\infty \Delta(\Gamma_\theta, \Pi_{\lambda,+}) Ch(\Pi_{\lambda,+}, \Gamma_\theta) d\lambda \\ + \int_0^\infty \Delta(\Gamma_\theta, \Pi_{\lambda,-}) Ch(\Pi_{\lambda,-}, \Gamma_\theta) d\lambda,\end{gathered}$$

where Γ_θ denotes the stable conjugacy class of $r(\theta)$ for $\theta \neq 0 \bmod \pi$. We may use Harish-Chandra's simple explicit formulas for the coefficients $\Delta(\Gamma_\theta, \Pi_n)$,

$\Delta(\Gamma_\theta, \Pi_{\lambda,+})$ and $\Delta(\Gamma_\theta, \Pi_{\lambda,-})$, along with the temperedness of the representations $\Pi_n, \Pi_{\lambda,\mp}$ to see that convergence of the series is absolute, uniform on compact subsets of $\theta \neq 0 \bmod \pi$,and similarly for the integrals.

26 Some heuristics

Now we explain some elementary and rather crude heuristics that do, however, lead us on from the general version of Harish-Chandra's first formula (namely, *Fourier inversion for stable orbital integrals*) to our final statement of stable-stable transfer. Without stating explicitly what we mean by the space $\mathbf{\Gamma}$ of stable orbital integrals nor describing the measure $d\Gamma$ on it, we write $O(\Gamma, m)$ for the (stable orbital) integral of a nice measure m along the stable conjugacy class Γ, and $\widehat{O}(\Gamma, m)$ for the normalized version via the usual discriminant function. On the spectral side we similarly use a space $\mathbf{\Pi}$ of tempered packets with measure $d\Pi$. Then $Tr(\Pi,*)$ denotes the stable trace for the packet Π, and $Ch(\Pi,*)$ is the real analytic function on the regular semisimple elements of $G(\mathbb{R})$ that represents the stable trace (via Harish-Chandra's Regularity Theorem). The normalized version is $\widehat{Ch}(\Pi,*)$.

Instead of G_1 and G_2, we label our two groups H and G, and attach subscript H or G to $\Pi, \mathbf{\Pi}$,Γ and $\mathbf{\Gamma}$, as needed. We do not assume that H is endoscopic for G but do insist that H and G have same rank and that we have primary datum ξ embedding ${}^L H$ in ${}^L G$. This determines a map $\mathbf{\Pi}_H \to \mathbf{\Pi}_G$, and we will write $\Pi_{H\to G}$ for the image of the packet Π_H.

What we seek are transfer identities of the following "shape":

for each nice measure m_G on $G(\mathbb{R})$ there exists a nice measure $m_H = (m_G)_H$ on $H(\mathbb{R})$ such that

$$\widehat{O}(\Gamma_H, (m_G)_H) = \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, m_G) d\Gamma_G$$

and

$$\widehat{Ch}(\Pi_{H\to G}, \Gamma_G) = \int_{\mathbf{\Gamma}_H} \widehat{Ch}(\Pi_H, \Gamma_H)\Theta(\Gamma_H, \Gamma_G) d\Gamma_H$$

for all (strongly) regular semisimple Γ_H, Γ_G.

Assume this is true (in some sense!). We will also change order of integration freely. Using the Weyl integration formula on $G(\mathbb{R})$ we write

$Tr(\Pi_{H\to G}, m_G)$ as

$$\int_{\mathbf{\Pi}_G} \widehat{Ch}(\Pi_{H\to G}, \Gamma_G)\widehat{O}(\Gamma_G, m_G) d\Gamma_G.$$

Then $Tr(\Pi_{H\to G}, m_G)$ is given by

$$\begin{aligned} &\int_{\mathbf{\Pi}_G} \int_{\mathbf{\Pi}_G} \widehat{Ch}(\Pi_H, \Gamma_H)\Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, m_G) d\Gamma_H d\Gamma_G \\ &= \int_{\mathbf{\Pi}_H} \widehat{Ch}(\Pi_H, \Gamma_H) \int_{\mathbf{\Pi}_G} \Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, m_G) d\Gamma_G d\Gamma_H \\ &= \int_{\mathbf{\Pi}_H} \widehat{Ch}(\Pi_H, \Gamma_H)\widehat{O}(\Gamma_H, (m_G)_H) d\Gamma_H \\ &= Tr(\Pi_H, (m_G)_H). \end{aligned}$$

This is stable-stable transfer at the level of traces, which we do expect as our final, emphatically not our initial, transfer formula.

Continuing in the same spirit, we also see functoriality emerging:

given a composition ${}^L J \to^L H \to^L G$ of L-homomorphisms, assume there are attached stable-stable transfer. Then m_G determines both $(m_G)_J$ and $((m_G)_H)_J$, and we see that these two measures are stably equivalent (same $\widehat{O}(\Gamma_J, *)$ for all Γ_J) provided

$$\Theta(\Gamma_J, \Gamma_G) = \int \Theta(\Gamma_J, \Gamma_H)\Theta(\Gamma_H, \Gamma_G) d\Gamma_H.$$

As a final comment, we write down our proposed "shape" for $\Theta(\Gamma_H, \Gamma_G)$, and concern ourselves just with our particular example, to make sense of this $\Theta(\Gamma_H, \Gamma_G)$ and verify the geometric stable-stable transfer.

Set

$$\Theta(\Gamma_H, \Gamma_G) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H)\widehat{Ch}(\Pi_{H\to G}, \Gamma_G) d\Pi_H,$$

where $\Delta(\Gamma_H, \Pi_H)$ is the coefficient in Fourier inversion of the stable orbital integral $\widehat{O}(\Gamma_H, *)$ on $H(\mathbb{R})$:

$$\widehat{O}(\Gamma_H, m_H) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H) Tr(\Pi_H, m_H) d\Pi_H$$

for all nice measures m_H on $H(\mathbb{R})$.

27 Back to the example

Here we write the proposed $\Theta(r(\theta), \Gamma_G)$ as $\sum_{n\in\mathbb{Z}} e^{in\theta}\widehat{Ch}(\Pi_{n^*}, \Gamma_G)$,.where $n^* > 0$ is attached to $n \in \mathbb{Z}$ using the Langlands' classification. We can make sense of $\Theta(r(\theta), \Gamma_G)$ as a distribution or as a generalized function. Calculation shows that we then get the desired transfer of orbital integrals, our stable-stable transfer.

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