FOUNDATIONS OF TWISTED ENDOSCOPY

R. KOTTWITZ AND D. SHELSTAD

In this paper we begin a study of the foundations for a theory of twisted endoscopy for reductive groups. While we build on standard endoscopy as developed, for example, in [K1], [K3], [L2], [LS1] and [S1], there are new features which will be described below and which in turn shed further light on the earlier theory.

In our setting F is a local or global field of characteristic zero, G is a connected reductive group defined over F, θ is an F-automorphism of G and ω is a quasicharacter on $G(F)$ if F is local or on $G(\mathbb{A})$ trivial on $G(F)$ if F is global. Endoscopy for (G, θ, ω) concerns the representations π of $G(F)$ or $G(\mathbb{A})$, as appropriate, for which $\pi \circ \theta$ is equivalent to $\omega \otimes \pi$. More generally, and for the most part conjecturally, we may consider L-packets or Arthur packets Π for which $\Pi \circ \theta = \omega \otimes \Pi$. Associated with such representations is a (θ, ω)-twisted invariant harmonic analysis: an Arthur trace formula, (θ, ω)-twisted characters, (θ, ω)-twisted orbital integrals and so on. Twisted endoscopy has played a role in a variety of problems. For example, the early paper [LL] of Labesse and Langlands on standard endoscopy for $\mathbf{SL}(2)$ is at the same time a study of a twisted endoscopy problem for $\mathbf{GL}(2)$: $\pi = \omega \otimes \pi$, and in the study of automorphic representations of unitary groups in three variables [R] we find the twisted endoscopy associated with base change.

We will begin by introducing endoscopic groups, or better endoscopic data, for $(G, \theta, \mathbf{a})$, where $\mathbf{a}$ is a Langlands parameter for ω. Our definitions were announced several years ago and indeed were used to recast the definitions for standard endoscopy in [LS1]. What remains perhaps as a surprise is the effort required in the general case to accommodate the possible lack of a suitable embedding of the L-group of an endoscopic group in the L-group of G. The basic theme of endoscopy is transfer from H to G, or more properly, transfer from a z-extension H_1 of H to G. At the level of F- or $\mathbb{A}$-points on the groups, examples such as base change or symmetric square for $\mathbf{GL}(2)$ have relied on concretely defined norm mappings. For the general case we take another more abstract approach, one which is well adapted to arguments involving the relevant systems of roots and restricted roots.

Now suppose that F is local. With the notion of norm mappings from sufficiently regular classes of elements in $G(F)$ to classes in $H_1(F)$ we can turn to the matching of (θ, ω)-twisted orbital integrals on $G(F)$ with stable orbital integrals on $H_1(F)$. The first goal of this paper will be to construct *transfer factors*. In analogy with standard endoscopy [LS1] these are the weighting factors for the (θ, ω)-twisted integrals needed to achieve the matching with the integrals on $H_1(F)$. Again as in the standard case

Partially supported by NSF Grants DMS-8913971, DMS-9203380, DMS-8903313 and DMS-9204484

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

they are quite elaborate for they must carry a great deal of information about the values of characters on the groups $G(F)$ and $H_1(F)$. There are new features. We need a slight generalization of the comparative study of the embeddings of L-groups of maximal tori in the L-group of a reductive group from [LS2] in order to construct one of our terms. We replace the Galois cohomology of standard endoscopy with Galois hypercohomology (for some complexes of tori of length 2) and introduce a pairing on hypercohomology that encompasses both the Langlands pairing for tori and Tate-Nakayama duality. We then gather all three cohomologically defined terms Δ_{I}, Δ_1, Δ_2 from the standard case into one term involving this pairing, although for the purposes of proof of canonicity and so on we have found it convenient to write this one term as a product of two, Δ_{I} and Δ_{III}.

The first five sections of the paper, which treat transfer factors, are organized as follows. The first section reviews results of Steinberg on semisimple automorphisms of semisimple groups. These results are used repeatedly in the rest of the paper, often without comment. At the end of this section one finds the definition of a-data and χ-data for twisted endoscopy.

The second section begins by giving the definition of endoscopic data $(H, \mathcal{H}, s, \xi)$ for $(G, \theta, \mathbf{a})$. The group $\mathcal{H}$ is an extension of W_F by $\hat{H}$ and ξ is an L-homomorphism from $\mathcal{H}$ to ${}^LG$. It is not always the case that the identity map from $\hat{H}$ to itself can be extended to an L-isomorphism from $\mathcal{H}$ to ${}^LH$, which forces us to use z-pairs (H_1, ξ_{H_1}), consisting of a z-extension H_1 of H and an L-embedding ξ_{H_1} of $\mathcal{H}$ in ${}^LH_1$ extending the natural inclusion of $\hat{H}$ in $\hat{H}_1$. The existence of such L-embeddings is proved in Lemma 2.2.A.

The third section introduces the abstract norm mapping which relates conjugacy classes in $H(\overline{F})$ and twisted conjugacy classes in $G(\overline{F})$. For this one first constructs a bijection from the set of twisted conjugacy classes in $G(\overline{F})$ to the analogous set for a quasi-split inner form of (G, θ). Unfortunately, unless the center of G is trivial, this map is not canonical, and there may in fact be no choice for it which is defined over F. For most of this paper we treat only the special case in which this difficulty does not occur (in other words we assume that the 1-cochain z_σ in (3.1) is trivial); then in (5.4) we explain the modifications needed in the general case.

The fourth section gives the definition of the relative transfer factor $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$, which should be thought of as the ratio

$$\Delta(\gamma_1, \delta)/\Delta(\bar{\gamma}_1, \bar{\delta})$$

of absolute transfer factors, these being canonical only up to a non-zero scalar (independent of γ_1, δ of course). In the case of standard endoscopy our relative transfer factor coincides with the one in [LS1], except that [LS1] takes z-extensions of G while our more general situation forces us to take z-extensions of H instead. The relative transfer factor is the product of four terms, the third of which, Δ_{III}, is the most complicated.

The fifth section uses the relative transfer factors to define absolute transfer factors $\Delta(\gamma_1, \delta)$ and lists their most important properties (see Lemmas 5.1.B, 5.1.C and Theorem 5.1.D). In case G is quasi-split and θ preserves an F-splitting $\mathbf{spl}_G$ there is a particular normalization $\Delta_0(\gamma_1, \delta)$ of the absolute transfer factor, depending only on $\mathbf{spl}_G$, just as in the standard case [LS1]. Let B_0 be the Borel subgroup appearing in the splitting $\mathbf{spl}_G$ and let λ be a θ-stable generic character on the F-points of the unipotent radical of B_0. Then one hopes [Sh] that the representations having Whittaker models for λ will serve as base points in tempered L-packets, and in (5.3) this leads us to multiply the absolute factor $\Delta_0(\gamma_1, \delta)$ by a suitable local ε-factor so as to obtain another absolute transfer factor $\Delta_\lambda(\gamma_1, \delta)$, depending only on the generic character λ. In (5.5) we give the definition of matching functions.

Before trying to understand the complicated factor Δ_{III} in the relative situation of (4.4), the reader may find it useful to study the absolute analogue of Δ_{III} given in (5.3). We now sketch this material under a number of simplying hypotheses, in the hope that the main idea will come through as clearly as possible. So let us assume for the moment that G is quasi-split, semisimple and simply connected. In particular $\mathbf{a}$ and ω are then trivial. Assume further that θ preserves some F-splitting of G. Let T be a θ-stable maximal F-torus of G that is contained in some θ-stable Borel subgroup B of G. Note that we do not assume that B is defined over F. Let T_θ denote the torus $T/(1-\theta)(T)$ (the coinvariants of θ on T). We think of the canonical surjection $N : T \to T_\theta$ as an abstract norm map. Let $\delta \in G(F)$ and $\gamma \in T_\theta(F)$, and assume that γ is sufficiently regular. We say that γ is a norm of δ if there exist elements $t \in T(\overline{F})$ and $g \in G(\overline{F})$ such that $N(t) = \gamma$ and $g\delta\theta(g)^{-1} = t$. Applying $\sigma \in \Gamma := \mathrm{Gal}(\overline{F}/F)$ to the equality $g\delta\theta(g)^{-1} = t$ and using that $N(t) = \gamma$ as well as the fact that γ is sufficiently regular (so that the twisted centralizer of t is the group of fixed points of θ on T) we see that the 1-cocycle $t_\sigma := g\sigma(g)^{-1}$ takes values in T and satisfies the equality

$$t \cdot \sigma(t)^{-1} = t_\sigma \theta(t_\sigma)^{-1}.$$

This equality simply says that the pair (t_σ^{-1}, t) is a 1-hypercocycle of Γ in the complex $T \xrightarrow{1-\theta} T$. The class $\mathrm{inv}(\gamma, \delta)$ of this 1-hypercocycle lies in the hypercohomology group $H^1(F, T \xrightarrow{1-\theta} T)$.

Now suppose that we are given a twisted endoscopic group H for (G, θ) and an admissible isomorphism over F from T_θ to a maximal F-torus T_H of H. Let γ_H be the element of $T_H(F)$ corresponding to γ under this isomorphism. The term $\Delta_{\mathrm{III}}(\gamma_H, \delta)$ in the absolute transfer factor is obtained by pairing the element $\mathrm{inv}(\gamma, \delta) \in H^1(F, T \xrightarrow{1-\theta} T)$ with the following element $\mathbf{A}$ in the dual hypercohomology group $H^1(W_F, \hat{T} \xrightarrow{1-\hat{\theta}} \hat{T})$.

Assume for simplicity that $\mathcal{H} = {}^L H$. Using χ-data we embed ${}^L T_H$ in ${}^L H$, and then we compose this with the embedding of ${}^L H$ in ${}^L G$ that is part of our endoscopic data, obtaining an embedding ξ_{T_H} of ${}^L T_H$ in ${}^L G$. Let ${}^L G^1$ denote the subgroup of ${}^L G$ given as the semidirect product of the Weil group W_F and the identity component of the group of fixed points of $\hat{\theta}$ on $\hat{G}$. Note that ${}^L G^1$ is the L-group of a twisted endoscopic group G^1 of G. Again using our χ-data, we embed ${}^L T_H$ in ${}^L G^1$, and then we compose this with the canonical inclusion ${}^L G^1 \hookrightarrow {}^L G$, obtaining another embedding ξ_1 of ${}^L T_H$ in ${}^L G$. Replacing ξ_{T_H} by a conjugate under $\hat{G}$ we may assume that ξ_{T_H} and ξ_1 agree on $\hat{T}_H$. Then the difference between ξ_{T_H} and ξ_1 is measured by a 1-cocycle A of W_F in $\hat{T}$, and $(1-\hat{\theta})(A^{-1})$ is the coboundary of an element $s_T \in \hat{T}$ coming from the element s appearing in our endoscopic data. The class of the hypercocycle (A^{-1}, s_T) is the desired element $\mathbf{A}$ in $H^1(W_F, \hat{T} \xrightarrow{1-\hat{\theta}} \hat{T})$.

Now we turn to our global results. In [L2] Langlands stabilized the elliptic regular terms in the trace formula; our second main goal in this paper is to do the same for the twisted trace formula (see [R] for the case of quadratic base change for unitary groups in three variables). Although the stabilization process is not difficult, it is surprisingly lengthy, partly because of the generality of the situation we consider. To ease the reader's task we will now summarize the main steps in the process.

Let F be a number field and G a connected reductive group over F. To make this introduction a little simpler we will assume that the center $Z(G)$ of G contains no non-trivial split torus. Let θ be an automorphism of G over F, and let $\mathbf{a}$ be an element of

$$H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})).$$

Note that $\mathbf{a}$ determines a quasicharacter ω on $G(\mathbb{A})$, trivial on $G(F)$. We assume that ω is unitary and trivial on $Z(G)^\theta(\mathbb{A})$. Consider the Hilbert space

$$L^2 := L^2(G(F)\backslash G(\mathbb{A}))$$

and let

$$f \in C_c^\infty(G(\mathbb{A})).$$

Then f gives us a convolution operator $R(f)$ on L^2. Moreover θ and ω give us unitary operators $R(\theta)$, $R(\omega)$ on L^2; $R(\theta)$ is given by composition with θ^{-1} and $R(\omega)$ is given by pointwise multiplication by ω. The composition

$$R(f)R(\theta)R(\omega)$$

is an integral operator with kernel

$$K(h,g) = \omega(g) \sum_{\delta \in G(F)} f(h^{-1}\delta\theta(g)).$$

Let $\delta \in G(F)$ be θ-semisimple and strongly θ-regular. Write I_δ for the θ-centralizer $\mathrm{Cent}_\theta(\delta, G)$ of δ. As in (3.3) we denote by T_δ the centralizer in G of I_δ^0; then T_δ is a maximal torus of G preserved by $\mathrm{Int}(\delta) \circ \theta$ and I_δ coincides with the fixed points of $\mathrm{Int}(\delta) \circ \theta$ on T_δ. We say that δ is *θ-elliptic* if the identity component of

$$I_\delta / Z(G)^\theta$$

is anisotropic over F.

Denote by $G(F)_e$ the set of $\delta \in G(F)$ that are θ-semisimple, strongly θ-regular and θ-elliptic. Denote by $K_e(h,g)$ the corresponding part of the kernel $K(h,g)$:

$$K_e(h,g) := \omega(g) \sum_{\delta \in G(F)_e} f(h^{-1}\delta\theta(g)).$$

We are interested in the part of the twisted trace formula coming from $G(F)_e$, namely

$$T_e(f) := \int_{G(F)\backslash G(\mathbb{A})} K_e(g,g)\, dg/dx.$$

As usual we can rewrite $T_e(f)$ as a sum of twisted orbital integrals (see (6.1.1))

$$T_e(f) = \sum_{\delta\in\Delta} c_G \cdot c_\delta \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f). \tag{1}$$

Here $\tau(I_\delta)$ denotes the Tamagawa number of the diagonalizable group I_δ, and $O_{\delta\theta}(f)$ denotes the twisted orbital integral

$$\int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt.$$

The numbers c_G and c_δ are defined in (6.1) (note that c_G is 1 since we assumed that $Z(G)^0$ is anisotropic). The sum is taken over a set Δ of representatives for the θ-conjugacy classes of elements $\delta \in G(F)_e$ such that ω is trivial on $I_\delta(\mathbb{A})$.

The next step (see (6.2)) is to rewrite (1) by combining the terms indexed by δ,δ' whenever δ,δ' are θ-conjugate under $G(\mathbb{A})$. Fix an element $\delta \in \Delta$. The set of $\delta' \in \Delta$ such that δ' is θ-conjugate to δ under $G(\mathbb{A})$ is in natural bijection with a certain finite abelian group (namely the dual finite abelian group to the group $B = B_\delta$ defined in (6.2)). Using that ω is trivial on $I(\mathbb{A})$, we associate to $\mathbf{a}$ an element $\beta(\mathbf{a}) \in B_\delta$. If $\beta(\mathbf{a})$ is non-trivial, then the total contribution of the θ-conjugacy class of δ under $G(\mathbb{A})$ is 0; otherwise it is $|B_\delta|$ times

$$c_G \cdot c_\delta \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f).$$

Therefore

$$T_e(f) = \sum_{\delta\in\Delta_1} c_G \cdot c_\delta \cdot |B_\delta| \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f), \tag{2}$$

where Δ_1 is a set of representatives for the θ-conjugacy classes under $G(\mathbb{A})$ of elements $\delta \in G(F)_e$ such that ω is trivial on $I_\delta(\mathbb{A})$ and the element $\beta(\mathbf{a}) \in B_\delta$ is trivial.

To proceed further we need to define an obstruction obs(δ). For standard endoscopy this obstruction is due to Langlands [L2,p. 137]. Let G^*, ψ, θ^* and g_θ be as in (1.2). Thus G^* is a quasi-split inner form of G, $\psi : G \to G^*$ is an inner twisting, θ^* is an F-automorphism of G^* preserving an F-splitting, and $g_\theta \in G^*_{\mathrm{sc}}$ has the property that

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}.$$

As in (3.1) we choose, for each $\sigma \in \Gamma$, an element $u(\sigma) \in G^*_{\mathrm{sc}}$ such that

$$\psi\sigma(\psi)^{-1} = \mathrm{Int}(u(\sigma)),$$

and we also define a morphism

$$m : G \to G^*$$

over $\overline{F}$ by

$$m(\delta) := \psi(\delta)g_\theta^{-1}.$$

Then, as in (3.1),

$$\sigma(m)(\delta) = u(\sigma)^{-1}m(\delta)z_\sigma\theta^*(u(\sigma))$$

for a 1-cochain z_σ of Γ in $Z^{\mathrm{sc}}(\overline{F})$, where Z^{sc} denotes the center of G^*_{sc}. Recall from Lemma 3.1.A that the image $\bar{z}_\sigma$ of z_σ under

$$Z^{\mathrm{sc}} \to Z^{\mathrm{sc}}_\theta := Z^{\mathrm{sc}}/(1-\theta^*)Z^{\mathrm{sc}}$$

is a 1-cocycle.

Let (B,T) be a θ^*-stable pair in G^* with T defined over F. Put $V := (1-\theta^*)T$ and $U := T/V$. Note that the map

$$\pi : G^*_{\mathrm{sc}} \to G^*$$

induces a map

$$Z^{\mathrm{sc}}_\theta \to U.$$

Let γ be an element of $U(\overline{\mathbb{A}})$ such that

$$\sigma(\gamma) = \gamma\bar{z}_\sigma \tag{3}$$

for all $\sigma \in \Gamma$. Let $\delta \in G(\mathbb{A})$. We say that γ is a *norm* of δ if there exist $\delta^* \in T(\overline{\mathbb{A}})$ and $g \in G^*_{\mathrm{sc}}(\overline{\mathbb{A}})$ such that

(4) the image of δ^* in $U(\overline{\mathbb{A}})$ equals γ, and
(5) $\delta^* = gm(\delta)\theta^*(g)^{-1}$.

Now let γ be an element of $U(\overline{F})$ satisfying (3), let $\delta \in G(\mathbb{A})$ and suppose that γ is a norm of δ. Suppose further that γ is fixed by no non-trivial θ^*-invariant element of the Weyl group of T, so that δ is strongly θ-regular. Then in (6.3) we define an element

$$\mathrm{obs}(\delta) \in H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V),$$

depending only on the θ-conjugacy class of δ under $G_{\mathrm{sc}}(\mathbb{A})$, and having the property that $\mathrm{obs}(\delta)$ is trivial if and only if δ is θ-conjugate under $G_{\mathrm{sc}}(\mathbb{A})$ to an element of $G(F)$.

Continue with $\gamma \in U(\overline{F})$ as above and assume further that $T_{\mathrm{sc}}^{\theta^*}$ is anisotropic over F. We denote by $T_e(f)_\gamma$ the part of the sum (2) indexed by elements $\delta \in \Delta_1$ for which γ is a norm. Define an abelian group $\mathfrak{K}(T,\theta,F)$ by

$$\mathfrak{K}(T,\theta,F) := H^1(W_F, \hat{V} \xrightarrow{\hat{\pi}\circ\phi} \hat{T}/Z(\hat{G}))$$

where

$$\hat{V} \xrightarrow{\phi} \hat{T}$$

is dual to

$$T \xrightarrow{1-\theta^*} V.$$

By duality (see Lemma C.2.C)

$$\mathfrak{K}(T,\theta,F) \simeq \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V), \mathbb{C}^\times).$$

There is a natural homomorphism (see (6.4))

$$\mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})). \tag{6}$$

If there is no element of Δ_1 having γ as norm, then $T_e(f)_\gamma = 0$. Otherwise we fix such an element $\delta_0 \in \Delta_1$, and we also fix an element

$$\kappa_0 \in \mathfrak{K}(T,\theta,F)$$

mapping to **a** under (6) (see (6.4) for a proof of the existence of κ_0). For any element $\delta \in G(\mathbb{A})$ having γ as norm we have (see (6.3)) an element

$$\mathrm{inv}(\delta_0,\delta) \in H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V),$$

and the map

$$\delta \mapsto \mathrm{inv}(\delta_0,\delta)$$

sets up a bijection from the set of θ-conjugacy classes under $G_{\mathrm{sc}}(\mathbb{A})$ of elements $\delta \in G(\mathbb{A})$ having γ as norm to the set

$$C_0 := \ker[H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathbb{A}, G_{\mathrm{sc}})]$$

(the map whose kernel we are taking is of course induced by the inclusion $T_{\mathrm{sc}} \to G_{\mathrm{sc}}$).

For any element $\delta \in G(\mathbb{A})$ having γ as norm we define the twisted orbital integral

$$O_{\delta\theta}(f) := \int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt.$$

Define a function Φ on C_0 by putting

$$\Phi(x) = \langle \mathrm{obs}(\delta), \kappa_0\rangle O_{\delta\theta}(f)$$

where $\delta \in G(\mathbb{A})$ has norm γ and is such that

$$\mathrm{inv}(\delta_0,\delta) = x.$$

Define an abelian group

$$\mathcal{E}(T,\theta,F) := \mathrm{im}[H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, T \xrightarrow{1-\theta^*} V)]$$

and a set

$$\mathcal{D}(T,\theta,F) := \ker[H^1(F, T \xrightarrow{1-\theta^*} V) \to H^1(F,G)].$$

It is not hard to see that $\mathcal{D}(T,\theta,F)$ is the image under

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, T \xrightarrow{1-\theta^*} V)$$

of

$$\ker[H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, G_{\mathrm{sc}})]$$

and consequently that $\mathcal{D}(T,\theta,F)$ is a subset of $\mathcal{E}(T,\theta,F)$. There is a natural bijection between $\mathcal{D}(T,\theta,F)$ and the set of θ-conjugacy classes under $G(F)$ of elements $\delta \in G(F)$ having γ as norm. Replacing F by F_v or $\mathbb{A}$ in these definitions, we get $\mathcal{E}(T,\theta,F_v)$, $\mathcal{D}(T,\theta,T_v)$, $\mathcal{E}(T,\theta,\mathbb{A})$, $\mathcal{D}(T,\theta,\mathbb{A})$ satisfying the analogs of the properties of $\mathcal{E}(T,\theta,F)$, $\mathcal{D}(T,\theta,F)$ mentioned above.

In (6.4) we check that $\Phi(x)$ depends only on the image of x in $\mathcal{E}(T,\theta,\mathbb{A})$, and therefore Φ may be regarded as a function on the image of C_0 in $\mathcal{E}(T,\theta,\mathbb{A})$, namely $\mathcal{D}(T,\theta,\mathbb{A})$. We extend Φ to a function on all of $\mathcal{E}(T,\theta,\mathbb{A})$ by making it 0 on the complement of $\mathcal{D}(T,\theta,\mathbb{A})$.

Since c_δ, $|B_\delta|$, $\tau(I_\delta)$ depend only on T, we obtain a rational number c_T depending only on T by putting

$$c_T := c_G \cdot c_{\delta_0} \cdot |B_{\delta_0}| \cdot \tau(I_{\delta_0}).$$

Since $\mathrm{obs}(\delta) = 1$ for any $\delta \in \Delta_1$ (see Lemma 6.3.A), we have

$$T_e(f)_\gamma = c_T \sum_{x \in S} \Phi(x),$$

where S is the set

$$\mathrm{im}[\mathcal{D}(T,\theta,F) \to \mathcal{E}(T,\theta,\mathbb{A})].$$

To simplify notation we now write E for $\mathcal{E}(T,\theta,\mathbb{A})$ and E_0 for

$$\mathrm{im}[\mathcal{E}(T,\theta,F) \to \mathcal{E}(T,\theta,\mathbb{A})].$$

In (6.4) we show that

$$S = \mathcal{D}(T,\theta,\mathbb{A}) \cap E_0.$$

Therefore

$$T_e(f)_\gamma = c_T \sum_{x \in E_0} \Phi(x). \tag{7}$$

Lemma 6.4.A says that the subgroup E_0 of E is discrete and that the quotient E/E_0 is compact. Moreover it says that the function Φ on E is locally constant and compactly supported. Applying the Poisson summation formula to E_0, E and Φ, we get

$$T_e(f)_\gamma = c_T \sum_{\xi} \int_E \Phi(e)\langle e, \xi\rangle\, de, \tag{8}$$

where the sum is taken over all characters ξ on the compact group E/E_0 and de is the unique Haar measure on E giving E/E_0 total measure 1.

In (6.4) we check that the kernel $\mathfrak{K}(T,\theta,F)_1$ of the map (6) maps onto the Pontryagin dual of E/E_0 and that the kernel of this surjection has order

$$d_T := |\ker^2(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)|.$$

In (6.4) we define the Tamagawa measure de_{Tam} on E and use it to define the Tamagawa number $\tau(\mathcal{E})$ of the compact group E/E_0. Then

$$T_e(f)_\gamma = c_T d_T^{-1} \tau(\mathcal{E})^{-1} \sum_{\kappa} O^{\kappa}_{\delta_0\theta}(f), \tag{9}$$

where κ runs over the inverse image under the map (6) of the element $\mathbf{a}$, and where

$$O^{\kappa}_{\delta_0\theta}(f) = \int_{\mathcal{D}(T,\theta,\mathbb{A})} \langle e,\kappa\rangle O_{\delta_e\theta}(f)\, de_{\mathrm{Tam}}$$

(here δ_e is such that $\mathrm{inv}(\delta_0,\delta_e) = e$). In Lemma 6.4.B we show that

$$c_T \cdot d_T^{-1} \cdot \tau(\mathcal{E})^{-1}$$

is a constant a_G independent of T, and we give a simple formula for a_G in terms of $Z(\hat{G})$.

In the last part of (6.4) we find a formula (see (6.4.16))

$$T_e(f)_\gamma = a_G \sum_{\kappa} O^{\kappa}_{\gamma}(f), \tag{10}$$

analogous to (9), but which holds even when there is no element $\delta_0 \in \Delta_1$ having γ as norm (in which case $T_e(f)_\gamma = 0$). Summing (10) over γ, we get Theorem 6.4.C, which says that

$$T_e(f) = a_G \sum_{(T,\gamma,\kappa)} O^{\kappa}_{\gamma}(f). \tag{11}$$

Our next goal is to rewrite the expression (11) for $T_e(f)$ in terms of stable trace formulas for endoscopic groups H associated to $(G,\theta,\mathbf{a})$ (see (7.4.4) for the final result). In (7.1) we give a different interpretation of elements in $\mathfrak{K}(T,\theta,F)$, which we then use in (7.2) to describe the index set appearing in (11) in terms of elliptic endoscopic data $(H,\mathcal{H},s,\xi)$ for $(G,\theta,\mathbf{a})$ and elliptic strongly G-regular elements $\gamma_H \in H(\overline{F})$ such that

$$\sigma(\gamma_H) = \gamma_H \bar{z}_\sigma \qquad (\sigma \in \Gamma).$$

In (7.3) we define absolute adelic transfer factors $\Delta_{\mathbb{A}}(\gamma_1,\delta)$ and show (Corollary 7.3.B) that for $\gamma_1 \in H_1(\overline{F})$ as above

$$\Delta_{\mathbb{A}}(\gamma_1,\delta) = \langle \mathrm{obs}(\delta), \kappa\rangle.$$

We then use the adelic transfer factor to define a global notion of functions with matching orbital integrals.

Finally, assuming that f admits matching functions f^{H_1}, we show in (7.4) that $T_e(f)$ can be written as the sum over $(H, \mathcal{H}, s, \xi)$ of (the θ_H-elliptic strongly G-regular part of) the stable θ_H-twisted trace formula for f^{H_1}. Here (see (7.3)) θ_H is an F-automorphism of H of the form $\mathrm{Int}(x)$ for some $x \in H_{\mathrm{ad}}(F)$ (if the 1-cochain z_σ is trivial, we may take $\theta_H = \mathrm{id}_H$). This concludes our summary of the stabilization process.

The appendices to this paper prove various facts about Galois hypercohomology for complexes

$$T \xrightarrow{f} U$$

of F-tori. The main point of Appendix A is to construct the local pairing (A.3.12)

$$H^1(F, T \xrightarrow{f} U) \otimes H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times.$$

The construction is complicated and unfortunately relies on lengthy cocycle calculations; we wish we knew a more conceptual approach. In addition this appendix develops properties of the pairing and of the hypercohomology groups themselves. Appendix B classifies inner forms of (G, θ) over p-adic fields. Although this result is not needed in the rest of the paper, it gives a clearer picture of the objects we are studying. Appendix C has the global duality theorems we need (Lemmas C.2.A, C.2.B, C.2.C, C.3.B) as well as a compactness theorem (Lemma C.2.D). Appendix D reviews Tate-Nakayama duality in the form we need. Appendix E generalizes work of Ono [O] and Oesterlé [Oe] by putting Tamagawa measures on the hypercohomology groups

$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)$$

and proving a formula (Lemma E.3.D) for

$$\mathrm{vol}(H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1)\,\mathrm{vol}(H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1)^{-1}.$$

The following notation is used throughout the paper: F denotes a local or global field of characteristic 0, $\overline{F}$ an algebraic closure of F, Γ the Galois group of $\overline{F}/F$, W_F the Weil group of $\overline{F}/F$, and (in the global case) $\mathbb{A} = \mathbb{A}_F$ the adele ring of F. Given a connected reductive F-group, we write $\hat{G}$ for the connected reductive complex group dual to G, and we write ${}^L G$ for the L-group

$${}^L G := \hat{G} \rtimes W_F.$$

For a connected reductive group G over any field we write G_{der} for the derived group of G and G_{sc} for the simply connected cover of G_{der}. We denote by π the natural map

$$G_{\mathrm{sc}} \to G.$$

We write G_{ad} for the quotient of G by its center $Z(G)$. Given a maximal torus T in G we put

$$T_{\text{der}} = T \cap G_{\text{der}}$$
$$T_{\text{sc}} = \pi^{-1}(T)$$
$$T_{\text{ad}} = T/Z(G) \subset G/Z(G) = G_{\text{ad}};$$

these are maximal tori in G_{der}, G_{sc}, G_{ad} respectively. We use superscripts to denote invariants and subscripts to denote coinvariants (for the action of a group—typically Γ—or the action of a single automorphism—typically θ). We write $\text{Int}(x)$ for the inner automorphism

$$g \mapsto xgx^{-1} \qquad (g \in G)$$

obtained from an element x in a group G. We write A_{tors} for the torsion subgroup of an abelian group A. We sometimes write A^D for the Pontryagin dual of a finite abelian group A. The notation we use for global hypercohomology groups is explained in Appendix C.

1. Automorphisms and L-groups

We collect some well-known results and add a few observations for which we have been unable to locate a reference. Our main object is then to define suitable a-data and χ-data [LS1] for twisted endoscopy.

(1.1) Automorphisms. We start with a connected reductive group G over a field F of characteristic zero. By a *pair* in G we mean a couple (B, T), with B a Borel subgroup of G and T a maximal torus in B, and by a *splitting* of G we mean a triple $(B, T, \{X\})$, where (B, T) is a pair in G and $\{X\}$ is a collection of root vectors, one for each simple root of T in B. Recall that G is quasi-split over F if and only if it has an F-splitting, *i.e.* one preserved by $\Gamma = \text{Gal}(\overline{F}/F)$ (*e.g.* [LS1]).

Suppose that θ is an automorphism of G defined over F. We call θ *quasi-semisimple* if the restriction of θ to the derived group of G is semisimple. Since F has characteristic zero θ is quasi-semisimple if and only if it preserves a pair (B, T) in G, *i.e.* we have $\theta(B) = B$ and $\theta(T) = T$ (see [St, Sect. 9]). Then write G^1 for the identity component of the group of fixed points of θ in G. We have the following structure result, due to Steinberg.

Theorem 1.1.A.

(1) *G^1 is reductive.*

(2) *Suppose that the pair (B, T) in G is θ-stable. Set $B^1 = B \cap G^1$ and $T^1 = T \cap G^1$. Then (B^1, T^1) is a pair in G^1. Denote by $R(B, T)$ the set of roots of T in B. Then $R(B^1, T^1)$ is contained in*

$$\{\alpha_{\text{res}} = \alpha|_{T^1} : \alpha \in R(B, T)\}.$$

If $\alpha \in R(B, T)$ then $\alpha_{\text{res}} \in R(B^1, T^1)$ if and only if there is a θ-fixed element in the span of the root spaces for those $\beta \in R(B, T)$ such that $\beta_{\text{res}} = \alpha_{\text{res}}$.

(3) *Let (B^1, T^1) be a pair in G^1. Then there exists a θ-stable pair in G such that $B^1 = B \cap G^1$ and $T^1 = T \cap G^1$.*

(4) *With T and T^1 as above, we have $T = \text{Cent}(T^\theta, G) = \text{Cent}(T^1, G)$.*

Most of this is proved in [St]. The rest is straightforward. See also (1.3).

We continue with θ-stable (B,T) and the attached (B^1,T^1). Then θ acts on the Weyl group $\Omega(G,T)$ of T in G. Denote by $\Omega(G,T)^\theta$ the group of fixed points of θ. Then by (4) above, $\mathrm{Norm}(T^1,G^1)$ coincides with $\mathrm{Norm}(T,G^1)$ and so $\Omega(G^1,T^1)$ embeds in $\Omega(G,T)^\theta$. We identify $\Omega(G^1,T^1)$ with its image. Note that the condition that $\omega \in \Omega(G,T)$ lie in $\Omega(G,T)^\theta$ can be given by any one of $\omega\theta = \theta\omega$, $\omega(T^\theta) = T^\theta$ or $\omega(T^1) = T^1$. If (C,U) is another θ-stable pair in G then up to G^1-conjugacy we have that U coincides with T and C is conjugate to B under $\Omega(G,T)^\theta$. Finally, a maximal torus which is θ-stable and contained in some θ-stable Borel subgroup, *i.e.* which is a component of a θ-stable pair, will be called *θ-admissible*. Observe that there exist θ-admissible maximal tori defined over F.

We will be particularly concerned with automorphisms which preserve splittings. Suppose then that θ preserves $\mathbf{spl} = (B,T,\{X\})$. This implies that the simple roots in $R(B^1,T^1)$ are exactly the restrictions to T^1 of the simple roots in $R(B,T)$. As a consequence, the embedding of the Weyl group $\Omega(G^1,T^1)$ in $\Omega(G,T)^\theta$ is surjective. We shall not distinguish in notation between $\Omega(G^1,T^1)$ and $\Omega(G,T)^\theta$. Further, (B^1,T^1) determines (B,T) uniquely and any two θ-stable pairs in G are conjugate under G^1. Thus for any θ-stable pair (C,U) in G there is a splitting $(C,U,\{Y\})$ preserved by θ.

If G is simply connected then the group G^θ of fixed points of θ in G is connected, for any semisimple θ [St]. If θ preserves a splitting $(B,T,\{X\})$ we have, by the comment on Weyl groups in the last paragraph, that for any G the group G^θ is connected if and only if T^θ is connected. If G is adjoint then

$$X^*(T^\theta) = X^*(T)_\theta := X^*(T)/(1-\theta)X^*(T)$$

is torsion-free and so T^θ is connected. This implies that for any G with θ preserving a splitting we have

$$G^\theta = G^1 Z(G)^\theta$$

where $Z(G)$ is the center of G.

(1.2) Quasi-split forms and L-groups. We now restrict our attention to a field F that is either local or global: θ is an arbitrary automorphism over F of a connected reductive group G over F. To fix quasi-split data it is useful to take a quasi-split group G^* over F with an inner class $\Psi = \{\psi = (\mathrm{Int}\, g) \circ \psi_0 : g \in G^*\}$ of inner twistings $\psi : G \to G^*$ and check that the choice of ψ within this class has no effect on our constructions. The group G^* has an F-splitting $\mathbf{spl}_{G^*} = (B^*,T^*,\{X^*\})$ and we may choose $g_\theta \in G^*_{\mathrm{sc}}$ such that the automorphism

$$\theta^* := \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}$$

of G^* preserves $\mathbf{spl}_{G^*}$. The automorphism θ^* is uniquely determined by $\mathbf{spl}_{G^*}$ and so $\sigma(\theta^*) = \theta^*$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, and θ^* is defined over F. We will see that the choice of $\mathbf{spl}_{G^*}$ has no effect on our constructions.

Suppose $\hat{G}, \rho, \eta_G$ are L-group data for G: $\hat{G}$ is a connected reductive group over $\mathbb{C}$, ρ is an L-action of Γ on $\hat{G}$ and $\eta_G : \Psi(G)^\vee \to \Psi(\hat{G})$ is a Γ-bijection between canonical based root data (see [K1]). The automorphism θ of G induces bijections $\theta : \Psi(G) \to \Psi(G)$ and $\theta^\vee : \Psi(G)^\vee \to \Psi(G)^\vee$. Then $\hat{\theta}$ will be an automorphism of $\hat{G}$ which induces the bijection $\eta_G \cdot \theta^\vee \cdot \eta_G^{-1}$ on $\Psi(\hat{G})$. For convenience we fix once and for all a Γ-splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of $\hat{G}$ and assume it is preserved by $\hat{\theta}$.

As always, ${}^LG$ will denote the semi-direct product $\hat{G} \rtimes W_F$ for the action given by ρ. The automorphism $\hat{\theta} \rtimes 1_{W_F}$ of ${}^LG$ will be denoted ${}^L\theta$.

(1.3) Restricted roots. Suppose G is quasi-split over F and θ preserves the F-splitting $\mathbf{spl} = (B, T, \{X\})$ of G. Here we review, somewhat repetitively, more detailed information about the root systems of θ-twisted centralizers of θ-semisimple elements, *i.e.* of the fixed points of quasi-semisimple automorphisms of the form $\mathrm{Int}\,\delta \circ \theta$ for $\delta \in G$. This will enable us to introduce a-data and χ-data in a convenient form.

Until we discuss a-data and χ-data we need only that F is of characteristic zero. There will be no harm in taking G *simply connected.* Then, in particular, G^θ and T^θ are connected. Set

$$R_{\mathrm{res}}(G,T) = \{\alpha_{\mathrm{res}} = \alpha|_{T^\theta} : \alpha \in R(G,T)\}.$$

For the purposes of this paper, α_{res} is a *restricted root.* Then:

(1.3.1) $R_{\mathrm{res}}(G,T)$ is a root system in 1-1 correspondence with the set of θ-orbits in $R(G,T)$,

(1.3.2) the irreducible components of $R_{\mathrm{res}}(G,T)$ are in 1-1 correspondence with θ-orbits of irreducible components of $R(G,T)$,

(1.3.3) an irreducible component of $R_{\mathrm{res}}(G,T)$ is reduced unless it corresponds to a θ-orbit of components of $R(G,T)$ with the following property: each component D in this θ-orbit is of type A_{2n} and for some positive integer m, θ^m preserves and acts nontrivially on D.

(1.3.4) The set of indivisible roots in $R_{\mathrm{res}}(G,T)$ coincides with $R(G^\theta, T^\theta)$.

We classify $\alpha_{\mathrm{res}} \in R_{\mathrm{res}}(G,T)$ as follows.

Type R_1 : $2\alpha_{\mathrm{res}}, \frac{1}{2}\alpha_{\mathrm{res}} \notin R_{\mathrm{res}}(G,T)$
Type R_2 : $2\alpha_{\mathrm{res}} \in R_{\mathrm{res}}(G,T)$
Type R_3 : $\frac{1}{2}\alpha_{\mathrm{res}} \in R_{\mathrm{res}}(G,T)$

Let l_α be the cardinality of the θ-orbit of $\alpha \in R(G,T)$ and

$$N\alpha = \sum_{i=0}^{l_\alpha - 1} \theta^i \alpha.$$

Let $\delta \in T$.

(1.3.5) if α_{res} is of type R_1 then the roots in the θ-orbit of α are mutually perpendicular and $\theta^{l_\alpha} X_\alpha = X_\alpha$; α_{res} is a root for G^θ and it is a root for the connected fixed points $G^{\delta\theta}$ of $\mathrm{Int}\,\delta \circ \theta$ if and only if $N\alpha(\delta) = 1$,

(1.3.6) if α_{res} is of type R_2 then in the notation of (1.3.3) $l_\alpha = 2m$ and $\alpha + \theta^m\alpha$ is a root; again $\theta^{l_\alpha} X_\alpha = X_\alpha$ and α_{res} is a root for G^θ, and it is a root for $G^{\delta\theta}$ if and only if $N\alpha(\delta) = 1$.

(1.3.7) if α_{res} is of type R_3 then $\alpha = \beta + \theta^m\beta$ with β of type R_2 and m as in (1.3.3); $l_\alpha = m$ and $\theta^m X_\alpha = -X_\alpha$; α_{res} is not a root for G^θ but it is a root of $G^{\delta\theta}$ if and only if $N\alpha(\delta) = -1$. Note also that $N\alpha = N\beta$.

The results (1.3.1), ... ,(1.3.7) are to be found mainly in [St]. See also [Spr, Chapter 11]. Some additional, but very simple, arguments are needed for the case θ permutes the irreducible components of $R(G,T)$ nontrivially. Details are left to the reader.

Now assume F is either local or global. We have also the dual $\hat{\theta}$ and $R_{\text{res}}(\hat{G}, \mathcal{T})$ which we may identify as

$$\{(\alpha^\vee)_{\text{res}} = \alpha^\vee|_{\hat{T}^{\hat{\theta}}} : \alpha^\vee \in R^\vee(G,T)\}.$$

Note that $\hat{T}^{\hat{\theta}}$ is connected because $\hat{G}$ is adjoint and $\hat{\theta}$ preserves a splitting. Then:

(1.3.8) $\alpha_{\text{res}} \mapsto (\alpha^\vee)_{\text{res}}$ is a well-defined Γ-bijection from $R_{\text{res}}(G,T)$ to $R_{\text{res}}(\hat{G}, \mathcal{T})$. This bijection preserves types: if $n\alpha_{\text{res}}$ is also a restricted root ($n = \frac{1}{2}, 2$) then so is $n(\alpha^\vee)_{\text{res}}$ and $n(\alpha^\vee)_{\text{res}}$ is the image of $n\alpha_{\text{res}}$.

Observe that $(\alpha^\vee)_{\text{res}} \in X^*(\hat{T}^{\hat{\theta}}) = X^*(\hat{T})_{\hat{\theta}}$ has its coroot in $X^*(T)^\theta$.

(1.3.9) If α_{res} is of types R_1 or R_3 then the coroot of $(\alpha^\vee)_{\text{res}}$ is $N\alpha$. Otherwise (type R_2) it is $2N\alpha$.

The proof of (1.3.8) and (1.3.9) follows easily from the arguments for the earlier assertions. For example, to compute the coroot of $(\alpha^\vee)_{\text{res}}$, see [Spr, p. 292] for type R_1 and α simple. This extends to all α of type R_1 by the Weyl group action. For types R_2 and R_3 we reduce quickly to an explicit calculation in a root system of type A_2.

Following [LS1, (2.2),(2.5)] we attach a-data and χ-data to $R_{\text{res}}(G,T)$, or to $R_{\text{res}}(\hat{G}, \mathcal{T})$ *via* $\alpha_{\text{res}} \leftrightarrow \alpha^\vee_{\text{res}}$ since this bijection respects Γ and $\alpha \mapsto -\alpha$. We may and shall assume that for all $\alpha_{\text{res}} \in R_{\text{res}}(G,T)$ we have:

(1.3.10) $a_{n\alpha_{\text{res}}} = a_{\alpha_{\text{res}}}$ and $\chi_{n\alpha_{\text{res}}} = \chi_{\alpha_{\text{res}}}$ if $n\alpha_{\text{res}} \in R_{\text{res}}(G,T)$.

2. Endoscopy

(2.1) Definitions. Again F will be either local or global and θ will be an automorphism over F of a connected reductive group G over F. Endoscopic data are attached to a triple $(G, \theta, \mathbf{a})$ where $\mathbf{a}$ is an element in $H^1(W_F, Z(\hat{G}))$ if F is local, or in the quotient of $H^1(W_F, Z(\hat{G}))$ by the everywhere locally trivial elements if F is global. We fix once and for all a 1-cocycle a of W_F in $Z(\hat{G})$ representing $\mathbf{a}$. This choice will have no effect on our definitions. From $\mathbf{a}$ we obtain a quasicharacter ω on $G(F)$ if F is local, or on $G(\mathbb{A})$ trivial on $G(F)$ if F is global.

We call the tuple $(H, \mathcal{H}, s, \xi)$ *endoscopic data* for $(G, \theta, \mathbf{a})$ if:

(2.1.1) H is a quasi-split group over F,

(2.1.2) $\mathcal{H}$ is a split extension of W_F by $\hat{H}$ such that the L-action $\rho_{\mathcal{H}}$ of W_F on $\hat{H}$ determined by this extension (see (2.2)) coincides with ρ_H (see (1.2) for a discussion of ρ_H),

(2.1.3) s is a $\hat{\theta}$-semisimple element of $\hat{G}$, *i.e.* the automorphism $\text{Int}(s) \circ \hat{\theta}$ is quasi-semisimple,

(2.1.4) $\xi : \mathcal{H} \to {}^LG$ is an L-homomorphism satisfying the following two conditions:

(2.1.4a) $\mathrm{Int}(s) \circ {}^L\theta \circ \xi = a' \cdot \xi$ where $a' : W_F \to Z(\hat{G})$ is a 1-cocycle which is equivalent to a if F is local, or is everywhere locally equivalent to a if F is global,

(2.1.4b) ξ maps $\hat{H}$ isomorphically onto the identity component of $\mathrm{Cent}_{\hat{\theta}}(s, \hat{G})$, the group of fixed points of $\mathrm{Int}(s) \circ \hat{\theta}$.

Observe that if we define the automorphism ${}^L\theta_{a'}$ of ${}^LG$ by

$$ {}^L\theta_{a'}(g \times w) = \hat{\theta}(g) a'(w)^{-1} \times w $$

for $g \in \hat{G}$ and $w \in W_F$, then the equation in (2.1.4a) may be replaced by the inclusion

$$ \xi(\mathcal{H}) \subset \mathrm{Cent}_{{}^L\theta_{a'}}(s, {}^LG). $$

Let us be more precise about the terminology used in this definition. In (2.1.2) when we say that $\mathcal{H}$ is an extension of W_F by $\hat{H}$ we mean that $\hat{H}$ is a normal subgroup of the topological group $\mathcal{H}$ and that we are given an isomorphism of topological groups $\mathcal{H}/\hat{H} \simeq W_F$. In particular $\hat{H}$ is closed in $\mathcal{H}$ and the composition of $\mathcal{H} \to \mathcal{H}/\hat{H}$ and the isomorphism $\mathcal{H}/\hat{H} \to W_F$ is a continuous, open homomorphism $p : \mathcal{H} \to W_F$. When we say that the extension $\mathcal{H}$ of W_F by $\hat{H}$ is split we mean that there exists a continuous homomorphism

$$ c : W_F \to \mathcal{H} $$

such that $p \circ c$ is the identity map on W_F. In (2.1.4), when we say that ξ is an L-homomorphism we mean that it is a continuous homomorphism $\xi : \mathcal{H} \to {}^LG$ such that the composition

$$ \mathcal{H} \xrightarrow{\xi} {}^LG \to W_F $$

is equal to $p : \mathcal{H} \to W_F$ and such that the restriction of ξ to $\hat{H}$ is a homomorphism of algebraic groups from $\hat{H}$ to $\hat{G}$. The map $(h, w) \mapsto c(w)h$ is a homeomorphism from $\hat{H} \times W_F$ to $\mathcal{H}$ and therefore $\mathcal{H}$ is locally compact and Hausdorff. Moreover $\xi \circ c$ splits $\xi(\mathcal{H}) \to W_F$, so that $(h, w) \mapsto \xi(c(w)h)$ is a homeomorphism from $\hat{H} \times W_F$ to $\xi(\mathcal{H})$, which shows that ξ induces an isomorphism of topological groups from $\mathcal{H}$ to $\xi(\mathcal{H})$. In particular $\xi(\mathcal{H})$ is locally compact, which implies that it is a closed subgroup of ${}^LG$.

We call $(H, \mathcal{H}, s, \xi)$ *elliptic* if $\xi(Z(\hat{H})^\Gamma)^0$ is contained in $Z(\hat{G})$. By *standard* endoscopic data for a group G we will mean data for $(G, \mathrm{id}, \mathrm{triv})$, *i.e.* the data of [LS1] (see also [L1],[L2],[S1],[K1]). In general endoscopic data for $(G, \theta, \mathbf{a})$ coincides with that for the *quasi-split triple* $(G^*, \theta^*, \mathbf{a})$, where (G^*, θ^*) is as in (1.2).

An isomorphism from $(H, \mathcal{H}, s, \xi)$ to $(H', \mathcal{H}', s', \xi')$ is an element $g \in \hat{G}$ such that

(2.1.5) $g\xi(\mathcal{H})g^{-1} = \xi'(\mathcal{H}')$ and

(2.1.6) $gs\hat{\theta}(g)^{-1} = s$ modulo $Z(\hat{G})$.

Let g be such an isomorphism. We use ξ and ξ' to identify $\mathcal{H}$ and $\mathcal{H}'$ with subgroups of ${}^LG$. Then (2.1.5) becomes

$$ g\mathcal{H}g^{-1} = \mathcal{H}', $$

and we write β for the isomorphism

$$ \beta : \hat{H} \to \hat{H}' $$

obtained by restricting $\mathrm{Int}(g)$ to $\hat{H}$. Let $w \in W_F$. We claim that there exists $x \in \hat{H}$ such that

(2.1.7) $\rho_{H'}(w) \circ \beta \circ \mathrm{Int}(x) = \beta \circ \rho_H(w)$.

To prove this we choose an element $h_w \in \mathcal{H}$ having image w in W_F. Then $\rho_H(w)$ and $\mathrm{Int}(h_w)|_{\hat{H}}$ differ by an inner automorphism of $\hat{H}$. By (2.1.5) $\rho_{H'}(w)$ and $\mathrm{Int}(gh_wg^{-1})|_{\hat{H}'}$ differ by an inner automorphism of $\hat{H}'$. It then follows from the trivial equality

$$\mathrm{Int}(gh_wg^{-1}) \circ \mathrm{Int}(g) = \mathrm{Int}(g) \circ \mathrm{Int}(h_w)$$

that there exists $x \in \hat{H}$ such that (2.1.7) holds. Pick F-splittings of H and H', and let $\alpha : H \to H'$ be the unique $\overline{F}$-isomorphism dual to β^{-1} and preserving F-splittings; it follows from (2.1.7) that α is defined over F.

By an automorphism of $(H, \mathcal{H}, s, \xi)$ we mean an isomorphism from it to itself; these form a group $\mathrm{Aut}(H, \mathcal{H}, s, \xi)$. As above an automorphism g determines automorphisms $\beta \in \mathrm{Aut}(\hat{H})$ and $\alpha \in \mathrm{Aut}_F(H)$ (with α preserving the chosen F-splitting of H). We write $\mathrm{Out}_F(H)$ for $\mathrm{Aut}_F(H)/H_{\mathrm{ad}}(F)$. The map $g \mapsto \alpha$ is a well-defined homomorphism

$$\mathrm{Aut}(H, \mathcal{H}, s, \xi) \to \mathrm{Out}_F(H). \tag{2.1.8}$$

We denote by $\mathrm{Out}(H, \mathcal{H}, s, \xi)$ the image of the homomorphism (2.1.8).

Let K denote the kernel of (2.1.8). We claim that $K = \hat{H}C$, where C is the subgroup of $Z(\hat{G})$ consisting of elements $z \in Z(\hat{G})$ such that

$$\sigma(z)z^{-1} \in Z(\hat{G}) \cap \hat{H}$$

for all $\sigma \in \Gamma$. It is easy to see that $\hat{H} \subset K$ and that $Z(\hat{G}) \cap K = C$. It remains to prove that any element $g \in K$ belongs to $\hat{H}Z(\hat{G})$. Since α is trivial, β is inner, and by modifying g by an element of $\hat{H}$ we may assume that g centralizes $\hat{H}$. Without loss of generality we may assume that s belongs to $\mathcal{T}$, part of a $\hat{\theta}$-stable pair $(\mathcal{B}, \mathcal{T})$ in $\hat{G}$. Since g centralizes $\hat{H}$, it centralizes $(\mathcal{T}^{\hat{\theta}})^0$, and therefore it centralizes the centralizer $\mathcal{T}$ of $(\mathcal{T}^{\hat{\theta}})^0$, which implies that $g \in \mathcal{T}$. Then (2.1.6) implies that $\hat{\theta}$ fixes the image of g in $\mathcal{T}/Z(\hat{G})$. Since $(\mathcal{T}/Z(\hat{G}))^{\hat{\theta}}$ is connected (see (1.1)), g is of the form tz for some $t \in (\mathcal{T}^{\hat{\theta}})^0$ and $z \in Z(\hat{G})$, and therefore $g \in \hat{H}Z(\hat{G})$.

It follows from the equality $K = \hat{H}C$ that

$$\begin{aligned} K/\hat{H} &= C/(C \cap \hat{H}) \\ &= C/(Z(\hat{G}) \cap \hat{H}) \\ &= (Z(\hat{G})/Z_1)^{\Gamma}, \end{aligned}$$

where Z_1 is defined by

$$\begin{aligned} Z_1 &:= Z(\hat{G}) \cap \hat{H} \\ &= Z(\hat{G}) \cap (\mathcal{T}^{\hat{\theta}})^0; \end{aligned}$$

in other words

$$K/\hat{H} = \overline{Z}^{\Gamma}$$

where

$$\overline{Z} := Z(\hat{G})/Z_1.$$

We see from this discussion that there is a natural exact sequence

$$1 \to \overline{Z}^{\Gamma} \to \mathrm{Aut}(H, \mathcal{H}, s, \xi)/\hat{H} \to \mathrm{Out}(H, \mathcal{H}, s, \xi) \to 1,$$

and from this it follows that there is a natural exact sequence

$$1 \to \pi_0(\overline{Z}^{\Gamma}) \to \pi_0(\mathrm{Aut}(H, \mathcal{H}, s, \xi)) \to \mathrm{Out}(H, \mathcal{H}, s, \xi) \to 1.$$

(2.2) z**-pairs.** Here we comment briefly on the role of the datum $\mathcal{H}$ and introduce the notion of z-pair needed to formulate transfer in twisted endoscopy. Thus suppose that $(H, \mathcal{H}, s, \xi)$ is a set of endoscopic data for $(G, \theta, \mathbf{a})$. Along with L-group data $(\hat{H}, \rho_H, \eta_H)$ for H we have fixed a Γ-splitting $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ for $\hat{H}$, *i.e.* $\mathbf{spl}_{\hat{H}}$ is preserved by ρ_H. By the definition of $\mathcal{H}$ there is a split exact sequence

$$1 \to \hat{H} \to \mathcal{H} \to W_F \to 1.$$

Let $\mathcal{Z}$ be the subgroup of $\mathcal{H}$ consisting of all elements $h \in \mathcal{H}$ such that the restriction of $\mathrm{Int}(h)$ to $\hat{H}$ preserves $\mathbf{spl}_{\hat{H}}$; it is clear that $\mathcal{Z}$ is closed in $\mathcal{H}$ and hence that $\mathcal{Z}$ is locally compact. Since the topologies of W_F and $\hat{H}$ have countable bases, so do those of $\mathcal{H}$ (since it is homeomorphic to the product of $\hat{H}$ and W_F) and its subgroup $\mathcal{Z}$. Since $\hat{H}/Z(\hat{H})$ acts simply transitively on the set of splittings for $\hat{H}$, the projection $p : \mathcal{H} \to W_F$ maps $\mathcal{Z}$ *onto* W_F with kernel $Z(\hat{H})$, so that p induces a bijective continuous homomorphism

$$\mathcal{Z}/Z(\hat{H}) \to W_F.$$

Since $\mathcal{Z}/Z(\hat{H})$ and W_F are locally compact and the topology of $\mathcal{Z}/Z(\hat{H})$ has a countable base, this homomorphism is a homeomorphism [Sm, Theorem 1.1]. Therefore $\mathcal{Z}$ is an extension of W_F by $Z(\hat{H})$.

Use ρ_H to form ${}^L H$. Then the identity map from $\hat{H}$ to $\hat{H}$ can be extended to an L-isomorphism $\mathcal{H} \to {}^L H$ if and only if the extension $\mathcal{Z}$ is a split extension of W_F by $Z(\hat{H})$, and any two such isomorphisms differ by a 1-cocycle of W_F in $Z(\hat{H})$. However, such an isomorphism does not always exist. In the case of standard endoscopy ($\theta = \mathrm{id}$, $\mathbf{a} = 1$) we may replace G by a suitable central extension, for example a z-extension in the terminology of [K2], and then such an isomorphism $\mathcal{H} \simeq {}^L H$ does always exist (see [L1]). In general, even if G is itself semisimple and simply connected, such an isomorphism $\mathcal{H} \simeq {}^L H$ need not exist. We will work instead with z-extensions of H.

If H_1 is a z-extension of H then we have an exact sequence

$$1 \to Z_1 \to H_1 \to H \to 1$$

with Z_1 a central torus in H_1. Dual to this is

$$1 \to \hat{H} \to \hat{H}_1 \to \hat{Z}_1 \to 1$$

and so we may regard $\hat{H}$ as a subgroup of $\hat{H}_1$. We may assume that the restriction of ρ_{H_1} to $\hat{H}$ is ρ_H and embed ${}^L H$ naturally in ${}^L H_1$. There is also a natural extension of $\hat{H}_1 \to \hat{Z}_1$ to ${}^L H_1 \to {}^L Z_1$.

Lemma 2.2.A. *The inclusion of $\hat{H}$ in $\hat{H}_1$ can be extended to an L-homomorphism*

$$\xi_{H_1} : \mathcal{H} \to {}^L H_1.$$

Any such L-homomorphism induces an isomorphism of topological groups from $\mathcal{H}$ to its image $\xi_{H_1}(\mathcal{H})$, which is necessarily a closed subgroup of ${}^L H_1$.

Recall the splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ used to form ${}^L G$. We are free to replace (s, ξ) by $(gs\hat{\theta}(g)^{-1}, \mathrm{Int}(g) \circ \xi)$ for any $g \in \hat{G}$. Therefore we may assume that $s \in \hat{T}$. Then $(\mathcal{B}, \mathcal{T})$ is an $\mathrm{Int}(s) \circ \hat{\theta}$-stable pair in $\mathrm{Cent}_{\hat{\theta}}(s, \hat{G})^0$ and by making another such replacement, this time with $g \in \mathrm{Cent}_{\hat{\theta}}(s, \hat{G})^0$, we may assume that $\xi(\mathcal{B}_H) = \mathcal{B} \cap \xi(\hat{H})$ and $\xi(\mathcal{T}_H) = \mathcal{T} \cap \xi(\hat{H}) = (\mathcal{T}^\theta)^0$.

Consider the subgroup $\mathcal{U}$ of $\mathcal{H}$ consisting of all $x \in \mathcal{H}$ such that the restriction of $\mathrm{Int}(x)$ to $\hat{H}$ preserves the pair $(\mathcal{B}_H, \mathcal{T}_H)$. Then $\mathcal{U}$ is closed in $\mathcal{H}$, and the projection $p : \mathcal{H} \to W_F$ maps $\mathcal{U}$ onto W_F with kernel $\mathcal{T}_H$. The natural map $\mathcal{U}/\mathcal{T}_H \to W_F$ is an isomorphism of topological groups (use the same proof as we used for $\mathcal{Z}$). Thus we have an exact sequence

$$1 \to \mathcal{T}_H \to \mathcal{U} \to W_F \to 1.$$

Let $x \in \mathcal{U}$. Then x normalizes $\mathcal{T}_H$ and hence normalizes $\mathcal{T}$ as well, since $\mathcal{T}$ is the centralizer of $\mathcal{T}_H$ in $\hat{G}$. Thus $\mathcal{U}$ acts (by conjugation) on $\mathcal{T}$, and the kernel of this action is closed. Moreover this kernel has finite index, since any element $x \in {}^L G$ that normalizes $\mathcal{T}$ must act on $\mathcal{T}$ by $\rho_G(w)$ times some element of the Weyl group, where w is the image of x in W_F. Therefore this kernel is open of finite index, hence contains $\mathcal{T}_H$, hence is the inverse image under $p : \mathcal{U} \to W_F$ of an open subgroup of finite index in W_F. We conclude that the action of $\mathcal{U}$ on $\mathcal{T}$ factors through

$$\mathcal{U} \to W_F \to \mathrm{Gal}(K/F)$$

for some finite Galois extension K/F.

By enlarging K we may assume that ρ_G also factors through $\mathrm{Gal}(K/F)$. Since $\mathcal{U}$ normalizes $\mathcal{T}$ we get a subgroup $\mathcal{U}_G$ of ${}^L G$ by taking $\mathcal{U}_G = \mathcal{U}\mathcal{T}$; clearly the projection ${}^L G \to W_F$ maps $\mathcal{U}_G$ onto W_F with kernel $\mathcal{T}$. Note that $\mathcal{U}_G$ contains all elements of ${}^L G = \hat{G} \rtimes W_F$ of the form $x = 1 \times w$ for $w \in W_K$. Indeed, let y be any element of $\mathcal{U}$ such that $p(y) = w$. Then both x and y centralize $\mathcal{T}$, so that $xy^{-1} \in \hat{G}$ does as well, which implies that $xy^{-1} \in \mathcal{T}$. Therefore $\mathcal{U}_G$ is the full inverse image under

$${}^L G \to \hat{G} \rtimes \mathrm{Gal}(K/F)$$

of a subgroup $\mathcal{U}'_G$ of $\hat{G} \rtimes \mathrm{Gal}(K/F)$ which maps onto $\mathrm{Gal}(K/F)$ with kernel $\mathcal{T}$. The extension $\mathcal{U}'_G$ of $\mathrm{Gal}(K/F)$ by $\mathcal{T}$ gives rise to a 2-cocycle of $\mathrm{Gal}(K/F)$ in $\mathcal{T}$. By a result of Langlands [L1, Lemma 4] (see also [La]) the inflation of this 2-cocycle to W_F is the coboundary of a continuous 1-cochain of $W_{K/F}$ in $\mathcal{T}$. This 1-cochain gives us a continuous homomorphism $W_F \to \mathcal{U}_G$ splitting the exact sequence

$$1 \to \mathcal{T} \to \mathcal{U}_G \to W_F \to 1,$$

or in other words the continuous 1-cochain gives us an isomorphism $\mathcal{U}_G \simeq {}^L U$, where U is the F-torus such that $X^*(U) = X_*(\mathcal{T})$ as $\mathrm{Gal}(K/F)$-modules, with $\mathrm{Gal}(K/F)$ acting on $X_*(\mathcal{T})$ in the way determined by the extension

$$1 \to \mathcal{T} \to \mathcal{U}'_G \to \mathrm{Gal}(K/F) \to 1.$$

Since $\mathcal{Z}$ is obviously a subgroup of $\mathcal{U}$ and hence of $\mathcal{U}_G$, we draw the following conclusion from the discussion above: there exists an F-torus U and an injective L-homomorphism $\mathcal{Z} \to {}^L U$.

Now we are ready to show that $\hat{H} \to \hat{H}_1$ can be extended to an L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^L H_1$. Consider the group

$$\mathcal{H}_1 := (Z(\hat{H}_1) \rtimes \mathcal{H})/Z(\hat{H})$$

and its closed subgroup

$$\mathcal{Z}_1 := (Z(\hat{H}_1) \rtimes \mathcal{Z})/Z(\hat{H})$$

(we embed $Z(\hat{H})$ diagonally in $Z(\hat{H}_1) \rtimes \mathcal{H}$). Of course $\mathcal{H}_1$ (respectively, $\mathcal{Z}_1$) is an extension of W_F by $\hat{H}_1$ (respectively, $Z(\hat{H}_1)$). There is an obvious L-homomorphism $\mathcal{H} \hookrightarrow \mathcal{H}_1$ extending $\hat{H} \hookrightarrow \hat{H}_1$, so that to construct ξ_{H_1} it is enough to show that the identity map $\hat{H}_1 \to \hat{H}_1$ can be extended to an L-homomorphism $\mathcal{H}_1 \to {}^L H_1$. Thus it is enough to show that there is a continuous homomorphism $W_F \to \mathcal{Z}_1$ splitting the exact sequence

$$1 \to Z(\hat{H}_1) \to \mathcal{Z}_1 \to W_F \to 1$$

(of course we used that $\mathcal{Z}_1$ is the subgroup of $\mathcal{H}_1$ consisting of all elements x such that the restriction of $\mathrm{Int}(x)$ to $\hat{H}_1$ preserves the splitting of $\hat{H}_1$ obtained from our fixed splitting of $\hat{H}$).

Recall that we have an injective L-homomorphism $\mathcal{Z} \to {}^L U$ and in particular an embedding $Z(\hat{H}) \hookrightarrow \hat{U}$. Define a complex torus $\hat{V}$ by $\hat{V} = (Z(\hat{H}_1) \times \hat{U})/Z(\hat{H})$. There is an obvious injective L-homomorphism $\mathcal{Z}_1 \to {}^L V$, induced by

$$Z(\hat{H}_1) \rtimes \mathcal{Z} \to Z(\hat{H}_1) \rtimes {}^L U = (Z(\hat{H}_1) \times \hat{U}) \rtimes W_F.$$

Restricting $\mathcal{Z}_1 \to {}^L V$ to $Z(\hat{H}_1)$ we get $Z(\hat{H}_1) \hookrightarrow \hat{V}$, and we put $\hat{W} = \hat{V}/Z(\hat{H}_1)$. The exact sequence of complex tori with Galois action

$$1 \to Z(\hat{H}_1) \to \hat{V} \to \hat{W} \to 1$$

is dual to an exact sequence of F-tori. The Langlands isomorphism of $H^1(W_F, \hat{V})$ with the group of quasi-characters on $V(F)$ (respectively, $(V(\mathbb{A})/V(\overline{F}))^\Gamma$) in the local (respectively, global) case shows that

$$H^1(W_F, \hat{V}) \to H^1(W_F, \hat{W})$$

is surjective (see the proof of Lemma 4 of [L1] and also [La]). The composition

$$\mathcal{Z}_1 \hookrightarrow {}^L V \to {}^L W$$

is trivial on $Z(\hat{H}_1)$ and hence induces an L-homomorphism $\varphi : W_F \to {}^LW$. By the discussion above there exists an L-homomorphism $\psi : W_F \to {}^LV$ whose composition with ${}^LV \to {}^LW$ coincides with φ. Clearly the image of W_F under ψ is contained in the image $\mathcal{Z}_2$ of $\mathcal{Z}_1$ in ${}^LV$. Therefore the existence of the continuous section ψ shows that $\mathcal{Z}_2$ is homeomorphic to $Z(\hat{H}_1) \times W_F$ and hence that $\mathcal{Z}_2$ is locally compact. Since $\mathcal{Z}_1$ is locally compact and its topology has a countable base, we conclude that $\mathcal{Z}_1 \to \mathcal{Z}_2$ is a homeomorphism. Therefore ψ can be viewed as the desired splitting of the exact sequence

$$1 \to Z(\hat{H}_1) \to \mathcal{Z}_1 \to W_F \to 1,$$

and the proof that ξ_{H_1} exists is finally complete.

It remains to prove that ξ_{H_1} induces an isomorphism of topological groups from $\mathcal{H}$ to $\xi_{H_1}(\mathcal{H})$ and that $\xi_{H_1}(\mathcal{H})$ is closed in ${}^LH$. This follows from the existence of a section $c : W_F \to \mathcal{H}$ of $p : \mathcal{H} \to W_F$ (use the same argument we used to show that $\xi : H \to \xi(\mathcal{H})$ is a homeomorphism and that $\xi(\mathcal{H})$ is closed in ${}^LG$).

By a *z-pair* for $\mathcal{H}$ we will mean a pair (H_1, ξ_{H_1}) with H_1 a z-extension of H and ξ_{H_1} an L-embedding of $\mathcal{H}$ in ${}^LH_1$ extending $\hat{H} \hookrightarrow \hat{H}_1$. Observe that ξ_{H_1} determines a character λ_{H_1} on $Z_1(F)$ (F local) or on $Z_1(F)\backslash Z_1(\mathbb{A})$ (F global). This character has parameter

$$W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\xi_{H_1}} {}^LH_1 \to {}^LZ_1,$$

where c is any section for $\mathcal{H} \to W_F$. We note in passing that λ_{H_1} is the *inverse* of the character constructed in [LS1, 4.4] for standard endoscopy.

To explain the significance of λ_{H_1} we recall that the purpose of endoscopic data for $(G, \theta, \mathbf{a})$ is to study representations π of $G(F)$ or $G(\mathbb{A})$ for which $\pi \circ \theta = \omega \otimes \pi$. We shall consider an example where we have available the Langlands parametrization, namely tempered L-packets for F archimedean. Here the automorphism θ preserves L-packets as does multiplication by ω (this may be checked from the definition of the parametrization [L3]). We see also that if Π has parameter $\varphi : W_F \to {}^LG$ then $\Pi \circ \theta = \{\pi \circ \theta : \pi \in \Pi\}$ has parameter ${}^L\theta \circ \varphi$ and $\omega \otimes \Pi = \{\omega \otimes \pi : \pi \in \Pi\}$ has parameter $a \cdot \varphi$. So we conclude that $\Pi \circ \theta \approx \omega \otimes \Pi$ if and only if

$$S_\varphi = \{s \in \hat{G} : \mathrm{Int}(s) \circ {}^L\theta \circ \varphi = a \cdot \varphi\}$$

is nonempty.

Any $s \in \hat{G}$ has a unique twisted Jordan decomposition

$$s = ut = t\theta(u)$$

with u unipotent in $\hat{G}$ and t $\hat{\theta}$-semisimple in $\hat{G}$. It is easy to see that if $s \in S_\varphi$ with twisted Jordan decomposition $s = ut = t\theta(u)$, then $t \in S_\varphi$ and $u \in \mathrm{Cent}(\varphi, \hat{G})^0$, a connected group which acts by left translations on S_φ; therefore every connected component of S_φ contains a $\hat{\theta}$-semisimple element.

Assume $s \in S_\varphi$ is $\hat{\theta}$-semisimple, so that $\hat{H} = \mathrm{Cent}_{\hat{\theta}}(s, \hat{G})^0$ is reductive. Set $\mathcal{H}$ equal to the subgroup of ${}^LG$ generated by $\hat{H}$ and the image of φ and give $\mathcal{H}$ the topology induced from ${}^LG$. Then we have a split exact sequence

$$1 \to \hat{H} \to \mathcal{H} \to W_F \to 1.$$

We define $\rho_{\mathcal{H}}$ as we must, after fixing a splitting of $\hat{H}$. As in the proof of Lemma 2.2.A we introduce $\mathcal{U}$ in order to see that the action of W_F on $\mathcal{T}$ factors through $\mathrm{Gal}(K/F)$ for some finite Galois extension K/F, and hence that the same is true of $\rho_{\mathcal{H}}$. We take H to be a quasi-split group over F with L-group $(\hat{H}, \rho_{\mathcal{H}})$. Then $(H, \mathcal{H}, s, \xi)$, where ξ is the inclusion of $\mathcal{H}$ in ${}^LG$, is a set of endoscopic data for $(G, \theta, \mathbf{a})$ and the image of φ is contained in $\mathcal{H}$. If (H_1, ξ_{H_1}) is a z-pair for $\mathcal{H}$ then $\xi_{H_1} \circ \varphi$ is a parameter φ_1 for H_1. The packet Π_{H_1} attached to φ_1 has the property that for each representation π in it, $Z_1(F)$ acts as λ_{H_1}. Conversely any packet for H_1 with this property defines a parameter φ for G with $\mathrm{Int}(s) \circ {}^L\theta \circ \varphi = a \cdot \varphi$. In line with the general Langlands conjectures, we expect Π_{H_1} to transfer to Π. This is known, for example, for standard endoscopy [S1], and for base change under the assumption $\mathcal{H} = {}^LH$ [B].

For the (dual) transfer of functions from $G(F)$ to $H_1(F)$ we shall therefore consider functions f^{H_1} on $H_1(F)$ for which

$$f^{H_1}(zh) = \lambda_{H_1}(z)^{-1} f^{H_1}(h)$$

for all $z \in Z_1(F)$ and all $h \in H_1(F)$. We will do this also for F nonarchimedean and for the adelic analogue in the global case, since conjecturally tempered L-packets (or more generally Arthur packets, whatever F) have similar properties.

3. Norm mappings

(3.1) The setting. We continue with $(G, \theta, \mathbf{a})$ as in Section 2, although most of the time we may work with any field F of characteristic zero. Recall (see (1.2)) that the automorphism θ^* of G^* preserves an F-splitting and is of the form $\mathrm{Int}(g_\theta)\psi\theta\psi^{-1}$. Since ψ is an inner twisting we may choose, for each $\sigma \in \Gamma$, an element $u(\sigma) \in G^*_{\mathrm{sc}}$ such that $\psi\sigma(\psi)^{-1} = \mathrm{Int}\, u(\sigma)$.

The θ-conjugacy class of $\delta \in G(\overline{F})$ is $\{g^{-1}\delta\theta(g) : g \in G(\overline{F})\}$. Denote by $Cl(G, \theta)$ the set of all such classes. Then the map $m : \delta \mapsto \psi(\delta)g_\theta^{-1}$ from $G(\overline{F})$ to $G^*(\overline{F})$ takes $g^{-1}\delta\theta(g)$ to

$$\psi(g^{-1})\psi(\delta)\psi(\theta(g))g_\theta^{-1} = \psi(g)^{-1}m(\delta)\theta^*(\psi(g))$$

and so induces a map

$$Cl(G, \theta) \to Cl(G^*, \theta^*) \tag{3.1.1}$$

which we also denote m. This map on classes is bijective but need not respect the action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$, *i.e.*, as we shall say, need not be defined over F.

Lemma 3.1.A.

(1) *Suppose $C \in Cl(G, \theta)$ and $\sigma \in \Gamma$. Then $\sigma(m)(C) = z_\sigma m(C)$, where z_σ is a 1-cochain of Γ in the center Z^{sc} of G^*_{sc} whose image in $Z^{\mathrm{sc}}_\theta = Z^{\mathrm{sc}}/(1 - \theta^*)Z^{sc}$ is a 1-cocycle. The class $\mathbf{z}$ of this cocycle in $H^1(F, Z^{\mathrm{sc}}_\theta)$ is independent of the choice of g_θ.*

(2) *If $\mathbf{z}$ is trivial we may so choose g_θ that m, as map on classes, is defined over F.*

Let $\delta \in G(\overline{F})$. A simple calculation shows that

$$\sigma(m)(\delta) = u(\sigma)^{-1} m(\delta) z_\sigma \theta^*(u(\sigma)),$$

where

$$z_\sigma = g_\theta u(\sigma) \sigma(g_\theta)^{-1} \theta^*(u(\sigma))^{-1}.$$

To see that z_σ is central just apply σ to the equation

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}.$$

Because $u(\sigma)$ involves a choice, z_σ is well-defined only up to $(1-\theta^*)Z^{\mathrm{sc}}$. Computing coboundaries, we find that

$$\partial z = (1-\theta^*)\partial u.$$

Thus the image of z_σ in Z^{sc}_θ is a 1-cocycle depending only on the choice of g_θ. Clearly this choice does not affect the class $\mathbf{z}$ of z_σ in $H^1(F, Z^{\mathrm{sc}}_\theta)$.

Finally, if $\mathbf{z}$ is trivial we may multiply g_θ by an element of Z^{sc} to obtain z_σ in $(1-\theta^*)Z_{\mathrm{sc}}$. Then $\sigma(m)(\delta)$ is θ^*-conjugate to $m(\delta)$ and (2) follows.

For most purposes it is enough to consider the case $\mathbf{z}$ is trivial. This includes standard endoscopy, base change, and transpose-inverse on $\mathbf{GL}(n)$, for example. That then will be our assumption until (5.4), as it saves a great deal of notation.

With $\mathbf{z}$ trivial we may continue the argument for (2) above and choose g_θ,$u(\sigma)$ so that z_σ is trivial and hence

$$\sigma(m)(\delta) = u(\sigma)^{-1} m(\delta) \theta^*(u(\sigma)). \tag{3.1.2}$$

(3.2) Abstract norms. If G is abelian we call the projection N_θ of G onto $G_\theta := G/(1-\theta)G$ the abstract norm map for G. Clearly N_θ maps $G(F)$ homomorphically to a subgroup of $G_\theta(F)$. If $(H, \mathcal{H}, s, \xi)$ is a set of endoscopic data for $(G, \theta, \mathbf{a})$ then the isomorphism $\xi : \hat{H} \to (\hat{G}^{\hat{\theta}})^0$ yields an isomorphism $G_\theta \to H$ over F. We call $\gamma \in H(F)$ the *norm* of $\delta \in G(F)$ if γ is the image of $N_\theta(\delta)$ under the latter isomorphism.

For general G we start with the elements δ of $G(\overline{F})$ which are θ-semisimple in the sense that the automorphism $\mathrm{Int}\,\delta \circ \theta$ is quasi-semisimple. Because

$$\mathrm{Int}\, m(\delta) \circ \theta^* = \psi\, \mathrm{Int}(\delta)\theta\psi^{-1},$$

m induces a bijection between the set $Cl_{\mathrm{ss}}(G,\theta)$ of θ-conjugacy classes of θ-semisimple elements in $G(\overline{F})$ and $Cl_{\mathrm{ss}}(G^*, \theta^*)$, the corresponding set for (G^*, θ^*).

Suppose that (B,T) is a θ^*-stable pair in G^*, Ω is the Weyl group of T in G^* and Ω^{θ^*} the subgroup of elements which commute with θ^*. Recall that each element of Ω^{θ^*} is represented by a θ^*-fixed element of G^* (see (1.1)).

Lemma 3.2.A.

(1) *Each* $\mathcal{O} \in Cl_{\mathrm{ss}}(G^*, \theta^*)$ *meets* T.
(2) *The image of* $\mathcal{O} \cap T$ *in* T_{θ^*} *is a single* Ω^{θ^*}*-orbit.*

If δ is θ^*-semisimple then there is a pair $(B(\delta), T(\delta))$ in G^* preserved by $\mathrm{Int}(\delta) \circ \theta^*$. Suppose $(B(\delta), T(\delta))^g = (B, T)$. Then we find that $g^{-1}\delta\theta(g)$ lies in T, and (1) follows.

If $\delta \in T$ then (B, T) is $\mathrm{Int}(\delta) \circ \theta^*$-stable. Thus if $g^{-1}\delta\theta^*(g)$ also lies in T we have that both $(T^\theta)^0$ and $g(T^\theta)^0 g^{-1}$ are maximal tori in $\mathrm{Cent}_{\theta^*}(\delta, G^*)^0$, and (2) follows.

We now map $\mathcal{O} \in Cl_{\mathrm{ss}}(G^*, \theta^*)$ to the Ω^{θ^*}-orbit in T_{θ^*} given by the lemma. Thus we have a bijection

$$Cl_{\mathrm{ss}}(G^*, \theta^*) \to T_{\theta^*}/\Omega^{\theta^*}. \tag{3.2.1}$$

If also (B', T') is θ^*-stable then we have a canonical bijection

$$T_{\theta^*}/\Omega^{\theta^*} \to T'_{\theta^*}/(\Omega')^{\theta^*}$$

induced by conjugation by a suitable θ^*-fixed element of G^*, and this bijection is compatible with the bijections (3.2.1) for (B, T) and (B', T').

The composition

$$Cl_{\mathrm{ss}}(G, \theta) \to Cl_{\mathrm{ss}}(G^*, \theta^*) \to T_{\theta^*}/\Omega^{\theta^*} \tag{3.2.2}$$

yields an *abstract norm map* N_θ for θ-semisimple θ-conjugacy classes in $G(\overline{F})$. We may assume that T is defined over F. Then it is easy to check that each arrow respects the action of Γ, *i.e.* N_θ is defined over F.

(3.3) Strongly G-regular norms. Returning now to the set $(H, \mathcal{H}, s, \xi)$ of endoscopic data for $(G, \theta, \mathbf{a})$, we shall first establish:

Theorem 3.3.A. *There is a canonical map*

$$\mathcal{A}_{H/G} : Cl_{\mathrm{ss}}(H) \to Cl_{\mathrm{ss}}(G, \theta)$$

from semisimple conjugacy classes in $H(\overline{F})$ to θ-semisimple θ-conjugacy classes in $G(\overline{F})$. This map is defined over F in the sense that it respects the action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

Suppose (B_H, T_H) is a pair in H and (B, T) is a θ^*-stable pair in G^*. To save notation, we will make use of the fixed splittings $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of $\hat{G}$ and $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ of $\hat{H}$; in addition we assume $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$, as we may. To (B_H, T_H) and $(\mathcal{B}_H, \mathcal{T}_H)$ is attached an isomorphism $\hat{T}_H \to \mathcal{T}_H$ and to (B, T) and $(\mathcal{B}, \mathcal{T})$ is attached an isomorphism $\hat{T} \to \mathcal{T}$. Because (B, T) is θ^*-stable the latter isomorphism induces

$$(T_{\theta^*})^\frown = (\hat{T}^{\hat{\theta}})^0 \simeq (\mathcal{T}^{\hat{\theta}})^0.$$

We have therefore attached to (B_H, T_H) and (B, T) a chain of isomorphisms

$$\hat{T}_H \simeq \mathcal{T}_H \xrightarrow{\xi} (\mathcal{T}^{\hat{\theta}})^0 \simeq (T_{\theta^*})^\frown$$

which then yields $T_H \simeq T_{\theta^*}$. This isomorphism transports $\Omega_H = \Omega(H, T_H)$ to a subgroup of $\Omega^{\theta^*} = \Omega(G^*, T)^{\theta^*}$ and so induces

$$T_H/\Omega_H \to T_{\theta^*}/\Omega^{\theta^*}. \tag{3.3.1}$$

We therefore have

$$Cl_{\rm ss}(H) \to T_H/\Omega_H \to T_{\theta^*}/\Omega^{\theta^*} \to Cl_{\rm ss}(G^*, \theta^*) \to Cl_{\rm ss}(G, \theta)$$

yielding a map

$$\mathcal{A}_{H/G} : Cl_{\rm ss}(H) \to Cl_{\rm ss}(G, \theta).$$

It is independent of all choices.

To see that $\mathcal{A}_{H/G}$ is defined over F it remains only to check that (3.3.1) is defined over F, and for this it is enough to prove the following.

Lemma 3.3.B. *If T_H is defined over F then we may choose $B_H \supset T_H$ and θ^*-stable (B,T) with T defined over F, so that the attached isomorphism $T_H \to T_{\theta^*}$ is defined over F.*

Suppose that T_H is defined over F. Let $B_H \supset T_H$ and let (B,T) be a θ^*-stable pair with T defined over F. Write η for the attached isomorphism $T_H \to T_{\theta^*}$. Reviewing the constructions, we see that $\sigma(\eta)\eta^{-1}$ is given by an element $\omega_\sigma \in \Omega^{\theta^*}$ which it is convenient to view as $\Omega((G^*_{\rm sc})^{\theta^*_{\rm sc}}, T^{\theta^*}_{\rm sc})$. A familiar argument using Steinberg's theorem on rational points in semisimple conjugacy classes in simply connected quasi-split groups then shows that there exists $g \in (G^*_{\rm sc})^\theta$ such that $\sigma(g)g^{-1}$ normalizes $T^{\theta^*}_{\rm sc}$ and induces ω_σ. We replace (B,T) by $(B,T)^g$ and then η by $\mathrm{Int}(g^{-1}) \circ \eta$ to obtain the desired isomorphism over F.

An F-isomorphism $T_H \to T_{\theta^*}$ as in the lemma will be called an *admissible embedding* of T_H (in G_θ, the set of θ-conjugacy classes in G), and T_H will be called a *norm group* for T.

To define norms in $H(F)$ we start with the most regular elements. Recall that $\mathrm{Cent}_\theta(\delta, G)$ is the group of fixed points of $\mathrm{Int}(\delta)\circ\theta$. Call θ-semisimple $\delta \in G$ *θ-regular* if $G^{\delta\theta} = \mathrm{Cent}_\theta(\delta, G)^0$ is a torus and *strongly θ-regular* if $\mathrm{Cent}_\theta(\delta, G)$ is abelian. In the latter case, $T_\delta = \mathrm{Cent}(G^{\delta\theta}, G)$ is a maximal torus in G stable under $\mathrm{Int}(\delta) \circ \theta$ and $\mathrm{Cent}_\theta(\delta, G)$ coincides with the set of fixed points of $\mathrm{Int}(\delta) \circ \theta$ in T_δ. Call $\gamma \in H$ *G-regular* if the image of the conjugacy class of γ under $\mathcal{A}_{H/G}$ consists of θ-regular elements, and *strongly G-regular* if the image consists of strongly θ-regular elements.

Lemma 3.3.C.

(1) *A G-regular element in H is regular.*
(2) *A strongly G-regular element in H is strongly regular.*

Let δ lie in the image of the class of γ under $\mathcal{A}_{H/G}$ and (B,T) be a θ^*-stable pair in G^*. Then there is δ^* in T such that δ^* is θ^*-conjugate to $m(\delta)$ and such that the image of δ^* in T_{θ^*} is equal to the image of γ under some admissible embedding $T_H \to T_{\theta^*}$. Because the twisted centralizers of δ and δ^* are isomorphic, δ is (strongly) θ-regular if and only if δ^* is (strongly) θ^*-regular. The condition for δ^* to be θ^*-regular is of course that $(G^*)^{\delta^*\theta^*}$ have no roots. We observe from (1.3) that this is equivalent to:

(3.3.2) for $\alpha \in R(G^*, T)$ we have

$$\begin{cases} N\alpha(\delta^*) \neq 1, & \text{for } \alpha \text{ of types } R_1, R_2 \\ N\alpha(\delta^*) \neq -1, & \text{for } \alpha \text{ of type } R_3. \end{cases}$$

The condition that γ be regular in H is that no $((\alpha^\vee)_{\text{res}})^\vee$ be a root of the centralizer of γ in H. By (1.3.9) this means:

(3.3.3) for $\alpha_H = ((\alpha^\vee)_{\text{res}})^\vee \in R(H, T_H)$ we have

$$\begin{cases} \alpha_H(\gamma) = N\alpha(\delta^*) \neq 1, & \text{for } \alpha \text{ of types } R_1, R_3 \\ \alpha_H(\gamma) = N\alpha(\delta^*)^2 \neq 1, & \text{for } \alpha \text{ of type } R_2. \end{cases}$$

But (3.3.2) implies (3.3.3) and so (1) is proved. For (2), suppose that δ^* is strongly θ^*-regular. This means that no nontrivial element of $\Omega^{\theta^*}(G^*, T)$ may be realized in $\text{Cent}_{\theta^*}(\delta^*, G^*)$. Recall that every element of $\Omega^{\theta^*}(G^*, T)$ may be realized in $(G^*)^{\theta^*}$ because θ^* preserves a splitting. If γ is not strongly regular in H then there is a nontrivial element ω_H of $\Omega(H, T_H)$ which fixes γ. Embedding Ω_H in Ω^{θ^*} we can then produce an element of $\text{Cent}_{\theta^*}(\delta^*, G^*)$ which realizes the (nontrivial) image of ω_H, a contradiction, and so the lemma is proved.

Turning to F-rational points, (2) of Lemma 3.3.C implies that the stable conjugacy class of strongly G-regular $\gamma \in H(F)$ is the intersection of its conjugacy class in $H(\overline{F})$ with $H(F)$. We define the *stable θ-conjugacy class* of strongly θ-regular $\gamma \in G(F)$ to be the intersection with $G(F)$ of its θ-conjugacy class in $G(\overline{F})$.

Now suppose $\gamma \in H(F)$ is strongly G-regular. We say that γ *is a norm of* $\delta \in G(F)$ if δ lies in the image of the conjugacy class of γ under $\mathcal{A}_{H/G}$. If this image contains no points of $G(F)$ then we say γ *is not a norm*. Observe that either γ is a norm of exactly one stable θ-conjugacy class of strongly θ-regular elements in $G(F)$ or it is not a norm. If (H_1, ξ_{H_1}) is a z-pair for $\mathcal{H}$ we say that $\gamma_1 \in H_1(F)$ is strongly G-regular if its image γ in $H(F)$ is, γ_1 is a norm of δ if γ is, and so on.

If strongly G-regular $\gamma \in T_H(F)$ is a norm of $\delta \in G(F)$ and $T_H \to T_{\theta^*}$ is an admissible embedding then we may choose $\delta^* \in T$ and $g \in G^*_{\text{sc}}$ such that

(3.3.4) γ has image $N_{\theta^*}(\delta^*)$ under $T_H \to T_{\theta^*}$ and

(3.3.5) $\delta^* = gm(\delta)\theta^*(g)^{-1}$.

In (3.3.5) $g \in G^*_{\text{sc}}$ has been identified with its image in G^*. Then

(3.3.6) $\text{Int}(g) \circ \psi : \text{Cent}_\theta(\delta, G) \to T^{\theta^*}$ is defined over F.

This is clear because

$$(\text{Int}(g)\psi)\sigma(\text{Int}(g)\psi)^{-1} = \text{Int}\, v(\sigma),$$

where $v(\sigma) = gu(\sigma)\sigma(g)^{-1}$ is readily seen (Lemma 4.4.A) to lie in T_{sc}.

4. Relative transfer factors

(4.1) The setting. Throughout Section 4, F will be *local.* Once again $(H, \mathcal{H}, s, \xi)$ is a set of endoscopic data for $(G, \theta, \mathbf{a})$ and (H_1, ξ_{H_1}) is a z-pair for $\mathcal{H}$. Our plan is to define a *relative transfer factor* $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$, for $\gamma_1, \bar{\gamma}_1$ strongly G-regular in $H_1(F)$ and norms of, respectively, $\delta, \bar{\delta} \in G(F)$, and then to show that it is canonical. This relative factor will be a product of four terms Δ_{I},Δ_{II},Δ_{III} and Δ_{IV}. In our setting the terms $\Delta_1 = \Delta_{\mathrm{III}_1}$ and $\Delta_2 = \Delta_{\mathrm{III}_2}$ of standard endoscopy [LS1] combine naturally as the single term Δ_{III}. Each of Δ_{I},Δ_{II},Δ_{IV} will be a quotient of the form $\Delta_{\mathrm{I}}(\gamma_1, \delta)/\Delta_{\mathrm{I}}(\bar{\gamma}_1, \bar{\delta})$, and so on. Each term will depend on additional data (admissible embeddings, a-data or χ-data). However, the main theorem of this section, Theorem 4.6.A, will assert that the effects of these choices cancel when we assemble the relative transfer factor.

We continue with γ_1,$\bar{\gamma}_1$ and δ,$\bar{\delta}$ as above. Let γ,$\bar{\gamma} \in H(F)$ be the images of γ_1,$\bar{\gamma}_1$ under $H_1 \to H$ and set $T_H = \mathrm{Cent}(\gamma, H)$, $\overline{T}_H = \mathrm{Cent}(\bar{\gamma}, H)$. Let $T_H \to T_{\theta^*}$ and $\overline{T}_H \to \overline{T}_{\theta^*}$ be admissible embeddings. Here we make the choices B_H,B,T and $\overline{B}_H$, *etc.* of Lemma 3.3.B and maintain the assumptions of the proof of Theorem 3.3.A, but only the resultant admissible embeddings will affect the terms in the transfer factor. Let $g, \bar{g} \in G^*_{\mathrm{sc}}$ and $\delta^* \in T$, $\bar{\delta}^* \in \overline{T}$ have the properties of (3.3.4) and (3.3.5). Let R_{res} denote the restricted roots of $R(G^*, T)$ and $R^\vee_{\mathrm{res}}$ those of $R(\hat{G}, \mathcal{T})$, equipped with the Galois action for T; similarly define $\overline{R}_{\mathrm{res}}$ and $\overline{R}^\vee_{\mathrm{res}}$ for $\overline{T}$. Finally we choose a-data and χ-data for R_{res} (or $R^\vee_{\mathrm{res}}$) and $\overline{R}_{\mathrm{res}}$ (or $\overline{R}^\vee_{\mathrm{res}}$) as in (1.3).

(4.2) The term Δ_{I}. We define $\Delta_{\mathrm{I}}(\gamma_1, \delta) = \Delta_{\mathrm{I}}(\gamma, \delta)$ by adapting a construction from [LS1]. The automorphism θ^* of G^* lifts to an automorphism θ^*_{sc} of G^*_{sc}. Let G^x be the group of fixed points of θ^*_{sc} and $T^x = T_{\mathrm{sc}} \cap G^x$. Recall that these groups are connected and $R(G^x, T^x)$ is contained in R_{res}. Thus we have a-data $\{a_\alpha : \alpha \in R(G^x, T^x)\}$. Fix an F-splitting of G^x. Then to T^x,$\{a_\alpha\}$ and this splitting we may attach the element $\lambda_{\{a_\alpha\}}(T^x)$ of $H^1(F, T^x)$ defined in [LS1, 2.3].

On the other hand, we may use the endoscopic datum s to define an element $\mathbf{s}_{T,\theta}$ of $\pi_0((\widehat{T^x})^\Gamma)$, as follows. The group $\widehat{T^x}$ is the group of coinvariants of $\hat{\theta}$ in $\hat{T}_{\mathrm{ad}} = \hat{T}/Z(\hat{G})$ which we may identify with $(\mathcal{T}_{\mathrm{ad}})_{\hat{\theta}_{\mathrm{ad}}}$ using $\mathcal{T} \simeq \hat{T}$. Since s lies in $\mathcal{T}$ we may project first to $\mathcal{T}_{\mathrm{ad}}$ and then to $(\mathcal{T}_{\mathrm{ad}})_{\hat{\theta}_{\mathrm{ad}}}$, so obtaining an element $s_{T,\theta}$.

Lemma 4.2.A.

(1) $s_{T,\theta}$ *is* Γ*-invariant and*
(2) $s_{T,\theta}$ *depends on the admissible embedding* $T_H \to T_{\theta^*}$ *but not on the choice of* B_H,B *leading to* $T_H \to T_{\theta^*}$.

Suppose $h \in \mathcal{H}$ normalizes $\mathcal{T}_H$ and write

$$\xi(h) = g \times w \in \hat{G} \rtimes W_F.$$

Then both $\xi(h)$ and w normalize $\mathcal{T}$ and preserve the subgroup $\mathcal{T}^1 := (\mathcal{T}^\theta)^0$; therefore the same is true of g, and the discussion in (1.1) implies that the image of g in $\Omega(\hat{G}, \mathcal{T})$ belongs to $\Omega(\hat{G}, \mathcal{T})^\theta$ and hence that g itself belongs to $\mathcal{T}\hat{G}^1$. Therefore $\xi(h)$ can be written as tx with $t \in \mathcal{T}$ and x fixed by ${}^L\theta$, and it follows immediately from (2.1.4a) that

$$\xi(h) s \xi(h)^{-1} = a'(w)^{-1} s \hat{\theta}(t) t^{-1}. \qquad (4.2.1)$$

Now we prove part (1) of the lemma. Let $\sigma \in \Gamma$ and write $\sigma_{\mathcal{T}_H}$ for the action of σ on $\mathcal{T}_H$ obtained from the natural action of σ on $\hat{T}_H$ *via* the isomorphism $\hat{T}_H \simeq \mathcal{T}_H$ determined by $B_H, \mathcal{B}_H$. Then $\sigma_{\mathcal{T}_H}$ differs from $\rho_{\mathcal{H}}(\sigma)$ by an element of the Weyl group of $\mathcal{T}_H$ in $\hat{H}$, and therefore there exists $h \in \mathcal{H}$ such that h normalizes $\mathcal{T}_H$ and acts on it by $\sigma_{\mathcal{T}_H}$. It is then immediate from (4.2.1) that $\sigma_{\mathcal{T}_H}$ fixes the image of s in $(\mathcal{T}_{\text{ad}})_{\hat{\theta}_{\text{ad}}}$.

Finally we prove part (2) of the lemma. In order to get s we have chosen B, B_H, $\mathbf{spl}_{\hat{G}}$, $\mathbf{spl}_{\hat{H}}$ and we have assumed that $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = \mathcal{T} \cap \xi(\hat{H})$ and $\xi(\mathcal{B}_H) = \mathcal{B} \cap \xi(\hat{H})$, which we can achieve after replacing (s, ξ) by

$$^g(s,\xi) := (gs\hat{\theta}(g)^{-1}, \text{Int}(g) \circ \xi)$$

for some $g \in \hat{G}$. These choices give us an admissible embedding $T_H \simeq T_\theta$, an element $s' \in \mathcal{T}_{\hat{\theta}}$ (project s into $\mathcal{T}_{\hat{\theta}}$), an isomorphism $\mathcal{T}_{\hat{\theta}} \simeq \hat{T}_{\hat{\theta}}$ and an element $s'' \in \hat{T}_{\hat{\theta}}$ (s'' is the image of s' under the isomorphism $\mathcal{T}_{\hat{\theta}} \simeq \hat{T}_{\hat{\theta}}$). We want to show that s'' depends only on the admissible embedding resulting from all these choices. This follows from the following assertion: if different choices cause the admissible embedding $T_H \simeq T_\theta$ to be replaced by its composition with $\omega \in \Omega(G^*, T)^\theta$ then s'' is replaced by $\omega^{-1}(s)$.

We now verify this assertion. Using conjugation by elements of $\hat{G}^1$ fixed by Γ we may assume that $\mathbf{spl}_{\hat{G}}$ remains the same. Similarly, using conjugation by elements of $\hat{H}$ fixed by Γ we may assume that $\mathbf{spl}_{\hat{H}}$ remains the same. Changing B to $\omega(B)$ for $\omega \in \Omega(G^*, T)$ fixed by θ and Γ obviously causes a change in $(T_H \simeq T_\theta, s'')$ of the form given above. Changing B_H to $\omega_H(B_H)$ for $\omega_H \in \Omega(H, T_H)$ fixed by Γ changes $T_H \simeq T_\theta$ by ω_H and does not change s''; however, it follows from (4.2.1) that ω_H fixes s'', so that our assertion is still correct. There is still one kind of change possible. We can replace (s, ξ) by $^g(s, \xi)$ for $g \in \text{Norm}(\mathcal{T}, \hat{G})$ whose image ω in $\Omega(\hat{G}, \mathcal{T})$ is fixed by θ and has the property that $\omega^{-1}(\mathcal{B}) \cap \xi(\hat{H}) = \mathcal{B} \cap \xi(\hat{H})$. Thus $T_H \simeq T_\theta$ is replaced by its composition with ω^{-1} and s'' is replaced by $\omega(s)$. This proves our assertion and concludes the proof of the lemma.

We may now define $\mathbf{s}_{T,\theta}$ to be the class of $s_{T,\theta}$ in $\pi_0((\widehat{T^x})^\Gamma)$. Let $\langle \cdot, \cdot \rangle$ denote the Tate-Nakayama pairing between $H^1(F, T^x)$ and $\pi_0((\widehat{T^x})^\Gamma)$ (see [K1]). Then we define

$$\Delta_{\text{I}}(\gamma, \delta) = \langle \lambda_{\{a_\alpha\}}(T^x), \mathbf{s}_{T,\theta} \rangle.$$

We need to understand how $\Delta_{\text{I}}(\gamma, \delta)$ depends on the choice of F-splitting for G^x. To simplify notation we write I for G^x. Suppose that we replace our chosen F-splitting $\mathbf{spl}_I$ for I by $\text{Int}(g)(\mathbf{spl}_I)$, where $g \in I(\overline{F})$ is such that $\text{Int}(g)$ is defined over F. Let $z \in H^1(F, Z(I))$ denote the image of g under the boundary map

$$I_{\text{ad}}(F) \to H^1(F, Z(I))$$

for the exact sequence

$$1 \to Z(I) \to I \to I_{\text{ad}} \to 1;$$

then z is represented by the 1-cocycle

$$\sigma \mapsto g^{-1}\sigma(g)$$

of Γ in $Z(I)$. It follows from (2.3.1) of [LS1] that $\lambda_{\{a_\alpha\}}(T^x)$ is multiplied by the image z' of z under

$$H^1(F, Z(I)) \to H^1(F, T^x).$$

Therefore $\Delta_{\mathrm{I}}(\gamma,\delta)$ is multiplied by $\langle z', \mathbf{s}_{T,\theta}\rangle$.

Although the expression $\langle z', \mathbf{s}_{T,\theta}\rangle$ appears to depend on T, it actually does not, as we will show by finding a more useful way to write it. As usual (see [L1]) the maximal torus T^x of I and the element $s_{T,\theta} \in (\widehat{T^x})^\Gamma$ determine an endoscopic group J of I together with an element $s_J \in Z(\hat{J})^\Gamma$. Let us recall the construction. We start by defining $\hat{J}$ to be the identity component of the centralizer of $s_{T,\theta}$ in $\hat{I}$ (pick an embedding $\widehat{T^x} \hookrightarrow \hat{I}$ belonging to the canonical conjugacy class of such embeddings). The natural action of Γ on $\widehat{T^x}$ preserves the root system of $\widehat{T^x}$ in $\hat{I}$, and since $s_{T,\theta}$ is fixed by Γ, the action of Γ preserves the root system of $\widehat{T^x}$ in $\hat{J}$ as well. Fix a splitting of $\hat{J}$ with $\widehat{T^x}$ as its torus component. Then for each $\sigma \in \Gamma$ we denote by ω_σ the unique element of the Weyl group of $\widehat{T^x}$ in $\hat{J}$ such that $\omega_\sigma\sigma$ preserves the positive Weyl chamber determined by our splitting of $\hat{J}$. Then there is a unique action ρ_J of Γ on $\hat{J}$, preserving the chosen splitting, such that $\rho_J(\sigma)$ coincides with $\omega_\sigma\sigma$ on $\widehat{T^x}$. We take J to be a quasi-split group whose dual group is $(\hat{J}, \rho_J)$. It is immediate that $s_{T,\theta}$ is an element of $Z(\hat{J})^\Gamma$; we denote this element by s_J.

We used T^x to get J, but in fact (J, s_J) depends only on $(H, \mathcal{H}, s, \xi)$ and not on T_H. To see this we need to compare the root systems of $\hat{J}$ and $\hat{H}$. Without loss of generality we may assume that G is semisimple and simply connected, so that $I = G^\theta$ and $T^x = T^\theta$. It follows from (1.3) that the map

$$\alpha_{\mathrm{res}} \mapsto (\alpha^\vee)_{\mathrm{res}}$$

induces a type-preserving bijection

$$R_{\mathrm{res}}(\hat{G}, \mathcal{T}) \to R_{\mathrm{res}}(G, T)$$

and thus also induces a bijection

$$R_{\mathrm{res}}(\hat{G}, \mathcal{T})_{\mathrm{red}} \to R(I, T^\theta),$$

where the subscript red indicates the subset of all reduced roots. Therefore

$$\alpha_{\mathrm{res}} \mapsto ((\alpha^\vee)_{\mathrm{res}})^\vee$$

is a bijection

(4.2.2) $$R_{\mathrm{res}}(\hat{G}, \mathcal{T})_{\mathrm{red}} \to R(\hat{I}, \mathcal{T}_\theta).$$

Recall also from (1.3) that $((\alpha^\vee)_{\mathrm{res}})^\vee$ is given by

$$\begin{cases} N\alpha & \text{if } \alpha_{\mathrm{res}} \text{ is of type } R_1 \\ 2N\alpha & \text{if } \alpha_{\mathrm{res}} \text{ is of type } R_2. \end{cases}$$

The root system of $\hat{H}$ (see (1.3)) consists of all the reduced elements α_{res} of $R_{\text{res}}(\hat{G}, \mathcal{T})$ such that $(N\alpha)(s) = 1$ together with all the non-reduced elements α_{res} of $R_{\text{res}}(\hat{G}, \mathcal{T})$ such that $(N\alpha)(s) = -1$. Replacing non-reduced roots α_{res} by $\alpha_{\text{res}}/2$, we get a bijection from the root system of $\hat{H}$ to the set of all α_{res} in $R_{\text{res}}(\hat{G}, \mathcal{T})_{\text{red}}$ such that

$$\begin{cases} (N\alpha)(s) = 1 & \text{if } \alpha_{\text{res}} \text{ is of type } R_1 \\ (2N\alpha)(s) = 1 & \text{if } \alpha_{\text{res}} \text{ is of type } R_2. \end{cases}$$

It is obvious that the bijection (4.2.2) carries this subset of $R_{\text{res}}(\hat{G}, \mathcal{T})_{\text{red}}$ over to the subset of $R(\hat{I}, \mathcal{T}_\theta)$ consisting of all elements β such that $\beta(s) = 1$; in other words we have constructed a natural bijection

$$R(\hat{H}, \mathcal{T}^\theta) \to R(\hat{J}, \mathcal{T}_\theta). \tag{4.2.3}$$

Since we have assumed that G is semisimple the automorphism θ has finite order, say order d. Let f be the isomorphism

$$f : X^*(\mathcal{T}^\theta)_{\mathbb{Q}} \to X^*(\mathcal{T}_\theta)_{\mathbb{Q}}$$

defined as follows: identify $X^*(\mathcal{T}^\theta)_{\mathbb{Q}}$ with $(X^*(\mathcal{T})_{\mathbb{Q}})_\theta$ and $X^*(\mathcal{T}_\theta)_{\mathbb{Q}}$ with $(X^*(\mathcal{T})_{\mathbb{Q}})^\theta$; then f becomes the map from the coinvariants to the invariants given by

$$x \mapsto \sum_{i=1}^{d} \theta^i(x).$$

Note that for α_{res} in $R_{\text{res}}(\hat{G}, \mathcal{T})$ the bijection (4.2.2) carries α_{res} into a positive rational multiple of $f(\alpha_{\text{res}})$, and the analogous statement holds for the bijection (4.2.3), which means that the isomorphism f carries Weyl chambers for $\hat{H}$ over to Weyl chambers for $\hat{J}$. The isomorphism f is obviously equivariant for the action of $\Omega(\hat{G}, \mathcal{T})^\theta$, and it carries the Weyl group of $\hat{H}$ over to the Weyl group of $\hat{J}$, yielding an isomorphism between these two groups.

We are finally ready to show that the L-action of Γ on $\hat{J}$ just depends on H, not on T_H. Suppose that T_H is replaced by T'_H. Then for $\sigma \in \Gamma$ the action of σ on T'_H differs from its action on T_H by an element of the Weyl group of H (identify T'_H with T_H over $\overline{F}$ using an inner automorphism of H over $\overline{F}$). Therefore the action of σ on the maximal torus $(T')^x$ of I differs from its action on T^x by an element of the Weyl group of J. Therefore the L-action of σ on $\hat{J}$ obtained from $(T')^x$ coincides with the one obtained from T^x.

It is convenient to pursue these considerations one step further, for use in (5.3). Consider the special case in which T_H is the torus component of an F-splitting of H, so that the Galois action on T_H preserves some Weyl chamber for T_H in H. As before we have T and $T_H \simeq T_\theta$ as in Lemma 3.3.B. Then T^θ is a maximal torus of I, and T^θ transfers to J (since we can use this particular T_H to construct J). It is clear from the remarks made earlier that the Galois action on T^θ preserves some Weyl chamber for T^θ in J, which means that T^θ is the torus component of some F-splitting of J.

Now we return to the factor $\Delta_{\rm I}(\gamma,\delta)$. Recall that replacing $\mathbf{spl}_I$ by $\mathrm{Int}(g)(\mathbf{spl}_I)$ has the effect of multiplying $\Delta_{\rm I}(\gamma,\delta)$ by $\langle z', s_{T,\theta}\rangle$. The torus T^x transfers to J, so that we can choose

$$T^x \to J.$$

Then $\langle z', s_{T,\theta}\rangle$ is equal to $\langle z_J, s_J\rangle$, where z_J denotes the image of z' under

$$H^1(F,T^x) \to H^1(F,J),$$

$s_J \in Z(\hat{J})^\Gamma$ is as before, and $\langle z_J, s_J\rangle$ is the usual pairing between $H^1(F,J)$ and $\pi_0(Z(\hat{J})^\Gamma)$. Clearly z_J can also be described as the image of z under the map

$$H^1(F,Z(I)) \to H^1(F,J)$$

coming from the canonical embedding of $Z(I)$ into the center of J. In particular it is now clear that $\langle z_J, s_J\rangle$ is independent of T_H. It follows that the relative factor

$$\Delta_{\rm I}(\gamma_1,\delta;\bar{\gamma}_1,\bar{\delta}) = \Delta_{\rm I}(\gamma,\delta)/\Delta_{\rm I}(\bar{\gamma},\bar{\delta})$$

is independent of the choice of F-splitting for G^x.

(4.3) The term $\Delta_{\rm II}$. The term $\Delta_{\rm II}(\gamma_1,\delta) = \Delta_{\rm II}(\gamma,\delta)$ will be a quotient, with the numerator a product over Galois orbits $\mathcal{O}$ in $R_{\rm res} = R_{\rm res}(G^*,T)$ and the denominator a product over Galois orbits $\mathcal{O}_H$ in $R_H = R(H,T_H)$. The a-data and χ-data of (1.3) serve for $R_{\rm res}$, $R_{\rm res}^\vee$ and $(R_{\rm res}^\vee)^\vee$ which contains R_H, so that they serve for R_H as well. Observe that Γ preserves the type R_1,R_2 or R_3 of an element of $R_{\rm res}$, and so we may speak of the *type* of a Galois orbit $\mathcal{O}$. Also recall that if we write a restricted root as $\alpha_{\rm res}$, with $\alpha \in R(G^*,T)$, then α is uniquely determined up to θ-orbit so that the term $N\alpha$ of (1.3) is well-defined.

Suppose $\mathcal{O}$ is of type R_1 or R_2. Then the contribution to the *numerator* of $\Delta_{\rm II}(\gamma,\delta)$ from the orbit $\mathcal{O}$ is

$$\chi_{\alpha_{\rm res}}\left(\frac{N\alpha(\delta^*)-1}{a_{\alpha_{\rm res}}}\right),$$

where $\alpha_{\rm res}$ is a representative for $\mathcal{O}$. Because $\sigma\chi_{\alpha_{\rm res}} = \chi_{\sigma\alpha_{\rm res}}$, $\sigma a_{\alpha_{\rm res}} = a_{\sigma\alpha_{\rm res}}$, $\sigma(N\alpha) = N(\sigma\alpha)$ and $\sigma\delta^* \equiv \delta^*$ modulo $(1-\theta^*)T$ (see (3.3.4)), we conclude that this term does not depend on the choice of $\alpha_{\rm res}$. Nor does it depend on the choice of δ^*.

If $\mathcal{O}$ is of type R_3 then the contribution is

$$\chi_{\alpha_{\rm res}}(N\alpha(\delta^*)+1).$$

Again the choice of representative $\alpha_{\rm res}$ for $\mathcal{O}$ does not matter.

The contribution to the *denominator* of $\Delta_{\rm II}(\gamma,\delta)$ from the orbit $\mathcal{O}_H$ in R_H is

$$\chi_{\alpha_H}\left(\frac{\alpha_H(\gamma)-1}{a_{\alpha_H}}\right),$$

where α_H is an(y) element of $\mathcal{O}_H$.

The term $\Delta_{\rm II}(\gamma,\delta)$ may be written in another way. Any $\alpha \in R$ with the property that $(\alpha^\vee)_{\rm res} = (\alpha_H)^\vee$, for some $\alpha_H \in R_H$, will be said to be *from H*. We may use the same terminology for the θ-orbit of α, for $\alpha_{\rm res}$ or for the Galois orbit of $\alpha_{\rm res}$. Recall that if $(\alpha^\vee)_{\rm res} = \alpha_H^\vee$ then:

$$\alpha_H(\gamma) = \begin{cases} N\alpha(\delta^*) & \text{if } \alpha_{\rm res} \text{ is of type } R_1 \text{ or } R_3 \\ (N\alpha(\delta^*))^2 & \text{if } \alpha_{\rm res} \text{ is of type } R_2. \end{cases} \tag{4.3.1}$$

Lemma 4.3.A. *$\Delta_{II}(\gamma,\delta)$ is the product over representatives α_{res} for the Galois orbits in R_{res} of:*

$$\chi_{\alpha_{\text{res}}}\left(\frac{N\alpha(\delta^*)-1}{a_{\alpha_{\text{res}}}}\right) \tag{4.3.2}$$

for orbits of type R_1 and not from H,

$$\chi_{\alpha_{\text{res}}}\left(\frac{(N\alpha(\delta^*)-1)(N\alpha(\delta^*)+1)}{a_{\alpha_{\text{res}}}}\right) \tag{4.3.3}$$

for orbits of type R_2 such that both α_{res} and $2\alpha_{\text{res}}$ are not from H,

$$\chi_{\alpha_{\text{res}}}(N\alpha(\delta^*)+1) \tag{4.3.4}$$

for orbits of type R_3 from H, and 1 for the remaining orbits.

If α_{res} is of type R_1 *and from H* then (4.3.1) shows that the contribution to $\Delta_{II}(\gamma,\delta)$ from the Γ-orbit of α_{res} is 1. So the contribution from all orbits of type R_1 is given by (4.3.2) over those orbits not from H.

Now suppose α_{res} is of type R_2. We have the three cases: (i) α_{res}, $2\alpha_{\text{res}}$ are not from H, (ii) α_{res} is from H, $2\alpha_{\text{res}}$ is not from H, (iii) α_{res} is not from H and $2\alpha_{\text{res}}$ is from H. For (i), the contribution to $\Delta_{II}(\gamma,\delta)$ from the *orbits of α_{res} and $2\alpha_{\text{res}}$* is given by (4.3.3). For (ii), (4.3.1) implies that the contribution is

$$\chi_{\alpha_{\text{res}}}\left(\frac{(N\alpha(\delta^*)-1)(N\alpha(\delta^*)+1)}{a_{\alpha_{\text{res}}}}\right)$$

divided by

$$\chi_{\alpha_{\text{res}}}\left(\frac{(N\alpha(\delta^*))^2-1}{a_{\alpha_{\text{res}}}}\right),$$

and so is trivial. For (iii), the contribution is given by (4.3.4), and the lemma is proved.

(4.4) The term Δ_{III}. This is the one genuinely relative term, *i.e.* in general, $\Delta_{III}(\gamma_1,\delta;\bar\gamma_1,\bar\delta)$ is not defined as a quotient. Further it is the only term where the dependence on γ_1,$\bar\gamma_1$ is *not* through γ,$\bar\gamma$. (Recall that γ is the image of γ_1 under $H_1(F)\to H(F)$, and so on.) In fact, in (5.1) we will prove that

$$\Delta_{III}(z_1\gamma_1,\delta;\bar\gamma_1,\bar\delta)=\lambda_{H_1}(z_1)^{-1}\Delta_{III}(\gamma_1,\delta;\bar\gamma_1,\bar\delta)$$

for $z_1\in Z_1(F)=\ker[H_1(F)\to H(F)]$, where λ_{H_1} is the character from (2.2). It is nevertheless useful to describe separately the rather special case $H_1=H$, for this reveals a structure on which we may build for the general case.

Thus we assume for the next several paragraphs that $H_1=H$. To begin, we recall the tori $S=S(T,\overline{T})$ and $U=U(T,\overline{T})$ from [LS2]. Because both $\hat{T}$ and $\hat{\overline{T}}$ are isomorphic to T we have isomorphisms $\hat{T}\to\hat{\overline{T}}$ and $T\to\overline{T}$. When convenient, we use these isomorphisms to identify T with $\overline{T}$ (over $\overline{F}$), $X^*(T)$ with $X^*(\overline{T})$, and so on. Then

$$S = (T \times \overline{T})/\{(z, z^{-1}) : z \in Z\}$$

and

$$U = (T_{\text{sc}} \times \overline{T}_{\text{sc}})/\{(z, z^{-1}) : z \in Z_{\text{sc}}\},$$

where $Z = Z(G)$ and $Z_{\text{sc}} = Z(G_{\text{sc}})$. Thus $X^*(S)$ is the lattice in $X^*(T) \times X^*(\overline{T})$ given as $\{(\lambda, \mu) : \lambda - \mu \in X^*(T_{\text{ad}})\}$ and $X^*(U)$ is the lattice in $X^*(T_{\text{sc}}) \times X^*(\overline{T}_{\text{sc}})$ given the same way.

Recall that we may identify $\hat{U}$ as

$$(\hat{T}_{\text{sc}} \times \hat{\overline{T}}_{\text{sc}})/\{(z, z) : z \in \hat{Z}_{\text{sc}}\},$$

where as usual the subscript sc indicates objects in the simply connected covering of the derived group of $\hat{G}$. There is another way of describing $\hat{U}$ that is convenient also when we come to $\hat{S}$. Namely,

$$\hat{U} \simeq \hat{T}_{\text{ad}} \times \hat{\overline{T}}_{\text{sc}},$$

where there is a twisted Galois action on this product: $\sigma \in \Gamma$ acts by

$$(t_{\text{ad}}, \bar{t}_{\text{sc}}) \mapsto (\sigma_T(t_{\text{ad}}), \alpha(\omega(\sigma))(\sigma_T(t_{\text{ad}})) \cdot \sigma_{\overline{T}}(\bar{t}_{\text{sc}})).$$

Here $\omega(\sigma)$ is the element $\sigma_{\overline{T}}\sigma_T^{-1}$ of the Weyl group of T and $\alpha(\omega(\sigma)) : \hat{T}_{\text{ad}} \to \hat{T}_{\text{sc}}$ is given by $t_{\text{ad}} \mapsto (\omega(\sigma)(t'_{\text{ad}}))(t'_{\text{ad}})^{-1}$, where t'_{ad} is an element of $\hat{T}_{\text{sc}}$ mapping to t_{ad}. The isomorphism from $\hat{U}$ to $\hat{T}_{\text{ad}} \times \hat{\overline{T}}_{\text{sc}}$ takes the element of $\hat{U}$ represented by $(t_{\text{sc}}, \bar{t}_{\text{sc}}) \in \hat{T}_{\text{sc}} \times \hat{\overline{T}}_{\text{sc}}$ to $(\tilde{t}_{\text{sc}}, \bar{t}_{\text{sc}} t_{\text{sc}}^{-1})$, where $\tilde{t}_{\text{sc}}$ denotes the image of t_{sc} in $\hat{T}_{\text{ad}}$. Similarly, $\hat{S} = \hat{T} \times \hat{\overline{T}}_{\text{sc}}$. See the proof of Lemma 3.5.A in [LS2].

By $1 - \theta : U \to S$ we will mean more precisely the map induced by

$$(t_{\text{sc}}, \bar{t}_{\text{sc}}) \mapsto ((1 - \theta^*)(t), (1 - \theta^*)(\bar{t})),$$

where $t, \bar{t}$ are the images of $t_{\text{sc}}, \bar{t}_{\text{sc}}$ in $T, \overline{T}$. The dual homomorphism $1 - \hat{\theta} : \hat{S} \to \hat{U}$ can be realized as the map

$$\hat{T} \times \hat{\overline{T}}_{\text{sc}} \to \hat{T}_{\text{ad}} \times \hat{\overline{T}}_{\text{sc}}$$

given by

$$(t, \bar{t}_{\text{sc}}) \mapsto ((1 - \hat{\theta}_{\text{ad}})t_{\text{ad}}, (1 - \hat{\theta}_{\text{sc}})\bar{t}_{\text{sc}}),$$

where t_{ad} is the image of t in T_{ad}.

We shall define an element

$$\mathbf{V} = \operatorname{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$$

of $H^1(F, U \xrightarrow{1-\theta} S)$ and an element $\mathbf{A}$ of $H^1(W_F, \hat{S} \xrightarrow{1-\hat{\theta}} \hat{U})$. Then we set

$$\Delta_{\text{III}}(\gamma, \delta; \bar{\gamma}, \bar{\delta}) = \langle \mathbf{V}, \mathbf{A} \rangle.$$

These hypercohomology groups and the pairing $\langle \cdot, \cdot \rangle$ are discussed in Appendix A.

We have $\delta^* = gm(\delta)\theta^*(g)^{-1}$ and that $N_{\theta^*}(\delta^*) \in T_{\theta^*}$ is equal to the image of γ under $T_H \to T_{\theta^*}$. Write $\psi\sigma(\psi)^{-1} = \operatorname{Int} u(\sigma)$ as in (3.1) and set $v(\sigma) = gu(\sigma)\sigma(g)^{-1}$ as in (3.3).

Lemma 4.4.A.

(1) *$v(\sigma)$ lies in T_{sc}.*
(2) $(1-\theta^*)v(\sigma) = \sigma(\delta^*)^{-1}\delta^*$.

We have arranged that

$$\sigma(m)(\delta) = u(\sigma)^{-1}m(\delta)\theta^*(u(\sigma))$$

(see (3.1.2)). From this and

$$gm(\delta)\theta^*(g)^{-1} = \delta^*$$

we conclude that

$$v(\sigma)^{-1}\delta^*\theta^*(v(\sigma)) = \sigma(\delta^*).$$

Since $\sigma(\delta^*) \equiv \delta^*$ modulo $(1-\theta^*)T$ we deduce that

$$(v(\sigma)t)^{-1}\delta^*\theta^*(v(\sigma)t) = \delta^*$$

for some $t \in T$. But $\mathrm{Cent}_{\theta^*}(\delta^*, G^*) = T^{\theta^*}$ and so we conclude that $v(\sigma)$ lies in T_{sc}, as asserted in (1). In particular all four elements in the equation

$$v(\sigma)^{-1}\delta^*\theta^*(v(\sigma)) = \sigma(\delta^*)$$

commute with each other, and (2) is then immediate.

Observe also that the coboundary ∂v of $v(\sigma)$ is $g(\partial u)g^{-1} = \partial u$ since ∂u takes values in Z_{sc}.

We make the same constructions for $\bar{\gamma},\bar{\delta}$ and thus have $\bar{\delta}^*,\bar{v}(\sigma)$ along with $\delta^*,v(\sigma)$. Let $V(\sigma)$ be the image in U of $(v(\sigma)^{-1}, \bar{v}(\sigma)) \in T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}}$ under the natural projection, and let D be the image of $(\delta^*, (\bar{\delta}^*)^{-1}) \in T \times \overline{T}$ in S. Because $(\partial v^{-1}, \partial \bar{v}) = (\partial u^{-1}, \partial u)$ we have that V is a 1-cocycle and (2) of Lemma 4.4.A shows that

$$(1-\theta)(V(\sigma)) = \sigma(D)D^{-1}$$

for all $\sigma \in \Gamma$. We conclude that (V, D) represents a class $\mathbf{V}$ in $H^1(F, U \xrightarrow{1-\theta} S)$. It is readily verified that as long as the embeddings $T_H \to T_{\theta^*}$, $\overline{T}_H \to \overline{T}_{\theta^*}$ remain fixed the element $\mathbf{V}$ is uniquely determined by $\gamma \in T_H(F)$, $\bar{\gamma} \in \overline{T}_H(F)$, δ and $\bar{\delta}$. Moreover the embedding $T_H \to T_{\theta^*}$ is determined by the element γ and the requirement that $\gamma \mapsto N_\theta(\delta) \in T_{\theta^*}$, and the analogous assertion is true for the other admissible embedding as well. For this reason we denote $\mathbf{V}$ by $\mathrm{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$.

We turn now to constructing $\mathbf{A}$. To the endoscopic datum s we attach the element s_U of $\hat{U}$ following the prescription of [LS1, Section 3.4], which we now review. Recall that we are working under the assumptions of the proof of Theorem 3.3.A. Now choose $\tilde{s} \in \mathcal{T}_{\mathrm{sc}}$ having the same image as s in $\mathcal{T}_{\mathrm{ad}}$ and use the isomorphisms $\hat{T} \simeq \mathcal{T}$, $\hat{\overline{T}} \simeq \mathcal{T}$ to obtain $\tilde{s}_T \in \hat{T}_{\mathrm{sc}}$ and $\tilde{s}_{\overline{T}} \in \hat{\overline{T}}_{\mathrm{sc}}$. Then the class s_U of $(\tilde{s}_T, \tilde{s}_{\overline{T}})$ in $\hat{U}$ is independent of the choice of $\tilde{s}$. In our present more general setting, s_U is not necessarily Γ-invariant. We will construct a 1-cocycle A of W_F in $\hat{S}$ such that

$$s_U w(s_U)^{-1} = (1-\hat{\theta})(A(w))$$

for all $w \in W_F$. Then $\mathbf{A}$ will be the class of (A^{-1}, s_U) in the dual hypercohomology group $H^1(W_F, \hat{S} \xrightarrow{1-\hat{\theta}} \hat{U})$ (see Appendix A).

Let $\hat{G}^1$ be the identity component of the fixed points of $\hat{\theta}$ in $\hat{G}$. Then $\hat{G}^1$ is preserved by W_F. Set ${}^L G^1 = \hat{G}^1 \rtimes W_F$ with this inherited action of W_F. Then ${}^L G^1$ is an L-group, for we may use the $\hat{\theta}$-stable Γ-splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \dots)$ to construct a Γ-splitting $\mathbf{spl}_{\hat{G}^1} = (\mathcal{B}^1 = \mathcal{B} \cap \hat{G}^1, \mathcal{T}^1 = \mathcal{T} \cap \hat{G}^1, \dots)$ of $\hat{G}^1$ (recall Section 1 and [St] on root vectors) and ${}^L G^1$ embeds naturally in ${}^L G$. Also, $\mathcal{T}^1$ is a maximal torus in both $\hat{G}^1$ and $\xi(\hat{H})$; both $R(\hat{G}^1, \mathcal{T}^1)$ and $R(\xi(\hat{H}), \mathcal{T}^1)$ have the actions Γ_T,$\Gamma_{\overline{T}}$ derived from the Galois actions on T,$\overline{T}$ and are contained in $R^\vee_{\text{res}}$, $\overline{R}^\vee_{\text{res}}$. Thus we have χ-data for them. To the data for $R(\xi(\hat{H}), \mathcal{T}^1)$ is attached an embedding ${}^L T_H \hookrightarrow {}^L H$ [LS1, Sect. 2.6]. We compose this with the given $\xi_H : {}^L H \to \mathcal{H}$ (recall our assumption $H_1 = H$) and $\xi : \mathcal{H} \hookrightarrow {}^L G$ to obtain an embedding $\xi_{T_H} : {}^L T_H \hookrightarrow {}^L G$ extending the isomorphism $\hat{T}_H \to \mathcal{T}^1$. On the other hand, if we identify $\mathcal{T}^1$ as $(\hat{T}^{\hat{\theta}})^0$ then $\hat{T}_H \to \mathcal{T}^1$ extends to an isomorphism ${}^L T_H \to {}^L(T_{\theta^*})$, and at the same time the χ-data for $R(\hat{G}^1, \mathcal{T}^1)$ yield an embedding

$$
{}^L(T_{\theta^*}) \hookrightarrow {}^L G^1 \hookrightarrow {}^L G,
$$

where the embedding of ${}^L G^1$ in ${}^L G$ is the natural one. Thus we obtain a second embedding $\xi_1 : {}^L T_H \hookrightarrow {}^L G$. But then $\xi_{T_H} = a_T \cdot \xi_1$, where a_T is a 1-cocycle of W_F in $\mathcal{T}$ with W_F acting *via* the Galois action on T. As usual [LS1] we transport a_T to $\hat{T}$ without change in notation. Replacing T by $\overline{T}$ we obtain also a 1-cocycle $a_{\overline{T}}$ of W_F in $\hat{\overline{T}}$. It will be convenient now to identify $\hat{\overline{T}}$ with $\hat{T}$ as tori over $\mathbb{C}$. Consider then $x(w) := a_{\overline{T}}(w)/a_T(w)$. Our constructions will be seen to yield naturally an element $x_{\text{sc}}(w)$ of $\hat{\overline{T}}_{\text{sc}}$ with $x(w)$ as image under the natural projection; this requires an extension of the Second Lemma of Comparison in [LS2] as we shall explain in the proof below. Then set

$$
A(w) = (a_T(w), x_{\text{sc}}(w)).
$$

This is an element of $\hat{S}$. We will be done with our constructions for Δ_{III} (in the case $H_1 = H$) as soon as we have checked the following.

Lemma 4.4.B.

(1) *A is a 1-cocycle of W_F in $\hat{S}$.*
(2) $(1 - \hat{\theta})(A(w)) = s_U w(s_U)^{-1}$ *for all* $w \in W_F$.

First we describe $x_{\text{sc}}(w)$ explicitly. Recall from (1.1) that

$$
\Omega(\xi(\hat{H}), \mathcal{T}^1) \subset \Omega(\hat{G}^1, \mathcal{T}^1) \subset \Omega(\hat{G}, \mathcal{T}),
$$

so that $\Omega(H, T_H)$ acts on $\hat{T}$. If $\sigma \in \Gamma$ then $\sigma_{\overline{T}} = \omega(\sigma)\sigma_T$, where $\omega(\sigma) \in \Omega(H, T_H)$. Thus $\Gamma_{\overline{T}} \subset \Omega(H, T_H) \rtimes \Gamma_T$ and we can proceed as in (3.4) of [LS2]. First we construct objects in $\hat{G}^1$ instead of in $\hat{G}$. Thus

$$
m_1(w) = r_1(w) n_1(w) \times w
$$

and so on. Again we shall index the corresponding objects for $\hat{H}$ by s. Thus

$$m_s(w) = r_s(w)n_s(w) \times w$$

and $a_T(w)$ is given by

$$\xi \circ \xi_H(m_s(w)) = a_T(w)m_1(w).$$

Similarly,

$$\xi \circ \xi_H(\bar{m}_s(w)) = a_{\overline{T}}(w)\bar{m}_1(w).$$

Set

$$c(w) = r_1(w)r_s(w)^{-1}$$

and

$$\bar{c}(w) = \bar{r}_1(w)\bar{r}_s(w)^{-1}.$$

We argue exactly as in the proof of Lemma 3.4.A of [LS2]—the Second Lemma of Comparison—to show that

$$a_{\overline{T}}(w)^{-1} = \bar{c}(w)\hat{b}(\omega)\omega(c(w)^{-1}a_T(w)^{-1})\hat{r}(\omega,\sigma)^{-1},$$

where $\omega = \omega(\sigma)$ and $\hat{b}$,$\hat{r}$ are as defined in (3.4) of [LS2]. We conclude then that $x(w) = a_{\overline{T}}(w)/a_T(w)$ is given by

$$\bar{c}(w)^{-1}\hat{b}(\omega)^{-1}\omega(c(w))\hat{r}(\omega,\sigma)(\omega - 1)(a_T(w)).$$

Each term of this product lifts naturally to $\hat{T}_{\text{sc}}$. Indeed, the first four of them are given in terms of elements constructed in $(\hat{G}^1)_{\text{sc}}$ or $\hat{H}_{\text{sc}}$ and projected into $\hat{G}$, and of course these projection maps factor through $\hat{G}_{\text{sc}}$. The last term lifts to $\alpha(\omega)(a_T(w))$ with $\alpha(\omega) = \alpha(\omega(\sigma))$ as before. Therefore we obtain $x_{\text{sc}}(w)$ in $\hat{T}_{\text{sc}}$ projecting to $x(w)$ in $\hat{T}$. Thus $A(w) \in \hat{S}$. To show that A is a 1-cocycle we follow the argument in the proof of Lemma 3.5.A of [LS2]; the analogue of their Lemma 4.2.A, which is needed for that argument, goes through without modification. So (1) of our lemma is proved.

For (2), we observe that (2.1.4a) implies that

$$s\hat{\theta}(a_T(w))m_1(w)s^{-1} = a'(w)a_T(w)m_1(w)$$

and thus

$$(1 - \hat{\theta})a_T(w) = sw_T(s)^{-1}a'(w)^{-1}. \tag{4.4.1}$$

On the other hand,

$$(1 - \hat{\theta}_{\text{sc}})x_{\text{sc}}(w) = (1 - \hat{\theta}_{\text{sc}})[\hat{b}(\omega)^{-1}\alpha(\omega)(a_T(w))]$$

as the remaining terms of $x_{\text{sc}}(w)$ are fixed by $\hat{\theta}_{\text{sc}}$. But $\hat{b}(\omega)^{-1} = n_s(\omega)n_1(\omega)^{-1}$, with $n_s(\omega)$ in the fixed points of $\text{Int}(s_{\text{sc}}) \circ \hat{\theta}_{\text{sc}}$ and $n_1(\omega)$ in the fixed points of $\hat{\theta}_{\text{sc}}$, where s_{sc} has image s_{ad} in $\hat{G}_{\text{ad}}$. Thus $(1 - \hat{\theta}_{\text{sc}})\hat{b}(\omega)^{-1} = \omega(s_{\text{sc}})^{-1}s_{\text{sc}} = \alpha(\omega)(s)^{-1}$. Next we compute $(1-\hat{\theta}_{\text{sc}})(\alpha(\omega)(a_T(w)))$ as $\alpha(\omega)((1-\hat{\theta})a_T(w)) = \alpha(\omega)(sw_T(s)^{-1})$, and so $(1-\hat{\theta}_{\text{sc}})x_{\text{sc}}(w) = \alpha(\omega)(w_T(s))^{-1}$.

Recalling the precise meaning of $1-\hat{\theta}$ as map from $\hat{S}$ to $\hat{U}$, we conclude that

$$(1-\hat{\theta})A(w) = (s_{\text{ad}}w_T(s_{\text{ad}})^{-1}, \alpha(\omega)(w_T(s))^{-1}).$$

Finally, when $\hat{U}$ is realized as $\hat{T}_{\text{ad}} \times \hat{\overline{T}}_{\text{sc}}$ the element s_U becomes $(s_{\text{ad}}, 1)$ and so $s_U w(s_U)^{-1}$ is also equal to $(s_{\text{ad}}w_T(s_{\text{ad}})^{-1}, \alpha(\omega)(w_T(s))^{-1})$ and we are done.

We turn now to the general case. Thus (H_1, ξ_{H_1}) is an arbitrary z-pair. Recall that Z_1 denotes the kernel of $H_1 \to H$. We shall modify our arguments for the special case $H_1 = H$ by replacing the torus S by a torus S_1 as follows. Start with the exact sequence

$$1 \to Z_1 \to T_{H_1} \xrightarrow{p} T_H \to 1.$$

Form a pull-back diagram from this exact sequence together with the homomorphism

$$N : T \to T_H$$

obtained by composing the projection map $T \to T_{\theta^*}$ with the isomorphism $T_H \simeq T_{\theta^*}$ provided by our admissible embedding; thus we get a commutative diagram

$$\begin{array}{ccc} T_1 & \longrightarrow & T \\ \downarrow & & \downarrow N \\ T_{H_1} & \xrightarrow{p} & T_H \end{array}$$

in which T_1 is the fiber product

$$\{(t, t_{H_1}) \in T \times T_{H_1} : N(t) = p(t_{H_1})\};$$

of course $T_1 \to T$ is surjective with kernel Z_1. Note that T_1 is a torus and that the automorphism θ^* of T lifts naturally to an automorphism θ_1 of T_1 that acts trivially on the subgroup Z_1 of T_1: for $(t, t_{H_1}) \in T_1$ the automorphism θ_1 is given by $\theta_1(t, t_{H_1}) = (\theta^*(t), t_{H_1})$. Set $\delta_1^* = (\delta^*, \gamma_1) \in T_1$ and $\bar{\delta}_1^* = (\bar{\delta}^*, \bar{\gamma}_1) \in \overline{T}_1$.

We identify $\overline{T}_1$ with T_1 over $\overline{F}$. Then S_1 is defined to be the quotient of $T_1 \times \overline{T}_1$ given by

$$X^*(S_1) = \{(\lambda, \mu) \in X^*(T_1) \times X^*(\overline{T}_1) : \lambda - \mu \in X^*(T_{\text{ad}})\}.$$

Thus $S_1 \simeq T_1 \times \overline{T}_{\text{ad}}$ with twisted Galois action on the second factor as before, and we have an obvious exact sequence

$$1 \to Z_1 \to S_1 \to S \to 1.$$

Recall the homomorphism

$$1 - \theta : U \to S.$$

There is a natural lifting of $1-\theta$ to a homomorphism

$$1-\theta : U \to S_1,$$

obtained as follows. The automorphism θ_1 of T_1 is trivial on Z_1, and therefore $1-\theta_1 : T_1 \to T_1$ has Z_1 in its kernel and hence yields $1-\theta_1 : T \to T_1$. The same is true for $\overline{T}$ so that we get a map

$$1-\theta_1 : T \times \overline{T} \to T_1 \times \overline{T}_1,$$

which carries the kernel of $T \times \overline{T} \to S$ into the kernel of $T_1 \times \overline{T}_1 \to S_1$ and hence induces a map

$$1-\theta_1 : S \to S_1,$$

which we compose with the obvious map

$$U \to S$$

to get the lifting

$$1-\theta : U \to S_1.$$

Recall that we have already defined a 1-cocycle V of Γ in U. Now define an element D_1 of S_1 by taking the image under

$$T_1 \times \overline{T}_1 \to S_1$$

of $(\delta_1^*, (\bar{\delta}_1^*)^{-1}) \in T_1 \times \overline{T}_1$. Recall from Lemma 4.4.A that

$$(1-\theta^*)v(\sigma) = \sigma(\delta^*)^{-1} \cdot \delta^*.$$

Therefore the image of $v(\sigma)$ under

$$T_{\mathrm{sc}} \to T \xrightarrow{1-\theta_1} T_1$$

is given by $(\sigma(\delta^*)^{-1} \cdot \delta^*, 1) \in T \times T_{H_1}$, and since $(\sigma(\delta^*)^{-1} \cdot \delta^*, 1) = \sigma(\delta_1^*)^{-1} \cdot \delta_1^*$ we conclude that (V, D_1) satisfies the hypercocycle condition

$$(1-\theta)V = \sigma(D_1) \cdot D_1^{-1}.$$

Thus we have produced an element

$$\mathbf{V}_1 \in H^1(F, U \xrightarrow{1-\theta} S_1).$$

We need to produce a class

$$\mathbf{A}_1 \in H^1(W_F, \hat{S}_1 \xrightarrow{1-\hat{\theta}} \hat{U})$$

so that we can define Δ_{III} by

$$\Delta_{\mathrm{III}}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta}) = \langle \mathbf{V}_1, \mathbf{A}_1 \rangle.$$

Recall that we already have defined an element $s_U \in \hat{U}$. However the construction of the 1-cocycle A of W_F in $\hat{S}$ breaks down since it used the assumption $H_1 = H$. Fortunately the construction can be modified so as to produce the desired 1-cocycle in $\hat{S}_1$.

We define

$$\xi_1 : {}^L T_H \to {}^L G$$

just as before. We no longer have the embedding

$$\xi_{T_H} : {}^L T_H \to {}^L G.$$

Instead we work with

$$\xi_{T_H} : {}^L T_H \to {}^L H_1$$

defined as the composition of the L-homomorphism

$${}^L T_H \to {}^L H$$

used before and the obvious inclusion

$${}^L H \hookrightarrow {}^L H_1.$$

We can no longer compare ξ_1 and ξ_{T_H} directly; we need the intermediary $\mathcal{H}$ and the L-homomorphisms

$$\xi : \mathcal{H} \hookrightarrow {}^L G$$

and

$$\xi_{H_1} : \mathcal{H} \hookrightarrow {}^L H_1.$$

Now consider the subgroup $\mathcal{U}$ of $\mathcal{H}$ consisting of all elements $h \in \mathcal{H}$ that normalize $\mathcal{T}_H$ and act on it by σ_{T_H}, where σ denotes the image of h under

$$\mathcal{H} \to W_F \to \Gamma$$

and σ_{T_H} denotes the action of σ on $\mathcal{T}_H$ given by the identification $\mathcal{T}_H \simeq \hat{T}_H$. Then it is easy to see that the projection $\mathcal{H} \to W_F$ maps $\mathcal{U}$ onto W_F. Since $\mathcal{U}$ is locally compact and its topology has a countable base we see that $\mathcal{U}/\mathcal{T}_H$ is isomorphic as topological group to W_F.

Use the splitting $(\mathcal{B}_H, \mathcal{T}_H, \dots)$ of $\hat{H}$ to get a splitting $(\mathcal{B}_{H_1}, \mathcal{T}_{H_1}, \dots)$ of $\hat{H}_1$. Then we have an isomorphism $\hat{T}_{H_1} \simeq \mathcal{T}_{H_1}$ extending $\hat{T}_H \simeq \mathcal{T}_H$ and there is a unique L-homomorphism

$$\xi'_{T_H} : {}^L T_{H_1} \hookrightarrow {}^L H_1$$

extending both ξ_{T_H} and the embedding $\hat{T}_{H_1} \simeq \mathcal{T}_{H_1} \hookrightarrow \hat{H}_1$. The map ξ'_{T_H} is a homeomorphism onto its image (see (2.1) for this kind of argument) and the image of $\mathcal{U}$ under ξ_{H_1} is contained in the image of ξ'_{T_H}, so that there exists a unique L-homomorphism

$$\alpha_0 : \mathcal{U} \to {}^L T_{H_1}$$

such that $\xi'_{T_H} \circ \alpha_0$ is equal to the restriction of ξ_{H_1} to $\mathcal{U}$. Let $\alpha : \mathcal{U} \to {}^L T_{H_1}$ be the composition of α_0 with the map ${}^L T_{H_1} \to {}^L T_{H_1}$ induced by the map $T_{H_1} \to T_{H_1}$ sending t to t^{-1}.

Let $t \in {}^L T_H$ and write $\xi_1(t) = x \times w \in \hat{G}^1 \rtimes W_F \subset \hat{G} \rtimes W_F$. Then $\xi_1(t)$ normalizes $\mathcal{T}^1 := (\mathcal{T}^\theta)^0$ and therefore normalizes $\mathcal{T}$ as well. We claim that $\xi_1(t)$ acts on $\mathcal{T}$ by σ_T, where σ is the image of w in Γ and σ_T denotes the action of σ on $\mathcal{T}$ coming from the identification $\mathcal{T} \simeq \hat{T}$. Indeed, the action of σ_T on $\mathcal{T}$ is given by some element $y \times w$ with y belonging to the normalizer of $\mathcal{T}^1$ (and hence to the normalizer of $\mathcal{T}$ as well). Since $x \times w$ and $y \times w$ both act by σ_{T_H} on $\mathcal{T}^1$, we conclude that x and y have the same action on $\mathcal{T}_1$ and therefore that they have equal images in $\Omega(\hat{G}, \mathcal{T})^\theta$. This proves our claim. It now follows immediately that $\xi_1({}^L T_H) \cdot \mathcal{T}$ is isomorphic to the L-group of T *via* the unique L-homomorphism

$$\xi_1' : {}^L T \hookrightarrow {}^L G$$

extending both ξ_1 (use the obvious inclusion ${}^L T_H \hookrightarrow {}^L T$ dual to $N : T \to T_H$) and $\hat{T} \simeq \mathcal{T} \hookrightarrow \hat{G}$.

Let $h \in \mathcal{U}$ and write $\xi(h) = g \times w \in \hat{G} \rtimes W_F$. Then $\xi(h)$ normalizes $\mathcal{T}^1$ and $\mathcal{T}$ and acts on $\mathcal{T}^1$ by σ_{T_H}. As above it follows that $\xi(h)$ acts on $\mathcal{T}$ by σ_T, where σ is the image of w in Γ. Therefore $\xi(\mathcal{U})$ is contained in the image of ${}^L T$ under ξ_1', and it follows that there exists a unique L-homomorphism

$$\beta : \mathcal{U} \to {}^L T$$

such that $\xi_1' \circ \beta$ is equal to the restriction of ξ to $\mathcal{U}$. Together α and β give an L-homomorphism

$$\alpha \times \beta : \mathcal{U} \to {}^L(T_{H_1} \times T).$$

The restriction of $\alpha \times \beta$ to $\hat{T}_H$ embeds $\hat{T}_H$ in $\hat{T}_{H_1} \times \hat{T}$, and the quotient of $\hat{T}_{H_1} \times \hat{T}$ by the image of $\hat{T}_H$ is equal to $\hat{T}_1$. Therefore the map

$$\varphi : W_F = \mathcal{U}/\hat{T}_H \to {}^L(T_{H_1} \times T)/\hat{T}_H = {}^L T_1$$

induced by $\alpha \times \beta$ gives us a 1-cocycle a_T of W_F in $\hat{T}_1$ defined by

$$\varphi(w) = a_T(w) \times w \in {}^L T_1.$$

Let $w \in W_F$. We can define $a_T(w)$ more concretely as follows. Pick $u(w) \in \mathcal{U}$ such that $u(w)$ maps to w under $\mathcal{U} \to W_F$. Write $\xi_1(1 \times w) = t(w)\xi(u(w))$ with $t(w) \in \hat{T}$ and write $\xi_{T_H}(1 \times w) = t_1(w)\xi_{H_1}(u(w))$ with $t_1(w) \in \hat{T}_{H_1}$. Then the element of $\hat{T}_1$ represented by $(t_1(w), t(w)^{-1}) \in \hat{T}_{H_1} \times \hat{T}$ is equal to $a_T(w)$.

Applying ${}^L\theta$ to the equation defining $t(w)$, we find that

$$(1 - \hat{\theta})(t(w)) = s^{-1} \cdot \sigma_T(s) \cdot a'(w).$$

It follows that

$$(1 - \hat{\theta})(a_T(w)) = s \cdot \sigma_T(s)^{-1} \cdot a'(w)^{-1} \in \hat{T}. \tag{4.4.2}$$

The rest of the construction is just the same as in the case $H_1 = H$. From the second torus $\overline{T}$ we get $a_{\overline{T}}(w)$ and the element $x(w) := a_{\overline{T}}(w)/a_T(w) \in \hat{\overline{T}}_1$ is the same as the previously considered $x(w) \in \hat{\overline{T}} \hookrightarrow \hat{\overline{T}}_1$. Therefore we use the same lifting $x_{\mathrm{sc}}(w) \in \hat{\overline{T}}_{\mathrm{sc}}$ as before. Again

$$A(w) = (a_T(w), x_{\mathrm{sc}}(w)) \in \hat{T}_1 \times \hat{\overline{T}}_{\mathrm{sc}} = \hat{S}_1$$

is a 1-cocycle of W_F in $\hat{S}_1$ and the class of (A^{-1}, s_U) is the desired element

$$\mathbf{A}_1 \in H^1(W_F, \hat{S}_1 \xrightarrow{1-\hat{\theta}} \hat{U}).$$

The next lemma will be needed in (5.5).

Lemma 4.4.C. *Let γ, γ_1, δ be as in (4.1), and let I denote the identity component of the twisted centralizer of δ in G. As usual let ω be the quasicharacter on $G(F)$ obtained from* $\mathbf{a}$. *Then the restriction of ω to $I(F)$ is trivial.*

The 1-cocycle $a'(w)$ appearing in (4.4.2) lies in the class $\mathbf{a}$. It follows from (4.4.2) that the image of the 1-cocycle $a'(w)$ becomes a coboundary when it is pushed forward using the inclusion of $Z(\hat{G})$ in $\hat{T}$ followed by the canonical surjection from $\hat{T}$ to $\hat{T}/(1-\hat{\theta})\hat{T}$. Let ω^* denote the quasicharacter on $G^*(F)$ obtained from $\mathbf{a}$. The pushforward of a' to $\hat{T}$ corresponds to the restriction of ω^* to $T(F)$, and its pushforward to $\hat{T}/(1-\hat{\theta})\hat{T}$ corresponds to its restriction to $(T^\theta)^0(F)$. But $(T^\theta)^0$ can be identified with I, and thus we see that the restriction of ω to $I(F)$ is trivial, as desired.

(4.5) The term Δ_{IV}. We define $\Delta_{\mathrm{IV}}(\gamma_1, \delta) = \Delta_{\mathrm{IV}}(\gamma, \delta)$ to be the quotient of

$$D_{G\theta}(\delta) = |\det[\mathrm{Ad}(\delta) \circ \theta - 1]; \mathrm{Lie}(G)/\mathrm{Lie}(\mathrm{Cent}(G^{\delta\theta}, G)|_F^{1/2}$$

by

$$D_H(\gamma) = |\det[\mathrm{Ad}(\gamma) - 1]; \mathrm{Lie}(H)/\mathrm{Lie}(T_H)|_F^{1/2}.$$

Observe that using the isomorphism $\mathrm{Int}(g) \circ \psi : G \to G^*$ provided by (3.3.6) we may rewrite $D_{G\theta}(\delta)$ as

$$D_{G^*\theta^*}(\delta^*) = |\det[\mathrm{Ad}(\delta^*) \circ \theta^* - 1]; \mathrm{Lie}(G^*)/\mathrm{Lie}(T)|_F^{1/2}.$$

The discriminant $D_{G^*\theta^*}(\delta^*)$ is easily computed using the root-space decomposition provided by a θ^*-splitting $(B, T, \dots)$. We obtain a product over θ-orbits of roots α in R, *i.e.* a product over $a_{\mathrm{res}} \in R_{\mathrm{res}}$. If α_{res} is of type R_1 or R_2 then the corresponding contribution is

$$|N\alpha(\delta^*) - 1|_{\overline{F}}^{1/2}$$

and if α_{res} is of type R_3 it is

$$|N\alpha(\delta^*) + 1|_{\overline{F}}^{1/2}.$$

On the other hand,

$$D_H(\gamma) = \prod_{\alpha_H} |\alpha_H(\gamma) - 1|_{\overline{F}}^{1/2}.$$

In these expressions we have written $|\cdot|_{\overline{F}}$ for the unique absolute value on $\overline{F}$ extending $|\cdot|_F$ on F. A now familiar comparison yields:

Lemma 4.5.A. $\Delta_{IV}(\gamma_1, \delta)$ *is the product of:*
(4.5.1) $\prod_1 |N\alpha(\delta^*) - 1|_F^{1/2}$,
where $\prod_1$ *indicates the product over* α_{res} *of type* R_1 *and not from* H,
(4.5.2) $\prod_2 |(N\alpha(\delta^*) - 1)(N\alpha(\delta^*) + 1)|_F^{1/2}$,
$\prod_2$ *indicating the product over* α_{res} *of type* R_2 *with both* α_{res} *and* $2\alpha_{res}$ *not from* H, *and*
(4.5.3) $\prod_3 |N\alpha(\delta^*) + 1|_F^{1/2}$,
where $\prod_3$ *indicates the product over* α_{res} *of type* R_3 *and from* H.

(4.6) Canonicity. We may now set $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$ equal to the product of the four terms:

$$\Delta_I(\gamma, \delta)/\Delta_I(\bar{\gamma}, \bar{\delta})$$
$$\Delta_{II}(\gamma, \delta)/\Delta_{II}(\bar{\gamma}, \bar{\delta})$$
$$\Delta_{III}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$$
$$\Delta_{IV}(\gamma, \delta)/\Delta_{IV}(\bar{\gamma}, \bar{\delta})$$

(recall that $\gamma, \bar{\gamma}$ are the images of $\gamma_1, \bar{\gamma}_1$ in $H(F)$). The terms depend variously on $(T_H \to T_{\theta^*}, \{a_\alpha\}, \{\chi_\alpha\})$ and $(\overline{T}_H \to \overline{T}_{\theta^*}, \{a_{\bar{\alpha}}\}, \{\chi_{\bar{\alpha}}\})$.

Theorem 4.6.A. $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$ *is canonical.*

We shall check the effect when:

(4.6.1) we replace $(T_H \to T_{\theta^*}, \{a_\alpha\}, \{\chi_\alpha\})$ by data $(T_H \to T'_{\theta^*}, \{a_{\alpha'}\}, \{\chi_{\alpha'}\})$ which are *conjugate* in the sense that there exists $g \in (G^*_{sc})^{\theta^*_{sc}}$ such that

1. $\mathrm{Int}(g^{-1})$ induces isomorphisms $B \to B'$, $T \to T'$ and thus $T_{\theta^*} \to T'_{\theta^*}$, the latter two over F,
2. $T_H \to T'_{\theta^*}$ is the composition of $T_H \to T_{\theta^*}$ with $T_{\theta^*} \to T'_{\theta^*}$,
3. $\mathrm{Int}(g)$ transports $\{a_\alpha\}$, $\{\chi_\alpha\}$ to $\{a_{\alpha'}\}$, $\{\chi_{\alpha'}\}$,

(4.6.2) we change the a-data alone, and

(4.6.3) we change the χ-data alone.

Assume (4.6.1). Then Lemma 3.2.B of [LS1] implies that $\Delta_I(\gamma, \delta)$ is multiplied by $\langle \mathbf{g}, \mathbf{s}_{T,\theta} \rangle^{-1}$ where $\mathbf{g}$ is the class of $\sigma \mapsto g\sigma(g)^{-1}$ in $H^1(F, T^x)$ (note that it is not necessary to pass to the simply-connected cover of the derived group of G^x). The only other term in Δ that is affected is Δ_{III}. Instead of pairing the class $\mathbf{V}_1$ of (V, D_1) with $\mathbf{A}_1$ (notation of (4.4)) we pair the class $\mathbf{V}'_1$ of (V', D_1) with the same $\mathbf{A}_1$, where $V'V^{-1}$ is the 1-cocycle $\sigma \mapsto (g\sigma(g)^{-1}, 1)$ of Γ in U. Thus $\mathbf{V}'\mathbf{V}^{-1}$ is the image of $\mathbf{g}$ under

$$H^1(F, T^x) \to H^1(F, U^\theta) \to H^1(F, U \xrightarrow{1-\theta} S_1).$$

It then pairs with $\mathbf{A}_1$ as $\langle \mathbf{g}, \mathbf{s}_{T,\theta} \rangle$. We conclude that Δ is unchanged by (4.6.1).

Now let us change the a-data alone (4.6.2). Only Δ_{I} and Δ_{II} are affected. If $\{a_\alpha\}$ is replaced by $\{a_\alpha b_\alpha\}$ then by Lemma 3.2.C of [LS1] $\Delta_{\mathrm{I}}(\gamma,\delta)$ is multiplied by

$$\prod \chi_\alpha(b_\alpha), \tag{4.6.4}$$

where the product is over representatives α for certain symmetric orbits of Γ in the roots of T^x in G^x. As in (4.2), we note that $s_{T,\theta}$ is a *standard* endoscopic datum for G^x, with endoscopic group, say, J. Our $\Delta_{\mathrm{I}}(\gamma,\delta)$ coincides with the (standard) Δ_{I} for (G^x, J) evaluated on appropriate elements of $J(F)$ and $T^x(F)$. Thus Lemma 3.2.C of [LS1] says that the product in (4.6.4) is over *symmetric* Γ*-orbits that are outside* J, and, as we saw in (4.2), these are the symmetric Γ-orbits in R_{res} represented by roots α_{res} that are either of type R_1 and *not from* H or else of type R_2 such that *both* α_{res} *and* $2\alpha_{\mathrm{res}}$ *are not from* H.

It remains to check that we get the inverse of (4.6.4) from the change in Δ_{II}. First, asymmetric orbits are easily seen to make no contribution to the change in Δ_{II} (we can argue as in Lemma 3.3.A of [LS1]). From (4.3.2),(4.3.3) and (4.3.4) we find the symmetric orbits contribute as we wish.

It remains to replace $\{\chi_\alpha\}$ by χ-data $\{\chi_\alpha\zeta_\alpha\}$ (4.6.3). We shall argue for the case $H_1 = H$, leaving to the reader the (immediate) extension to the general case. First we apply an argument for standard endoscopy to measure the effects of changing χ-data on the embeddings ${}^LT_H \hookrightarrow {}^LH$ and ${}^L(T_{\theta^*}) \hookrightarrow {}^LG^1$. In each case (2.6.3) of [LS1] applies and we conclude that our present cocycle a_T (notation of (4.4)) is multiplied by a quotient of cocycles, say c_s/c_1. Thus the term $\Delta_{\mathrm{III}}(\gamma,\delta;\bar\gamma,\bar\delta)$ is multiplied by

$$\langle c_s,\gamma\rangle/\langle c_1,\gamma_0\rangle$$

where γ_0 is the image of $\delta^* \in T$ in $T_{\theta^*}(F)$ and $\langle\cdot,\cdot\rangle$ is the Langlands pairing.

We evaluate separately the numerator and denominator following Lemma 3.5.A of [LS1]. The numerator is a product $\prod A_{\alpha_H^\vee}$ indexed by roots $\alpha_H^\vee$ representing pairs $\pm\mathcal{O}_H$ of Γ-orbits in $R_H^\vee$. If $\mathcal{O}_H$ is symmetric, attach to γ_H the element $\delta^{\alpha_H^\vee}$ of $T^{\alpha_H^\vee}(F_{\pm\alpha_H^\vee})$ as in the cited lemma; if $\mathcal{O}_H$ is asymmetric we have instead the element $\gamma^{\alpha_H^\vee}$. Then:

$$A_{\alpha_H^\vee} = \begin{cases} \zeta_{\alpha_H^\vee}(\alpha_H(\delta^{\alpha_H^\vee})) & \mathcal{O}_H \text{ symmetric} \\ \zeta_{\alpha_H^\vee}(\alpha_H(\gamma^{\alpha_H^\vee})) & \mathcal{O}_H \text{ asymmetric.} \end{cases}$$

For the denominator the index set is instead pairs $\pm\mathcal{O}_1$ of Γ-orbits in $R(\hat G^1,\hat T^1)$, *i.e.* in the elements of type R_1,R_2 in $R_{\mathrm{res}}^\vee$. If $\alpha_1^\vee$ represents $\pm\mathcal{O}_1$ then the corresponding contribution is $A_{\alpha_1^\vee}$.

To compare the numerator and denominator we set

$$B_{\alpha_{\mathrm{res}}^\vee} = \begin{cases} \zeta_{\alpha_{\mathrm{res}}^\vee}(N\alpha(\delta^{\alpha_{\mathrm{res}}^\vee})) & \mathcal{O} \text{ symmetric} \\ \zeta_{\alpha_{\mathrm{res}}^\vee}(N\alpha(\gamma^{\alpha_{\mathrm{res}}^\vee})) & \mathcal{O} \text{ asymmetric} \end{cases}$$

for $\alpha_{\mathrm{res}}^\vee$ representing a pair $\pm\mathcal{O}$ of Γ-orbits in $R_{\mathrm{res}}^\vee$. Then the contribution to

$$\langle c_s,\gamma\rangle/\langle c_1,\gamma_0\rangle$$

from $\pm\mathcal{O}$ is:

$$\begin{cases} B_{\alpha^\vee_{\rm res}}\cdot(B_{\alpha^\vee_{\rm res}})^{-1}=1 & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_1,\text{ from } H\\ (B_{\alpha^\vee_{\rm res}})^{-1} & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_1,\text{ not from } H\\ (B_{\alpha^\vee_{\rm res}})^{2}\cdot(B_{\alpha^\vee_{\rm res}})^{-2}=1 & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_2,\text{ from } H\\ (B_{\alpha^\vee_{\rm res}})^{-2} & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_2,\text{ not from } H\\ B_{\alpha^\vee_{\rm res}} & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_3,\text{ from } H\\ 1 & \text{if } \alpha^\vee_{\rm res} \text{ is of type } R_3,\text{ not from } H. \end{cases}$$

We compare this with the change in $\Delta_{\rm II}$. That change is calculated from (4.3.2), (4.3.3) and (4.3.4) using the method for the proof of Lemma 3.3.D of [LS1]. We see easily that the result cancels with the change in $\Delta_{\rm III}$, and so we are done.

From (4.6.1),(4.6.2),(4.6.3) we conclude that the choice of

$$(T_H \to T_{\theta^*}, \{a_\alpha\}, \{\chi_\alpha\})$$

has no effect on the relative transfer factor $\Delta(\gamma_1,\delta;\bar\gamma_1,\bar\delta)$. Nor does the choice of $(\overline{T}_H \to \overline{T}_{\theta^*}, \{a_{\bar\alpha}\}, \{\chi_{\bar\alpha}\})$ by similar arguments (only $\Delta_{\rm III}$ requires examination). Theorem 4.6.A is therefore proved.

THE NOTION OF TRANSFER

(5.1) Transfer factors: definition and first properties. We keep the assumptions and notation of (4.1). First we define the absolute transfer factor $\Delta(\gamma_1,\delta)$ for γ_1 strongly G-regular in $H_1(F)$ and δ strongly θ-regular in $G(F)$. If no such γ_1 is a norm then set $\Delta\equiv 0$. Otherwise, fix some (γ_1^0,δ^0) with γ_1^0 a norm of δ^0. Define $\Delta(\gamma_1^0,\delta^0)$ arbitrarily. Then set

$$\Delta(\gamma_1,\delta)=\Delta(\gamma_1,\delta;\gamma_1^0,\delta^0)\Delta(\gamma_1^0,\delta^0) \tag{5.1.1}$$

for all γ_1 strongly G-regular and δ strongly θ-regular. As we have not done so already, we define

$$\Delta(\gamma_1,\delta;\bar\gamma_1,\bar\delta)=0$$

if γ_1 is not a norm of δ *and* $\bar\gamma_1$ is a norm of $\bar\delta$. Then $\Delta(\gamma_1,\delta)=0$ unless γ_1 is a norm of δ.

We remark that for all γ_1,$\bar\gamma_1$ and δ,$\bar\delta$ we have

$$\Delta(\gamma_1,\delta)=\Delta(\gamma_1,\delta;\bar\gamma_1,\bar\delta)\Delta(\bar\gamma_1,\bar\delta)$$

provided that $\bar\gamma_1$ is a norm of $\bar\delta$. This is an immediate consequence of the transitivity property for relative transfer factors:

Lemma 5.1.A. $\Delta(\gamma_1^1,\delta^1;\gamma_1^2,\delta^2)\Delta(\gamma_1^2,\delta^2;\gamma_1^3,\delta^3)=\Delta(\gamma_1^1,\delta^1;\gamma_1^3,\delta^3)$ *provided* γ_1^i *is a norm of* δ^i $(i=2,3)$.

We may as well assume γ_1^1 is a norm of δ^1. Since $\Delta_{\rm I}$,$\Delta_{\rm II}$,$\Delta_{\rm IV}$ are naturally quotients we may replace Δ by $\Delta_{\rm III}$. Now we can follow closely the idea of the proof of Lemma 4.1.A of [LS1]. Because of the mountain of notation and the transparency of the argument we leave the details to the reader.

Lemma 5.1.B. *$\Delta(\gamma_1, \delta)$ is unchanged when γ_1 is replaced by a stably conjugate element in $H_1(F)$.*

γ_1 has image γ under $T_{H_1} \to T_H$. To define the various terms $\Delta_{\mathrm{I}}(\gamma_1, \delta)$, *etc.* we may choose any admissible embedding $T_H \to T_{\theta^*}$ we wish. Inspection of the terms Δ_{I},Δ_{II},Δ_{IV} shows that each term depends on γ_1 through the image δ^* of γ under this embedding. If we replace γ_1 by a stable conjugate γ_1', and thus γ by a stable conjugate γ' in $H(F)$, then we may form the composition of $T_H \to T_{\theta^*}$ with this stable conjugation to obtain an admissible embedding $\mathrm{Cent}(\gamma', H) \to T_{\theta^*}$ under which the image of γ' is δ^*. We use this embedding to define $\Delta_{\mathrm{I}}(\gamma_1', \delta)$, *etc.* Then $\Delta_{\mathrm{I}}(\gamma_1', \delta) = \Delta_{\mathrm{I}}(\gamma_1, \delta)$, and similarly for Δ_{II},Δ_{IV}. It remains then to compare $\Delta_{\mathrm{III}}(\gamma_1', \delta; \bar{\gamma}_1, \bar{\delta})$ to $\Delta_{\mathrm{III}}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$. Here we need further argument for the case $H_1 \neq H$. An examination of the construction of T_1 *etc.* shows that the stable conjugation of γ_1 and γ_1' provides natural isomorphisms between the tori associated to γ_1 and those associated to γ_1', and then that $\Delta_{\mathrm{III}}(\gamma_1', \delta; \bar{\gamma}_1, \bar{\delta})$ coincides with $\Delta_{\mathrm{III}}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$. This completes the proof.

Recall the character λ_{H_1} on $Z_1(F) = \ker(H_1(F) \to H(F))$ from (2.2).

Lemma 5.1.C. *$\Delta(z_1\gamma_1, \delta) = \lambda_{H_1}(z_1)^{-1}\Delta(\gamma_1, \delta)$ for $z_1 \in Z_1(F)$.*

We have only to prove that

$$\Delta_{\mathrm{III}}(z_1\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta}) = \lambda_{H_1}(z_1)^{-1}\Delta_{\mathrm{III}}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta}).$$

We recall the exact sequence

$$1 \to Z_1 \to S_1 \to S \to 1$$

of (4.4). The map $Z_1 \to S_1$ induces a homomorphism

$$Z_1(F) \to H^1(F, U \xrightarrow{1-\theta} S_1)$$

(see Appendix A) and dual

$$H^1(W_F, \hat{S}_1 \to \hat{U}) \to H^1(W_F, \hat{Z}_1).$$

Let $\mathbf{A}_2$ be the image of $\mathbf{A}_1$ under the latter map. Then we see that replacing γ_1 by $z_1\gamma_1$ multiplies $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$ by $\langle z_1, \mathbf{A}_2\rangle^{-1}$, where $\langle \cdot, \cdot \rangle$ indicates the Langlands pairing. Thus we have to show that $\mathbf{A}_2^{-1}$ is the Langlands parameter for $\lambda_{H_1}^{-1}$. Indeed, it is immediate from our definitions that $\mathbf{A}_2^{-1}$ is represented by the following 1-cocycle $z(w)$ of W_F in $\hat{Z}_1$: $z(w)$ is the image of $t_1(w)$ under the canonical surjection $\hat{T}_{H_1} \to \hat{Z}_1$, where $t_1(w)$ is as in the concrete definition of $a_T(w)$ given in (4.4). It then follows from the definition of $t_1(w)$ that $z(w)$ is a Langlands parameter for $\lambda_{H_1}^{-1}$.

There is a natural injection

$$Z(G)_\theta \to Z(H),$$

obtained as follows. Choose a θ^*-stable pair (B, T) in G^*. Then we have

$$\begin{aligned}\mathrm{Lie}(T_{\mathrm{sc}}^{\theta^*}) &= \mathrm{Lie}(T_{\mathrm{sc}})^{\theta^*}\\ &= \mathrm{Lie}(T_{\mathrm{ad}})^{\theta^*}\\ &= \mathrm{Lie}(T_{\mathrm{ad}}^{\theta^*});\end{aligned}$$

this, together with the fact that $T_{\mathrm{ad}}^{\theta^*}$ is connected, implies that

$$T_{\mathrm{sc}}^{\theta^*} \to T_{\mathrm{ad}}^{\theta^*}$$

is surjective and hence that

$$T^{\theta^*} \to T_{\mathrm{ad}}^{\theta^*}$$

is surjective as well. Applying the snake lemma to

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(G^*) & \longrightarrow & T & \longrightarrow & T_{\mathrm{ad}} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1-\theta^*} & & \downarrow{\scriptstyle 1-\theta^*} & & \downarrow{\scriptstyle 1-\theta^*} & & \\ 1 & \longrightarrow & Z(G^*) & \longrightarrow & T & \longrightarrow & T_{\mathrm{ad}} & \longrightarrow & 1 \end{array}$$

we conclude that the natural map

$$Z(G^*)_{\theta^*} \to T_{\theta^*}$$

is injective. Composing this map with any $\overline{F}$-embedding

$$j : T_{\theta^*} \to H$$

in the canonical $H(\overline{F})$-conjugacy class of such embeddings we get a map

$$Z(G^*)_{\theta^*} \to H$$

whose image is central in H. The resulting injection

$$Z(G)_\theta \to Z(H)$$

is independent of the choice of j and is defined over F.

Using the composed map

$$Z(G) \to Z(G)_\theta \to Z(H),$$

we form the fiber product C of $Z(H_1)$ and $Z(G)$ over $Z(H)$. An element of $C(F)$ is a pair

$$(z_1, z) \in Z(H_1)(F) \times Z(G)(F)$$

such that the images of z_1 and z in $Z(H)(F)$ are equal.

Lemma 5.1.C has the following generalization: there exists a quasicharacter λ_C on $C(F)$ such that

$$\Delta(z_1\gamma_1, z\delta) = \lambda_C(z_1, z)^{-1}\Delta(\gamma_1, \delta)$$

for all $(z_1, z) \in C(F)$. From this point of view the content of Lemma 5.1.C is that the restriction of λ_C to the subgroup $Z_1(F)$ of $C(F)$ is equal to λ_{H_1}.

It is enough to show that there exists a quasicharacter λ_C on $C(F)$ such that

$$\Delta_{\text{III}}(z_1\gamma_1, z\delta; \bar{\gamma}_1, \bar{\delta}) = \lambda_C(z_1, z)^{-1}\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$$

for all (z_1, z) in $C(F)$. For this we need to understand how

$$\mathbf{V}_1 \in H^1(F, U \xrightarrow{1-\theta} S_1)$$

changes when (γ_1, δ) is multiplied by (z_1, z). Recall from (4.4) the exact sequence

$$1 \to Z_1 \to T_1 \to T \to 1.$$

It is clear that C can be identified with the inverse image of $Z(G)$ under $T_1 \to T$ and that S_1 is equal to

$$(T_1 \times \overline{T}_1)/\{(c, c^{-1}) \mid c \in C\}.$$

Recall that $\mathbf{V}_1$ is the class of the 1-hypercocycle (V, D_1). Multiplying (γ_1, δ) by (z_1, z) does not change V and multiplies D_1 by the image of (z_1, z) under the map $T_1 \to S_1$ given by $t_1 \mapsto (t_1, 1)$ (think of S_1 as $(T_1 \times \overline{T}_1)/C$). Thus $\mathbf{V}_1$ is multiplied by the image of (z_1, z) under the homomorphism

$$C(F) \to T_1(F) \to H^1(F, U \xrightarrow{1-\theta} S_1),$$

which shows that λ_C does indeed exist.

We consider now the question of how $\Delta(\gamma_1, \delta)$ is affected when δ is replaced by a stable conjugate, say $\delta' = h^{-1}\delta\theta(h)$, with h in $G(\overline{F})$. Then $\sigma \mapsto \sigma(h)h^{-1}$ is a 1-cocycle of Γ in the (abelian reductive) group $\text{Cent}_\theta(\delta, G)$ and, as usual, we see that

$$\ker[H^1(F, \text{Cent}_\theta(\delta, G)) \to H^1(F, G)]$$

classifies the θ-conjugacy classes in the stable θ-conjugacy class of δ. But θ-conjugacy is too coarse an equivalence for our present setting and we are lead to a hypercohomology group instead.

Let T_δ denote the centralizer of $G^{\delta\theta}$ in G. Then T_δ is a maximal torus in G, whose inverse image under the canonical homomorphism $\pi : G_{\text{sc}} \to G$ we denote by T_δ^{sc}. Let θ^δ denote the automorphism $\text{Int}(\delta) \circ \theta$ of G. We write the element h above as $\pi(h_1)z$ with h_1 in G_{sc} and z in the center of G. Then $\sigma(h_1)h_1^{-1}$ lies in T_δ^{sc} and

$$(1 - \theta^\delta)\pi(\sigma(h_1)h_1^{-1}) = \sigma(\theta(z)z^{-1})(\theta(z)z^{-1})^{-1}.$$

Thus the pair

$$(\sigma \mapsto \sigma(h_1)h_1^{-1}, \theta(z)z^{-1})$$

defines an element $\mathrm{inv}(\delta, \delta')$ of

$$H^1(F, T_\delta^{\mathrm{sc}} \xrightarrow{(1-\theta^\delta)\circ\pi} V_\delta)$$

where V_δ denotes the subtorus $(1-\theta^\delta)(T_\delta)$ of T_δ.

Next we shall construct a class κ_δ in the dual

$$H^1(W_F, \hat{V}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}}).$$

Our first step is to pass from θ^δ-invariant T_δ in G to θ^*-invariant T in G^*. We do this by means of the isomorphism $\mathrm{Int}(g) \circ \psi$ of (3.3.6). Observe that this isomorphism maps T_δ to T over F and transports θ^δ to θ^*. It is thus sufficient to construct κ_δ in

$$H^1(W_F, \hat{V} \to \hat{T}_{\mathrm{ad}})$$

where V denotes the subtorus $(1-\theta)(T)$ of T.

Note that V is the kernel of $N : T \to T_H$. Therefore V embeds in T_1 as the kernel of the natural surjective homomorphism $T_1 \to T_{H_1}$, which means that we may identify $\hat{V}$ with the quotient $\hat{T}_1/\hat{T}_{H_1}$. In (4.4) we constructed a 1-cocycle a_T of W_F in $\hat{T}_1$ satisfying

$$(1-\hat{\theta})(a_T(w)) = s \cdot \sigma_T(s)^{-1} \cdot a'(w)^{-1}. \qquad (5.1.2)$$

Write b_T for the image of a_T under the canonical surjection

$$\hat{T}_1 \to \hat{T}_1/\hat{T}_{H_1} = \hat{V}.$$

Then, still writing $1-\hat{\theta}$ for the map $\hat{V} \to \hat{T}_{\mathrm{ad}}$ induced by

$$1-\hat{\theta} : \hat{T} \to \hat{T},$$

we find that

$$(1-\hat{\theta})(b_T(w)) = s \cdot \sigma_T(s)^{-1};$$

we define κ_δ to be the class in

$$H^1(W_F, \hat{V} \to \hat{T}_{\mathrm{ad}})$$

of the hypercocycle (b_T^{-1}, s).

Theorem 5.1.D.

(1) $\Delta(\gamma_1, \delta') = \langle \mathrm{inv}(\delta, \delta'), \kappa_\delta \rangle \Delta(\gamma_1, \delta)$.

(2) *If* $\delta' = h^{-1}\delta\theta(h)$ *with* $h \in G(F)$, *then*

$$\Delta(\gamma_1, \delta') = \omega(h)\Delta(\gamma_1, \delta),$$

where ω *is, as usual, the quasi-character on* $G(F)$ *obtained from* **a**.

We have only to consider Δ_{III}. First we replace δ by δ' in (3.3.4) and (3.3.5). Then g, δ^* are replaced by $g\psi(h_1)$ and $\delta^*\theta^*(z)z^{-1}$ (we use ψ to identify the centers of G and G^*). Thus $v(\sigma)$ becomes

$$v'(\sigma) = g\psi(h_1)u(\sigma)\sigma(\psi(h_1))^{-1}\sigma(g)^{-1}$$

which may be calculated as

$$[(\mathrm{Int}(g) \circ \psi)(h_1\sigma(h_1)^{-1})]gu(\sigma)\sigma(g)^{-1};$$

in other words $v'(\sigma)$ is the product of $v(\sigma)$ and the cocycle obtained from $h_1\sigma(h_1)^{-1}$ *via* the isomorphism $T_\delta \simeq T$. Moreover δ_1^* becomes

$$(\delta_1^*)' = (\delta^*\theta^*(z)z^{-1}, \gamma_1) = \delta_1^* \cdot (\theta^*(z)z^{-1}, 1).$$

Therefore $\mathbf{V}$ is replaced by $\mathbf{V}'$, the product of $\mathbf{V}$ and the element of

$$H^1(F, U \xrightarrow{1-\theta} S_1)$$

obtained as the image of

$$\mathrm{inv}(\delta, \delta') \in H^1(F, T_\delta^{\mathrm{sc}} \to V_\delta) = H^1(F, T_{\mathrm{sc}} \to V)$$

under the map

$$H^1(F, T_{\mathrm{sc}} \to V) \to H^1(F, U \to S_1)$$

induced by the obvious map of complexes from the complex $T_{\mathrm{sc}} \to V$ to the complex $U \to S_1$ (U is a quotient of $T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}}$, so that there is an obvious map $T_{\mathrm{sc}} \to U$; similarly there is an obvious map $T_1 \to S_1$ which we compose with the embedding of V in T_1 as the kernel of $T_1 \to T_{H_1}$ to get a map $V \to S_1$). Part (1) of the lemma now follows from the fact that the dual map

$$H^1(W_F, \hat{S}_1 \to \hat{U}) \to H^1(W_F, \hat{V} \to \hat{T}_{\mathrm{ad}})$$

sends $\mathbf{A}_1$ to κ_δ.

It remains to prove (2). We use Borovoi's method [Bo] of constructing the Langlands pairing between $G(F)$ and $H^1(W_F, Z(\hat{G}))$. There is a canonical homomorphism

$$d : G(F) \to H^1(F, T_\delta^{\mathrm{sc}} \xrightarrow{\pi} T_\delta) \tag{5.1.3}$$

defined as follows. Let $g \in G(F)$ and write $g = \pi(g_1)t$ for $g_1 \in G_{\mathrm{sc}}$ and $t \in T_\delta$. Then map g to the class of the hypercocycle

$$(\sigma(g_1)^{-1}g_1, t).$$

Let $\mathbf{b}$ be an element of $H^1(W_F, Z(\hat{G}))$ and let b be a 1-cocycle representing $\mathbf{b}$. Then $(b^{-1}, 1)$ is a 1-hypercocycle of W_F in

$$\hat{T}_\delta \xrightarrow{\hat{\pi}} (\hat{T}_\delta)_{\mathrm{ad}},$$

and the value of the Langlands pairing between $g \in G(F)$ and b is defined to be

$$\langle d(g), (b^{-1}, 1)\rangle$$

(use the pairing in (A.3)). We used b^{-1} rather than b in order to ensure that this Langlands pairing coincides with the usual one when G is a torus (note the minus sign in (A.3.13)).

There is an obvious homomorphism

$$H^1(F, T_\delta^{\mathrm{sc}} \xrightarrow{\pi} T_\delta) \to H^1(F, T_\delta^{\mathrm{sc}} \xrightarrow{(1-\theta^\delta)\circ\pi} V_\delta) \tag{5.1.4}$$

obtained from the map from the complex $T_\delta^{\mathrm{sc}} \xrightarrow{\pi} T_\delta$ to the complex $T_\delta^{\mathrm{sc}} \xrightarrow{(1-\theta^\delta)\circ\pi} V_\delta$ given by the identity map on T_δ^{sc} and by $1-\theta^\delta$ from T_δ to V_δ. A simple calculation shows that if $\delta' = h^{-1}\delta\theta(h)$ with $h \in G(F)$, then $\mathrm{inv}(\delta, \delta')$ is equal to the image of h^{-1} under the composition of the maps (5.1.3) and (5.1.4).

Therefore $\Delta(\gamma_1, \delta')$ is equal to $\Delta(\gamma_1, \delta)$ times

$$\langle d(h^{-1}), \kappa'_\delta\rangle$$

where κ'_δ is the image of κ_δ under the map (dual to (5.1.4))

$$H^1(W_F, \hat{V}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}}) \to H^1(W_F, \hat{T}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}})$$

induced by the map of complexes from $\hat{V}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}}$ to $\hat{T}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}}$ given by $1-\hat{\theta}^\delta : \hat{V}_\delta \to \hat{T}_\delta$ and the identity map from $(\hat{T}_\delta)_{\mathrm{ad}}$ to itself (the map $1-\hat{\theta}^\delta : \hat{T}_\delta \to \hat{T}_\delta$ is trivial on the kernel of the canonical surjection $\hat{T}_\delta \to \hat{V}_\delta$ and hence induces $1-\hat{\theta}^\delta : \hat{V}_\delta \to \hat{T}_\delta$).

It follows from what we have done that κ'_δ is represented by the hypercocycle

$$((1-\hat{\theta})(b_T^{-1}), s)$$

of W_F, which by (5.1.2) is equivalent to $(a', 1)$, or, in other words, the image of $\mathbf{a}$ under the canonical isomorphism

$$H^1(W_F, Z(\hat{G})) \to H^1(W_F, \hat{T}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}})$$

(use that $\hat{T}_\delta \to (\hat{T}_\delta)_{\mathrm{ad}}$ is surjective with kernel $Z(\hat{G})$). Therefore $\langle d(h^{-1}), \kappa'_\delta\rangle$ is the value of the Langlands pairing on $h^{-1} \in G(F)$ and $\mathbf{a}^{-1} \in H^1(W_F, Z(\hat{G}))$, and this value is equal to $\omega(h)$ by definition of ω. The proof of the lemma is now complete.

(5.2) Proof of a lemma needed in (5.3). In this section we assume temporarily that we are dealing with standard endoscopy, so that H is now an endoscopic group of G and s is an element of $Z(\hat{H})^\Gamma$. Moreover we assume that G is quasi-split and choose a pair (B, T) in G defined over F. We also choose a pair (B_H, T_H) in H defined over F. We write Z for the center of G. Let ρ denote half the sum of the B-positive coroots for T. Then $\rho \in X_*(T_{\mathrm{ad}})$ and of course 2ρ lies in the image of $X_*(T)$. Let

$a \in F^\times$. Then $t := \rho(a)$ is an element of $T_{\text{ad}}(F)$ whose square lies in the image of $T(F)$. Applying the boundary map for the exact sequence

$$1 \to Z \to T \to T_{\text{ad}} \to 1$$

to the element $t \in T_{\text{ad}}(F)$ we get an element

$$z \in H^1(F, Z)$$

whose square is trivial. The group Z embeds canonically into the center of H, inducing a map

$$H^1(F, Z) \to H^1(F, H),$$

and we denote by h the image of z under this map. There is a natural map (see [K3])

$$H^1(F, H) \to \pi_0(Z(\hat{H})^\Gamma)^D,$$

and thus we may pair h with $s \in Z(\hat{H})^\Gamma$, obtaining $\langle h, s\rangle \in \{\pm 1\}$.

The main result of this section, Lemma 5.2.A, gives a simple formula for $\langle h, s\rangle$. Let V_G (respectively, V_H) denote the complex representation $X^*(T) \otimes \mathbb{C}$ (respectively, $X^*(T_H) \otimes \mathbb{C}$) of Γ. Then the difference $V_H - V_G$ is a virtual representation W of Γ having dimension 0. The determinant $\det(W)$ is the character on Γ given by $\det(V_G)^{-1} \det(V_H)$, and by local classfield theory we may regard $\det(W)$ as a character on $F^\times$ (since the square of $\det(W)$ is trivial it does not matter how we normalize the isomorphism of local classfield theory).

Lemma 5.2.A. *There is an equality*

$$\langle h, s\rangle = (\det(W))(a).$$

A routine reduction step allows us to replace G by G_{sc} and thus assume that G itself is semisimple and simply connected. The first step is to use duality to understand the boundary map

$$T_{\text{ad}}(F) \to H^1(F, Z).$$

We have (Langlands duality)

$$\text{Hom}_{\text{cont}}(T_{\text{ad}}(F), \mathbb{C}^\times) = H^1(W_F, \hat{T}_{\text{sc}}),$$

where $\hat{T}_{\text{sc}}$ is the inverse image of $\hat{T}$ under $\hat{G}_{\text{sc}} \to \hat{G}$ (embed $\hat{T}$ as a maximal torus in $\hat{G}$), and we also have (duality for finite abelian F-groups)

$$\text{Hom}(H^1(F, Z), \mathbb{C}^\times) = H^1(F, \hat{Z}),$$

where $\hat{Z}$ denotes the kernel of

$$\hat{G}_{\text{sc}} \to \hat{G},$$

so that $\hat{Z}$ fits in a short exact sequence

$$1 \to \hat{Z} \to \hat{T}_{\mathrm{sc}} \to \hat{T} \to 1. \tag{5.2.1}$$

The boundary map $T_{\mathrm{ad}}(F) \to H^1(F, Z)$ is dual to the map

$$H^1(F, \hat{Z}) \to H^1(F, \hat{T}_{\mathrm{sc}})$$

induced by the inclusion map

$$\hat{Z} \to \hat{T}_{\mathrm{sc}},$$

or maybe its negative, depending on sign conventions; however, since $z^2 = 1$, this sign question is irrelevant for us. The conclusion of this discussion is that for any $\chi \in H^1(F, \hat{Z})$ the value of $\langle z, \chi \rangle$ is equal to $\langle a, \rho(\chi) \rangle$, where $\rho(\chi)$ denotes the image of χ under the map

$$H^1(F, \hat{Z}) \to H^1(W_F, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{cont}}(F^\times, \mathbb{C}^\times)$$

induced by

$$\rho : \hat{Z} \to \mathbb{C}^\times$$

(regard ρ as an element of $X^*(\hat{T}_{\mathrm{sc}}) = X_*(T_{\mathrm{ad}})$ and restrict it to the subgroup $\hat{Z}$ of $\hat{T}_{\mathrm{sc}}$).

The number $\langle h, s \rangle$ is equal to $\langle z', s' \rangle$, where z' denotes the image of z under

$$H^1(F, Z) \to H^1(F, T_H)$$

and s' denotes the image of s under

$$Z(\hat{H})^\Gamma \hookrightarrow \hat{T}_H^\Gamma.$$

Choose an isomorphism $T_H \to T$ over $\overline{F}$ such that

$$T_H \to T \hookrightarrow G$$

belongs to the canonical $G(\overline{F})$-conjugacy class of embeddings $T_H \to G$ and such that B_H-positive roots of T_H in H are carried into B-positive roots of T in G. Consider the analogue of (5.2.1) for T_H, viewed as maximal torus in G:

$$1 \to \hat{Z} \to (\hat{T}_H)_{\mathrm{sc}} \to \hat{T}_H \to 1. \tag{5.2.2}$$

Then $\langle z', s' \rangle = \langle z, \chi \rangle$, where χ denotes the image of s' under the boundary map

$$\hat{T}_H^\Gamma \to H^1(F, \hat{Z})$$

for (5.2.2). Combining this with what was done before we conclude that $\langle h, s \rangle$ is equal to $\langle a, \rho(\chi) \rangle$.

To proceed further we must determine $\rho(\chi)$. Choose $s'' \in (\hat{T}_H)_{\text{sc}}$ mapping to $s' \in \hat{T}_H^\Gamma$. Then χ is the class of the 1-cocycle

$$\sigma \mapsto (s'')^{-1}\sigma(s'')$$

of Γ in $\hat{Z}$. Our chosen isomorphism $T_H \to T$ allows us to view ρ as a character on $(\hat{T}_H)_{\text{sc}}$. Then $\rho(\chi)$ is the homomorphism

$$\Gamma \to \{\pm 1\}$$

given by

$$\sigma \mapsto \rho((s'')^{-1}\sigma(s'')) = (\sigma^{-1}(\rho) - \rho)(s'') = (\rho - \sigma^{-1}(\rho))(s'').$$

The character $\rho - \sigma^{-1}(\rho)$ on $(\hat{T}_H)_{\text{sc}}$ is equal to

$$\sum_{\alpha \in A(\sigma)} \alpha,$$

where $A(\sigma)$ denotes the set of positive roots of $\hat{T}_H$ in $\hat{G}$ such that $\sigma(\alpha)$ is negative (positive and negative with respect to B; use the isomorphism $\hat{T}_H \simeq \hat{T}$ dual to our chosen isomorphism $T_H \simeq T$). It is clear that the character $\rho - \sigma^{-1}(\rho)$ descends to a character on $\hat{T}_H$. Therefore $\langle h, s\rangle$, as a function of a, is the character $\Gamma \to \{\pm\}$ given by

$$\sigma \mapsto \prod_{\alpha \in A(\sigma)} \alpha(s),$$

viewed as a character on $F^\times$ by local classfield theory. Here we have written s rather than s' since we are now identifying $Z(\hat{H})$ with a subgroup of $\hat{T}_H$.

We are going to show that

$$\prod_{\alpha \in A(\sigma)} \alpha(s) = (-1)^{|A(\sigma)|}$$

where $|A(\sigma)|$ denotes the cardinality of $A(\sigma)$. We may as well fix $\sigma \in \Gamma$ and abbreviate $A(\sigma)$ to A. In order to prove this equality we consider the orbits $\mathcal{O}$ of σ in the set of roots of $\hat{T}_H$ in $\hat{G}$. If $\mathcal{O}$ is an orbit, so is $-\mathcal{O}$. If $\mathcal{O} = -\mathcal{O}$, we say that $\mathcal{O}$ is *symmetric;* otherwise we say that it is *asymmetric* . Since $\sigma(s) = s$, every element of $\mathcal{O}$ takes the same value on s. In particular, if $\mathcal{O}$ is symmetric, then $\alpha(s) = \pm 1$ for $\alpha \in \mathcal{O}$. Our isomorphism $T_H \simeq T$ was chosen so that every positive root of $\hat{T}_H$ in $\hat{H}$ is a positive root of $\hat{T}_H$ in $\hat{G}$, and, moreover, σ preserves the set of positive roots of $\hat{T}_H$ in $\hat{H}$; it then follows that no element of A is a root of $\hat{T}_H$ in $\hat{H}$, or in other words no element $\alpha \in A$ satisfies $\alpha(s) = 1$. It follows immediately that for any symmetric orbit $\mathcal{O}$

$$\prod_{\alpha \in A \cap \mathcal{O}} \alpha(s) = (-1)^{|A \cap \mathcal{O}|}.$$

If $\mathcal{O}$ is asymmetric, then $-\mathcal{O}$ is disjoint from $\mathcal{O}$ and we write $\pm\mathcal{O}$ for the union of $\mathcal{O}$ and $-\mathcal{O}$. We claim that

$$\prod_{\alpha \in A \cap (\pm\mathcal{O})} \alpha(s) = 1.$$

Since $\alpha(s) = \beta(s)^{-1}$ for $\alpha \in \mathcal{O}$, $\beta \in -\mathcal{O}$, it is enough to show that

$$|A \cap \mathcal{O}| = |A \cap (-\mathcal{O})|. \tag{5.2.3}$$

This is easy to see. Arrange the orbit $\mathcal{O}$ in a circle in the obvious way, so that σ rotates the circle, and indicate the positive roots in $\mathcal{O}$ by pluses and the negative roots by minuses. Then $|A \cap \mathcal{O}|$ is the number of pluses that σ takes into minuses, and $|A \cap (-\mathcal{O})|$ is the number of minuses that σ takes into pluses. These two numbers are equal (just think about it). Again using (5.2.3), we have that

$$(-1)^{|A \cap (\pm\mathcal{O})|} = 1,$$

which means that

$$\prod_{\alpha \in A \cap (\pm\mathcal{O})} \alpha(s) = (-1)^{|A \cap (\pm\mathcal{O})|}.$$

Putting together the contributions of all orbits, we conclude that

$$\prod_{\alpha \in A} \alpha(s) = (-1)^{|A|},$$

as claimed.

The last step of the argument is to recognize that $(-1)^{|A|}$ is a determinant. Until now we have been using the natural Galois action of σ on $\hat{T}_H$. We now write σ_H for this action. We also have the natural Galois action of σ on $\hat{T}$, which we denote by σ_G. Using our chosen isomorphism $\hat{T}_H \simeq \hat{T}$, we can compare σ_H and σ_G; indeed, there exists a unique element ω_σ of the Weyl group of $\hat{T}$ in $\hat{G}$ such that

$$\sigma_H = \sigma_G \omega_\sigma.$$

Since σ_G preserves the set of B-positive roots of $\hat{T}$ in $\hat{G}$, the set A can also be described as the set of positive roots α of $\hat{T}$ in $\hat{G}$ such that $\omega_\sigma \alpha$ is negative, and therefore $|A|$ is equal to the length of ω_σ and $(-1)^{|A|}$ is equal to

$$\det(\omega_\sigma; X^*(T))$$

which is of course also equal to

$$\det(\sigma_G; X^*(T))^{-1} \det(\sigma_H; X^*(T)).$$

This concludes the proof of the lemma.

(5.3) Whittaker normalization of transfer factors (quasi-split case). In this section we impose two further conditions on $(G,\theta,\mathbf{a})$: we assume that G is quasi-split and that θ preserves some F-splitting of G. We refer to an F-splitting that is preserved by θ as an (F,θ)-splitting. As before let G^x denote the group of fixed points of θ_{sc} on G_{sc} and recall from (1.1) that

$$(B,T)\mapsto(B^x,T^x)$$

sets up a bijection between the set of θ-stable pairs (B,T) in G and the set of pairs in G^x, where B^x (respectively, T^x) denotes the inverse image of B (respectively, T) under

$$G^x\to G.$$

Let $(B,T,\{X_\alpha\})$ be a θ-stable splitting of G. Let $\mathcal{O}$ be an orbit of θ in the set of simple roots of T in Lie(B). Then put

$$X_{\mathcal{O}}:=\sum_{\alpha\in\mathcal{O}}X_\alpha,$$

an element of the root space of G^x corresponding to the orbit $\mathcal{O}$. Thus, starting from a θ-stable splitting $(B,T,\{X_\alpha\})$ of G, we have constructed a splitting $(B^x,T^x,\{X_{\mathcal{O}}\})$ of G^x, and it is clear that this construction yields a Γ-equivariant bijection from the set of θ-stable splittings of G to the set of splittings of G^x. In particular, giving an (F,θ)-splitting of G is the same as giving a splitting of G^x.

In the theory of Whittaker models for representations of $G(F)$ one begins with a Borel subgroup B of G and a generic character

$$\lambda:N(F)\to\mathbb{C}^\times,$$

where N denotes the unipotent radical of B. Generic characters of $N(F)$ arise as follows. Let

$$\psi:F\to\mathbb{C}^\times$$

be a non-trivial additive character and let $(B,T,\{X_\alpha\})$ be an F-splitting of G. The choice of $\{X_\alpha\}$ yields a surjective homomorphism

$$N\to\prod_\alpha\mathbb{G}_a$$

over $\overline{F}$, with α running through the set of simple roots of T in Lie(B). Composing this with the map

$$\prod_\alpha\mathbb{G}_a\to\mathbb{G}_a$$

which sends (x_α) to $\sum_\alpha x_\alpha$, we get a homomorphism

$$N\to\mathbb{G}_a$$

defined over F. In particular we get a homomorphism

$$N(F) \to F$$

which when composed with ψ yields a generic character λ on $N(F)$. Every generic character on $N(F)$ arises in this way for some choice of splitting with B as its Borel component. We refer to (B, λ) as *Whittaker data* for G.

If B is θ-stable and $\lambda \circ \theta = \lambda$, then we say that (B, λ) is θ-stable. The significance of this is obvious: if a representation π of $G(F)$ has a Whittaker model for λ with (B, λ) θ-stable, then $\pi \circ \theta$ also has a Whittaker model for λ.

Let $(H, \mathcal{H}, s, \xi)$ be a set of endoscopic data for $(G, \theta, \mathbf{a})$ and let (H_1, ξ_{H_1}) be a z-pair for $\mathcal{H}$. Our goal in this section is to define transfer factors $\Delta_\lambda(\gamma_1, \delta)$ for $(H, \mathcal{H}, s, \xi, H_1, \xi_{H_1})$ depending only on the choice of θ-stable Whittaker data (B_0, λ) for G. One can hope that when these transfer factors are used, representations with Whittaker models for λ will serve as natural base points in tempered L-packets (see [Sh]).

First we define transfer factors $\Delta_0(\gamma_1, \delta)$ depending only on the choice of an (F, θ)-splitting $(B_0, T_0, \{X\})$, generalizing a definition in (3.7) of [LS1]. We had better be explicit about the norm map we are using. In the non-quasi-split case we had to choose G^*, θ^* and g_θ and then introduce the map

$$m : G \to G^*$$

over $\overline{F}$. In the case at hand we of course take $G^* = G$, $\theta^* = \theta$, $g_\theta = 1$, so that m is the identity map.

The definition of $\Delta_0(\gamma_1, \delta)$ is simpler than that of the relative transfer factor. As before we maintain the assumptions and notation of Theorem 3.3.A. Let γ_1 be a strongly G-regular element in $H_1(F)$, with image γ in $H(F)$, and suppose that γ is a norm of an element $\delta \in G(F)$. Let T_H denote the centralizer of γ in H and, as in Lemma 3.3.B, choose $B_H \supset T_H$ and θ-stable (B, T) with T defined over F, so that the attached admissible embedding $T_H \to T_\theta$ is defined over F.

As in §4 we fix a-data and χ-data for R_{res}. In §4 we have already defined factors $\Delta_{\text{I}}(\gamma, \delta)$, $\Delta_{\text{II}}(\gamma, \delta)$ and $\Delta_{\text{IV}}(\gamma, \delta)$. The factor Δ_{I} depends on the choice of a-data and on the choice of (F, θ)-splitting for G (recall that this is the same as the choice of F-splitting for G^x). The factor Δ_{II} depends on the choice of a-data and χ-data. The factor Δ_{IV} is independent of all choices.

Now that we are in a quasi-split situation we can also define an absolute factor $\Delta_{\text{III}}(\gamma_1, \delta)$, depending only on the choice of χ-data. First we will define an element

$$\text{inv}(\gamma_1, \delta) \in H^1(F, T_{\text{sc}} \xrightarrow{1-\theta_1} T_1),$$

where, as before, T_1 is the torus obtained as the fiber product of T_{H_1} and T over T_H. As before, $1 - \theta_1$ is trivial on $Z_1 \subset T_1$ and hence induces a map $T \to T_1$ whose composition with the natural map $T_{\text{sc}} \to T$ is the map denoted by

$$T_{\text{sc}} \xrightarrow{1-\theta_1} T_1$$

above. By definition of the norm map there exist $g \in G_{\mathrm{sc}}(\overline{F})$ and $\delta^* \in T$ such that

(1) γ has image $N_\theta(\delta^*)$ under $T_H \to T_\theta$ and
(2) $\delta^* = g\delta\theta(g)^{-1}$.

Define a 1-cocycle $v_0(\sigma)$ of Γ by

$$v_0(\sigma) := g\sigma(g)^{-1};$$

the proof of Lemma 4.4.A shows that $v_0(\sigma)$ lies in $T_{\mathrm{sc}}(\overline{F})$ and that

$$(1-\theta)v_0(\sigma) = \sigma(\delta^*)^{-1}\delta^*.$$

As in (4.4) we define an element $\delta_1^* \in T_1$ by $\delta_1^* = (\delta^*, \gamma_1)$. Then

$$(1-\theta_1)v_0(\sigma) = \sigma(\delta_1^*)^{-1}\delta_1^*,$$

which means that (v_0^{-1}, δ_1^*) satisfies the hypercocycle condition and hence yields the desired element $\mathrm{inv}(\gamma_1, \delta)$, independent of the choice of g and δ^*.

Of course we need to define an element $\mathbf{A}_0$ of the dual hypercohomology group

$$H^1(W_F, \hat{T}_1 \xrightarrow{1-\hat{\theta}_1} \hat{T}_{\mathrm{ad}})$$

so that we can define $\Delta_{\mathrm{III}}(\gamma_1, \delta)$ by

$$\Delta_{\mathrm{III}}(\gamma_1, \delta) = \langle \mathrm{inv}(\gamma_1, \delta), \mathbf{A}_0 \rangle.$$

Near the end of (4.4) we defined a 1-cocycle a_T of W_F in $\hat{T}_1$ and showed that

$$(1-\hat{\theta}_1)(a_T(w)) = s \cdot \sigma_T(s)^{-1} \in \hat{T}_{\mathrm{ad}}$$

(the factor $a'(w)^{-1}$ disappeared since it is central and we are now working with $\hat{T}_{\mathrm{ad}}$ rather than $\hat{T}$). Thus (a_T^{-1}, s) satisfies the hypercocycle condition and hence yields the desired element $\mathbf{A}_0$.

We now define the transfer factor Δ_0 by putting

$$\Delta_0(\gamma_1, \delta) = \Delta_{\mathrm{I}}(\gamma, \delta)\Delta_{\mathrm{II}}(\gamma, \delta)\Delta_{\mathrm{III}}(\gamma_1, \delta)\Delta_{\mathrm{IV}}(\gamma, \delta).$$

The arguments used in (4.6) show that Δ_0 is independent of the choice of admissible embedding $T_H \to T_\theta$ and the choice of a-data and χ-data.

Lemma 5.3.A. *The relative transfer factor is given in terms of* Δ_0 *by*

$$\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta}) = \Delta_0(\gamma_1, \delta)/\Delta_0(\bar{\gamma}_1, \bar{\delta}).$$

Clearly it is enough to prove that

$$\Delta_{\mathrm{III}}(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta}) = \Delta_{\mathrm{III}}(\gamma_1, \delta)/\Delta_{\mathrm{III}}(\bar{\gamma}_1, \bar{\delta}).$$

This follows easily from the definitions: use the obvious map from the complex

$$T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \xrightarrow{1-\theta_1} T_1 \times \overline{T}_1$$

to the complex

$$U \xrightarrow{1-\theta} S_1$$

as well as the dual map between the dual complexes.

It follows from this lemma that Δ_0 is a scalar multiple of the transfer factor Δ of (5.1). Therefore Δ_0 enjoys the same properties as Δ (see Lemmas 5.1.B, 5.1.C and Theorem 5.1.D). Moreover, the discussion at the end of (4.2) tells us how Δ_0 is affected by changing the choice of (F,θ)-splitting of G. As we have seen, giving an (F,θ)-splitting $\mathbf{spl}_G$ of G is the same as giving an F-splitting of $I := G^x$. Let $g \in I_{\mathrm{ad}}(F)$ and let z be the image of g under the boundary map

$$I_{\mathrm{ad}}(F) \to H^1(F, Z(I))$$

for the exact sequence

$$1 \to Z(I) \to I \to I_{\mathrm{ad}} \to 1.$$

Let (J, s_J) be the endoscopic group for I attached to $(H, \mathcal{H}, s, \xi)$ as in (4.2). Let z_J denote the image of z under the map

$$H^1(F, Z(I)) \to H^1(F, J)$$

induced by the canonical embedding of $Z(I)$ in the center of J. Then replacing $\mathbf{spl}_G$ by $\mathrm{Int}(g)\mathbf{spl}_G$ has the effect of multiplying Δ_0 by $\langle z_J, s_J \rangle$.

Recall the θ-stable Whittaker data (B_0, λ) from before. We are finally ready to define the transfer factor $\Delta_\lambda(\gamma_1, \delta)$. Choose a non-trivial additive character

$$\psi : F \to \mathbb{C}^\times$$

and then choose an (F,θ)-splitting $\mathbf{spl}_G = (B_0, T_0, \{X\})$ with B_0 as its Borel component such that ψ and $\mathbf{spl}_G$ determine the given θ-stable Whittaker data. Let $\Delta_0(\gamma_1, \delta)$ be the transfer factor obtained from $\mathbf{spl}_G$. Choose a splitting

$$(B_{H,0}, T_{H,0}, \{Y\})$$

for H. Let V_G denote the representation of Γ on

$$X^*(T_0)^\theta \otimes \mathbb{C},$$

where the superscript θ indicates fixed points under θ. Let V_H denote the representation of Γ on

$$X^*(T_{H,0}) \otimes \mathbb{C}.$$

Note that V_G and V_H have the same dimension, so that

$$V := V_G - V_H$$

is a virtual representation of dimension 0. Consider the local ε-factor $\varepsilon_L(V,\psi)$. Here we are following the notation of [T, (3.6)], so that $\varepsilon_L(V,\psi)$ is the normalization used by Langlands. For $a \in F^\times$ let ψ_a be the additive character defined by

$$\psi_a(x) = \psi(ax).$$

Then

$$\varepsilon_L(V,\psi_a) = (\det(V))(a)\varepsilon_L(V,\psi)$$

(see (3.6.6) in [T]).

One other property of $\varepsilon_L(V,\psi)$ is worth noting. Since V is defined over $\mathbb{R}$ (even $\mathbb{Q}$), property (3.6.8) of [T] implies that

$$(\varepsilon_L(V,\psi))^2 = (\det(V))(-1),$$

where $\det(V)$ is again being regarded as a character on $F^\times$.

We define $\Delta_\lambda(\gamma_1,\delta)$ by

$$\Delta_\lambda(\gamma_1,\delta) = \varepsilon_L(V,\psi)\Delta_0(\gamma_1,\delta).$$

We must show that Δ_λ is independent of the choice of ψ and $\mathbf{spl}_G$ giving rise to (B_0,λ). Keep ψ fixed for the moment. Then any two choices for $\mathbf{spl}_G$ are conjugate by an element of $N^x(F)$, where N^x denotes the unipotent radical of B_0^x. Therefore $\Delta_0(\gamma_1,\delta)$ is the same for the two splittings, and so is $\Delta_\lambda(\gamma_1,\delta)$. Now replace ψ by ψ_a for $a \in F^\times$. To compensate for this change in ψ we replace $\mathbf{spl}_G$ by $\mathrm{Int}(\rho(a))(\mathbf{spl}_G)$, where ρ is half the sum of the B_0-positive coroots of T_0, so that $\rho \in X_*(T_0/Z(G))$ and $\rho(a) \in (T_0/Z(G))(F)$ has the property that $\alpha(\rho(a)) = a$ for every simple root of T_0. Since the roots of T_0^x in I are the restrictions to T_0^x of the roots of T_0 in G, we have

$$T_0^x \cap Z(G) = Z(I),$$

so that $T_0^x/Z(I)$ embeds in $T_0/Z(G)$. Clearly ρ factors through $T_0^x/Z(I)$, and since $\langle \rho,\beta\rangle = 1$ for every root β we see that ρ can also be described as half the sum of the positive coroots for T_0^x in I.

Replacing $\mathbf{spl}_G$ by $\mathrm{Int}(\rho(a))\mathbf{spl}_G$ has the effect of multiplying Δ_0 by $\langle z_J, s_J\rangle$, for $z_J \in H^1(F,J)$ obtained from $\rho(a) \in I_{\mathrm{ad}}(F)$ as before. By Lemma 5.2.A, applied to I and its endoscopic group J, the number $\langle z_J, s_J\rangle$ is equal to $(\det(W))(a)$, where W is the virtual representation $V_J - V_I$. Replacing ψ by ψ_a has the effect of multiplying $\varepsilon_L(V,\psi)$ by $(\det(V))(a)$. To show that Δ_λ is left unchanged we must show that

$$\det(V)\det(W) = 1.$$

For this we may as well assume that G is semisimple and simply connected. The representations V_G and V_I of Γ are isomorphic: the former is $X^*(T_0)^\theta \otimes \mathbb{C}$ and the latter is $X^*(T_0)_\theta \otimes \mathbb{C}$. The representations V_H and V_J of Γ are also isomorphic: the former is $X^*(T_{H,0})\otimes\mathbb{C}$ and the latter is $X^*(T_J)\otimes\mathbb{C}$, where T_J is the torus component of some F-splitting of J. Choose T and an admissible embedding $T_{H,0} \simeq T_\theta$ as in

Lemma 3.3.B. Then $X^*(T_{H,0}) \otimes \mathbb{C}$ is equal to $X^*(T)^\theta \otimes \mathbb{C}$, while $X^*(T_J) \otimes \mathbb{C}$ is equal to $X^*(T)_\theta \otimes \mathbb{C}$ by the remarks near the end of (4.2). This shows that V_H and V_J are indeed isomorphic. Therefore V is the negative of W as virtual representation, and hence $\det(V)\det(W) = 1$, as desired.

We are now finished proving that Δ_λ depends only on (B_0, λ). Clearly Δ_λ enjoys the same properties as Δ_0 and Δ (see Lemmas 5.1.B, 5.1.C and Theorem 5.1.D). Replacing (B_0, λ_0) by $\text{Int}(g)(B_0, \lambda_0)$ for $g \in I_{\text{ad}}(F)$ multiplies Δ_λ by the same factor as it does Δ_0, namely $\langle z_J, s_J \rangle$.

As a check that our definition is the right one, let us examine the simplest case: standard endoscopy for $\mathbf{SL}_2$. Let E be a quadratic extension of F and choose $\tau \in E$ such that $\tau \notin F$. Suppose that $\tau^2 = u\tau + v$ with $u, v \in F$. Consider the embedding i of $E^\times$ in $\mathbf{GL}_2(F)$ given by sending $a + b\tau$ $(a, b \in F)$ to the matrix

$$\begin{pmatrix} a & bv \\ b & a+bu \end{pmatrix}$$

(the matrix of multiplication by $a + b\tau$ in the F-basis $\{1, \tau\}$ for E). Let T_E be the F-torus

$$\{y \in E^\times : N_{E/F}(y) = 1\}.$$

Then i gives an embedding

$$T_E \to \mathbf{SL}_2.$$

Consider the standard splitting (B, T, X) of $\mathbf{SL}_2$: B is the subgroup of upper triangular matrices, T is the subgroup of diagonal matrices, and X is the root vector

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let ψ be a non-trivial additive character; we get a character λ on the unipotent radical N of B by sending

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

to $\psi(x)$.

Of course T_E is an endoscopic group for $\mathbf{SL}_2$ (take s to be the unique non-trivial element of $\hat{T}_E^\Gamma$). If we use $\text{Gal}(E/F)$ to form the L-groups of $H = T_E$ and $G = \mathbf{SL}_2$ there is a unique $\hat{G}$-conjugacy class of L-homomorphisms

$$\xi : {}^LH \to {}^LG$$

such that the restriction of ξ to $\hat{T}_E$ belongs to the canonical conjugacy class of embeddings $\hat{T}_E \to \hat{G}$. Working through the definitions in [LS1], one finds that the transfer factor Δ_0, relative to ξ and the F-splitting above, is given by

$$\Delta_0(y, i(y)) = \omega_{E/F}((\bar{y} - y)/(\bar{\tau} - \tau)) \cdot \Delta_{\text{IV}}(y, i(y))$$

for regular elements $y \in T_E(F)$, where $\omega_{E/F}$ denotes the unique character on $F^\times$ of order 2 that is trivial on $N_{E/F}(E^\times)$.

We claim that

$$\varepsilon_L(V, \psi) = \lambda(E/F, \psi)\omega_{E/F}(-1).$$

Recall that $\lambda(E/F, \psi)$ is characterized by the following property: for any representation U of the Weil group W_E of E

$$\varepsilon_L(\mathrm{Ind}(U), \psi) = \varepsilon_L(U, \psi_E)\lambda(E/F, \psi)^{\dim(U)},$$

where $\mathrm{Ind}(U)$ is the representation of W_F induced by U and ψ_E is the additive character $\psi \circ \mathrm{tr}_{E/F}$ on E. Taking U to be the trivial one-dimensional representation of W_E and using that the factor ε_L is 1 for the trivial one-dimensional representation (and any additive character), we see that

$$\lambda(E/F, \psi) = \varepsilon_L(V', \psi)$$

where V' is the virtual representation $\omega_{E/F} - 1$ ($\omega_{E/F}$ is now being viewed as a character on Γ, and 1 stands for the trivial character on Γ). On the other hand it is obvious that V is the virtual representation $1 - \omega_{E/F}$, and therefore

$$\lambda(E/F, \psi) = \varepsilon_L(V, \psi)^{-1}.$$

Since V is defined over $\mathbb{R}$, we have

$$\varepsilon_L(V, \psi)^{-1} = \varepsilon_L(V, \psi)(\det(V))(-1),$$

and of course $\det(V) = \omega_{E/F}$. This proves that

$$\lambda(E/F, \psi) = \varepsilon_L(V, \psi)\omega_{E/F}(-1).$$

Combining the expressions we have found for Δ_0 and $\varepsilon_L(V, \psi)$, we find that for the Whittaker data (B, λ) above

$$\Delta_\lambda(y, i(y)) = \lambda(E/F, \psi)\omega_{E/F}((y - \bar{y})/(\bar{\tau} - \tau))\Delta_{\mathrm{IV}}(y, i(y))$$

for regular elements $y \in T_E(F)$.

Use the transfer factors Δ_λ to transfer characters θ on $T_E(F)$ to invariant distributions $\mathrm{Tran}(\theta)$ on $\mathbf{SL}_2(F)$. Then, as is remarked near the end of §2 of [LL], one has the equality

$$\mathrm{Tran}(\theta) = \chi_{\pi^+(\theta)} - \chi_{\pi^-(\theta)},$$

where $\chi_{\pi^+(\theta)}$, $\chi_{\pi^-(\theta)}$ are the characters of certain representations $\pi^+(\theta)$ and $\pi^-(\theta)$ of $\mathbf{SL}_2(F)$, obtained as follows. Extend θ to a character on $E^\times$. Attached to θ there is an irreducible representation $\pi(\theta)$ of $\mathbf{GL}_2(F)$. When restricted to

$$\{g \in \mathbf{GL}_2(F) : \det(g) \in N_{E/F}(E^\times)\},$$

it splits into two irreducible pieces, $\pi^+(\theta)$ and $\pi^-(\theta)$, and $\pi^+(\theta)$ can be characterized as the piece having a Whittaker model for λ (namely the representation $\pi(\theta, \psi)$ referred to in [LL]). Thus the normalization Δ_λ for the transfer factors has the effect of causing the coefficient of $\chi_{\pi^+(\theta)}$ in the character identity above to be 1, the significance of $\pi^+(\theta)$ being that it is the member of the L-packet $\{\pi^+(\theta), \pi^-(\theta)\}$ for the group

$$\{g \in \mathbf{GL}_2(F) : \det(g) \in N_{E/F}(E^\times)\}$$

having a Whittaker model for λ. Our reason for introducing Δ_λ is the hope that the analogous phenomenon occurs in general.

(5.4) Transfer factors in general (non-trivial z_σ). Recall from (3.1) the 1-cochain z_σ of Γ in the center Z^{sc} of G^*_{sc}. Let $\bar{z}_\sigma$ denote the image of z_σ in

$$Z^{\mathrm{sc}}_\theta := Z^{\mathrm{sc}}/(1-\theta^*)Z^{\mathrm{sc}}.$$

Recall (see Lemma 3.1.A) that $\bar{z}_\sigma$ is a 1-cocycle. Until now we have assumed that $z_\sigma = 1$. We now drop this assumption and see what changes are needed.

Let $(H, \mathcal{H}, s, \xi)$ be a set of endoscopic data for $(G, \theta, \mathbf{a})$ and let (H_1, ξ_{H_1}) be a z-pair for $\mathcal{H}$. Let (B_H, T_H), (B, T) and

$$T_H \simeq T_{\theta^*}$$

be as in Lemma 3.3.B. The obvious map

$$Z^{\mathrm{sc}} \to Z(G) \hookrightarrow T$$

induces a map on coinvariants for θ

$$Z^{\mathrm{sc}}_\theta \to T_{\theta^*} \simeq T_H,$$

whose image lies in the center $Z(H)$ of H, and the resulting map

$$Z^{\mathrm{sc}}_\theta \to Z(H) \tag{5.4.1}$$

is independent of all choices. We use (5.4.1) to view $\bar{z}_\sigma$ as a 1-cocycle in $Z(H)$.

Rather than elements of $T_H(F)$ we now consider elements $\gamma \in T_H(\overline{F})$ such that

$$\sigma(\gamma) = \gamma \bar{z}_\sigma \qquad (\sigma \in \Gamma).$$

We say that such an element γ is a norm of strongly θ-regular θ-semisimple $\delta \in G(F)$ if there exist $\delta^* \in T(\overline{F})$ and $g \in G^*_{\mathrm{sc}}(\overline{F})$ satisfying (3.3.4) and (3.3.5). Given such $\gamma, \delta, \delta^*, g$ we define (as in (3.3) and (4.4))

$$v(\sigma) := g u(\sigma) \sigma(g)^{-1}.$$

Just as in the proof of Lemma 4.4.A (but taking into account z_σ, which was assumed to be trivial in that lemma), we see that

(1) $v(\sigma)$ lies in $T_{\mathrm{sc}}(\overline{F})$, and
(2) $(1-\theta^*)v(\sigma) = \delta^* \sigma(\delta^*)^{-1} z_\sigma$.

Now suppose that $\bar{\gamma}$, $\bar{\delta}$ is another such pair of elements. Then, just as in (4.4), we use $v(\sigma)$, $\bar{v}(\sigma)$, δ^*, $\bar{\delta}^*$ to get a class

$$\mathbf{V} \in H^1(F, U \xrightarrow{1-\theta} S).$$

It is now clear how to define the relative transfer factor $\Delta(\gamma, \delta; \bar{\gamma}, \bar{\delta})$ in case $H_1 = H$. The terms Δ_{I}, Δ_{II}, Δ_{III}, Δ_{IV} are exactly the same as before; of course in defining

Δ_{III} we use the class $\mathbf{V}$ discussed above. Now consider the case $H_1 \neq H$. We may as well assume that there exists some $\gamma_0 \in H(\overline{F})$ such that

$$\sigma(\gamma_0) = \gamma_0 \bar{z}_\sigma \qquad (\sigma \in \Gamma)$$

(otherwise our problem is vacuous). Pick an element $\gamma_{01} \in H_1(\overline{F})$ such that $\gamma_{01} \mapsto \gamma_0$. Then put

$$z_1(\sigma) := \gamma_{01}^{-1}\sigma(\gamma_{01});$$

clearly $z_1(\sigma)$ is a 1-cocycle of Γ in $Z(H_1)$ and $z_1(\sigma) \mapsto \bar{z}_\sigma$ for all $\sigma \in \Gamma$.

Given γ, δ as before we choose an element $\gamma_1 \in H_1(\overline{F})$ such that

(1) $\sigma(\gamma_1) = \gamma_1 z_1(\sigma) \qquad (\sigma \in \Gamma)$

(2) $\gamma_1 \mapsto \gamma$ under $H_1 \to H$;

use the vanishing of $H^1(F, Z_1)$ to show that γ_1 exists. Suppose that $\bar{\gamma}$, $\bar{\delta}$, $\bar{\gamma}_1$ are another such triple. Then we define $\Delta(\gamma_1, \delta; \bar{\gamma}_1, \bar{\delta})$ as in §4. Only the term Δ_{III} requires comment. The point is that we can define δ_1^*, $\bar{\delta}_1^*$ and

$$\mathbf{V}_1 \in H^1(F, U \xrightarrow{1-\theta} S_1)$$

exactly as in §4 (the definitions are the same word-for-word).

The role played by the elements γ, γ_1 above is easy to understand once one makes the following remarks. We fix γ_0 and γ_{01} as before. Define maps

$$m : H \to H$$
$$m_1 : H_1 \to H_1$$

by

$$m(\delta) := \delta\gamma_0 \qquad (\delta \in H)$$
$$m_1(\delta_1) := \delta_1\gamma_{01} \qquad (\delta_1 \in H_1).$$

Define $\theta_H \in \mathrm{Aut}_F(H)$ and $\theta_{H_1} \in \mathrm{Aut}_F(H_1)$ by

$$\theta_H := \mathrm{Int}(\gamma_0)$$
$$\theta_{H_1} := \mathrm{Int}(\gamma_{01}).$$

Then m, m_1 induce bijections

$$H(F) \to \{\gamma \in H(\overline{F}) \,|\, \sigma(\gamma) = \gamma\bar{z}_\sigma\}$$
$$H_1(F) \to \{\gamma_1 \in H_1(\overline{F}) \,|\, \sigma(\gamma_1) = \gamma_1 z_1(\sigma)\}.$$

Moreover

$$m(x^{-1}\delta\theta_H(x)) = x^{-1}m(\delta)x \qquad (x \in H)$$
$$m_1(y^{-1}\delta_1\theta_{H_1}(y)) = y^{-1}m_1(\delta_1)y \qquad (y \in H_1).$$

In other words, working with conjugacy classes of $\gamma \in H(\overline{F})$ satisfying

$$\sigma(\gamma) = \gamma\bar{z}_\sigma$$

is equivalent to working with θ_H-conjugacy classes in $H(F)$.

(5.5) Definition of matching functions. Let $(H, \mathcal{H}, s, \xi)$ be a set of endoscopic data for $(G, \theta, \mathbf{a})$ and let (H_1, ξ_{H_1}) be a z-pair for $\mathcal{H}$. Let λ_{H_1} be the quasicharacter on $Z_1(F)$ constructed in (2.2). There is an exact sequence

$$1 \to Z_1(F) \to H_1(F) \to H(F) \to 1$$

(use the vanishing of $H^1(F, Z_1)$).

We keep the assumptions and notation of (5.4). Fix Haar measures dg on $G(F)$ and dh on $H(F)$. Let $f \in C_c^\infty(G(F))$ and let f^{H_1} be a smooth function on $H_1(F)$, whose support is compact modulo $Z_1(F)$ and that satisfies

$$f^{H_1}(zh) = \lambda_{H_1}(z)^{-1} f^{H_1}(h)$$

for all $z \in Z_1(F)$ and all $h \in H_1(F)$. For strongly θ-regular θ-semisimple $\delta \in G(F)$ such that ω is trivial on $I(F)$ we put

$$O_{\delta\theta}(f) := \int_{I(F)\backslash G(F)} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt.$$

Here $I = \mathrm{Cent}_\theta(\delta, G)$ and dt is a Haar measure on $I(F)$; the twisted orbital integral $O_{\delta\theta}(f)$ depends on the choice of dt. For strongly θ_{H_1}-regular θ_{H_1}-semisimple $\delta_H \in H_1(F)$ we put

$$O_{\delta_H\theta_H}(f^{H_1}) := \int_{T_H(F)\backslash H(F)} f^{H_1}(h^{-1}\delta_H\theta_H(h))\, dh/du.$$

Here $T_H = \mathrm{Cent}_{\theta_H}(\delta_H, H)$ and du is a Haar measure on $T_H(F)$. We also define

$$SO_{\delta_H\theta_H}(f^{H_1}) := \sum_{\delta'_H} O_{\delta'_H\theta_H}(f^{H_1}),$$

where the sum is taken over a set of representatives for the θ_H-conjugacy classes under $H(F)$ of elements $\delta'_H \in H_1(F)$ in the θ_H-conjugacy class of δ_H under $H(\overline{F})$.

We may as well assume that there exists $\gamma_0 \in H(\overline{F})$ such that

$$\sigma(\gamma_0) = \gamma_0 \bar{z}_\sigma \qquad (\sigma \in \Gamma)$$

and which is the norm of a strongly θ-regular θ-semisimple element $\delta_0 \in G(F)$ (otherwise H is irrelevant). As in (5.4) we choose $\gamma_{01} \in H_1(\overline{F})$ such that $\gamma_{01} \mapsto \gamma_0$. We say that $\delta_H \in H_1(F)$ is strongly G-regular if $m_1(\delta_H)$ is strongly G-regular, and we say that δ_H is a norm of $\delta \in G(F)$ if $m_1(\delta_H)$ is a norm of δ. As in (5.1) the relative transfer factor of (5.4) plus the choice of (γ_{01}, δ_0) allows us to define an absolute transfer factor by picking any non-zero complex number c and putting

$$\Delta(\gamma_1, \delta) := c\Delta(\gamma_1, \delta; \gamma_{10}, \delta_0),$$

which we also view as a function on $H_1(F) \times G(F)$ by putting

$$\Delta(\delta_H, \delta) := \Delta(m_1(\delta_H), \delta).$$

We say that f and f^{H_1} have *matching orbital integrals* if

$$SO_{\delta_H \theta_H}(f^{H_1}) = \sum_{\delta} \Delta(\delta_H, \delta) O_{\delta\theta}(f) \tag{5.5.1}$$

for every strongly G-regular $\delta_H \in H_1(F)$. The sum (which might be empty) is taken over a set of representatives for the θ-conjugacy classes under $G(F)$ of elements $\delta \in G(F)$ whose norm is δ_H. Note that for each δ in the sum the restriction of ω to $I(F)$ is trivial. This follows from Lemma 4.4.C (which remains valid even when z_σ is non-trivial). Note also that the product

$$\Delta(\delta_H, \delta) O_{\delta\theta}(f)$$

depends only on the θ-conjugacy class of δ under $G(F)$ (use Lemma 5.1.D(2)). In the equality above we use compatible measures on our twisted centralizers. As usual we have isomorphisms

$$T_H \simeq T_{\theta^*}$$

and

$$\mathrm{Cent}_\theta(\delta, G) \simeq T^{\theta^*}.$$

We see from the exact sequence

$$0 \to \mathrm{Lie}(T^{\theta^*}) \to \mathrm{Lie}(T) \xrightarrow{1-\theta^*} \mathrm{Lie}(T) \to \mathrm{Lie}(T_{\theta^*}) \to 0$$

that the top exterior powers of $\mathrm{Lie}(T^{\theta^*})$ and $\mathrm{Lie}(T_{\theta^*})$ are canonically isomorphic. Pick a non-zero element in the dual of these top exterior powers and use it to get Haar measures dt on $T^{\theta^*}(F)$ and du on $T_{\theta^*}(F)$. We use dt to form $O_{\delta\theta}(f)$ and du to form $SO_{\delta_H\theta_H}(f^{H_1})$. One hopes that for any f there exists some f^{H_1} having matching orbital integrals. Of course the notion of matching depends on the choices of dg, dh and $\Delta(\cdot,\cdot)$, each of which is well-defined only up to multiplication by non-zero complex numbers.

Let χ be a character on a closed subgroup Z_0 of $Z^\theta(F)$ (Z denotes $Z(G)$). In applications one must often consider smooth functions f on $G(F)$, compactly supported modulo Z_0, such that

$$f(zg) = \chi(z)^{-1} f(g)$$

for all $g \in G(F)$ and $z \in Z_0$. Consider the expression

$$\sum_{\delta} \Delta(\delta_H, \delta) O_{\delta\theta}(f) \tag{5.5.2}$$

on the right side of (5.5.1) (with $O_{\delta\theta}(f)$ defined by the same integral as before). Recall from (5.1) the group $C(F)$ and the quasicharacter λ_C on it. The group C contains

$$D := \ker[Z(G) \to Z(H)] = \ker[Z(G) \to Z(G)_\theta]$$

as a subgroup (see (5.1) for a proof of the injectivity of $Z(G)_\theta \to Z(H)$). For $z \in D(F) \cap Z_0$ we have

$$\Delta(\delta_H, z\delta) O_{z\delta\theta}(f) = \lambda_C(z)^{-1} \chi(z)^{-1} \Delta(\delta_H, \delta) O_{\delta\theta}(f).$$

Therefore the sum in (5.5.2) vanishes unless $\lambda_C \chi$ is trivial on $D(F) \cap Z_0$ (make the change of variable $\delta \to z\delta$ in the sum).

Now assume that $\lambda_C \chi$ is trivial on $D(F) \cap Z_0$. Let Z_{01} denote the inverse image under

$$Z(H_1)(F) \to Z(H)(F)$$

of the image of Z_0 under

$$Z(F) \to Z_\theta(F) \hookrightarrow Z(H)(F).$$

Let $C(F)_0$ denote the inverse image of Z_0 under

$$C(F) \to Z(F).$$

Then there is an exact sequence

$$1 \to D(F) \cap Z_0 \to C(F)_0 \to Z_{01} \to 1.$$

Note that χ can be viewed as a character on $C(F)_0$ (use $C(F)_0 \to Z_0$) and that our hypothesis that $\lambda_C \chi$ is trivial on $D(F) \cap Z_0$ implies that $\lambda_C \chi$ is a well-defined character χ_1 on Z_{01}. Note that the restriction of χ_1 to $Z_1(F)$ is λ_{H_1} (Lemma 5.1.C). It is now appropriate to consider smooth functions f^{H_1} on $H_1(F)$, compactly supported modulo Z_{01}, such that

$$f^{H_1}(zh) = \chi_1(z)^{-1} f^{H_1}(h)$$

for all $h \in H_1(F)$ and $z \in Z_{01}$. We again say that f, f^{H_1} have matching orbital integrals if (5.5.1) holds.

6. Beginning of the stabilization

(6.1) Preliminaries. Let F be a number field and G a connected reductive group over F. Denote by A_G the maximal split torus in the center $Z(G)$ of G. As usual we have the product decomposition

$$A_G(\mathbb{A}) = A_G(\mathbb{A})_1 \times \mathfrak{A}_G$$

where

$$\mathfrak{A}_G := \mathrm{Hom}(X^*(A_G), \mathbb{R}) = X_*(A_G) \otimes \mathbb{R}$$
$$A_G(\mathbb{A})_1 := \{a \in A_G(\mathbb{A}) \mid |\lambda(a)| = 1 \text{ for all } \lambda \in X^*(A_G)\}.$$

Let θ be an automorphism of G over F. Then θ acts on A_G and $\mathfrak{A}_G$, and we assume that the natural map

$$\mathfrak{A}_G^\theta \to (\mathfrak{A}_G)_\theta := \mathfrak{A}_G/(1-\theta)\mathfrak{A}_G$$

from θ-invariants to θ-coinvariants is an isomorphism (this is automatically the case if θ has finite order). Let $\mathbf{a}$ be an element of

$$H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})).$$

Note that $\mathbf{a}$ determines a quasicharacter ω on $G(\mathbb{A})$, trivial on $G(F)$. We assume that ω is unitary as well as trivial on $\mathfrak{A}_G$ (viewed as subgroup of $A_G(\mathbb{A})$) and $Z(G)^\theta(\mathbb{A})$.

There are unitary operators $R(\theta)$, $R(\omega)$ on

$$L^2 := L^2(G(F)\mathfrak{A}_G \backslash G(\mathbb{A}))$$

defined by

$$R(\theta)\psi = \psi \circ \theta^{-1}$$
$$R(\omega)\psi = \omega\psi \qquad \text{(pointwise multiplication)}$$

for $\psi \in L^2$. For

$$f \in C_c^\infty(G(\mathbb{A})/\mathfrak{A}_G^\theta)$$

there is the usual convolution operator $R(f)$ on L^2, given by

$$(R(f)\psi)(h) = \int_{G(\mathbb{A})/\mathfrak{A}_G^\theta} f(g)\psi(hg)\, dg/da,$$

where dg is the Tamagawa measure on $G(\mathbb{A})$ and da is the Haar measure on $\mathfrak{A}_G^\theta$ determined by the lattice

$$\mathrm{Hom}((X^*(G)^\Gamma)^\theta, \mathbb{Z})$$

in $\mathfrak{A}_G^\theta$. Define a function

$$\tilde{f} \in C_c^\infty(G(\mathbb{A})/\mathfrak{A}_G)$$

by

$$\tilde{f}(g) = \int_{\mathfrak{A}_G/\mathfrak{A}_G^\theta} f(ga)\, da_G/da$$

where da_G is the Haar measure on $\mathfrak{A}_G$ determined by the lattice

$$\mathrm{Hom}(X^*(G)^\Gamma, \mathbb{Z})$$

in $\mathfrak{A}_G$. The composition

$$R(f)R(\theta)R(\omega)$$

is an integral operator with kernel

$$K(h,g) = \omega(g) \sum_{\delta \in G(F)} \tilde{f}(h^{-1}\delta\theta(g));$$

in other words the value of the operator on $\psi \in L^2$ is the function

$$h \mapsto \int_{G(F)\mathfrak{A}_G\backslash G(\mathbb{A})} K(h,g)\psi(g)\,dg/dx,$$

where dx is the Haar measure on $G(F)\mathfrak{A}_G$ that induces da_G on the open subgroup $\mathfrak{A}_G$.

Let $\delta \in G(F)$ be θ-semisimple and strongly θ-regular. Write I_δ for the θ-centralizer $\mathrm{Cent}_\theta(\delta, G)$ of δ. As in (3.3) we denote by T_δ the centralizer in G of I_δ^0; then T_δ is a maximal torus of G preserved by $\mathrm{Int}(\delta)\circ\theta$ and I_δ coincides with the fixed points of $\mathrm{Int}(\delta)\circ\theta$ on T_δ. We say that δ is *θ-elliptic* if the identity component of

$$I_\delta/Z(G)^\theta$$

is anisotropic over F.

Denote by $G(F)_e$ the set of $\delta \in G(F)$ that are θ-semisimple, strongly θ-regular and θ-elliptic. Denote by $K_e(h,g)$ the corresponding part of the kernel $K(h,g)$:

$$K_e(h,g) := \omega(g)\sum_{\delta\in G(F)_e} \tilde{f}(h^{-1}\delta\theta(g)).$$

We are interested in the part of the twisted trace formula coming from $G(F)_e$, namely

$$T_e(f) := \int_{G(F)\mathfrak{A}_G\backslash G(\mathbb{A})} K_e(g,g)\,dg/dx.$$

As usual we can rewrite $T_e(f)$ as a sum of twisted orbital integrals. The first step is to rewrite $T_e(f)$ as

$$\sum_\delta \int_{I_\delta(F)\mathfrak{A}_G\backslash G(\mathbb{A})} \omega(g)\tilde{f}(g^{-1}\delta\theta(g))\,dg/dy,$$

where dy is the Haar measure on $I_\delta(F)\mathfrak{A}_G$ inducing da_G on the open subgroup $\mathfrak{A}_G$, and where the sum is taken over representatives δ for the θ-conjugacy classes in $G(F)_e$. By our hypothesis on θ the map

$$\theta - 1 : \mathfrak{A}_G/\mathfrak{A}_G^\theta \to \mathfrak{A}_G/\mathfrak{A}_G^\theta$$

is an isomorphism. Therefore $\tilde{f}$ is also given by

$$\tilde{f}(g) = c_G \int_{\mathfrak{A}_G/\mathfrak{A}_G^\theta} f(a^{-1}g\theta(a))\,da_G/da,$$

where

$$c_G := |\det(\theta - 1; \mathfrak{A}_G/\mathfrak{A}_G^\theta)|.$$

It follows that

$$\int_{I_\delta(F)\mathfrak{A}_G\backslash G(\mathbb{A})} \omega(g)\tilde{f}(g^{-1}\delta\theta(g))\,dg/dy = c_G \int_{I_\delta(F)\mathfrak{A}_G^\theta\backslash G(\mathbb{A})} \omega(g)f(g^{-1}\delta\theta(g))\,dg/dz,$$

where dz is the Haar measure on $I_\delta(F)\mathfrak{A}_G^\theta$ that induces da on the open subgroup $\mathfrak{A}_G^\theta$. This integral is 0 unless ω is trivial on $I_\delta(\mathbb{A})$, in which case it equals

$$c_G \operatorname{meas}_{dt/dz}(I_\delta(F)\mathfrak{A}_G^\theta\backslash I_\delta(\mathbb{A})) \int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g)f(g^{-1}\delta\theta(g))\,dg/dt,$$

where dt is the Tamagawa measure on $I_\delta(\mathbb{A})$ (see (E.2), where we defined the Tamagawa measure on $C(\mathbb{A})$ for the diagonalizable group $C = \ker[T \to U]$; the definition makes sense for any diagonalizable group and in particular for I_δ). We write $O_{\delta\theta}(f)$ for the twisted orbital integral

$$\int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g)f(g^{-1}\delta\theta(g))\,dg/dt.$$

Since δ is θ-elliptic the identity component of A_G^θ is the split component of I_δ. The measure

$$\operatorname{meas}_{dt/dz}(I_\delta(F)\mathfrak{A}_G^\theta\backslash I_\delta(\mathbb{A}))$$

would be the Tamagawa number

$$\tau(I_\delta) := \operatorname{meas}(I_\delta(F)\backslash I_\delta(\mathbb{A})_1)$$

of I_δ if we were using the canonical measure in Definition E.1.E on

$$\mathfrak{A}_G^\theta = \operatorname{Hom}(X^*(I_\delta)^\Gamma, \mathbb{R}),$$

namely $|(X^*(I_\delta)^\Gamma)_{\text{tors}}|^{-1}$ times the Haar measure determined by the lattice

$$\operatorname{Hom}(X^*(I_\delta)^\Gamma, \mathbb{Z}).$$

Instead we are using the Haar measure determined by the lattice

$$\operatorname{Hom}((X^*(G)^\Gamma)^\theta, \mathbb{Z}).$$

The ratio between these two measures is

$$c_\delta := |(X^*(I_\delta)^\Gamma)_{\text{tors}}|^{-1}|\operatorname{cok}[\operatorname{Hom}(X^*(I_\delta)^\Gamma, \mathbb{Z}) \to \operatorname{Hom}((X^*(G)^\Gamma)^\theta, \mathbb{Z})]|^{-1}.$$

Therefore

$$T_e(f) = \sum_{\delta\in\Delta} c_G \cdot c_\delta \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f) \tag{6.1.1}$$

where Δ is a set of representatives for the θ-conjugacy classes of elements $\delta \in G(F)_e$ such that ω is trivial on $I_\delta(\mathbb{A})$.

(6.2) Combining terms according to θ-conjugacy classes in $G(\mathbb{A})$. Our next goal is to rewrite (6.1.1) by combining the terms indexed by δ,δ' whenever δ,δ' are θ-conjugate under $G(\mathbb{A})$; this procedure will lead to (6.2.2). Fix an element $\delta \in \Delta$. To simplify notation we temporarily drop the subscript δ from T_δ and I_δ. We write θ_T for the automorphism $\mathrm{Int}(\delta) \circ \theta$. The map of complexes

$$[1 \to \hat{T}] \to [\hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T}]$$

induces a map

$$H^1(W_F, \hat{T}) \to H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})$$

which we compose with the map

$$H^1(W_F, Z(\hat{G})) \to H^1(W_F, \hat{T})$$

induced by the natural injection

$$Z(\hat{G}) \to \hat{T}$$

to get maps

$$H^1(W_F, Z(\hat{G})) \to H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})$$

$$\ker^1(W_F, Z(\hat{G})) \to \ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T}).$$

These in turn give rise to maps

$$\alpha : H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})) \to H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})/\ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})$$

$$\beta : \ker(\alpha) \to \mathrm{cok}[\ker^1(W_F, Z(\hat{G})) \to \ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})].$$

Using that the restriction of ω to $I(\mathbb{A})$ is trivial, we will now check that $\mathbf{a}$ belongs to $\ker(\alpha)$. Choose a representative $a \in H^1(W_F, Z(\hat{G}))$ for $\mathbf{a}$ and let b denote the image of a in $H^1(W_F, \hat{T})$. Then b is a global Langlands parameter for the restriction ω_T of ω to $T(\mathbb{A})$. By Lemma C.3.B the image of b in $H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})$ is locally trivial if and only if the restriction of ω_T to $I(\mathbb{A})$ is trivial. Therefore $\mathbf{a}$ belongs to $\ker(\alpha)$ if and only if the restriction of ω to $I(\mathbb{A})$ is trivial.

Applying the homomorphism β to $\mathbf{a}$, we get an element $\beta(\mathbf{a})$ of the finite abelian group

$$B := \mathrm{cok}[\ker^1(W_F, Z(\hat{G})) \to \ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})].$$

Denote by Δ_δ the set of $\delta' \in \Delta$ such that δ,δ' are θ-conjugate under $G(\mathbb{A})$. We will see that there is a natural bijection from Δ_δ to the finite abelian group

$$B^D := \mathrm{Hom}(B, \mathbb{C}^\times)$$

dual to B.

We begin by noting that if $\delta' \in G(F)_e$ is θ-conjugate to δ under $G(\mathbb{A})$, then $I_{\delta'}(\mathbb{A})$ is conjugate to $I(\mathbb{A})$ under $G(\mathbb{A})$, and therefore ω is automatically trivial on $I_{\delta'}(\mathbb{A})$. Let $\delta' \in \Delta_\delta$. Then δ' is θ-conjugate to δ under $G(\overline{F})$. Write V for the subtorus $(1-\theta_T)T$ of T and U for the quotient T/V. As in (5.1) the difference between δ' and δ is measured by an element

$$\mathrm{inv}(\delta, \delta') \in H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} V),$$

where T_{sc} denotes the inverse image of T under the natural map

$$\pi : G_{\mathrm{sc}} \to G.$$

This invariant depends on the θ-conjugacy classes of δ,δ' under $G_{\mathrm{sc}}(F)$. We get a cruder invariant

$$\mathrm{inv}'(\delta, \delta') \in H^1(F, T \xrightarrow{1-\theta_T} V) \simeq H^1(F, I)$$

depending only on the θ-conjugacy classes of δ,δ' under $G(F)$ by taking the image of $\mathrm{inv}(\delta, \delta')$ under the map on hypercohomology induced by the obvious map of complexes

$$[T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} V] \to [T \xrightarrow{1-\theta_T} V]$$

(use $\pi : T_{\mathrm{sc}} \to T$ and $1 : V \to V$). Note that it would also be easy to define $\mathrm{inv}'(\delta, \delta')$ directly as an element of $H^1(F, I)$. The map $\delta' \mapsto \mathrm{inv}'(\delta, \delta')$ is a bijection from Δ_δ to

$$\ker[\ker^1(F, I) \to \ker^1(F, G)].$$

Using the exact sequence

$$1 \to H^1(F, I) \to H^1(F, T \xrightarrow{1-\theta_T} T) \to U(F)$$

and its adelic analog (see (C.4) for the definition of $H^1(\mathbb{A}, I)$), we see that

$$\ker^1(F, I) = \ker^1(F, T \xrightarrow{1-\theta_T} T).$$

We have a commutative diagram whose vertical maps are bijective

$$\begin{array}{ccc} \ker^1(F, T \to T) & \longrightarrow & \ker^1(F, G) \\ \downarrow & & \downarrow \\ \ker^2(W_F, \hat{T} \to \hat{T})^D & \longrightarrow & \ker^1(W_F, Z(\hat{G}))^D. \end{array}$$

The left vertical arrow is the duality isomorphism of Lemma C.3.B, and the right vertical arrow is the bijection defined in [K1]. The commutativity of the diagram follows from that of

$$\begin{array}{ccc} \ker^1(F, T) & \longrightarrow & \ker^1(G) \\ \downarrow & & \downarrow \\ \ker^1(W_F, \hat{T})^D & \longrightarrow & \ker^1(W_F, Z(\hat{G}))^D. \end{array}$$

It follows that there is a natural bijection from Δ_δ to

$$\ker[\ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})^D \to \ker^1(W_F, Z(\hat{G}))^D],$$

which is indeed the finite abelian group dual to B.

Let $\delta' \in \Delta_\delta$. Then we can pair

$$\mathrm{inv}'(\delta, \delta') \in B^D$$

with

$$\beta(\mathbf{a}) \in B,$$

obtaining

$$\langle \mathrm{inv}'(\delta, \delta'), \beta(\mathbf{a}) \rangle \in \mathbb{C}^\times.$$

We need to express this number in terms of the character ω obtained from $\mathbf{a}$. Using the natural injections

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} V) \to H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} T)$$
$$H^1(F, T \xrightarrow{1-\theta_T} V) \to H^1(F, T \xrightarrow{1-\theta_T} T)$$

we view $\mathrm{inv}(\delta, \delta')$, $\mathrm{inv}'(\delta, \delta')$ as elements of

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} T)$$
$$H^1(F, T \xrightarrow{1-\theta_T} T)$$

respectively. Choose an element

$$t \in H^0(\mathbb{A}/F, T \xrightarrow{1-\theta_T} T)$$

whose image in

$$\ker^1(F, T \xrightarrow{1-\theta_T} T)$$

is $\mathrm{inv}'(\delta, \delta')$. By definition of the pairing between

$$\ker^1(F, T \xrightarrow{1-\theta_T} T)$$

and

$$\ker^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T})$$

we have

$$\langle \mathrm{inv}'(\delta, \delta'), \beta(\mathbf{a}) \rangle = \langle t, \beta(\mathbf{a}) \rangle,$$

where the second pairing is induced by the one between

$$H^0(\mathbb{A}/F, T \xrightarrow{1-\theta_T} T)$$

and

$$H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T}).$$

There is a distinguished triangle

$$[T_{\mathrm{sc}} \xrightarrow{\pi} T] \xrightarrow{\gamma_1} [T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} T] \xrightarrow{\gamma_2} [T \xrightarrow{1-\theta_T} T] \xrightarrow{\gamma_3} [T_{\mathrm{sc}} \xrightarrow{\pi} T][1] \tag{6.2.1}$$

where γ_1, γ_2, γ_3 are given by the vertical maps below

$$\begin{array}{ccccc} & & T_{\mathrm{sc}} & \xrightarrow{\pi} & T \\ & & {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1-\theta_T} \\ & & T_{\mathrm{sc}} & \xrightarrow{(1-\theta_T)\circ\pi} & V \\ & & {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ & & T & \xrightarrow{1-\theta_T} & V \\ & & {\scriptstyle -1}\big\downarrow & & \\ T_{\mathrm{sc}} & \xrightarrow{\pi} & T & & \end{array}$$

and dual to (6.2.1) is the distinguished triangle

$$[\hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})][-1] \to [\hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T}] \to [\hat{T} \xrightarrow{\hat{\pi}\circ(1-\hat{\theta}_T)} \hat{T}/Z(\hat{G})] \to [\hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})].$$

We have

$$\langle t, \beta(\mathbf{a})\rangle = \langle \gamma_3(t), \mathbf{a}\rangle^{-1},$$

where the second pairing is induced by the one between

$$H^0(\mathbb{A}/F, [T_{\mathrm{sc}} \xrightarrow{\pi} T][1]) = H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{\pi} T)$$

and

$$H^1(W_F, \hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})) = H^1(W_F, Z(\hat{G})).$$

There is a canonical homomorphism (see (5.1.3))

$$d : G(\mathbb{A}) \to H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{\pi} T).$$

Choose $h \in G(\mathbb{A})$ such that

$$\delta' = h^{-1}\delta\theta(h).$$

Then from the proof of Theorem 5.1.D(2) we have

$$\gamma_1(d(h^{-1})) = \mathrm{inv}_{\mathbb{A}}(\delta, \delta'),$$

where $\mathrm{inv}_{\mathbb{A}}(\delta,\delta')$ denotes the image of $\mathrm{inv}(\delta,\delta')$ under

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} T) \to H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta_T)\circ\pi} T).$$

The distinguished triangle (6.2.1) gives three long exact sequences, one each for F, $\mathbb{A}$ and $\mathbb{A}/F$. In this way we get a double complex of hypercohomology groups for F, $\mathbb{A}$, $\mathbb{A}/F$ and the three complexes in the triangle, the relevant portion of which is

$$\begin{array}{ccccccc}
 & & & & & & H^1(\mathbb{A}, R_1) \\
 & & & & & & \downarrow \\
 & & & & H^0(\mathbb{A}/F, R_3) & \xrightarrow{\gamma_3} & H^1(\mathbb{A}/F, R_1) \\
 & & & & \downarrow & & \\
 & & H^1(F, R_2) & \xrightarrow{\gamma_2} & H^1(F, R_3) & & \\
 & & \downarrow & & & & \\
H^1(\mathbb{A}, R_1) & \xrightarrow{\gamma_1} & H^1(\mathbb{A}, R_2) & & & &
\end{array}$$

where we have denoted by

$$R_1 \xrightarrow{\gamma_1} R_2 \xrightarrow{\gamma_2} R_3 \xrightarrow{\gamma_3} R_1[1]$$

the distinguished triangle (6.2.1). The elements we have been considering are

$$\begin{array}{ccccccc}
 & & & & t & \longrightarrow & \gamma_3(t) \\
 & & & & \downarrow & & \\
 & & \mathrm{inv}(\delta,\delta') & \longrightarrow & \mathrm{inv}'(\delta,\delta') & & \\
 & & \downarrow & & & & \\
d(h)^{-1} & \longrightarrow & \mathrm{inv}_{\mathbb{A}}(\delta,\delta') & & & &
\end{array}$$

By homological general nonsense the image of $d(h)$ under

$$H^1(\mathbb{A}, R_1) \to H^1(\mathbb{A}/F, R_1)$$

is equal to $\gamma_3(t)$ for some t as above. Therefore

$$\langle \gamma_3(t), \mathbf{a} \rangle = \omega(h)^{-1},$$

which when combined with our previous work yields the equality

$$\langle \mathrm{inv}'(\delta,\delta'), \beta(\mathbf{a}) \rangle = \omega(h),$$

expressing the value of the pairing in terms of ω, as desired.

From the equality

$$\delta' = h^{-1}\delta\theta(h)$$

we see that

$$\begin{aligned} O_{\delta'\theta}(f) &= \omega(h)^{-1}O_{\delta\theta}(f) \\ &= \langle \mathrm{inv}'(\delta, \delta'), \beta(\mathbf{a})\rangle^{-1}O_{\delta\theta}(f). \end{aligned}$$

Consider the part of the sum in (6.1.1) indexed by elements $\delta' \in \Delta_\delta$. Since $c_{\delta'} = c_\delta$ and $\tau(I_{\delta'}) = \tau(I_\delta)$ this part of the sum is equal to

$$c_G \cdot c_\delta \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f)$$

times

$$\sum_{\delta' \in \Delta_\delta} \langle \mathrm{inv}'(\delta, \delta'), \beta(\mathbf{a})\rangle^{-1} = \begin{cases} |B| & \text{if } \beta(\mathbf{a}) = 1, \\ 0 & \text{if } \beta(\mathbf{a}) \neq 1. \end{cases}$$

To indicate the dependence of B on δ we now denote it by B_δ. We conclude that

$$T_e(f) = \sum_{\delta \in \Delta_1} c_G \cdot c_\delta \cdot |B_\delta| \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f), \tag{6.2.2}$$

where Δ_1 is a set of representatives for the θ-conjugacy classes under $G(\mathbb{A})$ of elements $\delta \in G(F)_e$ such that

(1) ω is trivial on $I_\delta(\mathbb{A})$, and
(2) the element $\beta(\mathbf{a}) \in B_\delta$ is trivial.

(6.3) Definition of an obstruction. The next step in the stabilization of $T_e(f)$ requires that we introduce an obstruction obs(δ). For standard endoscopy this obstruction is due to Langlands [L2,p. 137] (see also [K3,§6] for a generalization to the case of arbitrary semisimple elements).

Let G^*, $\mathbf{spl}_{G^*}$, ψ, θ^* and g_θ be as in (1.2). Thus G^* is a quasi-split inner form of G with F-splitting $\mathbf{spl}_{G^*}$, and $\psi : G \to G^*$ is an inner twisting (an $\overline{F}$-isomorphism such that $\psi\sigma(\psi)^{-1}$ is inner for all $\sigma \in \Gamma$). Moreover θ^* is an F-automorphism of G^* preserving $\mathbf{spl}_{G^*}$, and $g_\theta \in G^*_{\mathrm{sc}}$ has the property that

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}.$$

As in (3.1) we choose, for each $\sigma \in \Gamma$, an element $u(\sigma) \in G^*_{\mathrm{sc}}$ such that

$$\psi\sigma(\psi)^{-1} = \mathrm{Int}(u(\sigma)),$$

and we also define a morphism

$$m : G \to G^*$$

over $\overline{F}$ by

$$m(\delta) := \psi(\delta)g_\theta^{-1}.$$

Then, as we saw in (3.1),

$$\sigma(m)(\delta) = u(\sigma)^{-1}m(\delta)z_\sigma\theta^*(u(\sigma)) \tag{6.3.1}$$

for the 1-cochain

$$z_\sigma := g_\theta u(\sigma)\sigma(g_\theta)^{-1}\theta^*(u(\sigma))^{-1}$$

of Γ in $Z^{\mathrm{sc}}(\overline{F})$, where Z^{sc} denotes the center of G^*_{sc}. Recall from Lemma 3.1.A that the image $\bar{z}_\sigma$ of z_σ under

$$Z^{\mathrm{sc}} \to Z^{\mathrm{sc}}_\theta := Z^{\mathrm{sc}}/(1-\theta^*)Z^{\mathrm{sc}}$$

is a 1-cocycle. We do not assume, as we did when defining transfer factors, that z_σ is trivial.

Let (B, T) be a θ^*-stable pair in G^* with T defined over F. Put $V := (1-\theta^*)T$ and $U := T/V$. Note that the map

$$\pi : G^*_{\mathrm{sc}} \to G^*$$

induces a map

$$Z^{\mathrm{sc}}_\theta \to U.$$

Let γ be an element of $U(\overline{\mathbb{A}})$ such that

$$\sigma(\gamma) = \gamma\bar{z}_\sigma \tag{6.3.2}$$

for all $\sigma \in \Gamma$. Let $\delta \in G(\mathbb{A})$. We say that γ is a *norm* of δ if there exist $\delta^* \in T(\overline{\mathbb{A}})$ and $g \in G^*_{\mathrm{sc}}(\overline{\mathbb{A}})$ such that

(6.3.3) the image of δ^* in $U(\overline{\mathbb{A}})$ equals γ, and
(6.3.4) $\delta^* = gm(\delta)\theta^*(g)^{-1}$.

Now let γ be an element of $U(\overline{F})$ satisfying (6.3.2), let $\delta \in G(\mathbb{A})$ and suppose that γ is a norm of δ. Suppose further that γ is fixed by no non-trivial θ^*-invariant element of the Weyl group of T, so that δ is strongly θ-regular. We are going to use δ,γ to get an element

$$\mathrm{obs}(\delta) \in H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Choose $\delta^* \in T(\overline{\mathbb{A}})$ and $g \in G^*_{\mathrm{sc}}(\overline{\mathbb{A}})$ satisfying (6.3.3) and (6.3.4). As in (3.3) and (4.4) we define

$$v(\sigma) := gu(\sigma)\sigma(g)^{-1} \qquad (\sigma \in \Gamma).$$

Just as in the proof of Lemma 4.4.A (but taking into account z_σ, which was assumed to be trivial at that time), we see that

(1) $v(\sigma)$ lies in $T_{\mathrm{sc}}(\overline{\mathbb{A}})$, and
(2) $(1-\theta^*)v(\sigma) = \delta^*\sigma(\delta^*)^{-1}z_\sigma$.

Observe that the coboundary ∂v of v is

$$g(\partial u)g^{-1} = \partial u$$

and takes values in $Z^{\mathrm{sc}}(\overline{F})$. Clearly $(v(\sigma)^{-1}, \delta^*)$ is a 1-hypercocycle of Γ in

$$T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F}) \xrightarrow{(1-\theta^*)\circ\pi} T(\overline{\mathbb{A}})/T(\overline{F}).$$

We need to understand the effect of the choices of ψ, $u(\sigma)$, g_θ, δ^*, g on the 1-hypercocyle; of course ψ is only allowed to vary within its inner class

$$\Psi := \{\psi = (\mathrm{Int}(x)) \circ \psi_0 \mid x \in G^*_{\mathrm{sc}}(\overline{F})\}.$$

For the moment we keep the same choices for ψ, $u(\sigma)$, g_θ. Using that γ is fixed by no non-trivial element of $\Omega(T, G^*)^{\theta^*}$, one sees that any other choice g' for g is of the form $g' = tg$ for some $t \in T_{\mathrm{sc}}(\overline{\mathbb{A}})$, and then as our new choice for δ^* we are forced to take

$$(\delta^*)' = \delta^* t \theta^*(t)^{-1}.$$

It follows that our 1-hypercocycle is multiplied by the 1-hypercoboundary

$$(\partial t, t\theta^*(t)^{-1}).$$

Suppose that we replace ψ, $u(\sigma)$, g_θ by ψ', $u'(\sigma)$, g'_θ. Choose $x \in G^*_{\mathrm{sc}}(\overline{F})$ such that

$$\psi' = \mathrm{Int}(x) \circ \psi.$$

Then

$$g'_\theta = \theta^*(x) g_\theta x^{-1} z$$

for some $z \in Z^{\mathrm{sc}}(\overline{F})$. Since x is well-defined up to $Z^{\mathrm{sc}}(\overline{F})$, the image $\bar{z}$ of z in $Z^{\mathrm{sc}}_\theta(\overline{F})$ is well-defined. It is not hard to check that the 1-cocycle $\bar{z}'_\sigma$ of Γ in $Z^{\mathrm{sc}}_\theta(\overline{F})$ obtained from our new choices is related to the old one by

$$\bar{z}'_\sigma = \bar{z}_\sigma (\partial \bar{z})^{-1}.$$

Define $\gamma' \in U(\overline{F})$ by

$$\gamma' = \gamma \bar{z}^{-1};$$

then

$$\sigma(\gamma') = \gamma' \bar{z}'_\sigma \qquad (\sigma \in \Gamma)$$

and γ' is a norm of δ relative to ψ', $u'(\sigma)$, g'_θ (take $(\delta^*)' = \delta^* z^{-1}$ and $g' = gx^{-1}$). For these choices of $(\delta^*)', g'$ the 1-hypercocycle remains unchanged.

We now fix the choices of ψ, $u(\sigma)$, g_θ once and for all. The class obs(δ) of our 1-hypercocycle in

$$H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} T)$$

is well-defined. The obvious short exact sequence of complexes

$$1 \to [T_{sc} \xrightarrow{(1-\theta^*)\circ\pi} V] \to [T_{sc} \xrightarrow{(1-\theta^*)\circ\pi} T] \to [1 \to U] \to 1$$

gives rise to a long exact sequence

$$1 \to H^1(\mathbb{A}/F, T_{sc} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathbb{A}/F, T_{sc} \xrightarrow{(1-\theta^*)\circ\pi} T) \to H^0(\mathbb{A}/F, U) \to \cdots$$

and since the image of $\mathrm{obs}(\delta)$ in $H^0(\mathbb{A}/F, U)$ is represented by $\gamma \in U(\overline{F})$ we conclude that $\mathrm{obs}(\delta)$ lies in the subgroup

$$H^1(\mathbb{A}/F, T_{sc} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Lemma 6.3.A. *The element* $\mathrm{obs}(\delta)$ *depends only on the* θ*-conjugacy class of* δ *under* $G_{sc}(\mathbb{A})$*, and* $\mathrm{obs}(\delta)$ *is trivial if and only if* δ *is* θ*-conjugate under* $G_{sc}(\mathbb{A})$ *to an element of* $G(F)$.

The (easy) proof of the first statement is left to the reader. Now we prove the second statement. Suppose that δ is θ-conjugate under $G_{sc}(\mathbb{A})$ to an element of $G(F)$. We want to show that $\mathrm{obs}(\delta)$ is trivial. Without loss of generality we may assume that $\delta \in G(F)$. Then it is possible to choose $\delta^* \in T(\overline{F})$ and $g \in G^*_{sc}(\overline{F})$ satisfying (6.3.3) and (6.3.4), and for such δ^*, g the corresponding 1-hypercocycle is trivial.

Conversely suppose that $\mathrm{obs}(\delta)$ is trivial. Then it is possible to choose δ^*, g satisfying (6.3.3) and (6.3.4) and such that $\delta^* \in T(\overline{F})$ and $v(\sigma) \in T_{sc}(\overline{F})$ for all $\sigma \in \Gamma$. Let ψ_{sc} denote the unique inner twisting

$$\psi_{sc} : G_{sc} \to G^*_{sc}$$

lifting ψ, and put

$$h := \psi_{sc}^{-1}(g) \in G_{sc}(\overline{\mathbb{A}})$$
$$x_\sigma := h\sigma(h)^{-1} \qquad (\sigma \in \Gamma).$$

Then an easy calculation shows that

$$\psi_{sc}(x_\sigma) = v(\sigma)u(\sigma)^{-1},$$

which means in particular that

$$\psi_{sc}(x_\sigma) \in G^*_{sc}(\overline{F})$$

and hence that

$$x_\sigma \in G_{sc}(\overline{F}).$$

It is immediate from the definition of x_σ that it is a 1-cocycle that is locally a 1-coboundary. By the Hasse principle

$$\ker^1(F, G_{sc}) = \{1\},$$

and therefore there exists $y \in G_{\mathrm{sc}}(\overline{F})$ such that

$$x_\sigma = y\sigma(y)^{-1}.$$

It follows that the element

$$h' := y^{-1}h$$

belongs to $G_{\mathrm{sc}}(\mathbb{A})$, and an easy calculation shows that

$$m(h'\delta\theta(h')^{-1}) = \psi_{\mathrm{sc}}(y)^{-1}\delta^*\theta^*(\psi_{\mathrm{sc}}(y)) \in G^*(\overline{F});$$

therefore

$$h'\delta\theta(h')^{-1}$$

belongs to

$$G(\overline{F}) \cap G(\mathbb{A}) = G(F),$$

and the proof is complete.

We continue with γ, δ as above, and choose δ^*, g satisfying (6.3.3) and (6.3.4). Then g is well-defined up to left multiplication by an element of $T_{\mathrm{sc}}(\overline{\mathbb{A}})$. Define

$$\psi' : G(\overline{\mathbb{A}}) \to G^*(\overline{\mathbb{A}})$$

by

$$\psi' := \mathrm{Int}(g) \circ \psi.$$

Write $T_\delta(\overline{\mathbb{A}})$ for the centralizer of $\mathrm{Cent}_\theta(\delta, G(\overline{\mathbb{A}}))$ in $G(\overline{\mathbb{A}})$. Then ψ' induces a Γ-equivariant isomorphism

$$T_\delta(\overline{\mathbb{A}}) \to T(\overline{\mathbb{A}}) \tag{6.3.5}$$

that carries

$$\theta^\delta := \mathrm{Int}(\delta) \circ \theta$$

over to θ^*; note that this isomorphism is independent of the choice of δ^*, g.

Let $\delta' \in G(\mathbb{A})$ and assume that δ' is θ-conjugate to δ under $G(\overline{\mathbb{A}})$. We now define an element

$$\mathrm{inv}(\delta, \delta') \in H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)$$

that depends only on the θ-conjugacy classes of δ, δ' under $G_{\mathrm{sc}}(\mathbb{A})$ and is trivial if and only if δ, δ' are θ-conjugate under $G_{\mathrm{sc}}(\mathbb{A})$. A 1-hypercocycle representing $\mathrm{inv}(\delta, \delta')$ is defined as follows. Choose $h \in G(\overline{\mathbb{A}})$ such that

$$\delta' = h^{-1}\delta\theta(h).$$

Write h as

$$h = t\pi(h_1)$$

with $t \in T_\delta(\overline{\mathbb{A}})$ and $h_1 \in G_{\mathrm{sc}}(\overline{\mathbb{A}})$. Then

$$(\sigma \mapsto \psi'(\sigma(h_1)h_1^{-1}), \theta^*(\psi'(t))\psi'(t)^{-1})$$

is the desired 1-hypercocycle. Of course the local components of $\mathrm{inv}(\delta, \delta')$ coincide with the local invariants defined in (5.1).

Lemma 6.3.B. *There is an equality*

$$\operatorname{obs}(\delta') = \operatorname{obs}(\delta)\, \operatorname{inv}_{\mathbb{A}/F}(\delta, \delta'),$$

where $\operatorname{inv}_{\mathbb{A}/F}(\delta, \delta')$ *denotes the image of* $\operatorname{inv}(\delta, \delta')$ *under*

$$H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

The elements

$$g' := g\psi(h_1)$$
$$(\delta^*)' := \theta^*(\psi'(t)) \cdot \psi'(t)^{-1} \cdot \delta^*$$

can serve as g, δ^* for δ'. For this g' we have

$$\begin{aligned} v'(\sigma) &= g'u(\sigma)\sigma(g')^{-1} \\ &= v(\sigma) \cdot \psi'(h_1\sigma(h_1)^{-1}). \end{aligned}$$

Therefore

$$(v'(\sigma)^{-1}, (\delta^*)') = (v(\sigma)^{-1}, \delta^*) \cdot (\psi'(\sigma(h_1)h_1^{-1}), \theta^*(\psi'(t))\psi'(t)^{-1}),$$

which proves the lemma.

In order to define $\operatorname{obs}(\delta)$ we needed to choose a θ^*-stable pair (B, T) with T defined over F and a norm $\gamma \in U(\overline{F})$ of δ. Suppose that (B', T') and γ' are another such triple. Then there exists $x \in (G^*_{\mathrm{sc}})^{\theta^*}(\overline{F})$ such that $xTx^{-1} = T'$ and $x\gamma x^{-1} = \gamma'$, and, since γ is fixed by no non-trivial element of

$$\Omega(T, G^*)^{\theta^*},$$

the element x is unique up to right multiplication by an element of $T_{\mathrm{sc}}(\overline{F})$ fixed by θ^*. Therefore the isomorphism

$$\operatorname{Int}(x) : T \simeq T'$$

is independent of the choice of x and in particular is defined over F. From T' we get V' and $\operatorname{obs}(\delta)'$. The isomorphism above induces an isomorphism

$$H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \simeq H^1(\mathbb{A}/F, T'_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V')$$

under which $\operatorname{obs}(\delta)$ goes over to $\operatorname{obs}(\delta)'$. Indeed, suppose that δ^*, g satisfy (6.3.3) and (6.3.4) relative to γ. Then $x\delta^*x^{-1}, xg$ satisfy them relative to γ'. The two 1-cocycles we get are related by

$$\begin{aligned} v'(\sigma) &= xv(\sigma)\sigma(x)^{-1} \\ &= x(v(\sigma) \cdot \sigma(x)^{-1}x)x^{-1} \end{aligned}$$

and since

$$\sigma(x)^{-1}x \in T_{\mathrm{sc}}(\overline{F}),$$

we see that the 1-hypercocycle $(v'(\sigma)^{-1}, x\delta^*x^{-1})$ representing $\operatorname{obs}(\delta)'$ is obtained by applying $\operatorname{Int}(x)$ to the 1-hypercocycle $(v(\sigma)^{-1}, \delta^*)$ representing $\operatorname{obs}(\delta)$.

(6.4) $\mathfrak{K}(T,\theta,F)$, $\mathcal{E}(T,\theta,F)$, $\mathcal{E}(T,\theta,\mathbb{A})$. We continue with ψ, $u(\sigma)$, g_θ, z_σ, B, T, U, V as in (6.3) and again consider an element $\gamma \in U(\overline{F})$ satisfying (6.3.2) and having trivial stabilizer in

$$\Omega(T,G^*)^{\theta^*}.$$

Moreover we assume that $T_{\mathrm{sc}}^{\theta^*}$ is anisotropic over F. We denote by $T_e(f)_\gamma$ the part of the sum (6.2.2) indexed by elements $\delta \in \Delta_1$ for which γ is a norm. We are headed toward (6.4.16), a formula for $T_e(f)_\gamma$. When summed over γ this formula yields the main result of this section, Theorem 6.4.C.

The obstruction defined in (6.3) lies in

$$H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Define an abelian group $\mathfrak{K}(T,\theta,F)$ by

$$\mathfrak{K}(T,\theta,F) := H^1(W_F, \hat{V} \xrightarrow{\hat{\pi}\circ\phi} \hat{T}/Z(\hat{G}))$$

where

$$\hat{V} \xrightarrow{\phi} \hat{T}$$

is dual to

$$T \xrightarrow{1-\theta^*} V.$$

Our hypothesis that $T_{\mathrm{sc}}^{\theta^*}$ is anisotropic implies that

$$H^1(W_F, \hat{V} \xrightarrow{\hat{\pi}\circ\phi} \hat{T}/Z(\hat{G})) = H^1(W_F, \hat{V} \xrightarrow{\hat{\pi}\circ\phi} \hat{T}/Z(\hat{G}))_{\mathrm{red}}.$$

By duality (see Lemma C.2.C)

$$\mathfrak{K}(T,\theta,F) \simeq \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V), \mathbb{C}^\times).$$

There is a distinguished triangle

$$[T_{\mathrm{sc}} \xrightarrow{\pi} T] \xrightarrow{\gamma_1} [T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V] \xrightarrow{\gamma_2} [T \xrightarrow{1-\theta^*} V] \xrightarrow{\gamma_3} [T_{\mathrm{sc}} \xrightarrow{\pi} T][1] \tag{6.4.1}$$

where γ_1, γ_2, γ_3 are given by the vertical maps below

$$\begin{array}{ccccc}
 & & T_{\mathrm{sc}} & \xrightarrow{\pi} & T \\
 & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1-\theta^*} \\
 & & T_{\mathrm{sc}} & \xrightarrow{(1-\theta^*)\circ\pi} & V \\
 & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle 1} \\
 & & T & \xrightarrow{1-\theta^*} & V \\
 & & {\scriptstyle -1}\downarrow & & \\
T_{\mathrm{sc}} & \xrightarrow{\pi} & T & &
\end{array}$$

and dual to (6.4.1) is

$$(6.4.2)\qquad [\hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})][-1] \to [\hat{V} \xrightarrow{\phi} \hat{T}] \to [\hat{V} \xrightarrow{\hat{\pi}\circ\phi} \hat{T}/Z(\hat{G})] \to [\hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})].$$

From (6.4.2) we get a long exact sequence of hypercohomology with respect to W_F, and since

$$H^1(W_F, \hat{T} \xrightarrow{\hat{\pi}} \hat{T}/Z(\hat{G})) = H^1(W_F, Z(\hat{G}))$$

part of this exact sequence is

$$(6.4.3)\qquad \cdots \to \mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G})) \to H^2(W_F, \hat{V} \xrightarrow{\phi} \hat{T}) \to \cdots .$$

In particular we get a natural map

$$(6.4.4)\qquad \mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G}))$$

by composing the map

$$\mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G}))$$

in (6.4.3) with the natural surjection

$$H^1(W_F, Z(\hat{G})) \to H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})).$$

If there is no element of Δ_1 having γ as norm, then $T_e(f)_\gamma = 0$. If there is such an element $\delta \in \Delta_1$, then $\mathbf{a}$ lies in the image of the map (6.4.4), as we now verify. Since $\delta \in G(F)$ there exist $\delta^* \in T(\overline{F})$ and $g \in G^*_{\mathrm{sc}}(\overline{F})$ satisfying (6.3.3) and (6.3.4). The restriction of $\mathrm{Int}(g)\circ\psi$ to T_δ gives an F-isomorphism $T_\delta \simeq T$ that carries $\theta_T = \mathrm{Int}(\delta)\circ\theta$ over to θ^*. By definition of Δ_1

(1) ω is trivial on $I_\delta(\mathbb{A})$, and
(2) the element $\beta(\mathbf{a}) \in B_\delta$ is trivial.

Therefore there is a representative $a \in H^1(W_F, Z(\hat{G}))$ for $\mathbf{a}$ whose image under

$$(6.4.5)\qquad H^1(W_F, Z(\hat{G})) \to H^2(W_F, \hat{T} \xrightarrow{1-\theta^*} \hat{T})$$

is trivial. Since the map

$$H^1(W_F, Z(\hat{G})) \to H^2(W_F, \hat{V} \xrightarrow{\phi} \hat{T})$$

in (6.4.3) factors through (6.4.5), we conclude that a lies in the image of

$$\mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G}))$$

and hence that $\mathbf{a}$ lies in the image of (6.4.4).

We now assume that there is an element of Δ_1 having γ as norm. We fix such an element $\delta_0 \in \Delta_1$, and we also fix an element

$$\kappa_0 \in \mathfrak{K}(T,\theta,F)$$

mapping to $\mathbf{a}$ under (6.4.4). For any element $\delta \in G(\mathbb{A})$ having γ as norm we have (see (6.3)) an element

$$\operatorname{inv}(\delta_0, \delta) \in H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)$$

and the map

$$\delta \mapsto \operatorname{inv}(\delta_0, \delta)$$

sets up a bijection from the set of θ-conjugacy classes under $G_{\mathrm{sc}}(\mathbb{A})$ of elements $\delta \in G(\mathbb{A})$ having γ as norm to the set

$$C_0 := \ker[H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathbb{A}, G_{\mathrm{sc}})]$$

(the map whose kernel we are taking is of course induced by the inclusion $T_{\mathrm{sc}} \to G_{\mathrm{sc}}$).

For any element $\delta \in G(\mathbb{A})$ having γ as norm we define $\mathbb{A}$-group schemes I_δ and T_δ by

$$I_\delta = \operatorname{Cent}_\theta(\delta, G)$$
$$T_\delta = \operatorname{Cent}_G(I_\delta^0).$$

As before (see (6.3.5)) there is a canonical isomorphism over $\mathbb{A}$

$$T_\delta \simeq T$$

that carries I_δ into $I := T^{\theta^*}$. We use this isomorphism to carry the Tamagawa measure on $I(\mathbb{A})$ over to a Haar measure dt on $I_\delta(\mathbb{A})$, and we put

$$O_{\delta\theta}(f) := \int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt.$$

Define a function Φ on C_0 by putting

$$\Phi(x) = \langle \operatorname{obs}(\delta), \kappa_0 \rangle O_{\delta\theta}(f)$$

where $\delta \in G(\mathbb{A})$ has norm γ and is such that

$$\operatorname{inv}(\delta_0, \delta) = x.$$

Suppose that δ, δ' are θ-conjugate under $G(\mathbb{A})$. We will now check that

$$\text{(6.4.6)} \qquad \langle \operatorname{obs}(\delta), \kappa_0 \rangle O_{\delta\theta}(f) = \langle \operatorname{obs}(\delta'), \kappa_0 \rangle O_{\delta'\theta}(f).$$

Choose $h \in G(\mathbb{A})$ such that

$$\delta' = h^{-1}\delta\theta(h).$$

Then

$$O_{\delta'\theta}(f) = \omega(h)^{-1} O_{\delta\theta}(f)$$

and

$$\langle \operatorname{inv}(\delta, \delta'), \kappa_0 \rangle = \omega(h)$$

(see the proof of Lemma 5.1.D(2)), and the equality (6.4.6) now follows from Lemma 6.3.B.

Define an abelian group

$$\mathcal{E}(T, \theta, F) := \operatorname{im}[H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, T \xrightarrow{1-\theta^*} V)]$$

and a set

$$\mathcal{D}(T, \theta, F) := \ker[H^1(F, T \xrightarrow{1-\theta^*} V) \to H^1(F, G)].$$

It is not hard to see that $\mathcal{D}(T, \theta, F)$ is the image under

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, T \xrightarrow{1-\theta^*} V)$$

of

$$\ker[H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(F, G_{\mathrm{sc}})]$$

and consequently that $\mathcal{D}(T, \theta, F)$ is a subset of $\mathcal{E}(T, \theta, F)$. There is a natural bijection between $\mathcal{D}(T, \theta, F)$ and the set of θ-conjugacy classes under $G(F)$ of elements $\delta \in G(F)$ having γ as norm. Replacing F by F_v or $\mathbb{A}$ in these definitions, we get $\mathcal{E}(T, \theta, F_v)$, $\mathcal{D}(T, \theta, F_v)$, $\mathcal{E}(T, \theta, \mathbb{A})$, $\mathcal{D}(T, \theta, \mathbb{A})$ satisfying the analogs of the properties of $\mathcal{E}(T, \theta, F)$, $\mathcal{D}(T, \theta, F)$ mentioned above.

From (6.4.6) it follows that $\Phi(x)$ depends only on the image of x in $\mathcal{E}(T, \theta, \mathbb{A})$, and therefore Φ may be regarded as a function on the image of C_0 in $\mathcal{E}(T, \theta, \mathbb{A})$, namely $\mathcal{D}(T, \theta, \mathbb{A})$. We extend Φ to a function on all of $\mathcal{E}(T, \theta, \mathbb{A})$ by making it 0 on the complement of $\mathcal{D}(T, \theta, \mathbb{A})$.

Since c_δ, $|B_\delta|$, $\tau(I_\delta)$ depend only on T, we obtain a rational number c_T depending only on T by putting

$$c_T := c_G \cdot c_{\delta_0} \cdot |B_{\delta_0}| \cdot \tau(I_{\delta_0}).$$

Since $\operatorname{obs}(\delta) = 1$ for any $\delta \in \Delta_1$ (see Lemma 6.3.A), we have

$$T_e(f)_\gamma = c_T \sum_{x \in S} \Phi(x),$$

where S is the set

$$\operatorname{im}[\mathcal{D}(T, \theta, F) \to \mathcal{E}(T, \theta, \mathbb{A})].$$

Obviously S is contained in the subset

$$S' := \mathcal{D}(T, \theta, \mathbb{A}) \cap \operatorname{im} \mathcal{E}(T, \theta, F)$$

of $\mathcal{E}(T, \theta, \mathbb{A})$, where $\operatorname{im} \mathcal{E}(T, \theta, F)$ denotes

$$\operatorname{im}[\mathcal{E}(T, \theta, F) \to \mathcal{E}(T, \theta, \mathbb{A})].$$

In fact $S = S'$, as we will now check. It is enough to prove the following stronger statement: if c is an element of $\mathcal{E}(T,\theta,F)$ whose image in $\mathcal{E}(T,\theta,\mathbb{A})$ belongs to $\mathcal{D}(T,\theta,\mathbb{A})$, then c belongs to $\mathcal{D}(T,\theta,F)$. Let c be such an element. By restriction of scalars we may assume that $F = \mathbb{Q}$. Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^1(\mathbb{Q}, T_{\text{sc}} \to T) & \xrightarrow{\gamma_1} & H^1(\mathbb{Q}, T_{\text{sc}} \to V) & \longrightarrow & \mathcal{E}(T,\theta,\mathbb{Q}) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^1(\mathbb{R}, T_{\text{sc}} \to T) & \xrightarrow{\gamma_1} & H^1(\mathbb{R}, T_{\text{sc}} \to V) & \longrightarrow & \mathcal{E}(T,\theta,\mathbb{R}) & \longrightarrow & 1 \end{array}$$

coming from the distinguished triangle (6.4.1). Choose an element c_0 of

$$H^1(\mathbb{Q}, T_{\text{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)$$

whose image in $\mathcal{E}(T,\theta,\mathbb{Q})$ is c, and let $c_0(\infty)$ denote the image of c_0 in

$$H^1(\mathbb{R}, T_{\text{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Consider the set X of elements

$$d_{\mathbb{R}} \in H^1(\mathbb{R}, T_{\text{sc}} \xrightarrow{\pi} T)$$

such that $\gamma_1(d_{\mathbb{R}})c_0(\infty)$ has trivial image in

$$H^1(\mathbb{R}, G_{\text{sc}}).$$

By our hypothesis that the image of c in $\mathcal{E}(T,\theta,\mathbb{R})$ belongs to $\mathcal{D}(T,\theta,\mathbb{R})$, the set X is non-empty. It is easy to see that X is open in

$$H^1(\mathbb{R}, T_{\text{sc}} \xrightarrow{\pi} T).$$

By Lemma C.5.A there exists

$$d \in H^1(\mathbb{Q}, T_{\text{sc}} \xrightarrow{\pi} T)$$

whose image $d(\infty)$ in

$$H^1(\mathbb{R}, T_{\text{sc}} \xrightarrow{\pi} T)$$

belongs to X. It follows that $\gamma_1(d)c_0$ has trivial image in

$$H^1(\mathbb{R}, G_{\text{sc}}).$$

From Kneser's vanishing theorem

$$H^1(\mathbb{Q}_p, G_{\text{sc}}) = \{1\}$$

together with the Hasse principle

$$\ker^1(\mathbb{Q}, G_{\rm sc}) = \{1\}$$

it follows that $\gamma_1(d)c_0$ has trivial image in

$$H^1(\mathbb{Q}, G_{\rm sc}),$$

which shows that c lies in $\mathcal{D}(T, \theta, \mathbb{Q})$.

To simplify notation we now write E for $\mathcal{E}(T, \theta, \mathbb{A})$ and E_0 for

$$\operatorname{im}[\mathcal{E}(T, \theta, F) \to \mathcal{E}(T, \theta, \mathbb{A})].$$

Since we defined Φ to be 0 on the complement of $\mathcal{D}(T, \theta, \mathbb{A})$, the equality $S = S'$ implies that

$$T_e(f)_\gamma = c_T \sum_{x \in E_0} \Phi(x). \tag{6.4.7}$$

For any finite place v of F at which T is unramified we put

$$\begin{aligned}\mathcal{E}(T, \theta, \mathcal{O}_v) : &= \operatorname{im}[H^1(\mathcal{O}_v, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathcal{O}_v, T \xrightarrow{1-\theta^*} V)] \\ &= \operatorname{im}[V(\mathcal{O}_v)/\operatorname{im} T_{\rm sc}(\mathcal{O}_v) \to V(\mathcal{O}_v)/\operatorname{im} T(\mathcal{O}_v)] \\ &= H^1(\mathcal{O}_v, T \xrightarrow{1-\theta^*} V).\end{aligned}$$

Here we used Lemma C.1.A, which also implies that $\mathcal{E}(T, \theta, \mathcal{O}_v)$ is a subgroup of $\mathcal{E}(T, \theta, F_v)$.

It follows from Lemma C.1.B that $\mathcal{E}(T, \theta, \mathbb{A})$ is the restricted direct product of the groups $\mathcal{E}(T, \theta, F_v)$ with respect to the subgroups $\mathcal{E}(T, \theta, \mathcal{O}_v)$. We give each finite group $\mathcal{E}(T, \theta, F_v)$ the discrete topology and give $\mathcal{E}(T, \theta, \mathbb{A})$ the restricted direct product topology. It is easy to see that this topology agrees with the one $\mathcal{E}(T, \theta, \mathbb{A})$ inherits as a closed subgroup of

$$H^1(\mathbb{A}, T \xrightarrow{1-\theta^*} V)$$

as well as the one it inherits as a quotient of

$$H^1(\mathbb{A}, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Lemma 6.4.A. *The subgroup E_0 of E is discrete and the quotient E/E_0 is compact. The function Φ on E is locally constant and compactly supported.*

The subgroup E_0 of E is a subgroup of

$$\operatorname{im}[H^1(F, T \xrightarrow{1-\theta^*} V) \to H^1(\mathbb{A}, T \xrightarrow{1-\theta^*} V)],$$

which is discrete in

$$H^1(\mathbb{A}, T \xrightarrow{1-\theta^*} V)$$

by Lemma C.3.A; therefore E_0 is discrete in E.

The quotient E/E_0 is equal to

$$\operatorname{cok}[\operatorname{cok}^1(F, T_{\mathrm{sc}} \xrightarrow{\pi} T) \to \operatorname{cok}^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)].$$

For any complex $T \xrightarrow{f} U$ the natural surjection

$$\operatorname{cok}^1(F, T \xrightarrow{f} U) \to \operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

has compact kernel by Lemmas C.2.D and C.3.A. Therefore the compactness of E/E_0 follows from the (obvious) surjectivity of

$$\operatorname{cok}[\mathfrak{A}_{T_{\mathrm{sc}}} \xrightarrow{\pi} \mathfrak{A}_T] \to \operatorname{cok}[\mathfrak{A}_{T_{\mathrm{sc}}} \xrightarrow{(1-\theta^*)\circ\pi} \mathfrak{A}_V].$$

Now we prove the statement about Φ. Without loss of generality we may assume that $F = \mathbb{Q}$ and that f is a product

$$f = \prod_v f_v$$

of functions

$$f_v \in \begin{cases} C_c^\infty(G(\mathbb{Q}_v)) & \text{if } v \text{ is finite,} \\ C_c^\infty(G(\mathbb{R})/\mathfrak{A}_G^\theta) & \text{if } v = \infty. \end{cases}$$

Write the Tamagawa measures dg on $G(\mathbb{A})$ and dt on $I(\mathbb{A})$ as products

$$dg = \prod_v dg_v$$
$$dt = \prod_v dt_v$$

in such a way that dg_v (respectively, dt_v) gives $G(\mathcal{O}_v)$ (respectively, $I(\mathcal{O}_v)$) measure 1 for all but finitely many finite places v of $\mathbb{Q}$ (pick $\mathbb{Z}$-structures on G and I). For $\delta \in G(\mathbb{A})$ having norm γ write $\delta(v)$ for the v-component of δ. We use the F_v-isomorphism

$$I_{\delta(v)} \simeq I$$

to carry dt_v over to $I_{\delta(v)}(F_v)$. Let ω_v denote the v-component of ω. Clearly

$$O_{\delta\theta}(f) = \prod_v O_{\delta(v)\theta}(f_v)$$

where $O_{\delta(v)\theta}(f_v)$ is defined by

$$\int_{I_{\delta(v)}(F_v)\backslash G(F_v)} \omega_v(g) f_v(g^{-1}\delta(v)\theta(g))\, dg_v/dt_v.$$

Moreover

$$\langle \text{obs}(\delta), \kappa_0 \rangle = \langle \text{inv}(\delta_0, \delta), \kappa_0 \rangle$$

by Lemmas 6.3.A and 6.3.B, and the v-component of

$$\text{inv}(\delta_0, \delta) \in H^1(\mathbb{A}, T_{\text{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)$$

is equal to

$$\text{inv}(\delta_0, \delta(v)) \in H^1(F_v, T_{\text{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

The product

$$\langle \text{inv}(\delta_0, \delta(v)), \kappa_0 \rangle O_{\delta(v)\theta}(f_v) \tag{6.4.8}$$

depends only on the θ-conjugacy class of $\delta(v)$ under $G(F_v)$, and we define a function Φ_v on $\mathcal{E}(T, \theta, F_v)$ as follows. On the complement of $\mathcal{D}(T, \theta, F_v)$ we define Φ_v to be 0 and for $x \in \mathcal{D}(T, \theta, F_v)$ we define $\Phi_v(x)$ to be the product (6.4.8) for any $\delta(v) \in G(F_v)$ having norm γ and such that the image of $\text{inv}(\delta_0, \delta(v))$ in $\mathcal{E}(T, \theta, F_v)$ equals x.

Clearly

$$\Phi = \prod_v \Phi_v.$$

What we must show is that for all but finitely many finite places v of F the function Φ_v is equal to the characteristic function of $\mathcal{E}(T, \theta, \mathcal{O}_v)$.

In fact at all but finitely many finite places v of F we are in the following situation. For simplicity we temporarily let F denote a p-adic field with valuation ring $\mathcal{O}$ and residue field k. We have a connected reductive group scheme G over $\mathcal{O}$ and an automorphism θ of G over $\mathcal{O}$. We have a character ω on $G(F)$, trivial on $G(\mathcal{O})$. We have a strongly θ-regular θ-semisimple element $\delta_0 \in G(\mathcal{O})$ whose image in $G(k)$ is strongly θ-regular. Thus the centralizer in G of the $\mathcal{O}$-group scheme

$$I = \text{Cent}_\theta(\delta_0, G)$$

is a maximal torus T in G over $\mathcal{O}$ stable under

$$\theta_T := \text{Int}(\delta_0) \circ \theta$$

and

$$I = T^{\theta_T}.$$

We also have Haar measures dg, dt on $G(F)$, $I(F)$ that give measure 1 to $G(\mathcal{O})$, $I(\mathcal{O})$. Finally we have a character κ_0 on $\mathcal{E}(T, \theta, F)$, trivial on $\mathcal{E}(T, \theta, \mathcal{O})$. Let δ be an element of $G(F)$ that is θ-conjugate to δ_0 under $G(\overline{F})$. Let f denote the characteristic function of $G(\mathcal{O})$. What we must show is that

$$\langle \text{inv}(\delta_0, \delta), \kappa_0 \rangle \int_{I(F)\backslash G(F)} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt$$

is 1 if the image of $\operatorname{inv}(\delta_0, \delta)$ in $\mathcal{E}(T, \theta, F)$ belongs to $\mathcal{E}(T, \theta, \mathcal{O})$ and is 0 otherwise.

The map

$$g \mapsto g^{-1}\delta_0\theta(g)$$

from G to G induces a closed immersion

$$I\backslash G \hookrightarrow G$$

over $\mathcal{O}$, whose image X is the θ-conjugacy class of δ_0. Since X is a closed subscheme of G we have

$$G(\mathcal{O}) \cap X(F) = X(\mathcal{O}).$$

Let $\mathcal{O}^{\mathrm{un}}$ denote the valuation ring in the maximal unramified extension F^{un} of F in $\overline{F}$. Then by smoothness of $G \to I\backslash G$

$$X(\mathcal{O}^{\mathrm{un}}) = I(\mathcal{O}^{\mathrm{un}})\backslash G(\mathcal{O}^{\mathrm{un}})$$

and therefore

$$X(\mathcal{O}) = [I(\mathcal{O}^{\mathrm{un}})\backslash G(\mathcal{O}^{\mathrm{un}})]^{\mathrm{Gal}(F^{\mathrm{un}}/F)}.$$

We conclude that as $G(\mathcal{O})$-space $X(\mathcal{O})$ is isomorphic to a disjoint union of copies of $I(\mathcal{O})\backslash G(\mathcal{O})$, one copy for each element of

$$\ker[H^1(\mathcal{O}, I) \to H^1(\mathcal{O}, G)].$$

Since $H^1(\mathcal{O}, G) = \{1\}$, this kernel is

$$\begin{aligned} H^1(\mathcal{O}, I) &= H^1(\mathcal{O}, T \xrightarrow{1-\theta_T} V) \\ &= \mathcal{E}(T, \theta, \mathcal{O}). \end{aligned}$$

The statement we needed follows easily from these last remarks, and the proof of the lemma is now complete.

By Lemma 6.4.A it is legitimate to apply the Poisson summation formula to E_0, E and Φ, and thus from (6.4.7) we get

$$T_e(f)_\gamma = c_T \sum_{\xi} \int_E \Phi(e)\langle e, \xi\rangle \, de, \tag{6.4.9}$$

where the sum is taken over all characters ξ on the compact group E/E_0 and de is the unique Haar measure on E giving E/E_0 total measure 1 (for the quotient of de by the discrete measure on E_0).

Let $\mathfrak{K}(T, \theta, F)_1$ denote the kernel of the map (6.4.4)

$$\mathfrak{K}(T, \theta, F) \to H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})).$$

It follows from Lemma C.3.B that the natural pairing between $\mathfrak{K}(T, \theta, F)$ and

$$H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)$$

induces a surjection

$$\mathfrak{K}(T,\theta,F) \to \operatorname{Hom}_{\rm cont}(\operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V), \mathbb{C}^\times)$$

with kernel

$$\operatorname{Hom}(\ker^2(F, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V), \mathbb{C}^\times).$$

Let $\kappa \in \mathfrak{K}(T,\theta,F)$. Then the corresponding homomorphism

$$\operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V) \to \mathbb{C}^\times$$

factors through

$$E/E_0 = \operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V)/\operatorname{im}\operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{\pi} T)$$

if and only if its pull-back to

$$\operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{\pi} T)$$

is trivial, and this is the case if and only if the image of κ under the map (6.4.4) is trivial, since

$$\begin{aligned}\operatorname{Hom}_{\rm cont}(\operatorname{cok}^1(F, T_{\rm sc} \xrightarrow{\pi} T), \mathbb{C}^\times) &= H^1(W_F, \hat{T} \xrightarrow{\hat{\pi}} (\hat{T})_{\rm ad})/\ker^1(W_F, \hat{T} \xrightarrow{\hat{\pi}} (\hat{T})_{\rm ad}) \\ &= H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})),\end{aligned}$$

where we have written $(\hat{T})_{\rm ad}$ for $\hat{T}/Z(\hat{G})$. We conclude that $\mathfrak{K}(T,\theta,F)_1$ maps onto the Pontryagin dual of E/E_0 and that the kernel of this surjection has order

$$d_T := |\ker^2(F, T_{\rm sc} \xrightarrow{(1-\theta^*)\circ\pi} V)|.$$

It then follows that

$$T_e(f)_\gamma = c_T d_T^{-1} \sum_\kappa \int_{\mathcal{D}(T,\theta,\mathbf{A})} \langle e, \kappa\rangle O_{\delta_e\theta}(f)\, de,$$

where κ runs over the inverse image under the map (6.4.4)

$$\mathfrak{K}(T,\theta,F) \to H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G}))$$

of the element

$$\mathbf{a} \in H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})),$$

and where δ_e denotes an element of $G(\mathbb{A})$ having norm γ and such that $\operatorname{inv}(\delta_0, \delta_e) = e$.

The measure de on E depends on the global group E_0. To define twisted κ-orbital integrals we need a Haar measure on E defined purely locally. This measure, which we will refer to as the Tamagawa measure on E, is defined as follows. We have seen that at unramified finite places

$$\mathcal{E}(T, \theta, \mathcal{O}_v) = H^1(\mathcal{O}_v, T \xrightarrow{1-\theta^*} V).$$

Therefore $\mathcal{E}(T, \theta, \mathbb{A})$ is an open subgroup of

$$H^1(\mathbb{A}, T \xrightarrow{1-\theta^*} V).$$

We define the Tamagawa measure de_{Tam} on E to be the restriction of the Tamagawa measure (see (E.2)) on

$$H^1(\mathbb{A}, T \xrightarrow{1-\theta^*} V)$$

to its open subgroup $\mathcal{E}(T, \theta, \mathbb{A})$. Recall that we are using I to denote the diagonalizable group T^{θ^*}. Define a finite algebraic group A by

$$A = I/I^0.$$

According to the prescription in (E.2), the Tamagawa measure de_{Tam} is equal to

$$\prod_v de_{\mathrm{Tam}}(v),$$

where $de_{\mathrm{Tam}}(v)$ is the Haar measure on the finite group

$$\mathcal{E}(T, \theta, F_v)$$

that gives points measure $|A(F_v)|^{-1}$.

Define $\tau(\mathcal{E})$ to be the measure of the compact group E/E_0 with respect to the quotient of the Tamagawa measure on E by the discrete measure on E_0. Define $O^{\kappa}_{\delta_0\theta}(f)$, a *twisted κ-orbital integral,* by putting

$$O^{\kappa}_{\delta_0\theta}(f) = \int_{\mathcal{D}(T,\theta,\mathbb{A})} \langle e, \kappa\rangle O_{\delta_e\theta}(f)\, de_{\mathrm{Tam}},$$

with δ_e as before (of course we lift e to an element of C_0 when defining $\langle e, \kappa\rangle$ and δ_e). It is immediate that our last expression for $T_e(f)_\gamma$ is equivalent to

$$T_e(f)_\gamma = c_T d_T^{-1} \tau(\mathcal{E})^{-1} \sum_{\kappa} O^{\kappa}_{\delta_0\theta}(f), \tag{6.4.10}$$

where κ runs over the inverse image under the map (6.4.4) of the element $\mathbf{a}$. We write a_T for the constant

$$\begin{aligned} a_T :&= c_T \cdot d_T^{-1} \cdot \tau(\mathcal{E})^{-1} \\ &= c_G \cdot c_{\delta_0} \cdot |B_{\delta_0}| \cdot \tau(I_{\delta_0}) \cdot d_T^{-1} \cdot \tau(\mathcal{E})^{-1} \end{aligned}$$

appearing in (6.4.10).

Lemma 6.4.B. *There is an equality*

$$a_T = |\pi_0(Z(\hat{G})^\Gamma)| \cdot |\ker^1(F, Z(\hat{G}))|^{-1} \cdot |\pi_0(\overline{Z}^\Gamma)|^{-1} \cdot |\pi_0(Z_1 \cap (Z(\hat{G})^\Gamma)^0)|,$$

where

$$Z_1 := Z(\hat{G}) \cap (\hat{T}^\theta)^0$$

(we abbreviate $\hat{\theta}^*$ to θ) and

$$\overline{Z} := Z(\hat{G})/Z_1.$$

Note that Z_1 is independent of T; therefore a_T is independent of T, and from now on we will denote it by a_G.

To simplify notation we denote the distinguished triangle (6.4.1) by

$$[T_1 \to U_1] \to [T_2 \to U_2] \to [T_3 \to U_3] \to [T_1 \to U_1][1]$$

or even just

$$\mathbf{S}_1 \to \mathbf{S}_2 \to \mathbf{S}_3 \to \mathbf{S}_1[1],$$

and for $i = 1, 2, 3$ we put

$$\mathfrak{A}_i := \ker[\mathfrak{A}_{T_i} \to \mathfrak{A}_{U_i}],$$
$$\mathfrak{A}_i' := \operatorname{cok}[\mathfrak{A}_{T_i} \to \mathfrak{A}_{U_i}].$$

We also abbreviate $\mathcal{E}(T, \theta, F)$ and $\mathcal{E}(T, \theta, \mathbb{A})$ to $\mathcal{E}(F)$ and $\mathcal{E}(\mathbb{A})$ respectively.

Consider the double complex

$$\begin{array}{cccccccccccc}
H^0(\mathbb{A}, \mathbf{S}_1)_1 & \to & H^0(\mathbb{A}, \mathbf{S}_2)_1 & \to & H^0(\mathbb{A}, \mathbf{S}_3)_1 & \to & H^1(\mathbb{A}, \mathbf{S}_1)_1 & \to & H^1(\mathbb{A}, \mathbf{S}_2)_1 & \to & \mathcal{E}(\mathbb{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\mathbb{A}, \mathbf{S}_1) & \to & H^0(\mathbb{A}, \mathbf{S}_2) & \to & H^0(\mathbb{A}, \mathbf{S}_3) & \to & H^1(\mathbb{A}, \mathbf{S}_1) & \to & H^1(\mathbb{A}, \mathbf{S}_2) & \to & \mathcal{E}(\mathbb{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}_1 & \to & \mathfrak{A}_2 & \to & \mathfrak{A}_3 & \to & \mathfrak{A}_1' & \to & \mathfrak{A}_2' & \to & \mathfrak{A}_3'
\end{array}$$

with the Tamagawa measures on the groups in the first two rows and the canonical measures (see (E.2)) on the real vector spaces in the bottom row. We will compute the t-value of the top row using Lemma E.1.D. The t-values of the columns are all 1 (note that $\mathfrak{A}_3'$ is trivial and has the discrete measure. Therefore the alternating product of the t-values of the rows is 1. The t-value of the middle row is 1 by Lemma E.2.A. The t-value of the bottom row is

$$u^{-1} v_1^{-1} v_2 v_3^{-1} w_1 w_2^{-1} w_3 \tag{6.4.11}$$

by Lemma E.2.C (we use the notation of that lemma: u, v_i, w_i, X_i, Y_i, C_i, W_i). Therefore the t-value of the top row is equal to the inverse of (6.4.11).

We have just seen that the t-value of the bottom row of the double complex

$$\begin{array}{ccccccccccc}
H^0(F,\mathbf{S}_1) & \to & H^0(F,\mathbf{S}_2) & \to & H^0(F,\mathbf{S}_3) & \to & H^1(F,\mathbf{S}_1) & \to & H^1(F,\mathbf{S}_2) & \to & \mathcal{E}(F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\mathbb{A},\mathbf{S}_1)_1 & \to & H^0(\mathbb{A},\mathbf{S}_2)_1 & \to & H^0(\mathbb{A},\mathbf{S}_3)_1 & \to & H^1(\mathbb{A},\mathbf{S}_1)_1 & \to & H^1(\mathbb{A},\mathbf{S}_2)_1 & \to & \mathcal{E}(\mathbb{A})
\end{array}$$

is equal to the inverse of (6.4.11). We put the discrete measures on the discrete groups in the top row; since the top row is exact its t-value is 1. Applying Lemma E.1.D to the double complex above we find that

$$u^{-1}v_1^{-1}v_2v_3^{-1}w_1w_2^{-1}w_3 = (\tau_1^0)^{-1}\tau_2^0(\tau_3^0)^{-1}\tau_1^1(\tau_2^1)^{-1}\tau(\mathcal{E})(k_1^1)^{-1}k_2^1l^{-1} \tag{6.4.12}$$

where

$$\begin{aligned}
\tau_i^j &:= \tau^j(T_i \to U_i) \\
k_i^j &:= |\ker^j(F, T_i \to U_i)| \\
l &:= |\ker[\mathcal{E}(F) \to \mathcal{E}(\mathbb{A})]|.
\end{aligned}$$

Lemma E.3.E implies that

$$\tau_i^0 k_i^1(\tau_i^1 k_i^2)^{-1} = v_i w_i^{-1};$$

using this to simplify (6.4.12), we find that

$$u^{-1}v_3^{-1}w_3 = (\tau_3^0)^{-1}\tau(\mathcal{E})l^{-1}(k_1^2)^{-1}k_2^2.$$

Since $W_3 = \{1\}$, both v_3 and w_3 are 1. Trivially we have

$$\begin{aligned}
\tau_3^0 &= \tau(I_{\delta_0}) \\
k_2^2 &= d_T
\end{aligned}$$

and (see the discussion in (6.2)) the finite abelian groups B_{δ_0} and

$$\ker[\mathcal{E}(F) \to \mathcal{E}(\mathbb{A})]$$

are dual, so that

$$|B_{\delta_0}| = l.$$

We conclude that

$$|B_{\delta_0}|\tau(I_{\delta_0})d_T^{-1}\tau(\mathcal{E})^{-1} = u(k_1^2)^{-1}.$$

The next step is to evaluate u and k_1^2. By Lemma C.3.B

$$\ker^2(F, T_{\mathrm{sc}} \xrightarrow{\pi} T)$$

is dual to

$$\ker^1(W_F, Z(\hat{G})) = \ker^1(F, Z(\hat{G}));$$

therefore

$$k_1^2 = |\ker^1(F, Z(\hat{G}))|.$$

To compute u we consider the long exact sequence of hypercohomology obtained from the distinguished triangle

$$[Y_3 \to X_3] \to [Y_2 \to X_2] \to [Y_1 \to X_1] \to [Y_3 \to X_3][1]$$

(recall that we are using the notation of Lemma E.2.C). The relevant portion of this long exact sequence is

(6.4.13)

$$H^0(F, Y_1 \to X_1) \to H^1(F, Y_3 \to X_3) \to H^1(F, Y_2 \to X_2) \to H^1(F, Y_1 \to X_1).$$

Clearly

$$H^0(F, Y_1 \to X_1) = X^*(G)^\Gamma,$$

and since the map $Y_3 \to X_3$ is injective with cokernel $X^*(I)$ it is also clear that

$$H^1(F, Y_3 \to X_3) = X^*(I)^\Gamma.$$

For $i = 1, 2$ the group

$$H^1(F, Y_i \otimes \mathbb{C} \to X_i \otimes \mathbb{C}) = \mathrm{cok}[(Y_i \otimes \mathbb{C})^\Gamma \to (X_i \otimes \mathbb{C})^\Gamma]$$

is trivial (for $i = 2$ use that $T_{\mathrm{sc}}^{\theta^*}$ is anisotropic) and therefore the global analogs of the exponential sequences at the beginning of (A.3) lead to the conclusion that

$$\pi_0(\ker[\hat{U}_i \to \hat{T}_i]^\Gamma) = H^1(F, Y_i \to X_i).$$

The exact sequence (6.4.13) yields an exact sequence

$$0 \to \mathrm{cok}[X^*(G)^\Gamma \to X^*(I)^\Gamma] \to \pi_0(\ker[\hat{V} \xrightarrow{\hat{\pi}\circ\phi} (\hat{T})_{\mathrm{ad}}]^\Gamma) \to \pi_0(Z(\hat{G})^\Gamma) \to K \to 1,$$

where $(\hat{T})_{\mathrm{ad}}$ denotes $\hat{T}/Z(\hat{G})$ and K denotes

$$\mathrm{cok}[H^1(F, Y_2 \to X_2) \to H^1(F, Y_1 \to X_1)].$$

Bearing in mind that $u = |K|$, we find that

$$u = |\pi_0(Z(\hat{G})^\Gamma)| \cdot |\mathrm{cok}[X^*(G)^\Gamma \to X^*(I)^\Gamma]| \cdot |\pi_0(\ker[\hat{V} \to \hat{T}/Z(\hat{G})]^\Gamma)|^{-1}.$$

We need to determine the kernel of

$$\hat{\pi} \circ \phi : \hat{V} \to \hat{T}/Z(\hat{G}).$$

Note that

$$\hat{V} = \hat{T}/(\hat{T}^\theta)^0$$

and that the map $\hat{\pi} \circ \phi$ is induced by

$$\hat{\pi} \circ (1-\theta) : \hat{T} \to \hat{T}/Z(\hat{G}).$$

As was remarked at the end of (1.1), the group

$$(\hat{T}/Z(\hat{G}))^\theta$$

is connected, and therefore $\hat{\pi}$ induces a surjection

$$(\hat{T}^\theta)^0 \to (\hat{T}/Z(\hat{G}))^\theta$$

(reduce to the case in which $\hat{G}$ is semisimple and then use that the induced map on Lie algebras is an isomorphism). From this it follows that the kernel of $\hat{\pi} \circ (1-\theta)$ is equal to $(\hat{T}^\theta)^0 Z(\hat{G})$, and hence that the kernel of

$$\hat{V} \to \hat{T}/Z(\hat{G})$$

is equal to

$$Z(\hat{G})/(Z(\hat{G}) \cap (\hat{T}^\theta)^0) = \overline{Z}.$$

At this point we know that a_T is the product of

$$|\pi_0(Z(\hat{G})^\Gamma)| \cdot |\ker^1(F, Z(\hat{G}))|^{-1} \cdot |\pi_0(\overline{Z}^\Gamma)|^{-1}$$

and

$$c_G \cdot c_{\delta_0} \cdot |\operatorname{cok}[X^*(G)^\Gamma \to X^*(I)^\Gamma]|. \tag{6.4.14}$$

It remains to show that the constant (6.4.14) is equal to

$$|\pi_0(Z_1 \cap (Z(\hat{G})^\Gamma)^0)|.$$

For the remainder of the proof we simplify notation by writing X for $X^*(G)^\Gamma$ and Y for $X^*(I)^\Gamma$. Consider the natural map

$$X^\theta \to X_\theta$$

from θ-invariants to θ-coinvariants. Our hypothesis that

$$\mathfrak{A}_G^\theta \to (\mathfrak{A}_G)_\theta$$

is an isomorphism implies that

$$X^\theta \otimes \mathbb{R} \to X_\theta \otimes \mathbb{R}$$

is an isomorphism; therefore, since X^θ is torsion-free, the map

$$X^\theta \to X_\theta$$

is injective with finite cokernel. We now note that

$$\begin{aligned} c_G &= |\det(\theta - 1; \mathfrak{A}_G/\mathfrak{A}_G^\theta)|, \\ &= |\det(\theta - 1; X/X^\theta)| \qquad \text{(duality)} \\ &= |\operatorname{cok}[X/X^\theta \xrightarrow{1-\theta} X/X^\theta]| \\ &= |\operatorname{cok}[X^\theta \to X_\theta]|. \end{aligned}$$

Note also that the map $X \to Y$ factors through the canonical surjection $X \to X_\theta$, yielding

$$X_\theta \to Y;$$

therefore

$$|\operatorname{cok}[X^*(G)^\Gamma \to X^*(I)^\Gamma]| = |\operatorname{cok}[X_\theta \to Y]|.$$

Composing $X^\theta \to X_\theta$ and $X_\theta \to Y$, we get

$$X^\theta \to Y,$$

which is also injective. Indeed $Y \otimes \mathbb{R}$ maps to $X^*(A_G^\theta) \otimes \mathbb{R} = X_\theta \otimes \mathbb{R}$ and the composition

$$X^\theta \otimes \mathbb{R} \to Y \otimes \mathbb{R} \to X_\theta \otimes \mathbb{R}$$

is the natural isomorphism

$$X^\theta \otimes \mathbb{R} \to X_\theta \otimes \mathbb{R}.$$

Therefore

$$X^\theta \to Y/Y_{\text{tors}}$$

is also injective, and

$$\begin{aligned} c_{\delta_0} &= |Y_{\text{tors}}|^{-1} |\operatorname{cok}[\operatorname{Hom}(Y, \mathbb{Z}) \to \operatorname{Hom}(X^\theta, \mathbb{Z})]|^{-1} \\ &= |Y_{\text{tors}}|^{-1} |\operatorname{cok}[X^\theta \to Y/Y_{\text{tors}}]|^{-1} \\ &= |\operatorname{cok}[X^\theta \to Y]|^{-1}. \end{aligned}$$

It follows that the constant (6.4.14) is equal to

$$|\operatorname{cok}[X^\theta \to X_\theta]| \cdot |\operatorname{cok}[X_\theta \to Y]| \cdot |\operatorname{cok}[X^\theta \to Y]|^{-1},$$

which in turn is equal to

$$|\ker[X_\theta \to Y]|$$

(use again that $X^\theta \to Y$ is injective). Since

$$X_\theta \otimes \mathbb{R} \to Y \otimes \mathbb{R}$$

is injective, we have

$$\begin{aligned}\ker[X_\theta \to Y] &= \ker[(X_\theta)_{\text{tors}} \to Y_{\text{tors}}] \\ &= \ker[(X_\theta)_{\text{tors}} \to (X^*(T)_\theta)_{\text{tors}}] \\ &= \ker[(X^*(S)_\theta)_{\text{tors}} \to (X^*(T)_\theta)_{\text{tors}}]\end{aligned}$$

where S is the split torus whose character group is X. It is easy to see that the dual (use $\mathbb{Q}/\mathbb{Z}$) of the finite abelian group $(X^*(T)_\theta)_{\text{tors}}$ is

$$(X^*(\hat{T})_\theta)_{\text{tors}},$$

and that the dual of this group (now use $\mathbb{C}^\times$) is $\pi_0(\hat{T}^\theta)$. Therefore

$$\begin{aligned}|\ker[X_\theta \to Y]| &= |\ker[\pi_0(\hat{S}^\theta) \to \pi_0(\hat{T}^\theta)]| \\ &= |(\hat{S}^\theta \cap (\hat{T}^\theta)^0)/(\hat{S}^\theta)^0|.\end{aligned}$$

Of course $\hat{S}$ is equal to

$$(Z(\hat{G})^\Gamma)^0,$$

and since $(\hat{S}^\theta)^0$ is the connected component of the identity in

$$\hat{S}^\theta \cap (\hat{T}^\theta)^0 = Z_1 \cap (Z(\hat{G})^\Gamma)^0,$$

we conclude that

$$|\ker[X_\theta \to Y]| = |\pi_0(Z_1 \cap (Z(\hat{G})^\Gamma)^0)|.$$

This completes the proof of the lemma.

So far we have shown the following. Suppose that there exists $\delta_0 \in \Delta_1$ having γ as norm. Then

$$T_e(f)_\gamma = a_G \sum_\kappa O^\kappa_{\delta_0\theta}(f), \tag{6.4.15}$$

where a_G is the constant defined in Lemma 6.4.B. We need an analog of (6.4.15) that remains valid even when there is no element $\delta_0 \in \Delta_1$ having γ as norm (in which case $T_e(f)_\gamma$ is 0). To accomplish this we define a generalization $O^\kappa_\gamma(f)$ of $O^\kappa_{\delta_0\theta}(f)$. If there is no element $\delta \in G(\mathbb{A})$ having γ as norm we define $O^\kappa_\gamma(f)$ to be 0. Otherwise we fix an element $\delta_0 \in G(\mathbb{A})$ having γ as norm, and for each $e \in C_0$ we pick an element $\delta_e \in G(\mathbb{A})$ having norm γ and such that

$$\operatorname{inv}(\delta_0, \delta) = e.$$

As before the product

$$\langle e, \kappa\rangle O_{\delta_e\theta}(f)$$

depends only on the image of e in $\mathcal{D}(T,\theta,\mathbb{A})$ and as function on $\mathcal{D}(T,\theta,\mathbb{A})$ is locally constant with compact support (see the proof of Lemma 6.4.A). Therefore it makes sense to define $O^\kappa_\gamma(f)$ by

$$O^\kappa_\gamma(f) := \int_{\mathcal{D}(T,\theta,\mathbb{A})} \langle \mathrm{obs}(\delta_0)e, \kappa\rangle O_{\delta_e\theta}(f)\, de_{\mathrm{Tam}},$$

and by Lemma 6.3.B the integral is independent of the choice of δ_0.

We will now check that

$$T_e(f)_\gamma = a_G \sum_\kappa O^\kappa_\gamma(f), \tag{6.4.16}$$

where, as before, the sum is taken over the set of $\kappa \in \mathfrak{K}(T,\theta,F)$ mapping to $\mathbf{a}$ under (6.4.4). If the set of such κ is empty, both sides of (6.4.16) are 0. Now assume that the set of such κ is non-empty. It follows from the exactness of (6.4.3) and the injectivity of

$$H^2(W_F, \hat{T} \xrightarrow{1-\hat{\theta}_T} \hat{T}) \to H^2(W_F, \hat{V} \xrightarrow{\phi} \hat{T})$$

that $\alpha(\mathbf{a})$ and $\beta(\mathbf{a})$ are trivial (recall that α, β were defined near the beginning of (6.2)) and hence that the restriction of ω to $I(\mathbb{A})$ is trivial (see the discussion following the definition of α and β). Therefore, if there is any element of $G(F)$ having norm γ, then there is an element $\delta_0 \in \Delta_1$ having norm γ; it follows from Lemma 6.3.A that

$$O^\kappa_\gamma(f) = O^\kappa_{\delta_0\theta}(f)$$

and thus (6.4.16) is a consequence of (6.4.15).

It remains to consider the case in which there is no element of $G(F)$ having norm γ. Then $T_e(f)_\gamma = 0$ and we must show that

$$\sum_\kappa O^\kappa_\gamma(f) = 0.$$

We may as well assume that there is an element $\delta_0 \in G(\mathbb{A})$ having norm γ (otherwise $O^\kappa_\gamma(f) = 0$ for all κ). Let K denote the kernel of the map from $\mathfrak{K}(T,\theta,F)_1$ to the Pontryagin dual of E/E_0. We have seen before that K is the finite abelian group dual to

$$\ker^2(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V),$$

or in other words dual to

$$\mathrm{cok}[H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V) \to H^1(\mathbb{A}/F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V)].$$

It is enough to show that for all κ

$$\sum_{\kappa' \in K} O^{\kappa\kappa'}_\gamma(f) = 0,$$

and for this it is enough to show that

$$\sum_{\kappa' \in K} \langle \mathrm{obs}(\delta_0), \kappa' \rangle = 0.$$

Suppose this last sum is non-zero. Then $\mathrm{obs}(\delta_0)$ is the image of some element

$$x \in H^1(\mathbb{A}, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V).$$

Modifying x by an element of

$$H^1(F, T_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V),$$

we may assume that x^{-1} has trivial image in

$$H^1(\mathbb{A}, G_{\mathrm{sc}})$$

(to prove this use Lemma C.5.A together with the vanishing of $H^1(F_v, G_{\mathrm{sc}})$ for finite places v). Then there exists $\delta \in G(\mathbb{A})$ having norm γ such that

$$\mathrm{inv}(\delta_0, \delta) = x^{-1}.$$

It follows from Lemma 6.3.B that $\mathrm{obs}(\delta)$ is trivial and from Lemma 6.3.A that δ is θ-conjugate under $G_{\mathrm{sc}}(\mathbb{A})$ to an element of $G(F)$, and this contradicts our assumption that no element of $G(F)$ has norm γ.

Now we can formulate the main result of this section. Consider the set of triples (T, γ, κ) consisting of

(1) a θ^*-stable maximal F-torus T in G^* such that $T_{\mathrm{sc}}^{\theta^*}$ is anisotropic and such that there exists a θ^*-stable Borel subgroup of G^* over $\overline{F}$ containing T,
(2) an element $\gamma \in U(\overline{F})$ (recall that $U = T_{\theta^*}$) satisfying (6.3.2) and such that the stabilizer of γ in $\Omega(T, G^*)^{\theta^*}$ is trivial,
(3) an element $\kappa \in \mathfrak{K}(T, \theta, F)$ whose image under (6.4.4) is $\mathbf{a}$.

Two such triples (T, γ, κ), (T', γ', κ') are said to be equivalent if there exists an element $g \in (G_{\mathrm{sc}}^*)^{\theta^*}(\overline{F})$ such that

(1) the restriction i of $\mathrm{Int}(g)$ to T maps T to T' and is defined over F,
(2) i maps γ to γ',
(3) the isomorphism from $\mathfrak{K}(T, \theta, F)$ to $\mathfrak{K}(T', \theta, F)$ induced by i maps κ to κ'.

Summing (6.4.16) over γ, we get the following result.

Theorem 6.4.C. *There is an equality*

$$T_e(f) = a_G \sum_{(T,\gamma,\kappa)} O_\gamma^\kappa(f),$$

where the sum is taken over a set of representatives for the equivalence classes of triples (T, γ, κ) as above.

We conclude this section by noting that for a given f there are only finitely many equivalence classes of triples (T, γ, κ) for which $O_\gamma^\kappa(f)$ is non-zero. Indeed an argument of the type used in the beginning of the proof of Proposition 8.2 in [K3] shows that there are only finitely many equivalence classes of pairs (T, γ) such that there exists $\delta \in G(\mathbb{A})$ having norm γ and lying in the support of f. Therefore it is enough to fix (T, γ) and show that there are only finitely many κ such that $O_\gamma^\kappa(f)$ is non-zero, and this follows easily from the proof of Lemma 6.4.A.

7. End of the stabilization

We continue with the assumptions of the previous section. Our goal in §7 is to rewrite the expression for $T_e(f)$ given in Theorem 6.4.C in terms of stable trace formulas for endoscopic groups H associated to $(G, \theta, \mathbf{a})$ (see (7.4.4) for the final result).

(7.1) $\mathfrak{K}(T, \theta, F)$ revisited. Let T and $\mathfrak{K}(T, \theta, F)$ be as in (6.4). As usual we write U for T_{θ^*}; note that $\hat{U} = (\hat{T}^\theta)^0$. There is a canonical $\hat{G}$-conjugacy class of embeddings $\hat{T} \to \hat{G}$; the restrictions of these to $\hat{U}$ form the *canonical* $\hat{G}$-conjugacy class of embeddings $\hat{U} \to \hat{G}$.

Consider the set $\mathcal{S}$ of pairs (η, s), where $\eta : {}^L U \to {}^L G$ is an L-homomorphism whose restriction to $\hat{U}$ belongs to the canonical $\hat{G}$-conjugacy class of embeddings $\hat{U} \to \hat{G}$, and where $s \in \hat{G}$ satisfies

$$\mathrm{Int}(s) \circ {}^L\theta \circ \eta = b \cdot \eta \tag{7.1.1}$$

for some (continuous) 1-cocycle b of W_F in $Z(\hat{G})$. We say that (η, s), (η', s') are equivalent if there exists $g \in \hat{G}$ such that

(1) $\mathrm{Int}(g) \circ (c \cdot \eta) = \eta'$ for some 1-cocycle c of W_F in $\hat{U}$, and
(2) $gs\hat{\theta}(g)^{-1} = s'$ modulo $Z(\hat{G})$

($c \cdot \eta$ denotes the L-homomorphism ${}^L U \to {}^L G$ defined by

$$(c \cdot \eta)(uw) = \eta(c_w uw)$$

for all $w \in W_F$, $u \in \hat{U}$). Write $\overline{\mathcal{S}}$ for the quotient of $\mathcal{S}$ by the equivalence relation above.

There is a canonical element of $\overline{\mathcal{S}}$, obtained as follows. As in (4.4) we write $\hat{G}^1$ for the identity component of the fixed points of $\hat{\theta}$ in $\hat{G}$ and form the semidirect product

$${}^L G^1 := \hat{G}^1 \rtimes W_F$$

for the inherited action of W_F on $\hat{G}^1$; there is an obvious embedding

$${}^L G^1 \hookrightarrow {}^L G.$$

Choose an embedding $\hat{U} \to \hat{G}^1$ lying in the canonical $\hat{G}^1$-conjugacy class of such embeddings. A familiar argument based on [L1,Lemma 4] shows that $\hat{U} \to \hat{G}^1$ can be extended to an L-homomorphism

$$\eta_1 : {}^L U \to {}^L G^1,$$

which we can also view as an L-homomorphism from ${}^L U$ to ${}^L G$. Then $(\eta_1, 1)$ lies in $\mathcal{S}$ and its equivalence class is independent of all choices.

We now define a map from $\overline{\mathcal{S}}$ to $\mathfrak{K}(T, \theta, F)$. Let $x \in \overline{\mathcal{S}}$ and choose a representative $(\eta, s) \in \mathcal{S}$ for x such that η coincides with η_1 on $\hat{U}$. Use η_1 to identify $\hat{U}$ with its image in $\hat{G}$ and let $\mathcal{T}$ denote the centralizer of $\hat{U}$ in $\hat{G}$; then $\mathcal{T}$ is $\hat{\theta}$-stable and the obvious isomorphism $\hat{U} \simeq (\mathcal{T}^{\hat{\theta}})^0$ extends canonically to an isomorphism $\hat{T} \simeq \mathcal{T}$. Then η and η_1 differ by a 1-cocycle t of W_F in $\hat{T}$:

$$\eta(w) = t_w \eta_1(w) \qquad (w \in W_F)$$

(we view t_w as element of $\mathcal{T}$ in this equality). Note that $\mathrm{Int}(s)$ fixes $\hat{U}$ pointwise and hence that $s \in \mathcal{T} \simeq \hat{T}$. The equation

$$\mathrm{Int}(s) \circ {}^L\theta \circ \eta = b \cdot \eta$$

yields the equation

$$(1 - \hat{\theta})(t^{-1}) = b \cdot \partial s.$$

It follows that (t^{-1}, s) is a 1-hypercocycle of W_F in

$$\hat{T} \xrightarrow{\hat{\pi} \circ (1 - \hat{\theta}^*)} \hat{T}/Z(\hat{G}).$$

Any other such representative for x is of the form $(\mathrm{Int}(t) \circ (u \cdot \eta), tsz\hat{\theta}(t)^{-1})$ for some $t \in \mathcal{T}$, some 1-cocycle u of W_F in $\hat{U}$ and some $z \in Z(\hat{G})$. Recalling the exact sequence

$$1 \to V \to T \to U \to 1$$

and its dual

$$1 \to \hat{U} \to \hat{T} \to \hat{V} \to 1,$$

we see that the class of (t^{-1}, s) in

$$H^1(W_F, \hat{V} \xrightarrow{\hat{\pi} \circ \phi} (\hat{T})_{\mathrm{ad}}) = \mathfrak{K}(T, \theta, F)$$

is independent of the choice of representative for x.

Lemma 7.1.A. *This construction yields a bijection from $\overline{\mathcal{S}}$ to $\mathfrak{K}(T, \theta, F)$.*

Suppose that $x, x' \in \overline{\mathcal{S}}$ map to the same element of $\mathfrak{K}(T, \theta, F)$. Choose representatives (η, s), (η', s') for x,x' as above. Conjugating (η, s) by suitable $t \in \mathcal{T}$ we may assume that $s = s'$ modulo $Z(\hat{G})$ and $t = t'$ modulo $\hat{U}$. It is then clear that (η, s) and (η', s') are equivalent. The surjectivity of the map is an easy consequence of the surjectivity of

$$H^1(W_F, \hat{T} \xrightarrow{\hat{\pi} \circ (1 - \hat{\theta}^*)} (\hat{T})_{\mathrm{ad}}) \to H^1(W_F, \hat{V} \xrightarrow{\hat{\pi} \circ \phi} (\hat{T})_{\mathrm{ad}}),$$

which in turn follows from the vanishing of $H^2(W_F, \hat{U})$.

(7.2) A map $(H, \gamma_H) \mapsto (T, \gamma, \kappa)$. Let $(H, \mathcal{H}, s, \xi)$ be a set of elliptic endoscopic data for $(G, \theta, \mathbf{a})$. As in (5.4) we use the natural map

$$Z_\theta^{\mathrm{sc}} \to Z(H)$$

to view $\bar{z}_\sigma$ as a 1-cocycle in $Z(H)$. Let γ_H be a strongly G-regular element of $H(\overline{F})$ such that

$$\sigma(\gamma_H) = \gamma_H \bar{z}_\sigma \qquad (\sigma \in \Gamma).$$

The centralizer T_H of γ_H in H is a maximal F-torus in H, and we assume that γ_H is *elliptic* in the sense that $T_H/Z(H)$ is anisotropic over F.

Choose a θ^*-stable pair (B, T) in G^* with T defined over F and an admissible isomorphism $T_H \simeq T_{\theta^*}$. As before we write U for T_{θ^*}. Use this admissible isomorphism to carry γ_H over to an element $\gamma \in U(\overline{F})$ satisfying

$$\sigma(\gamma) = \gamma \bar{z}_\sigma \qquad (\sigma \in \Gamma).$$

Let $\hat{U} \simeq \hat{T}_H \to \hat{H}$ be an embedding in the canonical $\hat{H}$-conjugacy class of such embeddings, and extend it to an L-homomorphism

$$\eta_H : {}^L U = {}^L T_H \to \mathcal{H}$$

(use [L1,Lemma 4] once again to prove existence of the extension). Then $(\xi \circ \eta_H, s)$ is an element of $\mathcal{S}$; in fact (use (2.1.4a))

$$\mathrm{Int}(s) \circ {}^L\theta \circ \xi \circ \eta_H = a' \cdot (\xi \circ \eta_H)$$

for a 1-cocycle a' of W_F in $Z(\hat{G})$ that represents

$$\mathbf{a} \in H^1(W_F, Z(\hat{G}))/\ker^1(W_F, Z(\hat{G})).$$

Since η_H is well-defined up to 1-cocycles in $\hat{U}$ and conjugation under $\hat{H}$, the equivalence class of $(\xi \circ \eta_H, s)$ is well-defined, and the construction in (7.1) produces a well-defined element $\kappa \in \mathfrak{K}(T, \theta, F)$ from $(\xi \circ \eta_H, s)$. Clearly κ maps to $\mathbf{a}$.

The triple (T, γ, κ) we have just constructed is of the type considered at the end of (6.4), and its equivalence class is independent of all choices. We say that $(H, \mathcal{H}, s, \xi, \gamma_H)$ and $(H', \mathcal{H}', s', \xi', \gamma'_H)$ are equivalent if there exists an isomorphism from $(H, \mathcal{H}, s, \xi)$ to $(H', \mathcal{H}', s', \xi')$ (see (2.1)) such that the associated isomorphism $\alpha : H \to H'$ (well-defined up to inner automorphisms) carries the $H(\overline{F})$-conjugacy class of γ_H into the $H'(\overline{F})$-conjugacy class of γ'_H.

Lemma 7.2.A. *The construction described above sets up a bijection from equivalence classes of quintuples $(H, \mathcal{H}, s, \xi, \gamma_H)$ to equivalence classes of triples (T, γ, κ).*

We simply describe the inverse construction, leaving it to the reader to check that it really is the inverse. Start with (T, γ, κ). Put $U = T_{\theta^*}$ and let (η, s) be an element of $\mathcal{S}$ whose class in $\overline{\mathcal{S}}$ corresponds to κ under the bijection in Lemma 7.1.A. It follows from the discussion in (7.1) that s is $\hat{\theta}$-conjugate to an element of $\mathcal{T}$ and hence that s is $\hat{\theta}$-semisimple. Define $\hat{H}$ by

$$\hat{H} := \mathrm{Cent}_{\hat{\theta}}(s, \hat{G})^0$$

and let $\mathcal{H}$ be the subgroup of ${}^L G$ generated by $\hat{H}$ and $\eta(W_F)$; it follows from (7.1.1) that $\eta(W_F)$ normalizes $\hat{H}$ and hence that we have a split exact sequence

$$1 \to \hat{H} \to \mathcal{H} \to W_F \to 1.$$

We define $\rho_{\mathcal{H}}$ as we must, after fixing a splitting of $\hat{H}$. It is obvious that $\rho_{\mathcal{H}}$ factors through $\mathrm{Gal}(K/F)$ for any finite Galois extension K of F in $\overline{F}$ that splits T. We take H to be a quasi-split group over F with L-group $(\hat{H}, \rho_{\mathcal{H}})$. Then $(H, \mathcal{H}, s, \xi)$, where ξ is the inclusion of $\mathcal{H}$ in ${}^L G$, is a set of endoscopic data for $(G, \theta, \mathbf{a})$. By the definition of $\mathcal{H}$ the map η factors through $\mathcal{H}$, yielding

$$\eta : {}^L U \to \mathcal{H}.$$

Dual to $\hat{U} \to \hat{H}$ are embeddings $i : U \to H$, well-defined up to $H(\overline{F})$-conjugacy. For all $\sigma \in \Gamma$ $\sigma(i)$ is conjugate to i under $H(\overline{F})$. A standard argument using Steinberg's theorem on the existence of rational elements in rational conjugacy classes shows that i can be chosen so that it is defined over F. Then put $T_H = i(U)$ and $\gamma_H = i(\gamma)$. Clearly γ_H is strongly G-regular. Since $U/\operatorname{im} Z_\theta^{\mathrm{sc}}$ is anisotropic, both $(H, \mathcal{H}, s, \xi)$ and T_H are elliptic. This completes the construction of $(H, \mathcal{H}, s, \xi, \gamma_H)$ from (T, γ, κ).

(7.3) Absolute transfer factors for global γ. Let $(H, \mathcal{H}, s, \xi)$ be a set of elliptic endoscopic data for $(G, \theta, \mathbf{a})$ and let (H_1, ξ_{H_1}) be a z-pair for $\mathcal{H}$ (see (2.2)). We consider elements $\gamma \in H(\overline{F})$ satisfying

$$\sigma(\gamma) = \gamma \bar{z}_\sigma \qquad (\sigma \in \Gamma). \tag{7.3.1}$$

We assume that there exists an elliptic strongly G-regular element $\gamma_0 \in H(\overline{F})$ satisfying (7.3.1) and arising as a norm of some element $\delta_0 \in G(\mathbb{A})$ (otherwise $(H, \mathcal{H}, s, \xi)$ is not needed for the stabilization of $T_e(f)$). We fix such an element γ_0 as well as an element $\gamma_{01} \in H_1(\overline{F})$ mapping to γ_0; this lets us define a 1-cocycle

$$z_1(\sigma) := \gamma_{01}^{-1} \sigma(\gamma_{01})$$

of Γ in $Z(H_1)$ (as in (5.4)). We are then interested in elements $\gamma_1 \in H_1(\overline{F})$ or $H_1(\overline{\mathbb{A}})$ such that

$$\sigma(\gamma_1) = \gamma_1 z_1(\sigma) \qquad (\sigma \in \Gamma). \tag{7.3.2}$$

Suppose that $\gamma_1 \in H_1(\overline{\mathbb{A}})$ satisfies (7.3.2) and that its image γ in $H(\overline{\mathbb{A}})$ is strongly G-regular and elliptic and arises as a norm of an element $\delta \in G(\mathbb{A})$. Then we define the adelic relative transfer factor

$$\Delta_{\mathbf{A}}(\gamma_1, \delta; \gamma_{01}, \delta_0) := \prod_v \Delta(\gamma_1(v), \delta(v); \gamma_{01}, \delta_0(v))$$

where the product is taken over all places v of F. The local relative transfer factors were defined in (5.4). Of course $\gamma_1(v)$, $\delta(v)$, $\delta_0(v)$ denote the v-components of γ_1, δ, δ_0 respectively. The statement "γ is strongly G-regular" means that $\gamma \in H_{\text{reg}}(\overline{\mathbb{A}})$, where H_{reg} denotes the Zariski open set of strongly G-regular elements in H; this condition on γ guarantees that the product defining $\Delta_{\mathbb{A}}$ has only finitely many terms different from 1.

The construction in (7.2) produces from $(H, \mathcal{H}, s, \xi)$ and γ_0 a triple $(T_0, \gamma_0, \kappa_0)$ (well-defined up to equivalence), and from δ_0 we get $\text{obs}(\delta_0)$, which can be paired with κ_0. We define an absolute transfer factor by

$$\Delta_{\mathbb{A}}(\gamma_1, \delta) := \Delta_{\mathbb{A}}(\gamma_1, \delta; \gamma_{01}, \delta_0)\langle \text{obs}(\delta_0), \kappa_0\rangle.$$

In the next lemma we make the additional assumption that γ_1 lies in $H_1(\overline{F})$. Then the construction in (7.2) produces from γ_1 a triple (T, γ, κ), and from δ we get $\text{obs}(\delta)$, which can be paired with κ.

Lemma 7.3.A. *For $\gamma_1 \in H_1(\overline{F})$ there is an equality*

$$\Delta_{\mathbb{A}}(\gamma_1, \delta; \gamma_{01}, \delta_0) = \langle \text{obs}(\delta), \kappa\rangle \langle \text{obs}(\delta_0), \kappa_0\rangle^{-1}.$$

Recall that the definitions of the individual factors $\Delta_{\text{I}}, \dots, \Delta_{\text{IV}}$ involve auxiliary choices, although their product does not. In proving this lemma it is convenient to choose global a-data and χ-data for $R_{\text{res}}(G^*, T_0)$ and $R_{\text{res}}(G^*, T)$ as well as a (global) F-splitting of G^x (recall that G^x is the group of θ^*_{sc}-fixed points in G^*_{sc}); we then use the localizations of the global objects to define the local factors $\Delta_{\text{I}}, \dots, \Delta_{\text{IV}}$.

With these choices $\Delta_{\text{I}}(\gamma_1, \delta)$, $\Delta_{\text{II}}(\gamma_1, \delta)$ and $\Delta_{\text{IV}}(\gamma_1, \delta)$ are all 1 (and the same is true for γ_{01}, δ_0). Indeed $\Delta_{\text{I}}(\gamma_1, \delta) = \Delta_{\text{I}}(\gamma, \delta)$ is of the form

$$\langle \lambda_{\mathbb{A}}, \mathbf{s}_{T,\theta}\rangle$$

with $\mathbf{s}_{T,\theta} \in \pi_0((\widehat{T^x})^\Gamma)$ and $\lambda_{\mathbb{A}}$ equal to the image under

$$H^1(F, T^x) \to H^1(\mathbb{A}, T^x)$$

of the class

$$\lambda_{\{a_\alpha\}}(T^x) \in H^1(F, T^x)$$

constructed from the a-data for T and the splitting for G^x, and of course the value of the pairing is 1 since $\lambda_{\mathbb{A}}$ comes from an element of $H^1(F, T^x)$.

Next consider $\Delta_{\text{II}}(\gamma_1, \delta) = \Delta_{\text{II}}(\gamma, \delta)$. The factors in the numerator of $\Delta_{\text{II}}(\gamma, \delta)$ involve δ^*, which we may assume lies in $T(\overline{F})$. In our present setup the image of δ^* in $T_{\theta^*}(\overline{F})$ need not be F-rational; however its image in $(T_{\text{ad}})_{\theta^*}$ is indeed F-rational. Let $\alpha_{\text{res}} \in R_{\text{res}}(G^*, T)$ and assume that α_{res} is of type R_1 or R_2. Then the total contribution of the Γ-orbit of α_{res} to the numerator of the adelic factor $\Delta_{\text{II}}(\gamma, \delta)$ is equal to

$$\chi_{\alpha_{\text{res}}}\left(\frac{N\alpha(\delta^*) - 1}{a_{\alpha_{\text{res}}}}\right),$$

where $\chi_{\alpha_{\mathrm{res}}}$ is an idele-class quasicharacter for the field of rationality E of α_{res}. Since

$$\frac{N\alpha(\delta^*)-1}{a_{\alpha_{\mathrm{res}}}} \in E^\times,$$

the idele-class quasicharacter takes the value 1 on it. The same reasoning applies to the other factors contributing to the numerator and denominator of $\Delta_{\mathrm{II}}(\gamma,\delta)$.

It is clear from the product formula that $\Delta_{\mathrm{IV}}(\gamma,\delta)=1$. To finish the proof of the lemma we must now show that

$$\Delta_{\mathrm{III}}(\gamma_1,\delta;\gamma_{01},\delta_0)=\langle \mathrm{obs}(\delta),\kappa\rangle\langle \mathrm{obs}(\delta_0),\kappa_0\rangle^{-1}. \tag{7.3.3}$$

For the rest of this proof we will write $\bar\gamma_1$, $\bar\gamma$, $\bar\delta$, $\overline{T}$, $\bar\kappa$ for γ_{01}, γ_0, δ_0, T_0, κ_0, so that our notation will be consistent with that of (4.4). Recall from (4.4) the tori

$$S=(T\times\overline{T})/Z$$
$$W=(T_{\mathrm{sc}}\times\overline{T}_{\mathrm{sc}})/Z_{\mathrm{sc}}$$

(we switched notation from U to W since we are already using U to denote T_{θ^*}), as well as the extension S_1 of S by Z_1.

The adelic factor $\Delta_{\mathrm{III}}(\gamma_1,\delta;\gamma_{01},\delta_0)$ is equal to the product

$$\prod_v \langle \mathbf{V}_1(v),\mathbf{A}_1(v)\rangle \tag{7.3.4}$$

with $\mathbf{A}_1(v)$ and $\mathbf{V}_1(v)$ as in (4.4) and (5.4). Since we chose global χ-data, the constructions used to form $\mathbf{A}_1(v)$ can be carried out globally, yielding

$$\mathbf{A}_1\in H^1(W_F,\hat S_1\xrightarrow{1-\hat\theta}\hat W)$$

such that $\mathbf{A}_1\mapsto\mathbf{A}_1(v)$ for all v. To get a 1-hypercocycle representing $\mathbf{V}_1(v)$ we must choose elements δ^*, $\bar\delta^*$. It is now convenient to assume (as we may) that $\delta^*\in T(\overline{F})$ and $\bar\delta^*\in\overline{T}(\overline{F})$. We will use this global pair δ^*,$\bar\delta^*$ at every place v of F. Let C denote the kernel of the surjection

$$T_1\times\overline{T}_1\to S_1.$$

It is not hard to see that the constructions in (4.4) and (5.4) lead to a 1-hypercocycle of Γ in the complex

$$\frac{(T_{\mathrm{sc}}\times\overline{T}_{\mathrm{sc}})(\overline{\mathbb{A}})}{Z_{\mathrm{sc}}(\overline{F})}\xrightarrow{1-\theta}\frac{(T_1\times\overline{T}_1)(\overline{\mathbb{A}})}{C(\overline{F})}.$$

We denote the class of this 1-hypercocycle by $\mathbf{V}_1$. For any place v the image of $\mathbf{V}_1$ in

$$H^1(F_v,W\xrightarrow{1-\theta}S_1)$$

is $\mathbf{V}_1(v)$; therefore the expression (7.3.4) is equal to

$$\langle \mathbf{V}_1',\mathbf{A}_1\rangle, \tag{7.3.5}$$

where $\mathbf{V}_1'$ denotes the image of $\mathbf{V}_1$ in

$$H^1(\mathbb{A}/F, W \xrightarrow{1-\theta} S_1).$$

Note also that $\mathbf{V}_1$ can be mapped to an element

$$\mathbf{V}_1'' \in H^1(\mathbb{A}/F, T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \xrightarrow{(1-\theta_1)\circ\pi} T_1 \times \overline{T}_1)$$

and that $\mathbf{V}_1'$ is simply the image of $\mathbf{V}_1''$ under the map on hypercohomology induced by the obvious map

$$[T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \xrightarrow{1-\theta} T_1 \times \overline{T}_1] \to [W \xrightarrow{1-\theta} S_1]. \tag{7.3.6}$$

Therefore (7.3.5) is equal to

$$\langle \mathbf{V}_1'', \mathbf{A}_1'' \rangle$$

where $\mathbf{A}_1''$ denotes the image of $\mathbf{A}_1$ in

$$H^1(W_F, \hat{T}_1 \times \hat{\overline{T}}_1 \to (\hat{T})_{\mathrm{ad}} \times (\hat{\overline{T}})_{\mathrm{ad}})$$

(use the dual of (7.3.6)).

Recall that T_1 was defined to be the fiber product of T_{H_1} and T over $T_H \simeq T_{\theta^*}$. Therefore the kernel V of $T \to T_{\theta^*}$ can be regarded as a subtorus of T_1. Moreover the map

$$1-\theta_1 : T \to T_1$$

of (4.4) is simply the composition

$$T \xrightarrow{1-\theta^*} V \hookrightarrow T_1.$$

Therefore we have a natural map of complexes

$$[T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \xrightarrow{(1-\theta^*)\circ\pi} V \times \overline{V}] \to [T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \xrightarrow{(1-\theta_1)\circ\pi} T_1 \times \overline{T}_1]. \tag{7.3.7}$$

To prove the lemma it is enough to show that $(\mathrm{obs}(\delta), \mathrm{obs}(\bar{\delta})^{-1})$ maps to $\mathbf{V}_1''$ under the map induced by (7.3.7) and that $\mathbf{A}_1''$ maps to $(\kappa, \bar{\kappa})$ under the map induced by the dual of (7.3.7). As the verification of these two facts is simply an exercise in using the relevant definitions, we leave it to the reader. The proof is now complete.

Corollary 7.3.B. *The absolute transfer factor $\Delta_{\mathbf{A}}(\gamma_1, \delta)$ is independent of the choice of (γ_{01}, δ_0), and if $\gamma_1 \in H_1(\overline{F})$*

$$\Delta_{\mathbf{A}}(\gamma_1, \delta) = \langle \mathrm{obs}(\delta), \kappa \rangle.$$

This follows from Lemmas 7.3.A and 5.1.A.

We now use the adelic transfer factor $\Delta_{\mathbb{A}}(\gamma_1, \delta)$ to define a global notion of functions with matching orbital integrals. Recall that we are dealing with a function

$$f \in C_c^\infty(G(\mathbb{A})/\mathfrak{A}_G^\theta).$$

The assumption (made in (6.1)) that

$$\mathfrak{A}_G^\theta \to (\mathfrak{A}_G)_\theta$$

is an isomorphism implies that

$$\mathfrak{A}_G^\theta \to Z(G)_\theta(\mathbb{A})$$

is injective and that its image in $Z(H)(\mathbb{A})$ is equal to $\mathfrak{A}_H$ (use also that $(H, \mathcal{H}, s, \xi)$ is elliptic). Let Z_{01} denote the inverse image under

$$H_1(\mathbb{A}) \to H(\mathbb{A})$$

of $\mathfrak{A}_G^\theta$.

Let C be as in (5.1). Then by the discussion following Lemma 5.1.C there is a quasicharacter λ_C on $C(\mathbb{A})$ such that

$$\Delta(z_1\gamma_1, z\delta) = \lambda_C(z_1, z)^{-1}\Delta(\gamma_1, \delta)$$

for all $(z_1, z) \in C(\mathbb{A})$, and by Lemma 5.1.C the restriction of λ_C to the subgroup $Z_1(\mathbb{A})$ of $C(\mathbb{A})$ is equal to λ_{H_1}. The injectivity of

$$\mathfrak{A}_G^\theta \to Z(G)_\theta(\mathbb{A})$$

implies that Z_{01} can be identified with a subgroup of $C(\mathbb{A})$. We consider smooth functions f^{H_1} on $H_1(\mathbb{A})$, compactly suported modulo Z_{01}, such that

$$f^{H_1}(zh) = \lambda_C(z)^{-1} f^{H_1}(h)$$

for all $h \in H_1(\mathbb{A})$ and $z \in Z_{01}$.

We want to say what it means for f, f^{H_1} as above to have matching orbital integrals. By restriction of scalars we may assume that $F = \mathbb{Q}$, so that f can be written as a linear combination of products of local functions. Since we want our notion of matching to be linear, we may as well assume that f is a product of local functions f_v. We also want our notion of matching to be local. Write the Tamagawa measure dg on $G(\mathbb{A})$ (respectively, dh on $H(\mathbb{A})$) as a product of local measures dg_v (respectively, dh_v). Write the adelic absolute transfer factor as a product of local absolute transfer factors Δ_v; for instance one could take (for local elements γ_1, δ)

$$\Delta_v(\gamma_1, \delta) = \Delta_v(\gamma_1, \delta; \gamma_{01}, \delta_0(v))$$

at every finite place v of $\mathbb{Q}$ and

$$\Delta_\infty(\gamma_1, \delta) = \Delta_\infty(\gamma_1, \delta; \gamma_{01}, \delta_0(\infty)) \cdot \langle \mathrm{obs}(\delta_0), \kappa_0 \rangle$$

at the infinite place of $\mathbb{Q}$. Using these local measures and local transfer factors, we now have a local notion of matching functions (see (5.5)) for each place v of $\mathbb{Q}$. Suppose that f^{H_1} as above is a product of local functions $f_v^{H_1}$ and that f_v, $f_v^{H_1}$ have matching orbital integrals for all v. Then we say that f and f^{H_1} have *matching orbital integrals.*

As in (5.4) we use the elements γ_0, γ_{01} to define automorphisms

$$\theta_H := \mathrm{Int}(\gamma_0) \in \mathrm{Aut}_F(H)$$
$$\theta_{H_1} := \mathrm{Int}(\gamma_{01}) \in \mathrm{Aut}_F(H_1)$$

and work with elements $\delta_H \in H_1(F)$ rather than elements $\gamma_1 \in H_1(\overline{F})$ satisfying (7.3.2).

Suppose that f, f^{H_1} have matching orbital integrals and that $\delta_H \in H_1(F)$ is strongly G-regular (in the sense of (5.5)). Let γ denote the corresponding element of $H(\overline{F})$ satisfying (7.3.1). Use the Tamagawa measure du on $T_H(\mathbb{A})$ to define the orbital integral

$$O_{\delta_H\theta_H}(f^{H_1}) := \int_{T_H(\mathbb{A})\backslash H(\mathbb{A})} f^{H_1}(h^{-1}\delta_H\theta_H(h))\, dh/du$$

and then define its stable analog by

$$SO_{\delta_H\theta_H}(f^{H_1}) := \sum_{\delta'_H} O_{\delta'_H\theta_H}(f^{H_1}),$$

where δ'_H runs over a set of representatives for the θ_H-conjugacy classes under $H(\mathbb{A})$ of elements $\delta'_H \in H_1(\mathbb{A})$ in the θ_H-conjugacy class of δ_H under $H(\overline{\mathbb{A}})$.

Lemma 7.3.C. *There is an equality*

$$SO_{\delta_H\theta_H}(f^{H_1}) = O_\gamma^\kappa(f).$$

By linearity (and restriction of scalars from F to $\mathbb{Q}$) we may assume that f and f^{H_1} are products of local functions with matching orbital integrals. We may as well suppose that there exists $\delta \in G(\mathbb{A})$ having norm γ (otherwise both sides of the equality we are trying to prove equal 0). Write the Tamagawa measure du on $T_{\theta^*}(\mathbb{A})$ as a product of local measures du_v and let dt_v^0 denote the Haar measure on $T^{\theta^*}(F_v)$ that is compatible with du_v in the sense of (5.5). Note that the Tamagawa measure dt on $T^{\theta^*}(\mathbb{A})$ is given by

$$dt = \prod_v dt_v$$

where $dt_v = |A(F_v)|^{-1} dt_v^0$ (recall that $A := T^{\theta^*}/(T^{\theta^*})^0$).

Clearly $SO_{\delta_H\theta_H}(f^{H_1})$ is the product of $\langle \text{obs}(\delta_0), \kappa_0\rangle$ and

$$\prod_v \sum_{e\in\mathcal{D}(T,\theta,F_v)} \Delta_v(\gamma_1, \delta(v)_e; \gamma_{01}, \delta_0(v))O_{\delta(v)_e\theta}(f_v),$$

where dt_v^0 is used to form $O_{\delta(v)_e\theta}(f_v)$ and $\delta(v)_e$ denotes an element of $G(F_v)$ having norm γ and such that

$$\text{inv}(\delta(v), \delta(v)_e) = e.$$

On the other hand it follows from Corollary 7.3.B and the definition of de_{Tam} that $O_\gamma^\kappa(f)$ is the product of $\langle \text{obs}(\delta_0), \kappa_0\rangle$ and

$$\prod_v |A(F_v)|^{-1} \sum_{e\in\mathcal{D}(T,\theta,F_v)} \Delta_v(\gamma_1, \delta(v)_e; \gamma_{01}, \delta_0(v))O_{\delta(v)_e\theta}(f_v),$$

where $O_{\delta(v)_e\theta}(f_v)$ is now formed using $dt_v = |A(F_v)|^{-1}dt_v^0$. This proves the lemma.

(7.4) Final step in the stablization of $T_e(f)$. Recall from Theorem 6.4.C the equality

$$T_e(f) = a_G \sum_{(T,\gamma,\kappa)} O_\gamma^\kappa(f).$$

By Lemma 7.2.A we may rewrite this equality as

$$T_e(f) = a_G \sum_{(H,\mathcal{H},s,\xi)} \sum_\gamma O_\gamma^\kappa(f), \tag{7.4.1}$$

where the first sum is taken over a set of representatives for the isomorphism classes of elliptic endoscopic data for $(G, \theta, \mathbf{a})$ and the second sum is taken over a set of representatives for the orbits of $\text{Out}(H, \mathcal{H}, s, \xi)$ (see (2.1)) on the set of $H(\overline{F})$-conjugacy classes of elliptic strongly G-regular $\gamma \in H(\overline{F})$ satisfying (7.3.1).

Let $\alpha \in \text{Out}(H, \mathcal{H}, s, \xi)$ and suppose that α fixes the $H(\overline{F})$-conjugacy class of γ. Modifying α by an inner automorphism over $\overline{F}$ (α is now defined over $\overline{F}$, but this does not matter), we may assume that α fixes γ. Then α preserves the centralizer T_H of γ and its action coincides with that of some element of $\Omega(G^*, T)^{\theta^*}$. Since γ is strongly G-regular, we conclude that α fixes T_H pointwise and hence that α is inner. Therefore

$$T_e(f) = \sum_{(H,\mathcal{H},s,\xi)} a_G \cdot \lambda(H, \mathcal{H}, s, \xi)^{-1} \sum_\gamma O_\gamma^\kappa(f), \tag{7.4.2}$$

where

$$\lambda(H, \mathcal{H}, s, \xi) := |\,\text{Out}(H, \mathcal{H}, s, \xi)|$$

and the second sum is now taken over a set of representatives for the $H(\overline{F})$-conjugacy classes of elliptic strongly G-regular $\gamma \in H(\overline{F})$ satisfying (7.3.1).

The contribution of $(H, \mathcal{H}, s, \xi)$ to $T_e(f)$ is 0 unless there exists γ_0 as above that arises as the norm of some element $\delta_0 \in G(\mathbb{A})$; we discard all $(H, \mathcal{H}, s, \xi)$ for which no such pair (γ_0, δ_0) exists. Now consider $(H, \mathcal{H}, s, \xi)$ for which (γ_0, δ_0) exists and fix such a pair. Choose a z-pair (H_1, ξ_{H_1}) for $\mathcal{H}$. As in (7.3) choose $\gamma_{01} \in H_1(\overline{F})$ and get θ_H, θ_{H_1}, $\Delta_{\mathbb{A}}(\gamma_1, \delta)$.

We now assume that for each $(H, \mathcal{H}, s, \xi)$ as above there exists a function f^{H_1} on $H_1(\mathbb{A})$ whose orbital integrals match those of f (in the sense of (7.3)). Then, using Lemma 7.3.C, the equality (7.4.2) becomes

$$T_e(f) = \sum_{(H,\mathcal{H},s,\xi)} a_G \cdot \lambda(H, \mathcal{H}, s, \xi)^{-1} \sum_{\delta_H} SO_{\delta_H \theta_H}(f^{H_1}), \tag{7.4.3}$$

where δ_H runs through a set of representatives for the θ_H-conjugacy classes under $H(\overline{F})$ of θ_{H_1}-elliptic strongly G-regular elements $\delta_H \in H(F)$. In order to make sense of $SO_{\delta_H \theta_H}(f^{H_1})$ we must lift δ_H to an element of $H_1(F)$ (since λ_C is trivial on $Z_1(F)$ the quantity $SO_{\delta_H \theta_H}(f^{H_1})$ is independent of this lifting). Now define a rational number $\iota(G, \theta, H)$ by

$$\iota(G, \theta, H) := a_G \cdot \lambda(H, \mathcal{H}, s, \xi)^{-1} \cdot |\pi_0(Z(\hat{H})^{\Gamma}|^{-1} \cdot |\ker^1(F, Z(\hat{H}))|.$$

Then (7.4.3) is equivalent to

$$T_e(f) = \sum_{(H,\mathcal{H},s,\xi)} \iota(G, \theta, H) ST_e^{**}(f^{H_1}), \tag{7.4.4}$$

where $ST_e^{**}(f^{H_1})$ is defined by

$$|\pi_0(Z(\hat{H})^{\Gamma})||\ker^1(F, Z(\hat{H}))|^{-1} \sum_{\delta_H} SO_{\delta_H \theta_H}(f^{H_1}).$$

Of course one hopes that there is a stable θ_H-twisted trace formula for f^{H_1} of which ST_e^{**} is the θ_H-elliptic strongly G-regular part.

A. Hypercohomology of complexes of tori over local fields

(A.1) Group hypercohomology. Let G be a group. Let $A^\bullet$ be a complex of G-modules with boundary map f:

$$\cdots \to A^{-1} \xrightarrow{f} A^0 \xrightarrow{f} A^1 \to \cdots .$$

Let

$$\cdots \to P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \to \mathbb{Z} \to 0$$

be the standard resolution of the trivial G-module $\mathbb{Z}$ by free G-modules P_i. Then the *group hypercohomology* $H^\bullet(G, A^\bullet)$ is the cohomology of the complex L^r associated to the double complex

$$K^{mn} = \mathrm{Hom}_G(P_m, A^n);$$

thus

$$L^r = \bigoplus_{m+n=r} K^{mn}$$

and the differential $d : L^r \to L^{r+1}$ is given by

$$d = f + (-1)^n \partial$$

on the subgroup K^{mn}. Note that K^{mn} can be identified with $C^m(G, A^n)$, the group of m-cochains of G in A^n. We refer to elements of L^r as *r-hypercochains* of G in $A^\bullet$ and denote L^r by $C^r(G, A^\bullet)$. Of course r-hypercochains on which d is trivial will be referred to as *r-hypercocycles* .

Suppose that G is finite. Then we have the standard complete resolution

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

of G, the resolution that yields the Tate cohomology groups $\tilde{H}^r(G, A)$ $(r \in \mathbb{Z})$ for any G-module A. Using this complete resolution to form a double complex

$$K^{mn} = \mathrm{Hom}_G(P_m, A^n) \qquad (m, n \in \mathbb{Z}),$$

we again take the cohomology of the associated complex, obtaining *Tate hypercohomology groups*

$$\tilde{H}^r(G, A^\bullet)$$

for any complex $A^\bullet$ of G-modules.

Suppose that G is profinite and that $A^\bullet$ is a complex of smooth G-modules (recall that smooth means each element has open stabilizer). Now $C^m(G, A^n)$ denotes the group of continuous m-cochains of G in A^n (continuous for the discrete topology on A^n), and we will take $C^r(G, A^\bullet)$ to be

$$\bigoplus_{m+n=r} C^m(G, A^n)$$

with differential

$$d : C^r(G, A^\bullet) \to C^{r+1}(G, A^\bullet)$$

as before. We refer to elements of $C^r(G, A^\bullet)$ as continuous r-hypercochains of G in $A^\bullet$.

Suppose that F is a field and that F_{sep} is a separable algebraic closure of F. Let Γ denote the profinite group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$. Let $A^\bullet$ be a complex of smooth Γ-modules. We write $H^r(F, A^\bullet)$ and $C^r(F, A^\bullet)$ instead of $H^r(\Gamma, A^\bullet)$ and $C^r(\Gamma, A^\bullet)$ and refer to elements of $C^r(F, A^\bullet)$ as r-hypercochains, it being understood that these are required to be continuous.

In all three theories there are restriction, corestriction and inflation maps, as well as Shapiro isomorphisms, just as for group cohomology. A quasi-isomorphism from one complex to another induces an isomorphism on hypercohomology. A short exact sequence of complexes, or more generally a distinguished triangle in the derived category of (smooth) G-modules, yields a long exact sequence of hypercohomology groups.

The applications of group hypercohomology to twisted endoscopy involve complexes of length 2. Suppose that A, B are (smooth) G-modules and that $f : A \to B$ is a G-map. We write $H^r(G, A \xrightarrow{f} B)$ for the r-th hypercohomology group of $A \xrightarrow{f} B$, regarded as a complex concentrated in degrees 0 and 1. In particular

$$H^0(G, A \xrightarrow{f} B) = \ker[A^G \to B^G]$$

and $H^1(G, A \xrightarrow{f} B)$ is the quotient of the group of 1-hypercocycles by the subgroup of 1-hypercoboundaries, a 1-hypercocycle being given by a pair (a, b) with a a 1-cocycle of G in A and b an element of B such that $f(a) = \partial b$, and a 1-hypercoboundary being given by a pair of the form $(\partial a, f(a))$ for $a \in A$ (we write ∂a for the 1-cocycle $\sigma \mapsto a^{-1}\sigma(a)$ of G in A).

For complexes of length two the two spectral sequences of hypercohomology reduce to long exact sequences. The first long exact sequence is

(A.1.1)
$$\cdots \to H^r(G, A \xrightarrow{f} B) \xrightarrow{i} H^r(G, A) \xrightarrow{f} H^r(G, B) \xrightarrow{j} H^{r+1}(G, A \xrightarrow{f} B) \to \cdots,$$

where the map i is given by

$$(a, b) \mapsto a$$

for any hypercocycle $(a, b) \in C^r(G, A) \oplus C^{r-1}(G, B)$, the map f is induced by

$$f : A \to B$$

and the map j is given by

$$b \mapsto (0, b)$$

for any cocycle b in $C^r(G, B)$. The second long exact sequence is

$$\cdots \to H^r(\ker(f)) \xrightarrow{i'} H^r(A \xrightarrow{f} B) \xrightarrow{j'} H^{r-1}(\mathrm{cok}(f)) \xrightarrow{k'} H^{r+1}(\ker(f)) \to \cdots$$

where we have abbreviated $H^i(G, \cdot)$ to $H^i(\cdot)$ and where the map i' is given by

$$a \mapsto (a, 0),$$

the map j' is given by

$$(a, b) \mapsto b,$$

and the map k' is given by the composition of the boundary maps

$$H^{r-1}(G, \mathrm{cok}(f)) \to H^r(G, \mathrm{im}(f))$$

and

$$H^r(G, \mathrm{im}(f)) \to H^{r+1}(G, \ker(f)).$$

(A.2) Tate-Nakayama pairing for hypercohomology. Let F be a local field of characteristic 0. As usual write Γ for $\mathrm{Gal}(\overline{F}/F)$. Let $T^\bullet$ be a complex of F-tori with boundary map f:

$$\cdots \to T^{-1} \xrightarrow{f} T^0 \xrightarrow{f} T^1 \to \cdots .$$

We get a complex $X^\bullet$ of smooth Γ-modules by putting

$$X^m := X^*(T^{-m})$$

and taking as boundary map

$$X^m \to X^{m+1}$$

the map

$$(-1)^m f^* : X^*(T^{-m}) \to X^*(T^{-(m+1)});$$

thus the complex $X^\bullet$ looks like

$$\cdots \to X^*(T^1) \xrightarrow{-f^*} X^*(T^0) \xrightarrow{f^*} X^*(T^{-1}) \to \cdots .$$

We write $C^n(F,T^\bullet)$ and $H^n(F,T^\bullet)$ instead of $C^n(F,T^\bullet(\overline{F}))$ and $H^n(F,T^\bullet(\overline{F}))$. For a single torus T there is a cup-product pairing

$$C^n(F,T) \otimes C^{n'}(F,X^*(T)) \to C^{n+n'}(F,\mathbb{G}_m),$$

denoted by $a \otimes b \mapsto a \cdot b$. This pairing satisfies

$$\partial(a \cdot b) = (\partial a) \cdot b + (-1)^n a \cdot \partial b.$$

We now define a cup-product pairing

$$C^r(F,T^\bullet) \otimes C^s(F,X^\bullet) \to C^{r+s}(F,\mathbb{G}_m),$$

denoted $a \smile b$, as follows: by linearity it is enough to define $a \smile b$ for

$$a \in C^j(F,T^k) \qquad (j+k=r)$$

and

$$b \in C^m(F,X^n) = C^m(F,X^*(T^{-n})) \qquad (m+n=s),$$

and for such a, b we put $a \smile b = 0$ unless $-n = k$, in which case we put

$$a \smile b = (-1)^{jk} a \cdot b,$$

an element of

$$C^{r+s}(F,\mathbb{G}_m)$$

since $j+m$ is then equal to $r+s$. As before we write d for the differentials in the complexes $C^r(F,T^\bullet)$ and $C^s(F,X^\bullet)$. Then for $a \in C^r(F,T^\bullet)$ and $b \in C^s(F,X^\bullet)$ one checks that

$$\partial(a \smile b) = (da) \smile b + (-1)^r a \smile db.$$

Therefore this pairing induces a cup-product

$$H^r(F,T^\bullet)\otimes H^s(F,X^\bullet)\to H^{r+s}(F,\mathbb{G}_m).$$

Combining this with the Hasse invariant

$$H^2(F,\mathbb{G}_m)\to \mathbb{Q}/\mathbb{Z}$$

we get a Tate-Nakayama pairing

$$H^r(F,T^\bullet)\otimes H^{2-r}(F,X^\bullet)\to \mathbb{Q}/\mathbb{Z}. \tag{A.2.1}$$

Suppose that K is a finite Galois extension of F and that $T^\bullet$ is a complex of tori split by K. In the same way as above we get a Tate-Nakayama pairing

$$\hat{H}^r(K/F,T^\bullet(K))\otimes \hat{H}^{2-r}(K/F,X^\bullet)\to \mathbb{Q}/\mathbb{Z}. \tag{A.2.2}$$

If, moreover, the complex $T^\bullet$ is bounded, then this pairing induces a perfect duality of finite abelian groups for all $r\in\mathbb{Z}$. To prove this use induction on the length of the complex. For a complex of length 1 the statement reduces to Tate-Nakayama duality for tori. For the induction step consider a complex of length $n+1$, which we may as well assume is concentrated in degrees 0 through n:

$$T^0\to T^1\to\cdots\to T^n.$$

Then the complex consisting of T^0 in degree 0 is a quotient of $T^\bullet$ with kernel

$$T^1\to\cdots\to T^n.$$

The induction step now follows from the induction hypothesis together with the long exact sequence of hypercohomology coming from the short exact sequence of complexes above (as well as the analogous long exact sequence involving the hypercohomology of $X^\bullet$).

It is harder to formulate duality theorems for the pairings (A.2.1). We will limit the discussion to the case of interest in this paper: a complex $T\xrightarrow{f}U$ of length 2, concentrated in degrees 0 and 1 (of course T,U are F-tori). We write $H^r(F,T\xrightarrow{f}U)$ for the hypercohomology of the complex $T(\overline{F})\to U(\overline{F})$, again placed in degrees 0 and 1. The corresponding complex $X^\bullet$ of character groups is given by

$$X^*(U)\xrightarrow{-f^*}X^*(T),$$

placed in degrees -1 and 0. The r-th hypercohomology group of this complex is canonically isomorphic to

$$H^{r+1}(F,X^*(U)\xrightarrow{f^*}X^*(T)),$$

with

$$X^*(U) \xrightarrow{f^*} X^*(T)$$

placed in degrees 0 and 1. Thus the Tate-Nakayama pairing becomes

$$H^r(F, T \xrightarrow{f} U) \otimes H^{3-r}(F, X^*(U) \xrightarrow{f^*} X^*(T)) \to \mathbb{Q}/\mathbb{Z}. \tag{A.2.3}$$

Now suppose that F is p-adic. Then $H^r(F, A) = 0$ for $r \geq 3$ and any smooth Γ-module A. Therefore $H^r(F, A \xrightarrow{f} B) = 0$ for $r \geq 4$ and any map $f : A \to B$ of smooth G-modules (use (A.1.1)). Thus the groups being paired in (A.2.3) are both 0 if $r \geq 4$. If F is archimedean and $r = 2, 3$, then $H^r(F, T \xrightarrow{f} U)$ coincides with $\tilde{H}^r(\overline{F}/F, T(\overline{F}) \xrightarrow{f} U(\overline{F}))$, so that we already have a duality theorem in this case.

Lemma A.2.A. *If F is p-adic and $r = 2, 3$ then the pairing* (A.2.3) *induces an isomorphism*

$$H^r(F, T \xrightarrow{f} U) \to \mathrm{Hom}(H^{3-r}(F, X^*(U) \xrightarrow{f^*} X^*(T)), \mathbb{Q}/\mathbb{Z}).$$

Moreover for $r = 2, 3$ the abelian group $H^{3-r}(F, X^*(U) \xrightarrow{f^*} X^*(T))$ is finitely generated; for $r = 3$ it is free as well.

It is part of one variant of Tate-Nakayama duality for tori that

$$H^r(F, T) \to \mathrm{Hom}(H^{2-r}(F, X^*(T)), \mathbb{Q}/\mathbb{Z})$$

is an isomorphism for $r = 1, 2$. The first statement of the lemma follows from this, the 5-lemma and the exact sequences (A.1.1) for $T \to U$ and $X^*(U) \to X^*(T)$. The last statement of the lemma follows from (A.1.1) for $X^*(U) \to X^*(T)$ together with the fact that $H^i(F, X^*(T))$ and $H^i(F, X^*(U))$ are free of finite rank for $i = 0$ and finite for $i = 1$.

In (A.3) we will prove a duality theorem involving $H^r(F, T \xrightarrow{f} U)$ for $r = 0, 1$ and all F, archimedean as well as non-archimedean. For this we will need a variant of the pairing (A.2.3).

(A.3) Another pairing. Consider the commutative diagram of exponential sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^*(U) & \longrightarrow & \mathrm{Lie}(\hat{U}) & \longrightarrow & \hat{U} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X^*(T) & \longrightarrow & \mathrm{Lie}(\hat{T}) & \longrightarrow & \hat{T} & \longrightarrow & 1, \end{array}$$

where the vertical maps are dual to f. We regard this diagram as a short exact sequence of complexes of length 2, and get boundary maps on hypercohomology:

$$H^r(F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to H^{r+1}(F, X^*(U) \xrightarrow{f^*} X^*(T)).$$

Combining this with the pairing (A.2.3) and using the exponential map to embed $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{C}^\times$, we get a pairing

$$H^r(F, T \xrightarrow{f} U) \otimes H^{2-r}(F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times, \tag{A.3.1}$$

independent of the choice of square-root of -1 in $\mathbb{C}$. However the pairing (A.3.1) is not adequate for defining transfer factors, which involve elements of a variant of

$$H^1(F, \hat{U} \to \hat{T})$$

that uses the absolute Weil group W_F of F rather than the Galois group Γ. What we need are hypercohomology groups

$$H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

that take into account the topology on W_F.

The first step is to define groups $C^m(W_F, \hat{T})$ for $m \geq 0$. Write $C^m_{\text{abs}}(W_F, \hat{T})$ for the m-cochains of W_F in $\hat{T}$ with W_F regarded as abstract group. For $m = 0$ we put

$$C^0(W_F, \hat{T}) = C^0_{\text{abs}}(W_F, \hat{T}).$$

For $m = 1$ we let $C^1(W_F, \hat{T})$ be the subgroup of $C^1_{\text{abs}}(W_F, \hat{T})$ consisting of 1-cochains t_w ($w \in W_F$) such that

(1) $w \mapsto t_w$ is a continuous map $W_F \to \hat{T}$, and
(2) t_w is a 1-cocycle.

Note that

$$C^1(W_F, \hat{T}) = \varinjlim_K C^1(W_{K/F}, \hat{T}),$$

where K ranges over the finite Galois extensions of F in $\overline{F}$ that split T, and

$$C^1(W_{K/F}, \hat{T})$$

denotes the group of abstract 1-cocycles t_w whose restriction to the subgroup $K^\times$ of $W_{K/F}$ is a continuous homomorphism

$$K^\times \to \hat{T}.$$

For $m \geq 2$ we put

$$C^m(W_F, \hat{T}) = \{1\}.$$

In this way we get a complex $(C^\bullet(W_F, \hat{T}), \partial)$

$$C^0(W_F, \hat{T}) \xrightarrow{\partial} C^1(W_F, \hat{T}) \to 1 \to 1 \to 1 \to \cdots$$

where ∂ is the usual coboundary map sending $t \in \hat{T}$ to the 1-cocycle $w \mapsto t^{-1}w(t)$. We denote by $H^m(W_F, \hat{T})$ the m-th cohomology group of this complex. Trivially

$$H^m(W_F, \hat{T}) = \{1\} \qquad (m \geq 2).$$

For $m = 0$ we have

$$H^0(W_F, \hat{T}) = H^0(F, \hat{T}) = \hat{T}^\Gamma.$$

For $m = 1$ we have

$$H^1(W_F, \hat{T}) = \mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times).$$

This theorem of Langlands is contained in his unpublished paper "Representations of abelian algebraic groups" and another proof can be found in [La].

Let

$$1 \to T \to U \to V \to 1$$

be an exact sequence of tori. Then

$$1 \to \hat{V} \to \hat{U} \to \hat{T} \to 1$$

is exact, and

$$1 \to C^\bullet(W_F, \hat{V}) \to C^\bullet(W_F, \hat{U}) \to C^\bullet(W_F, \hat{T}) \to 1$$

is an exact sequence of complexes. The only non-trivial point is the surjectivity of

$$C^1(W_F, \hat{U}) \to C^1(W_F, \hat{T}),$$

which is equivalent to the surjectivity of

$$H^1(W_F, \hat{U}) \to H^1(W_F, \hat{T}),$$

and this, by the theorem of Langlands mentioned above, is equivalent to the surjectivity of

$$\mathrm{Hom}_{\mathrm{cont}}(U(F), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times),$$

a well-known fact that we will review in the proof of Lemma A.3.A.

Now let $T \xrightarrow{f} U$ be a map of F-tori. We define the group of r-hypercochains of W_F in $\hat{U} \xrightarrow{\hat{f}} \hat{T}$ by putting

$$C^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = C^r(W_F, \hat{U}) \oplus C^{r-1}(W_F, \hat{T}).$$

Of course $C^\bullet(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$ forms a complex with differential as before, and we define

$$H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

to be the r-th cohomology group of this complex. Trivially

$$H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \{1\} \qquad (r \geq 3).$$

In an analogous way we define groups

$$H^r(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

for any finite Galois extension K of F in $\overline{F}$ that splits T and U. The inflation maps

$$H^r(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

are isomorphisms.

A map of complexes

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f'} & U' \end{array}$$

from $[T \xrightarrow{f} U]$ to $[T' \xrightarrow{f'} U']$ is a quasi-isomorphism if and only if

$$1 \to T \to T' \times U \to U' \to 1$$

(with the obvious maps) is a short exact sequence. Suppose we have such a quasi-isomorphism. Then

$$1 \to \hat{U}' \to \hat{T}' \times \hat{U} \to \hat{T} \to 1$$

is exact, and therefore

$$\begin{array}{ccc} \hat{U}' & \xrightarrow{\hat{f}'} & \hat{T}' \\ \downarrow & & \downarrow \\ \hat{U} & \xrightarrow{\hat{f}} & \hat{T} \end{array}$$

is a quasi-isomorphism. Moreover

$$1 \to C^\bullet(W_F, \hat{U}') \to C^\bullet(W_F, \hat{T}' \times \hat{U}) \to C^\bullet(W_F, \hat{T}) \to 1$$

is an exact sequence of complexes, which implies that the cone on

$$C^\bullet(W_F, \hat{U}' \xrightarrow{\hat{f}'} \hat{T}') \to C^\bullet(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

is an acyclic complex. It follows that

$$H^r(W_F, \hat{U}' \xrightarrow{\hat{f}'} \hat{T}') \simeq H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

An exact sequence of complexes or, more generally, any distinguished triangle

$$[T_1 \to U_1] \to [T_2 \to U_2] \to [T_3 \to U_3] \to [T_1 \to U_1][1]$$

induces a long exact hypercohomology sequence

$$\cdots \to H^r(W_F, \hat{U}_3 \to \hat{T}_3) \to H^r(W_F, \hat{U}_2 \to \hat{T}_2) \to H^r(W_F, \hat{U}_1 \to \hat{T}_1) \to \cdots.$$

Lemma A.3.A. *There is a canonical isomorphism*

$$H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \mathrm{Hom}_{\mathrm{cont}}(H^0(F, T \xrightarrow{f} U), \mathbb{C}^\times).$$

The analogue of (A.1.1) holds for $H^r(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$. Since $H^2(W_F, \hat{U})$ is trivial, we conclude that

$$H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \mathrm{cok}[H^1(W_F, \hat{U}) \to H^1(W_F, \hat{T})]$$

and hence by Langlands's duality theorem that

$$H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \mathrm{cok}[\mathrm{Hom}_{\mathrm{cont}}(U(F), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times)].$$

Finally, every quasi-character χ on $T(F)$ that is trivial on

$$\ker[T(F) \to U(F)]$$

is the composition with

$$T(F) \to U(F)$$

of a quasi-character on $U(F)$. Indeed, we may view χ as a character on an open subgroup of $V(F)$, where $V = \mathrm{im}[T \to U]$, and we need to extend χ to $U(F)$. It can be extended to χ_V on $V(F)$ since $\mathbb{C}^\times$ is an injective abelian group. Let K_V (respectively, K_U) denote the maximal compact subgroup of $V(F)$ (respectively, $U(F)$). We can write χ_V as the product of a unitary character and a quasi-character with positive real values. The former extends to $U(F)$ by Pontryagin theory and the latter extends to $U(F)$ since

$$V(F)/K_V \to U(F)/K_U$$

is the inclusion of one real vector space in another for archimedean F and the inclusion of one discrete group in another for non-archimedean F. Similarly we see that any quasi-character on $\ker[T(F) \to U(F)]$ can be extended to a quasi-character on $T(F)$. We conclude that

$$H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \mathrm{Hom}_{\mathrm{cont}}(\ker[T(F) \to U(F)], \mathbb{C}^\times),$$

as desired, since

$$H^0(F, T \xrightarrow{f} U) = \ker[T(F) \to U(F)].$$

The case $r = 1$ is more interesting, as it brings us to the main point of this appendix, the construction of a pairing $\langle \cdot, \cdot \rangle$ between

$$H^1(F, T \xrightarrow{f} U)$$

and

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}),$$

extending the pairing (A.3.1) between $H^1(F, T \xrightarrow{f} U)$ and the subgroup $H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$ of $H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$.

In order to motivate the definition of this pairing we recall the method used by Langlands to establish the isomorphism

$$H^1(W_{K/F}, \hat{T}) = \mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times)$$

for a torus T split by a finite Galois extension K of F in $\overline{F}$. The key point is to show that the homology group $H_1(W_{K/F}, X_*(T))$ (take the homology of $W_{K/F}$ as an abstract group) is canonically isomorphic to $T(F)$; then, because $\mathbb{C}^\times$ is an injective abelian group, one has

$$\begin{aligned}\mathrm{Hom}(H_1(W_{K/F}, X_*(T)), \mathbb{C}^\times) &= H^1_{\mathrm{abs}}(W_{K/F}, \mathrm{Hom}(X_*(T), \mathbb{C}^\times)) \\ &= H^1_{\mathrm{abs}}(W_{K/F}, \hat{T}),\end{aligned}$$

where the notation H^1_{abs} indicates that we regard $W_{K/F}$ as an abstract group when taking cohomology. Finally, it is easy to see that the resulting isomorphism

$$\mathrm{Hom}(T(F), \mathbb{C}^\times) = H^1_{\mathrm{abs}}(W_{K/F}, \hat{T})$$

induces an isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times) = H^1(W_{K/F}, \hat{T}).$$

The key isomorphism

$$H_1(W_{K/F}, X_*(T)) \to T(F)$$

is obtained as follows. The subgroup $K^\times$ of $W_{K/F}$ is of finite index, so that there is a restriction map

$$\mathrm{Res} : H_1(W_{K/F}, X_*(T)) \to H_1(K^\times, X_*(T)).$$

Since $K^\times$ acts trivially on $X_*(T)$, we have

$$\begin{aligned}H_1(K^\times, X_*(T)) &= X_*(T) \otimes_{\mathbb{Z}} H_1(K^\times, \mathbb{Z}) \\ &= X_*(T) \otimes_{\mathbb{Z}} K^\times \\ &= T(K).\end{aligned}$$

Thus the restriction map can be thought of as a map

$$H_1(W_{K/F}, X_*(T)) \to T(K),$$

and one then shows that the map is injective with image $T(F)$, yielding the desired isomorphism.

To see what to do for hypercohomology we should view the isomorphism above as an isomorphism

$$H_1(W_{K/F}, X_*(T)) \to H^0(K/F, T(K)). \tag{A.3.2}$$

Moreover we have the Tate-Nakayama isomorphism

$$\hat{H}^{-1}(K/F, X_*(T)) \to H^1(K/F, T(K)).$$

The Tate group $\hat{H}^{-1}(K/F, X_*(T))$ is the subgroup of the coinvariants of $\mathrm{Gal}(K/F)$ (equivalently, of $W_{K/F}$) on $X_*(T)$ consisting of elements whose norm (from K to F) is trivial. The coinvariants are the 0-th homology group $H_0(W_{K/F}, X_*(T))$, and we denote by $H_0(W_{K/F}, X_*(T))_0$ the subgroup of elements whose norm is trivial. Then the Tate-Nakayama isomorphism can also be thought of as an isomorphism

$$H_0(W_{K/F}, X_*(T))_0 \to H^1(K/F, T(K)). \tag{A.3.3}$$

The isomorphisms (A.3.2) and (A.3.3) suggest the possibility of defining an isomorphism

$$H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))_0 \to H^1(K/F, T(K) \to U(K)) \tag{A.3.4}$$

for a suitable subgroup

$$H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))_0$$

of the 0-th hyperhomology group

$$H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))$$

of $W_{K/F}$ with coefficients in the complex

$$X_*(T) \xrightarrow{f_*} X_*(U),$$

placed in degrees 0 and 1.

We need a concrete description of this hyperhomology group. We simplify notation by writing X for $X_*(T)$ and Y for $X_*(U)$. For $m \geq 0$ we write $C_m(X)$ for the group of m-chains of $W_{K/F}$ in X, so that $H_m(W_{K/F}, X)$ is the m-th homology group of the complex

$$\cdots \to C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X).$$

We then get a double complex

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_2(X) & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & C_2(Y) & \longrightarrow & C_1(Y) & \longrightarrow & C_0(Y)
\end{array}$$

with vertical maps induced by $f_*: X \to Y$, and from this double complex we get the complex

$$\cdots \to C_1(X) \oplus C_2(Y) \xrightarrow{\alpha} C_0(X) \oplus C_1(Y) \xrightarrow{\beta} C_0(Y),$$

with α given by

$$\alpha(x_1, y_2) = (\partial x_1, f_* x_1 - \partial y_2)$$

and β given by

$$\beta(x_0, y_1) = f_* x_0 - \partial y_1.$$

Then $H_0(W_{K/F}, X \to Y)$ is the quotient

$$\ker(\beta)/\operatorname{im}(\alpha),$$

and we refer to elements of $\ker(\beta)$ as 0-hypercycles. Of course $C_0(X) = X$, and we denote by $C_0(X)_0$ the subgroup of X consisting of elements whose norm (from K to F) is trivial. Note that $C_0(X)_0$ contains the image of

$$C_1(X) \xrightarrow{\partial} C_0(X),$$

so that $(\ker(\beta))_0$ contains $\operatorname{im}(\alpha)$, where by $(\ker(\beta))_0$ we mean those pairs (x_0, y_1) in $\ker(\beta)$ such that $x_0 \in C_0(X)_0$. We define $H_0(W_{K/F}, X \to Y)_0$ to be the subgroup

$$(\ker(\beta))_0/\operatorname{im}(\alpha)$$

of $H_0(W_{K/F}, X \to Y)$.

Now we are ready to begin defining the map (A.3.4). We write $C^0(T)$ for the group $C^0(K/F, T(K))$ of 0-cochains of $\operatorname{Gal}(K/F)$ in $T(K)$ and $Z^1(T)$ for the group $Z^1(K/F, T(K))$ of 1-cocycles of $\operatorname{Gal}(K/F)$ in $T(K)$. We are going to define maps

$$\phi : C_1(X) \to C^0(T)$$
$$\psi : C_0(X)_0 \to Z^1(T)$$

making the diagram

$$\text{(A.3.5)} \qquad \begin{array}{ccccc} C_2(X) & \xrightarrow{\partial} & C_1(X) & \xrightarrow{\partial} & C_0(X)_0 \\ \downarrow & & \phi\downarrow & & \psi\downarrow \\ 0 & \longrightarrow & C^0(T) & \xrightarrow{\partial} & Z^1(T) \end{array}$$

commute. Both ϕ and ψ will be functorial in T. Viewing (A.3.5) as a map between two complexes of length 3, we see that (A.3.5) induces maps

$$H_1(W_{K/F}, X) \to H^0(K/F, T(K))$$
$$H_0(W_{K/F}, X)_0 \to H^1(K/F, T(K)),$$

and ϕ, ψ will be so defined that these two maps are the isomorphisms (A.3.2) and (A.3.3) respectively.

Let us assume for the moment that we have maps ϕ,ψ as above and see how to define (A.3.4). Consider the diagram

$$\begin{array}{ccccc} C_1(X) \oplus C_2(Y) & \xrightarrow{\alpha} & C_0(X)_0 \oplus C_1(Y) & \xrightarrow{\beta} & C_0(Y)_0 \\ \phi\oplus 0 \downarrow & & \psi\oplus\phi \downarrow & & \psi \downarrow \\ C_0(T) \oplus 0 & \xrightarrow{\gamma} & Z^1(T) \oplus C^0(U) & \xrightarrow{\delta} & Z^1(U), \end{array}$$

where α,β are as before, γ is given by

$$\gamma(t_0, 0) = (\partial t_0, f(t_0))$$

and δ is given by

$$\delta(t_1, u_0) = f(t_1) - \partial u_0.$$

Since ϕ,ψ are functorial in T and the diagram (A.3.5) commutes, the diagram above also commutes, and hence induces a map

$$\ker(\beta)/\operatorname{im}(\alpha) \to \ker(\delta)/\operatorname{im}(\gamma).$$

Since we have used $C_0(X)_0$ rather than $C_0(X)$ in this diagram we have

$$H_0(W_{K/F}, X \to Y)_0 = \ker(\beta)/\operatorname{im}(\alpha).$$

It is clear from the definitions that

$$H^1(K/F, T(K) \to U(K)) = \ker(\delta)/\operatorname{im}(\gamma).$$

Thus we have constructed the desired map

$$H_0(W_{K/F}, X \to Y)_0 \to H^1(K/F, T(K) \to U(K)).$$

Let us check that this map is an isomorphism before entering into the definition of the maps ϕ,ψ. Indeed, it is enough to apply the 5-lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H_1(X) & \to & H_1(Y) & \to & H_0(X \to Y)_0 & \to & H_0(X)_0 & \to & H_0(Y)_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(T(K)) & \to & H^0(U(K)) & \to & H^1(T(K) \to U(K)) & \to & H^1(T(K)) & \to & H^1(U(K)) \end{array}$$

where to save space we have omitted $W_{K/F}$ from the notation in the top row and K/F from the notation in the bottom row.

It remains to define ϕ, ψ. In order to define ϕ we must recall how to define the restriction map Res on group homology. Let G be a group, let X be a G-module, and let H be a subgroup of finite index in G. Choose a section

$$s : H\backslash G \to G$$

for the projection map

$$p : G \to H\backslash G$$

(in other words choose representatives for the cosets Hg of H in G). Having chosen this section, we may identify the left H-set G with a disjoint union of copies of H, then get a map of left H-sets

$$t : G \to H$$

defined by taking the identity map on each copy of H inside G; explicitly $t(g)$ ($g \in G$) is determined by the equation

$$g = t(g)s(p(g)).$$

Since the standard resolution

$$\cdots \to P_2^G \to P_1^G \to P_0^G \to \mathbb{Z} \to 0$$

of the trivial G-module $\mathbb{Z}$ is by definition the chain complex associated to the simplicial complex consisting of a single simplex whose vertices are the elements of G, and since the same is true for H, the H-map t induces a map of complexes of left H-modules from $P_\bullet^G$ to $P_\bullet^H$, the standard resolution of the trivial H-module $\mathbb{Z}$. Explicitly the map

$$P_m^G \to P_m^H$$

is given as follows: P_m^G (respectively, P_m^H) is the free $\mathbb{Z}$-module with basis G^{m+1} (respectively, H^{m+1}) and the map sends an $(m+1)$-tuple $(g_0, \dots, g_m)$ of elements of G to the $(m+1)$-tuple $(t(g_0), \dots, t(g_m))$ of elements of H.

The H-map $P_m^G \to P_m^H$ induces a map

$$P_m^G \otimes_{\mathbf{Z}[H]} X \to P_m^H \otimes_{\mathbf{Z}[H]} X.$$

In addition we have the natural map

$$P_m^G \otimes_{\mathbf{Z}[G]} X \to P_m^G \otimes_{\mathbf{Z}[H]} X$$

given by

$$y \otimes x \mapsto \sum_{g \in H\backslash G} gy \otimes gx.$$

Composing these two maps, we get a map of complexes of abelian groups

$$\mathrm{Res} : C_\bullet(G, X) \to C_\bullet(H, X),$$

and the induced map on homology is the restriction map

$$\mathrm{Res}: H_\bullet(G,X) \to H_\bullet(H,X).$$

Let us make this explicit for $m=1$. Then an element of $C_1(G,X)$ is a map $g \mapsto x_g$ such that $x_g = 0$ for all but finitely many elements $g \in G$. Tracing through our definitions, we find that the map

$$\mathrm{Res}: C_1(G,X) \to C_1(H,X)$$

above is given by

$$(g \mapsto x_g) \mapsto (h \mapsto \prod (s(g')x_g))$$

where the product is taken over the set of pairs $(g,g') \in G \times (H\backslash G)$ such that

$$s(g')g = hs(g'g).$$

Now we return to our particular situation, in which $G = W_{K/F}$ and $H = K^\times$. As above we fix a section

$$s: \mathrm{Gal}(K/F) \to W_{K/F},$$

and, as usual, this section gives us a 2-cocycle $a_{\sigma,\tau}$ of $\mathrm{Gal}(K/F)$ in $K^\times$, defined by the equation

$$s(\sigma)s(\tau) = a_{\sigma,\tau}s(\sigma\tau)$$

for $\sigma, \tau \in \mathrm{Gal}(K/F)$.

We are almost ready to define ϕ. Recall that the Langlands map is given by the composition of the map

$$\mathrm{Res}: H_1(W_{K/F},X) \to H_1(K^\times,X)$$

and the isomorphism

$$H_1(K^\times,X) \to X \otimes_{\mathbb{Z}} K^\times = T(K).$$

The latter map sends a 1-cycle $a \mapsto x_a$ of $K^\times$ in X to the element

$$\prod_{a \in K^\times} x_a(a^{-1}) \in T(K),$$

where $x_a(a^{-1})$ denotes the value of the homomorphism $x_a : \mathbb{G}_m \to T$ on the element $a^{-1} \in \mathbb{G}_m(K) = K^\times$. Therefore it is natural to define

$$\phi: C_1(W_{K/F},X) \to C^0(K/F,T(K)) = T(K)$$

as the composition of the map

$$\mathrm{Res}: C_1(W_{K/F},X) \to C_1(K^\times,X)$$

(which depends on s) and the map

$$C_1(K^\times, X) \to T(K)$$

sending a 1-chain $a \mapsto x_a$ to

$$\prod_{a \in K^\times} x_a(a^{-1}) \in T(K).$$

Elementary manipulations then show that ϕ sends a 1-chain $w \mapsto x_w$ of $W_{K/F}$ in X to the element

$$\begin{aligned}\gamma &= \prod_{w,\sigma} \sigma(x_w)(s(\sigma w)w^{-1}s(\sigma)^{-1}) \\ &= \prod_{\sigma,\tau,a} \sigma(x_{as(\tau)})(a_{\sigma,\tau}^{-1}\sigma(a)^{-1})\end{aligned}$$

of $T(K)$, where the first product is taken over

$$(w, \sigma) \in W_{K/F} \times \mathrm{Gal}(K/F)$$

and the second product is taken over

$$(\sigma, \tau, a) \in \mathrm{Gal}(K/F) \times \mathrm{Gal}(K/F) \times K^\times.$$

Since we eventually need to define ψ and show that $\partial\phi = \psi\partial$, we might as well calculate $\partial\phi$ now. In other words for $\rho \in \mathrm{Gal}(K/F)$ and $\gamma \in T(K)$ as above we must calculate $\gamma^{-1}\rho(\gamma)$. Applying ρ to the second expression for γ, making the change of variable $\sigma \to \rho^{-1}\sigma$ in the product, and combining this with the inverse of the second expression for γ, we find that

$$\gamma^{-1}\rho(\gamma) = \prod_{\sigma,\tau,a} \sigma(x_{as(\tau)})(\rho(a_{\rho^{-1}\sigma,\tau})^{-1}a_{\sigma,\tau}).$$

The 2-cocycle relation for $a_{\sigma,\tau}$ gives

$$\rho(a_{\rho^{-1}\sigma,\tau})^{-1}a_{\sigma,\tau} = a_{\rho,\rho^{-1}\sigma\tau}a^{-1}_{\rho,\rho^{-1}\sigma},$$

and thus $\gamma^{-1}\rho(\gamma)$ is the product of

$$\prod_{\sigma,\tau,a} \sigma(x_{as(\tau)})(a_{\rho,\rho^{-1}\sigma\tau})$$

and

$$\prod_{\sigma,\tau,a} \sigma(x_{as(\tau)})(a^{-1}_{\rho,\rho^{-1}\sigma}).$$

Making the change of variable $\sigma \to \rho\sigma\tau^{-1}$ in the first of these expressions, making the change of variable $\sigma \to \rho\sigma$ in the second, and then recombining them, we find that

$$\gamma^{-1}\rho(\gamma) = \prod_{\sigma,\tau,a} [\rho\sigma\tau^{-1}(x_{as(\tau)}) - \rho\sigma(x_{as(\tau)})](a_{\rho,\sigma}).$$

Recall that the boundary $\partial(x_w)$ of the 1-chain x_w is the element

$$\sum_{w \in W_{K/F}} (w^{-1}x_w - x_w).$$

Comparing this with the expression for $\gamma^{-1}\rho(\gamma)$ we find that

$$\text{(A.3.6)} \qquad \partial\phi(x_w)(\rho) = \prod_{\sigma} \rho\sigma(\partial(x_w))(a_{\rho,\sigma}).$$

Now we turn to the map ψ. Recall that the Tate-Nakayama isomorphism

$$\tilde{H}^{-1}(K/F, X) \to H^1(K/F, T(K))$$

is given by the cup-product with the fundamental class in

$$H^2(K/F, K^\times).$$

In defining ϕ we chose a section of

$$W_{K/F} \to \mathrm{Gal}(K/F)$$

and thus obtained a particular 2-cocycle $a_{\rho,\sigma}$ lying in the fundamental class. A (-1)-cochain in X is a 0-chain in X, namely an element $\mu \in X$, and its cup-product with the 2-cocycle $a_{\rho,\sigma}$ is the 1-cochain of $\mathrm{Gal}(K/F)$ given by

$$\text{(A.3.7)} \qquad \rho \mapsto \prod_{\sigma \in \mathrm{Gal}(K/F)} \rho\sigma(\mu)(a_{\rho,\sigma}).$$

If $\mu \in X$ is a (-1)-cocycle, in other words if $N_{K/F}(\mu) = 0$, then the 1-cochain (A.3.7) is a 1-cocycle. Thus it is natural to define the map

$$\psi : C_0(X)_0 \to Z^1(T)$$

as follows: an element of $C_0(X)_0$ is an element $\mu \in X$ whose norm is 0, and we define $\psi(\mu)$ to be the 1-cocycle (A.3.7).

Looking back at the equality (A.3.6), we find that $\psi\partial = \partial\phi$, as desired. Let us now verify that ϕ,ψ satisfy all the conditions we want. It is obvious that ϕ,ψ are functorial in T, and by construction ϕ and ψ induce the isomorphisms (A.3.2) and (A.3.3) respectively. All that remains to check is that $\phi\partial = 0$ (part of the commutativity of the diagram (A.3.5)), and this follows immediately from the commutativity of

$$\begin{array}{ccc} C_2(W_{K/F}, X) & \xrightarrow{\partial} & C_1(W_{K/F}, X) \\ {\scriptstyle \mathbf{Res}}\downarrow & & {\scriptstyle \mathbf{Res}}\downarrow \\ C_2(K^\times, X) & \xrightarrow{\partial} & C_1(K^\times, X) \end{array}$$

plus the fact that the map

$$C_1(K^\times, X) \to T(K)$$

that we used to define ϕ is trivial on 1-boundaries.

We have seen that ϕ,ψ have all the desired properies and that ϕ,ψ together yield an isomorphism (A.3.4). However we chose a section s of

$$W_{K/F} \to \mathrm{Gal}(K/F)$$

while constructing ϕ,ψ, and we need to check that the isomorphism (A.3.4) is independent of this choice. Let s' be another section and let ϕ',ψ' be the corresponding maps. Let b_σ be the 1-cochain of $\mathrm{Gal}(K/F)$ in $K^\times$ defined by

$$s'(\sigma) = b_\sigma s(\sigma).$$

Define a homomorphism

$$k : C_0(X)_0 \to C^0(T)$$

by sending an element $\mu \in X$ whose norm is trivial to the element

$$k(\mu) := \prod_{\sigma \in \mathrm{Gal}(K/F)} (\sigma\mu)(b_\sigma)$$

of $T(K)$. Clearly k is functorial in T, and a routine calculation shows that

$$\phi' - \phi = k\partial$$

and

$$\psi' - \psi = \partial k.$$

It then follows easily that the isomorphism (A.3.4) does not change when s is replaced by s'.

The first step in the definition of the canonical pairing between $H^1(F, T \xrightarrow{f} U)$ and $H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$ is now complete. Fortunately the remaining steps are shorter. Since $\mathbb{C}^\times$ is an injective abelian group there are canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}(H_i(W_{K/F}, X), \mathbb{C}^\times) &= H^i_{\mathrm{abs}}(W_{K/F}, \mathrm{Hom}(X, \mathbb{C}^\times) \\ &= H^i_{\mathrm{abs}}(W_{K/F}, \hat{T}), \end{aligned}$$

where the subscript abs indicates that we regard $W_{K/F}$ as an abstract group when taking group cohomology. Indeed, the injectivity of $\mathbb{C}^\times$ implies that

$$\mathrm{Hom}(H_i(W_{K/F}, X), \mathbb{C}^\times)$$

is the cohomology of the complex

$$\mathrm{Hom}(P^i \otimes_{\mathbb{Z}[W_{K/F}]} X, \mathbb{C}^\times) = \mathrm{Hom}_{\mathbb{Z}[W_{K/F}]}(P^i, \mathrm{Hom}(X, \mathbb{C}^\times)),$$

where $P^\bullet$ is the standard resolution of the trivial $W_{K/F}$-module $\mathbb{Z}$, and this complex coincides with the one whose cohomology groups are

$$H^i_{\mathrm{abs}}(W_{K/F}, \mathrm{Hom}(X, \mathbb{C}^\times)).$$

It is immediate that the analogous statement holds for hyperhomology and hypercohomology; in particular there is a canonical isomorphism

$$\text{(A.3.8)} \qquad \mathrm{Hom}(H_0(W_{K/F}, X \xrightarrow{f_*} Y), \mathbb{C}^\times) \simeq H^1_{\mathrm{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

This gives a $\mathbb{C}^\times$-valued pairing between

$$H_0(W_{K/F}, X \xrightarrow{f_*} Y)$$

and

$$H^1_{\mathrm{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

and hence a pairing between the subgroups

$$H_0(W_{K/F}, X \xrightarrow{f_*} Y)_0 \subset H_0(W_{K/F}, X \xrightarrow{f_*} Y)$$

and

$$H^1(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \subset H^1_{\mathrm{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

Combining this last pairing with the canonical isomorphism (A.3.4), we get a pairing $\langle \cdot, \cdot \rangle$ between

$$H^1(K/F, T(K) \xrightarrow{f} U(K))$$

and

$$H^1(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

To make this more explicit we need to give a formula for the pairing between

$$H_0(W_{K/F}, X \xrightarrow{f_*} Y)$$

and

$$H^1_{\mathrm{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

Consider a 0-hypercycle (x, y_w) and a 1-hypercocycle (u_w, t). Thus $x \in X$, $(y_w) \in C_1(Y)$ with

$$f_* x = \partial(y_w) = \sum_{w \in W} (w^{-1} y_w - y_w)$$

and $t \in \hat{T}$, $(u_w) \in Z^1_{\text{abs}}(W_{K/F}, \hat{U})$ with

$$\hat{f}(u_w) = (\partial t)_w = t^{-1} w(t).$$

Then the value of the pairing on the classes of these two elements is given by

$$x(t) \prod_{w \in W_{K/F}} y_w(u_w^{-1}),$$

where we now view X, Y as the character groups of $\hat{T}$,$\hat{U}$ respectively.

At this point we have a pairing

$$H^1(K/F, T(K) \xrightarrow{f} U(K)) \otimes H^1(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times \tag{A.3.9}$$

for every finite Galois extension K of F in $\overline{F}$ such that T and U split over K. Suppose that $K' \supset K$ is another such Galois extension of F. Then we also have a pairing

$$H^1(K'/F, T(K') \xrightarrow{f} U(K')) \otimes H^1(W_{K'/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times. \tag{A.3.10}$$

We need to show that the pairings (A.3.9) and (A.3.10) are compatible with the inflation maps

$$\inf : H^1(K/F, T(K) \xrightarrow{f} U(K)) \to H^1(K'/F, T(K') \xrightarrow{f} U(K'))$$

and

$$\inf : H^1(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to H^1(W_{K'/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

This compatibility is an immediate consequence of two other compatiblilities. The first is the compatibility of the pairings

$$H_0(W_{K/F}, X \xrightarrow{f_*} Y) \otimes H^1_{\text{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times$$

and

$$H_0(W_{K'/F}, X \xrightarrow{f_*} Y) \otimes H^1_{\text{abs}}(W_{K'/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times$$

with the maps

$$p_* : H_0(W_{K'/F}, X \xrightarrow{f_*} Y) \to H_0(W_{K/F}, X \xrightarrow{f_*} Y)$$

and

$$\inf : H^1_{\text{abs}}(W_{K/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to H^1_{\text{abs}}(W_{K'/F}, \hat{U} \xrightarrow{\hat{f}} \hat{T}),$$

where p_* denotes the map induced by the canonical homomorphism

$$p : W_{K'/F} \to W_{K/F}.$$

This first compatibility is obvious.

The second compatibility is the commutativity of the diagram

$$\text{(A.3.11)} \qquad \begin{array}{ccc} H_0(W_{K'/F}, X \to Y)_0 & \longrightarrow & H^1(K'/F, T(K') \to U(K')) \\ {\scriptstyle p_*}\downarrow & & \uparrow{\scriptstyle \inf} \\ H_0(W_{K/F}, X \to Y)_0 & \longrightarrow & H^1(K/F, T(K) \to U(K)) \end{array}$$

where the horizontal maps are the isomorphisms (A.3.4). We will now sketch the proof of this commutativity. Consider the commutative square

$$\begin{array}{ccc} W_{K'/F} & \longrightarrow & \mathrm{Gal}(K'/F) \\ {\scriptstyle p}\downarrow & & \downarrow \\ W_{K/F} & \longrightarrow & \mathrm{Gal}(K/F). \end{array}$$

In order to define the isomorphism (A.3.4) for K/F we need to choose a section

$$s : \mathrm{Gal}(K/F) \to W_{K/F}$$

of

$$W_{K/F} \to \mathrm{Gal}(K/F)$$

and then define maps ϕ,ψ as before, and we must do the same for K'/F by choosing

$$s' : \mathrm{Gal}(K'/F) \to W_{K'/F}$$

and defining maps ϕ',ψ' analogous to ϕ,ψ. Since (A.3.4) is independent of the choice of section we may assume that s, s' are related as follows: first choose s', then choose for each $\sigma \in \mathrm{Gal}(K/F)$ an element $\bar{\sigma} \in \mathrm{Gal}(K'/F)$ mapping to σ under the canonical surjection

$$\mathrm{Gal}(K'/F) \to \mathrm{Gal}(K/F),$$

and finally define s by

$$s(\sigma) = p(s'(\bar{\sigma})).$$

Use s and s' to obtain fundamental 2-cocycles $a_{\sigma,\tau}$ and $a'_{\sigma',\tau'}$ for K/F and K'/F respectively. Consulting the proof of Theorem 6 in Chapter 13 of [AT] we find that

$$a_{\sigma,\tau} = \prod_{\rho \in \mathrm{Gal}(K'/K)} a'_{\bar{\sigma}\rho,\bar{\tau}} a'_{\rho,\bar{\sigma}} (a'_{\rho,\overline{\sigma\tau}})^{-1}.$$

Consider the diagrams

$$\begin{array}{ccc} C_0(W_{K'/F}, X)_0 & \xrightarrow{\psi'} & Z^1(K'/F, T(K')) \\ {\scriptstyle p_*}\big\downarrow & & \big\uparrow{\scriptstyle \inf} \\ C_0(W_{K/F}, X)_0 & \xrightarrow{\psi} & Z^1(K/F, T(K)) \end{array}$$

and

$$\begin{array}{ccc} C_1(W_{K'/F}, X) & \xrightarrow{\phi'} & C^0(K'/F, T(K')) \\ {\scriptstyle p_*}\big\downarrow & & \big\uparrow{\scriptstyle \inf} \\ C_1(W_{K/F}, X) & \xrightarrow{\phi} & C^0(K/F, T(K)), \end{array}$$

where p_* in the first diagram sends $x \in X$ (with $Nx = 0$) to itself and p_* in the second diagram sends a 1-cycle $x_{w'}$ of $W_{K'/F}$ in X to the 1-cycle

$$w \mapsto \sum_{w' \in p^{-1}(w)} x_{w'}$$

of $W_{K/F}$ in X. These two diagrams do not commute. Indeed, let us define a homomorphism

$$c : C_0(W_{K'/F}, X)_0 \to C^0(K'/F, T(K')),$$

functorial in T, by the formula

$$c(\mu) = \prod_{\sigma,\rho} (\sigma\mu)(a'_{\rho,\bar{\sigma}}),$$

where σ ranges over $\mathrm{Gal}(K'/F)$ and ρ ranges over $\mathrm{Gal}(K'/K)$. Then after lengthy calculations one finds that

$$\inf \circ \phi \circ p_* = \phi' + c\partial$$
$$\inf \circ \psi \circ p_* = \psi' + \partial c.$$

It now follows easily that the diagram (A.3.11) commutes.

We are done checking that the pairings (A.3.9) are compatible with inflation. We conclude that the pairings (A.3.9) induce a pairing

(A.3.12) $$H^1(F, T \xrightarrow{f} U) \otimes H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times.$$

This is the pairing $\langle \cdot, \cdot \rangle$ that we wanted to define.

It remains to discuss some properties of the pairing, starting with its compatibility with the exact sequence (A.1.1) for $H^1(F, T \xrightarrow{f} U)$ and its analogue for $H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$. The relevant portions of these sequences are

$$\cdots \to H^0(F, U) \xrightarrow{j} H^1(F, T \to U) \xrightarrow{i} H^1(F, T) \to \cdots$$

and

$$\cdots \to H^0(W_F, \hat{T}) \xrightarrow{\hat{\jmath}} H^1(W_F, \hat{U} \to \hat{T}) \xrightarrow{\hat{\imath}} H^1(W_F, \hat{U}) \to \cdots .$$

The first compatibility is that

$$\langle j(u), \hat{z}\rangle = \langle u, \hat{\imath}(\hat{z})\rangle^{-1} \tag{A.3.13}$$

for all $u \in H^0(F, U)$ and all $\hat{z} \in H^1(W_F, \hat{U} \to \hat{T})$; the pairing on the right side of the equality is Langlands's pairing between $U(F)$ and $H^1(W_F, \hat{U})$. The second compatibility is that

$$\langle z, \hat{\jmath}(\hat{t})\rangle = \langle i(z), \hat{t}\rangle \tag{A.3.14}$$

for all $z \in H^1(F, T \xrightarrow{f} U)$ and all $\hat{t} \in H^1(W_F, \hat{T})$; the pairing on the right side of the equality is obtained from the Tate-Nakayama isomorphism

$$(X_\Gamma)_0 \simeq H^1(F, T),$$

where $(X_\Gamma)_0$ denotes the torsion elements in the coinvariants X_Γ, together with the canonical pairing

$$(X_\Gamma)_0 \otimes H^0(W_F, \hat{T})$$

obtained by viewing elements of X as rational characters on the torus $\hat{T}$.

Before stating the next property of the pairing we need to topologize the group $H^1(F, T \xrightarrow{f} U)$. Recall the exact sequence

$$\cdots \to T(F) \to U(F) \to H^1(F, T \xrightarrow{f} U) \to H^1(F, T) \to \cdots .$$

We use the unique structure of topological group on $H^1(F, T \xrightarrow{f} U)$ for which the mapping

$$U(F) \to H^1(F, T \xrightarrow{f} U)$$

is continuous and open. Thus we have an isomorphism of topological groups from $U(F)/f(T(F))$ to an open subgroup of $H^1(F, T \xrightarrow{f} U)$, and in fact this subgroup is of finite index since $H^1(F, T)$ is finite.

Our pairing yields a map

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathrm{Hom}_{\mathrm{cont}}(H^1(F, T \xrightarrow{f} U), \mathbb{C}^\times). \tag{A.3.15}$$

Lemma A.3.B. *The map* (A.3.15) *is surjective and its kernel is the image of the identity component of* $\hat{T}^\Gamma$ *under the natural map*

$$\hat{\jmath} : H^0(W_F, \hat{T}) \to H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

The lemma follows immediately from the compatibilities discussed above together with the fact that

$$H^1(W_F, \hat{T}) \to \mathrm{Hom}_{\mathrm{cont}}(H^0(F, T), \mathbb{C}^\times)$$

is an isomorphism and

$$H^0(W_F, \hat{T}) \to \mathrm{Hom}(H^1(F,T), \mathbb{C}^\times)$$

is surjective with kernel equal to the identity component of $\hat{T}^\Gamma$ (use these facts for U as well as T).

The lemma suggests the introduction of the quotient group

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})/\,\mathrm{im}(\hat{T}^\Gamma)^0;$$

there is then an induced isomorphism

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \simeq \mathrm{Hom}_{\mathrm{cont}}(H^1(F, T \xrightarrow{f} U), \mathbb{C}^\times). \tag{A.3.16}$$

B. Inner twists of (G^*, θ^*)

Let F be a field of characteristic 0, $\overline{F}$ an algebraic closure of F, and write Γ for $\mathrm{Gal}(\overline{F}/F)$. Let G^* be a connected reductive group over F. Recall that an inner twist of G^* is a pair (G, ψ) consisting of a connected reductive group G over F and an $\overline{F}$-isomorphism $\psi : G \to G^*$ such that for all $\sigma \in \Gamma$ the isomorphisms $\sigma(\psi), \psi$ differ by an inner automorphism of G^* over $\overline{F}$; such an isomorphism ψ is referred to as an inner twisting. Two inner twists (G, ψ) and (G', ψ') are isomorphic if there exists an F-isomorphism $\alpha : G \to G'$ such that $\psi'\alpha$ and ψ differ by an inner automorphism of G^* over $\overline{F}$. Given an inner twist (G, ψ) we get a 1-cocycle $\sigma \mapsto \psi\sigma(\psi)^{-1}$ of Γ in $G^*_{\mathrm{ad}}(\overline{F})$, and this construction yields a bijection from the set of isomorphism classes of inner twists of G^* to the set $H^1(F, G^*_{\mathrm{ad}})$.

Consider the exact sequence

$$1 \to Z \to G^*_{\mathrm{sc}} \to G^*_{\mathrm{ad}} \to 1$$

where G^*_{sc} denotes the simply connected covering group of G^*_{ad} and Z denotes its center. Then we have the boundary map

$$H^1(F, G^*_{\mathrm{ad}}) \to H^2(F, Z).$$

For a p-adic field F this map is bijective, and thus in this case inner twists of G^* are classified by elements of $H^2(F, Z)$.

We are going to generalize the well-known results reviewed above. Now let θ^* be an F-automorphism of G^*. By an inner twist of (G^*, θ^*) we mean a triple (G, θ, ψ) consisting of a connected reductive group G over F, an F-automorphism θ of G, and an $\overline{F}$-isomorphism $\psi : G \to G^*$ such that

1. for all $\sigma \in \Gamma$ the isomorphisms $\sigma(\psi), \psi$ differ by an inner automorphism of G^* over $\overline{F}$, and
2. θ^* and $\psi\theta\psi^{-1}$ differ by an inner automorphism of G^* over $\overline{F}$.

We say that two inner twists (G,θ,ψ),(G',θ',ψ') are isomorphic if there exists an F-isomorphism $\alpha : G \to G'$ such that

(1) $\psi'\alpha$ and ψ differ by an inner automorphism of G^* over $\overline{F}$, and
(2) $\alpha^{-1}\theta'\alpha$ and θ differ by an inner automorphism induced by an element of $G(F)$.

Let Z denote the center of G^*. For each $\sigma \in \Gamma$ choose $u_\sigma \in G^*(\overline{F})$ such that $\psi\sigma(\psi)^{-1} = \mathrm{Int}(u_\sigma)$. Then $\sigma \mapsto \mathrm{Int}(u_\sigma)$ is a 1-cocycle of Γ in $G^*(\overline{F})/Z(\overline{F})$ and the coboundary $c_{\rho,\tau} := u_\rho\rho(u_\tau)u_{\rho\tau}^{-1}$ of u_σ is a 2-cocycle of Γ in $Z(\overline{F})$.

Choose $g_\theta \in G^*(\overline{F})$ such that

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}.$$

Applying $\sigma \in \Gamma$ to this equation, we find that

$$\mathrm{Int}(\sigma(g_\theta)u_\sigma^{-1}g_\theta^{-1}\theta^*(u_\sigma))$$

is trivial and hence that

$$z_\sigma := g_\theta u_\sigma \sigma(g_\theta)^{-1}\theta^*(u_\sigma^{-1})$$

belongs to $Z(\overline{F})$. It is not hard to verify that the coboundary of the 1-cochain z_σ is given by

$$\partial(z_\sigma) = (1-\theta^*)(c_{\rho,\tau}).$$

Thus $(c_{\rho,\tau}, z_\sigma)$ is a 2-hypercocycle of Γ in the complex

$$Z \xrightarrow{1-\theta^*} Z,$$

and it is easy to see that the class of this 2-hypercocycle in

$$H^2(F, Z \xrightarrow{1-\theta^*} Z)$$

is independent of the choices of u_σ and g_θ. It is also easy to see that if (G,θ,ψ), (G',θ',ψ') are isomorphic, then the associated elements of $H^2(F, Z \xrightarrow{1-\theta^*} Z)$ are equal.

Lemma B.1. *Suppose that G^* is semisimple and simply connected and that F is p-adic. Then the construction above sets up a bijection from the set of isomorphism classes of inner twists of (G^*,θ^*) to the set*

$$H^2(F, Z \xrightarrow{1-\theta^*} Z).$$

The proof involves routine but lengthy cocycle calculations, which we leave to the reader, and uses the following facts about Galois cohomology over p-adic fields: for every inner form G of the simply connected group G^* the set $H^1(F,G)$ is trivial and the natural maps

$$G_{\mathrm{ad}}(F)/\operatorname{im} G(F) \to H^1(F,Z)$$

and

$$H^1(F,G_{\mathrm{ad}}) \to H^2(F,Z)$$

are bijective.

C. Hypercohomology of complexes of tori over number fields

(C.1) Basic definitions. Let F be a number field. As usual write $\overline{F}$ for an algebraic closure of F and Γ for $\mathrm{Gal}(\overline{F}/F)$. For any finite extension K of F in $\overline{F}$ we write $\mathbb{A}_K$ for the adele ring of K; in case $K = F$ we write simply $\mathbb{A}$. We write $\overline{\mathbb{A}}$ for the direct limit of the rings $\mathbb{A}_K$ (the limit being taken over the directed set of finite extensions K of F in $\overline{F}$). Of course $\overline{\mathbb{A}}$ is a smooth Γ-module and $\mathbb{A}_K$ can be identified with the fixed points of $\mathrm{Gal}(\overline{F}/K)$ in $\overline{\mathbb{A}}$.

In this appendix we study global duality theorems for complexes $T \xrightarrow{f} U$ of length 2 located in degrees 0 and 1, where T,U are F-tori and f is a homomorphism defined over F. We start by defining groups

$$H^i(F, T \xrightarrow{f} U) := H^i(F, T(\overline{F}) \xrightarrow{f} U(\overline{F})),$$
$$H^i(\mathbb{A}, T \xrightarrow{f} U) := H^i(F, T(\overline{\mathbb{A}}) \xrightarrow{f} U(\overline{\mathbb{A}})),$$
$$H^i(\mathbb{A}/F, T \xrightarrow{f} U) := H^i(F, T(\overline{\mathbb{A}})/T(\overline{F}) \xrightarrow{f} U(\overline{\mathbb{A}})/U(\overline{F})),$$

where, as in Appendix A, we denote by $H^i(F, A^\bullet)$ the hypercohomology of Γ with coefficients in a complex $A^\bullet$ of smooth Γ-modules. There is an obvious long exact sequence

$$\text{(C.1.1)} \qquad \cdots \to H^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}, T \xrightarrow{f} U) \to H^i(\mathbb{A}/F, T \xrightarrow{f} U) \to \cdots$$

As usual $H^i(\mathbb{A}, T \xrightarrow{f} U)$ can be expressed as a restricted direct product of local groups. To accomplish this we begin by letting S be a finite set of places of F containing all infinite places and all finite places at which either T or U is ramified. For a finite place v of F we write $\mathcal{O}_v$ for the valuation ring of F_v and k_v for its residue field. For $v \notin S$ the tori T, U and the homomorphism f extend naturally to $\mathcal{O}_v$; for every finite extension K_w of F_v the group $T(\mathcal{O}_w)$ can be identified with the maximal compact subgroup of $T(K_w)$, and the analogous statement holds for U. For every place v of F we choose an algebraic closure $\overline{F}_v$ of F_v and an embedding $\overline{F} \to \overline{F}_v$, and for finite places v we write F_v^{un} for the maximal unramified extension of F_v in $\overline{F}_v$ and $\mathcal{O}_v^{\mathrm{un}}$ for the valuation ring of F_v^{un}. For any place v of F we put

$$H^i(F_v, T \xrightarrow{f} U) := H^i(F_v, T(\overline{F}_v) \xrightarrow{f} U(\overline{F}_v))$$

and for $v \notin S$ we put

$$H^i(\mathcal{O}_v, T \xrightarrow{f} U) := H^i(\mathrm{Gal}(F_v^{\mathrm{un}}/F_v), T(\mathcal{O}_v^{\mathrm{un}}) \xrightarrow{f} U(\mathcal{O}_v^{\mathrm{un}})).$$

Lemma C.1.A. *Consider a place $v \notin S$. Then the group*

$$H^i(\mathcal{O}_v, T \xrightarrow{f} U)$$

is given by

$$\begin{cases} \ker[T(\mathcal{O}_v) \xrightarrow{f} U(\mathcal{O}_v)] & i = 0 \\ \operatorname{cok}[T(\mathcal{O}_v) \xrightarrow{f} U(\mathcal{O}_v)] & i = 1 \\ \{1\} & i \geq 2. \end{cases}$$

Moreover the natural map

$$H^i(\mathcal{O}_v, T \xrightarrow{f} U) \to H^i(F_v, T \xrightarrow{f} U)$$

is injective for all i.

We begin by proving the first statement. It is enough to show that for any finite extension K_w of F_v in F_v^{un}

$$H^i(K_w/F_v, T(\mathcal{O}_w) \xrightarrow{f} U(\mathcal{O}_w))$$

is given by

$$\begin{cases} \ker[T(\mathcal{O}_v) \xrightarrow{f} U(\mathcal{O}_v)] & i = 0 \\ \operatorname{cok}[T(\mathcal{O}_v) \xrightarrow{f} U(\mathcal{O}_v)] & i = 1 \\ \{1\} & i \geq 2. \end{cases}$$

Using the long exact sequence (A.1.1) for the complex

$$T(\mathcal{O}_w) \xrightarrow{f} U(\mathcal{O}_w),$$

we see that it is enough to check that

$$\tilde{H}^i(K_w/F_v, T(\mathcal{O}_w)) = \{1\}$$

for all i (and the same for U). Using the obvious filtration on $T(\mathcal{O}_w)$, we reduce to proving that

$$\tilde{H}^i(k_w/k_v, T(k_w)) = \{1\}$$
$$\tilde{H}^i(k_w/k_v, (\operatorname{Lie} T)(k_w)) = \{1\}$$

for all i. By periodicity it is enough to prove this for $i = 0, 1$, and since in both cases the Herbrand quotient is trivial (the Galois modules being finite) it is enough to consider the case $i = 1$, which can be handled by Lang's Theorem.

The second statement of the lemma follows from the first except for $i = 1$, in which case we must show that

$$\operatorname{cok}[T(\mathcal{O}_v) \xrightarrow{f} U(\mathcal{O}_v)] \to \operatorname{cok}[T(F_v) \xrightarrow{f} U(F_v)]$$

is injective. In other words we must show that any element

$$u \in U(\mathcal{O}_v) \cap f(T(F_v))$$

lies in $f(T(\mathcal{O}_v))$. Put

$$C = \ker[T \to U]$$
$$V = \operatorname{im}[T \to U].$$

Clearly $u \in V(\mathcal{O}_v)$. Consider the natural isogeny

$$\bar{f} : T/C^0 \to V$$

induced by f. By our hypothesis on u there exists $x \in (T/C^0)(F_v)$ such that $\bar{f}(x) = u$; since x belongs to the compact subgroup $\bar{f}^{-1}(V(\mathcal{O}_v))$ it belongs to $(T/C^0)(\mathcal{O}_v)$. It is enough to show that x lies in the image of $T(\mathcal{O}_v)$; but this is clear since

$$T(\mathcal{O}_v) \to (T/C^0)(\mathcal{O}_v)$$

is in fact surjective (use smoothness of the map $T \to T/C^0$ of schemes over $\mathcal{O}_v$ plus surjectivity of $T(k_v) \to (T/C^0)(k_v)$, a consequence of Lang's Theorem for the connected group C^0).

Lemma C.1.B. *The group $H^i(\mathbb{A}, T \xrightarrow{f} U)$ is canonically isomorphic to the restricted direct product over all places v of F of the groups $H^i(F_v, T \xrightarrow{f} U)$, the restriction being with respect to the subgroups $H^i(\mathcal{O}_v, T \xrightarrow{f} U)$ for $v \notin S$. For $i \geq 2$ the restricted direct product is in fact a direct sum.*

Let K be a finite Galois extension of F in $\overline{F}$ that splits both T and U, and let S_K denote the union of the set of infinite places of F and the set of finite places of F that ramify in K. For each place v of F we choose a place w of K lying over v. The first step is to show that

$$H^i(K/F, T(\mathbb{A}_K) \xrightarrow{f} U(\mathbb{A}_K))$$

is the restricted direct product of the groups

$$H^i(K_w/F_v, T(K_w) \xrightarrow{f} U(K_w))$$

with respect to the subgroups

$$H^i(K_w/F_v, T(\mathcal{O}_w) \xrightarrow{f} U(\mathcal{O}_w)) \qquad (v \notin S_K)$$

(the proof of Lemma C.1.A shows that they are indeed subgroups). This is easy: use that taking hypercohomology for the finite group $\operatorname{Gal}(K/F)$ commutes with direct limits and products and then apply Shapiro's Lemma. The second step is to take the direct limit over K. This too is easy, but one must keep in mind that for any K and any $v \notin S_K$

$$H^i(K_w/F_v, T(\mathcal{O}_w) \xrightarrow{f} U(\mathcal{O}_w)) = H^i(\mathcal{O}_v, T \xrightarrow{f} U)$$

(a consequence of the proof of Lemma C.1.A). The second statement of the lemma follows immediately from the vanishing statement for $i \geq 2$ given in Lemma C.1.A.

Our next task is to topologize our hypercohomology groups. First consider the local case. Give

$$H^0(F_v, T \xrightarrow{f} U) = \ker[T(F_v) \xrightarrow{f} U(F_v)]$$

the topology it inherits as a closed subgroup of $T(F_v)$. Give

$$H^1(F_v, T \xrightarrow{f} U)$$

the unique structure of topological group for which the canonical injection

$$\operatorname{cok}[T(F_v) \xrightarrow{f} U(F_v)] \to H^1(F_v, T \xrightarrow{f} U)$$

is continuous and open (as we did in Appendix A). For $i \geq 2$ give the group

$$H^i(F_v, T \xrightarrow{f} U)$$

the discrete topology. Note that in the unramified situation considered earlier the subgroup $H^i(\mathcal{O}_v, T \xrightarrow{f} U)$ is compact and open in $H^i(F_v, T \xrightarrow{f} U)$ for all i.

Now consider the global case. We put the discrete topology on $H^i(F, T \xrightarrow{f} U)$ for all i. Give

$$H^0(\mathbb{A}, T \xrightarrow{f} U) = \ker[T(\mathbb{A}) \xrightarrow{f} U(\mathbb{A})]$$

the topology it inherits as a closed subgroup of $T(\mathbb{A})$. Give

$$H^1(\mathbb{A}, T \xrightarrow{f} U)$$

the unique structure of topological group for which the canonical injection

$$\operatorname{cok}[T(\mathbb{A}) \xrightarrow{f} U(\mathbb{A})] \to H^1(\mathbb{A}, T \xrightarrow{f} U)$$

is continuous and open. Note that $f(T(\mathbb{A}))$ is closed in $U(\mathbb{A})$, as follows easily from the equality

(C.1.2) $$U(\mathcal{O}_v) \cap f(T(F_v)) = f(T(\mathcal{O}_v))$$

for $v \notin S$ (see the proof of Lemma C.1.A) together with the compactness of

$$\prod_{v \notin S} f(T(\mathcal{O}_v)).$$

For $i \geq 2$ give $H^i(\mathbb{A}, T \xrightarrow{f} U)$ the discrete topology. It is easy to see that for all i the topology we have just given $H^i(\mathbb{A}, T \xrightarrow{f} U)$ agrees with its natural topology as a restricted direct product (Lemma C.1.B).

Finally we topologize the groups $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$. Give

$$H^0(\mathbb{A}/F, T \xrightarrow{f} U)$$

the topology it inherits as a closed subgroup of $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$ (the structure of topological group we use on $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$ is the unique one for which the subgroup $T(\mathbb{A})/T(F)$ is open and inherits its usual topology; recall that $T(\mathbb{A})/T(F)$ has finite index in $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$). Now consider the natural homomorphism

$$f : [T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma \to [U(\overline{\mathbb{A}})/U(\overline{F})]^\Gamma. \tag{C.1.3}$$

Write $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma_1$ for the kernel of the natural surjective homomorphism

$$H : [T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma \to (X_*(T) \otimes \mathbb{R})^\Gamma$$

defined by

$$\langle \lambda, H(t) \rangle = \log |\lambda(t)| \qquad (\lambda \in X^*(T)^\Gamma),$$

where $|\cdot|$ denotes the usual absolute value on $\mathbb{A}^\times$ (of course we use that $(\overline{\mathbb{A}}^\times/\overline{F}^\times)^\Gamma = \mathbb{A}^\times/F^\times$). Recall that there is a natural splitting of H (by restriction of scalars assume $F = \mathbb{Q}$; then $A(\mathbb{R})^0 \subset T(\mathbb{A})$ provides the splitting, where A denotes the biggest $\mathbb{Q}$-split subtorus of T), so that $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$ decomposes as the direct product of $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma_1$, a compact group, and the real vector space $(X_*(T) \otimes \mathbb{R})^\Gamma$, which we denote simply by $\mathfrak{A}_T$. One consequence of this product decomposition is that the image of (C.1.3) is closed in $[U(\overline{\mathbb{A}})/U(\overline{F})]^\Gamma$. We give

$$H^1(\mathbb{A}/F, T \xrightarrow{f} U)$$

the unique structure of topological group for which the map

$$\operatorname{cok}\left[[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma \to [U(\overline{\mathbb{A}})/U(\overline{F})]^\Gamma\right] \to H^1(\mathbb{A}/F, T \xrightarrow{f} U)$$

is continuous and open. For $i \geq 2$ give

$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)$$

the discrete topology.

(C.2) Duality for $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$. We keep the notation of the previous section. We write X for $X^*(T)$ and Y for $X^*(U)$. As in the local case (see (A.2)) there is a Tate-Nakayama pairing

$$H^r(\mathbb{A}/F, T \xrightarrow{f} U) \otimes H^{3-r}(F, Y \xrightarrow{f^*} X) \to \mathbb{Q}/\mathbb{Z}, \tag{C.2.1}$$

the $\mathbb{Q}/\mathbb{Z}$ coming from the canonical isomorphism

$$H^2(\mathbb{A}/F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

It is part of one variant of Tate-Nakayama duality for tori that

$$H^r(\mathbb{A}/F, T) = \{1\} \qquad (r \geq 3)$$

and that for $r = 1, 2$ the map

$$H^r(\mathbb{A}/F, T) \to \operatorname{Hom}(H^{2-r}(F, X), \mathbb{Q}/\mathbb{Z})$$

obtained from the Tate-Nakayama pairing is an isomorphism (see Appendix D for a review of this).

Lemma C.2.A. *For $r \geq 4$ the groups $H^r(\mathbb{A}/F, T \xrightarrow{f} U)$ vanish. For $r = 2,3$ the pairing* (C.2.1) *induces an isomorphism*

$$H^r(\mathbb{A}/F, T \xrightarrow{f} U) \to \mathrm{Hom}(H^{3-r}(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}).$$

Moreover for $r = 2,3$ the abelian group $H^{3-r}(F, Y \xrightarrow{f^*} X)$ is finitely generated; for $r = 3$ it is free as well.

The vanishing for $r \geq 4$ follows from the vanishing of $H^r(\mathbb{A}/F, T)$ for $r \geq 3$ using a long exact sequence of the form (A.1.1). The duality statement for $r = 2,3$ is proved the same way as in the p-adic case (Lemma A.2.A): use the facts reviewed above, the 5-lemma and the exact sequences (A.1.1) for $T \to U$ and $Y \to X$. The last statement of the lemma is also proved in the same way as in the p-adic case.

Of course we want a duality theorem for $r = 0,1$ as well, but for this, as in the local case, we need to use Weil groups and introduce another pairing. We write W_F for the Weil group of $\overline{F}/F$; it is the projective limit of the Weil groups $W_{K/F}$ for finite Galois extensions K of F in $\overline{F}$. We again need hypercohomology groups

$$H^m(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

that take into account the topology on W_F. We get cochain groups $C^m(W_F, \hat{T})$ and cohomology groups $H^m(W_F, \hat{T})$ by copying the definitions given in the local case (copy word-for-word with one exception: replace $K^\times$ in the local case by $\mathbb{A}_K^\times/K^\times$ in the global case). Trivially

$$H^m(W_F, \hat{T}) = \{1\} \qquad (m \geq 2).$$

For $m = 1$

$$H^1(W_F, \hat{T}) = \mathrm{Hom}_{\mathrm{cont}}(H^0(\mathbb{A}/F, T), \mathbb{C}^\times).$$

This theorem of Langlands is contained in his unpublished paper "Representations of abelian algebraic groups" and another proof can be found in [La]. Finally we define the group of m-hypercochains

$$C^m(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) := C^m(W_F, \hat{U}) \oplus C^{m-1}(W_F, \hat{T})$$

and define

$$H^m(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

to be the m-th cohomology group of the complex of hypercochains. Trivially

$$H^m(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = \{1\} \qquad (m \geq 3).$$

Now we can formulate a duality theorem for $r = 0$.

Lemma C.2.B. *There is a canonical isomorphism*

$$H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathrm{Hom}_{\mathrm{cont}}(H^0(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{C}^\times).$$

The proof of this lemma is essentially the same as that of Lemma A.3.A. To see that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{cont}}(H^0(\mathbb{A}/F, U), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cont}}(H^0(\mathbb{A}/F, T), \mathbb{C}^\times)& \\ \to \mathrm{Hom}_{\mathrm{cont}}(H^0(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{C}^\times) \to \{1\}&\end{aligned}$$

is exact use the direct product decomposition

$$[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma = [T(\overline{\mathbb{A}})/T(\overline{F})]_1^\Gamma \times \mathfrak{A}_T$$

discussed in (C.1).

It remains to give a duality theorem for $r = 1$. As in the local case we need a pairing

$$H^1(\mathbb{A}/F, T \xrightarrow{f} U) \otimes H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathbb{C}^\times. \tag{C.2.2}$$

The definition of the pairing is again the same as in the local case, and the global pairing satisfies the analogs of (A.3.13) and (A.3.14). Our pairing yields a map

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{C}^\times). \tag{C.2.3}$$

Lemma C.2.C. *The map* (C.2.3) *is surjective and its kernel is the image of the identity component of* $\hat{T}^\Gamma$ *under the natural map*

$$H^0(W_F, \hat{T}) \to H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

This is proved the same way as in the local case (Lemma A.3.B). Note that the lemma suggests the introduction of the quotient group

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) / \mathrm{im}(\hat{T}^\Gamma)^0;$$

there is then an induced isomorphism

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \simeq \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{C}^\times). \tag{C.2.4}$$

We finish the section by defining a closed subgroup

$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$$

of $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$, in analogy with the subgroup $[T(\overline{\mathbb{A}})/T(\overline{F})]_1^\Gamma$ of $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$, and showing that it is compact. Recall the groups $X = X^*(T)$ and $Y = X^*(U)$ as well as the real vector spaces

$$\mathfrak{A}_T = \mathrm{Hom}(X^\Gamma, \mathbb{R})$$

$$\mathfrak{A}_U = \mathrm{Hom}(Y^\Gamma, \mathbb{R})$$

from (C.1). The exact sequence

(C.2.5)
$$0 \to \mathrm{Hom}(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{R}) \to \mathrm{Hom}(X^\Gamma, \mathbb{R}) \to \mathrm{Hom}(Y^\Gamma, \mathbb{R}) \\ \to \mathrm{Hom}(\ker[Y^\Gamma \to X^\Gamma], \mathbb{R}) \to 0$$

shows that

$$\mathrm{Hom}(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{R}) = \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$\mathrm{Hom}(\ker[Y^\Gamma \to X^\Gamma], \mathbb{R}) = \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U].$$

We now define a homomorphism

$$H : H^0(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[\mathfrak{A}_T \to \mathfrak{A}_U].$$

By the discussion above it is equivalent to give a homomorphism

$$\mathrm{cok}[Y^\Gamma \to X^\Gamma] \to \mathrm{Hom}(H^0(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{R}).$$

Let $\lambda \in X^\Gamma$. Then this second homomorphism sends the class of λ in $\mathrm{cok}[Y^\Gamma \to X^\Gamma]$ to the following element of

$$\mathrm{Hom}(H^0(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{R}).$$

View the diagram

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ \lambda\big\downarrow & & \big\downarrow \\ \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

as a homomorphism from the complex $[T \xrightarrow{f} U]$ to the complex $[\mathbb{G}_m \to 1]$. This homomorphism of complexes yields a map

$$H^0(\mathbb{A}/F, T \xrightarrow{f} U) \to H^0(\mathbb{A}/F, \mathbb{G}_m) = \mathbb{A}^\times/F^\times$$

which we then compose with

$$\log|\cdot| : \mathbb{A}^\times/F^\times \to \mathbb{R}$$

to get the desired element of

$$\mathrm{Hom}(H^0(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{R}).$$

Similarly we define a homomorphism

$$H : H^1(\mathbb{A}/F, T \xrightarrow{f} U) \to \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

by using an element $\lambda \in \ker[Y^\Gamma \to X^\Gamma]$ to get a map of complexes from $[T \xrightarrow{f} U]$ to $[1 \to \mathbb{G}_m]$ given by

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ \downarrow & & \downarrow \lambda \\ 1 & \longrightarrow & \mathbb{G}_m \end{array}$$

which in turn yields a map

$$H^1(\mathbb{A}/F, T \xrightarrow{f} U) \to H^0(\mathbb{A}/F, \mathbb{G}_m) = \mathbb{A}^\times / F^\times$$

which, as before, we compose with $\log|\cdot|$ to obtain an element of

$$\mathrm{Hom}(H^1(\mathbb{A}/F, T \xrightarrow{f} U), \mathbb{R}).$$

It is easy to see that the following diagram commutes

(C.2.6)

$$\begin{array}{ccccccccccc} 1 & \to H^0(\mathbb{A}/F, T \to U) & \to H^0(\mathbb{A}/F, T) & \to H^0(\mathbb{A}/F, U) & \to H^1(\mathbb{A}/F, T \to U) & \to \cdots \\ & H \downarrow & H \downarrow & H \downarrow & H \downarrow & \\ 0 \to & \ker[\mathfrak{A}_T \to \mathfrak{A}_U] & \to \quad \mathfrak{A}_T & \to \quad \mathfrak{A}_U & \to \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U] & \to 0 \end{array}$$

The map

$$H^0(\mathbb{A}/F, T) \to H^0(\mathbb{A}/F, U)$$

is compatible with the splittings of

$$H : H^0(\mathbb{A}/F, T) \to \mathfrak{A}_T$$
$$H : H^0(\mathbb{A}/F, U) \to \mathfrak{A}_U,$$

and therefore the maps

$$H : H^0(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$H : H^1(\mathbb{A}/F, T \xrightarrow{f} U) \to \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

have natural splittings as well, yielding direct product decompositions

$$H^0(\mathbb{A}/F, T \xrightarrow{f} U) = H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1 \times \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$H^1(\mathbb{A}/F, T \xrightarrow{f} U) = H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1 \times \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U],$$

where $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ $(i = 0, 1)$ denotes the kernel of H on $H^i(\mathbb{A}, F, T \xrightarrow{f} U)$.

Similar considerations apply to H^2 and H^3. Write D_T for $\mathrm{Hom}(X^\Gamma, \mathbb{Q}/\mathbb{Z})$. The exact sequence

(C.2.7)
$$0 \to \mathrm{Hom}(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(X^\Gamma, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(Y^\Gamma, \mathbb{Q}/\mathbb{Z})$$
$$\to \mathrm{Hom}(\ker[Y^\Gamma \to X^\Gamma], \mathbb{Q}/\mathbb{Z}) \to 0$$

shows that

$$\mathrm{Hom}(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{Q}/\mathbb{Z}) = \ker[D_T \to D_U]$$
$$\mathrm{Hom}(\ker[Y^\Gamma \to X^\Gamma], \mathbb{Q}/\mathbb{Z}) = \mathrm{cok}[D_T \to D_U].$$

The Tate-Nakayama pairing for T induces an isomorphism (see Lemma D.2.A)

$$H : H^2(\mathbb{A}/F, T) \to D_T.$$

We now define a homomorphism

$$H : H^2(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[D_T \to D_U].$$

This works the same way as for H^0: given $\lambda \in X^\Gamma$ we get a map

$$H^2(\mathbb{A}/F, T \xrightarrow{f} U) \to H^2(\mathbb{A}/F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

Finally we define a homomorphism

$$H : H^3(\mathbb{A}/F, T \xrightarrow{f} U) \to \mathrm{cok}[D_T \to D_U].$$

This works the same way as for H^1: given $\lambda \in \ker[Y^\Gamma \to X^\Gamma]$ we get a map

$$H^3(\mathbb{A}/F, T \xrightarrow{f} U) \to H^2(\mathbb{A}/F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

For $i = 2, 3$ we let $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ denote the kernel of H on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ and for $i \geq 4$ we put

$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1 := H^i(\mathbb{A}/F, T \xrightarrow{f} U) = 1.$$

For T itself we of course write

$$H^0(\mathbb{A}/F,T)_1$$

for the subgroup $[T(\overline{\mathbb{A}})/T(\overline{F})]_1^\Gamma$ of $H^0(\mathbb{A}/F,T)$ and

$$H^2(\mathbb{A}/F,T)_1$$

for the kernel of the natural isomorphism

$$H : H^2(\mathbb{A}/F,T) \to D_T,$$

namely, the trivial subgroup of $H^2(\mathbb{A}/F,T)$; for $i \neq 0,2$ we put

$$H^i(\mathbb{A}/F,T)_1 = H^i(\mathbb{A}/F,T).$$

Note that $H^0(\mathbb{A}/F,T)_1$ is compact, $H^1(\mathbb{A}/F,T)_1$ is finite, and for $i \geq 2$ $H^i(\mathbb{A}/F,T)_1$ is trivial (see Appendix D).

It is easy to see that the following diagram commutes

(C.2.8)

$$\begin{array}{ccccccccc} \cdots \to & H^2(\mathbb{A}/F,T\to U) & \to & H^2(\mathbb{A}/F,T) & \to & H^2(\mathbb{A}/F,U) & \to & H^3(\mathbb{A}/F,T\to U) & \to 1 \\ & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} & \\ 0 \to & \ker[D_T \to D_U] & \to & D_T & \to & D_U & \to & \mathrm{cok}[D_T \to D_U] & \to 0 \end{array}$$

Since the middle two maps H are isomorphisms, the last map H is an isomorphism and the first map H is surjective. The surjectivity of all the maps H we have defined implies that the groups

$$\text{(C.2.9)} \qquad \cdots \to H^i(\mathbb{A}/F,T \xrightarrow{f} U)_1 \to H^i(\mathbb{A}/F,T)_1 \to H^i(\mathbb{A}/F,U)_1 \to \cdots$$

form a long exact sequence.

Suppose $T \xrightarrow{f} U$ is replaced by quasi-isomorphic $T' \xrightarrow{f'} U'$. Then $H^i(\mathbb{A}/F,T \xrightarrow{f} U)_1$ does not change for $i = 0,1,3$ but it can change for $i = 2$, which means that for $i = 2$ this subgroup is somewhat artificial. Nevertheless it will be useful later.

Lemma C.2.D. *The group $H^i(\mathbb{A}/F,T \xrightarrow{f} U)_1$ is compact for $i = 0,1$, finite for $i = 2$ and trivial for $i \geq 3$.*

The group $H^0(\mathbb{A}/F,T \xrightarrow{f} U)_1$ is closed in the compact group $H^0(\mathbb{A}/F,T)_1$ and is therefore compact. The group $H^1(\mathbb{A}/F,T \xrightarrow{f} U)_1$ contains as closed subgroup of finite index the compact group

$$H^0(\mathbb{A}/F,U)_1/f(H^0(\mathbb{A}/F,T)_1)$$

and is therefore compact. The statements for $i \geq 2$ follow from the analogous ones for $H^i(\mathbb{A}/F,T)$ $(i \geq 1)$ together with the long exact sequence (C.2.9).

(C.3) Duality for $\ker^i(F, T \xrightarrow{f} U)$ and $\mathrm{cok}^i(F, T \xrightarrow{f} U)$. We keep the notation of the previous sections. Consider the natural map

$$H^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}, T \xrightarrow{f} U).$$

We write $\ker^i(F, T \xrightarrow{f} U)$ for its kernel and $\mathrm{cok}^i(F, T \xrightarrow{f} U)$ for its cokernel, and we give $\mathrm{cok}^i(F, T \xrightarrow{f} U)$ its natural quotient topology. Since $H^i(\mathbb{A}, T \xrightarrow{f} U)$ is a restricted direct product of local groups we may also describe $\ker^i(F, T \xrightarrow{f} U)$ as

$$\ker[H^i(F, T \xrightarrow{f} U) \to \prod_v H^i(F_v, T \xrightarrow{f} U)].$$

Note that the long exact sequence (C.1.1) gives rise to short exact sequences

$$1 \to \mathrm{cok}^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker^{i+1}(F, T \xrightarrow{f} U) \to 1. \tag{C.3.1}$$

Lemma C.3.A. *For all i the image of $H^i(F, T \xrightarrow{f} U)$ in $H^i(\mathbb{A}, T \xrightarrow{f} U)$ is discrete. For all i the map*

$$\mathrm{cok}^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}/F, T \xrightarrow{f} U)$$

induces an isomorphism of topological groups from $\mathrm{cok}^i(F, T \xrightarrow{f} U)$ to an open subgroup of $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$.

For $i \geq 2$ both statements are trivial since in this case $H^i(\mathbb{A}, T \xrightarrow{f} U)$ and $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ have the discrete topology. Next consider $i = 0$. Let C denote the diagonalizable group $\ker[T \to U]$. Then

$$H^0(F, T \xrightarrow{f} U) = C(F)$$

is certainly discrete in

$$H^0(\mathbb{A}, T \xrightarrow{f} U) = C(\mathbb{A}).$$

To prove the second statement it is enough to show that

$$C(\mathbb{A}) \to \ker[T(\mathbb{A})/T(F) \to U(\mathbb{A})/U(F)]$$

is an open mapping (recall that $T(\mathbb{A})/T(F)$ is open in $[T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma$). Consider the closed subgroup

$$B := \{t \in T(\mathbb{A}) | f(t) \in U(F)\}$$

of $T(\mathbb{A})$. Since $U(F)$ is discrete in $U(\mathbb{A})$, the subgroup $C(\mathbb{A})$ of B is open in B. Therefore the mapping

$$C(\mathbb{A}) \to B/T(F) = \ker[T(\mathbb{A})/T(F) \to U(\mathbb{A})/U(F)]$$

is indeed open.

Now we consider $i = 1$. To prove the first statement it is enough to check that the intersection of the subgroup $\text{im}[H^1(F, T \xrightarrow{f} U)]$ of $H^1(\mathbb{A}, T \xrightarrow{f} U)$ with the open subgroup $U(\mathbb{A})/f(T(\mathbb{A}))$ of $H^1(\mathbb{A}, T \xrightarrow{f} U)$ is discrete, and since

$$\text{im}[U(F)/f(T(F)) \to U(\mathbb{A})/f(T(\mathbb{A}))]$$

has finite index in this intersection (use the finitenesss of $\ker^1(F, T)$) it is even enough to check that the image of $U(F)/f(T(F))$ in $U(\mathbb{A})/f(T(\mathbb{A}))$ is discrete. We have the product decomposition

$$T(\mathbb{A})_1 \times \mathfrak{A}_T \to U(\mathbb{A})_1 \times \mathfrak{A}_U$$

and therefore

$$U(\mathbb{A})/f(T(\mathbb{A})) = (U(\mathbb{A})_1/f(T(\mathbb{A})_1)) \times (\mathfrak{A}_U/f(\mathfrak{A}_T)).$$

Of course the image of $U(F)/f(T(F))$ in $U(\mathbb{A})/f(T(\mathbb{A}))$ lies in the subgroup

$$U(\mathbb{A})_1/f(T(\mathbb{A})_1).$$

Consider the discrete subgroup $f(T(F))$ of $U(\mathbb{A})_1$ and form the quotient group

$$U(\mathbb{A})_1/f(T(F));$$

it has the discrete subgroup

$$U(F)/f(T(F))$$

and the compact subgroup

$$f(T(\mathbb{A})_1)/f(T(F)).$$

Therefore the image of

$$U(F)/f(T(F))$$

in the quotient group

$$(U(\mathbb{A})_1/f(T(F)))\,/\,(f(T(\mathbb{A})_1)/f(T(F))) = U(\mathbb{A})_1/f(T(\mathbb{A})_1)$$

is discrete, as desired. Here we used the following fact about topological groups: let G be a Hausdorff topological group, let Γ be a discrete (hence closed) subgroup of G and let N be a compact normal subgroup of G; then $\Gamma N/N$ is discrete in G/N. To prove this it is enough to find a neighborhood $U \subset G$ of the identity such that $UN \cap \Gamma N = N$; in fact any neighborhood U of the identity such that $UN \subset G \setminus (\Gamma \setminus (\Gamma \cap N))$ will do; such U exist since N is compact and $G \setminus (\Gamma \setminus (\Gamma \cap N))$ is an open set containing N. In this proof we have used $A \setminus B$ to denote the complement of B in A.

Finally we prove the second statement of the lemma for $i = 1$. It is enough to show that

$$U(\mathbb{A}) \to [U(\overline{\mathbb{A}})/U(\overline{F})]^\Gamma / f([T(\overline{\mathbb{A}})/T(\overline{F})]^\Gamma)$$

is an open mapping, but this is obvious since $U(\mathbb{A})/U(F)$ is an open subgroup of $[U(\overline{\mathbb{A}})/U(\overline{F})]^\Gamma$ and for any subgroup H of any topological group G the natural map $G \to G/H$ is open.

Now we come to the main result of this section, a duality theorem involving $\ker^i(F, T \xrightarrow{f} U)$ and $\operatorname{cok}^i(F, T \xrightarrow{f} U)$. For this we will need the following groups:

$$\ker^i(F, Y \xrightarrow{f^*} X) := \ker\left[H^i(F, Y \xrightarrow{f^*} X) \to \prod_v H^i(F_v, Y \xrightarrow{f^*} X)\right]$$

$$\ker^i(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) := \ker\left[H^i(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to \prod_v H^i(W_{F_v}, \hat{U} \xrightarrow{\hat{f}} \hat{T})\right]$$

$$\ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := \ker\left[H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \to \prod_v H^1(W_{F_v}, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}\right].$$

In each case the product is taken over all places v of F.

Lemma C.3.B. *The groups* $\ker^i(F, T \xrightarrow{f} U)$ *are finite for all* i *and vanish except for* $i = 1, 2, 3$. *For* $i = 1, 2, 3$ *the dual finite abelian groups are given by*

$$\operatorname{Hom}\left(\ker^1(F, T \xrightarrow{f} U), \mathbb{C}^\times\right) = \ker^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

$$\operatorname{Hom}\left(\ker^2(F, T \xrightarrow{f} U), \mathbb{C}^\times\right) = \ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}$$

$$\operatorname{Hom}\left(\ker^3(F, T \xrightarrow{f} U), \mathbb{Q}/\mathbb{Z}\right) = \ker^1(F, Y \xrightarrow{f^*} X).$$

The groups $\operatorname{cok}^i(F, T \xrightarrow{f} U)$ *vanish for* $i \geq 4$. *For* $i \leq 3$ *there are duality isomorphisms*

$$\operatorname{Hom}_{\mathrm{cont}}\left(\operatorname{cok}^0(F, T \xrightarrow{f} U), \mathbb{C}^\times\right) = H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) / \ker^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

$$\operatorname{Hom}_{\mathrm{cont}}\left(\operatorname{cok}^1(F, T \xrightarrow{f} U), \mathbb{C}^\times\right) = H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} / \ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}$$

$$\operatorname{cok}^2(F, T \xrightarrow{f} U) = \operatorname{Hom}\left(H^1(F, Y \xrightarrow{f^*} X) / \ker^1(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right)$$

$$\operatorname{cok}^3(F, T \xrightarrow{f} U) = H^3(\mathbb{A}/F, T \xrightarrow{f} U) = \operatorname{Hom}\left(H^0(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right).$$

Moreover both $H^1(F, Y \xrightarrow{f^*} X)/\ker^1(F, Y \xrightarrow{f^*} X)$ *and* $H^0(F, Y \xrightarrow{f^*} X)$ *are finitely generated abelian groups* (*the second one is even free*).

We begin by proving that $\ker^i(F, T \xrightarrow{f} U)$ vanishes except for $i = 1, 2, 3$. This is obvious for $i = 0$. For $i \geq 4$ we consider the commutative square

$$\begin{array}{ccc} H^{i-2}(\mathbb{A}/F, U) & \xrightarrow{\ \alpha\ } & H^{i-1}(\mathbb{A}/F, T \to U) \\ \downarrow & & \downarrow{\scriptstyle\beta} \\ \ker^{i-1}(F, U) & \xrightarrow{\ \gamma\ } & \ker^i(F, T \to U). \end{array}$$

The map α is part of a long exact sequence of type (A.1.1); thus it follows from the vanishing of $H^{i-1}(\mathbb{A}/F,T)$ for $i \geq 4$ (see Lemma D.2.A) that α is surjective. Since β is also surjective (recall (C.3.1)) it follows that γ is surjective; since $\ker^{i-1}(F,U)$ vanishes for $i \geq 4$ (see the discussion preceding Lemma D.2.C) we conclude that $\ker^i(F,T \xrightarrow{f} U)$ vanishes, as desired.

The vanishing of $\operatorname{cok}^i(F,T \xrightarrow{f} U)$ for $i \geq 4$ follows from (C.3.1) and Lemma C.2.A. Moreover (C.3.1) and the vanishing of $\ker^4(F,T \xrightarrow{f} U)$ give the equality

$$\operatorname{cok}^3(F,T \xrightarrow{f} U) = H^3(\mathbb{A}/F,T \xrightarrow{f} U),$$

and the duality isomorphism for $\operatorname{cok}^3(F,T \xrightarrow{f} U)$ follows from the duality isomorphism for $H^3(\mathbb{A}/F,T \xrightarrow{f} U)$ (see Lemma C.2.A). The last statement of the lemma follows from the last statement of Lemma C.2.A.

It remains to consider $\operatorname{cok}^i(F,T \xrightarrow{f} U)$ and $\ker^{i+1}(F,T \xrightarrow{f} U)$ for $i = 0,1,2$. Of course we use the exact sequence (C.3.1). We begin with $i = 0,1$. It is easy to see that the maps

$$H : H^0(\mathbb{A}/F,T \xrightarrow{f} U) \to \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$H : H^1(\mathbb{A}/F,T \xrightarrow{f} U) \to \operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

remain surjective when restricted to the subgroup $\operatorname{cok}^i(F,T \xrightarrow{f} U)$ of $H^i(\mathbb{A}/F,T \xrightarrow{f} U)$. In other words $H^i(\mathbb{A}/F,T \xrightarrow{f} U)_1$ maps onto $\ker^{i+1}(F,T \xrightarrow{f} U)$ for $i = 0,1$; of course the kernel of this map is open (use Lemma C.3.A) in $H^i(\mathbb{A}/F,T \xrightarrow{f} U)_1$, a compact group by Lemma C.2.D. Therefore $\ker^{i+1}(F,T \xrightarrow{f} U)$ is finite for $i = 0,1$.

Applying the functor $\operatorname{Hom}_{\text{cont}}(\cdot,\mathbb{C}^\times)$ to the exact sequence (C.3.1) and keeping in mind that $\operatorname{cok}^i(F,T \xrightarrow{f} U)$ is open in $H^i(\mathbb{A}/F,T \xrightarrow{f} U)$, we find that

$$\operatorname{Hom}_{\text{cont}}\left(\ker^{i+1}(F,T \xrightarrow{f} U),\mathbb{C}^\times\right)$$

is equal to

$$\ker\left[\operatorname{Hom}_{\text{cont}}\left(H^i(\mathbb{A}/F,T \xrightarrow{f} U),\mathbb{C}^\times\right) \to \operatorname{Hom}_{\text{cont}}\left(H^i(\mathbb{A},T \xrightarrow{f} U),\mathbb{C}^\times\right)\right]$$

and

$$\operatorname{Hom}_{\text{cont}}\left(\operatorname{cok}^i(F,T \xrightarrow{f} U),\mathbb{C}^\times\right)$$

is equal to

$$\operatorname{Hom}_{\text{cont}}\left(H^i(\mathbb{A}/F,T \xrightarrow{f} U),\mathbb{C}^\times\right) / \operatorname{Hom}_{\text{cont}}\left(\ker^{i+1}(F,T \xrightarrow{f} U),\mathbb{C}^\times\right).$$

Using global duality (Lemmas C.2.B and C.2.C), local duality (Lemmas A.3.A and A.3.B), and the fact that $H^i(\mathbb{A},T \xrightarrow{f} U)$ is a restricted direct product of local groups

(Lemma C.1.B), we find that the duality isomorphisms for $\ker^{i+1}$ and cok^i $(i = 0, 1)$ given in the statement of the lemma are indeed valid.

Finally we consider $\operatorname{cok}^2(F, T \xrightarrow{f} U)$ and $\ker^3(F, T \xrightarrow{f} U)$. By global duality (Lemma C.2.A) and local duality (Lemma A.2.A and the discussion preceding it in the archimedean case) the map

$$\alpha : H^2(\mathbb{A}, T \xrightarrow{f} U) \to H^2(\mathbb{A}/F, T \xrightarrow{f} U)$$

is canonically isomorphic to the map

$$\beta : \bigoplus_v \operatorname{Hom}\left(\tilde{H}^1(F_v, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right) \to \operatorname{Hom}\left(H^1(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right),$$

where $\tilde{H}^1(F_v, Y \xrightarrow{f^*} X)$ denotes Tate hypercohomology (respectively, ordinary hypercohomology) in the archimedean (respectively, non-archimedean) case. Since the group $\ker^3(F, T \xrightarrow{f} U)$ is equal to $\operatorname{cok}(\alpha)$, we conclude that it is also equal to $\operatorname{cok}(\beta)$. It is easy to see that the natural map

$$H^1(F_v, Y \xrightarrow{f^*} X) \to \tilde{H}^1(F_v, Y \xrightarrow{f^*} X)$$

is surjective for each archimedean v, and therefore $\operatorname{cok}(\beta)$ is also equal to the cokernel of

$$\bigoplus_v \operatorname{Hom}(H_v, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}(H, \mathbb{Q}/\mathbb{Z}),$$

where we have written H_v (respectively, H) as an abbreviation for $H^1(F_v, Y \xrightarrow{f^*} X)$ (respectively, $H^1(F, Y \xrightarrow{f^*} X)$).

Let v_0 be any place of F. Then the kernel of $H \to H_{v_0}$ is finite. Indeed this follows easily from the finiteness of the groups $H^1(F, Y)$ and

$$\ker[\operatorname{cok}[Y^\Gamma \to X^\Gamma] \to \operatorname{cok}[Y \to X]].$$

For each of the finitely many elements in H that belong to the kernel of $H \to H_{v_0}$ but do not belong to the kernel of $H \to \prod_v H_v$ choose a place v of F such that the element has non-trivial image in H_v. Let S be the finite set of places obtained in this way, including v_0. Then the maps

$$H \to \prod_v H_v$$
$$H \to \prod_{v \in S} H_v$$

have the same kernel, call it K. It follows that

$$\bigoplus_{v \in S} \operatorname{Hom}(H_v, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}(H, \mathbb{Q}/\mathbb{Z})$$
$$\bigoplus_v \operatorname{Hom}(H_v, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}(H, \mathbb{Q}/\mathbb{Z})$$

have the same cokernel, namely $\mathrm{Hom}(K, \mathbb{Q}/\mathbb{Z})$, and we conclude that

$$\ker^3(F, T \xrightarrow{f} U) = \mathrm{Hom}(K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}\left(\ker^1(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right).$$

In the course of the proof we saw that K is finite, which shows that $\ker^3(F, T \xrightarrow{f} U)$ is finite. Applying the exact functor $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ to the exact sequence

$$0 \to \ker^1(Y \xrightarrow{f^*} X) \to H^1(Y \xrightarrow{f^*} X) \to H^1(Y \xrightarrow{f^*} X)/\ker^1(Y \xrightarrow{f^*} X) \to 0$$

(in which we have abbreviated $H^i(F, \cdot)$ to $H^i(\cdot)$) and comparing with (C.3.1), we find that the duality isomorphism for $\mathrm{cok}^2(F, T \xrightarrow{f} U)$ given in the statement of the lemma is indeed valid.

We finish by giving two alternative descriptions of $\ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}$ that use Γ rather than W_F. For this we define groups

$$H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})/\operatorname{im}(\hat{T}^\Gamma)^0$$

$$H^1(F_v, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := H^1(F_v, \hat{U} \xrightarrow{\hat{f}} \hat{T})/\operatorname{im}(\hat{T}^{\Gamma(v)})^0$$

$$\ker^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} := \ker\left[H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \to \prod_v H^1(F_v, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}\right]$$

where $\Gamma(v)$ denotes $\mathrm{Gal}(\overline{F}_v/F_v)$.

Lemma C.3.C. *The natural maps*

$$\ker^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \to \ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}$$

$$\ker^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \to \ker^2(F, Y \xrightarrow{f^*} X)$$

are isomorphisms.

The injectivity of the first map follows from the (obvious) injectivity of

$$H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) \to H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}).$$

To prove surjectivity of the first map consider a 1-hypercocycle (u_w, t) of W_F in $[\hat{U} \xrightarrow{\hat{f}} \hat{T}]$ that represents an element $x \in \ker^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}$. Then the 1-cocycle u_w of W_F in $\hat{U}$ represents the image of x under the natural map

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \to H^1(W_F, \hat{U}),$$

and it is evident that x belongs to

$$\ker^1(W_F, \hat{U}) := \ker\left[H^1(W_F, \hat{U}) \to \prod_v H^1(W_{F_v}, \hat{U})\right].$$

By Lemma D.2.C the natural map

$$\ker^1(F, \hat{U}) \to \ker^1(W_F, \hat{U})$$

is an isomorphism. Therefore by modifying (u_w, t) by the coboundary of a suitable 0-hypercochain we may assume that u_w comes from a 1-cocycle u_σ of Γ. Then (u_σ, t) represents an element of $\ker^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\text{red}}$ mapping to x.

Next we prove that the second map is an isomorphism. This map comes from the boundary map for the global analog of the commutative diagram of exponential sequences occurring at the beginning of (A.3), regarded as a short exact sequence of complexes of length 2. Since

$$H^2(F, \operatorname{Lie}(\hat{U}) \to \operatorname{Lie}(\hat{T})) = 0$$
$$H^1(F, \operatorname{Lie}(\hat{U}) \to \operatorname{Lie}(\hat{T})) = \operatorname{Lie}(\hat{T}^\Gamma)/\operatorname{im}\operatorname{Lie}(\hat{U}^\Gamma)$$

we see that

$$H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\text{red}} \simeq H^2(F, Y \xrightarrow{f^*} X),$$

and since this holds locally as well, the second map in the lemma is indeed an isomorphism.

(C.4) Diagonalizable groups. Recall that an abelian linear algebraic group over F is said to be *diagonalizable* if it can be realized as an algebraic subgroup of a torus over F (such groups are also called groups of *multiplicative type*). Our duality theorems for hypercohomology give duality theorems for the cohomology of diagonalizable groups C over F, simply by embedding C in a torus T over F and considering the complex $T \xrightarrow{f} U$ where U is the quotient torus T/C and f is the canonical surjection. It is then immediate that

$$H^i(F, C) = H^i(F, T \xrightarrow{f} U)$$
$$H^i(F_v, C) = H^i(F_v, T \xrightarrow{f} U).$$

We define groups $H^i(\mathbb{A}, C)$ and $H^i(\mathbb{A}/F, C)$ by

$$H^i(\mathbb{A}, C) := H^i(\mathbb{A}, T \xrightarrow{f} U)$$
$$H^i(\mathbb{A}/F, C) := H^i(\mathbb{A}/F, T \xrightarrow{f} U)$$

(it is easy to see that the groups on the right side obtained from various T are all canonically isomorphic to each other).

Note that

$$T(\overline{\mathbb{A}}) \to U(\overline{\mathbb{A}})$$
$$T(\overline{\mathbb{A}})/T(\overline{F}) \to U(\overline{\mathbb{A}})/U(\overline{F})$$

need not be surjective (they have the same cokernel in any case), as one sees already for

$$\mathbb{G}_m \xrightarrow{2} \mathbb{G}_m$$

over $\mathbb{Q}$ (an idele for $\mathbb{Q}$ which is a non-square unit at every odd prime cannot be a square in $\mathbb{A}_K^\times$ for any number field K because infinitely many primes of $\mathbb{Q}$ split completely in K). Therefore $H^i(F, C(\overline{\mathbb{A}}))$ need not coincide with $H^i(\mathbb{A}, C)$, and $H^i(F, C(\overline{\mathbb{A}})/C(\overline{F}))$ need not coincide with $H^i(\mathbb{A}/F, C)$. In fact for any finite abelian group C with trivial Γ-action we have, for any finite Galois extension K of F in $\overline{F}$,

$$\begin{aligned} H^1(K/F, C(\mathbb{A}_K)) &= \prod_v H^1(K_w/F_v, C) \\ &\subset \prod_v H^1(F_v, C) \end{aligned}$$

and therefore $H^1(F, C(\overline{\mathbb{A}}))$ is the union over K of these subgroups of $\prod_v H^1(F_v, C)$; this union is much smaller than $H^1(\mathbb{A}, C)$, which is the restricted direct product of the local groups $H^1(F_v, C)$, the restriction being with respect to the subgroups $H^1(F_v^{\mathrm{un}}/F_v, C)$ for finite places v. (Elements in $\prod_v H^1(K_w/F_v, C)$ are trivial at all places that split in K, hence are trivial infinitely often, while an element of the restricted direct product can be non-trivial locally everywhere, *e.g.* for $C = \{\pm 1\}$.)

It seems that $H^i(\mathbb{A}, C)$ is the "right" group and that $H^i(F, C(\overline{\mathbb{A}}))$ is the "wrong" one. In any case our duality theorems for $T \to U$ give duality theorems for C involving $H^i(\mathbb{A}, C)$ and $H^i(\mathbb{A}/F, C)$. Since we have the dual exact sequence of character groups

$$0 \to Y \xrightarrow{f^*} X \to X^*(C) \to 0,$$

duality statements involving $Y \xrightarrow{f^*} X$ can be converted to statements involving $X^*(C)$. For example it follows from Lemmas C.3.B and C.3.C that $\ker^2(F, C)$ and $\ker^1(F, X^*(C))$ are dual finite abelian groups (see [M,Theorem 4.20] for another proof of this). Another example is provided by $H^1(\mathbb{A}/F, C)$, as we now check. Recall that

$$\mathrm{Hom}_{\mathrm{cont}}(H^1(\mathbb{A}/F, C), \mathbb{C}^\times) = H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}}.$$

Using that the kernel of $\hat{U} \xrightarrow{\hat{f}} \hat{T}$ is finite, one sees easily that

$$H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T}) = H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$$

and hence that

$$\begin{aligned} H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} &= H^1(F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\mathrm{red}} \\ &= H^2(F, Y \xrightarrow{f^*} X) \qquad \text{(see proof of Lemma C.3.C)} \\ &= H^1(F, X^*(C)). \end{aligned}$$

Noting that $H^1(F, X^*(C))$ is finite and that $H^1(\mathbb{A}/F, C)$ is compact (use Lemma C.2.D), we conclude that $H^1(\mathbb{A}/F, C)$ and $H^1(F, X^*(C))$ are dual finite abelian groups (the pairing has values in $\mathbb{Q}/\mathbb{Z}$ since we used the exponential map to relate $H^1(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})_{\text{red}}$ to $H^1(F, X^*(C))$ and is in fact just the obvious cup-product pairing to $H^2(\mathbb{A}/F, \mathbb{G}_m)$). Since $\hat{U} \xrightarrow{\hat{f}} \hat{T}$ is in general neither surjective nor injective, the duality statements involving $H^2(W_F, \hat{U} \xrightarrow{\hat{f}} \hat{T})$ probably cannot be simplified (unless C is finite or a torus).

(C.5) Density of $H^1(\mathbb{Q}, T \to U)$ in $H^1(\mathbb{R}, T \to U)$. In this section we assume that $F = \mathbb{Q}$. Let T be a torus over $\mathbb{Q}$. Recall that

(1) $H^1(\mathbb{Q}, T) \to H^1(\mathbb{R}, T)$ is surjective (see [H,Thm. A.12]),

(2) $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$ (see [L2,Lemme 7.18] for a proof of this result of Serre).

These statements have a common generalization involving hypercohomology. Let $T \xrightarrow{f} U$ be a map of $\mathbb{Q}$-tori.

Lemma C.5.A. *The natural map*

$$H^1(\mathbb{Q}, T \xrightarrow{f} U) \to H^1(\mathbb{R}, T \xrightarrow{f} U)$$

has dense image.

Choose a surjective homomorphism $U' \to U$ where U' is of the form

$$U' = R_{K/\mathbb{Q}}(S)$$

for some finite extension K of $\mathbb{Q}$ and some split torus S over K ($R_{K/\mathbb{Q}}$ denotes restriction of scalars). Then use this homomorphism to form a pull-back diagram

$$\begin{array}{ccc} T' & \xrightarrow{f'} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & U \end{array}$$

which we regard as a quasi-isomorphism from $[T' \xrightarrow{f'} U']$ to $[T \xrightarrow{f} U]$. The lemma for $[T' \xrightarrow{f'} U']$ is equivalent to the lemma for $[T \xrightarrow{f} U]$, and thus we may as well assume that U is of the form

$$R_{K/\mathbb{Q}}(S).$$

Then $H^1(\mathbb{Q}, U)$ and $H^1(\mathbb{R}, U)$ are both trivial, and we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} T(\mathbb{Q}) & \longrightarrow & U(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, T \to U) & \longrightarrow & H^1(\mathbb{Q}, T) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ T(\mathbb{R}) & \longrightarrow & U(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, T \to U) & \longrightarrow & H^1(\mathbb{R}, T) & \longrightarrow & 1. \end{array}$$

The lemma follows from (1), applied to T, and (2), applied to U.

D. Duality for tori over number fields

(D.1) Preliminaries. Let F be a number field and let $\overline{F}$, Γ, $\overline{\mathbb{A}}$ and so on have the same meaning as in Appendix C. Let T be a torus over F and write X for its character group $X^*(T)$, a smooth Γ-module. We define groups

$$H^i(\mathbb{A},T) := H^i(F,T(\overline{\mathbb{A}}))$$
$$H^i(\mathbb{A}/F,T) := H^i(F,T(\overline{\mathbb{A}})/T(\overline{F})).$$

Recall that for $i \geq 1$

$$H^i(\mathbb{A},T) = \bigoplus_v H^i(F_v,T).$$

For $i \geq 0$ we define $\ker^i(F,T)$ (respectively, $\mathrm{cok}^i(F,T)$) to be the kernel (respectively, cokernel) of the natural map

$$H^i(F,T) \to H^i(\mathbb{A},T).$$

The long exact sequence of cohomology for

$$1 \to T(\overline{F}) \to T(\overline{\mathbb{A}}) \to T(\overline{\mathbb{A}})/T(\overline{F}) \to 1$$

gives rise to short exact sequences

$$1 \to \mathrm{cok}^i(F,T) \to H^i(\mathbb{A}/F,T) \to \ker^{i+1}(F,T) \to 1.$$

(D.2) Duality. We review a variant of Tate-Nakayama duality for T. This was done in [K1] as well, though at that time there was no convenient reference for that particular variant. Now there is, namely Milne's book [M], and we take this opportunity to give references for the statements we need.

Lemma D.2.A. *For $r \geq 3$ the groups $H^r(\mathbb{A}/F,T)$ vanish. For $r = 1,2$ the cup-product pairing*

$$H^r(\mathbb{A}/F,T) \otimes H^{2-r}(F,X) \to H^2(\mathbb{A}/F,\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

induces an isomorphism

$$H^r(\mathbb{A}/F,T) \to \mathrm{Hom}\left(H^{2-r}(F,X),\mathbb{Q}/\mathbb{Z}\right).$$

For $r = 0$ there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}\left(H^0(\mathbb{A}/F,T),\mathbb{C}^\times\right) = H^1(W_F,\hat{T}).$$

Moreover the group $H^1(\mathbb{A}/F,T)$ is finite.

The statement regarding $r \geq 3$ follows from [M,Corollary 4.21]. The statement regarding $r = 1,2$ follows from [M,Corollary 4.7]. The statement regarding $r = 0$ is a result of Langlands (see [La] and [M,I.8]). The last statement follows from duality and the (obvious) finiteness of $H^1(F,X)$.

Let A be any smooth Γ-module. We denote by $\ker^i(F,A)$ the kernel of the natural map

$$H^i(F,A) \to \prod_v H^i(F_v,A). \qquad \text{(D.2.1)}$$

Recall that $H^i(F_v,A)$ vanishes for $i \geq 3$ for finite places v.

Lemma D.2.B. *For $i \geq 3$ the map* (D.2.1) *is an isomorphism. Therefore the group* $\ker^i(F, A)$ *vanishes except for* $i = 1, 2$.

The second statement follows from the first since $\ker^0(F, A)$ obviously vanishes. Now we prove the first statement. For finite groups A the statement is a theorem of Tate [M,Theorem 4.10(c)], and since cohomology of Γ commutes with direct limits the statement holds for all torsion groups A. Therefore the statement holds for torsion-free groups A and $r \geq 4$, as one sees by considering the torsion group $B := (A \otimes_{\mathbb{Z}} \mathbb{Q})/A$, using the isomorphism

$$H^r(F, A) \simeq H^{r-1}(F, B)$$

and its local analogs. Now we handle the general case. Choose an exact sequence

$$0 \to K \to I \to A \to 0$$

where I is a direct sum of Γ-modules of the form $\mathbb{Z}[\mathrm{Gal}(K/F)]$ for finite Galois extensions K of F in $\overline{F}$. Note that K and I are torsion-free, so that the desired statement for $r \geq 4$ follows from the 5-lemma and our previous work. For $r = 3$ we have

$$H^3(F, I) = \prod_v H^3(F_v, I) = 0;$$

to see this use Shapiro's lemma to reduce to the case $I = \mathbb{Z}$ and then use [M,Corollary 4.17]. Once again we conclude from the 5-lemma that (D.2.1) is an isomorphism for $r = 3$.

Next we discuss duality for $\ker^i(F, T)$. Note that $\ker^r(F, T)$ and $\ker^r(F, T(\overline{F}))$ are defined differently: the first is the kernel of

$$H^r(F, T) \to \prod_v H^r(F_v, T(\overline{F}_v))$$

and the second is the kernel of

$$H^r(F, T) \to \prod_v H^r(F_v, T(\overline{F})).$$

Actually these two subgroups of $H^r(F, T)$ coincide for $r \geq 2$; for this it is enough to note that the natural map

$$H^r(F_v, T(\overline{F})) \to H^r(F_v, T(\overline{F}_v))$$

is an isomorphism for $r \geq 2$ (the quotient group $T(\overline{F}_v)/T(\overline{F})$ is a $\mathbb{Q}$-vector space and hence has trivial cohomology in positive degrees). We conclude from this discussion and Lemma D.2.B that $\ker^r(F, T)$ and $\ker^r(F, X)$ vanish except for $r = 1, 2$.

Lemma D.2.C. *For $r = 1,2$ the groups $\ker^r(F,T)$ and $\ker^{3-r}(F,X)$ are dual finite abelian groups. Moreover*

$$\ker^2(F,X) \simeq \ker^1(F,\hat{T}) \simeq \ker^1(W_F,\hat{T}),$$

where $\ker^1(W_F,\hat{T})$ of course denotes the kernel of

$$H^1(W_F,\hat{T}) \to \prod_v H^1(W_{F_v},\hat{T}).$$

For $r = 2$ the duality statement is given in [M,Theorem 4.20(a)]; it is also easy to prove directly (apply the functor $\mathrm{Hom}(\cdot,\mathbb{Q}/\mathbb{Z})$ to the exact sequence

$$H^1(\mathbb{A},T) \to H^1(\mathbb{A}/F,T) \to \ker^2(F,T) \to 1$$

and use duality for $H^1(\mathbb{A}/F,T)$ (Lemma D.2.A) together with local duality).

For $r = 1$ we begin by choosing a finite Galois extension K of F in $\overline{F}$ that splits T. We have the exact sequence

$$T(\mathbb{A}) \to H^0(K/F,T(\mathbb{A}_K)/T(K)) \to \ker^1(K/F,T) \to 1$$

where $\ker^1(K/F,T)$ denotes the kernel of the natural map

$$H^1(K/F,T(K)) \to H^1(K/F,T(\mathbb{A}_K)) = \bigoplus_v H^1(K_w/F_v,T(K_w))$$

(for each place v of F pick a place w of K lying over v). Since $H^1(K,T)$ and $H^1(K_w,T)$ are trivial, we have

$$\ker^1(K/F,T) \simeq \ker^1(F,T).$$

Dividing by $N_{K/F}(T(\mathbb{A}_K))$, we get another exact sequence

$$\bigoplus_v T(F_v)/N_{K_w/F_v}(T(K_w)) \to \tilde{H}^0(K/F,T(\mathbb{A}_K)/T(K)) \to \ker^1(K/F,T) \to 1.$$

Therefore $\mathrm{Hom}\left(\ker^1(F,T),\mathbb{Q}/\mathbb{Z}\right)$ is canonically isomorphic to the kernel of

$$\mathrm{Hom}\left(\tilde{H}^0(K/F,T(\mathbb{A}_K)/T(K)),\mathbb{Q}/\mathbb{Z}\right) \to \prod_v \mathrm{Hom}\left(\tilde{H}^0(K_w/F_v,T(K_w)),\mathbb{Q}/\mathbb{Z}\right).$$

Applying Tate-Nakayama duality in its standard form, we see that this kernel is canonically isomorphic to the kernel of

$$H^2(K/F,X) \to \prod_v H^2(K_w/F_v,X),$$

which we denote by $\ker^2(K/F,X)$. Note that $H^2(K/F,X)$ is finite and hence that $\ker^2(K/F,X)$ is also finite. Since $H^1(K,X)$ is trivial, we have the exact restriction-inflation sequence

$$0 \to H^2(K/F,X) \to H^2(F,X) \to H^2(K,X)$$

as well as its local analogs. Moreover $\ker^2(K,X)$ is trivial, since by classfield theory

$$H^2(K,\mathbb{Z}) = H^1(K,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{A}_K^\times/K^\times,\mathbb{Q}/\mathbb{Z})$$
$$H^2(K_w,\mathbb{Z}) = H^1(K_w,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(K_w^\times,\mathbb{Q}/\mathbb{Z}).$$

Therefore

$$\ker^2(K/F,X) \simeq \ker^2(F,X),$$

and we conclude that $\mathrm{Hom}\left(\ker^1(F,T),\mathbb{Q}/\mathbb{Z}\right)$ is canonically isomorphic to the finite abelian group $\ker^2(F,X)$. It then follows that $\ker^1(F,T)$ is also finite.

It remains to prove the last statement of the lemma. The isomorphism

$$\ker^2(F,X) \simeq \ker^1(F,\hat{T})$$

follows immediately from the isomorphism

$$H^1(F,\hat{T}) \simeq H^2(F,X)$$

obtained from the exponential sequence

$$0 \to X \to \mathrm{Lie}(\hat{T}) \to \hat{T} \to 1$$

as well as its local analogs. The isomorphism

$$\ker^1(F,\hat{T}) = \ker^1(W_F,\hat{T})$$

can be proved the same way as Lemma 11.2.2 of [K1].

E. Tamagawa numbers for $H^i(\mathbb{A}/F, T \to U)$

In this appendix we define Tamagawa measures on the groups

$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)$$
$$H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$$
$$H^i(\mathbb{A}, T \xrightarrow{f} U)$$
$$H^i(\mathbb{A}, T \xrightarrow{f} U)_1$$

and study the Tamagawa numbers

$$\mathrm{vol}(H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1)$$

(see Lemma E.3.D) as well as some variants (see Lemma E.3.E). In case $T \xrightarrow{f} U$ is surjective with finite kernel C, our results reduce to those of Oesterlé [Oe] for C; in case U is trivial they reduce to those of Ono [O] for T.

(E.1) Measured complexes. The results in this section are slight generalizations of results in [Oe,§§3–4]. Consider a complex

$$A^0 \to A^1 \to A^2 \to \cdots$$

of locally compact Hausdorff abelian topological groups. Put

$$\begin{aligned} B^i &= \operatorname{im}[A^{i-1} \to A^i] \qquad (B^0 = 0) \\ Z^i &= \ker[A^i \to A^{i+1}] \\ H^i(A^\bullet) &= Z^i/B^i. \end{aligned}$$

Of course we put the subspace topologies on B^i and Z^i and the quotient topology on $H^i(A^\bullet)$. We say that $A^\bullet$ is a *topological complex* if for all $i \geq 0$ B^{i+1} is closed in A^{i+1} and the canonical map

$$A^i/Z^i \to B^{i+1}$$

is an isomorphism of topological groups. We say that a topological complex $A^\bullet$ has *compact cohomology* if $H^i(A^\bullet)$ is compact for all i and trivial for all but finitely many i. By a *topological short exact sequence* we will mean a short exact sequence that is isomorphic to one of the form

$$0 \to B \to A \to A/B \to 0$$

for a closed subgroup B of A (we give B the subspace topology and A/B the quotient topology).

Lemma E.1.A. *Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a topological short exact sequence of complexes of locally compact Hausdorff abelian topological groups (in other words for all $i \geq 0$

$$0 \to A^i \to B^i \to C^i \to 0$$

is a topological short exact sequence). If $A^\bullet$,$B^\bullet$ are topological complexes with compact cohomology, then so is $C^\bullet$.

Fix $i \geq 0$ and put

$$\begin{aligned} A &= A^i \\ B &= \ker[B^i \to C^{i+1}] \\ D &= \operatorname{im}[B^{i-1} \to B^i] \\ E &= \ker[B^i \to B^{i+1}] \\ A_0 &= \operatorname{im}[A^{i-1} \to A^i]. \end{aligned}$$

Of course we interpret A^{-1},B^{-1},C^{-1} as 0. Our hypotheses imply the following statements:

(1) A,A_0,D,E are closed subgroups of B,
(2) $B \supset E \supset D \supset A_0$,
(3) $B \supset A \supset A_0$,
(4) E/D is compact,
(5) $(A \cap E)/A_0$ is compact,
(6) $A + E$ is closed in B,
(7) the natural map $A/(A \cap E) \to (A + E)/E$ is an isomorphism of topological groups,
(8) $B/(A + E)$ is compact.

Only the last statement deserves comment. Note that the natural injection $B/E \to B^{i+1}$ carries B/E into a closed subgroup of

$$\ker[A^{i+1} \to A^{i+2}],$$

and since $H^{i+1}(A^\bullet)$ is compact, we conclude that the quotient of B/E by $(A+E)/E$, namely $B/(A + E)$, is compact.

We need to show that the image of D in B/A is closed (equivalently, that $A + D$ is closed in B), that

$$D/(A \cap D) \to (A + D)/A$$

is a homeomorphism, and that $B/(A + D)$ is compact (note that $B/(A + D)$ is isomorphic as topological group to $H^i(C^\bullet)$). We will show that these three statements follow from (1)–(8). Dividing by A_0, we reduce immediately to the case in which $A_0 = 0$.

We begin by showing that the obvious surjective homomorphism

$$s : A \oplus E \to A + E$$

is proper. Let K be a compact subset of $A + E$. By (7) the set

$$K_A := \{a \in A | \exists e \in E \text{ such that } a + e \in K\}$$

has compact image in $A/(A \cap E)$, and since $A \cap E$ is compact by (5) it follows that K_A is compact. Let

$$K_E = (K - K_A) \cap E,$$

a compact subset of E (here $K - K_A$ denotes the set of elements of the form $x - y$ for $x \in K$ and $y \in K_A$). Then $s^{-1}(K)$ is closed in the compact set $K_A \oplus K_E$ and is therefore compact.

Since s is proper and $A + E$ is locally compact Hausdorff (by (6)), s is a closed mapping, and it follows that the image of $A \oplus D$, namely $A + D$, is closed in $A + E$ and hence in B (again by (6)). This was the first point we needed to check.

Consider the groups

$$B \supset A + E \supset A + D.$$

We have seen that both subgroups are closed in B. Furthermore $B/(A+E)$ is compact by (8), and $(A+E)/(A+D)$ is compact as well, being a quotient of the compact group E/D (use (4)). Therefore $B/(A+D)$ is compact; this was the third point we needed to check.

We have seen that s is closed; therefore s induces an isomorphism of topological groups

$$(A \oplus E)/(A \cap E) \to A + E,$$

and it follows that the natural map

$$A + E \to E/(A \cap E)$$

is continuous and hence that

$$\alpha : E/(A \cap E) \to (A+E)/A$$

is a homeomorphism. We want to verify the second point, namely that

$$j : D/(A \cap D) \to (A+D)/A$$

is a homeomorphism. For this it is enough to show that j is closed, and since $(A+D)/A$ is closed in $(A+E)/A$ by previous work, it is enough to check that

$$i : D/(A \cap D) \to E/(A \cap E) = (A+E)/A$$

is closed (use that α is a homeomorphism). Since $E/(A \cap E)$ is locally compact Hausdorff, it is enough to show that i is proper. Consider a compact subset K of $E/(A \cap E)$ and let K_E denote its inverse image in E. Then K_E is compact by (5). The inverse image of K in $D/(A \cap D)$ is $(D \cap K_E)/(A \cap D)$, a compact subset of $D/(A \cap D)$. Thus i is indeed proper, and we are done with the proof of the lemma.

Next we discuss complexes in which each group is equipped with a Haar measure. We say that a Haar measure on a discrete group is *discrete* if it gives points measure 1. Let $A^\bullet$ be a topological complex with compact cohomology having the further property that A^i is a discrete group for all but finitely many i. Assume that for each $i \geq 0$ we are given a Haar measure da_i on A^i and suppose that da_i is discrete for all but finitely many i. We refer to $A^\bullet$, with the given measures, as a *measured complex.*

Given a measured complex $A^\bullet$, we define a positive real number $t(A^\bullet)$ as follows. For each $i \geq 0$ pick a Haar measure dz_i on the subgroup Z^i of A^i with dz_i discrete for all but finitely many i; then take as Haar measure on $B^i \simeq A^{i-1}/Z^{i-1}$ the quotient measure $db_i := da_{i-1}/dz_{i-1}$ (for $i = 0$ take the discrete measure on $B^0 = 0$); then take as Haar measure on $H^i(A^\bullet) = Z^i/B^i$ the quotient measure dz_i/db_i; finally put

$$t(A^\bullet) = \prod_{i=0}^{\infty} \operatorname{meas}(H^i(A^\bullet))^{(-1)^i}.$$

Each term in the product is finite since $H^i(A^\bullet)$ is compact, and all but finitely many terms are equal to 1 since $H^i(A^\bullet)$ is 0 with the discrete measure for all but finitely many i. Moreover $t(A^\bullet)$ is independent of the choice of auxiliary measures dz_i. Note that if da_i is replaced by $c_i da_i$ (with $c_i = 1$ for all but finitely many i), then $t(A^\bullet)$ is multiplied by $\prod_{i=0}^{\infty} c_i^{(-1)^i}$.

We extend the definition of $t(A^\bullet)$ to finite topological complexes

$$A^\bullet = [A^0 \to A^1 \to \cdots \to A^n]$$

with compact cohomology and given Haar measures da_i on A^i by regarding $A^\bullet$ as an infinite complex

$$A^0 \to A^1 \to \cdots \to A^n \to 0 \to 0 \to \cdots$$

and taking the discrete measure on $A^m = 0$ for all $m > n$. Suppose that

$$1 \to A^0 \to A^1 \to A^2 \to 1$$

is a topological short exact sequence, and regard

$$A^0 \to A^1 \to A^2$$

as a complex with three terms. Put Haar measures da_i on A^i for $i = 0, 1, 2$. Then the t-value of our complex is 1 if and only if

$$da_2 = da_1/da_0.$$

Lemma E.1.B. *Suppose that A^i and da_i are discrete for all $i \geq 0$ and that $H^i(A^\bullet) = 0$ for all $i \geq 0$. Then $t(A^\bullet) = 1$.*

This is obvious.

Lemma E.1.C. *Let $A^\bullet$ be a finite measured complex such that A^i is compact for all i. Then*

$$t(A^\bullet) = \prod_{i=0}^{\infty} \mathrm{meas}_{da_i}(A^i)^{(-1)^i}.$$

Without loss of generality we may assume that $\mathrm{meas}_{da_i}(A^i) = 1$ for all i. We must then show that $t(A^\bullet) = 1$. To calculate $t(A^\bullet)$ we choose the Haar measure dz_i on the compact group Z_i so that $\mathrm{meas}_{dz_i}(Z_i) = 1$; then $\mathrm{meas}(H^i(A^\bullet)) = 1$ for all i and hence $t(A^\bullet) = 1$.

Lemma E.1.D. *Let A^{ij} $(i \geq 0, j \geq 0)$ be a double complex of locally compact Hausdorff abelian topological groups. Assume that each row and column of the double complex is a topological complex with compact cohomology. Assume further that we are given on each group A^{ij} a Haar measure da_{ij}. Finally we suppose that there exists a non-negative integer N such that for all pairs (i, j) with $i \geq N$ or $j \geq N$ the following three statements hold:*

(1) *A^{ij} and da_{ij} are discrete,*
(2) *the j-th cohomology group of the i-th row $A^{i\bullet}$ is trivial,*
(3) *the i-th cohomology group of the j-th column $A^{\bullet j}$ is trivial.*

Note that each row $A^{i\bullet}$ *and column* $A^{\bullet j}$ *is a measured complex. Then there is an equality*

$$\prod_{i=0}^{\infty} t(A^{i\bullet})^{(-1)^i} = \prod_{j=0}^{\infty} t(A^{\bullet j})^{(-1)^j}$$

(*the products are finite by Lemma* E.1.B).

We denote by

$$\begin{aligned} Z^{ij} &:= \ker[A^{ij} \to A^{i,j+1}] \\ B^{ij} &:= \operatorname{im}[A^{i,j-1} \to A^{ij}] \end{aligned}$$

the cycle and boundary groups for the rows of the double complex. We begin by reducing to the case in which the double complex has only finitely many non-trivial columns. Truncate each row at degree N to obtain a new double complex C^{ij} with

$$C^{ij} = \begin{cases} A^{ij} & j < N \\ Z^{iN} & j = N \\ 0 & j > N, \end{cases}$$

and put the discrete measure on each discrete group C^{ij} $(j \geq N)$. Note that the column Z^{iN} has trivial cohomology (calculate the total cohomology of the double complex A^{ij} $(i \geq 0, j \geq N)$ two ways). It is easy to see that every row and column of C^{ij} has the same t-value as the corresponding row or column of A^{ij}. Our reduction step is complete, and we now assume that $A^{ij} = 0$ for $j > N$.

We want to use induction on the number of non-trivial columns. To get a complex having fewer columns we consider

$$\begin{array}{ccccccc} A^{01}/B^{01} & \longrightarrow & A^{02} & \longrightarrow & A^{03} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{11}/B^{11} & \longrightarrow & A^{12} & \longrightarrow & A^{13} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{21}/B^{21} & \longrightarrow & A^{22} & \longrightarrow & A^{23} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \end{array}$$

To compare this with the original double complex we need three other double complexes, one having a single column and two having three columns:

$$\begin{array}{ccccccccccc}
Z^{00} & & Z^{00} & \longrightarrow & A^{00} & \longrightarrow & B^{01} & & B^{01} & \longrightarrow & A^{01} & \longrightarrow & A^{01}/B^{01} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z^{10} & & Z^{10} & \longrightarrow & A^{10} & \longrightarrow & B^{11} & & B^{11} & \longrightarrow & A^{11} & \longrightarrow & A^{11}/B^{11} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z^{20} & & Z^{20} & \longrightarrow & A^{20} & \longrightarrow & B^{21} & & B^{21} & \longrightarrow & A^{21} & \longrightarrow & A^{21}/B^{21} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

Note that Z^{i0} is compact for all i and trivial for all but finitely many i. Applying Lemma E.1.A to the two double complexes with three columns we find that $B^{\bullet 1}$ and $A^{\bullet 1}/B^{\bullet 1}$ are toplogical complexes with compact cohomology.

Pick Haar measures dz_{i0} on Z^{i0}, discrete for all but finitely many i; then as Haar measure on B^{i1} take the quotient measure $db_{i1} := da_{i0}/dz_{i0}$ and as Haar measure on A^{i1}/B^{i1} take the quotient measure da_{i1}/db_{i1}. With these measures all four of the double complexes obtained from A^{ij} also satisfy the hypotheses of the lemma. It is not hard to see that if the lemma is true for all four of these double complexes then it is true for the original double complex A^{ij}. Thus by induction it is enough to prove the lemma in the following two special cases:

(1) A^{ij} consists of a single column,
(2) A^{ij} consists of three columns $A^\bullet \to B^\bullet \to C^\bullet$, and each row of the double complex forms a topological short exact sequence

$$0 \to A^i \to B^i \to C^i \to 0.$$

The first special case is covered by Lemma E.1.C.

It remains to prove the lemma in the second special case. Applying the methods above to the rows rather than the columns, we reduce to proving the lemma in the case of a double complex with three rows and three columns, with all rows and columns forming topological short exact sequences:

$$\begin{array}{ccccc}
A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 \\
\downarrow & & \downarrow & & \downarrow \\
A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 \\
\downarrow & & \downarrow & & \downarrow \\
A^2 & \longrightarrow & B^2 & \longrightarrow & C^2
\end{array}$$

Pick Haar measures da_0, da_1, db_0, db_1 on A^0, A^1, B^0, B^1 respectively. Define Haar measures on A^2,B^2,C^0,C^1 by

$$\begin{aligned} da_2 &:= da_1/da_0 \\ db_2 &:= db_1/db_0 \\ dc_0 &:= db_0/da_0 \\ dc_1 &:= db_1/da_1. \end{aligned}$$

From these we get two Haar measures on C^2, namely $dx := db_2/da_2$ and $dy := dc_1/dc_0$. The lemma is equivalent to the statement that dx equals dy. Dividing by A_0, we reduce to the case in which A_0 is trivial. Then the closed subgroups A^1 and B^0 of B^1 have trivial intersection. Moreover, the map $A^1 \to B^1/B^0$ is a closed injection, which means that the kernel of $B^1 \to C^2$ is the direct product of A^1 and B^0. The measures dx and dy are both equal to the quotient measure $db_1/(da_1 \times db_0)$. This finishes the proof.

In (E.2) we will need the following construction. Let X be a finitely generated abelian group. In the real vector space $\operatorname{Hom}(X,\mathbb{R})$ we have the lattice $\operatorname{Hom}(X,\mathbb{Z})$.

Definition E.1.E. The canonical measure on $\operatorname{Hom}(X,\mathbb{R})$ is the Haar measure

$$|X_{\text{tors}}|^{-1}dx,$$

where X_{tors} is the torsion subgroup of X and dx is the Haar measure on $\operatorname{Hom}(X,\mathbb{R})$ that gives measure 1 to the compact group

$$\operatorname{Hom}(X,\mathbb{R})/\operatorname{Hom}(X,\mathbb{Z})$$

(use the discrete measure on $\operatorname{Hom}(X,\mathbb{Z})$).

Lemma E.1.F. *Let*

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

be a finite complex of finitely generated abelian groups, and assume that every homology group $H_i(X_\bullet)$ is finite. Let $A^\bullet$ be the measured complex

$$\operatorname{Hom}(X_0,\mathbb{R}) \to \cdots \to \operatorname{Hom}(X_n,\mathbb{R})$$

(for each i use the canonical measure on $\operatorname{Hom}(X_i,\mathbb{R})$*). Then*

$$t(A^\bullet)^{-1} = \prod_{i=0}^{n} |H_i(X_\bullet)|^{(-1)^i}.$$

Consider the double complex

$$\begin{array}{ccccccc} X_0^* & \longrightarrow & X_1^* & \longrightarrow \cdots \longrightarrow & X_n^* \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}(X_0,\mathbb{R}) & \longrightarrow & \operatorname{Hom}(X_1,\mathbb{R}) & \longrightarrow \cdots \longrightarrow & \operatorname{Hom}(X_n,\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ K_0 & \longrightarrow & K_1 & \longrightarrow \cdots \longrightarrow & K_n \end{array}$$

where X_i^* denotes the discrete group $\operatorname{Hom}(X_i, \mathbb{Z})$ and K_i denotes the compact group

$$\operatorname{Hom}(X_i, \mathbb{R})/X_i^*.$$

Put the discrete measures on the groups in the top row, put the canonical measures on the groups in the middle row, and put the Haar measures with total mass 1 on the groups in the bottom row. Lemma E.1.D implies that

$$t(A^\bullet)^{-1} = \prod_{i=0}^{n} \left(|(X_i)_{\text{tors}}|/|H^i(X_\bullet^*)|\right)^{(-1)^i}.$$

It remains to prove that

$$\prod_{i=0}^{n} \left(|(X_i)_{\text{tors}}|/|H^i(X_\bullet^*)|\right)^{(-1)^i} = \prod_{i=0}^{n} |H_i(X_\bullet)|^{(-1)^i}. \tag{E.1.1}$$

Assume for the moment that X_i is free for all i. Then

$$H^i(X_\bullet^*) = \operatorname{Ext}^i(X_\bullet, \mathbb{Z}).$$

From the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we get the long exact sequence

$$\cdots \to \operatorname{Ext}^i(X_\bullet, \mathbb{Z}) \to \operatorname{Ext}^i(X_\bullet, \mathbb{Q}) \to \operatorname{Ext}^i(X_\bullet, \mathbb{Q}/\mathbb{Z}) \to \cdots$$

and since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective we have

$$\operatorname{Ext}^i(X_\bullet, \mathbb{Q}) = \operatorname{Hom}(H_i(X_\bullet), \mathbb{Q}) = 0$$
$$\operatorname{Ext}^i(X_\bullet, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(H_i(X_\bullet), \mathbb{Q}/\mathbb{Z}).$$

Therefore $H^{i+1}(X_\bullet^*)$ and $H_i(X_\bullet)$ are dual finite abelian groups, and in particular they have the same cardinality. This proves (E.1.1) when the X_i are free.

Now we prove (E.1.1) in general. We can find a complex of abelian groups

$$Y_\bullet = [Y_n \to \cdots \to Y_0]$$

with each Y_i free of finite rank and a quasi-isomorphism

$$Y_\bullet \to X_\bullet$$

with $Y_i \to X_i$ surjective for all i. Put $Z_i := \ker[Y_i \to X_i]$. Then each Z_i is free of finite rank and $Z_\bullet$ is an acyclic complex. To construct $Y_\bullet$ start by picking any surjection $Y_0 \to X_0$ with Y_0 free of finite rank. Take Y_1 free of finite rank mapping onto $X_1 \times_{X_0} Y_0$. Then Z_1 maps onto Z_0, and by the snake lemma the kernels of $Z_1 \to Z_0$, $Y_1 \to Y_0$, $X_1 \to X_0$ form an exact sequence. By iterating this construction we get the desired complex.

Put the discrete measures on the groups X_i,Y_i,Z_i. Applying Lemma E.1.D to the double complex

$$Z_\bullet \to Y_\bullet \to X_\bullet$$

we find that

$$t(X_\bullet)t(Z_\bullet) = t(Y_\bullet).$$

We also have the dual double complex

$$X_\bullet^* \to Y_\bullet^* \to Z_\bullet^* \to E_\bullet$$

where

$$E_i := \mathrm{Ext}^1(X_i, \mathbb{Z}).$$

Again using discrete measures on all groups, and applying Lemma E.1.D, we find that

$$t(X_\bullet^*)t(Z_\bullet^*) = t(Y_\bullet^*)t(E_\bullet).$$

Moreover

$$t(Z_\bullet^*)t(Z_\bullet) = t(Y_\bullet^*)t(Y_\bullet) = 1$$

by the special case done before. Combining these equalities, we find that

$$t(X_\bullet^*)t(X_\bullet) = t(E_\bullet),$$

which is equivalent to (E.1.1) since by Lemma E.1.C

$$t(E_\bullet) = \prod_{i=0}^{n} |E_i|^{(-1)^i}$$

and E_i has the same cardinality as $(X_i)_{\mathrm{tors}}$.

(E.2) Tamagawa measures on $H^i(\mathbb{A}, T \to U)$ and $H^i(\mathbb{A}/F, T \to U)$. We use the same notation as in (C.1). In Appendix C we topologized the groups $H^i(\mathbb{A}, T \xrightarrow{f} U)$ and $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$. Now we define canonical Haar measures, called Tamagawa measures, on them. Put

$$\begin{aligned} C &:= \ker[T \to U] \\ V &:= \mathrm{im}[T \to U] \\ W &:= \mathrm{cok}[T \to U] \\ A &:= C/C^0. \end{aligned}$$

Then V, W are F-tori, C is a diagonalizable F-group, and A is a finite abelian F-group. First consider $H^0(\mathbb{A}, T \xrightarrow{f} U) = C(\mathbb{A})$. For the identity component C^0 of C

we have the usual Tamagawa measure (see [O]) on $C^0(\mathbb{A})$; write it as a product of local measures dc_v^0 on $C^0(F_v)$, and abuse notation by also writing dc_v^0 for the unique Haar measure on $C(F_v)$ that induces dc_v^0 on the open subgroup $C^0(F_v)$. Put

$$dc_v = |A(F_v)|^{-1} dc_v^0$$
$$dc = \prod_v dc_v.$$

We define the Tamagawa measure on $H^0(\mathbb{A}, T \xrightarrow{f} U)$ to be dc. Suppose that v is a finite place such that T and U are unramified and dc_v^0 gives measure 1 to $C^0(\mathcal{O}_v)$. We now verify that dc_v gives measure 1 to $C(\mathcal{O}_v)$, so that the product measure dc makes sense.

Note that $A(\mathcal{O}_v) = A(F_v)$, since A is proper over $\mathcal{O}_v$ (the groups C, A over F_v extend naturally to groups over $\mathcal{O}_v$), and note also that

$$C(\mathcal{O}_v) \to A(\mathcal{O}_v) = A(F_v)$$

is surjective (use smoothness of the map $C \to A$ of schemes over $\mathcal{O}_v$ plus surjectivity of $C(k_v) \to A(k_v)$, a consequence of Lang's Theorem for the connected group C^0). Therefore $C^0(\mathcal{O}_v)$ has index $|A(F_v)|$ in $C(\mathcal{O}_v)$, and we conclude that dc_v indeed gives measure 1 to $C(\mathcal{O}_v)$. At such a place v it is clear from what we have done that

$$C(F_v)/C^0(F_v) = A(F_v),$$

so that we could equally well have defined dc_v by dividing dc_v^0 by $|C(F_v)/C^0(F_v)|$ instead of $|A(F_v)|$. At ramified places this can lead to different local measures, and so we would then obtain an alternative Tamagawa measure on $H^0(\mathbb{A}, T \xrightarrow{f} U)$. There seems to be no reason to prefer one choice over the other, but we made a choice (randomly) and will stick to it throughout this paper.

Next consider $H^1(\mathbb{A}, T \xrightarrow{f} U)$. On $W(\mathbb{A})$ we have the usual Tamagawa measure dw; write it as a product of local measures dw_v. Consider the exact sequence (see (A.1))

$$1 \to H^1(F_v, C) \to H^1(F_v, T \xrightarrow{f} U) \to W(F_v) \to \cdots .$$

Let dw_v^0 denote the restriction of dw_v to the image of $H^1(F_v, T \xrightarrow{f} U)$ in $W(F_v)$, an open subgroup of finite index in $W(F_v)$. As topological group this subgroup is isomorphic to the quotient of $H^1(F_v, T \xrightarrow{f} U)$ by the discrete finite group $H^1(F_v, C)$, and so we may define a Haar measure dx_v on $H^1(F_v, T \xrightarrow{f} U)$ by requiring that its quotient by the discrete measure on $H^1(F_v, C)$ be equal to dw_v^0. Then put

$$dy_v = |A(F_v)|^{-1} dx_v$$
$$dy = \prod_v dy_v.$$

We leave it to the reader to check that if v is a finite place such that T, U are unramified and dw_v gives measure 1 to $W(\mathcal{O}_v)$, then dy_v gives measure 1 to $H^1(\mathcal{O}_v, T \xrightarrow{f} U)$; for this one uses the equality

$$|H^1(\mathcal{O}_v, C)| = |A(F_v)|.$$

We define the Tamagawa measure on $H^1(\mathbb{A}, T \xrightarrow{f} U)$ to be dy. Note that $|A(F_v)|^{-1}$ is the same factor that appeared before. Again we have an alternative: we could divide by $|C(F_v)/C^0(F_v)|$ instead of $|A(F_v)|$. What matters is that we use the same factor both times.

Finally, for all $i \geq 2$ we define the Tamagawa measure on the discrete group $H^i(\mathbb{A}, T \xrightarrow{f} U)$ to be the discrete measure. Note that for all i the Tamagawa measure on $H^i(\mathbb{A}, T \xrightarrow{f} U)$ does not change when $T \xrightarrow{f} U$ is replaced by a quasi-isomorphic complex $T' \xrightarrow{f'} U'$.

For a single torus T we define Tamagawa measures on $H^i(\mathbb{A}, T)$ as follows. On $H^0(\mathbb{A}, T) = T(\mathbb{A})$ we take the usual Tamagawa measure and for $i \geq 1$ we take the discrete measure on the discrete group $H^i(\mathbb{A}, T)$.

Lemma E.2.A. *Let*

$$[T_1 \to U_1] \to [T_2 \to U_2] \to [T_3 \to U_3] \to [T_1 \to U_1][1]$$

be a distinguished triangle. Let $A^\bullet$ denote the long exact sequence

$$H^0(\mathbb{A}, T_1 \to U_1) \to H^0(\mathbb{A}, T_2 \to U_2) \to H^0(\mathbb{A}, T_3 \to U_3) \to H^1(\mathbb{A}, T_1 \to U_1) \to \cdots .$$

Then $t(A^\bullet) = 1$ (use Tamagawa measures on all groups in the complex $A^\bullet$). In particular for any $T \xrightarrow{f} U$ the complex

$$H^0(\mathbb{A}, T \to U) \to H^0(\mathbb{A}, T) \to H^0(\mathbb{A}, U) \to H^1(\mathbb{A}, T \to U) \to \cdots$$

has t-value 1.

The second statement follows from the first, applied to the distinguished triangle

$$[1 \to U] \to [T \to U] \to [T \to 1] \to [U \to 1].$$

The first statement follows from the following local statement. Let v be a place of F. Define diagonalizable groups $C_1, \dots, C_6$ by

$$C_i = \ker[T_i \to U_i]$$
$$C_{i+3} = \operatorname{cok}[T_i \to U_i]$$

for $i = 1, 2, 3$. The distinguished triangle gives rise to an acyclic complex of diagonalizable F-groups

$$C_1 \to C_2 \to C_3 \to C_4 \to C_5 \to C_6.$$

Put

$$S_i = \ker[C_i \to C_{i+1}] = \operatorname{im}[C_{i-1} \to C_i]$$

(take $C_i = 1$ for i outside the range $1, ..., 6$). We have short exact sequences

$$1 \to S_i \to C_i \to S_{i+1} \to 1.$$

For all i pick a non-zero invariant differential form ds_i of top degree on S_i; assume $ds_i = 1$ for i outside the range $2, \ldots, 6$. We get a non-zero invariant differential form dc_i of top degree on C_i by putting

$$dc_i := ds_i \otimes ds_{i+1}.$$

Then dc_i (respectively, ds_i) gives us a Haar measure $|dc_i|$ (respectively, $|ds_i|$) on $C_i(F_v)$ (respectively, $S_i(F_v)$), and the restriction of $|ds_{i+1}|$ to the open subgroup

$$C_i(F_v)/S_i(F_v)$$

coincides with the quotient measure $|dc_i|/|ds_i|$.

Define groups $H_1, \ldots, H_6$ by

$$H_i = H^0(F_v, T_i \to U_i) = C_i(F_v)$$
$$H_{i+3} = H^1(F_v, T_i \to U_i)$$

for $i = 1, 2, 3$. Then the first six terms in the local analog of $A^\bullet$ are

$$H_1 \to H_2 \to \cdots \to H_6.$$

There is an obvious commutative diagram

$$\begin{array}{ccccccc} H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_6 \\ \downarrow & & \downarrow & & & & \downarrow \\ C_1(F_v) & \longrightarrow & C_2(F_v) & \longrightarrow & \cdots & \longrightarrow & C_6(F_v) \end{array}$$

in which the vertical arrows are isomorphisms for $i = 1, 2, 3$ and have finite kernel and cokernel for $i = 4, 5, 6$. Put a Haar measure dh_i on H_i by requiring that the quotient of dh_i by the discrete measure on

$$\ker[H_i \to C_i(F_v)]$$

be equal to the restriction of $|dc_i|$ to the open subgroup

$$\operatorname{im}[H_i \to C_i(F_v)].$$

We can finally formulate the local statement that we need to prove: the complex

$$H^0(T_1 \to U_1) \to H^0(T_2 \to U_2) \to H^0(T_3 \to U_3) \to H^1(T_1 \to U_1) \to \cdots$$

(in which we have abbreviated $H^i(F_v,\cdot)$ to $H^i(\cdot)$) has t-value 1, where we use the measures $dh_1,\dots,dh_6$ on the first six terms in the complex and the discrete measures on the remaining groups. To prove this we define groups

$$Z_i := \mathrm{im}[H_{i-1} \to H_i]$$

for $i = 1,\dots,6$ (take $H_0 = 1$). There are natural maps

$$Z_i \to S_i(F_v)$$

for $i = 1,\dots,6$, and these maps have finite kernel and cokernel. Put a Haar measure dz_i on Z_i by requiring that the quotient of dz_i by the discrete measure on

$$\ker[Z_i \to S_i(F_v)]$$

be equal to the restriction of $|ds_i|$ to the open subgroup

$$\mathrm{im}[Z_i \to S_i(F_v)].$$

To prove that the t-value of our local complex is 1 it suffices to prove that $dh_i/dz_i = dz_{i+1}$ for $i = 1,\dots,5$ and also that dh_i/dz_i is discrete for $i = 6$. Assume first that i is in the range $1,\dots,5$ and consider the measured double complex (whose rows are short exact sequences)

$$\begin{array}{ccccc} Z_i & \longrightarrow & H_i & \longrightarrow & Z_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ S_i(F_v) & \longrightarrow & C_i(F_v) & \longrightarrow & C_i(F_v)/S_i(F_v) \end{array}$$

(use the measures dz_i, dh_i, dz_{i+1}, $|ds_i|$, $|dc_i|$, $|dc_i|/|ds_i|$ and recall that $|dc_i|/|ds_i|$ is the restriction of $|ds_{i+1}|$). The kernels and cokernels of the vertical maps are finite, and by the snake lemma the alternating product of their orders is 1; therefore the alternating product of the t-values of the columns is 1. It then follows from Lemma E.1.D that the t-value of the first row is equal to the t-value of the second row, namely 1, which means that $dh_i/dz_i = dz_{i+1}$.

For $i = 6$ we have the measured double complex

$$\begin{array}{ccccc} Z_6 & \longrightarrow & H_6 & \longrightarrow & H_6/Z_6 \\ \downarrow & & \downarrow & & \downarrow \\ S_6(F_v) & \xrightarrow{\simeq} & C_6(F_v) & \longrightarrow & 1 \end{array}$$

and again the vertical maps have finite kernel and cokernel. Using the snake lemma as before, we conclude that the t-value of the top row is 1 if we use dz_6, dh_6 and the discrete measure on H_6/Z_6, which means that dh_6/dz_6 is discrete.

Next we need Tamagawa measures on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$. Let dy_i denote the Tamagawa measure on $H^i(\mathbb{A}, T \xrightarrow{f} U)$; we take its quotient by the discrete measure on the discrete subgroup

$$\operatorname{im}\left[H^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}, T \xrightarrow{f} U)\right],$$

obtaining a Haar measure $d\bar{y}_i$ on the open subgroup $\operatorname{cok}^i(F, T \xrightarrow{f} U)$ of $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$; finally we define the Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ to be the unique Haar measure whose restriction to $\operatorname{cok}^i(F, T \xrightarrow{f} U)$ coincides with $d\bar{y}_i$. Note that for $i \geq 2$ this Tamagawa measure is discrete. We follow the same procedure for a single torus T: as Tamagawa measure on $H^0(\mathbb{A}/F, T)$ we take the unique Haar measure inducing the usual Tamagawa measure on the open subgroup $T(\mathbb{A})/T(F)$ and for $i \geq 1$ we take the Tamagawa measure to be the discrete measure on the discrete group $H^i(\mathbb{A}/F, T)$.

Lemma E.2.B. *Fix $i \geq 0$ and let $A^\bullet$ denote the complex*

$$\begin{aligned}\ker^i(F, T \to U) \to & H^i(F, T \to U) \to H^i(\mathbb{A}, T \to U)\\ & \to H^i(\mathbb{A}/F, T \to U) \to \ker^{i+1}(F, T \to U).\end{aligned}$$

Make $A^\bullet$ into a measured complex by using the Tamagawa measures on $H^i(\mathbb{A}, T \xrightarrow{f} U)$ and $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ and the discrete measures on the other three groups. Then $t(A^\bullet) = 1$.

This is an immediate consequence of the definitions.

There are still more Tamagawa measures to define. In (C.2) we defined a compact subgroup $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ of $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$. For $i \geq 2$ we take as Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ the discrete measure on this discrete group. The groups $H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1$ and $H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1$ are the kernels of

$$H : H^0(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$H : H^1(\mathbb{A}/F, T \xrightarrow{f} U) \to \operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

respectively. We will put canonical measures on $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ and $\operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$ and then define the Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ $(i = 0, 1)$ by requiring that the quotient of the Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ by the Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ yield the canonical measure on $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ (resp., $\operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$) for $i = 0$ (resp., $i = 1$).

Recall that

$$\begin{aligned}\operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U] &= \operatorname{Hom}\left(\ker[Y^\Gamma \to X^\Gamma], \mathbb{R}\right)\\ &= \operatorname{Hom}\left(X^*(W)^\Gamma, \mathbb{R}\right);\end{aligned}$$

thus we have the lattice

$$\mathrm{Hom}(X^*(W)^\Gamma, \mathbb{Z})$$

inside the real vector space

$$\mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U].$$

Of course we take as canonical measure on $\mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$ the unique Haar measure for which the quotient of the vector space by the lattice gets measure 1 (for the quotient of the canonical measure by the discrete measure on the lattice); in other words we are viewing $\mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$ as $\mathrm{Hom}(X^*(W)^\Gamma, \mathbb{R})$ and using the canonical measure in Definition E.1.E.

For $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ the definition of canonical measure is less obvious. Recall that

$$\ker[\mathfrak{A}_T \to \mathfrak{A}_U] = \mathrm{Hom}\left(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{R}\right);$$

thus we have the lattice

$$\mathrm{Hom}\left(\mathrm{cok}[Y^\Gamma \to X^\Gamma], \mathbb{Z}\right)$$

inside the real vector space $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$. However there is a second natural lattice, namely the sublattice

$$\mathrm{Hom}(X^*(C)^\Gamma, \mathbb{Z}).$$

Just as above this sublattice determines a Haar measure da on the vector space $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$, and for our canonical measure on $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ we take the measure

$$|(X^*(C)^\Gamma)_{\mathrm{tors}}|^{-1} da;$$

in other words we are viewing $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ as $\mathrm{Hom}(X^*(C)^\Gamma, \mathbb{R})$ and using the canonical measure in Definition E.1.E.

Note that the kernel and cokernel of $\mathfrak{A}_T \to \mathfrak{A}_U$, as well as the canonical measures on them, do not change if $T \xrightarrow{f} U$ is replaced by a quasi-isomorphic complex $T' \xrightarrow{f'} U'$. That is why we preferred to use $X^*(C)^\Gamma$ rather than $\mathrm{cok}[Y^\Gamma \to X^\Gamma]$. Consequently the Tamagawa measure on $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ also remains unchanged for $i = 0, 1$ and trivially the same is true for $i \geq 3$ as well. However for $i = 2$ the group $H^2(\mathbb{A}/F, T \xrightarrow{f} U)_1$ itself can change when $T \to U$ is replaced by $T' \to U'$.

In $\mathfrak{A}_T$ we have the lattice $\mathrm{Hom}(X^\Gamma, \mathbb{Z})$, which we use as above to get a canonical measure on $\mathfrak{A}_T$ and a Tamagawa measure on $H^0(\mathbb{A}/F, T)_1$. The restriction of this measure to the open subgroup $T(\mathbb{A})_1/T(F)$ is of course the usual Tamagawa measure on that group.

Let

$$[T_1 \to U_1] \to [T_2 \to U_2] \to [T_3 \to U_3] \to [T_1 \to U_1][1]$$

be a distinguished triangle (of complexes of F-tori). Put the canonical measures on

$$\ker[\mathfrak{A}_{T_i} \to \mathfrak{A}_{U_i}]$$

and

$$\mathrm{cok}[\mathfrak{A}_{T_i} \to \mathfrak{A}_{U_i}]$$

for $i = 1, 2, 3$. Let $A^\bullet$ be the measured complex

$$\ker[\mathfrak{A}_{T_1} \to \mathfrak{A}_{U_1}] \to \ker[\mathfrak{A}_{T_2} \to \mathfrak{A}_{U_2}] \to \ker[\mathfrak{A}_{T_3} \to \mathfrak{A}_{U_3}] \to$$
$$\operatorname{cok}[\mathfrak{A}_{T_1} \to \mathfrak{A}_{U_1}] \to \operatorname{cok}[\mathfrak{A}_{T_2} \to \mathfrak{A}_{U_2}] \to \operatorname{cok}[\mathfrak{A}_{T_3} \to \mathfrak{A}_{U_3}]$$

with $\ker[\mathfrak{A}_{T_1} \to \mathfrak{A}_{U_1}]$ placed in degree 0.

Put

$$C_i = \ker[T_i \to U_i]$$

and

$$W_i = \operatorname{cok}[T_i \to U_i]$$

for $i = 1, 2, 3$. The distinguished triangle gives rise to an exact sequence

$$0 \to X^*(W_3) \to X^*(W_2) \to X^*(W_1) \to X^*(C_3) \to X^*(C_2) \to X^*(C_1) \to 0$$

which in turn gives rise to a complex

$$X^*(W_3)^\Gamma \to X^*(W_2)^\Gamma \to X^*(W_1)^\Gamma \to X^*(C_3)^\Gamma \to X^*(C_2)^\Gamma \to X^*(C_1)^\Gamma$$

with $X^*(W_3)^\Gamma$ placed in degree 0. Denote by H^i the i-th cohomology group of this complex.

Let $X_i := X^*(T_i)$ and $Y_i := X^*(U_i)$ for $i = 1, 2, 3$. There is a dual distinguished triangle

$$[Y_3 \to X_3] \to [Y_2 \to X_2] \to [Y_1 \to X_1] \to [Y_3 \to X_3][1].$$

Put

$$u = |\operatorname{cok}[H^1(F, Y_2 \to X_2) \to H^1(F, Y_1 \to X_1)]|$$

and for $i = 1, 2, 3$ put

$$\begin{aligned} v_i &= |\operatorname{cok}[H^1(F, Y_i \to X_i) \to X^*(C_i)^\Gamma]| \\ w_i &= |\ker[H^1(F, Y_i \to X_i) \to X^*(C_i)^\Gamma]| \\ &= |H^1(F, X^*(W_i))| \end{aligned}$$

(the map

$$H^1(F, Y_i \to X_i) \to X^*(C_i)^\Gamma$$

is part of a long exact sequence of the kind described at the end of (A.1)).

Lemma E.2.C. *There are equalities*

$$\begin{aligned} t(A^\bullet) &= \prod_{i=0}^{5} |H^i|^{(-1)^i} \\ &= u^{-1} v_1^{-1} v_2 v_3^{-1} w_1 w_2^{-1} w_3. \end{aligned}$$

The first equality follows immediately from Lemma E.1.F. We get $t(A^\bullet)$ rather than its inverse since we switched from homological degree i to cohomological degree $5-i$. Now we check the second equality. There is a commutative diagram

$$\begin{array}{cccccc} X^*(W_3)^\Gamma & \to X^*(W_2)^\Gamma & \to X^*(W_1)^\Gamma & \to H(3) & \to H(2) & \to H(1) \\ {\scriptstyle 1}\downarrow & {\scriptstyle 1}\downarrow & {\scriptstyle 1}\downarrow & \downarrow & \downarrow & \downarrow \\ X^*(W_3)^\Gamma & \to X^*(W_2)^\Gamma & \to X^*(W_1)^\Gamma & \to X^*(C_3)^\Gamma & \to X^*(C_2)^\Gamma & \to X^*(C_1)^\Gamma \end{array}$$

where we have written $H(i)$ as an abbreviation for

$$H^1(F, Y_i \to X_i).$$

Apply Lemma E.1.D to the commutative diagram above (use the discrete measure on each group). The top row is part of the long exact sequence coming from our distinguished triangle and is therefore exact except at the right end, which means that its t-value is

$$|\operatorname{cok}[H(2) \to H(1)]|^{-1} = u^{-1}.$$

The t-value of the bottom row is of course

$$\prod_{i=0}^{5} |H^i|^{(-1)^i}.$$

The t-value of each of the first three columns is 1, and the t-values of the last three columns are $w_3 v_3^{-1}$, $w_2 v_2^{-1}$, $w_1 v_1^{-1}$. We conclude from Lemma E.1.D that

$$\prod_{i=0}^{5} |H^i|^{(-1)^i} = u^{-1} v_1^{-1} v_2 v_3^{-1} w_1 w_2^{-1} w_3,$$

as desired.

Corollary E.2.D. *Let $T \to U$ be a complex of F-tori. Let $A^\bullet$ denote the complex*

$$\ker[\mathfrak{A}_T \to \mathfrak{A}_U] \to \mathfrak{A}_T \to \mathfrak{A}_U \to \operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

with $\ker[\mathfrak{A}_T \to \mathfrak{A}_U]$ *placed in degree 0 and use the canonical measures above to make $A^\bullet$ into a measured complex. Then*

$$t(A^\bullet) = d/e$$

where d (resp., e) is the cardinality of the kernel (resp., cokernel) of the natural map

$$\operatorname{cok}[Y^\Gamma \to X^\Gamma] \to X^*(C)^\Gamma.$$

Apply the first equality in Lemma E.2.C to the distinguished triangle

$$[1 \to U] \to [T \to U] \to [T \to 1] \to [U \to 1].$$

(E.3) Generalization of Ono's formula for $\tau(T)$. We write

$$\operatorname{vol}(H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1)$$

for the measure of the compact group $H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1$ with respect to Tamagawa measure, and we use parallel notation in the case of a single torus T. Thus

$$\operatorname{vol}\left(H^0(\mathbb{A}/F, T)_1\right) = \tau(T)|\ker^1(F, T)|,$$

where $\tau(T)$ denotes the Tamagawa number of T, since the group $T(\mathbb{A})_1/T(F)$ has index $|\ker^1(F,T)|$ in $H^0(\mathbb{A}/F,T)_1$. Ono [O] proved that

$$\begin{aligned}\tau(T) &= |\ker^1(F,T)|^{-1}|H^1(F, X^*(T))| \\ &= |\ker^1(F,T)|^{-1}|H^1(\mathbb{A}/F,T)| \qquad \text{(by duality).}\end{aligned}$$

Thus Ono's result can be reformulated as the equality

$$\operatorname{vol}\left(H^0(\mathbb{A}/F, T)_1\right)\operatorname{vol}(H^1(\mathbb{A}/F,T))^{-1} = 1. \tag{E.3.1}$$

Our next goal is to prove an analogous formula involving

$$\operatorname{vol}(H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1).$$

Lemma E.3.A. *Let $A^\bullet$ denote the complex*

$$H^0(\mathbb{A}/F, T \to U) \to H^0(\mathbb{A}/F, T) \to H^0(\mathbb{A}/F, U) \to H^1(\mathbb{A}/F, T \to U) \to \cdots .$$

Make $A^\bullet$ into a measured complex by putting the Tamagawa measure on each group. Then $t(A^\bullet) = 1$.

We apply Lemma E.1.D to the following double complex.

$$\begin{array}{ccccccccc}
\to \ker^i(F,T\to U) & \to & \ker^i(F,T) & \to & \ker^i(F,U) & \to & \ker^{i+1}(F,T\to U) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to H^i(F,T\to U) & \to & H^i(F,T) & \to & H^i(F,U) & \to & H^{i+1}(F,T\to U) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to H^i(\mathbb{A},T\to U) & \to & H^i(\mathbb{A},T) & \to & H^i(\mathbb{A},U) & \to & H^{i+1}(\mathbb{A},T\to U) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to H^i(\mathbb{A}/F,T\to U) & \to & H^i(\mathbb{A}/F,T) & \to & H^i(\mathbb{A}/F,U) & \to & H^{i+1}(\mathbb{A}/F,T\to U) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to \ker^{i+1}(F,T\to U) & \to & \ker^{i+1}(F,T) & \to & \ker^{i+1}(F,U) & \to & \ker^{i+2}(F,T\to U) & \to
\end{array}$$

We put the discrete measure on each group in the first, second and fifth rows. We put Tamagawa measures on the groups in the third and fourth rows. Then by Lemma E.2.B (and its analog for a single torus) the t-value of each column is 1. Therefore, by Lemma E.1.D the alternating product of the t-values of the rows is 1. The t-value of the first row is inverse to that of the fifth row, so that these two contributions cancel in the alternating product. The t-value of the second row is 1 by Lemma E.1.B, and that of the third row is 1 by Lemma E.2.A. We conclude that the t-value of the fourth row is 1, which is what we needed to prove.

Lemma E.3.B. *Let $A^\bullet$ denote the complex*

$$H^0(\mathbb{A}/F, T \to U)_1 \to H^0(\mathbb{A}/F, T)_1 \to H^0(\mathbb{A}/F, U)_1 \to H^1(\mathbb{A}/F, T \to U)_1 \to \cdots .$$

Make $A^\bullet$ into a measured complex by putting the Tamagawa measure on each group. Then

$$t(A^\bullet) = e/d,$$

where d, e are as in Corollary E.2.D.

Consider the double complex whose first row is the complex in this lemma, whose second row is the complex in Lemma E.3.A, and whose third row is

$$\begin{aligned} &\ker[\mathfrak{A}_T \to \mathfrak{A}_U] \to \mathfrak{A}_T \to \mathfrak{A}_U \to \operatorname{cok}[\mathfrak{A}_T \to \mathfrak{A}_U] \to 1 \to 1 \to \\ &\ker[D_T \to D_U] \to D_T \to D_U \to \operatorname{cok}[D_T \to D_U] \to 1 \to \cdots . \end{aligned}$$

We put Tamagawa measures on the groups in the first and second rows. We put the canonical measures of (E.2) (resp., discrete measures) on the real vector spaces (resp., discrete groups) in the third row. Clearly the t-value of each column is 1. Therefore by Lemma E.1.D the alternating product of the t-values of the three rows is 1. The t-value of the second row is 1 by Lemma E.3.A; therefore the t-value of the complex in this lemma is inverse to the t-value d/e (use Corollary E.2.D) of the third row.

Lemma E.3.C. *There is an equality*

$$\operatorname{vol}(H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1) \operatorname{vol}(H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1)^{-1} |H^2(\mathbb{A}/F, T \xrightarrow{f} U)_1| = e/d,$$

where d, e are as in Corollary E.2.D.

Consider once again the complex in Lemma E.3.B. It satisfies the hypotheses of Lemma E.1.C. Therefore the alternating product of the total masses of the groups in the complex is equal to e/d. Applying Ono's equality (E.3.1) to T and U, we get the statement of the lemma.

The equality of Lemma E.3.C is indeed a generalization of Ono's equality, but it has the defect that the quantities d,e,$|H^2(\mathbb{A}/F, T \xrightarrow{f} U)_1|$ can change when $T \xrightarrow{f} U$ is replaced by a quasi-isomorphic complex $T' \xrightarrow{f'} U'$. Our next result eliminates this defect.

Lemma E.3.D. *There is an equality*

$$\operatorname{vol}(H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1) \operatorname{vol}(H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1)^{-1} = vw^{-1},$$

where

$$\begin{aligned} v &:= |\operatorname{cok}[H^1(F, Y \xrightarrow{f^*} X) \to X^*(C)^\Gamma]| \\ w &:= |\ker[H^1(F, Y \xrightarrow{f^*} X) \to X^*(C)^\Gamma]| \\ &= |H^1(F, X^*(W))|. \end{aligned}$$

Let I denote

$$\operatorname{cok}[H^1(F, Y \xrightarrow{f^*} X) \to X^*(C)^\Gamma].$$

We have two exact sequences (see (A.1))

$$1 \to H^1(F, X^*(W)) \to H^1(F, Y \xrightarrow{f^*} X) \to X^*(C)^\Gamma \to I \to 1$$

$$1 \to \operatorname{cok}[Y^\Gamma \to X^\Gamma] \to H^1(F, Y \xrightarrow{f^*} X) \to \ker[H^1(F,Y) \to H^1(F,X)] \to 1.$$

Thus we have an exact sequence

$$1 \to D \to H^1(F, X^*(W)) \to \ker[H^1(F,Y) \to H^1(F,X)] \to E \to I \to 1,$$

where we have written D (resp., E) for the kernel (resp., cokernel) of the natural map

$$\operatorname{cok}[Y^\Gamma \to X^\Gamma] \to X^*(C)^\Gamma.$$

Since each group in the exact sequence is finite, we see that

$$1 = dw^{-1}|\ker[H^1(F,Y) \to H^1(F,X)]|e^{-1}v. \tag{E.3.2}$$

By duality the map

$$H^2(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[D_T \to D_U]$$

can be thought of as the map

$$\operatorname{Hom}\left(H^1(F, Y \xrightarrow{f^*} X), \mathbb{Q}/\mathbb{Z}\right) \to \operatorname{Hom}\left(\operatorname{cok}[Y^\Gamma \to X^\Gamma], \mathbb{Q}/\mathbb{Z}\right),$$

and therefore

$$\ker[H^1(F,Y) \to H^1(F,X)]$$

is dual to

$$H^2(\mathbb{A}/F, T \xrightarrow{f} U)_1.$$

Thus the equality in this lemma follows from (E.3.2) and the equality in Lemma E.3.C.

Next we reformulate our results in terms of $\mathbb{A}$ rather than $\mathbb{A}/F$. We put

$$\operatorname{cok}^i(F, T \xrightarrow{f} U)_1 := \operatorname{cok}^i(F, T \xrightarrow{f} U) \cap H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1.$$

By the Tamagawa measure dt on

$$\operatorname{cok}^i(F, T \xrightarrow{f} U)$$

we of course mean the quotient of the Tamagawa measure on

$$H^i(\mathbb{A}, T \xrightarrow{f} U)$$

by the discrete measure on

$$\mathrm{im}[H^i(F, T \xrightarrow{f} U) \to H^i(\mathbb{A}, T \xrightarrow{f} U)],$$

and by the Tamagawa measure on

$$\mathrm{cok}^i(F, T \xrightarrow{f} U)_1 \qquad (i = 0, 1)$$

we mean the Haar measure dt_1 having the property that dt/dt_1 is the canonical measure (see (E.2)) on

$$\begin{aligned} &\ker[\mathfrak{A}_T \to \mathfrak{A}_U] && \text{if } i = 0, \\ &\mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U] && \text{if } i = 1. \end{aligned}$$

For $i = 0, 1$ we define Tamagawa numbers

$$\tau^i(T \xrightarrow{f} U) := \mathrm{vol}(\mathrm{cok}^i(F, T \xrightarrow{f} U)_1).$$

Because the maps

$$H : H^0(\mathbb{A}/F, T \xrightarrow{f} U) \to \ker[\mathfrak{A}_T \to \mathfrak{A}_U]$$
$$H : H^1(\mathbb{A}/F, T \xrightarrow{f} U) \to \mathrm{cok}[\mathfrak{A}_T \to \mathfrak{A}_U]$$

remain surjective when restricted to

$$\mathrm{cok}^i(F, T \xrightarrow{f} U) \qquad (i = 0, 1)$$

the sequences

$$1 \to \mathrm{cok}^i(F, T \xrightarrow{f} U)_1 \to H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1 \to \ker^{i+1}(F, T \xrightarrow{f} U) \to 1$$

are exact for $i = 0, 1$, and therefore

$$\mathrm{vol}(H^i(\mathbb{A}/F, T \xrightarrow{f} U)_1) = \tau^i(T \xrightarrow{f} U) k(i+1) \qquad (i = 0, 1),$$

where $k(i)$ denotes

$$|\ker^i(F, T \xrightarrow{f} U)|,$$

which immediately yields the following reformulation of Lemma E.3.D.

Lemma E.3.E. *There is an equality*

$$\tau^0(T \xrightarrow{f} U) k(1) (\tau^1(T \xrightarrow{f} U) k(2))^{-1} = v w^{-1}.$$

One final remark is needed. When we defined Tamagawa measures on $H^i(\mathbb{A}, T \xrightarrow{f} U)$ and $H^i(\mathbb{A}/F, T \xrightarrow{f} U)$ for $i = 0, 1$ we noted that there is an alternative normalization. However this alternative normalization has the effect of multiplying the measures for $i = 0, 1$ by the same factor, so that the quantity

$$\mathrm{vol}(H^0(\mathbb{A}/F, T \xrightarrow{f} U)_1) \, \mathrm{vol}(H^1(\mathbb{A}/F, T \xrightarrow{f} U)_1)^{-1}$$

does not change, and Lemmas E.3.C, E.3.D and E.3.E remain valid.

References

[AT] E. Artin and J. Tate, *Class Field Theory*, Benjamin, New York, 1968.

[Bo] M. Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. (to appear).

[B] A. Bouaziz, *Relèvement des caractères d'un groupe endoscopique pour le changement de base* $\mathbb{C}/\mathbb{R}$, Asterisque **171** (1989), 163–194.

[H] G. Harder, *Über die Galoiscohomologie halbeinfacher Matrizengruppen. II*, Math. Zeit. **92** (1966), 396–415.

[K1] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), 611–650.

[K2] ———, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), 785–806.

[K3] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399.

[La] J-P. Labesse, *Cohomologie, L-groupes et fonctorialité*, Compositio Math. **55** (1984), 163–184.

[LL] J-P. Labesse and R. Langlands, *L-indistinguishability for* $\mathbf{SL}_2$, Can. J. Math. **31** (1979), 726–785.

[L1] R. Langlands, *Stable conjugacy: definitions and lemmas*, Can. J. Math. **31** (1979), 700–725.

[L2] ———, *Les débuts d'une formule des traces stable*, Publ. Math. Univ. Paris VII, vol. 13, Paris, 1983.

[L3] ———, *On the classification of irreducible representations of real algebraic groups*, Representation Theory and Harmonic Analysis on Semisimple Lie Groups, AMS Mathematical Surveys and Monographs 31, 1989, pp. 101-170.

[LS1] R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271.

[LS2] ———, *Descent for transfer factors*, The Grothendieck Festschrift, vol. II, Birkhauser, Boston, 1990, pp. 485–563.

[M] J. Milne, *Arithmetic Duality Theorems*, Academic Press, Boston, 1986.

[Oe] J. Oesterlé, *Compatibilité de la suite exacte de Poitou-Tate aux mesures de Haar*, Séminaire de Théorie des Nombres de Bordeaux, Année 1982–1983, exposé 19.

[O] T. Ono, *On Tamagawa numbers*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. 9, 1966, pp. 122–132.

[R] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Math. Studies, vol. 123, Princeton University Press, 1990.

[Sh] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), 273–330.

[S1] D. Shelstad, *L-indistinguishability for real groups*, Math. Ann. **259** (1982), 385–430.

[Sm] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.

[Spr] T. Springer, *Linear Algebraic Groups*, Birkhauser, Boston, 1981.

[St] R. Steinberg, *Endomorphisms of algebraic groups*, Mem. Amer. Math. Soc. **80** (1968).

[T] J. Tate, *Number theoretic background*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math. 33, part 2, 1979, pp. 165–184.