On elliptic factors in real endoscopic transfer I

D. Shelstad

Abstract

This paper is concerned with the structure of packets of representations and some refinements that are helpful in endoscopic transfer for real groups. It includes results on the structure and transfer of packets of limits of discrete series representations. It also reinterprets the Adams-Johnson transfer of certain nontempered representations via spectral analogues of the Langlands-Shelstad factors, thereby providing structure and transfer compatible with the associated transfer of orbital integrals. The results come from two simple tools introduced here. The first concerns a family of splittings of the algebraic group G under consideration; such a splitting is based on a fundamental maximal torus of G rather than a maximally split maximal torus. The second concerns a family of Levi groups attached to the dual data of a Langlands or an Arthur parameter for the group G. The introduced splittings provide explicit realizations of these Levi groups. The tools also apply to maps on stable conjugacy classes associated with the transfer of orbital integrals. In particular, they allow for a simpler version of the definitions of Kottwitz-Shelstad for twisted endoscopic transfer in certain critical cases. The paper prepares for spectral factors in twisted endoscopic transfer that are compatible in a certain sense with the standard factors discussed here. This compatibility is needed for Arthur's global theory. The twisted factors themselves will be defined in a separate paper.

Contents

1 Introduction

Our main purpose is to continue a study of the coefficients appearing in the spectral identities of endoscopic transfer for real groups. The coefficients carry information about the structure of packets of irreducible representations, and in the global theory of endoscopy this structure plays a central role in determining if certain irreducible representations are automorphic or not; see [Ar13].

Here we will consider both the standard and the more general twisted versions of endoscopic transfer. We focus on the *fundamental case* where the endoscopic group and the ambient group share, in a certain precise sense, fundamental maximal tori; see Section 3.3. It includes the case where the ambient group G is cuspidal and the endoscopic group H_1 is elliptic. We call this the cuspidal-elliptic setting; see Section 3.4. Then $G(\mathbb{R})$ and $H_1(\mathbb{R})$ share fundamental Cartan subgroups that are elliptic, *i.e.*, compact modulo the centers of $G(\mathbb{R})$ and $H_1(\mathbb{R})$ respectively. Thus there is a discrete series of representations for each of $G(\mathbb{R})$ and $H_1(\mathbb{R})$ [HC75], along with limits of discrete series representations (see [KZ82]).

Endoscopic transfer begins with the matching of orbital integrals, the so-called geometric side. In the standard version we use the transfer factors of Langlands-Shelstad ([LS87], see also [Sh14]) for the geometric side. Factors with a parallel definition appear in the tempered dual spectral transfer, *i.e.*, as coefficients in the dual spectral identities for tempered irreducible representations [Sh10, Sh08b]. Properties of these spectral factors simplify the related harmonic analysis; for example, inversion of the identities becomes a short exercise (see [Sh08b]).

In preparation for generalizing (in [ShII]) the definition of spectral factors to the twisted setting of Kottwitz-Shelstad [KS99] we will establish three refinements. First, we make use of an alternative simpler description of limits of discrete series packets in terms of elliptic data, *i.e.*, data attached to an elliptic Cartan subgroup (see Remark 5.6), to simplify transfer and structure in that setting.

Second, introducing the nontempered spectrum to our picture, we reinterpret the transfer of Adams-Johnson in terms of data attached directly to the associated Arthur parameters. Here we will consider only parameters that are elliptic in the sense of Arthur. Our new factors are related very simply to the tempered factors already defined, and we check that they do provide the transfer that is precisely dual (*i.e.*, there are no extraneous constants) to that of orbital integrals with the Langlands-Shelstad factors. The inversion properties of our spectral transfer are more delicate than for the tempered case ([AJ87], [Ar89] explain why this must be the case) and will be described in [ShII].

For the third refinement we turn to twisted transfer and the underlying definitions of [KS99]. The transfer of orbital integrals is based on an abstract norm correspondence (γ_1, δ) for suitably regular points γ_1 in endoscopic $H_1(\mathbb{R})$ and δ in $G(\mathbb{R})$. For the fundamental part of the correspondence that concerns us here we will see that we may limit the twisting automorphism to a family for which the norm correspondence is well-behaved. The standard spectral factors generalize readily for this family [ShII] and we have the standard-twisted compatibility needed in Arthur's global theory [Ar13].

To obtain these refinements we introduce two simple tools. The first involves fundamental splittings. These are particular splittings based on fundamental maximal tori and exist for any G. They work well with both elliptic data and Whittaker data; here Vogan's characterization of generic representations plays a critical role. See Sections 2.3, 6.1. The second tool involves a family of Levi groups in G. First we attach to a Langlands or Arthur parameter a family of L-groups and then we use fundamental splittings to identify their real duals as a family of (nonstandard) Levi groups in G and its inner forms; see Sections 5.2, 6.1.

We begin the paper with fundamental splittings and their properties. The main result is Lemma 2.5. In Part 3 we review the norm correspondence and see, in particular, that the fundamental part of geometric transfer is nonempty if and only if the twisting automorphism preserves a fundamental splitting up to a further twist by an element of $G(\mathbb{R})$. This may be expressed precisely in terms of the norm correspondence itself or in terms of nonvanishing of the geometric transfer factors of [KS99]; see Theorem 3.12, Corollary 3.13. We prove a spectral analogue, but not until Part 9 where we also finish the discussion of Part 4 on certain properties of endoscopic data that will be used in the definition of twisted spectral factors.

The rest of the paper concerns standard transfer and the first two refinements. In Part 5 we turn to the Langlands and Arthur parameters attached to the representations of interest to us here. In the cuspidal-elliptic setting these are the s-elliptic Langlands parameters and the elliptic u-regular Arthur parameters of Sections 5.5 - 5.7. Given a parameter, we generate data for the various attached packets of representations by means of pairs (G, η), where η is an inner twist of G to a given quasi-split form G^*.

For the limits of discrete series representations attached to an s-elliptic Langlands parameter we reformulate some well-known properties in terms of our attached Levi groups. For example, the critical Lemma 6.1 characterizes the pairs (G, η) for which we obtain a well-defined (*i.e.,* nonzero) representation. Then Lemma 6.2 gives a description of the packet that allows us to attach an elliptic invariant to each member; see Section 6.4. Lemmas 6.4 and 6.5 describe the application to endoscopic transfer.

Part 6 has further results on limits of discrete series representations that we will apply in various places. For example, as in Section 6.7, every s-elliptic parameter factors through a totally degenerate parameter for an attached Levi group. We will check in [ShII] that this gives a simple characterization of those pairs (G, η) for which the distribution character of the attached representation is elliptic.

The representations attached to elliptic u-regular Arthur parameters are the derived functor modules of Vogan and Zuckerman [Vo84] from the main setting in [AJ87]; we allow without harm a nontrivial split component in the center of G. In the case of regular infinitesimal character they are discussed in [Ar89, Ko90]; see the last paragraph of Section 7.1. Generalizing some familiar

L-group constructions we attach directly to the Arthur parameter the following: elliptic data, a family of Levi groups, and an s-elliptic Langlands parameter with same infinitesimal character and central behavior. We will use these again in [ShII] in the twisted setting. In the present paper we pursue only the case of regular infinitesimal character so that the attached Langlands parameter is elliptic. Lemma 7.5 describes the Arthur packet by means of pairs (G, η). In Section 8 we introduce spectral transfer factors for the Arthur packet by tethering the packet to the elliptic Langlands packet via *relative* factors with good transitivity properties [Sh10]. We then verify in Section 8.3 that the corresponding absolute factors are correct, in the sense already mentioned, for endoscopic transfer with Langlands-Shelstad factors on the geometric side.

To finish this brief sketch we refer to Sections 3.5, 4.1, 4.2, 6.4 and 6.5 where there are further remarks on the properties of transfer factors that are crucial for our approach to work.

Note: This paper is an expanded version of part of the preprint "On spectral transfer factors in real twisted endoscopy" posted on the author's website, May 2011.

2 Automorphisms and inner forms

This section introduces notation we will use throughout the paper, along with definitions and properties related to fundamental splittings. We finish with an application to the *inner forms of a quasi-split pair.*

2.1 Quasi-split pairs and inner forms

By a quasi-split pair we mean a pair (G^*, θ^*), where G^* is a connected, reductive algebraic group defined and quasi-split over $\mathbb{R}$, and θ^* is an $\mathbb{R}$-automorphism of G^* that preserves an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$ of G^*. We assume that the restriction of θ^* to the identity component of the center of G^* is semisimple or, equivalently, that θ^* has finite order.

Recall from [KS99, Appendix B] that (G, θ, η) is defined to be an inner form of (G^*, θ^*) if G is connected, reductive and defined over $\mathbb{R}$, η is an isomorphism from G to G^* that is an inner twist, θ is an $\mathbb{R}$-automorphism of G, and θ coincides with the transport of θ^* to G via η up to an inner automorphism. Notice that if θ^* is the identity then θ must be an inner automorphism of G defined over $\mathbb{R}$, *i.e.,* θ must act on G as an element of $G_{ad}(\mathbb{R})$.

Let (G, θ, η) be an inner form of (G^*, θ^*). By the *inner class* of (θ, η) we will mean the set of all pairs (θ', η') where (G, θ', η') is an inner form of (G^*, θ^*) such that (i) $\eta' \circ \eta^{-1}$ is inner and (ii) the automorphism $\theta' \circ \theta^{-1}$ of G, which is inner and acts as an element of $G_{ad}(\mathbb{R})$ by (i), is induced by an element of $G(\mathbb{R})$, *i.e.,* acts as an element of the image of $G(\mathbb{R})$ in $G_{ad}(\mathbb{R})$ under the natural projection. We will see that replacing (θ, η) by a member of its inner class has no effect on our final results.

Let (G, θ, η) be an inner form of (G^*, θ^*). Then we choose $u(\sigma) \in G^*_{sc}$ such that

$$\eta \circ \sigma(\eta)^{-1} = Int(u(\sigma)). \tag{2.1}$$

Here and throughout the paper we use σ to denote the nontrivial element of $\Gamma = Gal(\mathbb{C}/\mathbb{R})$. The action of σ on a Γ-set X will be denoted by σ_X or by σ itself when X is evident.

2.2 Fundamental splittings

While $\mathbb{R}$-splittings exist only for quasi-split groups, *fundamental splittings* may be constructed for any connected, reductive G defined over $\mathbb{R}$. We recall the definition (see [Sh12]).

Consider a pair (B, T), where T is a maximal torus in G defined over $\mathbb{R}$ and B is a Borel subgroup of G containing T. We call (B, T) a *fundamental pair* if (i) T is fundamental, *i.e.,* T is minimally $\mathbb{R}$-split or, equivalently, T has no roots fixed by σ_T and (ii) the set of (simple) roots of T in B is preserved by $-\sigma_T$. The existence of fundamental pairs is noted in [Ko86, Section 10.4].

Lemma 2.1 *The set of all fundamental pairs for G forms a single stable conjugacy class in the sense that another fundamental pair (B', T') is conjugate to (B, T) by an element g of G for which $Int(g) : T \to T'$ is defined over $\mathbb{R}$.*

Proof. Observe that (B', T') is conjugate to (B, T) under some element g of G, and then $g^{-1}\sigma(g)$ acts as an element of the Weyl group of T preserving the roots of B, *i.e.,* as the identity element. ∎

To prescribe a fundamental splitting we start with a fundamental pair (B, T) and pick an $\mathfrak{sl}_2$-triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ for each simple root α of T in B. Here we identify the Lie algebra of T with $X_*(T) \otimes \mathbb{C}$ and require H_α be the element identified with the coroot $\alpha^\vee$ of α; $X_\alpha, X_{-\alpha}$ are to be root vectors for $\alpha, -\alpha$ respectively. There is an attached splitting $spl = (B, T, \{X_\alpha\})$ for G. Conversely, each splitting for G determines uniquely a collection of $\mathfrak{sl}_2$-triples of the above form. We call *spl fundamental* if the Galois action satisfies: $\sigma X_\alpha = X_{\sigma_T \alpha}$ in the case $\sigma_T\alpha \neq -\alpha$, and $\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$ in the case $\sigma_T\alpha = -\alpha$, where $\varepsilon_\alpha = \pm 1$.

If $\sigma_T\alpha = -\alpha$ then such a triple $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ determines an $\mathbb{R}$-homomorphism from a real form of $SL(2)$ into G; examples are written in [Sh79a]. The isomorphism class of that real form, split or anisotropic, is uniquely determined by α. If the real form is split then $\varepsilon_\alpha = 1$ and α is called *noncompact.* If the real form is anisotropic then $\varepsilon_\alpha = -1$ and α is *compact.*

Lemma 2.2 *Each fundamental pair (B, T) extends to a fundamental splitting $spl = (B, T, \{X_\alpha\})$. Moreover, two fundamental splittings extending (B, T) are conjugate under $T_{sc}(\mathbb{R})$.*

Proof. For each simple root α of T in B, pick an $\mathfrak{sl}_2$-triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$. If $\sigma_T\alpha \neq -\alpha$ then we may arrange that $\sigma X_\alpha = X_{\sigma_T\alpha}$ and $\sigma X_{-\alpha} = X_{-\sigma_T\alpha}$ since $\alpha, \sigma_T\alpha$ are distinct. If $\sigma_T\alpha = -\alpha$ then calculation shows that $\sigma X_\alpha = \lambda X_{-\alpha}$ and

$\sigma X_{-\alpha} = \lambda^{-1} X_\alpha$, where λ is real. Then we can adjust the choice of X_α and $X_{-\alpha}$ to arrange that $\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$ and $\sigma X_{-\alpha} = \varepsilon_\alpha X_\alpha$, where $\varepsilon_\alpha = \pm 1$.

Suppose we have two such splittings. Then they are conjugate under $T_{ad}(\mathbb{R})$ since if X_α is replaced by $Int(t)X_\alpha$ for each B-simple root α, where $t \in T_{sc}$, then our requirements on the action of σ imply that $\alpha(\sigma(t)t^{-1}) = 1$. Because T_{sc}, T_{ad} are fundamental, $T_{sc}(\mathbb{R})$ and $T_{ad}(\mathbb{R})$ are connected; see [Ko86, Section 10] and [Sh12, Section 6]. Then the projection $T_{sc}(\mathbb{R}) \to T_{ad}(\mathbb{R})$ is surjective, and the desired conjugation exists. ■

Corollary 2.3 *Each fundamental splitting $spl' = (B', T', \{X'_\alpha\})$ is conjugate to spl by an element g of G for which $Int(g) : T \to T'$ is defined over $\mathbb{R}$.*

2.3 Fundamental splittings of Whittaker type

We return to a quasi-split pair (G^*, θ^*). Recall that θ^* preserves the $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$ of G^*. *From now on we will typically use the same notation $\{X_\alpha\}$ for the root vectors in any splitting.* We will say a fundamental pair (B, T), or a fundamental splitting $spl_f = (B, T, \{X_\alpha\})$ *of* G^*, is *of Whittaker type* if all imaginary simple roots of (B, T) are noncompact. We use this terminology because of Vogan's classification theorem [Vo78, Corollary 5.8, Theorem 6.2] for representations with Whittaker model, *i.e.*, for *generic* representations. It is not difficult to check directly that a group G has a fundamental pair of Whittaker type if and only if G is quasi-split over $\mathbb{R}$, although this characterization is naturally part of Vogan's classification.

Lemma 2.4 *(i) There exists a fundamental pair of Whittaker type preserved by θ^* and (ii) each fundamental pair of Whittaker type preserved by θ^* has an extension to a fundamental splitting spl_{Wh} of G^* preserved by θ^*.*

Proof. (i) We use Steinberg's structure theorems as described in [KS12, Section 3] and [KS99, Section 1.3]. First attach to spl^* an $\mathbb{R}$-splitting for $(G^*_{sc})^{\theta^*_{sc}}$. We may then find h in $(G^*_{sc})^{\theta^*_{sc}}$ conjugating the pair determined by this $\mathbb{R}$-splitting to a fundamental pair in $(G^*_{sc})^{\theta^*_{sc}}$ of Whittaker type; such a pair exists since $(G^*_{sc})^{\theta^*_{sc}}$ is quasi-split. This pair determines uniquely a pair (B, T) for G^* preserved by θ^*. Then (B, T) is fundamental because T can have no real roots; see the proof of Lemma 3.4.1 below. An examination of root vectors shows further that (B, T) is of Whittaker type.

(ii) Now attach to any fundamental (B, T) of Whittaker type a fundamental pair in $(G^*_{sc})^{\theta^*_{sc}}$ also of Whittaker type, and define h in $(G^*_{sc})^{\theta^*_{sc}}$ as in (i). Extend (B, T) to a fundamental splitting $spl_f = (B, T, \{X_\alpha\})$ for G^*. There is $t \in T^*_{sc}$ such that th transports spl^* to spl_f. Then

$$\theta_f = Int(th) \circ \theta^* \circ Int(th)^{-1} = Int(t\theta^*_{sc}(t)^{-1}) \circ \theta^* \qquad (2.2)$$

preserves spl_f and coincides with θ^* on T. A calculation on root vectors shows that $\sigma(\theta_f) = \theta_f$. For this, note that the Whittaker property of (B, T) implies that $\sigma X_\alpha = X_{-\alpha}$, for each imaginary root vector X_α in spl_f. Thus θ_f is defined over $\mathbb{R}$. Then $Int(t\theta^*_{sc}(t)^{-1})$ lies in $T_{ad}(\mathbb{R})$. Since $T_{sc}(\mathbb{R}) \to T_{ad}(\mathbb{R})$ is surjective, we may take $t\theta^*_{sc}(t)^{-1}$ in $V(\mathbb{R}) = T_{sc}(\mathbb{R}) \cap V$, where $V = [1 - \theta^*_{sc}](T_{sc})$. Now we claim that for fundamental T, the kernel of $H^1(\Gamma, (T_{sc})^{\theta^*_{sc}}) \to H^1(\Gamma, T_{sc})$ is trivial. From the Tate-Nakayama isomorphisms it is enough to show the kernel of $H^{-1}(\Gamma, [X_*(T_{sc})]^{\theta^*_{sc}}) \to H^{-1}(\Gamma, X_*(T_{sc}))$ is trivial. This is immediate since both $-\sigma_T$ and θ^*_{sc} preserve a base for the coroot lattice $X_*(T_{sc})$. Triviality of the kernel implies that $V(\mathbb{R})$ is connected. Thus we may assume $t \in T_{sc}(\mathbb{R})$. Then $\theta^* = Int(t^{-1}) \circ \theta_f \circ Int(t)$ preserves the splitting $Int(t^{-1})(spl_f)$ which is fundamental and of Whittaker type. ■

2.4 An application

We continue with an inner form (G, θ, η) of the quasi-split pair (G^*, θ^*). Following [KS99, Chapter 3] we say an element δ of $G(\mathbb{R})$ is θ-semisimple if $Int(\delta) \circ \theta$ preserves a pair (B, T). We will say that the θ-semisimple element δ of $G(\mathbb{R})$ is *θ-fundamental* if $Int(\delta) \circ \theta$ preserves a fundamental pair (B, T).

Recall that G is *cuspidal* if and only if a fundamental maximal torus T is *elliptic*, *i.e.,* T is anisotropic modulo the center Z_G of G. In a setting where G is assumed cuspidal we will use the term *θ-elliptic* interchangeably with *θ-fundamental*. For strongly θ-regular θ-semisimple elements there is another definition of θ-ellipticity (which does not require G to be cuspidal) in [KS99, Introduction]. We observe that a strongly θ-regular θ-semisimple element δ of cuspidal $G(\mathbb{R})$ is θ-elliptic in our present sense if and only if it is θ-elliptic in the sense of [KS99]; see Lemma 3.8(i). *In the general setting we will use exclusively the term θ-fundamental.* The strongly θ-regular θ-semisimple elements of $G(\mathbb{R})$ that are θ-elliptic in the sense of [KS99] are θ-fundamental; this is another consequence of the observation about real roots in the proof of Lemma 3.8(i).

Following Lemma 2.4 we choose a fundamental splitting spl_{Wh} of G^* of Whittaker type preserved by θ^*.

Lemma 2.5 *(i) There exists a θ-fundamental element in $G(\mathbb{R})$ if and only if there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G.*

(ii) If such (θ_f, η_f) exists and θ_f preserves the fundamental splitting spl_G then we may further assume η_f transports spl_G to spl_{Wh} and θ_f to θ^*.

Proof. Assume that there exists a θ-fundamental element in $G(\mathbb{R})$. Then we may multiply θ by an element of $Int(G(\mathbb{R}))$ to obtain an $\mathbb{R}$-automorphism θ' preserving a fundamental pair. Now apply Lemma 2.2.2 to extend this pair to a fundamental splitting spl_G. Since θ' carries spl_G to another fundamental splitting, the lemma also shows that a further multiplication by an element of $Int(G(\mathbb{R}))$ provides an $\mathbb{R}$-automorphism θ_f which preserves spl_G. We choose $\eta_f : G \to G^*$ in the inner class of η carrying spl_G to spl_{Wh}. Then $\eta_f \circ \theta_f \circ \eta_f^{-1}$ and θ^* are automorphisms of G^* which preserve spl_{Wh} and differ by an inner automorphism. Hence they coincide. The converse assertion in (i) is immediate, and so the lemma is proved. ■

For θ_f as in (ii) of the lemma, write $\eta_f \circ \sigma(\eta_f)^{-1}$ as $Int(u_f(\sigma))$. Then, applying σ to the equation

$$\eta_f \circ \theta_f \circ \eta_f^{-1} = \theta^*, \tag{2.3}$$

we see that $Int(u_f(\sigma))$ lies in the torus $(T_{ad})^{\theta^*_{ad}}$. Since $(T_{sc})^{\theta^*_{sc}} \to (T_{ad})^{\theta^*_{ad}}$ is surjective (both are connected; see [KS99, Section 1.1]) we may now assume

$$u_f(\sigma) \in (T_{sc})^{\theta^*_{sc}}. \tag{2.4}$$

3 Norms and the fundamental case

Here we include notation and review, and show that the norm correspondence is well-behaved in the fundamental case.

3.1 Endoscopic data

We now consider as quasi-split data, a triple (G^*, θ^*, a), where (G^*, θ^*) is a quasi-split pair as above, and a is a 1-cocycle of the Weil group $W_\mathbb{R}$ of $\mathbb{C}/\mathbb{R}$ in the center of the connected Langlands dual group $G^\vee$. Then ϖ will denote the character on $G^*(\mathbb{R})$, or on the real points of an inner form of G^*, attached to a. As always, and without harm, we provide an explicit transition of data between G^* and its Langlands dual ${}^LG = G^\vee \rtimes W_\mathbb{R}$ by the choice of $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$ of G^* preserved by θ^* and dual Γ-splitting $spl^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$. The action of $W_\mathbb{R}$ on $G^\vee$ factors through $W_\mathbb{R} \to \Gamma$. Then $\theta^\vee$ is the Γ-automorphism of $G^\vee$ that preserves $spl^\vee$ and is dual to θ^* as automorphism of the dual based root data. We write ${}^L\theta_a$ for the extension

$$g \times w \to a(w).\theta^\vee(g) \times w$$

of $\theta^\vee$ to an automorphism of ${}^LG$.

We assume $\mathfrak{e}_z$ is a supplemented set of endoscopic data (SED) for (G^*, θ^*, a) and its inner forms. The SED consists of a set $\mathfrak{e} = (H, \mathcal{H}, s)$ of endoscopic data for (G^*, θ^*, a) and a z-pair (H_1, ξ_1) for $\mathfrak{e}$ in the sense of [KS99], although we avoid the additional choice a' from Section 2.1 of [KS99] by adjusting s. There is no harm in assuming $\mathfrak{e}_z$ is bounded in the sense of [Sh14, Section 2]. Recall that H_1 is what we call the endoscopic group defined by the SED; Z_1 will denote the kernel of the z-extension $H_1 \to H$. We remark that SEDs exist for (G^*, θ^*, a) precisely when there are Langlands parameters preserved by ${}^L\theta_a$; see Section 9.1.

As noted in the introduction, we will be concerned mainly with the *fundamental case* for which we make an ad hoc definition in Section 3.3, and more particularly with the *cuspidal-elliptic setting* of Section 3.4.

3.2 Norm correspondence

A norm correspondence for $G(\mathbb{R})$ and an endoscopic group $H_1(\mathbb{R})$ is defined via maps on (twisted) conjugacy classes [KS99, Chapter 3, Section 5.4]. In general, the correspondence is not uniquely determined by (θ, η) and there are examples where it is empty on all or much of the *very regular set* defined in the paragraph following (3.1) below. In preparation for the fundamental case to be introduced in Section 3.3, we review two simpler settings indicated as I, II.

(I) Assume that θ preserves a fundamental splitting or, more precisely, that (θ, η) is of the form (θ_f, η_f) from (ii) in Lemma 2.5. The equation (2.3) allows us to attach a unique norm correspondence to (θ, η). To begin, there is no need to choose the datum g_θ of [KS99, Chapter 3]; in the formulas there, set $g_\theta = 1$. To compute the cochain z_σ of Lemma 3.1.A of [KS99], write $u(\sigma)$ from (2.1) above as $u_1(\sigma).z(\sigma)$, where $u_1(\sigma) \in (T_{sc})^{\theta^*_{sc}}$ as in (2.4) and $z(\sigma)$ is central in G^*_{sc}. Thus $z_\sigma = (1 - \theta^*_{sc})\ z(\sigma)$. Then, by (2) of Lemma 3.1.A in [KS99], η determines uniquely a Γ-equivariant bijective map from the set $Cl_{\theta\text{-}ss}(G, \theta)$ of θ-twisted conjugacy classes of θ-semisimple elements in $G(\mathbb{C})$ to the corresponding set $Cl_{\theta^*\text{-}ss}(G^*, \theta^*)$ for (G^*, θ^*). This map provides the first step in defining the norm correspondence. By restriction, we obtain a Γ-equivariant bijective map from the set $Cl_{str\ \theta\text{-}reg}(G, \theta)$ of θ-twisted conjugacy classes of strongly θ-regular elements in $G(\mathbb{C})$ to the corresponding set $Cl_{str\ \theta^*\text{-}reg}(G^*, \theta^*)$ for (G^*, θ^*).

For the second step, the endoscopic datum $\mathfrak{e}$ provides a unique Γ-equivariant surjective map from the set $Cl_{ss}(H)$ of semisimple conjugacy classes in $H(\mathbb{C})$ to $Cl_{\theta^*\text{-}ss}(G^*, \theta^*)$. The inverse image of $Cl_{str\ \theta^*\text{-}reg}(G^*, \theta^*)$ is, by definition, the set $Cl_{str\ G\text{-}reg}(H)$ of strongly G-regular conjugacy classes in $H(\mathbb{C})$; see [KS99, Lemma 3.3.C].

Third, the z-extension $H_1 \to H$ provides a Γ-equivariant surjective map from $Cl_{ss}(H_1)$ to $Cl_{ss}(H)$, and then by restriction, a Γ-equivariant surjective map from $Cl_{str\ G\text{-}reg}(H_1)$ to $Cl_{str\ G\text{-}reg}(H)$.

In summary, we have established the following diagram with all arrows Γ-equivariant.

$$\begin{array}{ccccc}
Cl_{str\text{-}\theta\text{-}reg}(G,\theta) & & & & \\
 & \searrow & & & \\
 & & Cl_{str\text{-}\theta^*\text{-}reg}(G^*,\theta^*) & & \\
 & & & \nwarrow & \\
 & & \uparrow & & Cl_{str\ G\text{-}reg}(H_1) \\
 & & & \swarrow & \\
 & & Cl_{str\ G\text{-}reg}(H) & &
\end{array} \tag{3.1}$$

Turning now to real points, by the *very regular set* in $H_1(\mathbb{R}) \times G(\mathbb{R})$ we mean the set of all pairs (γ_1, δ) where $\gamma_1 \in H_1(\mathbb{R})$ is strongly G-regular and $\delta \in G(\mathbb{R})$ is strongly θ-regular. Restricting to the real points of the classes in (3.2.1) we obtain maps from stable θ-twisted conjugacy classes of strongly θ-regular elements in $G(\mathbb{R})$ to stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$, from stable conjugacy classes of strongly G-regular elements in $H(\mathbb{R})$ to stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$, and from the set of stable classes of strongly regular elements in $H_1(\mathbb{R})$ to the stable conjugacy classes of strongly regular elements in $H(\mathbb{R})$. Because $H_1 \to H$ is a z-extension, the last map is surjective and remains surjective when we replace "strongly regular" by "strongly G-regular". As in [KS99, Section 3.3], we now define a *norm correspondence on the very regular set*: $\delta \in G(\mathbb{R})$ has norm γ_1 in $H_1(\mathbb{R})$, *i.e.,* (γ_1, δ) lies in the norm correspondence, if and only if the images of the respective stable classes of γ_1, δ have the same image among the stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$.

To attach data to the norm correspondence as in [KS99, Section 4.4], consider strongly θ-regular $\delta \in G(\mathbb{R})$. Then unraveling the definition of the last paragraph shows that δ has norm γ_1 in $H_1(\mathbb{R})$ if and only if there exist a θ^*-stable pair (B, T) in G^* with T defined over $\mathbb{R}$ and elements g in G^*_{sc}, δ^* in T such that

$$\delta^* = g.\eta(\delta).\theta^*(g)^{-1} \tag{3.2}$$

and the image γ of δ^* under some admissible $T \to T_{\theta^*} \to T_H$ coincides with the image of γ_1 under $H_1 \to H$. See [KS99, Section 3.3]. This is summarized in the following diagram, where N denotes the projection $T \to T_{\theta^*}$ to coinvariants.

$$\begin{array}{ccccc}
G(\mathbb{R}) \ni \delta & \longrightarrow & \delta^* \in T & & \\
 & & \downarrow & & \gamma_1 \in H_1(\mathbb{R}) \\
 & & & & \swarrow \\
 & & N\delta^* \in T_{\theta^*}(\mathbb{R}) & \longrightarrow & \gamma \in T_H(\mathbb{R})
\end{array}$$

As in [KS99, Section 3.3], we say a maximal torus T in G^* is θ^*-*admissible* if there exists a θ^*-stable pair (B, T) in G^*. Also, T_H and its inverse image T_1 in H_1 are θ^*-*norm groups for* T if there exists an admissible $T \to T_{\theta^*} \to T_H$. Then write T_1 for the inverse image of T_H in H_1. Every maximal torus over $\mathbb{R}$ in H_1 is a θ^*-norm group for some θ^*-admissible maximal torus T in G^* [KS99, Lemma 3.3.B].

In regard to (3.2) we note the following for use in calculations.

Remark 3.1 *Suppose (B_δ, T_δ) is preserved by $Int(\delta) \circ \theta$, where δ is a strongly θ-regular element of $G(\mathbb{R})$ as above. Then T_δ is the centralizer in G of the abelian reductive subgroup $Cent_\theta(\delta, G)$ of G. We may arrange that $Int(g) \circ \eta$ carries (B_δ, T_δ) to (B, T) and $Cent_\theta(\delta, G)$ to T^{θ^*} with the restriction of $Int(g) \circ \eta$ to T_δ defined over $\mathbb{R}$.*

(II) Here we consider the effect of replacing θ in (I) by θ' of the form $Int(g_{\mathbb{R}}) \circ \theta$, where $g_{\mathbb{R}} \in G(\mathbb{R})$. The norm correspondence is no longer canonical but there is a quick and transparent definition in this case. Namely, the map $\delta \to \delta.g_{\mathbb{R}}$ carries strongly θ'-regular elements in $G(\mathbb{R})$ to strongly θ-regular elements in $G(\mathbb{R})$, providing a bijection between the stable classes of strongly θ'-regular elements and the stable classes of strongly θ-regular elements. We then extend the definition of the norm correspondence on stable classes to this case in the obvious way. This norm for θ' depends on our choice of $g_{\mathbb{R}}$ and so we use the terminology $g_{\mathbb{R}}$-norm. The dependence is that $g_{\mathbb{R}}$ may be replaced by $z_{\mathbb{R}}g_{\mathbb{R}}$, with $z_{\mathbb{R}} \in Z_G(\mathbb{R})$. Then strongly G-regular γ_1 in endoscopic group $H_1(\mathbb{R})$ is a $g_{\mathbb{R}}$-norm of δ if and only if γ_1 is a $z_{\mathbb{R}}g_{\mathbb{R}}$-norm of $\delta z_{\mathbb{R}}$. See Section 4.2 for the role of $z_{\mathbb{R}}$ in transfer statements.

3.3 Fundamental case

Let (G^*, θ^*) be a quasi-split pair. A fundamental maximal torus T_1 in H_1 is a θ^*-norm group for some θ^*-admissible maximal torus T in G^*; see Section 3.2. It will be convenient to call $(G^*, \theta^*, \mathfrak{e}_z)$ *fundamental* if we may choose T to be fundamental; see Remark 3.4 below.

In general, write $StrReg(G^*, \theta^*)$ for the set of all strongly θ^*-regular elements in $G^*(\mathbb{R})$ and $StrReg(G^*, \theta^*)_f$ for the subset of θ^*-fundamental elements as defined in Section 2.4.

Lemma 3.2 *$StrReg(G^*, \theta^*)_f$ is nonempty and a union of stable θ^*-twisted conjugacy classes.*

Proof. There is a fundamental pair (B, T) in G^* preserved by θ^*; see the proof of Lemma 3.8 below. Then $T(\mathbb{R})$ contains (many) elements in $StrReg(G^*, \theta^*)_f$. The rest is immediate from definitions. ■

Write $StrReg_{G^*}(H_1)$ for the set of all strongly G^*-regular elements in $H_1(\mathbb{R})$ and $StrReg_{G^*}(H_1)_f$ for the subset of elements γ_1 such that the maximal torus $Cent(\gamma_1, H_1)$ is fundamental in H_1. We call (γ_1, δ) in the very regular set, *i.e.*, in $StrReg_{G^*}(H_1) \times StrReg(G^*, \theta^*)$, a *related pair* if it lies in the uniquely defined norm correspondence for (G^*, θ^*), *i.e.,* if γ_1 is a norm of δ.

Lemma 3.3 *(i) $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental if and only if*

$$StrReg_{G^*}(H_1)_f \ \times \ StrReg(G^*, \theta^*)_f$$

contains a related pair.

Now assume $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental. Then (ii) each δ in $StrReg(G^*, \theta^*)_f$ has a norm γ_1 in $H_1(\mathbb{R})$ and $\gamma_1 \in StrReg_{G^*}(H_1)_f$, (iii) if $\gamma_1 \in StrReg_{G^*}(H_1)_f$ is a norm of strongly θ-regular δ in $G^*(\mathbb{R})$ then $\delta \in StrReg(G^*, \theta^*)_f$.

Proof. (i) Assume that $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental and choose an admissible $T \to T_{\theta^*} \to T_H$ with both T, T_H fundamental. This provides related pairs in $StrReg_{G^*}(H_1)_f \times StrReg(G^*, \theta^*)_f$. Conversely, a related pair in $StrReg_{G^*}(H_1)_f \times StrReg(G^*, \theta^*)_f$ provides an admissible $T \to T_{\theta^*} \to T_H$ with both T, T_H fundamental, and so $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental. To check (ii), we may replace δ with a twisted conjugate by an element of $G^*(\mathbb{R})$ and assume that $T_\delta = T$. The result then follows easily; see [KS99, Lemma 4.4.A]. For (iii), suppose (γ_1, δ) is a related pair with attached T_H fundamental. Then Remark 3.1 implies that the stable class of attached θ^*-admissible T is uniquely determined by γ_1, and (iii) follows. ■

Remark 3.4 *The argument for (iii) shows that $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental if and only if every θ^*-admissible maximal torus T in G^* with a fundamental maximal torus in H_1 as θ^*-norm group is fundamental.*

Now consider an inner form (G, θ, η), and define $StrReg(G, \theta)_f$ in the same way as $StrReg(G^*, \theta^*)_f$. In general, we modify $StrReg_{G^*}(H_1)_f$ slightly as in Section 5.4 of [KS99]. Namely, we replace $H_1(\mathbb{R})$ by a suitable coset $H_1(\mathbb{R})^\dagger$ of $H_1(\mathbb{R})$ in $H_1(\mathbb{C})$. Then we define a subset $StrReg_{G^*}(H_1)_f^\dagger$ of this coset $H_1(\mathbb{R})^\dagger$ which may be empty. For (θ, η) as in (ii) of the next lemma we take, as we may, $H_1(\mathbb{R})^\dagger = H_1(\mathbb{R})$.

Lemma 3.5 *Assume that $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental. Then the following are equivalent for an inner form (G, θ, η) of (G^*, θ^*):*

(i) there exists a θ-fundamental element in $G(\mathbb{R})$,

(ii) there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G,

(iii) there exists a related pair in $StrReg_{G^*}(H_1)_f^\dagger \times StrReg(G, \theta)_f$.

Proof. We have proved (i) $\Rightarrow$ (ii) in Lemma 2.5. For (ii) $\Rightarrow$ (iii) we may further assume that $\theta = \theta_f$ and η transports θ to θ^*. Then the assertion follows easily. (iii) $\Rightarrow$ (i) is immediate. ■

Lemma 3.6 *Assume any one of the equivalent conditions from Lemma 3.5 is satisfied. Then:*

(i) each δ in $StrReg(G, \theta)_f$ has a norm γ_1 in $H_1(\mathbb{R})$ and γ_1 lies in $StrReg_{G^*}(H_1)_f$,

(ii) if $\gamma_1 \in StrReg_{G^*}(H_1)_f$ is a norm of strongly θ-regular δ in $G(\mathbb{R})$ then δ lies in $StrReg(G, \theta)_f$.

Proof. We may assume that θ preserves fundamental splitting spl_G, that θ^* preserves fundamental splitting spl_{Wh} of Whittaker type, and that η transports θ to θ^*. Recall that $Int(\delta) \circ \theta$ preserves the fundamental pair (B_δ, T_δ). Extend the pair to a fundamental splitting spl_δ. Then there is t_δ in $(T_\delta)_{sc}(\mathbb{R})$ such that $Int(t_\delta \delta) \circ \theta$ preserves spl_δ. Here, as usual, we have used the same notation t_δ for the image of t_δ in $(T_\delta)(\mathbb{R})$ under $G_{sc} \to G$. We now choose g in G_{sc} such that $Int(g)$ carries spl_δ to spl_G. Let T_G be the elliptic maximal torus specified by spl_G. Then $g_\sigma = g\sigma(g)^{-1}$ lies in $(T_G)_{sc}$, $t_G = gt_\delta^{-1}g^{-1}$ lies in $(T_G)_{sc}(\mathbb{R})$ and $\delta_G = g\delta\theta(g)^{-1}$ is of the form zt_G, where z is central. Also

$$\sigma(z)^{-1}z = \sigma(\delta_G)^{-1}\delta_G = (1-\theta)g_\sigma, \tag{3.3}$$

so that $N_\theta(z)$ lies in $(T_G)_\theta(\mathbb{R})$. Now apply the twist η which carries spl_G to spl_{Wh}. Then (i), (ii) follow; see Lemma 4.4.A of [KS99]. ■

Example 3.7 *For general (G^*, θ^*), consider a basic SED $\mathfrak{e}_z$, i.e., assume that $s = 1$. Then an argument along the same lines as that for Lemma 3.3 shows that $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental.*

3.4 Cuspidal-elliptic setting

By the cuspidal-elliptic setting we mean that G^*, or equivalently an inner form of G^*, is cuspidal and that the endoscopic datum $\mathfrak{e}$ is *elliptic* in the sense that the identity component of the Γ-invariants in the center of $H^\vee$ lies in the center of $G^\vee$ [KS99]. We then call H_1 an *elliptic endoscopic group.*

Lemma 3.8 *(i) Assume G^* is cuspidal. Then $(G^*)^{\theta^*}$ is cuspidal and there exists an elliptic θ^*-admissible maximal torus T in G^*.*

(ii) Assume also that $\mathfrak{e}$ is elliptic. Then H_1 is cuspidal and each elliptic T_1 in H_1 is a θ^*-norm group for each elliptic θ^*-admissible T in G^*.

Proof. There is no harm, for both (i) and (ii), in assuming that G^* is semisimple and simply-connected, so that $I = (G^*)^{\theta^*}$ is connected (as well as reductive) as algebraic group. Consider a pair (B^1, T^1), where T^1 is a fundamental maximal torus defined over $\mathbb{R}$ in I and B^1 is any Borel subgroup of I containing T^1. Set $T = Cent(T^1, G^*)$ and $B = Norm(B^1, G^*)$, so that (B, T) is a θ^*-stable pair for G^*. Then T must be fundamental, for otherwise T would have a real root and then a multiple of the restriction of this root to $T^1 = T^{\theta^*}$ would provide us with a real root for T^1 in I; no such root exists since T^1 is fundamental. (i) then follows. For (ii), let T_H be a fundamental maximal torus in H. Then there is some admissible isomorphism $T_H \to T_{\theta^*}$ associated to a θ^*-admissible T. Attach to H the standard endoscopic group J for I as in Section 4.2 of [KS99]. Then T^1 is (isomorphic to) a fundamental maximal torus in J, and moreover J is elliptic because H is. Thus, by (ii) in the case of standard endoscopy, T^1 is anisotropic modulo Z_I. Since T is then anisotropic modulo Z_{G^*} as in (i), T_H is anisotropic modulo Z_H, and (ii) follows. ■

Corollary 3.9 $(G^*, \theta^*, \mathfrak{e}_z)$ *is fundamental in the sense of Section 3.3.*

Consider an inner form (G, θ, η). We write $sr\text{-}ell(G, \theta)$ for the set of all θ-elliptic strongly θ-regular elements in $G(\mathbb{R})$ and $sGr\text{-}ell(H_1)^\dagger$ for the set of all strongly G-regular elliptic elements in $H_1(\mathbb{R})^\dagger$.

Corollary 3.10 *The following are equivalent:*

(i) there exists a θ-elliptic element in $G(\mathbb{R})$,

(ii) there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G,

(iii) there exists a related pair in $sGr\text{-}ell(H_1)^\dagger \times sr\text{-}ell(G, \theta)$.

Proof. By Lemma 3.8, this is a special case of Lemma 3.5. ■

Corollary 3.11 *Assume any one of the conditions of Corollary 3.10 is satisfied. Then:*

(i) each δ in $sr\text{-}ell(G, \theta)$ has a norm γ_1 in $H_1(\mathbb{R})$ and $\gamma_1 \in sGr\text{-}ell(H_1)$,

(ii) if $\gamma_1 \in sGr\text{-}ell(H_1)$ is a norm of strongly θ-regular δ in $G(\mathbb{R})$ then $\delta \in sr\text{-}ell(G, \theta)$.

Proof. By Lemma 3.8, this is a special case of Lemma 3.3. ■

3.5 Consequences for geometric transfer factors

We conclude by summarizing some of the results of Sections 3.3, 3.4 in terms of the transfer factor Δ of [KS99] (see also [KS12, Sh14]) for the matching of orbital integrals, *i.e.,* for geometric twisted transfer [Sh12]. The factor Δ is defined on the very regular set of Section 3.2. By construction, $\Delta(\gamma_1, \delta) \neq 0$ if and only if (γ_1, δ) is a related pair, *i.e.*, γ_1 is a norm of δ. We consider (i) transfer for quasi-split data (G^*, θ^*) with SED $\mathfrak{e}_z$ and (ii) transfer for an inner form of the quasi-split data in (i) when $\mathfrak{e}_z$ is fundamental. Then Lemmas 3.3, 3.6 imply that:

Theorem 3.12 *(i) There exists fundamental γ_1 and θ^*-fundamental δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental.*

(ii) Assume $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental and that (G, θ, η) is an inner form. Then there exist fundamental γ_1 and θ-fundamental δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if there exists a θ-fundamental element in $G(\mathbb{R})$.

From this and Lemma 3.4.1 we conclude:

Corollary 3.13 *In the cuspidal-elliptic setting:*

(i) there exist elliptic γ_1 and θ^*-elliptic δ such that $\Delta(\gamma_1, \delta) \neq 0$ and

(ii) for an inner form (G, θ, η), there exist elliptic γ_1 and θ-elliptic δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if there exists a θ-elliptic element in $G(\mathbb{R})$.

We will return to the results of Sections 3.3 and 3.4 in [ShII].

4 Formulating spectral factors

We turn now to some remarks on transfer statements in the setting from Lemma 2.5. We have checked that this setting captures all nontrivial geometric transfer on the fundamental very regular set. There is an analogous statement for the spectral side which we will introduce now but make precise and verify later; see Part 9. We will limit our discussion in the present section to the cuspidal-elliptic setting, as the general fundamental case follows quickly.

4.1 Transfer statements

For the main case I, we consider an inner form (G, θ, η) of (G^*, θ^*) for which (i) the transport of spl_{Wh} to G by η is fundamental and (ii) θ is the transport of θ^* to G by η. We have assumed for convenience that G is cuspidal and the endoscopic datum $\mathfrak{e}$ is elliptic. Also for convenience, we will discuss transfer for the tempered rather than the essentially tempered spectrum.

First recall geometric transfer. Test functions are Harish-Chandra Schwartz functions; we consider functions $f \in \mathcal{C}(G(\mathbb{R}), \theta)$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ [Sh12, Section 1]. We may also use $C_c^\infty(G(\mathbb{R}), \theta)$ and $C_c^\infty(H_1(\mathbb{R}), \varpi_1)$ by Bouaziz's Theorem (see [Sh12, Section 2]), as we will need in (8.2) for the generally non-tempered transfer of Adams-Johnson. Measures and integrals will be defined and normalized as in [Sh12]. To be more careful, we should use test measures in place of test functions throughout, in order to have the transfer depend only on the normalization of transfer factors. However, this will be ignored here; see instead the note [Sh].

Theorem 2.1 of [Sh12] shows that for all $f \in \mathcal{C}(G(\mathbb{R}), \theta)$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ such that

$$SO(\gamma_1, f_1) = \sum_{\delta,\ \theta\text{-}conj} \Delta(\gamma_1, \delta)\ O^{\theta,\varpi}(\delta, f) \tag{4.1}$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. Here $O^{\theta,\varpi}$ denotes a (θ, ϖ)-twisted orbital integral and SO denotes a standard (untwisted) stable orbital integral. We write $f_1 \in Trans_{\theta,\varpi}(f)$.

Suppose π_1 is a tempered irreducible admissible representation of $H_1(\mathbb{R})$ and Π_1 is its packet. We will assume, usually without further mention, that $\pi_1(Z_1(\mathbb{R}))$ acts by the character ϖ_1; recall Z_1 is the central torus $Ker(H_1 \to H)$. Let $St\text{-}Tr\ \pi_1$ be the stable tempered distribution

$$f_1 \to \sum_{\pi_1' \in \Pi_1} Trace \pi_1'(f_1).$$

Because $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ we have taken $\pi_1(f_1)$ as the operator

$$\int_{H_1(\mathbb{R})/Z_1(\mathbb{R})} f_1(h_1)\pi_1(h_1)\frac{dh_1}{dz_1}.$$

Following the case of standard endoscopic transfer we may consider the linear form $f \to St\text{-}Tr\ \pi_1(f_1)$ on $\mathcal{C}(G(\mathbb{R}), \theta)$, where f_1 is attached to f as in (4.1). For the present discussion we restrict the form to $C_c^\infty(G(\mathbb{R}), \theta)$. It is well-defined by Lemma 5.3 of [Sh79a], and results of Waldspurger [Wa14] (see also [Me13]) show that it is a linear combination of twisted traces of representations of $G(\mathbb{R})$. Our purpose is different. We want to describe certain coefficients closely related to the geometric factors and then later establish that they are correct for such a spectral transfer. Our interest in the spectral transfer statement (4.3) below is in certain constraints it places on our factors. With these constraints in mind we will verify various lemmas before making our definitions. For example, Lemma 9.2 will be the spectral analogue of Corollary 3.10, namely that our present assumption on (G, θ, η) captures all nonempty twistpackets of discrete series representations for inner forms of (G^*, θ^*). Other results require more effort and for these we will introduce further tools.

Let π be a tempered irreducible admissible representation of $G(\mathbb{R})$ and Π denote its packet. We use the same notation for a representation and its isomorphism class; we may also work with unitary representations and unitary isomorphisms. For a related pair of Langlands parameters (see Part 9) we consider the corresponding packets Π_1 for $H_1(\mathbb{R})$ and Π for $G(\mathbb{R})$. The construction of endoscopic data ensures that the packet Π is preserved under the map

$$\pi \to \varpi^{-1} \otimes (\pi \circ \theta).$$

This last property is a simple condition on the Langlands parameter of Π; whenever it is satisfied we call the attached packet (θ, ϖ)-*stable*. Thus we may define a twisted trace on $\oplus_{\pi' \in \Pi}\ \pi'$. Only those π' fixed by the map will contribute nontrivially. We then define $\Pi^{\theta,\varpi}$ to be the subset of Π consisting of such π' and call $\Pi^{\theta,\varpi}$ a *twistpacket* for (θ, ϖ).

Suppose π belongs to the twistpacket $\Pi^{\theta,\varpi}$ and that the unitary operator $\pi(\theta, \varpi)$ on the space of π interwines $\pi \circ \theta$ and $\varpi \otimes \pi$ or, more precisely, that

$$\pi(\theta(g)) \circ \pi(\theta, \varpi) = \varpi(g).(\pi(\theta, \varpi) \circ \pi(g)), \tag{4.2}$$

for $g \in G(\mathbb{R})$. Then by the twisted trace of π we mean the linear form

$$f \to Trace\ \pi(f) \circ \pi(\theta, \varpi).$$

Note that we have not fixed a normalization of the operator $\pi(\theta, \varpi)$. Also, if f is replaced by $g \to f(xg\theta(x)^{-1})$ then $Trace\ \pi(f)\ \pi(\theta, \varpi)$ is multiplied by $\varpi(x)$, for $x \in G(\mathbb{R})$.

Spectral transfer factors will be nonzero complex coefficients $\Delta(\pi_1, \pi)$ such that

$$St\text{-}Trace\ \pi_1(f_1) = \textstyle\sum_{\pi \in \Pi^{\theta,\varpi}} \Delta(\pi_1, \pi)\ Trace\ \pi(f)\ \pi(\theta, \varpi). \tag{4.3}$$

The factors $\Delta(\pi_1, \pi)$ depend on how we normalize the geometric factors $\Delta(\gamma_1, \delta)$ that prescribe the correspondence (f, f_1). Following the method for standard transfer we will introduce a geometric-spectral compatibility factor. For standard transfer this factor was canonical. In the twisted case there is a new dependence: the choice of normalization for the operators $\pi(\theta, \varpi)$, $\pi \in \Pi^{\theta,\varpi}$. We may multiply $\pi(\theta, \varpi)$ by a nonzero complex number λ (of absolute value one since we have required unitarity). In standard endoscopy the term Δ_{II} in $\Delta(\pi_1, \pi)$ comes from the explicit local representation of $f \to Trace\ \pi(f)$ around the identity. In the twisted case, we consider a similar twisted term for $f \to Trace\ \pi(f)\pi(\theta, \varpi)$ around a certain point, in general not the identity element. We will then see that multiplying $\pi(\theta, \varpi)$ by λ has the effect of dividing Δ_{II} by λ. No other term in $\Delta(\pi_1, \pi)$ will depend on $\pi(\theta, \varpi)$ and so

$$\Delta(\pi_1, \pi)\ Trace\ \pi(f)\ \pi(\theta, \varpi)$$

will be independent of the choice for $\pi(\theta, \varpi)$. Then the (geometric-spectral) compatibility factor $\Delta(\pi_1, \pi; \gamma_1, \delta)$ will depend on $\pi(\theta, \varpi)$ but the quotient

$$\Delta(\pi_1, \pi) \big/ \Delta(\pi_1, \pi; \gamma_1, \delta)$$

will not. We conclude that we may define geometric-spectral compatibility as in the standard case [Sh10, Section 12].

4.2 Additional twist by an element of $G(\mathbb{R})$

We now consider the setting II where we twist an automorphism θ as in I by an element $g_{\mathbb{R}}$ of $G(\mathbb{R})$. This yields no new twistpackets but it will be useful to have a precise formulation for transfer with the twisted automorphism.

Denote by $\Delta_{g_\mathbb{R}}$ the geometric transfer factors defined using $g_\mathbb{R}$-norms. Suppose we replace $g_\mathbb{R}$ by $z_\mathbb{R} g_\mathbb{R}$, where $z_\mathbb{R}$ lies in the center of $G(\mathbb{R})$. Then the relative factors

$$\Delta_{g_\mathbb{R}}(\gamma_1, \delta; \gamma_1', \delta')$$

and

$$\Delta_{z_\mathbb{R} g_\mathbb{R}}(\gamma_1, \delta z_\mathbb{R}; \gamma_1', \delta' z_\mathbb{R})$$

coincide. Indeed we see quickly from the definitions that the only difference between the two is that the element $z_\mathbb{R}$ is inserted in the element D constructed for Δ_{III} (see p. 33 of [KS99]) where it clearly has no effect. This property of the relative factors allows us to normalize absolute factors so that

$$\Delta_{z_\mathbb{R} g_\mathbb{R}}(\gamma_1, \delta z_\mathbb{R}) = \Delta_{g_\mathbb{R}}(\gamma_1, \delta)$$

for all very regular related pairs (γ_1, δ) for the $g_\mathbb{R}$-norm.

The choice of $z_\mathbb{R}$ affects the correspondence on test functions. If $f_1 \in Trans(f)$ for $g_\mathbb{R}$-norms then clearly $f_1 \in Trans(f_{z_\mathbb{R}})$ for $z_\mathbb{R} g_\mathbb{R}$-norms, where $f_{z_\mathbb{R}}$ denotes the translate of f by $(z_\mathbb{R})^{-1}$. The extended version of Lemma 5.1.C at the bottom of p.53 of [KS99] applies also to $g_\mathbb{R}$-norms since it is easily rewritten as a statement about relative factors. Thus if $z_1 \in Z_{H_1}(\mathbb{R})$ has image in $Z_H(\mathbb{R})$ equal to the image of $z_\mathbb{R}$ under N, *i.e.,* if $(z_1, z_\mathbb{R})$ belongs to the group $C(\mathbb{R})$ from (5.1) of [KS99], then there is quasicharacter ϖ_C on $C(\mathbb{R})$ such that

$$\Delta_{g_\mathbb{R}}(z_1\gamma_1, \delta z_\mathbb{R}) = \varpi_C(z_1, z_\mathbb{R})^{-1} \Delta_{g_\mathbb{R}}(\gamma_1, \delta).$$

A calculation with (4.1) now shows that

$$\varpi_C(z_1, z_\mathbb{R}).(f_1)_{z_1} \in Trans(f)$$

for $z_\mathbb{R} g_\mathbb{R}$-norms.

In Lemma 9.5 we will prove that the central characters $\varpi_{\pi_1}, \varpi_\pi$ for a related pair (π_1, π) have the property that

$$\varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1} = \varpi_C(z_1, z_\mathbb{R}) \tag{4.4}$$

for all $(z_1, z_\mathbb{R})$ in $C(\mathbb{R})$. This and (4.2) imply that if the spectral factors $\Delta_{g_\mathbb{R}}(\pi_1, \pi)$ and $\Delta_{z_\mathbb{R} g_\mathbb{R}}(\pi_1, \pi)$ are compatible with geometric $\Delta_{g_\mathbb{R}}$ and $\Delta_{z_\mathbb{R} g_\mathbb{R}}$ respectively, then

$$\Delta_{g_\mathbb{R}}(\pi_1, \pi) = \Delta_{z_\mathbb{R} g_\mathbb{R}}(\pi_1, \pi)$$

for all pairs (π_1, π) as in Section 4.1. Here if $\pi(\theta, \varpi)$ is used in the definition on the left then $\varpi_\pi(z_\mathbb{R}).\pi(\theta, \varpi)$ is to be used on the right. Our conclusion is then that the spectral factors will be independent of the choice for $g_\mathbb{R}$.

5 Packets and parameters I

Next we review briefly Langlands parameters and Arthur parameters for real groups [La89, Ar89]. We make a construction in Section 5.2 that attaches a *c-Levi group* to a parameter. We will show in subsequent sections how this group provides useful additional information about the parameters we are concerned with. Twisting will be ignored until Part 9.

5.1 Langlands parameters, Arthur parameters

Consider a homomorphism of the form

$$\psi = (\varphi, \rho) : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \rightarrow^{L} G,$$

where $\varphi : W_{\mathbb{R}} \rightarrow {}^{L}G$ is an essentially bounded admissible homomorphism and ρ is a continuous homomorphism of $SL(2, \mathbb{C})$ into $G^{\vee}$. The conditions on φ mean that $\varphi(w) = \varphi_0(w) \times w$, $w \in W_{\mathbb{R}}$, where φ_0 is a continuous 1-cocycle of $W_{\mathbb{R}}$ in $G^{\vee}$ and $\varphi_0(W_{\mathbb{R}})$ is a group of semisimple elements in $G^{\vee}$ that is bounded mod center in the sense that the image of $\varphi_0(W_{\mathbb{R}})$ in the adjoint form $G^{\vee}_{ad}$ under the natural projection $G^{\vee} \rightarrow G^{\vee}_{ad}$ is bounded.

An element g of $G^{\vee}$ acts on the set of such φ by conjugation: $\varphi \rightarrow Int(g) \circ \varphi$. The $G^{\vee}$-orbits are the essentially bounded Langlands parameters for G^*; see [La89]. Similarly, $G^{\vee}$ acts on the set of such ψ and the orbits are the Arthur parameters for G^*; see [Ar89].

When we replace G^* by an inner twist (G, η) in our considerations, we will often limit our attention to Langlands parameters which are *relevant to* (G, η) in the usual sense that the image of a representative is contained only in parabolic subgroups of ${}^{L}G$ relevant to (G, η) [La89]. The essentially bounded Langlands parameters relevant to (G, η) parametrize the essentially tempered packets of irreducible admissible representations of $G(\mathbb{R})$ [La89].

Notation: occasionally we distinguish between a homomorphism φ, ψ and its $G^{\vee}$-orbit $\boldsymbol{\varphi}, \boldsymbol{\psi}$ respectively, but much of the time we use the symbols φ or ψ for both.

Let $\psi = (\varphi, \rho)$ be a Arthur parameter and let $S = S_{\psi}$ denote the centralizer in $G^{\vee}$ of the image of ψ. Recall that Arthur calls ψ *elliptic* if the identity component of S is central in $G^{\vee}$ and that this is equivalent to requiring that the image of ψ be contained in no proper parabolic subgroup of ${}^{L}G$ [Ar89].

For calculations with the Weil group $W_{\mathbb{R}}$ we fix an element w_{σ} of $W_{\mathbb{R}}$ such that w_{σ} maps to σ under $W_{\mathbb{R}} \rightarrow \Gamma$ and $(w_{\sigma})^2 = -1$.

5.2 c-Levi group attached to a parameter

Let $\psi = (\varphi, \rho)$ be an Arthur parameter. Set

$$M^{\vee}_{\psi} = Cent(\varphi(\mathbb{C}^{\times}), G^{\vee}).$$

Then $M^{\vee}_{\psi} = M^{\vee}$ is a connected reductive subgroup of $G^{\vee}$. Because $\varphi(\mathbb{C}^{\times})$ is a torus, $M^{\vee}$ is *Levi* in the sense that there is a parabolic subgroup of $G^{\vee}$ with $M^{\vee}$ as Levi subgroup. Notice that $\varphi(W_{\mathbb{R}})$ normalizes $M^{\vee}$. We define $\mathcal{M}$ to be the subgroup of ${}^{L}G$ generated by $M^{\vee}$ and $\varphi(W_{\mathbb{R}})$. Then $\mathcal{M}$ is a split extension of $W_{\mathbb{R}}$ by $M^{\vee}$. Notice that $\mathcal{M}$ contains S_{ψ}.

While $\mathcal{M}$ is typically not endoscopic, *i.e.,* it is not the group $\mathcal{H}$ in some set $(H, \mathcal{H}, s)$ of standard endoscopic data for G, we may extract an L-action on $M^{\vee}$ in the same way as for the endoscopic case. For this, recall the fixed splitting $spl^{\vee} = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^{\vee}}\})$ for $G^{\vee}$. There is no harm in assuming that $\varphi_0(\mathbb{C}^{\times})$ lies in

$\mathcal{T}$ and that $\varphi_0(w_\sigma)$ normalizes $\mathcal{T}$ [La89]. Then $\mathcal{T} \subset M^\vee$ and a simple root $\alpha^\vee$ for $M^\vee \cap \mathcal{B}$ is also simple for $\mathcal{B}$. We then use the same root vector $X_{\alpha^\vee}$. Write $spl_M^\vee$ for this splitting $(M^\vee \cap \mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $M^\vee$. To define an L-action on $M^\vee$ we need only to specify an automorphism σ_M of $M^\vee$ that preserves $spl_M^\vee$ and has order at most two. Since $Int\ \varphi(w_\sigma)$ preserves $M^\vee$ and has order at most two as automorphism of $\mathcal{T}$, it is clear that there is a unique such σ_M of the form $Int[m_\sigma.\varphi(w_\sigma)]$, with $m_\sigma \in M^\vee$. For the L-action itself, $W_\mathbb{R}$ acts through $W_\mathbb{R} \to \Gamma$; in particular, w_σ acts as σ_M and $\mathbb{C}^\times$ acts trivially.

Write ${}^L M_\psi = {}^L M$ for the corresponding L-group $M_\psi^\vee \rtimes W_\mathbb{R}$ and M_ψ for a group defined and quasi-split over $\mathbb{R}$ that is dual to ${}^L M_\psi$. In Section 5.4 we will describe explicitly the L-isomophisms $\xi_M : {}^L M_\psi \to \mathcal{M}$ in the critical case, that where we have the property $\bullet$ of the next section. In Section 6.2 we will define an embedding over $\mathbb{R}$ of the quasi-split group M_ψ in the quasi-split form G^* in that case (and the general case follows quickly). For now, however, the following observation will be sufficient: in the same sense and by the same arguments as for an endoscopic group (see [KS99, Lemma 3.3.B]), the group M_ψ shares various maximal tori over $\mathbb{R}$ with an inner form G of G^* and all maximal tori over $\mathbb{R}$ in M_ψ are shared with G^*.

The groups $\mathcal{M}$ and M_ψ attached to Arthur parameter $\psi = (\varphi, \rho)$ depend only on φ. We may make the same definitions for any Langlands parameter φ and then we use the notation M_φ. We call the group M_φ a *c-Levi group* of G^*. We will define c-Levi groups in an inner form via inner twists; see Section 6.2. In Section 7.4 we will see M_ψ in a more familiar setting, namely as a Levi subgroup defined over $\mathbb{R}$ of a parabolic subgroup preserved by a Cartan involution.

The group M_φ also appears indirectly in the dual version of the Knapp-Zuckerman decomposition of unitary principal series (see [Sh82, Sections 4, 5]), as we will recall briefly in Part 6. For certain Arthur parameters ψ, a family of inner forms of M_ψ is introduced in [Ko90, Section 9].

5.3 A property for Arthur parameters

Let $\psi = (\varphi, \rho)$ be an Arthur parameter. As above, we choose a representative ψ such that $\varphi_0(\mathbb{C}^\times)$ lies in $\mathcal{T}$ and $\varphi_0(w_\sigma)$ normalizes $\mathcal{T}$. Consider the property:

- there is an element of $\mathcal{M} \cap (G^\vee \times w_\sigma)$ that normalizes $\mathcal{T}$ and acts as -1 on all roots of $G^\vee$.

Notice that $\bullet$ is true if and only if both G^*, M_ψ are cuspidal and share an elliptic maximal torus T, *i.e.*, a maximal torus that is anisotropic modulo the center of G^*. Here we may replace G^* by an inner form (G, η) if we wish; we then write T_G in place of T and assume harmlessly that η maps T_G to T over $\mathbb{R}$.

5.4 L-isomorphisms for attached c-Levi group

We describe next the L-isomorphisms $\xi_M : {}^L M \to \mathcal{M}$ for the case that $\bullet$ is true. There will be no harm in working with standard χ-data and we do so as it returns us to a familiar setting. In particular, Lemma 5.4.1 below is well known; much of it is stated in [AJ87], [Ar89] without details of proof. We give a quick proof based on some explicit calculations we will need. These calculations also pinpoint dependence on the critical Lemma 3.2 in [La89].

The element described in • may be written as $n \times w_\sigma$, where $n \in G^\vee$ normalizes $\mathcal{T}$ and represents the longest element of the Weyl group of $\mathcal{T}$ in $G^\vee$. Since $n \times w_\sigma \in \mathcal{M}$, our construction of ${}^LM = M^\vee \rtimes W_\mathbb{R}$ yields $n_M \times w_\sigma$ in the group ${}^LM$ normalizing $\mathcal{T}$ and acting as -1 on the roots of $M^\vee$. Then $n_M \in M^\vee$ normalizes $\mathcal{T}$ and represents the longest element of the Weyl group of $\mathcal{T}$ in $M^\vee$. Because $M^\vee$ is Levi, we may multiply n by an element of $\mathcal{T} \cap (M^\vee)_{der} \subseteq \mathcal{T} \cap (G^\vee)_{der}$ to obtain n' such that the action of $n' \times w_\sigma \in {}^LG$ on the entire group $M^\vee$ coincides with that of $n_M \times w_\sigma \in {}^LM$. Notice that

$$n'\sigma(n') = n\sigma(n), \tag{5.1}$$

where σ denotes the action of $1 \times w_\sigma \in {}^LG$ on $G^\vee$, and that the action of $1 \times w_\sigma \in {}^LM$ on $M^\vee$ is given by conjugation by

$$(n_M)^{-1}n' \times w_\sigma \in \mathcal{M}. \tag{5.2}$$

Returning to the construction of an L-isomorphism $\xi_M : {}^LM \to \mathcal{M}$, we require ξ_M to act as the identity on $M^\vee$. It remains to define ξ_M on $W_\mathbb{R}$. There is no harm in assuming that the element n above, and thus also n', belongs to $G^\vee_{der}$, and that n_M belongs to $M^\vee_{der}$. Then set

$$\xi_M(w_\sigma) = (n_M)^{-1}n' \times w_\sigma,$$

Let ι denote one-half the sum of the coroots for $\mathcal{T}$ in $\mathcal{B}$, and let ι_M be the corresponding term for the coroots in $M^\vee \cap \mathcal{B}$. Notice that because $M^\vee$ is Levi, we have

$$\langle \iota - \iota_M, \alpha^\vee \rangle = 0 \tag{5.3}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $M^\vee$. This, together with •, implies that σ_M acts on $\iota - \iota_M$ as -1. For $z \in \mathbb{C}^\times$, define the element $\xi_M(z)$ of $\mathcal{T} \times z$ by

$$\xi_M(z) = (z/\overline{z})^{\iota - \iota_M} \times z.$$

Lemma 5.1 *(i) The map ξ_M extends to a well-defined homomorphism $\xi_M : W_\mathbb{R} \to \mathcal{M}$ and thence to an L-isomophism $\xi_M : {}^LM \to \mathcal{M}$.*

(ii) An L-isomorphism $\xi'_M : {}^LM \to \mathcal{M}$ extending the identity on $M^\vee$ is of the form $\xi'_M = a \otimes \xi_M$, where a is a 1-cocycle of $W_\mathbb{R}$ in the center $Z_{M^\vee}$ of $M^\vee$.

Proof. Following (5.1) and (5.2) in our construction above, we see that

$$[(n_M)^{-1}n' \times w_\sigma]^2$$

may be rewritten as

$$(n_M\sigma_M(n_M))^{-1}.n\sigma(n) \times (-1).$$

By [La89, Lemma 3.2], this is

$$(-1)^{-2\iota_M}(-1)^{2\iota} \times (-1),$$

and (i) then follows. Also, (ii) is immediate. ■

5.5 u-regular Arthur parameters

We continue with an Arthur parameter $\psi = (\varphi, \rho)$. Notice that the image of ρ lies in $M^\vee$. From now on we will limit our attention to *u-regular* Arthur parameters. By this we mean a parameter ψ for which the image of ρ contains a regular unipotent element of $M^\vee$. Then ρ maps regular unipotent elements of $SL(2, \mathbb{C})$ to regular unipotent elements of $M^\vee$. We include the case that $M^\vee$ is abelian. Then ρ is trivial so that $\psi = (\varphi, triv)$, where φ is a Langlands parameter that is regular in the sense of [Sh10, Section 2]. Representations in the attached essentially tempered packet have regular infinitesimal character. On the other hand, if $M^\vee$ is nonabelian then there are *u-regular* Arthur parameters where representations in the attached Arthur packet have singular infinitesimal character; see Lemma 7.4.

Observe that for a u-regular Arthur parameter ψ, the centralizer S_ψ of the image of ψ in $G^\vee$ consists exactly of the σ_M-invariants in $Z_{M^\vee}$.

Lemma 5.2 *A u-regular Arthur parameter ψ is elliptic if and only if $\bullet$ is true.*

Proof. There is no harm in assuming G is simply-connected and semisimple, so that $Z_{G^\vee}$ is trivial. Then a nontrivial torus in the σ_M-invariants of $Z_{M^\vee}$ determines a nontrivial $\mathbb{R}$-split torus in a fundamental maximal torus of M, and conversely. ■

In the next lemma we assume ψ is elliptic since we have yet to describe ξ_M in general (see [ShII]). By construction, we have $\varphi(W_\mathbb{R})$ contained in $\mathcal{M}$, and so we may factor φ through $\xi_M : {}^LM \to \mathcal{M}$. Define the Langlands parameter φ_M by $\varphi = \xi_M \circ \varphi_M$. Set ${}^LZ_M = Z_{M^\vee} \rtimes W_\mathbb{R} \subseteq {}^LM$.

Lemma 5.3 *The Langlands parameter φ_M factors through ${}^LZ_M$.*

Proof. By (5.3), $\varphi_M(\mathbb{C}^\times)$ lies in ${}^LZ_M$. Because $1 \times w_\sigma \in {}^LM$ preserves the splitting $spl_M^\vee$ of $M^\vee$ we may adjust ψ by an element of $M^\vee$ to arrange also that $\rho(SL(2, \mathbb{C}))$ contains a regular unipotent element of $M^\vee$ that is fixed by $1 \times w_\sigma$. Writing $\varphi_M(w_\sigma)$ as $m(w_\sigma) \times w_\sigma$, we have then that the semisimple element $m(w_\sigma)$ commutes with a regular unipotent element and hence is central in $M^\vee$. ■

Remark 5.4 *By (ii) of Lemma 5.1 we may replace ξ_M by φ itself. Then φ factors through the trivial parameter $w \to 1 \times w$; this factoring is used for the parameters in [Ko90, Section 9].*

5.6 Langlands parameters for discrete series

Discrete series parameters are defined in [La89, Section 3]. They are precisely the Langlands parameters $\boldsymbol{\varphi}$ that are elliptic as Arthur parameters, *i.e.,* such that $\psi = (\varphi, triv)$ is elliptic, and then M_φ is just the elliptic torus T of Section 5.3.

We recall that there is a representative φ for $\boldsymbol{\varphi}$ such that

$$\varphi_0(z) = z^{\mu}\overline{z}^{\sigma_T \mu},\ z \in \mathbb{C}^{\times}, \tag{5.4}$$

and

$$\varphi_0(w_\sigma) = e^{2\pi i \lambda} n. \tag{5.5}$$

Here n is the element of the derived group of $G^\vee$ constructed from $spl^\vee$ to represent the longest element of the Weyl group of $\mathcal{T}$ (see [LS87, Section 2.6]). Then $\varphi(w_\sigma)$ acts on $\mathcal{T}$ as σ_T. Also $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$ and

$$\frac{1}{2}(\mu - \sigma_T \mu) - \iota \equiv \lambda + \sigma_T \lambda \mod X_*(\mathcal{T}), \tag{5.6}$$

where ι is one-half the sum of the coroots for $\mathcal{T}$ in the Borel subgroup $\mathcal{B}$ provided by $spl^\vee$. Notice that the congruence implies that

$$\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$. We require μ to be strictly dominant for $spl^\vee$. Thus μ is determined uniquely by $\boldsymbol{\varphi}$, while λ is determined uniquely modulo $\mathcal{K}$, where

$$\mathcal{K} = X_*(\mathcal{T}) + \{\nu \in X_*(\mathcal{T}) \otimes \mathbb{C} : \sigma_T \nu = -\nu\}.$$

Finally, the representative $\varphi(\mu, \lambda)$ for $\boldsymbol{\varphi}$ is determined uniquely up to the action of $\mathcal{T}$.

5.7 Langlands parameters for limits of discrete series

As in [Sh82], we may use (5.4) and (5.5) above to construct a Langlands parameter $\boldsymbol{\varphi}$ with representative $\varphi = \varphi(\mu, \lambda)$, where (μ, λ) satisfy (5.6) but the strict dominance condition on μ is relaxed to dominance.

Notice that if $\mu_0 \in X_*(\mathcal{T})$ is dominant then we obtain another such parameter $\varphi(\mu + \mu_0, \lambda + \frac{1}{2}\mu_0)$. This will provide a translation by the rational character μ_0 in the character data of Parts 6 and 7.

Also, because $\langle \mu + \sigma_T \mu, \alpha^\vee \rangle = 0$ for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$, the image of $\varphi_0 = \varphi_0(\mu, \lambda)$ is bounded mod center. Thus $\boldsymbol{\varphi}$ is essentially bounded.

Remark 5.5 *We will now use the term* ***s-elliptic*** *for any Langlands parameter $\boldsymbol{\varphi}$ with representative of the form $\varphi = \varphi(\mu, \lambda)$. By construction, • is true, so that the attached c-Levi group M_φ is cuspidal and shares an elliptic maximal torus T with G.*

Remark 5.6 *Langlands' definition of the packet attached to relevant $\boldsymbol{\varphi}$ involves components of principal series representations [La89]. Theorem 4.3.2 of [Sh82] shows that $\varphi(\mu, \lambda)$ may be used directly to identify these components as limits of discrete series characters, nondegenerate or not.*

6 Packets of limits of discrete series

We pause for a more detailed analysis of the packet of tempered irreducible representations attached to an s-elliptic parameter $\boldsymbol{\varphi}$ with representative $\varphi = \varphi(\mu, \lambda)$. Namely, we expand on Remark 5.6 using the attached c-Levi group M_φ and the strong generic base-point property from [Sh08b, Section 11] which is based on Vogan's classification of generic representations.

6.1 Character data, Whittaker data

Let $\mathcal{C}$ denote the closed Weyl chamber in $X_*(\mathcal{T}) \otimes \mathbb{C}$ dominant for $spl^\vee$. Recall that $\mu \in \mathcal{C}$. We choose an inner automorphism of G^* carrying $spl^* = (spl^\vee)^\vee$ to a fundamental splitting $spl_{Wh} = (B, T, \{X_\alpha\})$ for G^* of Whittaker type. We thus have a transport to T of $(\mu, \lambda, \mathcal{C})$. Then $(\mu, \lambda, \mathcal{C})$ becomes character data for a generic discrete series or limit of discrete series representation π^* of $G^*(\mathbb{R})$.

To fix Whittaker data for G^*, by which we mean a $G^*(\mathbb{R})$-conjugacy class of the pairs (B, λ) of [KS99, Section 5.3], we choose an additive character $\psi_\mathbb{R}$ for $\mathbb{R}$ and use the conjugacy class of the pair determined by $\psi_\mathbb{R}$ and spl^*. We may adjust the fundamental splitting spl_{Wh} to arrange that π^* is generic for the chosen Whittaker data. We then say that the splitting is *aligned* with the data. This determines spl_{Wh} uniquely up to $G^*(\mathbb{R})$-conjugacy.

Let (G, η) be an inner twist and let spl_f be a fundamental splitting for G. We use a twist η' in the inner class of η to transport spl_f to spl_{Wh}. This provides us with a further transport of $(\mu, \lambda, \mathcal{C})$ to the maximal torus specified by spl_f. The transported triple serves as character data $(\mu_\pi, \lambda_\pi, \mathcal{C}_\pi)$ for a discrete series or limit of discrete series representation π of $G(\mathbb{R})$ which is determined uniquely by the $G(\mathbb{R})$-conjugacy class of spl_f. In the case of limits of discrete series *we must now allow* $\pi = 0$, *i.e.,* that the distribution attached to the character data is zero. We write $spl_f = spl_\pi = (B_\pi, T_\pi, \{X_\alpha\})$, $\eta' = \eta_\pi$ and the character data for π as $(\mu_\pi, \lambda_\pi, \mathcal{C}_\pi)$.

By Lemma 2.1 and the theorem cited in Remark 5.6, as spl_f varies we generate the packet of essentially tempered representations attached to $\varphi = \varphi(\mu, \lambda)$ and possibly some zeros. By the same theorem we obtain all zeros if and only if φ is irrelevant to G.

6.2 Characterizing nonzero limits

Our first concern will be to detect when $\pi = 0$. There is a well-known characterization, in terms of roots, for a limit of discrete series to be nonzero; see [KZ82, Theorem 1.1 (b)]. The precise statement is noted in the next proof. We want a characterization in terms of the c-Levi group M_φ.

Consider the subgroup M^* generated by T and the root vectors X_α from spl_{Wh} for which α is the transport to T of the coroot of a simple root of $\mathcal{T}$ in $M^\vee \cap \mathcal{B}$. Because spl_{Wh} is of Whittaker type, *i.e.,* the simple roots are all noncompact, the group M^* is quasi-split over $\mathbb{R}$. Moreover, we may identify the L-group of M^* with the L-group ${}^LM_\varphi$ constructed in Section 5.2. For this, we reverse the construction of Section 6.1 to determine an $\mathbb{R}$-splitting spl_M^* for M^* from the fundamental splitting $spl_{Wh,M}$ attached to spl_{Wh} and the additive character $\psi_{\mathbb{R}}$. Then spl_M^* is unique up to $M^*(\mathbb{R})$-conjugacy. Each such spl_M^* determines a unique isomorphism from ${}^L(M^*)$ to ${}^LM_\varphi$. In summary, M^* provides a concrete realization of the quasi-split group M_φ.

By definition, η_π carries spl_π to spl_{Wh}. Let $M_\pi = \eta_\pi^{-1}(M^*)$. Then $\eta_\pi : M_\pi \to M$ is an inner twist in the inner class of η that carries T_π to T^* over $\mathbb{R}$. We say that M_π is an *elliptic c-Levi group for* (G, η).

Lemma 6.1 *π is nonzero if and only if the inner twist $\eta_\pi : M_\pi \to M^*$ is an $\mathbb{R}$-isomorphism.*

Proof. The characterization cited (in sufficient generality for our setting) is that $\pi \neq 0$ if and only if all $\mathcal{C}_\pi$-simple roots α such that $\langle \mu_\pi, \alpha^\vee \rangle = 0$ are noncompact. In other words, $\pi \neq 0$ if and only if the splitting of M_π determined by spl_π is of Whittaker type. Let $\eta_\pi \sigma(\eta_\pi)^{-1} = Int(u_\pi(\sigma))$, where $u_\pi(\sigma) \in T_{sc}$. Then because spl_{Wh} is of Whittaker type and spl_π is fundamental, a calculation with root vectors shows that this is the same as requiring that $\alpha(u_\pi(\sigma)) = 1$ for all B^*-simple roots α of T in M^*, *i.e.,* that $u_\pi(\sigma)$ lies in the center of $M^*_{(sc)}$. Here $M^*_{(sc)}$ denotes the inverse image of M^* in G^*_{sc} under the natural projection $G^*_{sc} \to G^*$. The lemma then follows. ■

6.3 Generating packets

Fix an inner form (G, η) and assume s-elliptic $\varphi = \varphi(\mu, \lambda)$ is relevant to (G, η). We consider the packet Π of representations of $G(\mathbb{R})$ attached to φ.

Let $\pi \in \Pi$ and assume $\pi \neq 0$. Then:

Lemma 6.2 *(i) $spl_{\pi'}$ yields character data for nonzero π' in Π if and only if*

$$\eta_{\pi'} = Int(g_*) \circ \eta_\pi \circ Int(g),$$

where $g \in G(\mathbb{R})$ and where $g_* \in G^*_{sc}$ normalizes T and is such that the restriction of $Int(g_*)$ to M^* is defined over $\mathbb{R}$.

(ii) Further, $\pi' = \pi$ if and only if $\eta_{\pi'}$ is of the form $\eta_\pi \circ Int(g)$, where $g \in G(\mathbb{R})$.

Proof. The second assertion is just a restatement of a well-known property of limits of discrete series; see [Sh82, Section 4] or [KZ82, Theorem 1.1(c)]. The first assertion follows from Lemma 6.1. ■

6.4 An elliptic invariant

We return now to using the notation π *only for nonzero* representations. Again fix an inner form (G, η) for which s-elliptic $\varphi = \varphi(\mu, \lambda)$ is relevant and consider π in the attached packet Π of representations of $G(\mathbb{R})$. Recall $u_\eta(\sigma)$, $u_\pi(\sigma) \in G^*_{sc}$;

we have $\eta\sigma(\eta)^{-1} = Int(u_\eta(\sigma))$ and $\eta_\pi\sigma(\eta_\pi)^{-1} = Int(u_\pi(\sigma))$. Since η_π is in the inner class of η, we write $\eta_\pi = Int(x_\pi) \circ \eta$ and

$$u_\pi(\sigma) = x_\pi[u_\eta(\sigma)]\sigma(x_\pi)^{-1},$$

where $x_\pi \in G^*_{sc}$.

If we may choose $u_\eta(\sigma)$, and thence $u_\pi(\sigma)$, to be a cocycle then we will say that (G, η) is of quasi-split type because (G, η) then occurs as a component of an extended group of quasi-split type. An extended group (introduced by Kottwitz) consists of several pairs (G, η), (G', η'), ... and conditions on the twists η, η' ensure the property (6.3) below; see [Sh08b] for a review and examples. There is a quasi-split component if and only the coboundaries in (6.3) are trivial. Then we say the extended group is of quasi-split type. The quasi-split component, if it exists, is unique [Sh08b].

For pairs (G, η), (G', η') in the same extended group, relative factors for tempered spectral transfer are defined in [Sh08b] (the relative geometric factors were introduced by Kottwitz). When the extended group is of quasi-split type our already chosen Whittaker data provides a unique normalization Δ_{Wh} of the absolute transfer factors for each component (G, η); see [KS99]. The spectral factors Δ_{Wh} possess the strong base-point property [Sh08b]. In particular, we have the formula (6.4) for discrete series representations. For a general extended group, the results of Kaletha [Ka13] provide a natural normalization for the absolute factors. The setting, and in particular the definition of extended group, is modified with additional structure. For our purposes it is convenient to work with the minimal extended groups of the present setting, and we will allow any normalization of the absolute factors that possesses geometric-spectral compatibility in the sense of [Sh10, Sh08b]. The extended groups will play a more central role when we come to finer structure on packets in [ShII].

We begin our definition of elliptic invariants with the case that (G, η) is of quasi-split type. In this setting we define an absolute invariant $inv(\pi)$ in $H^1(\Gamma, T)$. Recall that T is the elliptic maximal torus in G^* specified by spl_{Wh}. First we have by Lemma 6.3.1 that $u_\pi(\sigma)$ lies in the center $Z_{M^*_{(sc)}}$ of $M^*_{(sc)}$ and so defines an element of $H^1(\Gamma, Z_{M^*_{(sc)}})$. It depends only on π, *i.e.,* only on the $G(\mathbb{R})$-conjugacy class of spl_π. Now $inv(\pi)$ is defined to be the image of this class under

$$H^1(\Gamma, Z_{M^*_{(sc)}}) \to H^1(\Gamma, Z_{M^*}) \to H^1(\Gamma, T) \tag{6.1}$$

given by the composition of the obvious map $Z_{M^*_{(sc)}} \to Z_{M^*}$ and inclusion $Z_{M^*} \to T$. From the diagram

$$\begin{array}{ccc} Z_{M^*_{(sc)}} & \longrightarrow & T_{sc} \\ \downarrow & & \downarrow \\ Z_{M^*} & \longrightarrow & T \end{array} \tag{6.2}$$

we conclude that $inv(\pi)$ lies in the image $\mathcal{E}(T)$ of $H^1(\Gamma, T_{sc})$ in $H^1(\Gamma, T)$.

We will also make use of the following.

Lemma 6.3 *Suppose that T is a fundamental maximal torus in a connected reductive group G over $\mathbb{R}$. Then $H^1(\Gamma, Z_G) \to H^1(\Gamma, T)$ is injective.*

Proof. Because T is fundamental, both $T_{sc}(\mathbb{R})$, $T_{ad}(\mathbb{R})$ are connected and hence $T_{sc}(\mathbb{R}) \to T_{ad}(\mathbb{R})$ is surjective; see the proof of Lemma 2.2 for references. A calculation then shows that the kernel of $H^1(\Gamma, Z_G) \to H^1(\Gamma, T)$ is trivial. ■

In general, we define a relative invariant $inv(\pi, \pi')$ when (G, η), (G', η') are components of the same extended group and π, π' belong to packets Π, Π' for $G(\mathbb{R}), G'(\mathbb{R})$ attached to relevant s-elliptic parameters $\varphi = \varphi(\mu, \lambda), \varphi' = \varphi(\mu', \lambda')$ respectively. We follow the method introduced in [LS87, LS90]; see also [KS99, Section 4.4]. First, recall that

$$\partial u_\pi = \partial u_\eta = \partial u_{\eta'} = \partial u_{\pi'} \tag{6.3}$$

takes values in $Z_{G^*_{sc}}$ as subgroup of T, the elliptic maximal torus in G^* specified by spl_{Wh}. As in the references, set

$$U_{sc} = U(T_{sc}, T_{sc}) = T_{sc} \times T_{sc} \diagup \{(z^{-1}, z) : z \in Z_{G^*_{sc}}\}$$

and

$$U = U(T, T) = T \times T \diagup \{(z^{-1}, z) : z \in Z_{G^*}\}.$$

Also consider

$$U(Z_{M^*_{(sc)}}) = Z_{M^*_{(sc)}} \times Z_{M^*_{(sc)}} \diagup \{(z^{-1}, z) : z \in Z_{G^*_{sc}}\}$$

and

$$U(Z_{M^*}) = Z_{M^*} \times Z_{M^*} \diagup \{(z^{-1}, z) : z \in Z_{G^*}\}.$$

Then we replace (6.1) above by

$$H^1(\Gamma, U(Z_{M^*_{(sc)}})) \to H^1(\Gamma, U(Z_{M^*})) \to H^1(\Gamma, U),$$

and use the cocycle that is the image in $U(Z_{M^*_{(sc)}})$ of the pair $(u_\pi(\sigma)^{-1}, u_{\pi'}(\sigma))$ in $Z_{M^*_{(sc)}} \times Z_{M^*_{(sc)}}$. Then we obtain $inv(\pi, \pi')$ in the image of $H^1(\Gamma, U_{sc})$ in $H^1(\Gamma, U)$. If the extended group is of quasi-split type then $inv(\pi, \pi')$ is the image of $(inv(\pi)^{-1}, inv(\pi'))$ under the evident homomorphism $H^1(\Gamma, T) \times H^1(\Gamma, T) \to H^1(\Gamma, U)$.

6.5 Application to endoscopic transfer

We consider standard endoscopic transfer in the cuspidal elliptic setting. If φ_1 is an s-elliptic Langlands parameter for H_1 then its transport φ to G is also s-elliptic; see [Sh08a, Section 11] for how to transport attached character data from the z-extension H_1. If φ_1 is elliptic then φ may, of course, fail to be elliptic, but φ is at least s-elliptic and moreover the associated triples of nonzero character data are nondegenerate. It is then straightforward to define spectral transfer factors via the Zuckerman translation principle; this is recalled in Section 14 of [Sh10].

For a general s-elliptic related pair (φ_1, φ), however, neither side of the spectral transfer statement has support on the regular elliptic set. Then we have defined the associated transfer factors via an L-group version of the Knapp-Zuckerman (nondegenerate) decomposition of unitary principal series; see [Sh10, Sh08b]. In that form, the factors display desired structure on the packet; see [Sh08b, Section 11].

Our purpose now is to note a simpler description, based on the elliptic invariant of Section 6.4, of the transfer factors for a general s-elliptic related pair. Whittaker data for G^* has been fixed. First we transport a Γ_T-invariant s_T in the complex dual $T^\vee$ of T to an element s of the maximal torus $\mathcal{T}$ in $G^\vee$ via the method of Section 6.1. To the pair (s, φ) we attach the endoscopic data $\mathfrak{e}(s)$ of Section 7 of [Sh08b], now writing $\mathfrak{e}_z(s)$ since it is already supplemented, as well as the related pair of parameters (φ^s, φ). The attached endoscopic group will be denoted $H^{(s)}$. When φ is singular we have used a different representative, say φ', to display the structure on the packet via Knapp-Zuckerman theory. The conjugacy of φ and φ' under $G^\vee$ determines a canonical isomorphism of the attached abelian groups $\mathbb{S}_\varphi$ and $\mathbb{S}_{\varphi'}$ (see Section 6.6 or 6.7). To examine the effect of this isomorphism on transfer, see [Sh08b, Section 2] for passage to isomorphic endoscopic data and [Sh08b, Section 11] for related results. For present needs, the results of Sections 6.6 - 6.8 will be sufficient.

Recall from Section 6.4 that our Whittaker data also determine absolute transfer factors Δ_{Wh} for any inner form (G, η) of quasi-split type. We use π^s to denote a representation in the packet for $H^{(s)}(\mathbb{R})$ attached to φ^s; the choice within the packet will not matter. Finally, $\langle -, - \rangle_{tn}$ will be the Tate-Nakayama pairing between $H^1(\Gamma, T)$ and $\pi_0((T^\vee)^\Gamma)$, and the image of s_T in $\pi_0((T^\vee)^\Gamma)$ will again be written s_T.

Lemma 6.4 *Suppose (G, η) is of quasi-split type and $\varphi(\mu, \lambda)$ is an s-elliptic parameter relevant to (G, η). Then*

$$\Delta_{Wh}(\pi_s, \pi) = \langle inv(\pi), s_T \rangle_{tn} \tag{6.4}$$

for each limit of discrete series representation π of $G(\mathbb{R})$ attached to $\varphi(\mu, \lambda)$.

Proof. Although not necessary, we reduce easily to the case that the derived group of G is simply-connected as this allows us to refer directly to the first half of the argument for the proof of Theorem 11.5 in [Sh08b]. There a totally degenerate parameter as in Section 6.6 was needed; now we apply the coherent continuation argument to any relevant s-elliptic φ, so obtaining the transfer identity in the middle of p. 400. The formula (6.4) then follows from its truth in the case φ is elliptic. ■

Returning to the notation of Section 6.4, recall from [LS90] that we identify $(U_{sc})^\vee$ with $\mathcal{T}_{sc} \times \mathcal{T}_{sc} / \{(z, z) : z \in Z_{G_{sc}^\vee}\}$ and define s_U as there. Now $\langle \_, \_ \rangle_{tn}$ will denote the Tate-Nakayama pairing for U. The following requires a minor variant of the last proof but it will be convenient to have a separate statement.

Lemma 6.5 *Suppose $(G,\eta),(G',\eta')$ are components of an extended group and that Δ is an absolute transfer factor for the extended group. Then*

$$\Delta(\pi_s,\pi)\diagup\Delta(\pi_s,\pi') = \langle inv(\pi,\pi'), s_U\rangle_{tn}$$

for all limits of discrete series representations π,π' of $G(\mathbb{R}),G'(\mathbb{R})$.

Remark 6.6 *We use transfer factors Δ for the classic version of the Langlands correspondence for real groups. See [Sh14] for (simple) transition to the alternate factors Δ_D.*

6.6 Example: totally degenerate parameters

First, the notion of *totally degenerate* character data of Carayol and Knapp [CK07] extends to reductive groups, and since our data are generated by a Langlands parameter we consider the parameter instead. We call an s-elliptic parameter $\varphi=\varphi(\mu,\lambda)$ *totally degenerate* if $\langle\mu,\alpha^\vee\rangle=0$ for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$; see [Sh08b, Section 12].

This definition implies that a totally degenerate parameter is relevant to (G,η), *i.e.*, there is a packet for $G(\mathbb{R})$ attached to the parameter, if and only if G is quasi-split. Thus we may as well assume that $G=G^*$ and $\eta=id$.

Further, an examination of the congruences for μ,λ shows that totally degenerate parameters exist only for certain cuspidal quasi-split groups. For example, if G_{der} is simply-connected then such (μ,λ) do exist: they are the data for an extension of the rational character ι on T_{der}, regarded as character on $T_{der}(\mathbb{R})$, to a continuous quasicharacter on $T(\mathbb{R})$; see [Sh08b]. Then an elliptic endoscopic group for G also has totally degenerate parameters [Sh08b]. So also does each cuspidal standard or c-Levi group X for G because X_{der} is also simply-connected. A z-extension G_z of any cuspidal quasi-split group G has totally degenerate characters for the same reason.

Suppose now that $\varphi=\varphi(\mu,\lambda)$ is a totally degenerate parameter for $G=G^*$. The congruences for μ,λ further show that the parameter $\boldsymbol{\varphi}$ is uniquely determined by G up to multiplication by element of $H^1(W_\mathbb{R},Z_{G^\vee})$, and hence that the attached packet is uniquely determined up to twisting by a quasi-character on $G(\mathbb{R})$.

To describe the packet Π attached to totally degenerate φ in terms of the elliptic character data provided by (μ,λ) and the Whittaker data, we may proceed as follows. Recall the fixed $\mathbb{R}$-splitting $spl^*=(B^*,T^*,\{X_\alpha\})$ for G. There is another representative $\overline{\varphi}$ for $\boldsymbol{\varphi}$ attached to the maximally split maximal torus T^*. We obtain it by applying a sequence of dual Cartan transforms to $\varphi=\varphi(\mu,\lambda)$; the sequence is prescribed by a suitable set of strongly orthogonal roots and the transforms are defined as in the proof of Lemma 4.3.5 in [Sh82]. Write $\overline{\varphi}=\varphi(\mu,\overline{\lambda})$ relative to T^*. These data determine an essentially unitary minimal principal series representation for $G(\mathbb{R})$. By definition of the Langlands correspondence, Π consists of the components of this representation. By Vogan's classification of generic representations [Vo78], these components include generic π^* with attached fundamental splitting spl_{Wh}. Then Lemma 6.2 shows that we obtain the other components by applying $Int(g_*)$ to spl_{Wh}, where $g_* \in G_{sc}$ and the automorphism $Int(g_*)$ of G is defined over $\mathbb{R}$. Each such element g_* determines an element of $H^1(\Gamma, Z_{sc})$, where Z_{sc} denotes the center of G_{sc}. Conversely, each element of $H^1(\Gamma, Z_{sc})$ has trivial image in $H^1(\Gamma, G_{sc})$ [Sh08b, Lemma 12.3] and so determines g_* such that $Int(g_*)$ is defined over $\mathbb{R}$. Finally, two elements of $H^1(\Gamma, Z_{sc})$ determine the same component if and only if they differ by an element of $Ker[H^1(\Gamma, Z_{sc}) \to H^1(\Gamma, Z)]$, where Z denotes the center of G, so that we have bijections

$$\Pi \leftrightarrow G_{ad}(\mathbb{R})/Int(G(\mathbb{R})) \leftrightarrow Image[H^1(\Gamma, Z_{sc}) \to H^1(\Gamma, Z)]. \tag{6.5}$$

If we map the image of π in $H^1(\Gamma, Z)$ to $H^1(\Gamma, T)$ under the injective $H^1(\Gamma, Z) \to H^1(\Gamma, T)$ then we recover the elliptic invariant $inv(\pi)$ defined in Section 6.4.

The group $S_{\overline{\varphi}} = Cent(\overline{\varphi}(W_{\mathbb{R}}), G^\vee)$ consists of the fixed points in $G^\vee$ for the action of $\sigma \in \Gamma$ by $\overline{\sigma} = Int(\overline{\varphi}(w_\sigma))$. Thus

$$\mathbb{S}_{\overline{\varphi}} := S_{\overline{\varphi}}/[(Z_{G^\vee})^\Gamma.S^0_{\overline{\varphi}}] = (G^\vee)^{\overline{\Gamma}}/[(Z_{G^\vee})^\Gamma.((G^\vee)^{\overline{\Gamma}})^0].$$

Notice that $\mathbb{S}_{\overline{\varphi}}$ is isomorphic to Langlands' R-group $R_{\overline{\varphi}}$ for $\boldsymbol{\varphi}$ in this setting; see [Sh82, Section 5.3]. Combining this with the pairing obtained via nondegenerate Knapp-Zuckerman theory (see [Sh08b, Sh10]), we have that Π determines a perfect pairing of

$$Image[H^1(\Gamma, Z_{sc}) \to H^1(\Gamma, Z)] \tag{6.6}$$

with

$$(G^\vee)^{\overline{\Gamma}}/(Z_{G^\vee})^\Gamma.((G^\vee)^{\overline{\Gamma}})^0.$$

In particular, if G is semisimple and simply-connected then our pairing for the unique totally degenerate packet for $G(\mathbb{R})$ exhibits a perfect pairing of $H^1(\Gamma, Z)$ with $\pi_0[(G^\vee)^{\overline{\Gamma}}]$.

6.7 General limits: factoring parameters

We return to general s-elliptic $\varphi = \varphi(\mu, \lambda) : W_{\mathbb{R}} \to {}^L G$. Since the image of φ lies in $\mathcal{M}$, we factor φ through ${}^L M$, and write $\varphi = \xi_M \circ \varphi_M$, where φ_M is the s-elliptic parameter $\varphi(\mu_M, \lambda_M)$ for M^*, with

$$\mu_M = \mu - (\iota - \iota_M), \lambda_M = \lambda.$$

Clearly φ_M is totally degenerate. In summary:

Lemma 6.7 *An s-elliptic parameter φ determines a well-defined totally degenerate parameter for the c-Levi group attached to φ.*

Turning to packets, we start with the quasi-split form G^* and generic π^* whose character data is the transport of $(\mu, \lambda, \mathcal{C})$ to T provided by spl_{Wh}. Our realization of M_φ as M^* in Section 6.2 determines a fundamental splitting $spl_{Wh,M}$ and chamber $\mathcal{C}_M$ for T. We use the same notation for the inverse transport of this chamber to $M^\vee$. The transport by $spl_{Wh,M}$ of dual data $(\mu_M, \lambda_M, \mathcal{C}_M)$ attached to φ_M determines a totally degenerate limit of discrete series representation π_M^* of $M^*(\mathbb{R})$. By construction, π_M^* is generic relative to the Whittaker data attached to $\psi_\mathbb{R}$ and the $\mathbb{R}$-splitting $spl_M^* = (\overline{B}_M, \overline{T}_M, \{X_\alpha\})$ for M^* from Section 6.1.

Consider now general (G, η) for which $\varphi(\mu, \lambda)$ is relevant. Let Π be the attached packet and consider $\pi \in \Pi$. Recall that $\eta_\pi : M_\pi \to M^*$ is an $\mathbb{R}$-isomorphism. Define the representation π_M of $M_\pi(\mathbb{R})$ by transport: $\pi_M = \pi_M^* \circ \eta_\pi$. Then π_M lies in the totally degenerate packet Π_{M_π} of representations of $M_\pi(\mathbb{R})$ attached to φ_M.

We return to the elliptic invariants of (6.4) and consider the subgroup

$$Image(H^1(\Gamma, Z_{M^*_{sc}}) \to H^1(\Gamma, T))$$

of

$$Image(H^1(\Gamma, Z_{M^*_{(sc)}}) \to H^1(\Gamma, T)).$$

From (6.5) and Lemma 6.3, we have an isomorphism of this subgroup with $M_{ad}(\mathbb{R}) / Int(M(\mathbb{R}))$.

On the other hand, notice that $S_\varphi = Cent(\varphi(W_\mathbb{R}), G^\vee)$ is contained in $M^\vee$ and hence

$$S_\varphi = S_{\varphi_M} = Cent(\varphi_M(W_\mathbb{R}), M^\vee)$$

which is the group of fixed points of $M^\vee$ under either of the automorphisms $Int(\varphi(w_\sigma)), Int(\varphi_M(w_\sigma))$; we arranged in Section 5.4 that these automorphisms act the same way on $M^\vee$. Again write $\mathbb{S}_\varphi$ for the quotient $S_\varphi / (Z_{G^\vee})^\Gamma S_\varphi^0$. Then

$$\mathbb{S}_{\varphi_M} = S_\varphi / (Z_{M^\vee})^\Gamma S_\varphi^0$$

and since $(Z_{M^\vee})^\Gamma \cap ((M^\vee)^\Gamma)^0$ is contained in $(Z_{G^\vee})^\Gamma$ we have an exact sequence

$$1 \to (Z_{M^\vee})^\Gamma / (Z_{G^\vee})^\Gamma \to \mathbb{S}_\varphi \to \mathbb{S}_{\varphi_M} \to 1.$$

6.8 General limits: companion standard Levi group

We continue with the packet Π of the last section. It consists of the components of several essentially tempered principal series representations of $G(\mathbb{R})$. To describe them we return to the representative $\overline{\varphi}_M = \varphi(\mu_M, \overline{\lambda}_M)$ for $\boldsymbol{\varphi}_M$ in Section 6.6 and set $\overline{\varphi} = \xi_M \circ \overline{\varphi}_M$. Then $\overline{\varphi}$ also represents $\boldsymbol{\varphi}$.

We may replace spl_{Wh} by a $G^*(\mathbb{R})$-conjugate and then M^* by its conjugate relative to the same element to arrange that the maximal torus $\overline{T}_M$ in M^* provided by spl_M^* is a standard maximal torus in G^*. We then drop the subscript M in notation for this torus. Here by *standard* we mean that the maximal split torus $\overline{S}$ in $\overline{T}$ is contained in T^* provided by $spl^* = (spl^\vee)^\vee$. Let $\overline{M}$ be the standard Levi group $Cent(\overline{S}, G^*)$. Then ${}^L\overline{M}$ will denote the dual standard Levi group in ${}^LG$, naturally embedded by inclusion.

Lemma 6.8 *The image of $\overline{\varphi}$ lies in ${}^L\overline{M}$ and defines an elliptic parameter for $\overline{M}$.*

Proof. We return to the notation of Section 5.4. We have arranged that $\sigma_{\overline{T}} = \sigma_M$ on $\mathcal{T}$. Then the element $n_{\overline{M}} \times w_\sigma$ in ${}^LG$ coincides with $\xi_M(w_\sigma)$ up to an element of $\mathcal{T} \cap G^\vee_{der}$. It follows that $\overline{\varphi}(w_\sigma) \in {}^L\overline{M}$ and then that $\overline{\varphi}(W_\mathbb{R}) \subseteq {}^L\overline{M}$. Since $\sigma_M \alpha^\vee = \sigma_{\overline{T}}\alpha^\vee = -\alpha^\vee$ for each root $\alpha^\vee$ of $\mathcal{T}$ in $\overline{M}^\vee$, it is clear that $\overline{\varphi}$ is s-elliptic as Langlands parameter for $\overline{M}$. If we write $\overline{\varphi} = \varphi(\mu, \overline{\lambda})$ relative to $\overline{M}$, then μ is $\overline{M}$-regular for otherwise $\overline{T}$ would have an imaginary root in M^*. Thus the lemma is proved. ■

We continue with $\overline{\varphi} = \varphi(\mu, \overline{\lambda})$ and consider the quasi-split form G^*. The Whittaker data Wh for G^* determines, by restriction, Whittaker data $Wh_{\overline{M}}$ for $\overline{M}$. We choose a corresponding fundamental splitting $spl_{Wh_{\overline{M}}} = (B_{\overline{M}}, \overline{T}, \{X_\alpha\})$ of Whittaker type for $\overline{M}$, and then transport $(\mu, \overline{\lambda})$ to discrete series character data on $\overline{T}$. Via unitary parabolic induction, each discrete series representation in the packet for $\overline{M}(\mathbb{R})$ attached to $\overline{\varphi}$ determines an essentially tempered principal series representation of $G^*(\mathbb{R})$. Then Π consists of the irreducible components of all principal series representations so obtained. Consider next an inner form (G, η) for which φ is relevant. Recall that for $\pi \in \Pi$, η_π transports elliptic character data for π to that for π^*. By (6.5) and Lemma 6.2 we may choose π so that $\pi_M = \pi^*_M \circ \eta_\pi$ is isomorphic to π^*_M. We then adjust our discussion for G^* to describe the packet for $G(\mathbb{R})$; we will not need details here.

From definitions (recalled in [Sh82, Section 5.3]) it is clear that Langlands' version of the R-group is unchanged by passage from ${}^LM$ to ${}^LG$:

$$R_{\overline{\varphi}} = R_{\overline{\varphi}_M}.$$

Also there is a surjective homomorphism $\mathbb{S}_{\overline{\varphi}} \to R_{\overline{\varphi}}$ with kernel that may be identified with the dual of $\mathcal{E}(\overline{T})$ (see [Sh82, Sections 5.3, 5.4]). Because φ_M is totally degenerate we have that $\mathbb{S}_{\overline{\varphi}_M} \to R_{\overline{\varphi}_M} = R_{\overline{\varphi}}$ is an isomorphism. Then by the discussion around (6.6.3) we have a perfect pairing of $R_{\overline{\varphi}}$ with

$$Image(H^1(\Gamma, Z_{M^*_{sc}}) \to H^1(\Gamma, T)) \simeq M_{ad}(\mathbb{R}) / Int(M(\mathbb{R})).$$

7 Packets and parameters II

7.1 Data for elliptic u-regular parameters

Suppose that $\psi = (\varphi, \rho)$ is an elliptic u-regular Arthur parameter. Continuing from (5.5), we may assume that φ takes the following form:

$$\varphi(z) = z^\mu \overline{z}^{\sigma_M \mu} \times z$$

for $z \in \mathbb{C}^\times$, and$\varphi(w_\sigma) = e^{2\pi i \lambda}.\xi_M(w_\sigma)$.Here

$$\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$$

and

$$\langle \mu, \alpha^\vee \rangle = 0, \ \ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \tag{7.1}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $M^\vee$. The element μ is uniquely determined by the $\mathcal{T}$-conjugacy class of the representative φ, and λ is determined uniquely modulo

$$\mathcal{K}_M = X_*(\mathcal{T}) + \{\nu \in X_*(\mathcal{T}) \otimes \mathbb{C} : \sigma_M \nu = -\nu\}.$$

We will use the notation $\varphi = \varphi[\mu, \lambda]$. Notice that in the case $M_\psi = T$, where φ is elliptic, we return to the pair (μ, λ) from Section 5.6.

From our construction of ξ_M and the equation $\varphi(w_\sigma)^2 = \varphi(-1)$ we have immediately the following congruence:

$$\frac{1}{2}(\mu - \sigma_M \mu) - (\iota - \iota_M) \equiv \lambda + \sigma_M \lambda \mod X_*(\mathcal{T}). \tag{7.2}$$

The properties (7.1) allow us to replace σ_M by σ_T in (7.2) and then to rewrite the congruence as

$$\frac{1}{2}[\mu + \iota_M - \sigma_T(\mu + \iota_M)] - \iota \equiv \lambda + \sigma_T \lambda \mod X_*(\mathcal{T}). \tag{7.3}$$

For the second component ρ of ψ we turn to Section 5.5 and the u-regularity property. With first component φ prescribed as above we may assume that $\rho : SL(2, \mathbb{C}) \to M^\vee$ is in standard form with cocharacter $2\iota_M$. Then

$$\rho(diag(|w|^{1/2}, |w|^{-1/2})) = (z\overline{z})^{\iota_M}, w \in W_{\mathbb{R}},$$

where $w = z$ or zw_σ, $z \in \mathbb{C}^\times$, as in [Ar89]. We write $\rho = \rho(\iota_M)$.

We observe that (7.3) implies that $\mu + \iota_M \in X_*(\mathcal{T}) \otimes \mathbb{C}$ is integral, *i.e.,*

$$\langle \mu + \iota_M, \alpha^\vee \rangle \in \mathbb{Z}, \tag{7.4}$$

for all roots of $\mathcal{T}$ in $G^\vee$.

Remark 7.1 *μ is at least half-integral; μ is integral if the derived group of G is simply-connected since ι_M is integral in that case.*

Recall that $\mathcal{B}$ denotes the Borel subgroup that is part of $spl^\vee$.

Lemma 7.2 *Let $\boldsymbol{\psi}$ be an elliptic u-regular Arthur parameter. Then there exists a representative $\psi = (\varphi, \rho)$ for $\boldsymbol{\psi}$, where $\varphi = \varphi[\mu, \lambda]$ and $\rho = \rho(\iota_M)$, with both μ and $\mu + \iota_M$ $\mathcal{B}$-dominant.*

Proof. First, we observe that it is sufficient to arrange that $\mu + \iota_M$ is dominant. Since $M^\vee$ is Levi we have that $\langle \iota_M, \alpha^\vee \rangle \leq 0$ for each $\mathcal{B}$-simple $\alpha^\vee$ that is not a root of $\mathcal{T}$ in $M^\vee$. Then dominance of $\mu + \iota_M$ implies $\langle \mu, \alpha^\vee \rangle \geq 0$ for all such $\alpha^\vee$ and so by (7.1.3), μ is dominant.

Second, suppose we pick $\varphi = \varphi[\mu, \lambda]$, $\rho = \rho(\iota_M)$ as in the paragraphs above. There is ω in the Weyl group of $\mathcal{T}$ in $G^\vee$ such that $\omega(\mu + \iota_M)$ is $\mathcal{B}$-dominant. Let $x \in G^\vee$ normalize $\mathcal{T}$ and act on $\mathcal{T}$ as ω. Set $M_\omega^\vee = xM^\vee x^{-1}$ and $\psi_x = Int(x) \circ \psi$. If $\alpha^\vee$ is a $\mathcal{B}$-positive root of in $\mathcal{T}$ in $M_\omega^\vee$ then $\omega^{-1}\alpha^\vee$ is a root in $M^\vee$ and so

$$\langle \iota_M, \omega^{-1}\alpha^\vee \rangle = \langle \mu + \iota_M, \omega^{-1}\alpha^\vee \rangle = \langle \omega(\mu + \iota_M), \alpha^\vee \rangle \geq 0.$$

Then $\omega^{-1}\alpha^\vee$ must be $\mathcal{B}$-positive. It now follows that $\iota_{M_\omega} = \omega\iota_M$. Then after multiplying x by an element of $\mathcal{T}$ we may replace ψ by ψ_x in our constructions to complete the proof. ■

From now on we choose representative $\psi = (\varphi, \rho)$ as in Lemma 7.3.

From (7.3) we conclude that:

Lemma 7.3 *$\mu + \iota_M, \lambda$ are data for an s-elliptic Langlands parameter*

$$\widehat{\varphi} = \varphi(\mu + \iota_M, \lambda).$$

Finally, we set

$$\mu_M = \mu - (\iota - \iota_M), \ \lambda_M = \lambda. \tag{7.5}$$

As one of the ingredients [La89] of the Langlands correspondence for M^*, the parameter $\varphi_M : W_{\mathbb{R}} \to {}^L Z_M$ from Section 5.5 defines a quasicharacter χ_{M^*} on $M^*(\mathbb{R})$. Because of (5.3) and (7.1) the restriction of χ_{M^*} to each Cartan subgroup in $M^*(\mathbb{R})$ takes the form

$$\Lambda(\mu_M, \lambda_M) \tag{7.6}$$

in the Langlands correspondence for real tori [La89]; see [Sh81, Section 9] for a discussion and [Sh10, Section 7] for notation. Further, for an inner twist $\eta : M_\eta \to M^*$ we may replace M^* by $M_\eta = \eta^{-1}(M^*)$. Then the new quasicharacter χ_η on $M_\eta(\mathbb{R})$ depends only on the inner class of η.

On the other hand, $\widehat{\varphi}$ factors through the discrete series parameter

$$\widehat{\varphi}_M = \varphi(\mu_M + \iota_M, \lambda_M)$$

for M^*.

From (7.5) and (7.4) we see that μ_M is integral for $G^\vee$. Clearly:

Lemma 7.4 *(i) $\mu + \iota_M$ is regular, i.e., $\widehat{\varphi}$ is elliptic, if and only if μ_M is $\mathcal{B}$-dominant.*

(ii) $\mu + \iota_M$ is singular if and only if $\langle \mu_M, \alpha^\vee \rangle = -1$ for some $\mathcal{B}$-simple root $\alpha^\vee$ of $\mathcal{T}$.

Assume that $\mu + \iota_M$ is regular and G has anisotropic center; this is the setting of [Ar89, Section 5], [Ko90, Section 9]. Here we recover the same parameters, but now with data for use in canonical transfer factors; see, for example, Section 8.2. Our parameter $\widehat{\varphi}$ coincides with the discrete series parameter constructed slightly differently in [Ko90, Section 9].

7.2 Character data and elliptic u-regular parameters

We combine the setting of Section 7.1 with that of Section 6.1. Thus G^* is cuspidal, and we have fixed Whittaker data for G^* together with an aligned fundamental splitting $spl_{Wh} = (B_{Wh}, T, \{X_\alpha\})$ for G^*. We now transport the data $(\mu + \iota_M, \lambda, \mathcal{C})$ for $\mathcal{T} \subseteq G^\vee$ of Section 7.1 to data for $T \subseteq G^*$, by the means described in Section 6.1. Recall that M^* is the subgroup of G^* generated by T and the root vectors $\{X_\alpha\}$, for $\alpha^\vee$ a simple root of $\mathcal{T}$ in $M^\vee \cap \mathcal{B}$. We use the same notation for the transported data, except that now we write ι_{M^*} for the transport of ι_M, *i.e.*, for one-half the sum of the roots of T in $B_{Wh} \cap M^*$.

7.3 Elliptic u-regular data: attached s-elliptic packet

We start with the case that $\mu + \iota_M$ is regular. Consider an inner form (G, η). Replacing η by a member of its inner class if necessary, we assume that the transport $spl_\eta = \eta^{-1}(spl_{Wh})$ of spl_{Wh} to G is a fundamental splitting. As in Section 6.2, to each $G(\mathbb{R})$-conjugacy class of fundamental splittings for G is attached a discrete series representation $\widehat{\pi}$ of $G(\mathbb{R})$ in the packet $\widehat{\Pi}_G$ for $\widehat{\varphi}$, and conversely. Again write $spl_{\widehat{\pi}}$ for a representative of this conjugacy class and $\eta_{\widehat{\pi}} = Int(x_{\widehat{\pi}}) \circ \eta$ for the inner twist carrying $spl_{\widehat{\pi}}$ to spl_{Wh}. Set $M_{\widehat{\pi}} = \eta_{\widehat{\pi}}^{-1}(M^*)$. By definition, $\eta_{\widehat{\pi}}$ transports character data $(\mu_{\widehat{\pi}} + \iota_{M_{\widehat{\pi}}}, \lambda_{\widehat{\pi}}, \mathcal{C}_{\widehat{\pi}})$ for $\widehat{\pi}$ to the data $(\mu + \iota_{M^*}, \lambda, \mathcal{C})$ for the elliptic torus T in G^* that is part of spl_{Wh}. Recall that the latter triple serves as character data for the Wh-generic discrete series representation of $G^*(\mathbb{R})$ in the packet $\widehat{\Pi}_{G^*}$ attached to $\widehat{\varphi}$.

Now allow $\mu + \iota_M$ to be singular. Then we assume that $\widehat{\varphi} = \varphi(\mu + \iota_M, \lambda)$ is relevant to G so that $\widehat{\Pi}_G$ is nonempty. As in Section 6.2, there is attached to $\widehat{\varphi}$ a c-Levi group which we will call E^*. Notice that $E^* \cap M^* = T$. Each $G(\mathbb{R})$-conjugacy class of fundamental splittings of G again has a representative $spl_{\widehat{\pi}}$, but now $\widehat{\pi}$ (or, more precisely, the attached distribution character) may be zero. We obtain precisely the members $\widehat{\pi}$ of $\widehat{\Pi}_G$ by requiring that $\eta_{\widehat{\pi}} : E_{\widehat{\pi}} \to E^*$ be defined over $\mathbb{R}$ (Lemma 6.1).

7.4 Elliptic u-regular data: attached Arthur packet

For the rest of Parts 7 and 8 we will limit our attention to the case that $\mu + \iota_M$ is regular as we will need it to structure our arguments for the singular case. For convenience we could also require the center of G to be anisotropic, but the general case requires no extra notation and so we will at least write it here. Finally there is the matter of how we treat z-extensions. We will continue to use the construction needed for the twisted case (see Section 3.1) but defer checking that the Adams-Johnson results may be extended in this manner until we come to the general twisted case.

Consider an inner form (G, η), where $spl_\eta = (B_\eta, T_\eta, \{X_\alpha\})$ is fundamental and η carries spl_η to spl_{Wh}. We may fix a Cartan involution c on G of the form $Int(t_\eta)$, where $t_\eta \in T_\eta(\mathbb{R})$ and $(t_\eta)^2$ is central in G. Then B_η, M_η together generate a c-stable parabolic subgroup P_η of G with M_η as Levi subgroup defined over $\mathbb{R}$. We have the quasi-character $\chi = \chi_\eta$ on $M_\eta(\mathbb{R})$ described in Section 7.1. Because of (7.6), it is clear that χ_η is unitary modulo the center of $G(\mathbb{R})$. As usual, we will identify a representation with its (appropriate) isomorphism class. Define $\pi(\eta)$ to be the irreducible essentially unitary representation of $G(\mathbb{R})$ attached to χ_η by the method of [Vo84, Theorems 1.2, 1.3]; see Lemma 2.10 of [AJ87].

Suppose we replace η by $\eta^\dagger$ within its inner class and that $\eta^\dagger$ also carries a fundamental splitting of G to spl_{Wh}. It is convenient to write $\eta^\dagger$ in the form

$$\eta^\dagger = Int(m^*) \circ \eta \circ Int(g),$$

where $m^* \in M^*_{sc}$ and $g \in G_{sc}$. Let $m = \eta_{sc}^{-1}(m^*)$. Then we insist also that $Int(m)$ transports spl_η to another fundamental splitting of G. Since we are concerned with splittings only up to $G(\mathbb{R})$-conjugacy there is no harm in considering only those $\eta^\dagger$ for which $T_{\eta^\dagger} = T_\eta$, and requiring that both $Int(m)$ and $Int(g)$ preserve T_η.

We define $\pi(\eta^\dagger)$ by replacing $B_\eta, M_\eta, \chi_\eta$ from the definition of $\pi(\eta)$ with $B_{\eta^\dagger}, M_{\eta^\dagger}, \chi_{\eta^\dagger}$. Then:

Lemma 7.5 *(i) $\pi(\eta^\dagger)$ lies in same Arthur packet Π_G prescribed by Adams-Johnson (enlarged packet in their terminology) as $\pi(\eta)$, and all members of the packet are so obtained.*

(ii) $\pi(\eta^\dagger) = \pi(\eta)$ if and only if $\eta^\dagger$ is of the form $Int(m^*) \circ \eta \circ Int(g)$, where $m^* \in M^*_{sc}$ and $g \in G(\mathbb{R})$.

Let ω_M, ω_G be the elements of the complex Weyl group $\Omega(G, T_\eta)$ of T_η in G defined by the restrictions of $Int(m)$, $Int(g)$ to T_η. Then (ii) says $\pi(\eta^\dagger) = \pi(\eta)$ if and only if we may arrange that ω_G lies in the subgroup $\Omega_{\mathbb{R}}(G, T_\eta)$ of $\Omega(G, T_\eta)$ consisting of those elements that are realized in $G(\mathbb{R})$.

Proof. To compare explicitly with Lemma 2.10 of [AJ87], first note that since we do not assume that the center of G is anisotropic our elliptic data have an extra component, namely λ as above. The "λ, ρ" of [AJ87] are our μ_M, ι. Note that (7.5) says that

$$\mu_M + \iota = \mu + \iota_M.$$

We further have the alternative short definition after Remark 7.4 for the quasicharacter χ_η, but it is clear from calculations of Section 2 of [AJ87] (or see [Sh79b, Lemma 9.2], [Sh81, Section 9]) that we obtain the same character when we require the center of G to be anisotropic. The claim (i) now follows. More accurately, we have adapted the definitions of Adams-Johnson to the case where there is no restriction on the center for G while retaining (i) of their Lemma 2.10. Because $\mu + \iota_M$ is regular, the claim (ii) follows easily from the character formulas we will recall in Section 8.3. ■

8 Standard factors for elliptic u-regular packets

Here by standard factors we mean the spectral transfer factors for standard endoscopy. We introduce these, with a two-fold purpose, for the elliptic u-

regular Arthur packets Π_G of the last section. First, we will check that the Adams-Johnson transfer can be recast in terms of these factors and thereby made compatible with the transfer of orbital integrals using the canonical factors of [LS87]. Second, we will write the factors in a way that allows quick generalization to the twisted setting [ShII].

8.1 Canonical relative factor: setting

We continue with the setting at the end of Part 7. In summary, $\psi = (\varphi, \rho)$ is an elliptic u-regular Arthur parameter with $\varphi = \varphi[\mu, \lambda]$ and $\rho = \rho(\iota_M)$ as in Lemma 7.2. We assume $\mu + \iota_{M^*}$ is regular as well as dominant. Then $\widehat{\varphi}$ is the attached elliptic parameter $\varphi(\mu + \iota_{M^*}, \lambda)$.

To introduce elliptic endoscopic groups as in Section 6.5, we turn to the Γ-invariants in the maximal torus $\mathcal{T}$ from $spl^\vee$. We use the elliptic action of Γ, so that σ acts by $Int\ \widehat{\varphi}(w_\sigma)$. Consider the elliptic SED $\mathfrak{e}_z(s)$ as in Section 6.5, using the notation $(H, \mathcal{H}, s)$ for the endoscopic data and $H^{(s)}$ for the endoscopic group. It will be sufficient for our purposes in [ShII] to consider the case that the Γ-invariant s lies in the center $Z_{M^\vee}$ of $M^\vee$. This is the same as requiring that $H^\vee := Cent(s, G^\vee)^0$ contain $M^\vee$. Thus we place ourselves in the setting of Adams-Johnson; see [AJ87, 2.16].

Because ψ is elliptic, the subgroup $\mathcal{M} = \mathcal{M}_\psi$ of ${}^LG$ may be generated by $M^\vee$ and either $\varphi(W_\mathbb{R})$ or $\widehat{\varphi}(W_\mathbb{R})$. Thus $\mathcal{H}$, generated by $H^\vee$ and $\widehat{\varphi}(W_\mathbb{R})$, contains $\mathcal{M}$. Since the endoscopic group $H^{(s)}$ is a z-extension of the endoscopic datum H, we will need to thicken $\mathcal{M}$.

Recall that $(Z_{M^\vee})^\Gamma = S_\psi := Cent(Image(\psi), G^\vee)$. Thus the image of ψ lies in $\mathcal{M} \subseteq \mathcal{H}$. As a component of the SED $\mathfrak{e}_z(s)$, we have $\xi^{(s)} : \mathcal{H} \to {}^LH^{(s)}$ and thus an elliptic u-regular Arthur parameter for $H^{(s)}$ represented by $\psi^{(s)} = \xi^{(s)} \circ \psi$. Now we attach $\mathcal{M}^{(s)}$ to $\psi^{(s)}$ in the same way we attached $\mathcal{M}$ to ψ. Then $\mathcal{M}^{(s)}$ is what we mean by the thickened version of $\mathcal{M}$. We will thicken various other subgroups when needed, again using the super- or subscript (s) to indicate this.

We now describe transfer factors attached to the pair $(\psi^{(s)}, \psi)$; see [Sh10, Section 9], [Sh08b, Sections 7, 11] for the tempered analogue. Recall $\psi = (\varphi, \rho)$. Then we write $\psi^{(s)}$ as $(\varphi^{(s)}, \rho)$.

First, $\varphi^{(s)} = \varphi[\mu^{(s)}, \lambda^{(s)}]$ and

$$\mu^{(s)} = \mu - \mu^*,\ \lambda^{(s)} = \lambda - \lambda^*. \tag{8.1}$$

The pair (μ^*, λ^*) is from [Sh81]; it is typically nontrivial and is critical for a well-defined transfer of orbital integrals. Here we need its construction for general standard transfer with z-extensions; see Section 11 of [Sh08a]. Also see Section 9.3 below for a detailed contruction in the general twisted case. The formula (8.1) follows from combining the construction with that in Lemma 7.2.

Second, the component ρ of $\psi^{(s)}$ may be written again as $\rho(\iota_M)$. For this we recall the splittings involved in our constructions: we have $spl^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$ with attached $spl^\vee_M = (\mathcal{B} \cap M^\vee, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $M^\vee$, along with $spl^\vee_H = (\mathcal{B} \cap H^\vee, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $H^\vee$ and thickened $spl^\vee_{(s)} = (\mathcal{B}^{(s)}, \mathcal{T}^{(s)}, \{X_{\alpha^\vee}\})$ for $(H^{(s)})^\vee$.

Then ι_M is one-half the sum of the coroots of $\mathcal{T}$ in $\mathcal{B} \cap M^\vee = \mathcal{B} \cap \mathcal{M}$. Each such coroot is naturally identified as a coroot of $\mathcal{T}^{(s)}$ in $\mathcal{B}^{(s)} \cap \mathcal{M}^{(s)}$ and conversely, which justifies our use of ι_M for ρ as component of $\psi^{(s)}$.

The c-Levi group $M^{(s)}$ in $H^{(s)}$ is the analogue for $\psi^{(s)}$ of the c-Levi group M^* in G^* attached to ψ. There is a c-Levi group M_H in H such that $M^{(s)} \to M_H$ is a z-extension with kernel Z_1, *i.e.*, with same kernel as the z-extension $H^{(s)} \to H$ provided by the SED $\mathfrak{e}_z(s)$.

Our next step is to define an $\mathbb{R}$-isomorphism $M_H \to M^*$ uniquely up to composition with an element of $Int[M^*(\mathbb{R})]$, and thence a surjective homomorphism $M_H^{(s)} \to M^*$ with kernel Z_1. For this, recall that in the construction of M^* at the beginning of (6.2) we also determined an $\mathbb{R}$-splitting for M^* uniquely up to $M^*(\mathbb{R})$-conjugacy. The same is then true for M_H. There is a unique $\mathbb{R}$-isomorphism $M_H \to M^*$ transporting the latter splitting to the former. We may further assume the isomorphism carries chosen elliptic maximal torus T_H in H to chosen T in G^*; recall that each torus is part of an appropriate fundamental splitting of Whittaker type. If $T^{(s)}$ is the inverse image of T_H in $M_H^{(s)}$ then we have now have a well-defined transport to T of our various data attached to $T^{(s)}$.

8.2 Canonical relative factor: definition

We now define a relative transfer factor in preparation for a nontempered supplement to Section 6.5. Thus (G, η) is an inner form of G^* such that η transports fundamental splitting spl_η, of G to spl_{Wh}. Let $\pi \in \Pi_G$ (Arthur packet for $G(\mathbb{R})$ attached to ψ) and let $\widehat{\pi} \in \widehat{\Pi}_G$ (discrete series packet for $G(\mathbb{R})$ attached to ψ). Also let $\pi_s \in \Pi_{H^{(s)}}$ and $\widehat{\pi}_s \in \widehat{\Pi}_{H^{(s)}}$ (packets for $H^{(s)}(\mathbb{R})$ attached to $\psi^{(s)}$). Then our first concern will be a relative factor $\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi})$.

Attach the cochain $x_{\widehat{\pi}}(\sigma) \in T_{sc}$ to the discrete series representation $\widehat{\pi}$ as in Section 6.4; recall that $\eta_{\widehat{\pi}} = Int(x_{\widehat{\pi}}) \circ \eta$ and $x_{\widehat{\pi}}(\sigma) = x_{\widehat{\pi}}.u_\eta(\sigma).\sigma(x_{\widehat{\pi}})^{-1}$. Again we write $\mathcal{E}(T)$ for the image of $H^1(\Gamma, T_{sc})$ in $H^1(\Gamma, T)$ under the homomorphism induced by $T_{sc} \to T$. Then if (G, η) is a component of an extended group of quasi-split type, so that $u_\eta(\sigma)$ is a cocycle, we map the class of $x_{\widehat{\pi}}(\sigma)$ in $H^1(\Gamma, T_{sc})$ to $H^1(\Gamma, T)$ to obtain the element $inv(\widehat{\pi})$ of $\mathcal{E}(T)$.

Turning to π in the Arthur packet for $G(\mathbb{R})$, we pick a twist $\eta^\dagger$ such that $\pi = \pi(\eta^\dagger)$ as in Section 7.4. We write $\eta^\dagger$ as $Int(x^\dagger) \circ \eta$ and form the cochain $x^\dagger(\sigma) = x^\dagger.u_\eta(\sigma).\sigma(x^\dagger)^{-1}$. Recall the torus U_{sc} from Section 6.4. The image in U_{sc} of the cochain $(x^\dagger(\sigma)^{-1}, x_{\widehat{\pi}}(\sigma))$ in $T_{sc} \times T_{sc}$ is a cocycle whose class in $H^1(\Gamma, U_{sc})$ we denote by $\mathbf{x}_{sc}(\eta^\dagger, \widehat{\pi})$. Then $\mathbf{x}(\eta^\dagger, \widehat{\pi})$ is the image of this class in $H^1(\Gamma, U)$. Recall s_U from Section 6.4 and that in the present setting we assume that the Γ-invariant s lies in the center of $M^\vee$.

Lemma 8.1 $\left\langle \mathbf{x}(\eta^\dagger, \widehat{\pi}),\ s_U \right\rangle_{tn}$ *depends only on* $\pi, \widehat{\pi}$.

Then we define

$$pair_{(s)}(\pi, \widehat{\pi}) := \left\langle \mathbf{x}(\eta^\dagger, \widehat{\pi}),\ s_U \right\rangle_{tn}.$$

Before proving Lemma 8.1 we examine $x^{\dagger}(\sigma)$ in the case that (G,η) is a component of an extended group of quasi-split type. Then $x^{\dagger}(\sigma)$ is a cocycle and so defines an element $\mathbf{x}(\eta^{\dagger})$ of $\mathcal{E}(T)$. We have $T \subseteq M^* \subseteq G^*$. Then $\mathcal{E}_{M^*}(T)$ is the image of $H^1(\Gamma, T_{M^*_{sc}}) \to H^1(\Gamma, T)$. It is a subgroup of $\mathcal{E}(T)$.

Lemma 8.2 *The image of* $\mathbf{x}(\eta^{\dagger})$ *in* $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ *depends only on* π.

Proof. There is no harm in replacing $x^{\dagger}(\sigma)$ by its inverse. The twist $\eta^{\dagger}$ may be replaced only by $Int(m^*) \circ \eta \circ Int(g)$, where m^*, g are as specified in Section 7.4. Then $x^{\dagger}(\sigma)^{-1}$ is replaced by $\sigma(m^*)(m^*)^{-1}.m^*x^{\dagger}(\sigma)^{-1}.(m^*)^{-1}$. Our assumptions on m^* ensure that $\sigma(m^*)(m^*)^{-1}$ is a cocycle in T_{sc}; its class then has image in $\mathcal{E}_{M^*}(T)$. Finally, the $\mathbb{R}$-automorphism $Int(m^*) : T_{sc} \to T_{sc}$ induces a homomorphism $H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T_{sc})$. Passing to T, we may then define a homomorphism $\mathcal{E}(T) \to \mathcal{E}(T)/\mathcal{E}_{M^*}(T)$. From the Tate-Nakayama isomorphism of $H^1(\Gamma, T_{sc})$ with $H^{-1}(\Gamma, X_*(T_{sc}))$, we see that the homomorphism coincides with the natural projection, and the lemma follows. ■

Now define

$$inv(\pi) := \mathbf{x}(\eta^{\dagger}).\mathcal{E}_{M^*}(T).$$

Because s is a Γ-invariant in the center of $M^{\vee}$, we have that

$$\langle \mathcal{E}_{M^*}(T),\ s_T \rangle_{tn} = 1,$$

and so the Tate-Nakayama pairing for T determines a well-defined sign we will write as

$$\langle inv(\pi),\ s_T \rangle .$$

We may view $\langle \_, \_ \rangle$ as a pairing between $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ and $(Z_{M^{\vee}})^{\Gamma}$ or, better, between $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ and $(Z_{M^{\vee}})^{\Gamma}/(Z_{G^{\vee}})^{\Gamma}$. In the latter case we identify s_T with its image in $(Z_{M^{\vee}})^{\Gamma}/(Z_{G^{\vee}})^{\Gamma}$ without change in notation. We will say more about the pairing in [ShII].

Notice that Lemma 8.1 is now proved in this setting, *i.e.*, for an extended group of quasi-split type, because

$$\left\langle \mathbf{x}(\eta^{\dagger}, \widehat{\pi}),\ s_U \right\rangle_{tn} = pair_{(s)}(\pi, \widehat{\pi}) = \langle inv(\pi), s_T \rangle^{-1} . \langle inv(\widehat{\pi}), s_T \rangle_{tn} . \tag{8.2}$$

Proof. [of Lemma 8.1] A factoring via the method for the proof of Lemma 8.2, but now in U_{sc} instead of T_{sc}, may be applied to the cocycle defining $\mathbf{x}_{sc}(\eta^{\dagger}, \widehat{\pi})$. Then we follow closely the rest of the argument to complete the proof. ■

Next, we recall the sign

$$\varepsilon(G) := (-1)^{q(G)\ -\ q(G^*)},$$

where $2q(G)$ is the rank of the symmetric space attached to G_{sc}. It is well-defined in general and appears in the tempered character identities for transfer from the inner form (G,η) to G^*; see [Sh79a, Theorem 6.3]. This sign is recast by Kottwitz in [Ko83, p.295] in terms of Galois cohomology. Notice that the choice of inner twist does not matter; see [Ko83, p.292]. In our present setting we have $\pi = \pi(\eta^\dagger)$. Let $M_{\eta^\dagger} = (\eta^\dagger)^{-1}(M^*)$. Then it is clear from either definition that $\varepsilon(M_{\eta^\dagger})$ is independent of the various choices for $\eta^\dagger$ and so we write it as $\varepsilon_M(\pi)$.

We conclude then that the relative factor

$$\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}) := \varepsilon_M(\pi).pair_{(s)}(\pi, \widehat{\pi}) \tag{8.3}$$

is well-defined, *i.e.*, depends only on s, π and $\widehat{\pi}$. This factor and others similarly defined have useful transitivity properties (see [LS87, Section 4.1], [Sh10, Section 4]). We will ignore them for now except to remark that if the discrete series representation $\widehat{\pi}$ has the property that $\eta_{\widehat{\pi}}$ serves as $\eta^\dagger$ then

$$\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}) = \varepsilon_M(\pi). \tag{8.4}$$

To define an absolute factor $\Delta(\pi_s, \pi)$, assume that we have absolute geometric factors and absolute spectral factors for the essentially tempered spectrum that are compatible in the sense of [Sh10, Section 12]. This notion of compatibility is defined via another canonical relative factor, and compatible factors are easily shown to exist for all inner forms (G, η); see [Sh10, Section 4]. We then set

$$\Delta(\pi_s, \pi) := \Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}).\Delta(\widehat{\pi}_s, \widehat{\pi}). \tag{8.5}$$

In particular if M^* is a torus, so that (π_s, π) is a related pair of discrete series representations, we return the original constructions for the (essentially) tempered spectrum; see [Sh10, Section 9].

Consider an extended group of quasi-split type and use the Whittaker normalization Δ_{Wh} of absolute factors attached to our choice of Whittaker data [KS99, Section 5.3]. Then (8.5), (8.3), (8.2) and the strong base-point property of Whittaker normalization [Sh08b, Theorem 11.5] (recall Section 6.5) imply:

Lemma 8.3

$$\Delta_{Wh}(\pi_s, \pi) = \varepsilon_M(\pi). \langle inv(\pi),\ s_T \rangle .$$

8.3 Application to the transfer of Adams-Johnson

Continuing in the same setting, we write the correspondence of test functions (more precisely, test measures) as $(f, f^{(s)})$. Then

$$SO(\gamma, f^{(s)}) = \sum_{\delta,\ conj} \Delta(\gamma, \delta)\ O(\delta, f) \tag{8.6}$$

for all strongly G-regular γ in $H^{(s)}(\mathbb{R})$ and

$$St\text{-}Trace\ \widehat{\pi}_s(f^{(s)}) = \textstyle\sum_{\widehat{\pi}} \Delta(\widehat{\pi}_s, \widehat{\pi})\ Trace\ \widehat{\pi}(f). \tag{8.7}$$

Now to consider the pair (π_s, π), we observe that the Adams-Johnson stable combination [AJ87, Theorem 2.13] agrees with

$$St - Trace\pi_s(f^{(s)}) := \textstyle\sum_{\pi'_s \in \Pi_{H^{(s)}}} \varepsilon_M(\pi'_s) Trace\pi'_s(f^{(s)}),$$

up to the sign $(-1)^{\gamma(M^*)}$ defined in [AJ87, 2.12].

Next we claim the following transfer for (π_s, π) :

$$St\text{-}Trace\ \pi_s(f^{(s)}) = \textstyle\sum_{\pi \in \Pi_G} \Delta(\pi_s, \pi)\ Trace\ \pi(f). \tag{8.8}$$

Here $(f, f^{(s)})$ is any pair of test functions related by the geometric transfer (8.6) and $\Delta(\pi_s, \pi)$ is given by (8.3), (8.5) (or by (8.9) below).

Suppose G has anisotropic center, so that we may apply the main transfer theorem of Adams-Johnson directly. We recast the geometric transfer of [AJ87, Section 2] as the correspondence $(f, f^{(s)})$ above; see [LS90, Theorem 2.6.A]. Also, because we must work with C_c^∞-functions, we have applied Bouaziz's Theorem as in [Sh12, Sections 1, 2]. From [AJ87, Theorem 2.21] we then have that the transfer (8.8) is true for some choice of the coefficients, say $\Delta'(\pi_s, \pi)$. With a little more effort we may show that our choice of $\Delta(\pi_s, \pi)$ is correct up to a constant, but we will not need that. Instead, we turn to the transfer (8.7) in the case of the discrete series pairs $(\widehat{\pi}_s, \widehat{\pi})$ from Sections 7.3 and 8.2.

For each pair (π_s, π), where $\pi = \pi(\eta^\dagger)$, we consider all pairs $(\widehat{\pi}_s, \widehat{\pi})$ such that $\eta_{\widehat{\pi}}$ serves as $\eta^\dagger$. From (8.4) and (8.5) we have that

$$\Delta(\pi_s, \pi) = \varepsilon_M(\pi).\Delta(\widehat{\pi}_s, \widehat{\pi}). \tag{8.9}$$

Now we choose $(f, f^{(s)})$ with support within the very regular elliptic set (see Section 3.4). We follow the comparison in [Ko90, Section 9] of the Vogan-Zuckerman character formula for π on the regular elliptic set with the Harish-Chandra formulas for the discrete series characters $\widehat{\pi}$ attached to π. From this we deduce that

$$Trace\ \pi(f) = (-1)^{q(M_{\eta^\dagger})} \textstyle\sum_{\widehat{\pi}} Trace\ \widehat{\pi}(f) \tag{8.10}$$

for our particular pairs $(f, f^{(s)})$. Multiply across (8.7) by $(-1)^{q(M^*)}$. From that identity, together with (8.9) and (8.10), we then have that

$$\textstyle\sum_{\pi \in \Pi_G} [\Delta(\pi_s, \pi) - \Delta'(\pi_s, \pi)] Trace \pi(f) = 0$$

for all f supported in the strongly regular elliptic set. It now follows that the coefficients $\Delta(\pi_s, \pi) - \Delta'(\pi_s, \pi)$ are all zero; we could also argue this directly with the transfer of characters as functions. We conclude then that our choice of the constants $\Delta(\pi_s, \pi)$ in (8.8) is correct.

9 Parameters and twistpackets

We now return to the general twisted setting of Section 3.1 and finish the proof of various assertions made earlier.

9.1 Twistpackets

Attached to the triple (G^*, θ^*, a) is the automorphism ${}^L\theta_a$ of ${}^LG$. We are interested in Langlands parameters $\boldsymbol{\varphi}$ preserved by ${}^L\theta_a$, *i.e.,* those $\boldsymbol{\varphi}$ for which

$$S_\varphi^{tw} := \{s \in G^\vee :{}^L \theta_a \circ \varphi = Int(s) \circ \varphi\}$$

is nonempty, for some, and hence any, representative φ. Then we may construct supplemented endoscopic data for (G^*, θ^*, a) following the last paragraphs of [KS99, Chapter 2]; see Section 3.1.

Let (G, θ, η) be an inner form of (G^*, θ^*). It follows quickly from the Langlands classification, at least in the essentially tempered case, that the L-packet Π for $G(\mathbb{R})$ attached to φ is stable under the operation $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$. As in Section 4.1, we then say Π is (θ, ϖ)-stable. Conversely the parameter for a (θ, ϖ)-stable packet is preserved by ${}^L\theta_a$. In general, this operation on a (θ, ϖ)-stable packet Π need have no fixed points, *i.e.,* the twistpacket $\Pi^{\theta,\varpi}$ introduced in Section 4.1 may be empty. We examine this further for φ elliptic.

Suppose φ is elliptic and preserved by ${}^L\theta_a$. We use the standard representative $\varphi = \varphi(\mu, \lambda)$ from (5.6) and define data (μ_a, λ_a) for the cocycle a in the usual manner: $a(z) = z^{\mu_a} \overline{z}^{\sigma \mu_a}$ for $z \in \mathbb{C}^\times$, and $a(w_\sigma) = e^{2\pi i \lambda_a}$. First we observe that because φ is regular, each element s of S_φ^{tw} must normalize $\mathcal{T}$. Then because $\theta^\vee$ preserves $spl^\vee$, s lies in $\mathcal{T}$, so that $S_\varphi^{tw} \subseteq \mathcal{T}$. We conclude then that

$$\theta^\vee \mu = \mu + \mu_a \text{ and } \theta^\vee \lambda \equiv \lambda + \lambda_a \bmod \mathcal{K}_f, \tag{9.1}$$

where

$$\mathcal{K}_f = X_*(\mathcal{T}) + [1 - \varphi(w_\sigma)] X_*(\mathcal{T}) \otimes \mathbb{C}.$$

Returning to Section 6.1, we now assume the chosen Whittaker data for G^* is θ^*-stable (see [KS99, Section 5.3]). We have a uniquely defined transport of $(\mu, \lambda, \mathcal{C})$ to character data for the generic discrete series representation π^* attached to φ. Then (9.1) implies that $\pi^* \circ \theta^* \approx \varpi \otimes \pi^*$, or we could argue this directly from Whittaker properties.

9.2 Nonempty fundamental twistpackets

Since the general fundamental case requires only a trivial modification, we continue with the elliptic setting of Section 9.1. We have attached a fundamental splitting spl_π to a discrete series representation π of $G(\mathbb{R})$ in Section 6.1. It is unique up to $G(\mathbb{R})$-conjugacy.

Lemma 9.1 *Suppose that π is a discrete series representation of $G(\mathbb{R})$ such that $\pi \circ \theta \approx \varpi \otimes \pi$. Then there exists $\delta_\pi \in G(\mathbb{R})$ such that $Int(\delta_\pi) \circ \theta$ preserves spl_π. If spl_π is replaced by another fundamental splitting $Int(x).spl_\pi$, where $x \in G(\mathbb{R})$, then δ_π is replaced by an element δ'_π of the form $zx\delta_\pi\theta(x)^{-1}$, where $z \in Z_G(\mathbb{R})$.*

Proof. Since θ transports spl_π to a fundamental splitting for $\pi \circ \theta$ and we may use spl_π as splitting for $\varpi \otimes \pi$, the existence of δ_π is clear. Now, with spl_π fixed, δ_π may be replaced only by an element of $Z_G(\mathbb{R})\delta_\pi$. Next replace spl_π by $Int(x).spl_\pi$, where $x \in G(\mathbb{R})$. Then

$$Int(x) \circ (Int(\delta_\pi) \circ \theta) \circ Int(x)^{-1} = Int(x\delta_\pi\theta(x)^{-1}) \circ \theta$$

preserves $Int(x).spl_\pi$, and the lemma follows. ∎

Lemma 9.2 *If there exist nonempty twistpackets of discrete series (or fundamental series) representations of* $G(\mathbb{R})$ *then there is* (θ_f, η_f) *in the inner class of* (θ, η) *such that* θ_f *preserves a fundamental splitting for* G*. The converse is also true in the case that there exists an elliptic (fundamental) Langlands parameter preserved by* ${}^L\theta_a$*.*

Proof. A nonempty twistpacket provides us with a θ-fundamental element δ_π, and so Lemma 2.5 applies. For the converse, we may assume that θ is as in (ii) of Lemma 2.5. We then apply the remarks of Section 9.1 using the transport of data to G provided by the inner twist η. ■

We see then that, as on the geometric side, to capture the elliptic (fundamental) contribution we may assume that, up to a twist by an element of $G(\mathbb{R})$, θ is the transport of θ^* to G by an inner twist η which also carries spl_{Wh} to a fundamental splitting for G, *i.e.*, that we are in the setting I of Section 3.2. We will need further information about the element δ_π of Lemma 9.1.

Lemma 9.3 *(i)* $\delta_\pi \in G(\mathbb{R})$ *has a norm* γ_π *in* $H_1(\mathbb{R})$*. (ii)* γ_π *lies in* $Z_{H_1}(\mathbb{R})$ *and its image in* $Z_H(\mathbb{R})$ *under the projection* $H_1 \to H$ *is determined uniquely by* δ_π*. (iii) If* δ_π *is replaced by* $\delta'_\pi = zx\delta_\pi\theta(x)^{-1}$*, where* $z \in Z_G(\mathbb{R})$ *and* $x \in G(\mathbb{R})$*, then* γ_π *is replaced by an element* $\gamma'_\pi = z_1\gamma_\pi$*, where* $(z_1, z) \in C(\mathbb{R})$*.*

We will explain what we mean by (i) in the proof. The group $C(\mathbb{R})$ is from Section 5.1 of [KS99]; it was recalled in Section 4.2.

Proof. We may as well assume that we are in the setting I of Section 3.2 since the modifications for a further twist by an element of $G(\mathbb{R})$ are immediate. Suppose $Int(x_\pi) \circ \eta$ carries spl_π to spl_{Wh}. Then a calculation shows that $\delta^*_\pi = x_\pi\eta(\delta_\pi)\theta^*(x_\pi)^{-1}$ has the property that $Int(\delta^*_\pi) \circ \theta^*$ preserves spl_{Wh}. Because θ^* also preserves spl_{Wh}, we conclude that δ^*_π lies in Z_{G^*}. Further, we calculate that $\sigma(\delta^*_\pi)^{-1}.\delta^*_\pi \in (1 - \theta^*)T$. As in (5.1) of [KS99], we regard the coinvariants $(Z_{G^*})_{\theta^*}$of Z_{G^*} as a subgroup of the coinvariants T_{θ^*} of T. Then under the projection $N : T \to T_{\theta^*}$, δ^*_π maps into $(Z_{G^*})_{\theta^*}$. Since $N(\sigma(\delta^*_\pi)^{-1}.\delta^*_\pi) = 1$, we have that $N(\delta^*_\pi) \in (Z_{G^*})_{\theta^*}(\mathbb{R})$. We identify $(Z_{G^*})_{\theta^*}(\mathbb{R})$ as a subgroup of $Z_H(\mathbb{R})$ and then as a subgroup of $T_H(\mathbb{R})$. Let γ_π be an element of $Z_{H_1}(\mathbb{R})$ whose image under $p : H_1 \to H$ coincides with the image of $N(\delta^*_\pi)$ in $Z_H(\mathbb{R})$. Then γ_π is a T_1-norm of δ_π in the sense of Section 6 of [Sh12]. In general, γ_π is determined up to stable conjugacy by δ_π [Sh12]. Since $\gamma_\pi \in Z_{H_1}(\mathbb{R})$, it is uniquely determined by δ_π. The rest is immediate. ■

9.3 Elliptic related pairs of parameters

Let $\mathfrak{e}_z$ be a supplemented set of endoscopic data for (G^*, θ^*, a) as in Section 3.1. We may define *related pairs* of essentially tempered parameters (φ_1, φ) as in Section 2 of [Sh10] for the standard case. The arguments there, and accompanying definitions, apply word for word apart from the shift in notation to ϖ_1 for the character on the central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$.

We return to the cuspidal-elliptic setting of Section 3.4 since the general fundamental case follows quickly from this. If an elliptic parameter φ_1 for the endoscopic group H_1 satisfies the stronger requirement of G-regularity then there is an elliptic parameter φ for G^* providing us with a related pair (φ_1, φ). Here it is assumed that φ_1 factors, in the sense of [Sh10, Section 2], through the group $\mathcal{H}$ included in the chosen SED.

We now recall explicit data attached to such pairs (φ_1, φ). To the $\theta^\vee$-stable Γ-splitting $spl_{G^\vee} = (\mathcal{B}, \mathcal{T}, \{X\})$ of $G^\vee$ we attach a Γ-splitting $spl_{G^\vee}^{\theta^\vee}$ for the identity component of $(G^\vee)^{\theta^\vee}$ in the standard manner (see, for example, p.61 of [KS99]). We adjust the endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s)$ within its isomorphism class so that $s \in \mathcal{T}$, and then fix a Γ-splitting $spl_{H^\vee} = (\mathcal{B}_{H^\vee}, \mathcal{T}_{H^\vee}, \{Y\})$ for $H^\vee$, where $\mathcal{B}_{H^\vee} = \mathcal{B} \cap H^\vee$ and $\mathcal{T}_{H^\vee} = \mathcal{T} \cap H^\vee = (\mathcal{T}^{\theta^\vee})^0$. Embed $H^\vee$ in $H_1^\vee$ and extend $spl_{H^\vee}$ to $spl_{H_1^\vee} = (\mathcal{B}_1, \mathcal{T}_1, \{Y\})$ by taking $\mathcal{B}_1 = Norm(\mathcal{B}_{H^\vee}, H_1^\vee)$ and $\mathcal{T}_1 = Cent(\mathcal{T}_{H^\vee}, H_1^\vee)$. None of these choices will matter for transfer factors. Nor will the choice of χ-data (this choice does matter for the construction of geometric Δ_{II} and Δ_{III}). We will thus define all Langlands data μ_1, λ_1, *etc.* for packets in familiar terms [La89]; this amounts to the choice of χ-data such that $\chi_{(\alpha^\vee)_{res}} = (z/\overline{z})^{1/2}$, where $(\alpha^\vee)_{res}$ denotes the restriction to $(\mathcal{T}^{\theta^\vee})^0$ of a root $\alpha^\vee$ of $\mathcal{T}$ in $\mathcal{B}$.

We follow the approach of Section 11 of [Sh08a] for standard endoscopy. To $spl_{H^\vee}$ we attach the representative $\varphi_1 = \varphi(\mu_1, \lambda_1)$ as in Section 5.6. Now consider an elliptic parameter $\varphi = \varphi(\mu, \lambda)$ for G^*. We alter the construction slightly. To fix an element of $G_{der}^\vee \rtimes W_\mathbb{R}$ acting on $\mathcal{T} \cap G_{der}^\vee$ as $t \to t^{-1}$, we may use either $n_G \times w_\sigma$ defined relative to $spl_{G^\vee}$ or $n_{G,\theta} \times w_\sigma$ defined relative to $spl_{G^\vee}^{\theta^\vee}$. It is more convenient to choose the latter. Thus $\varphi = \varphi(\mu, \lambda)$ will mean that

$$\varphi(w_\sigma) = e^{2\pi i \lambda}.n_{G,\theta} \times w_\sigma.$$

We may also drop the dominance requirement on μ. We do require that μ_1 is $\mathcal{B}_1$-dominant. While G-regularity of φ_1 requires that μ be regular when $\varphi(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$ are related, $\mathcal{B}_1$-dominance of μ_1 does not ensure that μ is $\mathcal{B}$-dominant. That case, however, is the only one that will matter to us (in general, an extra sign is needed in transfer factors, see Sections 7, 9 of [Sh10]). Thus we call φ_1 *well-positioned relative to* φ if μ is $\mathcal{B}$-dominant, and make that our assumption throughout. Given φ we can always find such φ_1 and it is unique up to $\mathcal{T}$-conjugacy. It is not difficult to check that this notion is independent of the choices made in its formulation; again see [Sh10].

Finally, we determine the conditions on (μ_1, λ_1) and (μ, λ) for $\varphi_1(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$ to be related. First, for $w \in W_\mathbb{R}$ pick $u(w) \in \mathcal{H}$ projecting to w, as follows. For $z \in \mathbb{C}^\times$, $u(z)$ is to act trivially and $u(zw_\sigma)$ is to act on $\mathcal{T}_H$ and $\mathcal{T}_1$ as $n_H \times w_\sigma \in {}^L H$. Since ξ_1 (part of the chosen SED) embeds $\mathcal{H}$ in ${}^L H_1$ we may define

$$\xi_1(u(zw_\sigma)) = t_{\xi_1}(zw_\sigma).n_H \times zw_\sigma$$

and

$$\xi_1(u(z)) = t_{\xi_1}(z) \times z,$$

where each $t_{\xi_1}(w)$ lies in $\mathcal{T}_1$. On the other hand, in ${}^L G$ we have that $u(w_\sigma)$ acts as $n_{G,\theta} \times zw_\sigma$. Write

$$u(w) = t(w).u'(w),$$

where

$$u'(z) = 1 \times z, u'(zw_\sigma) = n_{G,\theta} \times w_\sigma$$

for $z \in \mathbb{C}^\times$. Then $t(w) \in \mathcal{T}$. Let

$$\mathcal{T}_2 = (\mathcal{T}_1 \times \mathcal{T})/\mathcal{T}_H,$$

where $\mathcal{T}_H$ is embedded by $t \to (t^{-1}, t)$. On $\mathcal{T}_2$ we use the elliptic action σ_2 of Γ inflated to $W_\mathbb{R}$: σ_2 acts as $n_H \times w_\sigma$ on the first component and as $n_{G,\theta} \times w_\sigma$ on the second. Let $t_2(w)$ denote the image in $\mathcal{T}_2$ of $(t_{\xi_1}(w)^{-1}, t(w)) \in \mathcal{T}_1 \times \mathcal{T}$. Then we define

$$(\mu^*, \lambda^*) \in (X_*(\mathcal{T}_2) \otimes \mathbb{C})^2$$

by

$$t_2(z) = z^{\mu^*}.\overline{z}^{\sigma_2\mu^*} \times z$$

for $z \in \mathbb{C}^\times$, and

$$t_2(w_\sigma) = e^{2\pi i\lambda^*} \times w_\sigma.$$

Notice that we have constructed (μ^*, λ^*) independently of φ_1, φ. The cochain $t_2(w)$ is *not* the cocycle $a_T(w)$ of (4.4) in [KS99]; $a_T(w)$ requires a ρ-shift (ι-shift in our notation) to be applied to the datum μ^*. See Section 11 of [Sh08a] for the case of standard endoscopy, where the torus $\mathcal{T}_2$ collapses to $\mathcal{T}_1$.

Recall that $\varphi_1(W_\mathbb{R})$ is assumed to lie in $\xi_1(\mathcal{H})$. Identify μ_1, μ with their images in $X_*(\mathcal{T}_2) \otimes \mathbb{C}$ under the componentwise embeddings. We may write

$$\varphi_1(w) = t_H(w).\xi_1(u(w))$$

and

$$\varphi(w) = t_H(w).t(w).u'(w),$$

where $t_H(w) \in \mathcal{T}_H$. We now conclude that:

Lemma 9.4 *An elliptic pair $(\varphi_1(\mu_1, \lambda_1)$, $\varphi(\mu, \lambda))$ as above is related if and only if*

$$\mu_1 + \mu^* = \mu \text{ and } \lambda_1 + \lambda^* \equiv \lambda \bmod \mathcal{K}_f.$$

9.4 An application

We finish with a proof of the formula (4.4) from our discussion on properties required of spectral factors.

Lemma 9.5

$$\varpi_C((z_1, z)) = \varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1},$$

for all $(z_1, z) \in C(\mathbb{R})$.

Proof. There is no harm in arguing in G^* since ϖ_{π_1}, ϖ_π may be calculated there. Thus we embed Z_{G^*} in fundamental T and write $z \in Z_{G^*}(\mathbb{R})$ in the form $z = \exp Y. \exp i\pi\lambda^\vee$, where Y lies in the Lie algebra $\mathfrak{z}_{G^*}(\mathbb{R})$ viewed as a subspace of $X_*(T) \otimes \mathbb{C}$ and $\lambda^\vee \in X_*(T)$ is σ_T-invariant. Then it follows easily from the Langlands parametrization that

$$\varpi_\pi(z) = e^{<\mu,Y>} e^{2i\pi<\lambda,\lambda^\vee>}.$$

Here we have, as usual, identified $X^*(T) \otimes \mathbb{C}$ with $X_*(\mathcal{T}) \otimes \mathbb{C}$. Similarly, for $z_1 = \exp Y_1. \exp i\pi\lambda_1^\vee$, with $Y_1 \in \mathfrak{z}_{H_1}(\mathbb{R}) \subset X_*(T_1) \otimes \mathbb{C}$ and $\lambda_1^\vee \in X_*(T)^{\sigma_{T_1}}$, we have

$$\varpi_{\pi_1}(z_1) = e^{<\mu_1,Y_1>} e^{2i\pi<\lambda_1,\lambda_1^\vee>}.$$

Now identify the torus $\mathcal{T}_2$ from Section 9.2 as the dual of the torus T_2. Then if z, z_1 have the same image in $Z_H(\mathbb{R})$, *i.e.,* if $(z_1, z) \in C(\mathbb{R})$, it follows from our remarks in Section 9.2 that

$$\varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1} = e^{-\ <\mu^*,Y_2>} e^{-\ 2i\pi<\lambda^*,\lambda_2^\vee>},$$

where $Y_2 = (Y_1, Y)$ and $\lambda_2^\vee = (\lambda_1^\vee, \lambda^\vee)$. On the other hand, it is clear from the definitions of ϖ_C and (μ^*, λ^*), and from the relation of the cochain $t_2(w)$ to the cocycle $a_T(w)$ of p. 45 of [KS99] (see Section 9.2), that this last expression is the same as $\varpi_C((z_1, z))$. ■

References

[AJ87] Adams, J. and Johnson, J. Endoscopic groups and packets of nontempered representations, Compositio Math. 64 (1987), 271-309.

[Ar89] Arthur, J. Unipotent automorphic representations: conjectures, Astérisque, 171-172 (1989), 13-71.

[Ar13] Arthur, J. *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*, AMS Colloq. Publ., vol. 61, 2013.

[CK07] Carayol, H. and Knapp, A. Limits of discrete series with infinitesimal character zero, Trans. Amer. Math Soc., 359 (2007), 5611-5651.

[HC75] Harish-Chandra, Harmonic analysis on real reductive groups I, J. Funct. Analysis, 19 (1975), 104-204.

[Ka13] Kaletha, T. Rigid inner forms of real and p-adic groups, preprint (2013), arXiv 1309.3731v5

[Ko83] Kottwitz, R. Sign changes in harmonic analysis on reductive groups, Trans. Amer. Math. Soc., 278 (1983), 289-297.

[Ko86] Kottwitz, R. Stable trace formula: elliptic singular terms, Math. Ann., vol. 275 (1986), 365-399.

[Ko90] Kottwitz, R. Shimura varieties and λ-adic representations, in *Automorphic forms, Shimura varieties, and L-functions, Vol. I*, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[KS99] Kottwitz, R. and Shelstad, D. *Foundations of Twisted Endoscopy*, Astérisque, 255, 1999.

[KS12] Kottwitz, R. and Shelstad, D. On splitting invariants and sign conventions in endoscopic transfer, preprint (2012), arXiv 1201.5658

[KZ82] Knapp, A. and Zuckerman, G. Classification of Irreducible Tempered Representations of Semisimple Groups, Annals of Math., 116 (1982), 389-455.

[La89] Langlands, R. On the classification of irreducible representations of real algebraic groups, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, AMS Math Surveys and Monographs, 31, 1989, 101-170.

[LS87] Langlands, R. and Shelstad, D. On the definition of transfer factors, Math. Ann., 278 (1987), 219-271.

[LS90] Langlands, R. and Shelstad, D. Descent for transfer factors, in *The Grothendieck Festschrift II*, Birkhauser, 1991, 485-563.

[Me13] Mezo, P. *Character identities in the twisted endoscopy of real reductive groups*, Memoirs AMS, Nr. 1042, 2013.

[Sh79a] Shelstad, D. Characters and inner forms of a quasisplit group over $\mathbb{R}$, Compositio Math., 39 (1979), 11-45.

[Sh79b] Shelstad, D. Orbital integrals and a family of groups attached to a real reductive group, Ann. Sc. Ec. Norm. Sup., 12 (1979), 1-31.

[Sh81] Shelstad, D. Embeddings of L-groups, Canad. J. Math., 33 (1981), 513-558.

[Sh82] Shelstad, D. L-indistinguishability for real groups, Math. Ann., 259 (1981), 385-430.

[Sh08a] Shelstad, D. Tempered endoscopy for real groups I: geometric transfer with canonical factors, Contemp. Math., 472 (2008), 215-246.

[Sh08b] Shelstad, D. Tempered endoscopy for real groups III: inversion of transfer and L-packet structure, Representation Theory, 12 (2008), 369-402.

[Sh10] Shelstad, D. Tempered endoscopy for real groups II: spectral transfer factors, in *Automorphic Forms and the Langlands Program,* Higher Ed. Press/Int'l Press, 2009/2010, 236-276.

[Sh12] Shelstad, D. On geometric transfer in real twisted endoscopy, Annals of Math., 176 (2012), 1919-1985.

[Sh14] Shelstad, D. On the structure of endoscopic transfer factors, to appear. arXiv 1412:4649

[Sh] Shelstad, D. A note on real endoscopic transfer and pseudo-coefficients, preprint posted Nov 2010 on author's website.

[ShII] Shelstad, D. On elliptic factors in real endoscopic transfer II, in preparation.

[Vo78] Vogan, D. Gelfand-Kirillov dimension for Harish-Chandra modules, Invent. Math., 48 (1978), 75-98.

[Vo84] Vogan, D. Unitarizability of Certain Series of Representations, Annals of Math., 120 (1984), pp. 141-187.

[Wa14] Waldspurger, J-L. Stabilisation de la formule des traces tordue IV: transfert spectral archimédien, preprint (2014), arXiv 1403.1454

Mathematics Department, Rutgers University-Newark
shelstad@rutgers.edu