Proceedings of Symposia in Pure Mathematics
Vol. **33** (1979), part 1, pp. 107–110

ORBITAL INTEGRALS FOR $\mathrm{GL}_2(\boldsymbol{R})$*

D. SHELSTAD

We report briefly on the characterization of orbital integrals of smooth (C^∞) functions of compact support on $\mathrm{GL}_2(\boldsymbol{R})$, following [3]. A similar argument applies to $\mathrm{GL}_2(\boldsymbol{C})$ [3]. We begin by recalling some well-known properties of these integrals in a form convenient for the characterization, indicating the proof afterwards; a more elegant formulation is given in [3].

We fix an invariant 4-form ω_G on $G = \mathrm{GL}_2(\boldsymbol{R})$. If T is a Cartan subgroup of G we take ω_T to be the form $C_T d\gamma_1 d\gamma_2/\gamma_1\gamma_2$ where γ_1, γ_2 are the eigenvalues of γ under some order (prescribed by a diagonalization of T) and C_T is a constant as follows:

$C_T = 1$ if T is split, and $C_T = i$ otherwise. If $f \in C_c^\infty(G)$ and $\gamma \in T_{\mathrm{reg}}$, the set of regular elements of G lying in T, we set

$$\Phi_f^T(\gamma) = \int_{G/T} f(g\gamma g^{-1}) \frac{dg}{dt}$$

where dg, dt are the Haar measures defined by ω_G, ω_T respectively. Then Φ_f^T is a well-defined C^∞ function on T_{reg}, invariant under the Weyl group and vanishing off some set relatively compact in T. Let Z be the group of scalar matrices in G; thus $Z = T - T_{\mathrm{reg}}$. The behavior of Φ_f^T near $z \in Z$ is described as follows: there exist a neighborhood N_z of z in T (invariant under the Weyl group) and C^∞ functions $\Lambda_f^0(z, \ )$ and $\Lambda_f^1(z, \ )$, each defined on N_z and invariant under the Weyl group, such that

$$\Phi_f^T(\gamma) = \Lambda_f^0(z, \gamma) + \Lambda_f^1(z, \gamma) \mid D(\gamma)|^{-1/2} \tag{1}$$

for $\gamma \in N_z \cap T_{\mathrm{reg}}$. Here $D(\gamma) = (\gamma_1 - \gamma_2)^2/\gamma_1\gamma_2$ where, as before, γ_1 and γ_2 are the eigenvalues of γ. The functions $\Lambda_f^0(z, \ )$ and $\Lambda_f^1(z, \ )$ depend on T although we omit this in notation. Note that the equation (1) determines uniquely the restriction to $Z \cap N_z$ of $\Lambda_f^i(z, \ )$, and of all its derivatives. Thus we may set $\Lambda_f^i(z) = \Lambda_f^i(z', z)$ for any z' such that $z \in N_{z'}$, with a similar definition for derivatives. Further

(a) if T is split then we may take $\Lambda_f^0(\ , \ ) \equiv 0$

and if X_T denotes the image under the Harish-Chandra isomorphism of the operator X in the center of the universal enveloping algebra of $\mathfrak{gl}_2(\boldsymbol{C})$ then

(b) for each $z \in Z$, $X_T \Lambda_f^1(z)$ is independent of T.

To determine the restriction to $Z \cap N_z$ of the derivatives of $\Lambda_f^i(z, \ )$ it is suffi-

---

AMS (MOS) subject classifications (1970). Primary 22E30.

*This work has been partially supported by the National Science Foundation under Grant MCS76-08218.

cient to compute $X_T\Lambda^i_f(z)$ for each X; $X_T\Lambda^1_f(z)$, which we will not need explicitly, is the appropriately defined integral of Xf over the conjugacy class of $\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ and if T is not split then $X_T\Lambda^0_f(z) = c_G Xf(z)$, where c_G is a constant depending only on our choice of Haar measure on G.

We recall the proof. It is sufficient to consider the Cartan subgroups A, the diagonal group, and

$$B = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 \neq 0\right\}.$$

We find it more convenient to write an element of B_{reg} as

$$\gamma(\lambda, \theta) = \begin{pmatrix} \lambda\cos\theta & \lambda\sin\theta \\ -\lambda\sin\theta & \lambda\cos\theta \end{pmatrix}, \qquad \lambda > 0, \theta \not\equiv 0(\pi).$$

Proceeding formally, we may choose ω_G so that

$$|D(\gamma)|^{1/2}\Phi^A_f(\gamma) = \frac{1}{2}\left|\frac{\gamma_2}{\gamma_1}\right|^{1/2}\int_N f_0(n\gamma)\,dn \tag{2}$$

where

$$N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}, \qquad dn = dx, \qquad f_0(x) = \int_{K_0} f(kxk^{-1})\,dk,$$
$$K_0 = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\}, \quad dk = d\theta$$

and

$$\Phi^B_f(\gamma(\lambda, \theta)) = \frac{1}{4}\int_0^\infty (H_\lambda(e^t\theta, e^{-t}\theta) + H_\lambda(-e^t\theta, -e^{-t}\theta))\,(e^t - e^{-t})\,dt \tag{3}$$

where

$$H_\lambda(u, v) = \int_{K_0} f\left(\lambda k \exp\begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} k^{-1}\right) dk.$$

We obtain (2) from the Iwasawa decomposition of $\mathrm{SL}_2(\boldsymbol{R})$ and (3) from the Cartan decomposition. The function on the right-hand side of (3) can be analyzed as in [**1**]. It is easy then to see that this function can be expanded as on the right-hand side of (1). The proof is now straightforward. To compute $X_T\Lambda^i_f(z)$ note that $|D|^{1/2}\Phi^T_{Xf} = X_T(|D|^{1/2}\Phi^T_f)$. This is essentially Harish-Chandra's formula $F^T_{Xf} = X_T F^T_f$ [**2**]; $|D|^{1/2}\Phi^A_f$ is the function F^A_f and $(e^{i\theta} - e^{-i\theta})\,\Phi^B_f(\gamma(\lambda, \theta)) = F^B_f(\gamma(\lambda, \theta))$.

We come then to the characterization. Suppose that for each Cartan subgroup T we are given a function Φ^T, defined and C^∞ on T_{reg}, invariant under the Weyl group and vanishing off some set relatively compact in T. Suppose that Φ^T and $\Phi^{T'}$ satisfy the obvious consistency requirements when T and T' are conjugate. Finally, suppose that for each T and $z \in Z$ there exist a neighborhood N_z of z in T invariant under the Weyl group and C^∞ functions $\Lambda^0(z, \ )$ and $\Lambda^1(z, \ )$ on N_z, also invariant under the Weyl group, such that

$$\Phi^T(\gamma) = \Lambda^0(z, \gamma) + \frac{\Lambda^1(z, \gamma)}{|D(\gamma)|^{1/2}} \tag{4}$$

for $\gamma \in N_z \cap T_{\text{reg}}$; the functions $\Lambda^i(z, \ )$ are assumed to have the following two properties:

(a) $\Lambda^0(\ ,\ ) \equiv 0$ if T is split and

(b) for each X in the center of the universal enveloping algebra of $\mathfrak{gl}_2(\boldsymbol{C})$ the restriction to $Z \cap N_z$ of $X_T\Lambda^1(z,\ )$ is independent of T.

Then Lemma 4.1 of [3] asserts that there exists $f \in C_c^\infty(G)$ such that $\Phi_f^T = \Phi^T$ for each T. We sketch the argument.

Let $G_r = G - Z$ and $Y = \{(x_1, x_2) \in \boldsymbol{R}^2;\ x_2 \neq 0\}$. Define $\pi: G_r \to Y$ by $\pi(g) =$ (trace g, det g); π is submersive and each fiber is a conjugacy class in G. Let $S = \{(2x_1, x_1^2);\ x_1 \neq 0\}$. Then we define a function ψ on $Y - S$ by $\psi(\pi(\gamma)) = |D(\gamma)|^{1/2}\Phi^T(\gamma)$, $\gamma \in T_{\text{reg}}$, allowing T to vary among the Cartan subgroups of G. If ψ vanishes near S, that is, if each Φ^T vanishes in a neighborhood of Z, then it is easy to find $f \in C_c^\infty(G)$ such that $\Phi_f^T = \Phi^T$ for all T (via the coverings $T_{\text{reg}} \times G/T \to T_{\text{reg}}^G$). Suppose now that ψ extends to a smooth function on Y. Since ψ has compact support we may apply a partition of unity argument on Y and assume that ψ has support in some neighborhood (to be specified) of a point in S.

Fix $a \in S$ and $g \in \pi^{-1}(a)$. We can find a neighborhood N_1 of g in G_r with a coordinate system $y_1, \cdots, y_4$ such that $y_1 = x_1 \circ \pi$, $y_2 = x_2 \circ \pi$; we may as well assume that (y_i) maps N_1 to a cube in $\boldsymbol{R}^4$. Set $N_2 = \pi(N_1)$ and assume that $\psi \in C_c^\infty(N_2)$. We lift the form $|x_2|^{-3/2}dx_1dx_2$ to N_1. Using this and the invariant form ω_G we construct a G-invariant measure on each fiber of π. It is easy to find $f \in C_c^\infty(N_1)$ such that $\int_{\pi^{-1}(x)} f = \psi(x)$, $x \in N_2$. On the other hand, suppose that $\gamma \in N_1 \cap T$ and that $x = \pi(\gamma)$. Then we find that $\int_{\pi^{-1}(x)} h = |D(\gamma)|^{1/2}\,\Phi_h^T(\gamma)$, $h \in C_c^\infty(N_1)$. This is a straightforward computation with coordinates. Hence $\psi(x) = |D(\gamma)|^{1/2}\Phi_f^T(\gamma)$ and our argument is complete in the case that ψ is smooth.

We observe next that ψ extends smoothly to Y when the functions $\Lambda^0(z,\ )$ attached to the Cartan subgroup B satisfy

$$\lim_{\theta\to 0;(\theta\to\pi)} \frac{d^n}{d\theta^n}\left(|D(\gamma(\lambda, \theta))|^{1/2}\,\Lambda^0(\pm\lambda, \gamma(\lambda, \theta))\right) = 0$$

for each n or, more simply, $X_B\Lambda^0(z', z) \equiv 0$ for each X, z' and z. As before we suppress z' and write just $X_B\Lambda^0(z)$. To compute $X_B\Lambda^0(z)$ in general we resort to rapidly decreasing functions and their orbital integrals.

On G, or any real reductive group, we may introduce the space of rapidly decreasing (Schwartz) functions, as defined by Harish-Chandra [2]; [2], together with earlier papers listed there, contains an extensive analysis of the "F_f" (normalized orbital integral) transform on this space. If now f is rapidly decreasing on G then Φ_f^T has the properties listed earlier, except that in place of the statement about the support of Φ_f^T we have that $|D|^{1/2}\Phi_f^T$ is "rapidly decreasing on T_{reg}" [2]. For the characterization we take $\{\Phi^T\}$ as before, but allow $|D|^{1/2}\Phi^T$ to be just rapidly decreasing on T_{reg}. The argument is straightforward (since there are many rapidly decreasing functions with computable orbital integrals [2]), but lengthy. Here is our procedure. Consider the rapidly decreasing function $|D|^{1/2}\Phi^A$ on A. Using wave-packets [2] we find f_1 such that $F_{f_1}^A = |D|^{1/2}\Phi^A$; then $\Phi_{f_1}^A = \Phi^A$. Consider $\sin\theta\,\Phi^B(\gamma(\lambda, \theta)) - F_{f_1}^B(\gamma(\lambda, \theta))$. From (1) we see that this function is C^∞ on B (and rapidly decreasing as a function of λ). Then using essentially matrix coefficients of the discrete series representations of $\{x \in G\colon |\det x| = 1\}$ we find f_2 such that $F_{f_2}^B = \sin\theta\,\Phi^B - F_{f_1}^B$ and $F_{f_2}^A \equiv 0$. If $f = f_1 + f_2$ then $\Phi_f^T = \Phi^T$ for all T (... for a general group this argument characterizes only stable orbital integrals). We refer to [4] for details.

Returning to our original family $\{\Phi^T\}$, where Φ^T vanishes off some set relatively compact in T, we can find a rapidly decreasing function f such that $\Phi_f^T = \Phi^T$. Then $X_B\Lambda^0(z) = c_G Xf(z)$, $z \in Z$. Multiplying f, if necessary, by a suitable function of det, we may assume that $\{x \in \boldsymbol{R}^\times;\, x = \det g,\, f(g) \neq 0\}$ is relatively compact in $\boldsymbol{R}^\times$. This allows us to find in $C_c^\infty(G)$ a function f_1 which coincides with f on a neighborhood of Z. Then $Xf(z) = Xf_1(z)$ for all z and X, and the function ψ attached to $\{\Phi^T - \Phi_{f_1}^T\}$ is smooth on Y. We now argue as earlier and the proof of the characterization is complete.

Finally, fix a quasi-character χ on Z. Suppose that $f \in C^\infty(G)$ satisfies $f(zg) = \chi(z)f(g)$ for $z \in Z$, $g \in G$ and has support compact modulo Z. Then Φ_f^T is well defined and has the properties listed earlier, with the necessary modifications concerning support and transformation under Z. To characterize $\{\Phi_f^T\}$ we can argue as follows. Let $\{\Phi^T\}$ have those properties. We can easily find Φ_0^T, C^∞ on T_{reg}, invariant under the Weyl group, vanishing off a set relatively compact in T, satisfying (4) and such that $\Phi^T(\gamma) = \int_Z \chi(z^{-1})\Phi_0^T(z\gamma)\,dz$, $\gamma \in T_{\text{reg}}$. For example, we may take Φ_0^T as Φ^T multiplied by a suitable function of $|\det|$. We pick $f_0 \in C_c^\infty(G)$ such that $\Phi_{f_0}^T = \Phi_0^T$ for each T. Then f defined by $f(g) = \int_Z \chi(z^{-1})f_0(zg)\,dz$, $g \in G$, satisfies $\Phi_f^T = \Phi^T$ for each T and is of the desired form.

References

1. Harish-Chandra, *Some results on an invariant integral on a semi-simple Lie algebra*, Ann. of Math. (2) **80** (1964), 457–508.

2.———, *Harmonic analysis on real reductive groups*. I, II, III, J. Functional Analysis **19** (1975), 104–204; Invent. Math. **36** (1976), 1–55; Ann. of Math. (2) **104** (1976), 117–201.

3. R. P. Langlands, *Base change for* GL(2) (to appear).

4. D. Shelstad, *Characters and inner forms of a quasi-split group over* $\boldsymbol{R}$, Compositio Math. (to appear).

Columbia University