SOME RESULTS IN ENDOSCOPIC TRANSFER (NOTES FOR WORKSHOP, JUNE 2011)

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Abstract. We follow the theme of stabilization, and start with Arthur's paradigm for the invariant trace formula, *geometric side = spectral side*, in the case $G = SL(2)$ over a number field. A simple canonical sign, an adelic transfer factor, provides a measure of instability in the invariant trace formula from the geometric side. If we write a good product formula, over all places, for the factor then we can find another simple canonical sign, an adelic spectral transfer factor, giving a *spectral* interpretation of instability. This offers some motivation for a more recent look at endoscopy, twisted or not, for general connected, reductive G defined over $\mathbb{R}$, for extended groups (K-groups) even. Internal motivation is that a structure for tempered spectral factors comes *almost* for free once the geometric transfer factors have been defined. We will discuss some of the theorems and describe tools used in their proof that may be helpful as well for an approach to some questions at the infinite places in Langlands' program for stable transfer (or *stable-stable* transfer for emphasis that it lies beyond endoscopy). We also discuss results, some only partial, useful in Arthur's endoscopic classification for classical groups.

1. A SETTING FOR REAL ENDOSCOPY

• Let G be a connected reductive algebraic group defined over $\mathbb{R}$. Then $G(\mathbb{C})$ is connected but $G(\mathbb{R})$ is not, in general. For our statements, it is crucial that we use the entire group $G(\mathbb{R})$, as it is in this setting that some basic formulas of Harish Chandra from connected semisimple Lie groups with finite center attain their simplest form. In particular, we consider full special orthogonal groups.

• By means of twisted analysis we can introduce disconnected $G(\mathbb{C})$ if we work one component at a time. For example, we may discuss orthogonal groups in terms of twisted analysis on special orthogonal groups.

• There are two kinds of endoscopy, ordinary and twisted. The first is a special case of the second, but we will single it out when it illustrates a remark adequately. More parts of the theory are incomplete when there is twisting.

• We use the Weil form of the L-group. Thus ${}^LG$ is a semidirect product $G^\vee \rtimes W_\mathbb{R}$, with $W_\mathbb{R}$ acting through $W_\mathbb{R} \to \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$. By construction, Γ preserves some splitting (épinglage) of $G^\vee$. We work with a fixed Γ-splitting $spl^\vee$; this will be harmless for our results.

• In a **general twisted setting**, we use both an automorphism θ of G defined over $\mathbb{R}$ and a character ϖ on $G(\mathbb{R})$ with central parameter a (which is used also for inner twists of G). There will be no harm in fixing a 1-cocycle of $W_\mathbb{R}$ in the center of $G^\vee$ that determines this parameter.

• As **quasi-split case**, we consider a quasi-split group G^* with $\mathbb{R}$-splitting spl^*, an automorphism θ^* of G^* defined over $\mathbb{R}$ and preserving spl^*, along with a cocycle a. There exists a dual Γ-automorphism $\theta^\vee$ of $G^\vee$ preserving $spl^\vee$.

• We call (G, θ, ψ) an **inner twist** of (G^*, θ^*) if $\psi : G \to G^*$ is an inner twist transporting θ to θ^* *up to an inner automorphism*.

• In ordinary endoscopy we consider all Langlands (or Arthur) parameters. In twisted endoscopy we are interested only in those fixed under the action of the automorphism ${}^L\theta_a$ of ${}^LG$ given by $g \times w \to \theta^\vee(g).a(w) \times w$.

• Endoscopy starts with injective L-homomorphisms ${}^LH \to {}^LG$, where $H^\vee$ is the identity component of the $\theta^\vee$-twisted centralizer of a $\theta^\vee$-semisimple element s in $G^\vee$. Here we may as well take s in the maximal torus provided by $spl^\vee$. This distinguished element s is required to possess appropriate Galois-invariance (it generates various κ's as in *κ-orbital integral*). In general, there are not enough such homomorphisms ${}^LH \to {}^LG$.

•• Instead we may work with a diagram

$$\begin{array}{ccc} & & {}^LH_1 \\ & \nearrow & \\ \mathcal{H} & & \\ & \searrow & \\ & & {}^LG \end{array}$$

of injective homomorphisms, where $\mathcal{H}$ is a split extension of $W_\mathbb{R}$ by the subgroup $H^\vee = Cent_{\theta^\vee}(s, G^\vee)^0$ of $G^\vee$ and H_1 is as follows.

•• From $\mathcal{H}$ we can extract an L-action on $H^\vee$ and thus an L-group ${}^LH$. Let H denote a dual quasi-split group over $\mathbb{R}$. Then H_1 is a z-extension of H : the derived group of H_1 is simply-connected and the kernel of $H_1 \to H$ is a central induced torus.

•• This setup is directly useful for endoscopy. For example, consider Langlands' original construction in ordinary endoscopy for **factoring parameters** for G.

•• Thus let $\varphi : W_\mathbb{R} \to {}^LG$ be admissible (so that its conjugacy class under $G^\vee$ is a Langlands parameter). Write S_φ for the centralizer in $G^\vee$ of the image of

φ. Let s be a semisimple element in S_φ. Define $\mathcal{H} = \mathcal{H}(\varphi, s)$ as the subgroup of ${}^LG$ generated by $Cent(s, G^\vee)^0$ and the image of φ. Since φ factors through $\mathcal{H}$, a diagram provides us with admissible $\varphi^s : W_\mathbb{R} \to {}^LH_1$.

•• We will write ${}^LH_1 \xrightarrow{z} {}^LG$ for the diagram and call H_1 an endoscopic group.

•• Norms, or images in the case of ordinary endoscopy, of appropriate elements in $G(\mathbb{R})$ lie naturally in the group $H(\mathbb{R})$. Since $H_1(\mathbb{R}) \to H(\mathbb{R})$ is surjective and the structure of stable classes is the same for both these groups, we also find good norm correspondences between $G(\mathbb{R})$ and $H_1(\mathbb{R})$.

•• On the spectral side, we are interested in those Langlands parameters for H_1 which factor through $\mathcal{H}$ (which makes sense at the level of parameters). This is the same as prescribing a character on $Z_1(\mathbb{R}) = Ker(H_1(\mathbb{R}) \to H(\mathbb{R}))$, nontrivial if ${}^LH \to {}^LG$ does not exist. Thus, via parameters at least, packets of representations of $H_1(\mathbb{R})$ for which $Z_1(\mathbb{R})$ acts by this character, call it ϖ_1, transfer to packets for $G(\mathbb{R})$.

2. SOME THINGS WE USE

• By a Cartan subgroup of $G(\mathbb{R})$ we mean one in the sense of Harish Chandra for reductive Lie groups. It coincides with the real points on a maximal torus T in G defined over $\mathbb{R}$. We call G cuspidal if it has a maximal torus T that is $\mathbb{R}$-anisotropic modulo the center Z_G of G, so that $G(\mathbb{R})$ has a discrete series if and only if G is cuspidal. For now, we ignore twisting data θ, ϖ.

• Assume T is a maximal torus over $\mathbb{R}$ in G (cuspidal or not). If we use the Galois form for L-groups there is of course an obstruction to embedding ${}^LT$ in ${}^LG$, a 2-cocycle of Γ in $T^\vee$ which we may write explicitly. Since its inflation to $W_\mathbb{R}$ splits, there always exists an embedding of ${}^LT$ in ${}^LG$ for the Weil form of L-groups.

• An explicit description of the splitting, and then of ${}^LT \hookrightarrow {}^LG$, is provided by *$\chi$-data* for the action of Γ on the set $R(T, G)$ of roots of T in G. These χ-data also play a role in the harmonic analysis for endoscopy.

• An orbit O of Γ in $R(T, G)$ is either **symmetric** ($O = -O$) or **asymmetric** ($O \cap -O = \varnothing$). Here, as we are considering only $\mathbb{C}/\mathbb{R}$, the possibilities are simply the following. Symmetric O is of the form $\{\pm\alpha\}$, with α an imaginary root in Lie group terminology, and asymmetric O is either $\{\alpha\}$, where α is a real root, or $\{\alpha, \sigma\alpha\}$, with α a complex root.

• To a symmetric orbit $\{\pm\alpha\}$ we attach two characters $\chi_\alpha, \chi_{-\alpha}$ of $\mathbb{C}^\times$ such that $\chi_{-\alpha} = \chi_\alpha^{-1}$ and each character extends the *sign* character on $\mathbb{R}^\times$. To the asymmetric $\{\alpha, \sigma\alpha\}$ we attach any two characters $\chi_\alpha, \chi_{\sigma\alpha}$ of $\mathbb{C}^\times$ such that $\chi_{\sigma\alpha} = \overline{\chi_\alpha}$, and to asymmetric $\{\alpha\}$ we attach any character χ_α on $\mathbb{R}^\times$. These various characters form a set $\{\chi_\alpha\}$ of χ-data.

•• **An example of an embedding** ${}^LT \hookrightarrow {}^LG$. Assume that G is semisimple and T is $\mathbb{R}$-anisotropic. Each O is symmetric. Identify $T^\vee$ with the maximal torus in $G^\vee$ provided by $spl^\vee$ (use the inner twist and an inner automorphism of G^* to identify T with the maximal torus provided by spl^* equipped with transported

Galois action; then take duals). Define $\chi_\alpha(z) = (z/\overline{z})^{1/2}$ if $\alpha^\vee$ is a root of the Borel subgroup provided by $spl^\vee$, and then take $\chi_{-\alpha}$ as we must. Let ι be one-half the sum of all such roots α. Then $(z/\overline{z})^\iota = \prod_O [\chi_\alpha(z)]^\alpha$ is an element of $T^\vee$. The embedding attached to $\{\chi_\alpha\}$ is given by

$$\begin{aligned} t \times (z \times 1) &\rightarrow t.(z/\overline{z})^\iota \times (z \times 1) \\ t \times (1 \times \sigma) &\rightarrow t.n \times (1 \times \sigma), \end{aligned}$$

where $t \in T^\vee, z \in \mathbb{C}^\times$, and $n \in G^\vee$ represents the longest element of the Weyl group of $T^\vee$ in $G^\vee$ (ambiguity in the choice of n does not matter).

• Along with χ-data we have a-data $\{a_\alpha\}$: $a_\alpha \in \mathbb{C}^\times$, $a_{-\alpha} = -a_\alpha$ and $a_{\sigma\alpha} = \overline{a_\alpha}$, for all $\alpha \in R(T, G)$. The a-data appear first in another problem which we skip here. Next we have a use for both types of data that prepares for the geometric side of endoscopy.

• For regular $\gamma \in T(\mathbb{R})$, $D(\gamma) = \det[Ad(\gamma) - I]_{\mathfrak{g}/\mathfrak{t}}$ lies in $\mathbb{R}^\times$ and is stably invariant. We will include $|D(\gamma)|^{1/2}$ in the definition of **an orbital integral**:

$$O(\gamma, fdg) = |D(\gamma)|^{1/2} \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}\gamma g) \frac{dg}{dt}$$

for a suitable measure fdg on $G(\mathbb{R})$ (dg is a Haar measure on $G(\mathbb{R})$, f is of C_c^∞- or Schwartz type) and Haar measure dt on $T(\mathbb{R})$. Then, for γ *strongly* regular, the stable orbital integral is

$$SO(\gamma,\ fdg) = \sum_{\gamma'} SO(\gamma',\ fdg),$$

with the summation over representatives γ' for the conjugacy classes in the stable class of γ. Here we use related Haar measures dt, dt' ... prescribed by invariant differential forms of highest degree.

• To define a variant Ψ_T of Harish Chandra's F_f-transform we note that

$$\chi_\alpha(\frac{\alpha(\gamma)\ -\ 1}{a_\alpha})$$

is independent of the choice of α within its Galois orbit O, and set

$$\Psi_T(\gamma) = \prod_O \chi_\alpha(\frac{\alpha(\gamma) - 1}{a_\alpha}).SO(\gamma, fdg).$$

Then Ψ_T extends smoothly everywhere on $T(\mathbb{R})$ off the walls $\alpha = 1$ for certain (in the quasi-split case, *all*) imaginary roots α.

• Across such walls Ψ_T exhibits **jump behavior**. This is the key ingredient in a characterization of stable orbital integrals. The wall $\alpha = 1$ is shared with an adjacent and more split Cartan subgroup $T'(\mathbb{R})$ for which the wall is not problematic. Assume $\gamma_0 \in G(\mathbb{R})$ lies on this and no other wall. Then $\Psi_{T'}(\gamma_0)$ is well-defined. In $T(\mathbb{R})$ we may cross through the wall at γ_0 along the curve

$$\mathbb{R} \ni \nu \rightarrow \gamma_\nu = \gamma_0 \exp \nu(a_\alpha H_\alpha).$$

Here H_α is more familiar notation for the coroot $\alpha^\vee$ as element of the complex Lie algebra; $a_\alpha H_\alpha$ lies in the Lie algebra of $T(\mathbb{R})$ and is independent of the choice of α

in its orbit. Because we use stable integrals it is clear, given existence of the limits (Harish Chandra), that

$$lim_{\nu \to 0-} \Psi_T(\gamma_\nu) = - \; lim_{\nu \to 0+} \Psi_T(\gamma_\nu).$$

We now require certain easily satisfied symmetry properties relative to α of the a-data and χ-data for T, T'. Then we also find, after some quite long arguments if we start with the original F_f-transform, the simple formula

$$lim_{\nu \to 0+} \Psi_T(\gamma_\nu) = \Psi_{T'}(\gamma_0).$$

Here Haar measures dt, dt', suppressed in notation, are also to be related appropriately.

• There are also formulas for derivatives. Consider the action of operators in the symmetric algebra on $\mathfrak{t}$ either symmetric or antisymmetric relative to the Weyl reflection for α, and apply a suitable twist. In the antisymmetric case the jump is zero, while in the symmetric case we adapt (quickly) Harish Chandra's differential equations for the F_f-transform to get the same simple formulas as above when a related pair of operators on T, T' are applied to $\Psi_T, \Psi_{T'}$ respectively.

• How do we use this? We start with twisted orbital integrals for a measure fdg on the group $G(\mathbb{R})$, and find that suitably weighted sums of these integrals factor through the norm map to the endoscopic group $H_1(\mathbb{R})$, at least for sufficently regular elements. The suitable weights are the *geometric transfer factors* which we come to next. Then we will use our characterization of stable orbital integrals on $H_1(\mathbb{R})$ via the transforms Ψ_{T_1} to prove geometric transfer, *i.e.* to verify that this factoring is through the stable orbital integrals of a suitable measure $f_1 dh_1$. See Theorem 1 for a more precise (and stronger) statement.

3. **RE MOTIVATION FROM STABLE TRACE FORMULA**

• Stabilization of the regular elliptic term on the geometric side of Arthur's invariant trace formula offers guidelines for defining transfer factors; in particular, these factors are needed at all places and they must satisfy a product formula. The motivation from $SL(2)$ mentioned in the abstract is a refinement which requires more preparation, and so we will save it until later.

4. **STRUCTURE OF TRANSFER FACTORS**

• Transfer factors are defined first on **very regular** pairs of points or representations. For example, for the geometric factor Δ_{geom} we consider pairs of points (γ_1, δ), where $\gamma_1 \in H_1(\mathbb{R})$ is strongly G-regular and $\delta \in G(\mathbb{R})$ is strongly θ-regular. We set $\Delta_{geom}(\gamma_1, \delta) = 0$ unless γ_1 is a norm of δ. Rather than pick a normalization for Δ_{geom}, we will define a *canonical relative factor* $\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$ and then call $\Delta_{geom}(-,-)$ a **geometric transfer factor** if

$$\Delta_{geom}(\gamma_1, \delta)/\Delta_{geom}(\gamma_1', \delta') = \Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$$

whenever strongly G-regular γ_1, γ_1' is a norm of δ, δ' respectively.

• Consider now a spectral pair (π_1, π). Here we will limit our attention to tempered irreducible admissible representations. Call a parameter φ for $G(\mathbb{R})$ **regular** if $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ is abelian. Because this centralizer is always connected, we can ignore *strongly* in the definition. Recall that for the endoscopic group $H_1(\mathbb{R})$ we consider parameters φ_1 with image in $\mathcal{H}$. The diagram ${}^L H_1 \xrightarrow{z} {}^L G$ attaches to φ_1 a parameter φ for $G(\mathbb{R})$. We call φ_1 **G-regular** if φ is regular. Then φ_1 is regular as parameter for $H_1(\mathbb{R})$, and we call (φ_1, φ) a G-regular related pair.

• Fix (π_1, π) with G-regular related pair (φ_1, φ) as parameters. By the endoscopic constructions, φ is preserved by the automorphism ${}^L\theta_a$. Then the packet Π of π is preserved under $\pi' \to \varpi^{-1} \otimes (\pi' \circ \theta)$. We write $\Pi^{\theta,\varpi}$ for those $\pi' \in \Pi$ fixed by this map, *i.e.* for those $\pi' \in \Pi$ such that $\pi' \circ \theta \approx \varpi \otimes \pi'$. These are the only members of the packet contributing nontrivially to (θ, ϖ)-twisted traces. We call $\Pi^{\theta,\varpi}$ a **twist-packet**. A twist-packet may be empty but not in the setting we will consider. We call (π_1, π) a *G-regular related pair* (or a **very regular related pair**) if also π lies in $\Pi^{\theta,\varpi}$.

• If (π_1, π) is a very regular pair that is not related then set $\Delta_{spec}(\pi_1, \pi) = 0$. For any two very regular related pairs (π_1, π), (π_1', π'), we will define canonical $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ and then call $\Delta_{spec}(-,-)$ a **spectral transfer factor** if

$$\Delta_{spec}(\pi_1, \pi)/\Delta_{spec}(\pi_1', \pi') = \Delta_{spec}(\pi_1, \pi; \pi_1', \pi').$$

In the general twisted case there are two cheats here: (i) in some cases our method misses a few pairs, and (ii) there is a hidden dependence in relative Δ_{spec} we will discuss later (it will be harmless for the transfer statement).

• We consider now the structure of $\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$ in some detail, and then $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ will come almost for free.

• Relative Δ_{geom} comes as a product $\Delta_I . \Delta_{II} . \Delta_{III}$.

• Start with ordinary endoscopy. First, we remove a small piece of Δ_{III} (usually written Δ_{III_2}) and put it with Δ_{II}, a term related to the harmonic analysis we have already discussed. The new Δ_{II} involves χ-data but that choice doesn't matter. The choice of a-data is, however, important.

• The remaining piece Δ_{III_1} of the Δ_{III} term now measures the relative positions of the conjugacy classes of δ, δ' in their stable classes. It is critical that *this measurement is done via the quasi-split form*. We can define, via Tate-Nakayama pairing in Galois cohomology, relative positions of the two classes, but not absolute position, in general. However, even the relative position depends on how we identify maximal tori in $H = H_1/Z_1$ as maximal tori in G^*. Such identifications are given by a choice of *toral data* (for the quasi-split form).

• A local analysis of Δ_{III} using a geometric method of Langlands shows us that we can introduce a term Δ_I which cancels the dependence on toral data. It does introduce a dependence on a-data, but that is cancelled by the use of a-data in the term Δ_{II} needed for harmonic analysis. Thus we have canonical $\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$.

• In twisted endoscopy we cannot separate a term Δ_{III_2} from Δ_{III} in general: the pairings in Δ_{III} become a single pairing in Galois hypercohomology. There is then no separate relative positional term in general. Nevertheless the same principle applies: the positional contribution to Δ_{III} depends on toral data in a way that cancels with the dependence in Δ_I at the expense of introducing dependence on a-data. That cancels with the dependence in Δ_{II}. The choice of χ-data does not matter for the product $\Delta_{II}\Delta_{III}$, and so we again have canonical $\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta')$.

• We can use the same strategy for spectral factors. Again start with the ordinary case. First we make a relative positional term Δ_{III} for representations within a packet. This uses the same Galois cohomology constructs as in the geometric case and, in particular, depends in the same way on toral data. Thus we introduce Δ_I as before. Again we need Δ_{II} depending correctly on a-data. We will find it in local character expansions, as described shortly. There is no dependence on χ-data. Thus we have canonical $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ for any two tempered G-regular related pairs (π_1, π), (π_1', π'). Harish Chandra's explicit formulas for discrete series characters show $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ to be a fourth root of unity, and then a little further analysis shows it to be a sign.

• Before extending this to the twisted case, we review our setting. We will start with the case G is cuspidal as our main interest will be discrete series packets for $G(\mathbb{R})$. Suppose then that T is a maximal torus in G that is $\mathbb{R}$-anisotropic modulo the center Z_G of G, so that $T(\mathbb{R})/Z_G(\mathbb{R})$ is compact and connected.

• Start with the quasi-split data (G^*, θ^*). Recall that θ^* preserves the $\mathbb{R}$-splitting spl^* based on a maximally $\mathbb{R}$-split torus in G^*. Consider a splitting spl based instead on T, say $(B, T, \{X_\alpha\})$. Here B is a Borel subgroup containing T and, for each root α in a base for $R(T, B)$, X_α is a root vector for α. Complete each X_α, H_α to a simple triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$. We call spl **fundamental** if $\sigma X_\alpha = \pm X_{-\alpha}$ for all α in the base. Such splittings exist for G^* *and* for each of its inner forms (same definitions apply). Moreover, any automorphism of G that preserves a fundamental splitting is defined over $\mathbb{R}$ as long as its restriction to the center Z_G is defined over $\mathbb{R}$. For G^* alone, we can find splittings with $\sigma X_\alpha = X_{-\alpha}$ for all α in the base; this will concern us later. Finally, our quasi-split datum θ^* preserves some fundamental splitting of G^*.

• Now take an inner twist (G, θ, ψ). We can adjust ψ within its inner class to transport a θ^*-stable fundamental splitting of G^* to a fundamental splitting of G. So $\theta_f = \psi^{-1} \circ \theta^* \circ \psi$ is defined over $\mathbb{R}$, and coincides with θ up to an inner automorphism defined over $\mathbb{R}$. The twist-packets are nonempty if the inner automorphism is realized by an element of $G(\mathbb{R})$. To save time and notation, we will assume here that θ itself preserves a fundamental splitting. The terms $\Delta_I(\pi_1, \pi)$, $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ are defined by adapting the machinery from the geometric case to spectral information. For Δ_{III}, fundamental splittings are the key new ingredient: under our assumption on θ, we can attach to each π (in a discrete series twist-packet) a fundamental splitting spl_π preserved by θ.

5. SPECTRAL FACTORS AND LOCAL CHARACTER FORMULAS

• Here we focus on the term $\Delta_{II}(\pi_1, \pi)$ from harmonic analysis, in the case that π is a discrete series representation. This term will depend on a-data for T in exactly the way we need.

• We start with the endoscopic group $H_1(\mathbb{R})$ and the ordinary stable traces attached to a discrete series packet. Write Π_1 for the packet of π_1. By $\pi_1(f_1 dh_1)$ we mean the operator

$$\int_{H_1(\mathbb{R})/Z_1(\mathbb{R})} f_1(h_1)\pi_1(h_1)dh$$

(in $f_1 dh_1$, only $dh = \frac{dh_1}{dz_1}$ is variable), and then $St\text{-}Trace\ \pi_1(f_1 dh_1)$ means the sum over $\pi_1' \in \Pi_1$ of $Trace\ \pi_1'(f_1 dh_1)$. Fix Haar measures and drop them from notation. According to Harish Chandra's regularity theorem, the stable tempered distribution $f_1 \to St\text{-}Trace\ \pi_1(f_1 dh_1)$ is represented by a locally L^1 function we will call $St\text{-}Ch_{\pi_1}$ that is real analytic on the regular set of $H_1(\mathbb{R})$. It will be enough to examine the Harish Chandra formula for $St\text{-}Ch_{\pi_1}$ near the identity on the regular set of a Cartan subgroup $T_1(\mathbb{R})$ that is compact modulo $Z_{H_1}(\mathbb{R})$. Such a T_1 is provided by the choice of toral data for T as in the last step. The choice of toral data will not matter for $\Delta_{II}(\pi_1, \pi)$.

• Consider a regular element $\gamma_1 = \exp X_1$,with $X_1 \in \mathfrak{t}_1(\mathbb{R})$ close to 0. Then we may write $St\text{-}Ch_{\pi_1}(\gamma_1)$ as

$$\upsilon_{H_1}(a_1).\Delta_{a_1}(X_1).\sum_{w_1} det(w_1).e^{w_1\mu_1(X_1)}.$$

Here $\Delta_{a_1}(X_1)$ is, by definition,

$$\prod_{\mathcal{O}_1}\ sign\ (\frac{e^{\alpha_1(X_1)/2}\ -\ e^{-\alpha_1(X_1)/2}}{a_{\alpha_1}}).\left|D(\exp X_1)\right|^{-1/2},$$

where the product is over representatives α_1 for the Galois orbits $\mathcal{O}_1 = \{\pm\alpha_1\}$ of roots of T_1 in H_1. We use a-data $a_1 = \{a_{\alpha_1}\}$ provided by a-data for T. The summation $\sum_{w_1}$ is over the full (complex) Weyl group of T_1 in H_1. Next, $\mu_1 \in X^*(T_1)\otimes\mathbb{C}$ is a regular linear form on $\mathfrak{t}_1 = X_*(T_1)\otimes\mathbb{C}$ *uniquely determined* by the choice of toral data. The constant $\upsilon_{H_1}(a_1)$ is computed, using Harish Chandra's character formula, to be a fourth root of unity.

•• The formula we have written for $St\text{-}Ch_{\pi_1}(\gamma_1)$ is purely local. We may use χ-data to make it global: each choice of χ-data provides us with a different way of writing the (same) extension of this local formula to Harish Chandra's global formula on the regular set in $T_1(\mathbb{R})$.

• We turn to the twisted character of π. First define a unitary (bounded) operator $\pi(\theta, \varpi)$ on the space of π to interwine $\pi \circ \theta$ and $\varpi \otimes \pi$:

$$\pi(\theta(g)) \circ \pi(\theta, \varpi) = \varpi(g).(\pi(\theta, \varpi) \circ \pi(g)),$$

for $g \in G(\mathbb{R})$. Then consider the distribution

$$f \to Trace[\pi(f) \circ \pi(\theta, \varpi)]$$

on $G(\mathbb{R})$. By Bouaziz's extension of Harish Chandra's regularity theorem and a little further argument if the twisting character ϖ is nontrivial, this distribution is represented by a locally L^1 function $Tw\text{-}Ch_\pi$ real analytic on the twisted regular set of $G(\mathbb{R})$.

• We will define $\Delta_{II}(\pi_1, \pi)$ so that

$$\Delta_{II}(\pi_1, \pi).Trace[\pi(f) \circ \pi(\theta, \varpi)]$$

is independent of the normalization of $\pi(\theta,\varpi)$.

• Recall that θ fixes the fundamental splitting $spl_\pi = (B_\pi, T_\pi, \{X_\alpha\})$. We consider a local expression for $Tw\text{-}Ch_\pi$ around the identity. Let T^1 be the identity component of the fixed points of θ in T_π. For strongly θ-regular $\delta = \exp X$, with $X \in \mathfrak{t}^1(\mathbb{R})$ sufficiently close to zero, we may write

$$Tw\text{-}Ch_\pi(\delta) = \upsilon_G(a_{res}).\Delta_a(X).\sum_{w,\mathbb{R}} \det w.e^{w\mu(X)}.$$

Here:

(i) The summation $\sum_{w,\mathbb{R}}$ is over the quotient

$$Norm(T^1, G(\mathbb{R}))/Cent(T^1, G(\mathbb{R})) = Norm(T^1, G(\mathbb{R}))/T_\pi(\mathbb{R}).$$

Since θ preserves a (fundamental) splitting, this quotient embeds naturally in the complex Weyl group of T^1 in $(G^\theta)^0$.

(ii) The factor $\Delta_a(X)$ is

$$\prod\nolimits_{\mathcal{O}_{red}} sign(\frac{e^{N\alpha(X)/2} - e^{-N\alpha(X)/2}}{a_{\alpha_{res}}}).\left|\det[Ad(\exp X)\circ\theta - I]_{\mathfrak{g}/\mathfrak{t}_\pi}\right|^{-1/2},$$

where the product is over the set of Galois orbits $\mathcal{O}_{red} = \{\pm\alpha_{res}\}$ of the reduced roots among the restrictions to T^1 of the roots α of T_π in G, and $N\alpha$ is the sum of all roots in the θ-orbit of α. Also $a_{res} = \{a_{\alpha_{res}}\}$ are a-data in this setting.

(iii) The constant $\upsilon_G(a_{res})$ depends on our choice of the operator $\pi(\theta,\varpi)$, but clearly

$$\upsilon_G(a_{res})^{-1} Trace[\pi(f)\circ\pi(\theta,\varpi)]$$

does not.

(iv) The linear form μ on $\mathfrak{t}^1(\mathbb{R})$ is uniquely determined by the choice of toral data for T.

•• Once again we may use χ-data to globalize the formula, now to the θ-regular θ-elliptic set in $G(\mathbb{R})$. Care is needed for the orbits of those α_{res} for which $2\alpha_{res}$ is a restricted root (the sign above is misleading).

• The linear forms μ_1, μ attached to the parameters φ_1, φ for π_1, π have dominance properties relative to the fixed splittings for $H^\vee, G^\vee$. In general, μ_1 is not dominant for $G^\vee$. It is, however, dominant for $G^\vee$, or **well-positioned**, in the case that φ_1 is obtained from φ by Langlands' factoring, the case of main interest to us here.

• For φ_1 well-positioned, we define

$$\Delta_{II}(\pi_1,\pi) = (-1)^{q_G - q_{G^*}}.\upsilon_{H_1}(a_1).\upsilon_G(a_{res})^{-1},$$

where q_G is one-half the dimension of the symmetric space attached to the simply-connected cover G_{sc} of the derived group of G. For general φ_1 there is an additional, easily described, sign.

• We have now finished describing the terms in $\Delta(\pi_1,\pi;\pi_1',\pi')$. For ordinary endoscopy this relative spectral factor is a canonical sign. In the twisted setting, it is nonzero but depends on the normalization of operators $\pi(\theta,\varpi), \pi'(\theta,\varpi)$.

• We reduce this last choice to normalization of a single operator per twist-packet. Consider a map $\mathcal{P}_\Pi : \pi \to \pi(\theta, \varpi)$ defined on the twist-packet in Π. Call $\mathcal{P}_\Pi$ *balanced* if, with this choice of operators $\pi(\theta, \varpi)$, we obtain

$$\Delta_{II}(\pi_1, \pi) = \Delta_{II}(\pi_1, \pi'),$$

for all π, π' in the twist-packet. This is easily arranged for $\mathcal{P}_\Pi$ in the setting we discuss here, and so we assume it from now on.

6. GEOMETRIC-SPECTRAL COMPATIBILITY

• The geometric transfer theorem will depend on the normalization of Δ_{geom}. Once that is fixed there can be only one normalization of Δ_{spec} that works for the dual transfer. Suppose we pick a very regular pair (π_1, π). Also pick a very regular geometric pair (γ_1, δ). Neither choice will matter (because of transitivity properties of the various relative factors). We call $\Delta_{geom}, \Delta_{spec}$ **compatible** if

$$\Delta_{geom}(\gamma_1, \delta)/\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\gamma_1, \delta; \pi_1, \pi),$$

where the (almost) canonical factor Δ_{comp} is constructed as a product $\Delta_I.\Delta_{II}.\Delta_{III}$ using the various pieces of the parallel geometric and spectral constructions.

• We now define *Whittaker normalizations* for $\Delta_{geom}, \Delta_{spec}$ and see that they are compatible. We will assume here that (G, θ, ψ) is the trivial inner twist ($G = G^*$, $\theta = \theta^*$, $\psi = identity$), although there is a useful extended (K-group) setting.

• First we may define absolute terms $\Delta_{III}(\gamma_1, \delta)$ and $\Delta_{III}(\pi_1, \pi)$ for very regular pairs. Then absolute $\Delta_0 = \Delta_I \Delta_{II} \Delta_{III}$ makes sense for both the geometric and spectral versions. We verify easily that each is a transfer factor and then that they are compatible, almost by definition. There are two dependences in this normalization: in spectral Δ_{II} there is the choice of balanced $\mathcal{P}_\Pi$ (in the twisted case), and in both Δ_I terms there is our choice (fixed throughout) of θ^*-stable $\mathbb{R}$-splitting *spl**.

• For Whittaker normalization, the insertion of an epsilon factor shifts the dependence on *spl** to dependence on Whittaker data.

• **Whittaker data** consist of a pair (B, λ), where B is a θ^*-stable Borel subgroup of G^* defined over $\mathbb{R}$ and λ is a θ^*-invariant generic character on $N(\mathbb{R})$, where N is the unipotent radical of B. There will be no harm in assuming that B is part of *spl**, and that λ is the character determined by *spl** and the choice of an additive character $\psi_\mathbb{R}$ on $\mathbb{R}$. Consider the representation of Γ on $V_G = X(T^*)^{\theta^*} \otimes \mathbb{C}$, where T^* comes from *spl**. Similarly, although we have not mentioned it explicitly, we work with an $\mathbb{R}$-splitting *spl*$_H^*$ for H (determining *spl*$_1^*$ for H_1) with torus T_H^*. Set $V_H = X(T_H^*) \otimes \mathbb{C}$. Then $X(T^*)^{\theta^*}$, $X(T_H^*)$ are isomorphic but not as Γ-modules in general. Set $V = V_G - V_H$ and define $\varepsilon_L(V, \psi_\mathbb{R})$ as in Section 3 of Tate's Corvallis article.

• The **transfer factors with Whittaker normalization** (relative to (B, λ)) are defined for very regular pairs (γ_1, δ), (π_1, π) by

$$\Delta_{Wh}(\gamma_1, \delta) = \varepsilon_L(V, \psi_\mathbb{R}).\Delta_0(\gamma_1, \delta)$$

and

$$\Delta_{Wh}(\pi_1, \pi) = \varepsilon_L(V, \psi_{\mathbb{R}}).\Delta_0(\pi_1, \pi).$$

It is immediate from our earlier comments that they are compatible. In the case of $SL(2)$, this normalization coincides with that in Labesse-Langlands (for all local fields of characteristic zero).

• In the case of Whittaker normalization there is also a natural choice for balanced $\mathcal{P}_\Pi$: the representation π_{Wh} generic relative to (B, λ) lies in the twist-packet $\Pi^{\theta,\varpi}$, and then $\mathcal{P}_\Pi$ is determined by the familiar requirement that $\pi_{Wh}(\theta, \varpi)$ fix one, and hence every, Whittaker functional.

7. SOME THEOREMS FOR REAL ENDOSCOPIC TRANSFER

• We start with ordinary endoscopy: G is arbitrary and H_1 is endoscopic for G.

• Conventions on measures are to make choices irrelevant as far as transfer factors are concerned: see references. There is also some analysis regarding central characters, which will become critical after twisting is introduced. Here we will ignore this also: see references. Recall, however, we consider only representations of $H_1(\mathbb{R})$ for which the central subgroup $Z_1(\mathbb{R})$ acts by the fixed character λ_1.

THEOREM I, Part 1 (Geometric transfer, ordinary endoscopy)

Let Δ_{geom} be a geometric transfer factor. Then for each measure fdg on $G(\mathbb{R})$ there exists a measure $f_1 dh_1$ on $H_1(\mathbb{R})$ such that

$$SO(\gamma_1, f_1 dh_1) = \sum\nolimits_\delta \Delta_{geom}(\gamma_1, \delta)\ O(\delta, fdg)$$

for all strongly G-regular $\gamma_1 \in H_1(\mathbb{R})$.

• If f is a Schwartz function then we take f_1 Schwartz modulo λ_1,while if f is C_c^∞ then we may take f_1 to be C_c^∞ modulo λ_1 (Bouaziz).

• The easiest proof appears to be as a special case of the one we will discuss (briefly) for the twisted case.

• The very regular geometric transfer extends in various ways to other conjugacy classes. The very regular case is sufficient to determine uniquely the stable tempered traces *St-Trace* $\pi_1(f_1 dh_1)$.

• We now break up the results on dual transfer to highlight the various challenges for the twisted case.

THEOREM I, Part 2a (Dual very regular tempered spectral transfer)

Let Δ_{spec} be the spectral transfer factor compatible with Δ_{geom}. Then

$$St\text{-}Trace\ \pi_1(f_1 dh_1) = \sum\nolimits_\pi \Delta_{spec}(\pi_1, \pi)\ Trace\ \pi(fdg)$$

for all G-regular tempered irreducible representations π_1 of $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1.

• In this setting, the critical case is when π_1 is a G-regular discrete series representation. Then $\Delta_{spec}(\pi_1, \pi)$ is nonzero only for π in the corresponding packet of discrete series representations on $G(\mathbb{R})$. Here we use Harish Chandra's characterization of discrete series representations to prove the following:

$$f \to \textit{St-Trace}\ \pi_1(f_1 dh_1)$$

and

$$f \to \sum\nolimits_\pi \Delta_{spec}(\pi_1, \pi)\ \textit{Trace}\ \pi(fdg)$$

are tempered invariant eigendistributions which agree on the regular elliptic set and hence coincide. The remaining cases follow from putting together the pieces of parabolic descent.

THEOREM I, Part 2b (Extension to full tempered spectral transfer)

Let Δ_{spec} be the spectral transfer factor compatible with Δ_{geom}. Then there is a unique extension of Δ_{spec} such that

$$\textit{St-Trace}\ \pi_1(f_1 dh_1) = \sum\nolimits_\pi \Delta_{spec}(\pi_1, \pi)\ \textit{Trace}\ \pi(fdg)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1.

• This extension comes from the translation principle (coherent continuation) in the tempered setting, and uses a uniform L-group version for decomposing unitary principal series representations of real reductive groups.

• Variants of the Weyl integration formula allow us to rewrite the results as a set of *character identities,* if desired.

THEOREM I, Part 3 (Converse)

If fdg and $f_1 dh_1$ satisfy

$$\textit{St-Trace}\ \pi_1(f_1 dh_1) = \sum\nolimits_\pi \Delta_{spec}(\pi_1, \pi)\ \textit{Trace}\ \pi(fdg)$$

for all tempered irreducible representations π_1 on $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1 then

$$SO(\gamma_1, f_1 dh_1) = \sum\nolimits_\delta \Delta_{geom}(\gamma_1, \delta)\ O(\delta, fdg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

• Proof is easy given existence of the geometric transfer (Part 1) and dual transfer attached to it (Part 2b): if $f_1' dh_1$ is provided by geometric transfer, all stable tempered traces agree on $f_1 dh_1$ and $f_1' dh_1$, and so then do all stable orbital integrals.

• Consider now the twisted setting. For the proof of geometric transfer, both G and θ, ϖ are (almost) arbitrary. We do have to allow for a slight twisting in defining stable classes on $H_1(\mathbb{R})$ but we will ignore this in notation. This slight twisting arises in cases where twist-packets are empty.

THEOREM II, Part 1 (Geometric transfer, twisted endoscopy)

Let Δ_{geom} be a geometric transfer factor. Then for each measure fdg on $G(\mathbb{R})$ there exists a measure $f_1 dh_1$ on $H_1(\mathbb{R})$ such that

$$SO(\gamma_1, f_1 dh_1) = \sum\nolimits_{\delta} \Delta_{geom}(\gamma_1, \delta)\ O^{\theta,\varpi}(\delta, fdg)$$

for all strongly G-regular $\gamma_1 \in H_1(\mathbb{R})$.

• Proof: Is long, but less so if the twisting is trivial. A study of norm correspondences and some descent arguments reduce the problem to an analysis of the relative transfer factor across the problematic walls. Here the canonicity is critical, as we make different choices of a-data and χ-data at the various walls, choices that are *not* globally consistent in general.

• A twisted analogue of the statement in THEOREM I, Parts 2a and 2b, is:

Let Δ_{spec} be the spectral transfer factor compatible with Δ_{geom}. Then

$$St\text{-}Trace\ \pi_1(f_1 dh_1) = \sum\nolimits_{\pi} \Delta_{spec}(\pi_1, \pi)\ Trace\ [\pi(fdg) \circ \pi(\theta, \varpi)].$$

• Here again π_1 is a tempered irreducible representation of the endoscopic group $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1. We start with the G-regular case, and more particularly with π_1 a G-regular discrete series representation.

• First we reduce to the case θ preserves a fundamental splitting. Then we apply a characterization lemma: if a transfer exists for π_1 then the coefficients must coincide with those we have defined. Thus Mezo's results for discrete series representations can be applied to our setting, and the transfer statement is proved for π_1.

• Then we handle most of the G-regular case via an inductive step: an analysis of parabolic descent for the norm map gives a straightforward generalization of the argument for ordinary endoscopy.

• Extension to general tempered case: this is taken up by Mezo.

• Then the analogue of the converse statement in THEOREM I, Part 3 follows, as before.

• We have not discussed settings with empty twist-packets.

8. STRUCTURE ON PACKETS AND AN APPLICATION

• Here we limit our attention to tempered parameters φ and look first at structure on the full packet Π attached to φ. We return to factoring φ. Write S for S_φ, the centralizer in $G^\vee$ of the image of φ. Then to semisimple $s \in S$ there are attached an ordinary endoscopic group, now denoted H^s, and a well-positioned parameter φ^s for $H^s(\mathbb{R})$; π^s will denote a representation in the packet for φ^s.

• We need a slight variant of this factoring. First, let S^{ad} be the image of S in the adjoint form $G^\vee_{ad}$ and S^{sc} be the preimage of S^{ad} in the simply-connected covering $G^\vee_{sc}$ of the derived group of $G^\vee$. Thus we have surjective homomorphisms

$$\begin{array}{ccc} S^{sc} & & S \\ & \searrow \quad \swarrow & \\ & S^{ad} & \end{array}$$

Also

$$\pi_0(S^{ad}) \simeq S/Z^\Gamma S^0 \simeq S^{sc}/Z_{sc}(S^{sc})^0.$$

• The group $\pi_0(S^{ad})$ can be paired with the packet Π but, following Arthur's *Note*, we work instead with $\pi_0(S^{sc})$ as this will give us better information for global multiplicity formulas outside the quasi-split setting.

• We will write s for the image in $G^\vee$ of an element s_{sc} in S^{sc}. Although s need not lie in S, it is still an endoscopic datum and the factoring construction works. We use the same notation as for the case $s \in S$.

• Suppose first that Π consists of discrete series representations. We choose the (almost) canonical representative for φ attached to $spl^\vee$. Then S consists of the Γ-invariants in the maximal torus for $spl^\vee$, where the action of Γ is that given by the tori in G that are anisotropic mod center. Thus S^0 is central and both S^{ad}, S^{sc} are finite and abelian, so that

$$\pi_0(S^{ad}) = S^{ad} \simeq S/Z^\Gamma \simeq S^{sc}/Z_{sc}.$$

•• [For $SL(2)$, S^{ad} is cyclic of order two and S^{sc} is cyclic of order four. In contrast, for p-adic $SL(2)$ and parameters that factor through the image of a ${}^LT$, T anisotropic, if the parameter is regular but not strongly regular then S^{ad} is the four-group while S^{sc} is the quaternion group: see Arthur's *Note.* The strongly regular p-adic case is like real $SL(2)$.]

• Arthur's refinement in *Note* of Langlands' original conjecture requires a pairing

$$\langle -, - \rangle : \Pi \times S^{sc} \to \mathbb{C}^\times$$

with the following properties (since S^{sc} is abelian).

(i) The function $\varepsilon_\pi : s_{sc} \to \langle \pi, s_{sc} \rangle$ *is a character on* S^{sc} *for each* $\pi \in \Pi$.
(ii) [A condition on the restriction of ε_π *to* Z_{sc} *we almost ignore here.]*
(iii) The characters $\varepsilon_\pi, \varepsilon_{\pi'}$ *are distinct for distinct* $\pi, \pi' \in \Pi$.
(iv) For all $s_{sc} \in S^{sc}$ *and* $\pi, \pi' \in \Pi$, *we have*

$$\langle \pi, s_{sc} \rangle \ / \ \langle \pi', s_{sc} \rangle = \Delta(\pi^s, \pi) \ / \ \Delta(\pi^s, \pi').$$

• Recall that $\Delta(\pi^s, \pi) \,/\, \Delta(\pi^s, \pi')$ coincides with $\Delta(\pi^s, \pi; \pi^s, \pi')$. Our definition of this canonical relative factor makes

$$s_{sc} \to \Delta(\pi^s, \pi; \pi^s, \pi')$$

an easily described character trivial on Z_{sc}, *i.e.* a character on S^{ad}. We use Kottwitz's version of the Tate-Nakayama pairings $(-,-)$ for the description.

•• Thus let T be a maximal torus over $\mathbb{R}$ in G that is anisotropic mod center. Then π, π' determine (the real Weyl group orbits of) Weyl chambers C, C' for T. We take a Weyl group element mapping C to C' and attach to it an element $inv_G(\pi, \pi')$ of

$$\mathcal{D}(T) = Ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$$

in a familiar way: if $n \in G$ represents the Weyl element then $\sigma \to n\sigma(n)^{-1}$ represents $inv_G(\pi, \pi')$. Recall that $\mathcal{D}(T)$ is contained in

$$\mathcal{E}(T) = \mathrm{Im}(H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T)).$$

Define $inv(\pi, \pi')$ by choosing n in G_{sc} instead. Then $inv(\pi, \pi')$ maps to $inv_G(\pi, \pi')$ under $H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T)$. [Notice $inv(\pi, \pi')$ is the inverse of the term $inv(\pi', \pi)$ from references.]

•• On the other hand, there are unique toral data making C' dominant for $spl^\vee$. Use these data to regard S as the Γ-invariants in the torus $T^\vee$ dual to T. Since the homomorphism $(T^\vee)^\Gamma \to (T^\vee_{ad})^\Gamma$ factors through the projection $(T^\vee)^\Gamma \to \pi_0[(T^\vee)^\Gamma]$ we can find $\overline{s} \in \pi_0[(T^\vee)^\Gamma]$ with same image s_{ad} in $(T^\vee_{ad})^\Gamma$ as $s_{sc} \in S^{sc}$.

•• Then

$$\begin{aligned} \Delta(\pi^s, \pi; \pi^s, \pi') &= (inv_G(\pi, \pi'), \overline{s}) \\ &= (inv(\pi, \pi'), s_{ad}). \end{aligned}$$

Thus $\Delta(\pi^s, \pi; \pi^s, \pi')$ depends only on s_{ad} and is evidently a character.

• Now pick one (base) character ε_b satisfying (ii) and one (base) element π_b of Π. Then there is a unique pairing $\langle -,- \rangle : \Pi \times S^{sc} \to \mathbb{C}^\times$ satisfying $\varepsilon_b = \varepsilon_{\pi_b}$ plus the conditions (i) - (iv), namely

$$\begin{aligned} \langle \pi, s_{sc} \rangle &= \Delta(\pi^s, \pi; \pi^s, \pi_b).\varepsilon_b(s_{sc}) \\ &= (inv(\pi, \pi_b), s_{ad}).\varepsilon_b(s_{sc}) \end{aligned}$$

for $\pi \in \Pi$, $s \in S^{sc}$.

•• The pairing generalizes to all tempered packets Π, with S^{sc} replaced by $\pi_0(S^{sc})$ which is abelian. The proof relies on precise information in the extension of $\Delta(\pi^s, \pi; \pi^s, \pi')$ to the general tempered setting as in Theorem I, Part 2b.

• We stay with discrete series for the rest of our discussion here. We also return to the Whittaker normalization. Thus we assume $G = G^*$ and the inner twist ψ is the identity. Fix Whittaker data, and define transfer factors Δ_{Wh} with Whittaker normalization. We can take Arthur's constraint (ii) above as: each ε_π is trivial on Z_{sc} and so defines a character on S^{ad}. Then we may as well return to working with S^{ad} as S/Z^Γ, and define s_{ad} as the image of $s \in S$. Let π_{Wh} be the unique member of the packet that is generic relative to the given Whittaker data. We take $\varepsilon_b = \varepsilon_{\pi_{Wh}}$ to be the trivial character. Then the pairing is given by

$$\begin{aligned}\langle \pi, s_{ad} \rangle &= \Delta(\pi^s, \pi; \pi^s, \pi_{Wh}) = \Delta_{Wh}(\pi^s, \pi) \\ &= (inv(\pi, \pi_{Wh}), s_{ad}) \\ &= \pm 1,\end{aligned}$$

and spectral transfer becomes simply

$$St\text{-}Trace\ \pi^s(f^s dh^s) = \sum\nolimits_{\pi\, \in\, \Pi}\ \langle \pi, s_{ad} \rangle\ Trace\ \pi(fdg).$$

• The key property $\Delta(\pi^s, \pi; \pi^s, \pi_{Wh}) = \Delta_{Wh}(\pi^s, \pi)$ follows immediately from a theorem (the strong base-point property for Whittaker normalization):

$$\Delta_{Wh}(\pi^s, \pi_{Wh}) = 1.$$

•• The pairing is perfect in the sense that it identifies Π as the dual of S^{ad} if we replace $G = G^*$ by an extended group (K-group) $\amalg_j G^j$ and take Π as an extended packet $\amalg_j \Pi^j$. The class $inv(\pi^j, \pi_{Wh})$ now lies in $\amalg_j \mathcal{D}(T^j)$ which the K-group construction identifies with $\mathcal{E}(T)$. This construction is nontrivial, *i.e.* $\amalg_j G^j$ is more than just G^* itself, if and only if $H^1(\Gamma, G_{sc}) \neq 1$.

•• [Inversion of spectral transfer in this setting is now trivial.]

•• [comments on global motivation, elsewhere]

• Finally we reintroduce twisting data $(\theta = \theta^*, \varpi)$ for $G = G^*$, and put structure on the twist-packet $\Pi^{tw} = \Pi^{\theta, \varpi}$. The set

$$S_\varphi^{tw} = S^{tw} = \{s \in G^\vee :\ {}^L\theta_a \circ \varphi = Int(s^{-1}) \circ \varphi\}$$

is nonempty because Π is assumed (θ, ϖ)-stable.

• Let $s \in S^{tw}$ and factor φ through parameter φ^s as in the ordinary case. Again let π^s denote an element of the packet attached to φ^s.

• Fix θ-stable Whittaker data, and let π_{Wh} denote the unique element of the twist-packet Π^{tw} in Π that is generic for the given data. Then we define

$$\langle \pi, s \rangle_{tw} = \Delta(\pi^s, \pi; \pi^s, \pi_{Wh})$$

for all $\pi \in \Pi^{tw}$, because we take $\langle \pi_{Wh}, s \rangle_{tw} \equiv 1$.

• Assuming the (tentative) strong base-point property in this twisted setting, we have

$$\Delta_{Wh}(\pi^s, \pi_{Wh}) = 1,$$

so that

$$\langle \pi, s \rangle_{tw} = \Delta_{Wh}(\pi^s, \pi),$$

and we have a simple transfer statement also in the twisted case. In any case,

$$\langle \pi, s \rangle_{tw} = (inv_\theta(\pi, \pi_{Wh}), s_\theta).$$

Here we use the Tate-Nakayama pairing after defining the element $inv_\theta(\pi, \pi_{Wh})$ of $H^1(\Gamma,\ (T_{sc})^{\theta_{sc}})$ via conjugacy of splittings and prescribing $s_\theta \in ((T_{ad}^\vee)_{\theta_{ad}^\vee})^\Gamma$ as below.

• We introduce also ordinary (*i.e.* untwisted) transfer for Π. Then S^{tw} is replaced by the subgroup S from before. To $s_0 \in S$ attach parameter φ^{s_0}; π^{s_0} denotes an element of the packet attached to φ^{s_0}. We use the Whittaker data already specified to define Whittaker normalizations of the ordinary geometric and spectral transfer factors. Then

$$\langle \pi, s_0 \rangle = \Delta_{Wh}(\pi^{s_0}, \pi)$$

defines our pairing between Π and S/Z^Γ.

• Suppose π lies in the twist-packet $\Pi^{tw} \subseteq \Pi$. Notice that if $s_1, s_2 \in S^{tw}$ then $s_0 = s_1(s_2)^{-1}$ lies in S, so that $\langle \pi, s_0 \rangle$ is well-defined. Then we have that

$$\langle \pi, s_1 \rangle_{tw} = \langle \pi, s_0 \rangle \,.\, \langle \pi, s_2 \rangle_{tw} \,.$$

•• To check this, we calculate

$$\begin{aligned} \langle \pi, s_1 \rangle_{tw} &= \langle \pi, s_0 s_2 \rangle_{tw} = (inv_\theta(\pi, \pi_{Wh}),\ (s_0 s_2)_\theta) \\ &= (inv_\theta(\pi, \pi_{Wh}),\ s_\theta).(inv_\theta(\pi, \pi_{Wh}),\ (s_2)_\theta), \end{aligned}$$

where s_θ denotes the image of s_0 as Γ-invariant in $T^\vee$ under

$$(T^\vee)^\Gamma \to (T^\vee_{ad})^\Gamma \to ((T^\vee_{ad})_{\theta^\vee_{ad}})^\Gamma,$$

$(T^\vee_{ad})_{\theta^\vee_{ad}}$ being the dual of $(T_{sc})^{\theta_{sc}}$. We can just as well project only as far as $(T^\vee_{ad})^\Gamma$ and project $inv_\theta(\pi, \pi_{Wh})$ to untwisted $inv(\pi, \pi_{Wh})$ in $H^1(\Gamma, T_{sc})$, to compute $(inv_\theta(\pi, \pi_{Wh}), s_\theta)$ as untwisted $\langle \pi, s_0 \rangle$, and the claim follows.

• We can interpret this claim slightly differently. Assume for convenience that ϖ is trivial.

• Replace the dual group $G^\vee$ by $G^\vee \rtimes \langle \theta^\vee \rangle$ and consider instead the group

$$\overline{S} = Cent(\varphi(W_{\mathbb{R}}), G^\vee \rtimes \langle \theta^\vee \rangle),$$

where $\theta^\vee$ acts on $\varphi(W_{\mathbb{R}})$ by its action on the first component, *i.e.* by ${}^L\theta$.

• Then both S and S^{tw} embed in $\overline{S}$. First, $s \mapsto s \times 1$ embeds S as

$$Cent(\varphi(W_{\mathbb{R}}), G^\vee \times 1),$$

and then $t \mapsto t \times \theta^\vee$ embeds S^{tw} as

$$Cent(\varphi(W_{\mathbb{R}}), G^\vee \times \theta^\vee).$$

• For each π in the twist-packet, the function

$$t.Z^\Gamma \mapsto \langle \pi, t \rangle_{tw}$$

on the quotient set $S^{tw} /\ Z^\Gamma$ determines (uniquely) a sign character on the group $\overline{S}\ /\ Z^\Gamma$ extending both itself and the character

$$s.Z^\Gamma \mapsto \langle \pi, s \rangle$$

on the group $S\ /\ Z^\Gamma$.

• In particular, in the case of quasi-split special orthogonal groups, we see that we have the correct pairing for discrete series packets in the statement of Theorem 2.2.4 in Arthur's book.

9. **REFERENCES FOR 7 AND 8**

J. Arthur
A note on L-packets, Pure and Appl. Math. Quart., Coates vol. (2006), 199-217.
The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups, (book) preprint.
http://www.claymath.org/cw/arthur/pdf/Book.pdf

P. Mezo
Character identities in the twisted endoscopy of real reductive groups, preprint.
http://mathstat.carleton.ca/~mezo/twisted2.pdf

D. Shelstad
Tempered endoscopy for real groups I: geometric transfer with canonical factors, Contemp. Math., 472 (2008), 215-246.
Tempered endoscopy for real groups II: spectral transfer factors, in Automorphic Forms and the Langlands Program, Higher Ed Press/International Press, 2009/2010, 236-276.
Tempered endoscopy for real groups III: inversion of transfer and L-packet structure, Represent. Theory, 12 (2008), 369-402.
On geometric transfer in real twisted endoscopy, preprint.
On spectral transfer factors in real twisted endoscopy, preprint.
A note on real endoscopic transfer and pseudo-coefficients, preprint.
http://andromeda.rutgers.edu/~shelstad

email: shelstad rutgers edu

June 24, 2011

6/30 fixed typo: page 4, line 3