Transfer in endoscopy (and beyond) for real groups

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Abstract

We consider real reductive groups and describe some theorems on endoscopic transfer in this setting.

In preparation we review the notion of stabilization first from a more elementary perspective and then briefly from the global perspective of the Arthur-Selberg trace formula.

If time permits, we also discuss very briefly the stable transfer for orbital integrals on real groups envisaged by Langlands within the theme of Beyond Endoscopy and describe the complementary nature of the two transfers via examples we carry throughout the talk.

Transfer in endoscopy (and beyond) for real groups

transfer: of basic objects in invariant harmonic analysis from one group to another

basic objects: orbital integrals *along strongly regular conjugacy classes* OR traces (characters) *of irreducible admissible representations*

real group: Lie group $G(\mathbb{R})$, all real points on a reductive algebraic group G defined over $\mathbb{R}$

complex points $G(\mathbb{C})$ connected (almost)

Transfer in endoscopy (and beyond) for real groups

Endoscopic transfer

is motivated by stabilization problems

- locally, *i.e.* for real groups
- globally, *i.e.* for adelic groups

Beyond Endoscopy

involves transfer of stabilized basic objects

(ET) Langlands, 1970± *(BET) Langlands, 2010±*

1. Stabilization problem for real groups

G($\mathbb{R}$) compact, connected : Weyl, Peter-Weyl

- irreducible unitary π is finite-dimensional
- Weyl character formula for $Char(\pi, g) \stackrel{\text{def}}{=} Trace\ \pi(g)$

  is valid for all regular elements g in G($\mathbb{R}$)
- orbital integrals appear in the Weyl integration formula

Characters and orbital integrals : stable and smooth

Example: SU(2)

1. Stabilization problem for real groups

$\mathrm{G}(\mathbb{R}) = \mathrm{SU}(2)$ $\mathrm{T}(\mathbb{R})$ = diagonal subgroup

π_n for $n = 1, 2, 3, \ldots$ $\gamma_\theta = \mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$

$$\boldsymbol{Char}(\boldsymbol{\pi_n}, \boldsymbol{\gamma_\theta}) = \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} \qquad \theta \neq 0 \bmod \pi$$

Harish-Chandra normalization of orbital integrals applied to smooth function f on $\mathrm{SU}(2)$:

$$\boldsymbol{F_f}(\boldsymbol{\gamma_\theta}) \stackrel{\text{def}}{=} \left(e^{i\theta} - e^{-i\theta}\right) \int_{cl(\gamma_\theta)} f$$

1. Stabilization problem for real groups

General setting via Harish-Chandra theorems

- Orbital integral as Schwartz distribution $\boldsymbol{f \to F_f(\gamma)}$

$F_f(\gamma)$ *as function of* γ ...

- Irreducible trace as distribution $\boldsymbol{f \to Trace\, \pi(f)}$

$\pi(f) \stackrel{\text{def}}{=} \int_{G(\mathbb{R})} \pi(g)\, f(g)\, dg$

- Harish-Chandra regularity theorem:

$$Trace\, \pi(f) = \int_{G(\mathbb{R})} Char(\pi, g)\, f(g)\, dg$$

smooth on regular semisimple set

- Examples for $\mathrm{SL}(2, \mathbb{R})$ ...

1. Stabilization problem for real groups

$G(\mathbb{R}) = SL(2,\mathbb{R})$ $\quad$ $T(\mathbb{R}) = \left\{\gamma_\theta \stackrel{\text{def}}{=} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right\}$

Discrete series π_{n+} : $\quad Char(\pi_{n+}, \gamma_\theta) = \dfrac{-e^{in\theta}}{e^{i\theta} - e^{-i\theta}}$

Discrete series π_{n-} : $\quad Char(\pi_{n-}, \gamma_\theta) = \dfrac{e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$

$\theta \neq 0 \ mod \ \pi$

- $f \to [Trace\ \pi_{n+}(f) + Trace\ \pi_{n-}(f)]$ is stable
- $f \to [Trace\ \pi_{n+}(f) - Trace\ \pi_{n-}(f)]$ is supported off the regular hyperbolic set

… interpret now as transfer from $\mathrm{T}(\mathbb{R})$

1. Stabilization problem for real groups

Stable conjugacy class of γ_θ is $cl(\gamma_\theta) \cup cl(\gamma_{-\theta})$

$$f \to SO(\gamma_\theta, f) = |e^{i\theta} - e^{-i\theta}|[\int_{cl(\gamma_\theta)} f + \int_{cl(\gamma_{-\theta})} f]$$

is a stable distribution

- $\theta \to (e^{i\theta} - e^{-i\theta}) [\int_{cl(\gamma_\theta)} f - \int_{cl(\gamma_{-\theta})} f]$ extends to a smooth function f_T on $\mathrm{T}(\mathbb{R})$
- $Trace\, \pi_{n+}(f) - Trace\, \pi_{n-}(f)$ is the Fourier coefficient of f_T at n (... trace on rep. $e^{in\theta}$)

Labesse-Langlands (early 1970's): nonarchimedean case also

2. General remarks on endoscopic transfer

Endoscopic group $\mathrm{H}(\mathbb{R})$ *

- Set of conjugacy classes in stable class of a strongly regular element of $\mathrm{G}(\mathbb{R})$ has structure of a finite abelian group

  may need to include some inner forms with G

- A character $\boldsymbol{\kappa}$ on this group identifies a semisimple element $\mathbf{s}$ in the complex dual $\mathrm{G}^{\vee}$ of G ... $\mathbf{H}^{\vee} \stackrel{\text{def}}{=} \mathbf{Cent}(\mathbf{s},\ \mathbf{G}^{\vee})^{\mathbf{0}}$

- Galois action on $\mathrm{H}^{\vee}$ determined by sharing of maximal tori
  Thm: inclusion of $\mathrm{H}^{\vee}$ in $\mathrm{G}^{\vee}$ extends to L−morphism ${}^{\mathbf{L}}\mathbf{H} \to {}^{\mathbf{L}}\mathbf{G}$

  may need slight adjustment to setting, ignore here

  ... result critical for existence of transfer

- $\mathbf{s}$ propagates $\boldsymbol{\kappa}$ (nonuniquely) to each strongly regular stable class in $\mathrm{G}(\mathbb{R})$ meeting a maximal torus shared with $\mathrm{H}(\mathbb{R})$

**** Langlands: Stable conjugacy ... (1970's)***

2. General remarks on endoscopic transfer

A twist: $\mathbf{G}(\mathbb{R}) = \mathbf{SL}(2, \mathbb{C})$ **with automorphism** $\boldsymbol{\theta} : \boldsymbol{g} \mapsto \overline{\boldsymbol{g}}$

- *Consider only* π *with* $\pi \circ \theta \approx \pi$
- *Same definitions as standard endoscopy but on non-identity component of* $SL(2, \mathbb{C}) \rtimes \langle\theta\rangle$
- *Endoscopic groups turn out to be standard endoscopic groups for* $SL(2, \mathbb{R})$, *but with different* L*-morphisms*
- π *is transfer from* $SL(2, \mathbb{R})$ *if central character is trivial*
- π *is transfer from* $T(\mathbb{R})$ *if central character is nontrivial*

- *Point correspondence (sharing tori) now involves norm maps*

General twisted setting* $\pi \circ \theta \approx \varpi \otimes \pi$

- ϖ *character on* $\mathrm{G}(\mathbb{R})$
- $\boldsymbol{e}$ *set of endoscopic data, includes* L*-morphism*

****Kottwitz-Shelstad : Foundations of twisted endoscopy***

2. General remarks on endoscopic transfer

- Replace κ−orbital integrals by terms of the form

$$\sum_{cl(\gamma)} \Delta(\gamma_H, \gamma)\, O(\gamma, f)$$

where $\Delta(\gamma_H, \gamma)$ is a factor determined uniquely up to constant by endoscopic data e

- For spectral transfer, consider terms of the form

$$\sum_{\pi} \Delta(\pi_H, \pi)\, Trace\, \pi(f)$$

with similar factor $\Delta(\pi_H, \pi)$, require compatibility

- To define factors, start with **very regular pair** (γ_H, γ) and **tempered very regular pair** (π_H, π)

very regular: (strongly) regular relative to G data

3. Transfer factors

Existence of such factors $\Delta(\gamma_H, \gamma)$ in standard case is suggested by

- *results for real groups in implicit form*
- *Langlands' stabilization of regular elliptic term in the Arthur-Selberg trace formula (Les débuts …)*
- *Kottwitz's approach to the global hypothesis in Les débuts*

Example of $SL(2)$ over number field F:

- Labesse-Langlands has factor Δ_v for each place v of F

  see Whittaker normalization as in K-S

- Product formula over all places involves simpler adelic factor

  factor tests if certain adelic conjugacy classes have points in $G(F)$

3. Transfer factors

In general, there is a uniform definition of factors over all places, with product formula:

- *hint for shape from behavior of κ-orbital integrals near regular unipotent set using a construction of Langlands* valid at all places*

- define relative factor $\Delta(\gamma_H, \gamma; \gamma'_H, \gamma')$, determined uniquely by e, when each pair of points is **related**

- then for absolute factor $\Delta(\gamma_H, \gamma)$ require:

$$\Delta(\gamma_H, \gamma)/\Delta(\gamma'_H, \gamma') = \Delta(\gamma_H, \gamma; \gamma'_H, \gamma')$$

- *References: Langlands-Shelstad (1988, 90) for definitions in standard case with beginning of descent theory, Kottwitz-Shelstad (1999) for definitions in twisted case.*

** Orbital integrals on forms of SL(3) I*

3. Transfer factors

Remarks

- Relative factors have the form

$$\Delta = \Delta_{\boldsymbol{I}}\Delta_{\boldsymbol{II}}\Delta_{\boldsymbol{III}}$$

- *Each term involves two out of three additional choices … effects cancel*

- *$\Delta_I\Delta_{III}$ is defined in terms of pairings in Galois (hyper-)cohomology*
- *Only Δ_{III} persists globally*
- *Δ_{II} comes from harmonic analysis, given explicitly in terms of the root systems for G, H*
- *For general G: Δ_{III} is genuinely relative term, others are quotients*
- *For G of quasi-split type: all terms are quotients*

3. Transfer factors

Spectral factors for real groups:

- **Same form** $\Delta = \Delta_I \Delta_{II} \Delta_{III}$ (some cases to finish in twisted case)
- Δ_I, Δ_{III} in terms of same groups, pairings as before
- Δ_{II} now from local formula for characters around the identity
- *SL(2) example (L-L): spectral factors at all places, adelic version tests which representations in certain packets are automorphic*

Compatibility for geometric and spectral factors:

- make relative factor $\Delta(\gamma_H, \gamma;\ \pi_H, \pi)$ from related pair (γ_H, γ) and related pair (π_H, π), again canonical
- enough to check compatibility of absolute factors on one set:

$$\Delta(\boldsymbol{\gamma}_H, \boldsymbol{\gamma}) / \Delta(\boldsymbol{\pi}_H, \boldsymbol{\pi}) = \Delta(\boldsymbol{\gamma}_H, \boldsymbol{\gamma};\ \boldsymbol{\pi}_H, \boldsymbol{\pi})?$$

In standard case: fix compatible pair of factors for use in theorems.

First theorem (geometric transfer) applies in general twisted case …

4. Theorems on real endoscopic transfer

- *test measures* $f\,dg$ *on* $\mathrm{G}(\mathbb{R})$ *and* $f_H dh$ *on* $\mathrm{H}(\mathbb{R})$
- f, f_H *Harish-Chandra Schwartz functions*
- *compatible Haar measures on related maximal tori*

Theorem: For each $f\,dg$ there exists $f_H dh$ such that

$$SO(\gamma_H, f_H dh) = \sum_{cl(\gamma)} \Delta(\gamma_H, \gamma)\, O(\gamma, f\,dg)$$

for all strongly G-regular γ_H in $\mathrm{H}(\mathbb{R})$.

For (long) proof: Use Harish-Chandra's Plancherel theory to characterize stable orbital integrals on $\mathrm{H}(\mathbb{R})$ *by stability, behavior near semi-regular elements. Find version with data used in transfer factors and apply to* $\gamma_H \to \sum_{cl(\gamma)} \Delta(\gamma_H, \gamma)\, O(\gamma, f\,dg)$. *Semi-regular descent for the integrals reduces the problem to verifying properties of norms and transfer factors across walls in* $\mathrm{H}(\mathbb{R})$ *…*

4. Theorems on real endoscopic transfer

Again:

Theorem: For each $f dg$ there exists $f_H dh$ such that

$$SO(\gamma_H, f_H dh) = \sum_{cl(\gamma)} \Delta(\gamma_H, \gamma)\, O(\gamma, f dg)$$

for all strongly G-regular γ_H in $\mathrm{H}(\mathbb{R})$.

Corollary: *… via theorem of A. Bouaziz (Invent. 1994, AENS 1994)*

If f has compact support then we may take f_H with compact support.

4. Theorems on real endoscopic transfer

Dual transfer

Suppose $\boldsymbol{\Theta_H}$ is a stable character on $\mathrm{H}(\mathbb{R})$ with an infinitesimal character. Its transfer to $\mathrm{G}(\mathbb{R})$ is by definition:

$$\boldsymbol{\Theta} : \boldsymbol{f} \longmapsto \boldsymbol{\Theta_H(f_H)}$$

... an invariant eigendistribution on $G(\mathbb{R})$ with shifted infinitesimal character

- *shift is determined by L-morphism from endoscopic data **e***
- *Θ_H is tempered $\Longrightarrow$ Θ is tempered (can assume **e** is bounded).*

Start dual spectral transfer with tempered very regular pair (π_H, π) and stable trace for L-packet Π_H of π_H :

$$\boldsymbol{\Theta_H = St\text{–}Trace\,\pi_H}$$

4. Theorems on real endoscopic transfer

- **Spectral theorems:** here, discuss standard endoscopy

General twisted case: results analogous, but incomplete. A critical step is existence of tempered character identities (Mezo, Mem. 2012)

Theorem (tempered very regular case):

$$St\text{–}Trace\,\pi_H(f_H) = \sum_{\pi}^{o} \Delta(\pi_H, \pi)\; Trace\,\pi(f)$$

For proof : Main case is where left side is a stable discrete series character and right side has nonzero contributions only from a discrete series packet. Apply Harish-Chandra's characterization of discrete series characters. Use compatibility property of geometric and spectral transfer factors to organize and cancel …

4. Theorems on real endoscopic transfer

General tempered (π_H, π): *After parabolic descent, can assume Π_H consists of discrete series. Then only a packet of limits of discrete series will contribute nontrivially to the right.* **Define** $\Delta(\pi_H, \pi)$ *by coherent continuation to the wall (Zuckerman translation)*

Theorem (general tempered case):

$$St\text{–}Trace\,\pi_H(f_H) = \sum_{\pi}^{o} \Delta(\pi_H, \pi)\; Trace\,\pi(f)$$

Proof uses L-group analogue of coherent continuation, Hecht-Schmid character identities … *Then converse follows:*

f_H, f **match on tempered traces**

$\Rightarrow f_H, f$ **match on orbital integrals**

4. Theorems on real endoscopic transfer

L-packet structure

- *a first motivation: stabilization problem for real groups …*
- *identities to invert: select via Langlands parameter for Π*
  … a $G^\vee$-conjugacy class of bounded
  semi-simple L-morphisms $\varphi : W \to {}^L G$

Langlands factoring*

$$\boldsymbol{S} \stackrel{\text{def}}{=} \textbf{centralizer in } \boldsymbol{G}^\vee \textbf{ of the image of } \boldsymbol{\varphi}$$

- *to $s \in S$ attach endoscopic data e^s and factor φ through (well-positioned) parameter φ^s*
- *so have packet Π^s with representative π^s*

**L-L, Notes on K-Z theory (1977)*

4. Theorems on real endoscopic transfer

$\mathbb{S} \stackrel{\text{def}}{=}$ **group of components of the image of S in the adjoint form of $G^\vee$**

- *$\mathbb{S}$ is finite abelian, a sum of groups of order two*

> **Theorem: For distinct π, π' in Π**
>
> $$s \rightarrow \Delta(\pi^s, \pi;\ \pi^s, \pi')$$
>
> **defines a nontrivial character on $\mathbb{S}$ ... and all so obtained.**

(Long) proof reduces to calculations in L-group based on main results of Knapp-Zuckerman for limits of discrete series (PNAS 1976, Annals 1982)

Case G quasi-split and Whittaker normalization:

- *$\Delta = \Delta_\lambda$ where λ is $G(\mathbb{R})$-conjugacy class of Whittaker characters*
- *tempered Π has natural base-point π_λ generic for λ*

4. Theorems on real endoscopic transfer

Theorem (strong base-point property): $\Delta_\lambda(\pi^s, \pi_\lambda) = 1$

Proof uses transfer theorems and is based on classification of generic representations by Kostant (Invent. 1978) and Vogan (Invent. 1978).

Corollary: The pairing $(s, \pi) \rightarrow \Delta_\lambda(\pi^s, \pi\ )$ **identifies** Π **canonically as dual of the finite abelian group** $\mathbb{S}$**.**

- *Inversion of trace identities by Fourier inversion in* $\mathbb{S}$
- *Calculate pairing explicitly via Tate-Nakayama duality*