A FORMULA FOR REGULAR UNIPOTENT GERMS

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The purpose of this note is to give a simple formula for the Shalika germs corresponding to the regular unipotent conjugacy classes in a reductive p-adic group. It is based on some constructions and a theorem in [2].

Suppose then that F is a nonarchimedean local field of characteristic zero and G is a connected reductive algebraic group defined over F. For γ regular semisimple in $G(F)$ we denote by $\Phi(\gamma, f)$ the integral of $f \in C_c^\infty(G(F))$ along the conjugacy class of γ. For u unipotent in $G(F)$ we use instead the notation $a_u(f)$ for the orbital integral. Measures are to be chosen as in [2]. For γ near 1 let

$$D(\gamma)\Phi(\gamma, f) = \sum_u \Gamma_u(\gamma) a_u(f)$$

be the Shalika germ expansion of $\Phi(\gamma, f)$ normalized by the usual discriminant function ([5], or [6] where D is written d). The sum is over representatives u for the unipotent conjugacy classes in $G(F)$ and Γ_u is the Shalika germ for the class of u.

We assume that there exist regular unipotent elements in $G(F)$, that is, that G is quasi-split over F (e.g. [2, §5.1]), and from now on require that u be regular. There are three ingredients in the formula for $\Gamma_u(\gamma)$. Each is an element of $\mathcal{E}(T)$, where $T(F)$ is the Cartan subgroup containing γ. Recall that $\mathcal{E}(T)$ is the image of the Galois cohomology group $H^1(T_{sc})$ in $H^1(T)$ ([1]). We call the classes $inv(\gamma), inv_T(u)$ and $inv(T)$.

First, if γ is near 1 and α is a root of T in G then $\alpha(\gamma)^{1/2}$ is well-defined for we may write $\gamma = exp X$ and set $\alpha(\gamma)^{1/2} = exp(\alpha(X)/2)$. Suppose $\{a_\alpha\}$ are a-data for the action of $\Gamma = Gal(\bar{F}/F)$ on the roots of T [2, §2.2]. Then $(\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2})/a_\alpha$ lies in the fixed field $F_{\pm\alpha}$ of the stabilizer of $\pm\alpha$ in Γ. Thus, by Lemma 2.2.B of [2],

$$\sigma \to \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0}} \left[\frac{\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}}{a_\alpha}\right]^{\alpha^\vee}$$

is a 1-cocycle of Γ in $T(\bar{F})$. Its class $inv(\gamma)$ in $H^1(T)$ lies in $\mathcal{E}(T)$. The choice of ordering, or more simply of a gauge, on the roots does not affect $inv(\gamma)$ [2, Lemma 2.2.C]. The a-data do of course affect $inv(\gamma)$ but they will appear again in the definition of $inv(T)$ and as long as we use the same data in both places the choice will be of no consequence.

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The class $inv_T(u)$ was defined in §5.1 of [2]. It requires the choice of an F-splitting spl for G. Briefly, to u is attached an F- splitting $spl(u)$. If $spl(u)^g = spl$, where $g \in G_{sc}$, then $\sigma \to g\sigma(g)^{-1}$ is a 1-cocycle of Γ in the center of G_{sc}. This cocycle defines the class $inv_T(u)$ in $\mathcal{E}(T)$.

Finally the class $inv(T)$ will be the image in $H^1(T)$ of the class $\lambda(T_{sc})$ of [2, §2.3]. We recall that the definition $\lambda(T_{sc})$ is less immediate and requires the choice of a-data — we use that for $inv(\gamma)$ — and an F-splitting — we use the *opposite* to that for $inv_T(u)$ (see [2, §5.1]).

Theorem : For γ near 1 we have

$$\Gamma_u(\gamma) = \begin{cases} 1 & if\ inv(\gamma) = inv_T(u)/inv(T) \\ 0 & otherwise. \end{cases}$$

Proof: We fix u regular unipotent and suppose γ is sufficiently close to 1 in $T(F)$. To each character κ on $\mathcal{E}(T)$ we attach an endoscopic group $H = H(T, \kappa)$ and an admissible embedding $T_H \to T$ carrying, say, γ_H to γ (see [1]). There is no harm in assuming that each ${}^L H$ embeds admissibly in ${}^L G$. We apply Theorem 5.5.A of [2] to a function $f_u \in C_c^\infty(G(F))$ supported on the regular set of $G(F)$ for which $a_u\,(f_u) = 1$ and $a_{u'}(f_u) = 0$ if u' is not conjugate to u (e.g. [6]). Thus $D(\gamma)\Phi(\gamma, f_u) = \Gamma_u(\gamma)$. Because f_u is supported on the regular set we can omit the limit from the formula of Theorem 5.5.A and write

$$D_H(\gamma_H)D_G(\gamma)^{-1}\sum_{\gamma_G}\Delta(\gamma_H, \gamma_G)\Gamma_u(\gamma_G) = \Delta(u)$$

for γ_H near 1 in $T_H(F)$. We may write each element γ_G as $\gamma(\omega) = g^{-1}\gamma g$ where $\sigma \to \sigma(g)g^{-1}$ represents the element ω of $\mathcal{E}(T)$. Then $\Delta(\gamma_H, \gamma_G) = \kappa(\omega)\Delta(\gamma_H, \gamma)$ and so

$$\sum_{\omega}\kappa(\omega)\Gamma_u\big(\gamma(\omega)\big) = \frac{\Delta(u)\ \ D_G(\gamma)}{D_H(\gamma_H)\ \ \Delta(\gamma_H, \gamma)}.$$

Recalling the definitions of $\Delta(u)$ and $\Delta(\gamma_H, \gamma)$ [2,§§5.1,3.7], we see that this is

$$\kappa(inv_T(u))\ /\ \kappa(inv(T))\Delta_{II}(\gamma_H, \gamma)\Delta_2(\gamma_H, \gamma).$$

But $\Delta_2(\gamma_H, \gamma)$ is a character evaluated at γ and so takes the value 1 for γ near 1. Also

$$\Delta_{II}(\gamma_H, \gamma) = \prod_{\alpha}\chi_\alpha\left(\frac{\alpha(\gamma) - 1}{a_\alpha}\right)$$

where the product is over representatives α for the orbits of Γ which are not from H. For γ near 1 this coincides with the product over symmetric orbits not from H and then with $\kappa(inv(\gamma))$, by [2, Lemmas 2.2.C, 3.2.D]. Thus

$$\sum_{\omega}\kappa(\omega)\Gamma_u\big(\gamma(\omega)\big) = \kappa\big(inv_T(u)/inv(T)inv(\gamma)\big).$$

We sum over κ and reverse the order of summation on the left to obtain the theorem.

Note that if $G(F)$ contains just one conjugacy class of regular unipotent elements, for example if G is adjoint, the formula becomes

$$\Gamma_{reg}(\gamma) = \begin{cases} 1 & \text{if } inv(\gamma) = inv(T)^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

If $G = GL(n)$ then $\Gamma_{reg}(\gamma) = 1$ for all γ near 1; the constant which appears in [3] is that dictated by the different choice of measures. In the case of $SL(2)$ our formula gives a way of computing the characteristic function which appears in [5], and for $SL(n)$ we recover Theorem 6.3 of [4], again up to a constant which can be computed directly. For the case of $Sp(4)$ see also [7].

REFERENCES

[1] R. Langlands, Les débuts d'une formule des traces stable, Pub. Math. Univ. Paris VII, vol. 13 (1983).

[2] R. Langlands and D. Shelstad, "On the definition of transfer factors," to appear in Math. Annalen.

[3] J. Repka, "Shalika's germs for p-adic $GL(n)$: the leading term", Pac. J. Math., 113 (1984), 165–172.

[4] J. Repka, "Germs associated to regular unipotent classes in p-adic $SL(n)$," Cand. Math. Bull., 28(1985), 257–266.

[5] J. Shalika, "A theorem on p-adic semisimple groups", Ann. of Math.,95(1972), 226–242.

[6] M. -F. Vignéras, "Charactérisation des intégrales orbitales sur un groupe réductif p-adique", J. Fac. Sci., Tokyo 28(1981), 945–961.

[7] Kim Yandon, "Regular germs for p-adic $Sp(4)$", Ph.D. thesis, Univ. of Toronto (1987).

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