Tempered Endoscopy for Real Groups II: spectral transfer factors

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Abstract

This is the second of three papers reinterpreting old theorems in endoscopy, or L-indistinghuishability, for real groups in terms of the canonical transfer factors of Langlands and Shelstad. The a priori definition of those factors provides an explicit geometric transfer theorem. The present paper introduces a parallel definition for spectral transfer factors. The author uses various simple properties of these factors and their relation to the geometric factors to prove an explicit version of the tempered spectral transfer theorem. This prepares for an explicit inversion of the transfer for several groups simultaneously and related results in the third paper.

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1. Introduction

Endoscopy is an example of Langlands' functoriality principle. It arises when representations with same L-functions appear with different multiplicities in automorphic forms. For SL(2), Labesse and Langlands determined the multiplicity of certain representations $\pi = \otimes_v \pi_v$ in terms of the *position* of each π_v amongst representations with same local L-factor, *i.e.* amongst representations within its L-packet [LL]. Their method is based on stabilization of the adelic trace formula for $SL(2)$. Langlands' proposed generalization of needed local analysis, in particular, transfer of orbital integrals and characters, is accessible for the tempered spectrum of real groups through the Plancherel theory of Harish Chandra. The Arthur conjectures carry the program to the nontempered spectrum.

We start then with a connected reductive algebraic group G defined over $\mathbb{R}$. Endoscopy has two *sides:* geometric and spectral. First, there are two standard constructions of *endoscopic data* for G, one geometric and one spectral, each introduced by Langlands. For the purposes of this introduction, we will assume an arbitrary set of data has been fixed, and mention specifically only the attached

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endoscopic group H_1. Modulo a central subgroup $Z_1(\mathbb{R})$, the Cartan subgroups of $H_1(\mathbb{R})$ are Cartan subgroups of $G(\mathbb{R})$, and the coroots form a subsystem of the coroots for G.

On the geometric side, the basic notion is *stable conjugacy.* For the very regular elements this is simply conjugacy under the complex points of the group. We are concerned first with comparing very regular stable conjugacy classes in $H_1(\mathbb{R})$ and $G(\mathbb{R})$. Then geometric transfer amounts to matching certain precise combinations of the integrals of a test function on $G(\mathbb{R})$ along the conjugacy classes in a stable class with a stable combination of the integrals for a test function on $H_1(\mathbb{R})$. This matching will define a correspondence on test functions (initially, Harish Chandra's Schwartz functions) for $G(\mathbb{R})$ and $H_1(\mathbb{R})$.

As the example of $G = SL(2)$ and H_1 a compact torus quickly shows, there are several ways to view the Cartan subgroups of $H_1(\mathbb{R})$ as Cartan subgroups of $G(\mathbb{R})$, and we shall want a transfer statement independent of these choices. Introduction of the canonical transfer factors from [LS] allows us to formulate such a statement, as in the first part of Theorem 5.1. A direct proof of geometric transfer in this form was given in [S3]. A previous proof combined the older implicit methods of [S1] with results from [LS] and [LS2].

In this paper we turn to the spectral side where, for tempered representations at least, the basic notion is an *L-packet.* It has an attached stable tempered character. We examine transfer to $G(\mathbb{R})$, dual to the geometric transfer, of a stable tempered character on $H_1(\mathbb{R})$. Our discussion continues on from the introductory discussion of Sections 15 and 16 of [S3]. This transfer yields a tempered virtual character on $G(\mathbb{R})$ [S1], an *endoscopic character.* The irreducible characters contributing to an endoscopic character form the L-packet predicted by Langlands' functoriality principle (applied to the map attached to the choice of endoscopic data). It is the coefficients that concern us, as these were given only implicitly in [S1]. Arthur [A] has labeled them as *spectral transfer factors.*

Our main purpose is to give a simple explicit formula for the spectral transfer factors in terms of the geometric transfer factors or, more precisely, to introduce *independent, but parallel, definitions* of canonical spectral transfer factor and compatibility factor (relative to the geometric factor), and then *show directly* that these describe the tempered transfer correctly. The arguments for the transfer follow closely those in [S1] in many places, but now we structure them explicitly in parallel with the geometric side. In particular, we start by looking for canonical definitions in a (strongly) G-regular setting.

We first prepare in Sections 2, 3 and 4 for the statement of the transfer theorem in Section 5. We then return to an introductory discussion of spectral transfer factors in Section 6, along the same lines as our discussion of the geometric factors of [LS] in Section 8 of [S3]. We prepare in Section 7 for defining the terms in the spectral factors. We define the spectral factors term by term in Sections 8, 9 and 10 for the G-regular setting, and summarize our results in Section 11. We see, in particular, that we can always arrange that the spectral factors are simply signs. This happens, for example, when we use a Whittaker normalization in the quasi-split case.

In Section 12 we turn to definition of the compatibility factors discussed in

Section 4. We then devote Section 13 to the proof of the transfer theorem, Theorem 5.1, in the G-regular case. The calculation there, while elementary, is given step by step, as it explains how our definitions untangle the various contributions to the signs in [S1]. Certain signs now cancel naturally, thanks to the generalized Weyl denominators introduced in Section 7c. In Section 14 we describe how to extend the definition of the spectral factors to the general tempered setting, and complete the proof of Theorem 5.1.

The argument in Section 13 for the G-regular case of the transfer theorem rests on Harish Chandra's characterization of discrete series characters, while for the general case we show, with the aid of the character identities of Hecht and Schmid, that coherent continuation of the identities from the G-regular case works correctly for functoriality and so yields the desired transfer. We do not need the full strength of the Knapp-Zuckerman classification of the tempered spectrum at this stage.

We have yet to show that we have assembled sufficiently many character identities to retrieve each tempered irreducible character as a combination of endoscopic characters. We could quote Section 5 of [S1] directly. Instead we prefer to work with our newly defined spectral factors, and will discuss very briefly in Section 15 the attendant inversion formula of [S2]. Once again the spectral factors appear as coefficients.

2. The setting of endoscopy

Let G be a connected reductive algebraic group defined over $\mathbb{R}$, and G^* be a quasisplit inner form of G with $\mathbb{R}$-splitting

$$spl_{G^*} = (\mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\}).$$

Let $\psi : G \to G^*$ be an inner twist and $u(\sigma)$ be an element in the simply connected covering G^*_{sc} of the derived group of G^* such that

$$\psi\sigma(\psi)^{-1} = Intu(\sigma),$$

where $\Gamma = \{1, \sigma\}$ denotes the Galois group of $\mathbb{C}/\mathbb{R}$. Let $G^\vee$ denote the complex dual of G, with splitting

$$spl_{G^\vee} = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$$

preserved by the algebraic dual $\sigma_{G^\vee}$ of the Galois action. By the L-group ${}^LG$ we mean $G^\vee \rtimes W_\mathbb{R}$, where the Weil group $W_\mathbb{R}$ of $\mathbb{C}/\mathbb{R}$ acts through $W_\mathbb{R} \to \Gamma$. The transfer factors will be independent of the choice of splittings, although spl_{G^*} is involved directly in the construction of the term Δ_I, and $spl_{G^\vee}$ in that of Δ_{II}.

Suppose

$$(H, \mathcal{H}, \mathfrak{s}, \xi)$$

is a set of endoscopic data for G as in [LS]. We choose a z-pair (H_1, ξ_1) following [KS]. Thus H_1 is a z-extension of H and ξ_1 is an embedding of extensions

$$\mathcal{H} \to {}^LH_1$$

that extends the embedding $H^\vee \to H_1^\vee$ dual to $H_1 \to H$. Let λ_1 be the attached quasicharacter on $Z_1(\mathbb{R})$, where

$$Z_1 = Ker(H_1 \to H).$$

We may assume λ_1 unitary. See [S3] for further discussion and notation.

The set $\Phi_{temp}(G)$ of Langlands parameters for the L-packets of equivalence classes of tempered irreducible admissible representations of $G(\mathbb{R})$ is the set of $G^\vee$-conjugacy classes of continuous homomorphisms

$$\varphi(w) = \varphi_0(w) \times w$$

of $W_{\mathbb{R}}$ into ${}^LG = G^\vee \ltimes W_{\mathbb{R}}$ for which $\varphi_0(W_{\mathbb{R}})$ is bounded and consists of semisimple elements, and for which φ is relevant to G. We will discuss *relevance* in more detail in Section 7b; it depends on the choice of inner twist ψ from G to G^*. For the quasisplit form G^* all parameters are relevant, and in general we may identify $\Phi_{temp}(G)$ as a subset of $\Phi_{temp}(G^*)$, although we will often *not identify* φ with its image φ^*in notation. We write $\boldsymbol{\varphi}$ for the class of φ when we need to distinguish between the canonical $\boldsymbol{\varphi}$ and the various choices of representatives φ for it. This is particularly relevant in Section 14. Also, $\boldsymbol{\varphi}(\pi)$ will denote the parameter of the L-packet of a tempered irreducible representation π.

Regarding the parameters

$$\Phi_{temp}(H_1, \lambda_1)$$

for the L-packets of the representations $\pi_1$of $H_1(\mathbb{R})$ which transform by the character λ_1 on $Z_1(\mathbb{R})$, we recall first the Langlands parameter for λ_1. Let $c : W_{\mathbb{R}} \to \mathcal{H}$ be a splitting of $\mathcal{H} \to W_{\mathbb{R}}$ as specified in the definition of endoscopic data. Then the composition of c with ξ_1 and the natural map ${}^LH_1 \to {}^LZ_1$ provides us with a homomorphism $a : W_{\mathbb{R}} \to {}^LZ_1$. The $Z_1^\vee$-conjugacy class of a is the Langlands parameter for λ_1. Now suppose $\boldsymbol{\varphi}_1 \in \Phi_{temp}(H_1)$. Then composing any φ_1 in $\boldsymbol{\varphi}_1$ with ${}^LH_1 \to {}^LZ_1$ to obtain

$$\overline{\varphi_1} : W_{\mathbb{R}} \to^L Z_1$$

also yields a unique parameter for a character on $Z_1(\mathbb{R})$, and $\Phi_{temp}(H_1, \lambda_1)$ consists of those $\boldsymbol{\varphi}_1$ for which that character is λ_1. We will write $\boldsymbol{\lambda}_1$ for λ_1 whenever there is a notational conflict.

To the pair of embeddings $\xi : \mathcal{H} \to {}^LG$, $\xi_1 : \mathcal{H} \to {}^LH_1$ we attach a map

$$\Phi_{temp}(H_1, \lambda_1) \to \Phi_{temp}(G^*)$$

as follows. Take any splitting $c : W_{\mathbb{R}} \to \mathcal{H}$ of $\mathcal{H} \to W_{\mathbb{R}}$ as above, and again use ξ_1 to form a parameter $a : W_{\mathbb{R}} \to^L Z_1$ for λ_1. Now pick φ_1 in $\boldsymbol{\varphi}_1$ such that $\overline{\varphi_1} = a$. This ensures that $\varphi_1(W_{\mathbb{R}})$ is contained in $\mathcal{H}$, so that we can apply ξ_1^{-1}. Also, if we take another φ_1' in $\boldsymbol{\varphi}_1$ such that $\overline{\varphi_1'} = a$ then because $Z(H_1^\vee)^\Gamma \to (Z_1^\vee)^\Gamma$ is surjective (recall Z_1 is induced) we may check that φ_1' is conjugate to φ_1 under $H^\vee$. We then conclude that

$$\varphi_1 \;\to\; \varphi = \xi \circ \xi_1^{-1} \circ \varphi_1$$

gives a welldefined map $\Phi_{temp}(H_1, \lambda_1) \to \Phi_{temp}(G^*)$ on conjugacy classes of homomorphisms.

Suppose now π_1 is a tempered irreducible representation of $H_1(\mathbb{R})$ which transforms by the character λ_1 on $Z_1(\mathbb{R})$ and π is a tempered irreducible representation of $G(\mathbb{R})$.

Definition 2.1 *We call* (π_1, π) *a related pair* if $\boldsymbol{\varphi}(\pi)$ *is the image of* $\boldsymbol{\varphi}(\pi_1)$ *under the map* $\Phi_{temp}(H_1, \lambda_1) \to \Phi_{temp}(G^*)$ *associated to* ξ, ξ_1.

Given any such π_1, there are always related pairs (π_1, π^*) for the quasisplit form G^*, and there are related pairs (π_1, π) for G if and only if $\boldsymbol{\varphi}(\pi^*)$ lies in $\Phi_{temp}(G)$.

Definition 2.2 *A related pair* (π_1, π) *is* G*-regular if the parameter* $\boldsymbol{\varphi}(\pi_1)$ *is* G*-regular in the sense that for some, and hence any,* φ *in its image* $\boldsymbol{\varphi}(\pi)$ *we have that*

$$Cent(\varphi(\mathbb{C}^\times), G^\vee)$$

is abelian.

Since the group $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ is connected a separate notion of *strong* G-regularity is unnecessary in this setting.

3. Spectral transfer factors

Continuing from [S3], we write the spectral transfer factors, so far defined only implicitly, as $\Delta(\pi_1, \pi)$. For all $f \in \mathcal{C}(G(\mathbb{R}))$ there is $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ with orbital integrals matched via the geometric transfer factors, and then

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) Tr\ \pi(f),$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ which transform by the character λ_1 on $Z_1(\mathbb{R})$. The summation on the right is over equivalence classes of tempered irreducible representations π of $G(\mathbb{R})$. Here $St\text{-}Tr\ \pi_1$ is our notation for the stable character attached to (the L-packet of) π_1. By definition, $\Delta(\pi_1, \pi) = 0$ unless (π_1, π) is a related pair, so that the sum on the right is either zero or has nonzero contributions exactly from those π in the single L-packet predicted by the functoriality principle.

Now we start afresh and define explicit candidates for $\Delta(\pi_1, \pi)$. We first restrict our attention to those tempered, irreducible π_1 that are G-regular in the sense we have defined at the end of Section 2. If (π_1, π) and (π_1', π') are G-regular related pairs then each factor $\Delta(\pi_1, \pi)$, $\Delta(\pi_1', \pi')$ will be nonzero. We will give an explicit formula for the quotient

$$\Delta(\pi_1, \pi)/\Delta(\pi_1^{'}, \pi') = \Delta(\pi_1, \pi; \pi_1^{'}, \pi')$$

that, like the formula for the relative geometric factor on the very regular set, is canonical.

Recall that any normalization for the geometric transfer factors can be recaptured as follows. On the geometric side, strongly G-regular related pairs are of the form (γ_1, δ), with γ_1 strongly G-regular in $H_1(\mathbb{R})$ and an image of (strongly regular) δ in $G(\mathbb{R})$. Fix a strongly G-regular related pair $(\overline{\gamma}_1, \overline{\delta})$ and specify $\Delta(\overline{\gamma}_1, \overline{\delta})$ as desired. Then for all γ_1 strongly G-regular in $H_1(\mathbb{R})$ and all strongly regular δ in $G(\mathbb{R})$ we have

$$\Delta(\gamma_1, \delta) = \Delta(\overline{\gamma}_1, \overline{\delta})\Delta(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta}),$$

where $\Delta(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta})$ is the canonical relative transfer factor of [LS].

We may proceed in a similar way on the spectral side. Thus fix a G-regular tempered related pair $(\overline{\pi}_1, \overline{\pi})$. Since $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ has been prescribed by geometric transfer for $f \in \mathcal{C}(G(\mathbb{R}))$, we will have to specify $\Delta(\overline{\pi}_1, \overline{\pi})$ in terms of $\Delta(\overline{\gamma}_1, \overline{\delta})$ for the dual transfer. Then we set

$$\Delta(\pi_1, \pi) = \Delta(\overline{\pi}_1, \overline{\pi})\Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi}),$$

for all π_1 G-regular, using the canonical $\Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi})$ that we will describe shortly.

In Section 14 we will see how to extend our formula for $\Delta(\pi_1, \pi)$ in the G-regular case to all tempered pairs (π_1, π).

4. Compatible normalization

We now write the transfer factors as Δ_{geom} and Δ_{spec} respectively. They are determined uniquely up to normalization, and for the transfer theorem the normalizations must satisfy a compatibility condition. To describe this, we start with the device of chosen related pairs. Suppose $(\overline{\pi}_1, \overline{\pi})$ and $(\overline{\gamma}_1, \overline{\delta})$ are chosen. Then the quotient

$$\Delta_{spec}(\overline{\pi}_1, \overline{\pi})/\Delta_{geom}(\overline{\gamma}_1, \overline{\delta})$$

of compatible normalizing factors will have an interpretation as a canonical relative form

$$\Delta_{comp}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta}),$$

as we will describe in Section 12.

Continuing with chosen related pairs, we observe that we may then normalize spectral factors first if we wish. For example, we may choose a G-regular tempered related pair $(\overline{\pi}_1, \overline{\pi})$ first, and require $\Delta(\overline{\pi}_1, \overline{\pi}) = \pm 1$. We will show independently that the canonical factor $\Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi})$ is a sign. Thus this normalization yields

$$\Delta(\pi_1, \pi) = \pm 1$$

for all G-regular tempered related pairs (π_1, π). Then for a compatible normalization on the geometric side we take a strongly G-regular related pair $(\overline{\gamma}_1, \overline{\delta})$ and normalize the geometric transfer factor $\Delta(\gamma_1, \delta)$ by setting

$$\Delta(\overline{\gamma}_1, \overline{\delta}) = \Delta_{comp}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta})^{-1}\Delta(\overline{\pi}_1, \overline{\pi}).$$

With this compatible normalization, the local form for the geometric transfer factor also simplifies. See Lemma 12.6.

We may avoid explicit mention of chosen related pairs. Factors Δ_{geom} and Δ_{spec}, with any normalization, are compatible if and only if

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)\Delta_{geom}(\gamma_1, \delta)$$

for all G-regular pairs (π_1, π) of tempered irreducible representations and all strongly G-regular pairs (γ_1, δ) of $\mathbb{R}$-rational points (Lemma 12.2). For example, suppose that G is the quasisplit form and the inner twist ψ is the identity. Then geometric factors $\Delta_0(\gamma_1, \delta)$ are introduced in [LS] that depend on an $\mathbb{R}$-splitting for G. Using the same splitting we will define spectral factors $\Delta_0(\pi_1, \pi)$. These normalizations are compatible (Lemma 12.3). We find however that

$$\Delta_0(\pi_1, \pi) = \pm i$$

is possible (recall $SL(2)$), in which case this is true for all tempered G-regular pairs (π_1, π). The Whittaker normalization (Sections 5.2 and 5.3 of [KS]) remedies this. The geometric factor $\Delta_\lambda(\gamma_1, \delta)$ of [KS] depends instead on Whittaker data (B, λ). We will introduce a Whittaker normalization for the spectral factor the same way, i.e. we multiply spectral Δ_0 by the same epsilon factor. Then we obtain, just as for $SL(2, F)$, that

$$\Delta_\lambda(\pi_1, \pi) = \pm 1$$

for all G-regular pairs (π_1, π) of tempered irreducible representations (Lemma 11.4).

5. Transfer theorem

Tempered endoscopic transfer for the real reductive group G is now summarized in the following:

Theorem 5.1 *Let $(H, \mathcal{H}, \mathfrak{s}, \xi)$ be a set of endoscopic data for G, and (H_1, ξ_1) be a z-pair for H with attached character λ_1 on the central subgroup $Z_1(\mathbb{R})$, where $Z_1 = Ker(H_1 \to H)$. Let Δ_{geom} and Δ_{spec} be transfer factors attached to these endoscopic data and z-pair, with compatible normalization. Then for each $f \in \mathcal{C}(G(\mathbb{R}))$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ such that*

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \sum_{\delta, conj} \Delta_{geom}(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. Moreover, there is a dual transfer of stable tempered characters given by

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi) Tr\ \pi(f)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ which transform under $Z_1(\mathbb{R})$ according to λ_1, and, conversely, if $f \in \mathcal{C}(G(\mathbb{R}))$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ satisfy

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi) Tr\ \pi(f)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ which transform under $Z_1(\mathbb{R})$ according to λ_1 then

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \sum_{\delta, conj} \Delta_{geom}(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

Here measures dh_1, dg are arbitrary: the measures dt_1, dt must be related by transport ([S3]).

Outline of proof: See [S3] for a discussion and proof of the geometric transfer, and the converse matching statement. What remains now is to describe the proposed explicit spectral transfer factors $\Delta_{spec}(\pi_1, \pi)$ and to show they may be used in place of the implicit values for which the spectral transfer has been proved ([S1], Theorem 4.1.1). This will occupy the rest of the paper. First, we will introduce the various ingredients and some of their properties. Only the term Δ_{II} of Section 9 is not familiar from geometric transfer. Once we are done with the canonical factors in the G-regular case, we complete the proof of transfer in this setting in Section 13. Then we extend the factors to all tempered pairs (π_1, π) and finish the proof in Section 14.

Notice that in the statement of the theorem we do not mention explicitly the twisting data (ψ, u) of Section 2. We will examine the role of this choice in Section 4 of [S2]. This will then lead us to an extended version of Theorem 5.1 for K-groups ([S2], Theorem 6.2) or, equivalently, a simultaneous version for several triples (G, ψ, u).

6. Canonical spectral transfer factors: outline

Suppose that (π_1, π) and (π_1', π') are related pairs, with π_1, π_1' both G-regular. Let $\varphi_1 = \varphi_1(\pi_1)$, $\varphi_1' = \varphi_1(\pi_1')$, $\varphi = \varphi(\pi)$ and $\varphi' = \varphi(\pi')$. The factor $\Delta(\pi_1, \pi; \pi_1^{'}, \pi')$ will be a product of three terms:

$$\Delta(\pi_1, \pi; \pi_1^{'}, \pi') = \Delta_I(\pi_1, \pi; \pi_1^{'}, \pi')\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')\Delta_{III}(\pi_1, \pi; \pi_1^{'}, \pi').$$

We give a brief summary of the construction of each term.

First, $\Delta_I(\pi_1, \pi; \pi_1^{'}, \pi')$ is an exact analogue of the term Δ_I of the geometric factor in the sense that it is constructed in the same way from the same type of data. The starting point is the splitting invariant of a torus from Section 2.3 of [LS] (see Section 8 of [S3] for some comments). It, Δ_I, is a quotient $\Delta_I(\varphi_1)/\Delta_I(\varphi_1')$. Each term in the quotient depends only on the L-packet of the representation of the endoscopic group, rather than on the representations themselves. Each term does depend on the choice of $\mathbb{R}$-splitting for the quasisplit form G^*, but the quotient does not. On the other hand, each term in the quotient, like the quotient itself, does depend on the choice of toral data and a-data.

Second, $\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')$ is again a quotient, which we denote

$$\Delta_{II}(\varphi_1, \varphi)/\Delta_{II}(\varphi_1', \varphi'),$$

each term depending just on the L-packet of each representation. This spectral Δ_{II} parallels the term

$$\Delta_{II+} = \Delta_{II}\Delta_{III_2}\Delta_{IV}$$

of the geometric transfer factor. That term is a product of an ι-shift character (Δ_{III_2}) with a quotient of generalized Weyl denominators for G and H_1 (Δ_{II} Δ_{IV}). Construction of the terms $\Delta_{II}, \Delta_{III_2}$ involves χ-data, but the product is independent of that choice. Similarly, construction of the spectral Δ_{II} involves generalized Weyl denominators and χ-data, this time in the local formulas for stable traces, and again the choice of χ-data will be be seen not to matter. The choice of toral data also does not matter, but the effect of a change in a-data is to provide a term cancelling that produced by the change in $\Delta_I(\pi_1, \pi; \pi_1^{'}, \pi')$. The product

$$\Delta_I(\pi_1, \pi; \pi_1^{'})\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')$$

thus depends only on toral data.

Finally, the term

$$\Delta_{III}(\pi_1, \pi; \pi_1^{'}, \pi')$$

is the only genuinely relative term, and the only term depending on the representations π, π' of $G(\mathbb{R})$ themselves, rather than simply on the parameters. It is an exact analogue of the term Δ_{III_1} of the geometric factor. It depends, as it must for a canonical product, only on toral data.

Regarding the dependence on parameters, Δ_I is the same for φ^* and φ. In the case of Δ_{II}, however, we have chosen to include a global sign that would otherwise be included in the compatibility factor. In particular, we will have that

$$\Delta_{II}(\varphi_1, \varphi) = (-1)^{q_G^* - q_G}\Delta_{II}(\varphi_1, \varphi^*),$$

where $2(q_{G^*} - q_G)$ is the (even) difference in the dimension of the symmetric spaces attached to G^*_{sc}, G_{sc}.

We will wait until Section 3 of [S2] to observe (in more precise terms) that the effect of the choice of z-pair, and in particular the choice of z-extension $H_1 \to H$, is negligible.

Now we consider briefly the numerical values of each term. As with the corresponding geometric terms Δ_I and Δ_{III_1}, we have

$$\Delta_I(\pi_1, \pi; \pi_1^{'}, \pi')^2 = \Delta_{III}(\pi_1, \pi; \pi_1^{'}, \pi')^2 = 1,$$

since each term is defined by a pairing in (abelian) Galois cohomology for $\mathbb{C}/\mathbb{R}$.

We will observe during the construction in Section 10 that while the terms $\Delta_{II}(\varphi_1, \varphi)$ and $\Delta_{II}(\varphi_1^{'}, \varphi')$ are each fourth roots of unity, their quotient must be a sign. Thus

$$\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')^2 = 1,$$

and then we conclude that

$$\Delta(\pi_1, \pi; \pi_1^{'}, \pi')^2 = 1.$$

7. Various tools

a. Toral data

To transport data among shared Cartan subgroups in $G(\mathbb{R})$, $G^*(\mathbb{R})$ and $H_1(\mathbb{R})$ we recall first the notion of toral data. We will also use these data to transport data from the dual groups. First we can save some notation by assuming that the embedding ξ of $\mathcal{H}$ in ${}^LG$ is inclusion on the subgroup $H^\vee$, and then taking the endoscopic datum $\mathfrak{s}$ in $\mathcal{T}$, where

$$spl_{G^\vee} = (\mathcal{B}, \mathcal{T}, \{X\}).$$

We also take

$$spl_{H^\vee} = (\mathcal{B} \cap \mathcal{H}, \mathcal{T}, \{Y\})$$

and extend it to

$$spl_{H_1^\vee} = (\mathcal{B}_1, \mathcal{T}_1, \{Y\}),$$

although none of these choices affects our final results. If B_1 is a Borel subgroup of H_1 containing the maximal torus T_1 defined over $\mathbb{R}$, then attached to the pairs (B_1, T_1) and $(\mathcal{B}_1, \mathcal{T}_1)$ is an isomorphism $T_1^\vee \to \mathcal{T}_1$ and similarly $T^{*\vee} \to \mathcal{T}$ is attached to (B^*, T^*), for T^*defined over $\mathbb{R}$ in G^*.

Given T_1 over $\mathbb{R}$ in H_1 we can always find B_1 and a pair (B^*, T^*) in G^* with the composition

$$\eta^* = \eta_{B_1, B^*} : T_1 \to T_H \to T^*$$

defined over $\mathbb{R}$ and dual to $T^{*\vee} \to T_1^\vee$ constructed from maps

$$T^{*\vee} \to \mathcal{T} \hookrightarrow \mathcal{T}_1 \to T_1^\vee$$

as above. Such a map η^* is what we mean by toral data for G^*. The choice of Borel subgroups determining this map does not affect our constructions (see Sections 2.3, 2.6 of [LS], in particular). Since we have fixed $spl_{G^\vee}$ we will use $\mathcal{B}$ to specify positivity for roots and coroots of T_1 and T^*. Then, for example, by based χ-data for T^* we will mean the set of χ-data $\{\chi_\alpha\}$, where

$$\chi_\alpha(z) = (z/\overline{z})^{1/2}$$

for α positive imaginary and χ_α is trivial if α is not imaginary. We can also use η^* to define roots *from* H_1, or *outside* H_1 etc. If T is a maximal torus over $\mathbb{R}$ in G and

$$\psi_T = Int(x) \circ \psi : T \to T^*$$

is defined over $\mathbb{R}$ then the composition $\eta = \psi_T^{-1} \circ \eta^*$ defines toral data for G. Then we take $\psi_T(B^*) = B$ as Borel subgroup containing T, and say T *originates in* H_1.

We recall briefly parabolic descent (for endoscopic data) attached to a choice of toral data. Consider a cuspidal Levi group

$$M = M_T = Cent(S_T, G),$$

where T originates in H_1. Choose toral data

$$\eta_{B_1,B} : T_1 \to T_H \to T^* \to T.$$

We may assume that S_{T^*} is contained in $S_{\mathbf{T}^*}$, and set $\psi_M = Int\ g \circ \psi$ acting on T as the inverse of $T^* \to T$. Then ψ_M carries S_T to S_{T^*} and M to

$$M^* = Cent(S_{T^*}, G^*)$$

which will serve as quasisplit inner form for M, with inner twist ψ_M. For splitting spl_{M^*} we use

$$(M^* \cap \mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\}),$$

with the root vectors X_α for simple roots in M^*. Then we realize $M^\vee$ as the $\sigma_{G^\vee}$-invariant Levi group in $G^\vee$ with dual splitting s

$$pl_{M^\vee} = (M^\vee \cap \mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\}).$$

For ${}^LM$ we take $M^\vee \ltimes W_{\mathbb{R}}$, with the action of $W_{\mathbb{R}}$ on $M^\vee$ inherited from ${}^LG$.

Given endoscopic data $(H, \mathcal{H}, \mathfrak{s}, \xi)$ and z-pair (H_1, ξ_1), we may now attach descent data

$$(M_H\ , \mathcal{H}_M, \mathfrak{s}_M, \xi_M)$$

and pair $(M_{H_1}, \xi_{1,M})$ for M. First, we will assume $\mathfrak{s} \in \mathcal{T}$, and then set $\mathfrak{s}_M = \mathfrak{s}$. We have also assumed that ξ is inclusion, so that $\mathcal{H}$ is a subgroup of ${}^LG$. We set $\mathcal{H}_M = \mathcal{H} \cap {}^LM$ and take ξ_M to be inclusion also; $\mathcal{H}_M$ is a split extension of $W_{\mathbb{R}}$ by $M^\vee \cap H^\vee$. For M_H we may take a dual Levi group in H and we choose specifically

$$M_H = Cent(S_{T_H}, H),$$

where T_H is the image of T_1 under $H_1 \to H$. Let $M_{_1}$ be the inverse image of M_H under $H_1 \to H$, so that

$$M_{_1} = Cent(S_{T_1}, H_1)$$

and

$$1 \to Z_1 \to M_{_1} \to M_H \to 1$$

is exact. For embedding $\xi_{1,M}$ of $\mathcal{H}_M$ in ${}^LM_{_1}$ we take the restriction of ξ_1 to $\mathcal{H}_M$. The attached character on $Z_1(\mathbb{R})$ is again λ_1. If $T \preceq T'$ then, replacing T' by a $G(\mathbb{R})$-conjugate if necessary, we assume $S_{T'}$ contains S_T and descend to $M_{T'}$ through M_T when convenient. Finally, toral data for $M_{T'}$ or M_T in this setting serve also as toral data for G.

b. Character data (G-regular case)

We come then to the map $\Phi_{temp}(H_1, \lambda_1) \to \Phi_{temp}(G^*)$. Since we have another temporary use for the notation λ_1 we will switch this one temporarily to $\boldsymbol{\lambda}_1$. Suppose first that G is cuspidal, *i.e.* G contains Cartan subgroups that are compact modulo the center of G, that H_1 is elliptic, *i.e* these Cartan subgroups originate in H_1, and that $\boldsymbol{\varphi}_1 \in \Phi_{temp}(H_1, \boldsymbol{\lambda}_1)$ is discrete, *i.e.* $\boldsymbol{\varphi}_1$ factors through no proper parabolic subgroups of ${}^LH_1$. We will refer to this as the *cuspidal-elliptic-discrete* case from now on. Since we are assuming G-regularity, the image $\boldsymbol{\varphi}^*$ of $\boldsymbol{\varphi}_1$ in $\Phi_{temp}(G^*)$ is also discrete and relevant to G, so that $\boldsymbol{\varphi}$ is welldefined. Choose toral data

$$T_1 \to T^* \to T,$$

where each torus is compact modulo center.

We observe from [L] how to attach to any discrete $\boldsymbol{\varphi}_1$ the data needed for character formulas for the representations in the attached L-packet. First, we use the chosen splitting $spl_{H_1^\vee}$ to fix an essentially unique representative for $\boldsymbol{\varphi}_1$ as follows. Let ι_1 be one half the sum of the positive roots of $\mathcal{T}_1$ in $H_1^\vee$ relative to the splitting. Then there is a representative

$$\varphi_1 = \varphi_1(\mu_1, \lambda_1)$$

for $\boldsymbol{\varphi}_1$ given by

$$\varphi_1(z \times 1) = z^{\mu_1}\overline{z}^{\sigma_{T_1}(\mu_1)}$$

and

$$\varphi_1(1 \times \sigma) = e^{2\pi i \lambda_1} n(\sigma_{T_1}) \times (1 \times \sigma),$$

with $n(\sigma_{T_1})$ the element of $H_1^\vee$ attached by $spl_{H_1^\vee}$ as in [LS] to the Weyl group element $w(\sigma_{T_1})$, where σ_{T_1} acts as $w(\sigma_{T_1})\sigma_{H_1}$. Here

$$\mu_1, \lambda_1 \in X_*(\mathcal{T}_1) \otimes \mathbb{C}.$$

Because $\boldsymbol{\varphi}_1$ is discrete, μ_1 must be regular so that it is uniquely determined once we observe that $< \mu_1, \alpha_1^\vee >$ is integral for all roots $\alpha_1^\vee$ of $H_1^\vee$ (by the next displayed formula) and require that μ_1 be dominant. On the other hand, λ_1 is determined only modulo

$$X_*(\mathcal{T}_1) + \{\nu - \sigma_{T_1}\nu : \nu \in X_*(\mathcal{T}_1) \otimes \mathbb{C}\}$$

by the conjugacy class $\boldsymbol{\varphi}_1$ of φ_1. The crucial property from [L] (see Lemma 3.2) that we have for (μ_1, λ_1) is the following: if ι_1 is one half the sum of the positive roots of $\mathcal{T}_1$ in $H_1^\vee$ then the pair (μ_1', λ_1), where

$$\mu_1' = \mu_1 - \iota_1,$$

satisfies

$$\frac{1}{2}(\mu_1' - \sigma_{T_1}\mu_1') + (\lambda_1 + \sigma_{T_1}\lambda_1) \in X^*(T_1).$$

This provides us with a welldefined Langlands parameter $\varphi_{T_1}(\mu_1 - \iota_1, \lambda_1)$ for the Cartan subgroup $T_1(\mathbb{R})$ and hence a character

$$\Lambda(\mu_1 - \iota_1, \lambda_1)$$

on $T_1(\mathbb{R})$. Then for each w_1 in the Weyl group $\Omega(H_1^\vee, \mathcal{T}_1)$ of $\mathcal{T}_1$ in $H_1^\vee$, the character

$$\Lambda(w_1^{-1}\mu_1 - \iota_1, \lambda_1)$$

is also welldefined. Notice that the Weyl group acts trivially on λ_1 modulo equivalence under

$$X_*(\mathcal{T}_1) + \{\nu - \sigma_{T_1}\nu : \nu \in X_*(\mathcal{T}_1) \otimes \mathbb{C}\}.$$

It is this collection

$$\{\Lambda(w_1^{-1}\mu_1 - \iota_1, \lambda_1)\}$$

of characters, along with the usual Harish Chandra denominator, that we use for the character formulas on $T_1(\mathbb{R})$ of the representations in the L-packet attached to $\boldsymbol{\varphi}_1$.

Next, from the construction of geometric transfer factors, we recall that to χ-data for T_1 and $spl_{H_1^\vee}$ is attached an embedding $\xi_{T_1}$of ${}^LT_1$ in ${}^LH_1$ ([LS], Section 2.6). This gives us then a map

$$\Phi_{temp}(T_1) \to \Phi_{temp}(H_1).$$

Collecting definitions, we see that if we use the based χ-data and denote the attached embedding $\xi_{T_1}^{base}$ then

$$\varphi_1(\mu_1, \lambda_1) = \xi_{T_1}^{base} \circ \varphi_{T_1}(\mu_1 - \iota_1, \lambda_1).$$

Since we want the freedom to change χ-data we now check the effect on our parameters. Of course, for given $\boldsymbol{\varphi}_1$ the pair (μ_1, λ_1) is unchanged. On the other hand, ι_1 will be replaced, as will the denominator in the character formulas (by the denominator to be defined shortly in 7c). Suppose α is positive for $spl_{H_1^\vee}$ and the based χ-datum

$$\chi_\alpha^{base}(z) = (z/\overline{z})^{1/2}$$

is replaced by

$$\chi_\alpha(z) = (z/\overline{z})^{\frac{1}{2}+n_\alpha},$$

where n_α is integral. Set

$$\iota_\chi = \iota_1 + \sum_{\alpha>0} n_\alpha \alpha.$$

Then if ξ_{T_1} is the embedding attached to the new χ-data we see from the definitions that

$$\varphi_1(\mu_1, \lambda_1) = \xi_{T_1} \circ \varphi_{T_1}(\mu_1 - \iota_\chi, \lambda_1).$$

In Section 9 we will observe that the effects of a new choice of χ-data all cancel in our construction of Δ_{II}. At that point we will return to the based χ-data and use the familiar parametrization for the rest of the paper.

Turning now to character formulas for the representations attached to $\boldsymbol{\varphi}^*$ and $\boldsymbol{\varphi}$, we first examine the data from Langlands parameters. We have the two embeddings

$$\xi_1 : \mathcal{H} \to^L H_1,\ \xi : \mathcal{H} \to^L G.$$

While it is true that often ξ_1 is the identity map, and there is no harm in assuming that ξ is inclusion, we have some useful data to gather. We may assume that the image of the parameter $\varphi_1(\mu_1, \lambda_1)$ is contained in $\mathcal{H}$ and then map it to a welldefined representative φ^*. It will not in general be the representative we attach to $spl_{G^\vee}$ as in the last paragraph. Following Section 11 of [S3] we attach a pair (μ^*, λ^*) in

$$(X_*(\mathcal{T}_1) \otimes \mathbb{C})^2$$

(this pair is discussed at length in [S4] when ξ_1 is the identity map). The pair depends on the character $\boldsymbol{\lambda}_1$ on $Z_1(\mathbb{R})$. Notice that because we consider only parameters in $\Phi_{temp}(H_1, \boldsymbol{\lambda}_1)$ we have that both

$$\mu^* + \mu_1, \lambda^* + \lambda_1$$

lie in the subspace $X_*(\mathcal{T}) \otimes \mathbb{C}$ of $X_*(\mathcal{T}_1) \otimes \mathbb{C}$ (see Remark 11.3 in [S3]). Then we see that $\varphi^* = \varphi^*(\mu, \lambda)$, where $\mu = \mu^* + \mu_1$ and $\lambda = \lambda^* + \lambda_1$. Also μ is regular by the G-regularity assumption on $\boldsymbol{\varphi}_1$.

The characters on $T^*(\mathbb{R})$ needed for the local character formulas for the L-packet of $\boldsymbol{\varphi}^*$can be retrieved from μ just as well as from the dominant form in the orbit of μ. These characters are of course just

$$\Lambda(w^{-1}\mu - \iota, \lambda),$$

for w in the full Weyl group. For $\boldsymbol{\varphi}$ we transport these characters to $T(\mathbb{R})$ by the chosen toral data. We list the irreducible representations in the L-packets by real Weyl group cosets. Thus we denote by $\pi^* = \pi^*(1)$ the representation involving $\Lambda(w_0^{-1}\mu - \iota, \lambda)$ for w_0 in the real Weyl group of $T^*(\mathbb{R})$, and more generally, by $\pi^*(w)$ the representation involving the characters

$$\Lambda(w_0^{-1}w^{-1}\mu - \iota, \lambda).$$

We do the same in G, thus defining $\pi(w)$. Then we may define

$$inv(\pi(1), \pi(w))$$

to be the element of $H^1(\Gamma, T_{sc})$ represented by the cocycle

$$\sigma(w_{sc})w_{sc}^{-1},$$

where the image of $w_{sc} \in G_{sc}$ in G acts on T as w. When we pair this element with the image in

$$\pi_0 = \pi_0((T_{ad}^\vee)^\Gamma)$$

of the transport to $T^\vee$ of the endoscopic datum $\mathfrak{s}$ via the given toral data we write the result as

$$< inv(\pi(1), \pi(w)), \mathfrak{s}_{\pi(1)} > .$$

On the geometric side, we write $< inv(\delta, \delta^w), \mathfrak{s}_\delta >$ in the analogous setting.

Remark 7.1 If, in either pairing, w is replaced by $w_1 w$, where w_1 is from H_1 with respect to the given toral data then the value of the pairing is unchanged. This is an easy consequence of an observation of Langlands (see [S5], Propositions 2.1, 3.3). Note that here we use the fact that the Cartan subgroup $T(\mathbb{R})$ is compact modulo the center of $G(\mathbb{R})$.

Remark 7.2 (i) In this same setting (cuspidal-elliptic-discrete) we define w_* in the Weyl group of T^* by requiring that $w_*\mu$ be dominant. Notice first that $\det w_*$ is independent of the choice for $spl_{G^\vee}$.

Remark 7.2 (ii) Also, we may drop the requirement that μ_1 be dominant, *i.e.* allow μ_1 to be arbitrary in its Weyl orbit, but now define both $w_*(G)$ and $w_*(H_1)$. Then the original $\det w_*$ coincides with

$$\det w_*(G)/\det w_*(H_1)$$

since now we have

$$\mu = \mu^* + w_1^{-1}\mu_1' = w_1^{-1}(\mu^* + \mu_1'),$$

where μ_1' is the H_1-dominant form and $w_1 = w_*(H_1)$. Thus

$$w_*(G)w_1^{-1}(\mu^* + \mu_1')$$

is G-dominant, and so w_* coincides with $w_*(G)w_1^{-1}$, giving the claimed result.

Remark 7.2 (iii) It is also evident that we may rewrite $\det w_*$ as the sign of

$$\prod_{\alpha^\vee>0,\ outside\ H^\vee} <\mu, \alpha^\vee>,$$

where the ordering on the roots is that of $spl_{G^\vee}$. For most of the calculations in Section 13 we prefer to work with dominant μ_1 and $\det w_*$.

We consider now arbitrary G-regular $\boldsymbol{\varphi}_1 \in \Phi_{temp}(H_1, \lambda_1)$. Following [L] we choose a cuspidal Levi group M_1 in H_1 such that $\boldsymbol{\varphi}_1$ factors discretely (*i.e.* minimally) through ${}^LM_1$. We have then a discrete parameter $\boldsymbol{\varphi}_1^M$ for M_1. We may assume that M_1 is endoscopic for a cuspidal Levi group M^* in G^* by descent of endoscopic data, as recalled in Section 7a. Let $\boldsymbol{\varphi}^{*M}$ be the parameter for M^* so attached. Because $\boldsymbol{\varphi}_1$ is G-regular, $\boldsymbol{\varphi}^{*M}$ is discrete. On the other hand, we can continue to G, defining Levi group M and discrete parameter $\boldsymbol{\varphi}^M$ if and only if $\boldsymbol{\varphi}^*$ is relevant to G (i.e $\boldsymbol{\varphi}$ is welldefined) or, equivalently, if and only if we can extend given toral data $T_1 \to T^*$ for G^*, where T_1 is compact modulo center in M_1, to toral data $T_1 \to T^* \to T$ for G. Suppose that is the case and choose such toral data for the elliptic cuspidal endoscopic pair (M_1, M). Then we are back in the cuspidal elliptic discrete setting, and so we attach to $\boldsymbol{\varphi}_1^M$ a pair (μ_1, λ_1) with μ_1 dominant for M_1, and to $\boldsymbol{\varphi}^{*M}$ and $\boldsymbol{\varphi}^M$ the pair (μ, λ), where

$$\mu = \mu^* + \mu_1,\ \lambda = \lambda^* + \lambda_1.$$

The pair $(M^*, \boldsymbol{\varphi}^{*M})$ is determined uniquely up to $G^*(\mathbb{R})$-conjugacy by $\boldsymbol{\varphi}_1$, and $(M, \boldsymbol{\varphi}^M)$ up to $G(\mathbb{R})$-conjugacy, in the obvious sense for discrete parameters. Notice that the set of characters $\Lambda(w^{-1}\mu - \iota, \lambda)$, where w is in the subgroup of the Weyl group of T^* in G^* consisting of all elements which commute with σ, accounts for $G^*(\mathbb{R})$-conjugacy as well as $M^*(\mathbb{C})$-conjugacy, and is uniquely determined by $\boldsymbol{\varphi}_1$ and choice of pair (M^*, T^*), and similarly in G.

c. Generalized Weyl denominators

It will be helpful to introduce generalized Weyl denominators in a way that works for any local field of characteristic zero. We did this in [S3] for the geometric side. Now we write Δ'_{right} for the term defined in Section 7 of [S3], and introduce a left version for local character formulas. Motivation is provided by Lemma 7.3 below. Thus, given a maximal torus T over $\mathbb{R}$ in a connected reductive group G over $\mathbb{R}$ and a-data $\{a_\alpha\}$, χ-data $\{\chi_\alpha\}$ for T, we define

$$\Delta'_{left}(\gamma) = \Delta'_{left}(\gamma, \{a_\alpha\}, \{\chi_\alpha\})$$

to be the product

$$\left|\det(I - Ad(\gamma^{-1}))_{\mathfrak{g}/\mathfrak{t}}\right|^{1/2} \prod_{\mathcal{O}} \chi_\alpha(-a_\alpha(1 - \alpha(\gamma)^{-1})).$$

The choice of root α within a Galois orbit $\mathcal{O}$ of roots of T in G does not matter, and the product is over all Galois orbits of roots. If we use based χ-data (for a given ordering on the roots) then $\Delta'_{left}(\gamma, \{a_\alpha\}, \{\chi_\alpha\})$ is a multiple of the usual Harish Chandra denominator

$$\Delta'(\gamma) = \left|\det(I - Ad(\gamma^{-1}))_{\mathfrak{g}/\mathfrak{m}}\right|^{1/2} \prod_{\alpha>0,\ imag} (1 - \alpha(\gamma)^{-1}),$$

namely,

$$\Delta'_{left}(\gamma, \{a_\alpha\}, \{\chi_\alpha\}) = \Big(\prod_{\alpha>0,\ imag} \Big(\frac{-a_\alpha}{|a_\alpha|}\Big)\Big)\ \Delta'(\gamma).$$

In general, our definition for

$$\Delta'_{right}(\gamma, \{a_\alpha\}, \{\chi_\alpha\})$$

is

$$\left|\det(I - Ad(\gamma^{-1}))_{\mathfrak{g}/\mathfrak{m}}\right|^{1/2} \prod_{\mathcal{O}} \chi_\alpha((\alpha(\gamma) - 1)/a_\alpha).$$

In preparation for calculations with the Weyl integration formula we set

$$J_{G/T}(\gamma) = \left|\det(I - Ad(\gamma^{-1}))_{\mathfrak{g}/\mathfrak{t}}\right| = \left|\det(I - Ad(\gamma))_{\mathfrak{g}/\mathfrak{t}}\right|,$$

and observe the following:

Lemma 7.3

$$\Delta'_{left}(\gamma, \{a_\alpha\}, \{\chi_\alpha\}) \Delta'_{right}(\gamma, \{a_\alpha\}, \{\chi_\alpha\}) = J_{G/T}(\gamma)$$

Proof: We have to show

$$\prod_{\mathcal{O}} \chi_\alpha(-a_\alpha(1 - \alpha(\gamma)^{-1})) \prod_{\mathcal{O}} \chi_\alpha((\alpha(\gamma) - 1)/a_\alpha) = 1.$$

If $\mathcal{O}$ is symmetric, then

$$\chi_\alpha(-a_\alpha(1 - \alpha(\gamma)^{-1}))\chi_\alpha((\alpha(\gamma) - 1)/a_\alpha)$$
$$= \chi_\alpha((\alpha(\gamma)^{-1} - 1)(\alpha(\gamma) - 1)) = 1$$

since $\alpha(\gamma)^{-1}-1)$ and $(\alpha(\gamma)-1)$ are complex conjugates and χ_α is trivial on norms. If $\mathcal{O}$ is asymmetric then the two orbits $\pm\mathcal{O}$ contribute

$$\chi_\alpha((\alpha(\gamma)^{-1}-1)(\alpha(\gamma)-1))\chi_{-\alpha}((\alpha(\gamma)-1)(\alpha(\gamma)^{-1}-1)).$$

Since

$$\chi_{-\alpha}=\chi_\alpha^{-1}$$

this equals 1 also.

8. Spectral transfer factor: first term

Throughout Sections 8 - 11, suppose that (π_1,π) and (π_1',π') are G-regular related pairs of tempered irreducible representations. Write $\boldsymbol{\varphi}_1=\boldsymbol{\varphi}_1(\pi_1)$, $\boldsymbol{\varphi}=\boldsymbol{\varphi}(\pi)$ and $\boldsymbol{\varphi}_1'=\boldsymbol{\varphi}_1(\pi_1')$, $\boldsymbol{\varphi}'=\boldsymbol{\varphi}(\pi')$ for the attached Langlands parameters, and $\boldsymbol{\varphi}^*$ and $\boldsymbol{\varphi}'^*$ for the corresponding parameters for the quasisplit form G^*. We choose appropriate toral data, *i.e.* data as in Section 7a, and write T_1, T^*, T, etc., T_1', T'^*, T', etc. for the Cartan subgroups.

As mentioned already, we define

$$\Delta_I(\pi_1,\pi;\pi_1',\pi')=\Delta_I(\varphi_1)/\Delta_I(\varphi_1'),$$

where $\Delta_I(\varphi_1)$ is the term $<\lambda(T_{sc}^*),\,\mathbf{s}_{T^*}>$ introduced in Section 3.2 of [LS] for geometric factors. Here $\lambda(T_{sc}^*)$ is the splitting invariant for T_{sc}^*, an element of $H^1(\Gamma,T_{sc}^*)$, $\mathbf{s}_{T^*}$ is the element of $\pi_0((T_{ad}^\vee)^\Gamma)$ obtained by transport and projection of the endoscopic datum $\mathfrak{s}$, and $<,>$ denotes the Tate-Nakayama pairing. We also write $\Delta_I(\pi_1,\pi)$ in place of $\Delta_I(\varphi_1)$ when convenient.

To examine the dependence of $\Delta_I(\varphi_1)$ on spl_{G^*}, toral data and a-data we will refer directly to Section 3.2 of [LS]. First, notice that the calculations are done in the quasi-split form, there denoted G in place of our present G^*.

The splitting spl_{G^*} may be replaced by its conjugate under $g\in G_{sc}^*(\mathbb{C})$, where $g\sigma(g)^{-1}$ lies in the center $Z_{sc}(\mathbb{C})$. Then g defines an element $\mathbf{g}_{T^*}$ with which we may pair $\mathbf{s}_{T^*}$. Lemma 3.2.A shows that $\Delta_I(\varphi_1)$ is multiplied by $<\mathbf{g}_{T^*},\mathbf{s}_{T^*}>$ and then that this factor is independent of T^*, i.e. $\Delta_I(\varphi_1')$ is multiplied by the same number, so that the quotient is independent of the choice of spl_{G^*}.

The effect of a change in toral data, with a related change in a-data, is given in Lemma 3.2.B. We will return to this explicitly in our discussion of Δ_{III}.

We come then to the choice of a-data. Suppose $\{a_\alpha\}$ is replaced by $\{a_\alpha'\}$. Then $b_\alpha=a_\alpha'/a_\alpha$ lies in $\mathbb{R}^\times$, and Lemma 3.2.C shows that $\Delta_I(\varphi_1)$ is replaced by

$$\Delta_I(\varphi_1)\prod_{\mathcal{O}}\chi_\alpha(b_\alpha)\;=\Delta_I(\varphi_1)\prod_{\mathcal{O}}sign(b_\alpha).$$

Here the product is over symmetric Galois orbits, i.e. pairs $\pm\alpha$ of imaginary roots, of G^* (or G) outside H_1. The choice of representative α for $\mathcal{O}$ does not matter since $b_{-\alpha}=b_\alpha$. We could also include the asymmetric orbits in the product since each such pair $\pm\mathcal{O}$ contributes

$$\chi_\alpha(b_\alpha)\chi_{-\alpha}(b_{-\alpha})=1.$$

9. Spectral transfer factor: second term

Here we define

$$\Delta_{II}(\varphi_1, \varphi) = \Delta_{II}(\pi_1, \pi)$$

and then set

$$\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi') = \Delta_{II}(\pi_1, \pi)/\Delta_{II}(\pi_1', \pi').$$

The term $\Delta_{II}(\varphi_1, \varphi)$ comes from normalizing stable tempered distributions on G^* and H_1. We attach toral data and start with based χ-data. The a-data can be arbitrary.

We now write $\Delta_{II}(\varphi_1, \varphi)$ as $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ until we are done with choosing representatives. We start with the cuspidal-elliptic-discrete case. Then T_1 is compact modulo the center of H_1 and T^* is compact modulo the center of G^* and the G-regularity of $\boldsymbol{\varphi}_1$ requires that $\boldsymbol{\varphi}^*$, $\boldsymbol{\varphi}$ are also discrete parameters. Suppose $\varphi_1(\mu_1, \lambda_1) \in \boldsymbol{\varphi}_1$. Recall that (μ_1, λ_1) are the data for a character $\Lambda(\mu_1 - \iota_1, \lambda_1)$ on $T_1(\mathbb{R})$ and regular μ_1 is assumed dominant for convenience. Then the toral data and based χ-data also attach to φ^* the data (μ, λ), where $\mu = \mu_1 + \mu^*$ and $\lambda = \lambda_1 + \lambda^*$. The form μ is regular but not necessarily dominant for the chosen toral data, and the characters $\Lambda(w^{-1}\mu - \iota, \lambda)$, for w in the complex Weyl group of T^*, are well-defined.

The stable tempered character attached to discrete $\boldsymbol{\varphi}^*$ is a multiple of Harish Chandra's stable distribution Θ^* which may be described in terms of the regular data (μ, λ). On the regular set of $T^*(\mathbb{R})$ the formula for Θ^* may be written as

$$\Theta^*(\delta) = \frac{\sum_w \ (\det w) \ \Lambda(w^{-1}\mu \ - \ \iota, \ \lambda)(\delta)}{\det w_* \prod_{\alpha>0}(1 \ - \ \alpha(\delta)^{-1})},$$

where w_* is the Weyl element we apply to μ to obtain a dominant form, and of course the summation is over the full (complex) Weyl group. We replace the denominator by the generalized Weyl denominator from Section 7c to obtain the multiple

$$\frac{\sum_w \ (\det w) \ \Lambda(w^{-1}\mu \ - \ \iota, \ \lambda)(\delta)}{\Delta'_{left}(\delta, \{a_\alpha\}, \{\chi_\alpha\})}$$

of $\Theta^*(\delta)$. Since we are using the based χ-data, all we have done is to divide Θ^* by a factor $(\det w_*)(\pm i)^n$, where n is the number of positive roots. Recall Remark 7.2 regarding $\det w_*$. Note that because the a-data are arbitrary we have to allow

$$a_\alpha / |a_\alpha| = \pm i,$$

for each positive root α.

Now we consider arbitrary χ-data in the definition of $\Theta^*(\gamma, \{a_\alpha\}, \{\chi_\alpha\})$. Recall that χ-data is involved in the needed characters on $T^*(\mathbb{R})$, and so a change affects both numerator and denominator. We will see, however, that the effects cancel. Recall the linear form ι_χ attached to $\{\chi_\alpha\}$ in Section 7b: $\iota - \iota_\chi$ is an integral linear combination of the roots of T^*. Then, for each w in the complex Weyl group of T^*,

$$\Lambda(w^{-1}\mu_\chi - \iota_\chi, \lambda)$$

is a welldefined character on $T^*(\mathbb{R})$, and we may define

$$\Theta^*(\delta, \{a_\alpha\}, \{\chi_\alpha\}) = \frac{\sum_w \ (\det w)\ \Lambda(w^{-1}\mu \ - \ \iota_\chi,\ \lambda)(\delta)}{\Delta'_{left}(\delta, \{a_\alpha\}, \{\chi_\alpha\})}.$$

Lemma 9.1 *If both* $\{\chi_\alpha\}$ *and* $\{\chi'_\alpha\}$ *are* χ*-data for* T^* *then*

$$\Theta^*(\delta, \{a_\alpha\}, \{\chi_\alpha\}) = \Theta^*(\delta, \{a_\alpha\}, \{\chi'_\alpha\})$$

for all regular δ *in* $T^*(\mathbb{R})$ *and* a*-data* $\{a_\alpha\}$ *for* T^*.
Proof: We may assume $\{\chi'_\alpha\}$ are the based data, so that $\iota_{\chi'} = \iota$. Then because

$$\mu_\chi + \iota_\chi = \mu + \iota$$

the quotient of the two numerators is $(\iota - \iota_\chi)(\delta)$. Let $\alpha > 0$. Then

$$(\chi_\alpha/\chi'_\alpha)(z) = (z/\overline{z})^{n_\alpha},$$

where n_α is an integer. Recall that

$$\iota - \iota_\chi = -\sum_\alpha n_\alpha \alpha.$$

Thus the contribution to $(\iota - \iota_\chi)(\delta)$ from α is $(\alpha(\delta))^{-n_\alpha}$. On the other hand, the quotient of the denominators is

$$\prod_\alpha (z_\alpha/\overline{z}_\alpha)^{n_\alpha},$$

where

$$z_\alpha = -a_\alpha(1 - \alpha(\delta)^{-1})$$

so that

$$z_\alpha/\overline{z}_\alpha = -(1 - \alpha(\delta)^{-1})/(1 - \alpha(\delta)) = \alpha(\delta)^{-1},$$

and we are done. Notice the role of the a-datum: $a_\alpha/\overline{a}_\alpha = -1$ provides a crucial sign.
Lemma 9.2 *If both* $\{a_\alpha\}$ *and* $\{a'_\alpha = a_\alpha b_\alpha\}$ *are* a*-data for* T^* *then*

$$\Theta^*(\delta, \{a'_\alpha\}, \{\chi_\alpha\}) = \prod_{\mathcal{O}} \chi_\alpha(b_\alpha)^{-1} \ \ \Theta^*(\delta, \{a_\alpha\}, \{\chi_\alpha\})$$

for all regular δ *in* $T^*(\mathbb{R})$ *and* χ*-data* $\{\chi_\alpha\}$ *for* T^*.
Proof: This is immediate since only the Weyl denominator depends on a-data.

In conclusion, whatever the choice of χ-data, the formula for the distribution $\Theta^*(\delta, \{a_\alpha\}, \{\chi_\alpha\})$ determines the same multiple of the Harish Chandra distribution Θ^*. We now write $\Theta^*(-, \{a_\alpha\})$ for this distribution and its local formula.

Let $\chi_{\boldsymbol{\varphi}^*}$ denote the stable character attached to $\boldsymbol{\varphi}^*$. Then we define the number $\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})$ by

$$\chi_{\boldsymbol{\varphi}^*} = \upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})\ \Theta^*(-, \{a_\alpha\}).$$

As our notation suggests, a change in toral data does not change $\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})$ (see the next proof). Next we do the same in H_1 for $\boldsymbol{\varphi}_1$. Given a-data for T^* we will, as always, use the a-data for T_1 obtained by transport under the toral data, using the same notation. Then we obtain $\Theta_1^*(-, \{a_\alpha\})$ and $\upsilon(\boldsymbol{\varphi}_1, \{a_\alpha\})$ such that

$$\chi_{\boldsymbol{\varphi}_1} = \upsilon(\boldsymbol{\varphi}_1, \{a_\alpha\})\Theta_1^*(-, \{a_\alpha\}).$$

We set

$$\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}^*) = \upsilon(\boldsymbol{\varphi}_1, \{a_\alpha\}/\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\}).$$

We recall (from [S6]) that

$$\chi_{\boldsymbol{\varphi}} = (-1)^{q_G - q_{G^*}}\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})\Theta^*(-, \{a_\alpha\}),$$

where $\Theta^*(-, \{a_\alpha\})$ is identified with its transport to $G(\mathbb{R})$ by the chosen toral data. Thus we set

$$\upsilon(\boldsymbol{\varphi}, \{a_\alpha\}) = (-1)^{q_G - q_{G^*}}\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})$$

to obtain

$$\chi_{\boldsymbol{\varphi}} = \upsilon(\boldsymbol{\varphi}, \{a_\alpha\})\Theta^*(-, \{a_\alpha\}).$$

Observe that Lemmas 9.1 and 9.2 remain true when G^*, T^* are replaced by G, T using our toral data. We set

$$\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}) = \upsilon(\boldsymbol{\varphi}_1, \{a_\alpha\}/\upsilon(\boldsymbol{\varphi}, \{a_\alpha\}).$$

Lemma 9.3 *(i) $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ is independent of the choice of toral data and splitting for $G^\vee$, (ii) if a-data $\{a_\alpha\}$ is replaced by $\{a'_\alpha = a_\alpha b_\alpha\}$ then $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ is multiplied by*

$$\prod_{\mathcal{O}} \chi_\alpha(b_\alpha)^{-1} = \prod_{\mathcal{O}} sign(b_\alpha),$$

where the product is over all Galois orbits of roots of T^* outside H_1, and (iii) we have

$$\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}) = (-1)^{q_{G^*} - q_G}\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}^*).$$

Proof: If we change the splitting but not the toral data then there is no change in the terms in $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ since any two Γ-splittings of $G^\vee$ are conjugate by an element fixed by Γ ([L], Lemma 2.6). Also, all terms in $\Theta^*(\gamma, \{a_\alpha\})$ are unchanged when toral data are modified by a conjugation defined over $\mathbb{R}$, and $\chi_{\boldsymbol{\varphi}}(\gamma)$ is unchanged since $\chi_{\boldsymbol{\varphi}}$ is, by construction, stable. The same is true in H_1, and so (i) follows. Our requirement on a-data for H_1 ensures (ii), and (iii) is immediate.

Remark 9.4 To write $\Theta^*(-, \{a_\alpha\})$ as a sum of distributions attached to the irreducible representations contributing to Θ^* we set

$$\Theta_w(\delta, \{a_\alpha\}) = \frac{\sum_{w_0}(\det ww_0)\Lambda(w_0^{-1}w^{-1}\mu \;-\; \iota,\; \lambda)(\delta)}{\Delta'_{left}(\delta, \{a_\alpha\}, \{\chi_\alpha\})},$$

where the summation is over real Weyl group elements w_0 while w is a fixed element in the full Weyl group. Then $\upsilon(\boldsymbol{\varphi}^*, \{a_\alpha\})$ plays the same role as in the stable case, *i.e.* if $\pi = \pi(w)$ in the notation of Section 7b then

$$\chi_{\boldsymbol{\pi}} = \upsilon(\boldsymbol{\varphi}, \{a_\alpha\})\Theta_w(-, \{a_\alpha\}).$$

This is used in the main calculation in Section 13.

For a concrete formula we recall that

$$\varphi(\pi_1) = \varphi_1(\mu_1, \lambda_1) \text{ } and \text{ } \varphi(\pi) = \varphi(\mu, \lambda),$$

where

$$\mu = \mu^* + \mu_1, \lambda = \lambda^* + \lambda_1.$$

We will no longer require μ_1 dominant.

Lemma 9.5

$$\Delta_{II}(\pi_1, \pi) = (-1)^{q_{H_1} - q_G} \prod_{\alpha^\vee > 0, \text{ } outside \text{ } H^\vee} \frac{-|a_\alpha| \text{ } sign < \mu, \alpha^\vee >}{a_\alpha}$$

Proof: We may as well compute with based χ-data. Then we see that

$$\upsilon(\boldsymbol{\varphi}, \{a_\alpha\}) = (-1)^{q_G} \det w_*(G) \prod_{\alpha^\vee > 0} (-a_\alpha / |a_\alpha|),$$

in the notation of Remark 7.2, and so the lemma follows from (ii) and (iii) of Remark 7.2.

We now consider the general G-regular case. We factor $\boldsymbol{\varphi}_1$ minimally through the L-group ${}^L M_1$ of a standard cuspidal Levi group M_1 in H_1, obtaining a discrete parameter $\boldsymbol{\varphi}_1^M$. Attaching toral data as in Section 7a, we then obtain a standard cuspidal Levigroup M^* in G^* and M in G such that $\boldsymbol{\varphi}^*, \boldsymbol{\varphi}$ factor through ${}^L M$. Because $\boldsymbol{\varphi}_1$ is G-regular, we again obtain discrete parameters $\boldsymbol{\varphi}^{*M}, \boldsymbol{\varphi}^M$. We consider the related pair $(\boldsymbol{\varphi}_1^M, \boldsymbol{\varphi}^M)$ for the endoscopic group M_1 for M. Both terms $\upsilon(\boldsymbol{\varphi}^M, \{a_\alpha\})$ and $\upsilon(\boldsymbol{\varphi}_1^M, \{a_\alpha\})$ are well-defined, and we set

$$\Delta_{II}(\boldsymbol{\varphi}_1) = \Delta_{II}^M(\boldsymbol{\varphi}_1^M) = \upsilon(\boldsymbol{\varphi}_1^M, \{a_\alpha\}) / \upsilon(\boldsymbol{\varphi}^M, \{a_\alpha\}).$$

This is independent of the choice of factoring, and the three assertions of Lemma 9.3 remain true. As noted at the end of Section 8, the asymmetric orbits of roots, i.e. those orbits outside M, contribute trivially to the product in (ii). For (iii) we recall that

$$(-1)^{q_{M^*} - q_M} = (-1)^{q_{G^*} - q_G}$$

[S6].

Notice that because we make definitions in M we ignore a-data for the asymmetric orbits. We could just as well make our definition in terms of an induced stable character on $G(\mathbb{R})$ that included all a-data and χ-data. To show that the choice of χ-data does not matter, *i.e.* that the extra terms in the Weyl denominator cancel with the change in the character formula, we observe that Lemma 3.5.A of [LS] measures the change in the geometric factor Δ_{III_2}(take $\varsigma_\alpha = \chi_\alpha$ for asymmetric orbits and $\varsigma_\alpha = 1$ for symmetric orbits) which changes the numerator in the character formula. This provides the cancellation with the extra terms. Then Lemma 9.3 again applies and yields the same result as we have now since asymmetric orbits contribute trivially to the product in part (ii) of the lemma.

Remark 9.6 The formula of Lemma 9.5 remains true provided we adjust the q-sign and the product is taken over positive *imaginary* roots outside H_1.

This completes the definition of $\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')$ in the general G-regular case. We conclude now:

Lemma 9.7 *The product*

$$\Delta_I(\pi_1, \pi; \pi_1^{'}, \pi')\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')$$

is independent of the choice of a-data.

Recall that Δ_I depends also on the choice of toral data while Δ_{II} does not. Finally, we have that $\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')$ is simply a sign:

Lemma 9.8

$$\Delta_{II}(\pi_1, \pi; \pi_1^{'}, \pi')^2 = 1.$$

Proof: We use based a-data (and χ-data) to compute $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ as in Remark 9.4. All we have do now is to observe that as we apply a Cayley transform with respect to a root α from H_1 we change the number of positive imaginary roots among the roots outside H_1 by an even number: if $\pm\beta$ are imaginary roots outside H_1 so also are $\pm w_\alpha(\beta)$. So imaginary roots outside H_1 can change only to complex pairs up to sign. Then as we change $\boldsymbol{\varphi}_1$ we change $\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ by at most a sign, *i.e.*

$$\Delta_{II}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})/\Delta_{II}(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}')$$

is a sign at each step in a sequence of Cayley transforms, and the lemma is proved.

10. Spectral transfer factor: third term

Again we start with just the cuspidal-elliptic-discrete setting although the general G-regular case takes an analogous form. We have chosen two sets of toral data, one for (π_1, π), and one for $(\pi_1^{'}, \pi')$. We start with representative φ_1 for $\boldsymbol{\varphi}_1$ and use the toral data to produce T^*-data (μ, λ) for representation π^* as in Section 7b. Also π determines T-data (μ_π, λ_π), with μ_π regular, uniquely up real Weyl group conjugacy. Similarly, π' determines $T^{*\prime}$-data (μ', λ') and T'-data $(\mu_{\pi'}, \lambda_{\pi'})$.

We proceed now as in Section 3.4 of [LS] for the geometric term Δ_{III_1}. Recall that $\psi : G \to G^*$ is our chosen inner twist and that we have fixed $u(\sigma)$ in G^*_{sc} so that $\psi\sigma(\psi)^{-1} = Int\ u(\sigma)$. We may find $g, g' \in G^*_{sc}$ such that $Int\ g \circ \psi$ transports T to T^* over $\mathbb{R}$ and (μ_π, λ_π) to (μ, λ), while $Int\ g' \circ \psi$ transports T' to $T^{*\prime}$ over $\mathbb{R}$ and $(\mu_{\pi'}, \lambda_{\pi'})$ to (μ', λ'). Then

$$v(\sigma) = gu(\sigma)\sigma(g)^{-1}$$

lies in T^*_{sc}, and

$$v'(\sigma) = g'u(\sigma)\sigma(g')^{-1}$$

lies in $T^{*\prime}_{sc}$ and as cochains of Γ they are uniquely determined up to coboundaries because of the G-regularity of φ_1 and φ_1'. Moreover, the coboundary of each coincides with the coboundary of $u(\sigma)$, all three taking values in the center Z^*_{sc} of G^*_{sc}. We define the torus $U = U(T^*, T^{*\prime})$ as the quotient of $T^*_{sc} \times T^{*\prime}_{sc}$ by $\{(z^{-1}, z);\ z \in Z^*_{sc}\}$. Then

$$(v(\sigma)^{-1}, v'(\sigma))$$

is a 1-cocycle of Γ in U. We write

$$inv(\frac{\pi_1, \pi}{\pi_1', \pi'})$$

for its class in $H^1(\Gamma, U)$. As described in [LS], the endoscopic datum $\mathfrak{s}$ and the toral data determine an element $\mathbf{s}_U$ in the component group $\pi_0((U^\vee)^\Gamma)$ of the Galois invariants in the dual torus $U^\vee$. Now we use the Tate-Nakayama pairing to define

$$\Delta_{III}(\pi_1, \pi; \pi_1', \pi') = \ < inv(\frac{\pi_1, \pi}{\pi_1', \pi'}),\ \mathbf{s}_U > .$$

Notice that if G is quasisplit and ψ is the identity, so that $u(\sigma) = 1$, we may instead define $inv(\pi_1, \pi)$ in $H^1(\Gamma, T_{sc})$ and pair it with $\mathbf{s}_T$. Then

$$\Delta_{III}(\pi_1, \pi; \pi_1', \pi') = \ < inv(\pi_1, \pi), \mathbf{s}_T >^{-1} < inv(\pi_1', \pi'), \mathbf{s}_{T'} >,$$

which coincides with the previous definition.

A change in toral data replaces g by hg, where $h\sigma(h)^{-1}$ lies in T^*_{sc} and so defines an element $\mathbf{h}_{T^*}$ of $H^1(\Gamma, T^*_{sc})$, and we similarly obtain an element $\mathbf{h}_{T^{*\prime}}$ of $H^1(\Gamma, T^{*\prime}_{sc})$ from the second set of toral data. Then, as in Lemma 3.4.A of [LS], $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ is multiplied by

$$< \mathbf{h}_{T^*}, \mathbf{s}_{T^*} > \ \ < \mathbf{h}_{T^{*\prime}}, \mathbf{s}_{T^{*\prime}} >^{-1} .$$

On the other hand, Lemma 3.2.B shows that this cancels with the change in $\Delta_I(\pi_1, \pi; \pi_1', \pi')$.

Now we drop the assumption that G is cuspidal and H is elliptic. Thus the Cartan subgroups $T^*, T^{*\prime}$are now arbitrary. We again define $inv(\frac{\pi_1, \pi}{\pi_1', \pi'})$ in $H^1(\Gamma, U)$, where $U = U(T^*, T^{*\prime})$ as above. Because φ_1 is G-regular, π is an irreducible unitary principal series representation, and so the internal structure of the inducing L-packet is all we need. More precisely,we factor φ_1 minimally through the L-group ${}^LM_1$ of a standard cuspidal Levi group M_1 in H_1, obtaining a discrete parameter φ_1^M. For the given toral data, we obtain a standard cuspidal Levi group M^* in G^* such that φ^* factors through ${}^LM^*$. Because φ_1 is G-regular, we obtain discrete parameters φ^{*M} for $M^*$and φ^M for M. Although we need not insist that $(\varphi_1^M, \varphi^{*M})$ is a related pair for the endoscopic group M_{H_1} of M^*, we may attach data (μ, λ) to φ^{*M} as in the first paragraph, and attach (μ_π, λ_π) to the inducing representation π_M for π with parameter φ^M as in Section 7b, and we again may again define the cochain $v(\sigma)$. Similarly we have $v'(\sigma)$, and again $inv(\frac{\pi_1, \pi}{\pi_1', \pi'})$ is a welldefined element. We then define $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ as in the cuspidal elliptic case, with the same remark for the quasisplit case. The lemmas

from [LS] used in the discrete case again apply, and so we conclude:
Lemma 10.1 *The product*

$$\Delta_I(\pi_1,\pi;\pi_1^{'},\pi')\Delta_{III}(\pi_1,\pi;\pi_1^{'},\pi')$$

is independent of the choice of toral data.

We will also remark the analogue of the transitivity property of geometric transfer factors ([LS], Lemma 4.1.A). It follows from the next lemma which can be argued as in the geometric case.
Lemma 10.2

$$\Delta_{III}(\pi_1,\pi;\pi_1^{'},\pi')\Delta_{III}(\pi_1',\pi';\pi_1^{"},\pi^{"}) = \Delta_{III}(\pi_1,\pi;\pi_1^{"},\pi^{"}).$$

11. Canonical spectral transfer factor

We now gather up the results of last three sections to conclude that the relative spectral transfer factor is canonical.

Theorem 11.1 *Suppose that (π_1,π) and (π_1',π') are G-regular related pairs of tempered irreducible representations. Then*

$$\Delta(\pi_1,\pi;\pi_1^{'},\pi') = \Delta_I(\pi_1,\pi;\pi_1^{'},\pi')\Delta_{II}(\pi_1,\pi;\pi_1^{'},\pi')\Delta_{III}(\pi_1,\pi;\pi_1^{'},\pi')$$

is independent of choices made during the construction of $\Delta_I,\Delta_{II},\Delta_{III}$. Moreover,

$$\Delta(\pi_1,\pi;\pi_1^{'},\pi')^2 = 1,$$

and if $(\pi_1^{"},\pi^{"})$ is also a G-regular related pair then

$$\Delta(\pi_1,\pi;\pi_1^{'},\pi')\Delta(\pi_1',\pi';\pi_1^{"},\pi^{"}) = \Delta(\pi_1,\pi;\pi_1^{"},\pi^{"}).$$

In general, we will normalize the spectral transfer factor by choice of related pair. Thus we fix some G-regular related pair $(\overline{\pi}_1,\overline{\pi})$ and choose $\Delta(\overline{\pi}_1,\overline{\pi})$ arbitrarily. Then

$$\Delta(\pi_1,\pi) = \Delta(\overline{\pi}_1,\overline{\pi})\Delta(\pi_1,\pi;\overline{\pi}_1,\overline{\pi}).$$

In the case G is quasisplit and ψ is the identity there are two normalizations that we may write without explicit mention of a chosen related pair. They both exploit the fact that Δ_{III} is a quotient in this case. Thus, in the notation of Sections 8 - 10, set

$$\Delta_0(\pi_1,\pi) = \Delta_I(\pi_1,\pi)\Delta_{II}(\pi_1,\pi) < inv(\pi_1,\pi),\mathbf{s}_T >^{-1} .$$
$$= \Delta_{II}(\pi_1,\pi) < \lambda(T_{sc})inv(\pi_1,\pi)^{-1},\mathbf{s}_T > .$$

We may rewrite this as

$$\Delta_0(\pi_1,\pi) = \Delta_{I+}(\pi_1,\pi)\Delta_{II}(\pi_1,\pi),$$

where

$$\Delta_{I+}(\pi_1, \pi) = \ < \lambda(T_{sc}) inv(\pi_1, \pi)^{-1}, \mathbf{s}_T > .$$

Lemma 11.2 *For given (π_1, π), the product $\Delta_0(\pi_1, \pi)$ depends only on the choice of an $\mathbb{R}$-splitting of G (through $\Delta_{I+}(\pi_1, \pi)$), while $\Delta_{I+}(\pi_1, \pi)$ and $\Delta_{II}(\pi_1, \pi)$ each depend on the choice of a-data as well but not on the choice of toral data.*
Proof: Here is what affects the definition of each term in $\Delta_0(\pi_1, \pi)$. For

$$\Delta_I(\pi_1, \pi) = \ < \lambda(T_{sc}), \mathbf{s}_T >:$$

an $\mathbb{R}$-splitting, toral data, a-data. For $\Delta_{II}(\pi_1, \pi)$: a-data. For $< inv(\pi_1, \pi), \mathbf{s}_T >$: toral data. We have seen that the effects of changing toral data or a-data cancel appropriately. Thus the lemma is proved.

The effect of a change in $\mathbb{R}$-splitting on $\Delta_I(\pi_1, \pi)$, and hence on $\Delta_{I+}(\pi_1, \pi)$, was noted in Section 8.

Lemma 11.3 *(i)* $\Delta_0(\pi_1, \pi)$ *is a spectral transfer factor. (ii)* $\Delta_0(\pi_1, \pi)^4 = 1$.
Proof: Choose any G-regular tempered related pair $(\overline{\pi}_1, \overline{\pi})$ and set

$$\Delta(\overline{\pi}_1, \overline{\pi}) = \Delta_0(\overline{\pi}_1, \overline{\pi}).$$

Since

$$\Delta_0(\pi_1, \pi) = \Delta_0(\overline{\pi}_1, \overline{\pi}) \Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi})$$

by the factoring of $\Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi})$ in Section 10, we are done with (i). For (ii), both $\Delta_0(\pi_1, \pi)$ and $< inv(\pi_1, \pi), \mathbf{s}_T >$ are signs. We observed in Section 9 that a simple calculation shows that $\Delta_{II}(\pi_1, \pi)^4 = 1$, and so (ii) is proved.

Since

$$\Delta_0(\pi_1, \pi) / \Delta_0(\overline{\pi}_1, \overline{\pi}) = \Delta(\pi_1, \pi; \overline{\pi}_1, \overline{\pi}) = \pm 1$$

we may multiply $\Delta_0(\pi_1, \pi)$ by a constant to obtain a spectral transfer factor that takes just the values ± 1.

We introduce some temporary notation. Consider Whittaker data (B, λ) for G: B is a Borel subgroup of G defined over $\mathbb{R}$ and λ is a generic character on $N(\mathbb{R})$, where N is the unipotent radical of B. In Section 5.3 of [KS] the geometric transfer factor with Whittaker normalization $\Delta_\lambda(\gamma_1, \delta)$ was introduced in a general twisted setting. Although defined as a multiple of $\Delta_0(\gamma_1, \delta)$, it was shown to depend only on the choice of Whittaker data (B, λ).

We follow the same procedure now to define a Whittaker normalization for the spectral factor. Given Whittaker data (B, λ) we choose an $\mathbb{R}$-splitting with B as its Borel subgroup and an additive character ψ on $\mathbb{R}$ (temporary notation again) such that λ is the generic character on $N(\mathbb{R})$ attached to ψ by the splitting in the usual manner. We use the splitting to specify the spectral transfer factor $\Delta_0(\pi_1, \pi)$. Now choose toral data for a maximally split torus T_1 in H_1, and set $T_H = T_1/Z_1$. Then

$$\Delta_\lambda(\pi_1, \pi) = \varepsilon_L(V, \psi) \Delta_0(\pi_1, \pi),$$

where the subscript L indicates the Langlands normalization and V is a virtual representation $V_G - V_H$ of the Galois group Γ. First, the space V_H is $X^*(T_H) \otimes \mathbb{C}$ which we identify with $V_G = X^*(T) \otimes \mathbb{C}$ using the toral data. The action of $\tau \in \Gamma$

on V_H is by $\tau_H = \tau_{T_H}$ while its action on V_G is by τ_G which is not the same as $\tau_T = \tau_{T_H}$ unless the maximally split tori in $H = H_1/Z_1$ are maximally split in G. The choice of toral data does not matter as a change does not change the equivalence class of the representations, and now the argument that $\Delta_\lambda(\pi_1, \pi)$ depends only on the Whittaker data is the same as that for the geometric case (pp. 65, 66 of [KS]).

Lemma 11.4 *For all G-regular related pairs of tempered irreducible representations (π_1, π) we have*

$$\Delta_\lambda(\pi_1, \pi)^2 = 1.$$

Proof: We will show that for every G-regular parameter φ_1 in $\Phi_{temp}(H_1, \lambda_1)$, we have

$$\Delta_{II}(\varphi_1, \varphi)^2 = \varepsilon(V, \psi)^2.$$

Thus we have to show that when we normalize Harish Chandra's distribution Θ^* for the cuspidal Levi groups in H_1 and G, the results on corresponding groups in H_1 and G differ by a sign up to this fixed ε-factor. By Lemma 9.8 we just have to show this for a single parameter φ_1. We choose G-regular φ_1 factoring through ${}^L T_1$, i.e. associated to a minimal unitary principal series representation of $H_1(\mathbb{R})$, and apply Lemma 9.5 to $M_T = Cent(S_T, G)$. Because T_1 is maximally split in H_1, all imaginary roots of T in G are outside H_1. Thus we conclude that

$$\Delta_{II}(\varphi_1)^2 = (-1)^n,$$

where n is the number of positive imaginary roots of T in G.

To compute $\varepsilon(V, \psi)^2$ we note as in [KS] that it is $(\det V)(-1)$, where $\det V = \det[V_G - V_H]$ is regarded as a character on $\mathbb{R}^\times$. Now we observe that because the imaginary roots of T all lie outside H_1 (if they exist), the sum of any two of their coroots cannot be a coroot. Thus they must form a system of type $(A_1)^n$. Consider the positive roots among them. None of these roots can be (totally) compact since G is quasisplit, and so to pass from T to a maximally split torus in G we must apply n Cayley transforms with respect to these roots. The rest is elementary. Notice that the space V_G therefore has subspace V_1 with the following properties: V_1 has dimension n; V_1 is invariant under both σ_H and σ_G; σ_H acts on V_1as $-I$ while σ_G acts as I; and σ_H and σ_G have the same action on V_G/V_1. We conclude then that $\det[V_G - V_H] = (-1)^n$ and so

$$\varepsilon(V, \psi)^2 = (-1)^n = \Delta_{II}(\varphi_1, \varphi)^2,$$

as claimed. We could also argue as at the end of Section 5.2 of [KS]. The lemma is thus proved.

12. Canonical compatibility factor

Now we take up the plan outlined in Section 4. Given G-regular pair (π_1, π) of tempered irreducible representations and strongly G-regular pair (γ_1, δ) of $\mathbb{R}$-rational points, we define a compatibility factor

$$\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$$

as the product of three terms:

$$\Delta_I(\pi_1,\pi)/\Delta_I(\gamma_1,\delta)$$

$$\Delta_{II}(\pi_1,\pi)/\Delta_{II+}(\gamma_1,\delta)$$

$$\Delta_{III}(\pi_1,\pi;\gamma_1,\delta).$$

Only the last term has yet to be defined. We return to the definition of the relative terms

$$\Delta_{III}(\pi_1,\pi;\pi_1',\pi')$$

and

$$\Delta_{III}(\gamma_1,\delta;\gamma_1',\delta').$$

Notice that once we have chosen toral data they are defined in the same way with a cocycle

$$(v(\sigma)^{-1},v'(\sigma))$$

of Γ in $U=U(T,T')$. Now we take $v(\sigma)$ to be the cochain attached to (π_1,π) and $v'(\sigma)$ to be the cochain attached to (γ_1,δ), and so define

$$inv(\frac{\pi_1,\pi}{\gamma_1,\delta})$$

in $H^1(\Gamma,U)$. Then

$$\Delta_{III}(\pi_1,\pi;\gamma_1,\delta)= \quad < inv(\frac{\pi_1,\pi}{\gamma_1,\delta}),s_U > .$$

The first two quotients are so written to exploit the fact that numerator and denominator behave the same way under change of data used in the definitions: $\mathbb{R}$-splitting of G^*, toral data and a-data. Thus we conclude:

Lemma 12.1 *(i) the compatibility factor*

$$\Delta_{comp}(\pi_1,\pi;\gamma_1,\delta)$$

is independent of all choices, (ii) there is right transitivity with geometric factors:

$$\Delta_{comp}(\pi_1,\pi;\gamma_1',\delta')=\Delta_{comp}(\pi_1,\pi;\gamma_1,\delta)\Delta_{geom}(\gamma_1,\delta;\gamma_1',\delta'),$$

(iii) there is left transitivity with spectral factors

$$\Delta_{spec}(\pi_1,\pi;\pi_1',\pi')\Delta_{comp}(\pi_1',\pi';\gamma_1,\delta)=\Delta_{comp}(\pi_1,\pi;\gamma_1,\delta),$$

and (iv) if G is quasisplit and the inner twist ψ is the identity then

$$\Delta_{comp}(\pi_1,\pi;\gamma_1,\delta)=\Delta_0(\pi_1,\pi)/\Delta_0(\gamma_1,\delta).$$

Proof: For (i): compare definitions. For (ii), (iii) the crucial fact is the transitivity of the relative term, which is a rewording of Lemma 4.1.A of [S3]. Finally, (iv) is clear from the same statement for Δ_{geom} and Δ_{spec}.

Now we make the definition indicated in Section 4. Suppose that Δ_{geom} and Δ_{spec} are normalized by choice of related pairs $(\overline{\gamma}_1, \overline{\delta})$ and $(\overline{\pi}_1, \overline{\pi})$. Then Δ_{geom} and Δ_{spec} have *compatible normalization* if

$$\Delta_{spec}(\overline{\pi}_1, \overline{\pi}) = \Delta_{comp}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta})\Delta_{geom}(\overline{\gamma}_1, \overline{\delta}).$$

Lemma 12.2 *Suppose that Δ_{geom} and Δ_{spec} are any pair of geometric and spectral transfer factors. Then Δ_{geom} and Δ_{spec} are normalized compatibly if and only if $\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)\Delta_{geom}(\gamma_1, \delta)$ for all strongly G-regular pairs (γ_1, δ) of $\mathbb{R}$-rational points and G-regular pairs (π_1, π) of tempered irreducible representations.*

Proof: By (ii) and (iii) of Lemma 12.1 this statement is true for one choice of (γ_1, δ) and (π_1, π) if and only if it is true for all choices, and the lemma follows.

Lemma 12.3 *(i) If G is quasisplit and the inner twist ψ is the identity then $\Delta_0(\pi_1, \pi)$ and $\Delta_0(\gamma_1, \delta)$ have compatible normalization. (ii) If G is quasisplit and the inner twist ψ is the identity then the factors $\Delta_\lambda(\pi_1, \pi)$ and $\Delta_\lambda(\gamma_1, \delta)$ with Whittaker normalization are compatible.*

Proof: This is clear from (iv) in Lemma 12.1 and the observation that multiplying both factors by the same constant preserves compatibility.

We next check geometric-spectral compatibility against compatibility for descent to a cuspidal Levi group. This compatibility was recalled in Section 14 of [S3] for geometric factors, and so we return to that setting. Thus we have toral data $\eta = \eta_{B_1,B} : T_1 \to T$ defined over $\mathbb{R}$, and $M_1 = Cent(S_{T_1}, H_1)$ is endoscopic for $M = Cent(S_T, G)$; P is a parabolic subgroup of G defined over $\mathbb{R}$ and containing M as Levi subgroup, and N is its unipotent radical. Similarly, we define P_1 and N_1 in H_1. Working within the endoscopic pair M_1, M we choose a related pair $(\overline{\gamma}_1^M, \overline{\delta}^M)$ of elements such that $\overline{\gamma}_1^M$ is strongly G-regular, rather than just strongly M-regular. We have called the normalizations of geometric factors Δ_M and Δ compatible if $\Delta_M(\overline{\gamma}_1^M, \overline{\delta}^M)$ is chosen so that

$$\Delta_M(\overline{\gamma}_1^M, \overline{\delta}^M) = \left|\det(Ad(\overline{\delta}^M) - I)_{\mathfrak{g}/\mathfrak{m}}\right|^{-1/2} \Delta(\overline{\gamma}_1^M, \overline{\delta}^M).$$

Then we get the same formula for all related pairs (γ_1, δ) in M with γ_1 strongly G-regular; this rests on the property of the relative geometric transfer factor that we check term by term in Lemma 14.1 of [S3]. The same simple step will work for the spectral factors we have defined. Thus to normalize a spectral factor for M we choose a related pair $(\overline{\pi}_1^M, \overline{\pi}^M)$ with $\overline{\pi}_1^M$ now G-regular. Then

$$\overline{\pi}_1 = I(\overline{\pi}_1^M) = Ind(\overline{\pi}_1^{M_1} \otimes I_{N_1(\mathbb{R})}; P_1(\mathbb{R}), H_1(\mathbb{R}))$$

is tempered irreducible, with correct character on $Z_1(\mathbb{R})$. Similarly, let $\overline{\pi} = I(\overline{\pi}^M)$. We call spectral Δ_M compatible with given spectral Δ for G if $\Delta_M(\overline{\pi}_1^M, \overline{\pi}^M)$ coincides with the welldefined number

$$\Delta(I(\overline{\pi}_1^M), I(\overline{\pi}^M)).$$

Lemma 12.4 *If spectral Δ_M is compatible with spectral Δ then*

$$\Delta_M(\pi_1^M, \pi^M) = \Delta(I(\pi_1^M), I(\pi^M))$$

for all related pairs (π_1^M, π^M) in M for which π_1^M is G-regular.
Proof: We have to show the the relative spectral factor

$$\Delta_M(\pi_1^M, \pi^M; \overline{\pi}_1^M, \overline{\pi}^M)$$

for M coincides with the relative spectral factor

$$\Delta(I(\pi_1^M), I(\pi^M); I(\overline{\pi}_1^M), I(\overline{\pi}^M))$$

for G when $\pi_1^M, \overline{\pi}_1^M$ are G-regular. We write each as a product

$$\Delta_I \Delta_{II} \Delta_{III}.$$

To calculate those terms for G we use, as we may, toral data inside M, and a-data that is trivial for roots outside M. The remarks in the proof of Lemma 14.1 of [S3] apply to Δ_I and Δ_{III}. For Δ_{II}, we return to its definition and observe by inducing in stages that the required property is built into the definition.
Lemma 12.5 *In the same setting, suppose that Δ_{geom} and Δ_{spec} are normalized compatibly and that $\Delta_{M,geom}$, $\Delta_{M,spec}$ are compatible with Δ_{geom}, Δ_{spec} respectively. Then $\Delta_{M,geom}$ and $\Delta_{M,spec}$ are normalized compatibly.*
Proof: We just have to show that Δ_{comp} may be computed either inside M or in G with induced representations for sufficiently regular parameters. This is now clear from our arguments above.

Finally, we may now prove the following:
Lemma 12.6 *Suppose that Δ_{geom} and Δ_{spec} are normalized compatibly and that*

$$\Delta_{spec}(\pi_1, \pi) = \pm 1$$

for some, and hence every, G-regular related pair (π_1, π). Then if we remove the term Δ_{IV} from Δ_{geom} we obtain

$$\Delta_{geom}(\gamma_1, \delta) = \pm \varepsilon e^{\mu^*(X)}$$

for all strongly G-regular related pairs (γ_1, δ) with $\gamma_1 = \exp X$, where X is sufficiently close to the origin in the Lie algebra of $H_1(\mathbb{R})$ and ε is a constant fourth root of unity.
Proof: We work with the device of chosen related pairs. Since we have that

$$\Delta_{spec}(\overline{\pi}_1, \overline{\pi}) = \pm 1$$

we must have that

$$\Delta(\overline{\gamma}_1, \overline{\delta}) = \Delta_{comp}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta})^{-1}$$

up to a sign. Ignoring all evident signs and Δ_{IV} in $\Delta_{geom}(\gamma_1, \delta)$ we find that we are left with

$$\Delta_{II}(\gamma_1, \delta) \Delta_{III_2}(\gamma_1, \delta) / \Delta_{II}(\overline{\pi}_1, \overline{\pi}).$$

From Lemma 9.9 we have

$$\Delta_{II}(\overline{\pi}_1, \overline{\pi})^4 = 1.$$

On the other hand, the local form around the identity for

$$\Delta_{II}(\gamma_1,\delta)\Delta_{III_2}(\gamma_1,\delta)$$

is $e^{\mu^*(X)}$ times the product over positive imaginary roots α outside H_1 of the terms

$$sign((e^{\alpha(X)/2}-e^{-\alpha(X)/2})/a_\alpha).$$

The lemma is then proved.

Recall from Section 9 that μ^* provides a shift in infinitesimal character. Also if we use the Whittaker normalization then the proof shows that we may take the factor $\varepsilon(V,\psi)$ for ε.

We finish with some simple observations that will be useful in [S2]. Again Δ_{IV} is to be removed from Δ_{II+} and thus also from Δ_{comp} as well as Δ_{geom}.

Lemma 12.7

$$\left|\Delta_{geom}(\gamma_1,\delta;\gamma_1',\delta')\right|=1,$$
$$|\Delta_{spec}(\pi_1,\pi;\pi_1',\pi')|=1,$$
$$|\Delta_{comp}(\pi_1,\pi;\gamma_1,\delta)|=1$$

for all strongly G-regular related pairs (γ_1,δ), (γ_1',δ') of $\mathbb{R}$-rational points and G-regular related pairs (π_1,π), (π_1',π') of tempered irreducible representations.

Proof: Our (harmless) unitarity assumption on μ^* ensures that

$$|\Delta_{III_2}(\gamma_1,\delta)|=1.$$

The rest is immediate.

Corollary 12.8 *(i) For each normalization we have that*

$$\|\Delta_{geom}\|=|\Delta_{geom}(\gamma_1,\delta)|$$

is independent of the choice for strongly G-regular related pair (γ_1,δ), and similarly

$$\|\Delta_{spec}\|=|\Delta_{spec}(\pi_1,\pi)|$$

is independent of the choice for G-regular related pair (π_1,π). (ii) If $\Delta_{geom},\Delta_{spec}$ are compatible then

$$\|\Delta_{geom}\|=\|\Delta_{spec}\|.$$

13. Proof of Theorem 5.1 (G-regular case)

We follow the same procedure as in [S1] to prove that the spectral transfer factors are correct in the G-regular case. Because of the many constants involved, it is easier to start from scratch. First, we reduce by parabolic descent to the main case: G cuspidal, H_1 elliptic, and parameter φ_1 discrete as well as G-regular. We continue with the setting of Lemmas 12.4 and 12.5. For $f\in\mathcal{C}(G(\mathbb{R}))$ we have Harish Chandra's $f^{(P)}\in\mathcal{C}(M(\mathbb{R}))$ and for $f_1\in\mathcal{C}(H_1(\mathbb{R}),\lambda_1)$ and parabolic subgroup P_1 of H_1, defined over $\mathbb{R}$ and with M_1 as Levi subgroup, we have $f_1^{(P_1)}\in$

$\mathcal{C}(M_1(\mathbb{R}),\lambda_1)$. Here the modular function is inserted in $f^{(P)}$ and measures have been normalized so that for given dm, dg we have

$$O_\delta(f,dt,dg)=\left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\delta)-I\right|^{-1/2} O_\delta(f^{(P)},dt,dm)$$

for all δ in $M(\mathbb{R})$ that are strongly regular in G, and then also

$$Tr\ I(\pi^M)(f)=Tr\ \pi^M(f^{(P)}).$$

Compatible choices of normalizations for the various factors now give the reduction: Given f find f_1 from geometric transfer, descend to get both geometric matching for $f^{(P)}$ and $f_1^{(P_1)}$ (Lemma 14.3 of [S3]) and spectral matching by assumption (G-regular parameter $\varphi_1$factors discretely through M_1). Then lift the spectral matching back to G by Lemma 12.4. Notice that we no longer need to deal with a number of signs introduced in [S1].

Assume then for the rest of this section that G is cuspidal, H_1 is elliptic, and that the tempered parameter φ_1 is discrete as well as G-regular. Again we follow [S1], but now the argument for the elementary but crucial Lemma 4.2.4 is structurally much simpler. We will include the details after outlining the steps. We recall first the setting. The parameter φ_1 is fixed. We have two tempered invariant eigendistributions with same regular infinitesimal character:

$$f\to\sum_{\pi,temp}\Delta_{spec}(\pi_1,\pi)Tr\ \pi(f)$$

which is a finite sum of discrete series characters, and

$$f\to St\text{-}Tr\ \pi_1(f_1)$$

for which we can exploit the geometric matching of f and f_1 to get an explicit formula in terms of stable discrete series characters on $H_1(\mathbb{R})$. According to Harish Chandra's uniqueness theorem, applied with care regarding support since the theorem is stated for connected semisimple Lie groups (see Lemma 4.4.6 (iii) of [S1]), these two distributions are equal if they are represented by the same formula on the regular elements of a Cartan subgroup $T(\mathbb{R})$ that is compact modulo the center of $G(\mathbb{R})$. We start with the second distribution and pick toral data including T and a torus T_1 in H_1. We take the formula for the stable character χ_{φ_1} (representing $St\text{-}Tr\ \pi_1$) on the strongly G-regular elements of

$$T_H(\mathbb{R})=T_1(R)/Z_1(\mathbb{R}),$$

and integrate it against the stable orbital integrals of f_1 according to the Weyl integration formula. We then use the geometric transfer to transport this to the integral of a function against orbital integrals of f over strongly regular elements in $T(\mathbb{R})$. This function involves geometric transfer factors. We gather the terms we do not need for the harmonic analysis and then use the compatibility factor to transform them into spectral factors. We find then that we have exactly the local formula for

$$\sum_{\pi,temp} \Delta_{spec}(\pi_1,\pi) Tr\ \pi(f)$$

on the strongly regular elements in $T(\mathbb{R})$. Extension to all regular elements in $T(\mathbb{R})$ is immediate, and so the theorem is proved. Here are the steps of the calculation.

We may assume that f, f_1 are supported on the strongly G-regular elliptic sets. Then according to the Weyl integration formula we may evaluate St-Tr $\pi_1(f_1)$ as

$$[\Omega(H,T_H)]^{-1} \int_{T_H(\mathbb{R})} \chi_{\varphi_1}(\gamma_1) SO_{\gamma_1}(f_1) J_H(\gamma) \ \ d\gamma,$$

where the product

$$\chi_{\varphi_1}(\gamma_1) SO_{\gamma_1}(f_1)$$

is welldefined as a function of the image γ of γ_1 in $T_H(\mathbb{R})$, we have used the invariance of the usual Weyl integral formula under

$$\gamma \to w^{-1}\gamma w,\ w \in \Omega(H,T_H),$$

and $J_H = J_{H/T_H}$ is the Jacobian from Section 7. We take δ to be the image of γ_1 under $T_1 \to T$. Then

$$SO_{\gamma_1}(f_1) = \Delta_{geom}(\gamma_1,\delta) \sum_{w,G} < inv(\delta,\delta^w), \mathfrak{s}_\delta > O_{\delta^w}(f),$$

where w, G indicates summation over representatives w for $\Omega(G,T)/\Omega_{\mathbb{R}}(G,T)$, and

$$\chi_{\varphi_1}(\gamma_1) = \upsilon(\varphi_1, \{a_\alpha\}) \sum_{w_1 \in \Omega(H,T_H)} \frac{(\det w_1)\ \Lambda(w_1^{-1}\mu_1 \ - \ \iota_1,\ \lambda_1)(\gamma_1)}{\Delta'_{left}(\gamma,\{a_\alpha\},\{\chi_\alpha\})}.$$

We could drop $\{\chi_\alpha\}$ from notation since we are using the based choice of χ-data in writing the character formula this way. We now apply Lemma 7.3 to rewrite the product of the denominator of $\chi_{\varphi_1}(\gamma_1)$, the terms

$$\Delta_{II}(\gamma_1,\delta)\Delta_{IV}(\gamma_1,\delta)$$

of Δ_{geom}, and $J_H(\gamma)$ as

$$\Delta'_{left}(\delta,\{a_\alpha\},\{\chi_\alpha\})^{-1} J_G(\delta).$$

At the same time we observe that the product of the numerator of $\chi_{\varphi_1}(\gamma_1)$ with

$$\Delta_{III_2}(\gamma_1,\delta) = \Lambda(\mu^* + \iota_1 - \iota, \lambda^*)(\gamma_1)$$

is welldefined as a function of γ and hence of δ (note Section 9 of [S3]). More explicitly, recall from Section 7 that

$$\mu = \mu^* + \mu_1, \lambda = \lambda^* + \lambda_1,$$

that $(\mu - \iota,\, \lambda)$ are character data on $T(\mathbb{R})$, and that for w_1 in $\Omega(H, T_H)$ we have

$$w_1^{-1}\mu - \iota = (w_1^{-1}\mu_1 - \iota_1) + (\mu^* + \iota_1 - \iota).$$

We write Δ_{geom}^{rest} for the terms in Δ_{geom} that we have not mentioned so far. At this stage we can transport the original expression to $T(\mathbb{R})$ as

$$[\Omega(H, T_H)]^{-1} \int_{T(\mathbb{R})} \frac{\upsilon(\varphi_1, \{a_\alpha\}) \sum_{w_1 \in \Omega(H,T_H)} (\det w_1)\ \Lambda(w_1^{-1}\mu\ -\ \iota,\ \lambda)(\delta)}{\Delta'_{left}(\delta, \{a_\alpha\}, \{\chi_\alpha\})}$$

times

$$\Delta_{geom}^{rest}(\gamma_1, \delta) \sum_{w \in \Omega(G,T)/\Omega_{\mathbb{R}}(G,T)} < inv(\delta, \delta^w), \mathfrak{s}_\delta > O_{\delta^w}(f)$$

times

$$J_G(\delta)\ d\delta.$$

We expand the sum in the middle term to one over $\Omega(G, T)$, and divide the integral by $[\Omega_{\mathbb{R}}(G, T)]$. We will see below that $\Delta_{geom}^{rest}(\gamma_1, \delta)$ is a constant. Thus we may apply invariance of the whole integral under $\delta \to {}^w\delta$, to now write $St\text{-}Tr\ \pi_1(f_1)$ as

$$[\Omega_{\mathbb{R}}(G, T)]^{-1} \int_{T(\mathbb{R})} F(\delta) O_\delta(f) J_G(\delta)\ d\delta,$$

where $F(\delta)$ is the product of

$$[\Omega(H, T_H)]^{-1} \Delta_{geom}^{rest}(\gamma_1, \delta) \upsilon(\varphi_1, \{a_\alpha\})$$

times the sum over $w \in \Omega(G, T)$ of the terms

$$\frac{< inv(\delta, \delta^w), \mathfrak{s}_\delta > \sum_{w_1 \in \Omega(H,T_H)} (\det w_1)\ \Lambda(w_1^{-1}\mu\ -\ \iota,\ \lambda)({}^w\delta)}{\Delta'_{left}({}^w\delta, \{a_\alpha\}, \{\chi_\alpha\})}.$$

We regroup to write $F(\delta)$ as

$$\Delta_{geom}^{rest}(\gamma_1, \delta) \upsilon(\varphi_1, \{a_\alpha\})$$

times

$$\sum_{w \in \Omega(G,T)} < inv(\delta, \delta^w), \mathfrak{s}_\delta > \frac{\det w\ \Lambda(w^{-1}\mu\ -\ \iota,\ \lambda)(\delta)}{\Delta'_{left}(\delta, \{a_\alpha\}, \{\chi_\alpha\})}.$$

Here we have used

$$< inv(\delta, \delta^w), \mathfrak{s}_\delta > \ = \ < inv(\delta, \delta^{w_1 w}), \mathfrak{s}_\delta >$$

for w_1 from H_1 (see Remark 7.1). We regroup once again, this time with respect to $\Omega_{\mathbb{R}}(G, T)$, and so rewrite this last sum as

$$\upsilon(\varphi, \{a_\alpha\})^{-1} \sum_{w \in \Omega(G,T)/\Omega_{\mathbb{R}}(G,T)} < inv(\pi, \pi'), \mathfrak{s}_\pi > \ \chi_{\pi'}(\delta),$$

where $\pi = \pi(1)$ relative to the chosen toral data and π' denotes the representation $\pi(w)$, so that we have

$$inv(\pi, \pi') = inv(\delta, \delta^w),\ \mathfrak{s}_\delta = \mathfrak{s}_\pi.$$

See also Remark 9.4. Since

$$\Delta(\pi_1, \pi') = \Delta(\pi_1, \pi) < inv(\pi, \pi'), \mathfrak{s}_\pi >$$

it remains to show that

$$\Delta^{rest}_{geom}(\gamma_1, \delta) v(\varphi_1, \{a_\alpha\}) v(\varphi, \{a_\alpha\})^{-1}$$

or, equivalently,

$$\Delta^{rest}_{geom}(\gamma_1, \delta) \Delta_{II}(\pi_1, \pi)$$

coincides with the (constant for this calculation) spectral factor

$$\Delta(\pi_1, \pi).$$

This is an exercise with constants and compatibility:

$$\Delta^{rest}_{geom}(\gamma_1, \delta) = \Delta(\overline{\gamma}_1, \overline{\delta}) \Delta_I(\gamma_1, \delta) \Delta_{III}(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta}) / \Delta_I(\overline{\gamma}_1, \overline{\delta}) \Delta_{II+}(\overline{\gamma}_1, \overline{\delta}),$$

and, by our choices for π and δ, we have

$$\Delta_{III}(\pi_1, \pi; \gamma_1, \delta) = 1.$$

Thus

$$\begin{aligned} \Delta_{III}(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta}) &= \Delta_{III}(\pi_1, \pi; \overline{\gamma}_1, \overline{\delta}) \\ &= \Delta_{III}(\pi_1, \pi; \overline{\pi}_1, \overline{\pi}) \Delta_{III}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta}). \end{aligned}$$

Now we may use geometric-spectral compatibility again to replace

$$\Delta(\overline{\gamma}_1, \overline{\delta}) \Delta_{III}(\overline{\pi}_1, \overline{\pi}; \overline{\gamma}_1, \overline{\delta}) / \Delta_I(\overline{\gamma}_1, \overline{\delta}) \Delta_{II+}(\overline{\gamma}_1, \overline{\delta})$$

by

$$\Delta(\overline{\pi}_1, \overline{\pi}) / \Delta_I(\overline{\pi}_1, \overline{\pi}) \Delta_{II}(\overline{\pi}_1, \overline{\pi}),$$

and so $\Delta^{rest}_{geom}(\gamma_1, \delta)$ is the product of

$$\Delta(\overline{\pi}_1, \overline{\pi}) / \Delta_I(\overline{\pi}_1, \overline{\pi}) \Delta_{II}(\overline{\pi}_1, \overline{\pi})$$

and

$$\Delta_{III}(\pi_1, \pi; \overline{\pi}_1, \overline{\pi}) \Delta_I(\gamma_1, \delta).$$

This product simplifies to

$$\Delta(\pi_1, \pi) \Delta_I(\gamma_1, \delta) / \Delta_I\ (\pi_1, \pi) \Delta_{II}\ (\pi_1, \pi).$$

Since

$$\Delta_I(\gamma_1, \delta) = \Delta_I(\pi_1, \pi)$$

we have that

$$\Delta_{geom}^{rest}(\gamma_1, \delta)\Delta_{II}\ (\pi_1, \pi) = \Delta\ (\pi_1, \pi),$$

as desired.

We then conclude that the distributions

$$f \to \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi) Tr\ \pi(f)$$

and

$$f \to St\text{-}Tr\ \pi_1(f_1)$$

coincide. Since we have dealt with the inductive step already, this finishes the proof of the transfer theorem in the G-regular case.

14. Completion of proof of Theorem 5.1

To complete the proof of the transfer theorem we have two tasks. The first is, given π_1 without assumption of G-regularity, to define $\Delta_{spec}(\pi_1, \pi)$ for all tempered irreducible representations π of $G(\mathbb{R})$.The second is then to prove that the distributions

$$f \to \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi) Tr\ \pi(f)$$

and

$$f \to St\text{-}Tr\ \pi_1(f_1)$$

coincide for our choice of π_1. We will again start with the case G cuspidal, H_1 elliptic and φ_1 discrete, and complete the two tasks in this setting. Then we will be able to complete both tasks in general by an application of parabolic induction.

Assume then we are in the cuspidal-elliptic-discrete setting, and pick toral data for both G^* and G. The parameter $\boldsymbol{\varphi}_1$ is H_1-regular but no longer necessarily G-regular, and while $\boldsymbol{\varphi}^*$ is defined, it is not necessarily discrete. Further, a parameter $\boldsymbol{\varphi}$ for G need not exist. Notice, however, that $Cent(\varphi^*(\mathbb{C}^\times), G^\vee)$, while no longer a torus, can only be of type $(A_1)^r$ since its roots form a system $R(\varphi^*)$ in which, by the H_1-regularity of $\boldsymbol{\varphi}_1$, every root takes the value -1 on the endoscopic datum $\mathfrak{s}$.

We start again with the representative $\varphi_1(\mu_1, \lambda_1)$ for $\boldsymbol{\varphi}_1$, with μ_1 dominant regular in H_1, and attach representative $\varphi^*(\mu, \lambda)$ for $\boldsymbol{\varphi}^*$. Recall that the representations in the L-packet for $\boldsymbol{\varphi}^*$ are defined as constituents of certain unitary principal series. Notice that $R(\varphi^*)$, of type $(A_1)^n$, consists of the coroots $\alpha^\vee$ for which $< \mu, \alpha^\vee >$ vanishes, and so it will be a straightforward exercise with Hecht-Schmid character identities to find the appropriate cuspidal Levi group for the principal series and also to test if φ^* is relevant to G. To recall those arguments (Lemmas 4.3.5 and 4.3.7 of [S1]) we first assemble tempered distributions associated to the Weyl group orbit of (μ, λ) by coherent continuation of discrete series characters.

We may translate the given H_1-regular-dominant μ_1 by a form $\eta \in X^*(T) \subseteq X^*(T_1)$ with the properties that (i) $\mu_1 + \eta$ is a G-regular H_1-dominant element in $X^*(T_1) \otimes \mathbb{C}$, (ii)

$$\mu + \eta = \mu^* + \mu_1 + \eta$$

remains in a G-chamber containing μ, and (iii), to preserve unitarity, η is trivial on the maximal split torus in T. For this, recall μ^* is perpendicular to the roots from H_1. Then $\varphi^*(\mu + \eta, \lambda)$ is a discrete parameter for G^* and $\varphi(\mu + \eta, \lambda)$ a discrete parameter for G. So we have L-packets

$$\{\pi^*(w) = \pi^*(w^{-1}(\mu + \eta), \lambda)\}$$

and

$$\{\pi(w) = \pi(w^{-1}(\mu + \eta), \lambda)\}.$$

Write Ψ_η for the positive system relative to which $\mu + \eta$ is dominant. Then we rewrite the character of $\pi^*(w^{-1}(\mu + \eta), \lambda)$ as

$$\Phi^*(w^{-1}(\mu + \eta), \lambda, w^{-1}\Psi_\eta),$$

and that of $\pi(w^{-1}(\mu + \eta), \lambda)$ as

$$\Phi(w^{-1}(\mu + \eta), \lambda, w^{-1}\Psi_\eta).$$

(We have already used the usual notation Θ for the Harish Chandra distribution and Θ^* for the stable version). We then define the tempered distributions

$$\Phi^*(w^{-1}\mu, \lambda, w^{-1}\Psi_\eta)$$

and

$$\Phi(w^{-1}\mu, \lambda, w^{-1}\Psi_\eta)$$

by coherent continuation.

Now the first step (Lemma 4.3.5 of [S1]) is to show that $\Phi(w^{-1}\mu, \lambda, w^{-1}\Psi_\eta)$ vanishes, *i.e.* is the zero distribution, for all w in the Weyl group, if $\boldsymbol{\varphi}^*$ is not relevant to G. In that case, of course, we define the transfer factor

$$\Delta_{spec}(\pi_1, \pi) = 0$$

for all tempered irreducible representations π of $G(\mathbb{R})$. To complete the proof of the transfer theorem for π_1 we then have to show that $f \to St\text{-}Tr\ \pi_1(f_1)$ vanishes as well. For that we just have to believe we may put $\eta = 0$ in the character identities established for the G-regular case.

Suppose then that $\boldsymbol{\varphi}^*$ is not relevant to G. First we will find

$$M^* = Cent(S_{\overline{T}^*}, G^*)$$

or, equivalently, $\overline{T}^*$, such that $\boldsymbol{\varphi}^*$ factors discretely through M^*. We have chosen representative $\varphi^* = \varphi^*(\mu, \lambda)$ for $\boldsymbol{\varphi}^*$. Set

$$R(\varphi^*) = \{\pm\alpha_1^\vee, ..., \pm\alpha_r^\vee\}.$$

Then we construct easily a homomorphism $\overline{\varphi}^*$ of the form $Int(s)\circ\ \varphi^*$ such that φ^* and $\overline{\varphi}^*$ agree on $\mathbb{C}^\times$ while $\overline{\varphi}^*(1\times\sigma)$ acts on $\mathcal{T}$ as

$$w_{\alpha_1^\vee}...w_{\alpha_r^\vee}\varphi^*(1\times\sigma) = w_{\alpha_1^\vee}...w_{\alpha_r^\vee}\sigma_{T^*}$$

(see top of p. 407 of [S1] for the exact choice). Because G^* is quasisplit there are toral data for G^*, including torus $\overline{T}^*$ with $S_{\overline{T}^*}$ contained in the chosen maximal split torus, such that $\sigma_{\overline{T}^*}$ acts as $w_{\alpha_1}...w_{\alpha_r}\sigma_{T^*}$. Then $\boldsymbol{\varphi}^*$ factors through the conjugacy class of the homomorphism $\overline{\varphi}^* = \overline{\varphi}^*(\mu,\overline{\lambda})$ for M^*. This parameter is discrete since $R(\overline{\varphi}^*)$ consists of coroots that are real for $\overline{T}^*$. We have assumed that $\boldsymbol{\varphi}^*$ is not relevant to G, and so we cannot continue the toral data for M^* to G. Given that is the case, at least one root among $\alpha_1,...,\alpha_r$ must become totally compact in G (as recalled in the context of orbital integrals in [S3]) along the way, when we pass from T^* to $\overline{T}^*$ by Cayley transforms from the Weyl orbits of the roots $\alpha_1,...,\alpha_r$. Then we can argue on T that for each w in the Weyl group, $\{w^{-1}\alpha_1,...,w^{-1}\alpha_r\}$ contains a compact root. Since one of each pair $\pm w^{-1}\alpha_i$ is evidently $w^{-1}\Psi_\eta$-simple we conclude from a wellknown result of Hecht and Schmid (but in the disconnected case) that for each w, the distribution $\Phi(w^{-1}\mu,\lambda,w^{-1}\Psi_\eta)$ vanishes.

Suppose now that $\boldsymbol{\varphi}^*$ is relevant to G, so that $\boldsymbol{\varphi}$ is welldefined. Then we can extend the toral data to G, obtaining now $\overline{T}$ and M, as well as discrete parameter $\overline{\varphi}(\mu,\overline{\lambda})$ for M. We may and will adjust the toral data so that $\alpha_1,...,\alpha_r$ are all noncompact on T, and then $\Phi(\mu,\lambda,\Psi_\eta)$ does not vanish. An argument with K-types shows that if

$$\Phi(w^{-1}\mu,\lambda,w^{-1}\Psi_\eta)$$

is also nonvanishing then it coincides with $\Phi(\mu,\lambda,\Psi_\eta)$ if and only if w lies in the real Weyl group of T. A routine argument (see bottom of p. 408 of [S1]) using the characterization of noncompact roots in a Weyl orbit shows that $\Phi(w^{-1}\mu,\lambda,w^{-1}\Psi_\eta)$ vanishes unless, modulo right multiplication by an element of the real Weyl group $\Omega_\mathbb{R}$, w lies in the subgroup Ω_μ generated by reflections with respect to $\alpha_1,...,\alpha_r$ and the roots perpendicular to each of $\alpha_1,...,\alpha_r$. It remains then to show that

$$\{\Phi(w^{-1}\mu,\lambda,w^{-1}\Psi_\eta) : w\in\Omega_\mu\Omega_\mathbb{R}/\Omega_\mathbb{R}\}$$

consists precisely of the characters of the constituents of the induced representations attached to $\overline{\varphi}(\mu,\overline{\lambda})$. That can be argued directly with the character identities of Hecht and Schmid, once we observe that $\overline{\lambda}$ has the correct form for these identities to exist. We get that from remarking that $\overline{\varphi}^*(1\times\sigma)$ was chosen expressly to act on a root vector $X_{\alpha_i^\vee}$ by (-1), and then comparing this with a formula of Langlands (in an appendix to [A2]). The formula tells us which multiple of $X_{\alpha_i^\vee}$ we get in terms of the parity of $<\rho_i,\alpha_i^\vee>$, where ρ_i is one half the sum of the roots which restrict to a positive multiple of α_i on $S_{\overline{T}}$. Here we could just as well take one half the sum of all positive roots. The needed result is that the parity matches that of $<2\overline{\lambda},\alpha_i^\vee>$. The main step is the inductive argument to be found starting at the bottom of p.84 of [A2].

Notice that the characters

$$\Phi(w^{-1}\mu,\lambda,w^{-1}\Psi_\eta),$$

for w in $\Omega_\mu\Omega_{\mathbb{R}}$, are *nondegenerate* in the sense of Knapp and Zuckerman because the Weyl reflection with respect to a noncompact root outside H_1 cannot be realized in $G(\mathbb{R})$. This same property of noncompact roots outside H_1 is decisive in the transfer of orbital integrals (see Section 14 of [S3]).

Finally, we define $\Delta_{spec}(\pi_1, \pi)$ for

$$\pi = \Phi(w^{-1}\mu, \lambda, w^{-1}\Psi_\eta)$$

in the L-packet of $\boldsymbol{\varphi}$ by

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{spec}(\pi_1(\eta), \pi(\eta)),$$

where $\pi_1(\eta)$ is any representation attached to

$$\varphi_1(\mu_1 + \eta, \lambda_1)$$

and

$$\pi(\eta) = \pi(w^{-1}(\mu + \eta), \lambda).$$

For any other tempered irreducible representation π we set

$$\Delta_{spec}(\pi_1, \pi) = 0.$$

Now to complete the transfer theorem for π_1 in this case, as well as in the case that $\boldsymbol{\varphi}^*$ is not relevant to G, we argue by coherent continuation to the wall. Since we are dealing only with tempered representations, we work somewhat informally with character formulas and identities. We start with the identity proved when π_1 replaced by $\pi_1(\eta)$:

$$St\text{-}Tr\ \pi_1(\eta)(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1(\eta), \pi(\eta))\ Tr\ \pi(\eta)(f).$$

The right side is coherent (to a G-wall) in η, and so is the left side as distribution on $H_1(\mathbb{R})$, *i.e.* as function of f_1. Also we argue, for example by use of the Weyl integration formula on the various Cartan subgroups, that transport to $G(\mathbb{R})$ does not destroy this coherence. Thus we may put $\eta = 0$ on each side to obtain the transfer statement. Further, since the left side is independent of the choice for η, so also is the right side. By the linear independence of characters we conclude then that $\Delta_{spec}(\pi_1, \pi)$ is independent of the choice of η and toral data in the last paragraph. We can also check this with explicit computations as in [S1].

The last case to consider is that where we drop the cuspidal-elliptic-discrete assumption. Thus the parameter $\boldsymbol{\varphi}_1$ is an arbitrary tempered parameter for H_1. We choose appropriate toral data and factor $\boldsymbol{\varphi}_1$ minimally through a cuspidal Levi group M_1, and thus return to the cuspidal-elliptic-discrete setting for M_1 as endoscopic group for M^*. Assume $\boldsymbol{\varphi}_1^M$ and $\boldsymbol{\varphi}^{*M}$ are so defined and that $\boldsymbol{\varphi}^*$ is relevant to G, so that $\boldsymbol{\varphi}$ and a pair $M, \boldsymbol{\varphi}^M$ are defined, and otherwise set the spectral transfer factor to be zero. We describe the L-packet for $\boldsymbol{\varphi}^M$ as limits of discrete series representations π^M, as above. Suppose π_1 is a constituent of $I(\pi_1^M)$ and π is a constituent of $I(\pi^M)$. Then we set

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{spec}(\pi_1^M, \pi^M),$$

where we use a compatible factor on $M(\mathbb{R})$. We now argue as in first paragraph of Section 13 for the G-regular case to obtain the desired transfer result for π_1. As in the G-regular case, we see that $\Delta_{spec}(\pi_1, \pi)$ is independent of the factoring choice. For this we could also argue by coherent continuation of the relevant induced representations. This completes the proof of Theorem 5.1.

The following is an immediate consequence of the proof.

Corollary 14.1 *The assertions of Lemma 11.3(ii), Lemma 11.4 remain true for all tempered related pairs* (π_1, π).

Notice also that, in general, we have

$$\Delta(\pi_1', \pi') = \pm\, \Delta(\pi_1, \pi)$$

for any two tempered related pairs (π_1, π) and (π_1', π').

15. Conclusion

Given an endoscopic group we have now defined the spectral factors $\Delta(\pi_1, \pi)$ for all tempered related pairs (π_1, π) and proved the transfer identity

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi) Tr\ \pi(f).$$

This completes our study of tempered spectral transfer factors as functions of π_1, although in [S2] we will check the effect on the transfer theorem of an isomorphism of endoscopic data and of a change in z-pair. Our main focus in [S2], however, will be with the factors as functions of π.

We know by Section 5 of [S1] that we can invert the transfer identities, *i.e.* given a tempered irreducible admissible representation π of $G(\mathbb{R})$, we can find endoscopic groups and related pairs (π_1, π) such that the character $Tr\ \pi(f)$ is a linear combination of the endoscopic characters

$$f \to St\text{-}Tr\ \pi_1(f_1).$$

The coefficients in this combination are sufficiently explicit to display a structure on tempered L-packets conjectured by Langlands, but leave us with several concerns. We know, originally by results of Adams, Barbasch and Vogan, that it is desirable to consider several inner forms at once. Also a recent conjecture of Arthur [A3] places precise requirements on the coefficients in terms of transfer factors. So for the inversion we will start, once again, directly from our definition of the spectral transfer factors and work with several inner forms simultaneously.

There is no harm in assuming

$$\Delta(\pi_1, \pi) = \pm 1$$

for all tempered related pairs (π_1, π) as, for example, in the Whittaker normalization. This allows us to write the inversion formula simply as

$$Tr\ \pi(f) = \frac{1}{n(\pi)} \sum_{\pi_1} \Delta(\pi_1, \pi) St\text{-}Tr\ \pi_1(f_1),$$

where $n(\pi)$ is the cardinality of the extended L-packet of π. We have hidden in the notation $\sum_{\pi_1}$ a description, intrinsic to the extended L-packet of π, of appropriate related pairs (π_1, π). This description allows us to display a structure on tempered L-packets satisfying Arthur's requirements. See Sections 1 and 7 of [S2] for a more detailed outline of our approach.

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