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NOTES ON L-INDISTINGUISHABILITY

(BASED ON A LECTURE OF R. P. LANGLANDS)*

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These notes are intended as a brief discussion of the results of [4]. Although we consider essentially just groups G which are inner forms of SL_2, we emphasize formulations (cf. [7]) which suggest possible generalizations. We assume that F is a field of characteristic zero. In the case that F is local there are only finitely many irreducible admissible representations of $G(F)$ which are "L-indistinguishable" from a given representation. We structure this set, an L-packet, by considering not the characters of the members but rather sufficiently many linear combinations of these characters. In the case that F is global we consider certain L-packets of representations of $G(A)$ and describe the multiplicity in the space of cusp forms of a representation $\pi = \bigotimes_v \pi_v$ in terms of the position of the local representations π_v in their respective L-packets.

1. $\mathfrak{A}(T)$, $\mathfrak{D}(T)$ and $\mathscr{E}(T)$. Suppose that G is a connected reductive group defined over F, any field of characteristic zero, and that T is a maximal torus in G, also defined over F. Fix an algebraic closure $\bar{F}$ of F. Then we set $\mathfrak{A}(T) = \{g \in G(\bar{F})$: $\operatorname{ad} g^{-1}/T$ is defined over $F\}$ and $\mathfrak{D}(T) = T(\bar{F})\backslash\mathfrak{A}(T)/G(F)$. If $g \in \mathfrak{A}(T)$ then $\sigma \to g_\sigma = \sigma(g)g^{-1}$ is a continuous 1-cocycle of $\mathfrak{G} = \operatorname{Gal}(\bar{F}/F)$ in $T(\bar{F})$. The map $g \to (\sigma \to g_\sigma)$ induces an injection of $\mathfrak{D}(T)$ into a subgroup $\mathscr{E}(T)$ of $H^1(\mathfrak{G}, T(\bar{F}))$ defined as follows. Let T_{sc} be the preimage of T in the simply-connected covering group G_{sc} of the derived group of G. Then $\mathscr{E}(T)$ is the image of the natural homomorphism of $H^1(\mathfrak{G}, T_{sc}(\bar{F}))$ into $H^1(\mathfrak{G}, T(\bar{F}))$. If $H^1(\mathfrak{G}, G_{sc}(\bar{F})) = 1$ and so, in particular, if F is local and nonarchimedean then $\mathfrak{D}(T)$ coincides with $\mathscr{E}(T)$.

L-indistinguishability appears when G contains a torus T such that $\mathfrak{D}(T)$ is nontrivial.

2. Groups attached to G (local case). Assume now that F is local. Fix a finite Galois extension K of F over which T splits. We replace $\bar{F}$ by K and $\mathfrak{G}$ by $\mathfrak{G}_{K/F} = \operatorname{Gal}(K/F)$ in the definitions of the last section. An application of Tate-Nakayama duality then allows us to identify $\mathscr{E}(T)$ with the quotient of $\{\lambda \in X_*(T_{sc})$: $\sum_{\sigma\in\mathfrak{G}_{K/F}} \sigma\lambda = 0\}$ by

$$\left\{\lambda \in X_*(T_{sc}) \colon \lambda = \sum_{\sigma\in\mathfrak{G}_{K/F}} \sigma\mu_\sigma - \mu_\sigma,\ \mu_\sigma \in X_*(T)\right\},$$

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$X_*(\ )$ denoting $\mathrm{Hom}(\mathrm{GL}_1,\ )$. A quasi-character κ on $X_*(T_{sc})$ trivial on this latter module defines, by restriction, a character on $\mathscr{E}(T)$. In [7] there is attached to each triple (G, T, κ) a quasi-split group over F (there denoted H). We will pursue this just in the case that G is an inner form of SL_2.

3. Groups attached to an inner form of SL_2 (local case). Suppose that G is an inner form of SL_2 and that F is local; we will continue with this assumption until §14. We have then two groups to consider: SL_2 and the group of elements of norm one in a quaternion algebra over F. We may take $\mathrm{PGL}_2(\boldsymbol{C})$ as ${}^LG^\circ$ (notation as in [1]) and the diagonal subgroup as distinguished maximal torus ${}^LT^\circ$.

Fix a maximal torus T in G, defined over F. A quasi-character κ from the last section is just a $\mathfrak{G}_{K/F}$-invariant quasi-character on $X_*(T)$. We fix an isomorphism between $X_*(T)$ and $X^*({}^LT^\circ) = \mathrm{Hom}({}^LT^\circ, \boldsymbol{C}^\times)$ as follows. If G is SL_2 and T the diagonal subgroup we use the map defined by the pairing between $X^*(T)$ and $X^*({}^LT^\circ)$; if T is arbitrary in SL_2 we choose a diagonalization and compose the induced map on $X^*(T)$ with that already prescribed. If G is the anisotropic form we may still regard T as a torus in SL_2 and proceed in the same way.

Using this isomorphism between $X_*(T)$ and $X^*({}^LT^\circ)$ we transfer κ to a quasi-character on $X^*({}^LT^\circ)$; using the canonical isomorphism between $\mathrm{Hom}(X^*({}^LT^\circ), \boldsymbol{C}^\times)$ and ${}^LT^\circ$ we then regard κ as an element of ${}^LT^\circ$. At the same time we transfer the action of $\mathfrak{G}_{K/F}$ on $X_*(T)$ to $X^*({}^LT^\circ)$ and ${}^LT^\circ$, writing σ_T for the new action of $\sigma \in \mathfrak{G}_{K/F}$.

Here are the possibilities. If T is split then $\mathfrak{G}_{K/F}$ acts trivially and κ is an arbitrary element of ${}^LT^\circ$. If T is anisotropic, suppose that T is defined by the quadratic extension E of F. We shall assume that K is some fixed large but finite Galois extension of F containing, in particular, E; $\mathfrak{G}_{K/F}$ acts on T through $\mathfrak{G}_{E/F}$. Let σ° be the nontrivial element of $\mathfrak{G}_{E/F}$ and $\alpha^\vee$ be a coroot for T in G. Then $\sigma^\circ\alpha^\vee = -\alpha^\vee$ so that $(\kappa(\alpha^\vee))^2 = 1$. Since $\alpha^\vee$ generates $X_*(T)$ there are then just two possibilities for κ. The nontrivial κ defines the element $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)_*$ of ${}^LT^\circ$; here, and throughout these notes, we use $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)_*$ to denote the image in $\mathrm{PGL}_2(\boldsymbol{C})$ of the matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $\mathrm{GL}_2(\boldsymbol{C})$. The action of σ_T on ${}^LT^\circ$ is described as follows: if $\sigma \in \mathfrak{G}_{K/F}$ maps to the trivial element in $\mathfrak{G}_{E/F}$ under $\mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$ then σ_T acts trivially and if σ maps to σ° then σ_T acts by

$$\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}_* \longrightarrow \begin{pmatrix} b & 0\\ 0 & a\end{pmatrix}_*.$$

As an element of ${}^LT^\circ$, κ is σ_T-invariant, $\sigma \in \mathfrak{G}_{K/F}$. We define ${}^LH^\circ$ to be the connected component of the identity in the centralizer of κ in ${}^LG^\circ$. Whatever T, if κ is trivial then ${}^LH^\circ = {}^LG^\circ$ and if κ is nontrivial then ${}^LH^\circ = {}^LT^\circ$. Let $\sigma \in \mathfrak{G}_{K/F}$. Then since both ${}^LH^\circ$ and ${}^LT^\circ$ are invariant under σ_T we may multiply σ_T by an inner automorphism of ${}^LH^\circ$ to obtain an automorphism σ_H stabilizing ${}^LT^\circ$ and fixing each root, if any, of ${}^LT^\circ$ in ${}^LH^\circ$. The collection $\{\sigma_H, \sigma \in \mathfrak{G}_{K/F}\}$ defines a semidirect product ${}^LH = {}^LH^\circ \rtimes W_{K/F}$ where $W_{K/F}$, the Weil group of K/F, acts through $\mathfrak{G}_{K/F}$. In duality, we obtain a quasi-split group H over F. Specifically:

PROPOSITION. (a) *If κ is trivial (whatever T) then ${}^LH = {}^LG = {}^LG^\circ \times W_{K/F}$ and $H = \mathrm{SL}_2$.*

(b) *If T is split and κ nontrivial then ${}^LH = {}^LT^\circ \times W_{K/F}$ and $H = T$.*

(c) *If T is anisotropic and κ nontrivial then ${}^LH = {}^LT^\circ \rtimes W_{K/F}$ where $w \in W_{K/F}$ acts trivially on ${}^LT^\circ$ if w maps to 1 under $W_{K/F} \to \mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$, and w acts by $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)_* \to \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)_*$ if w maps to σ° (E, σ° as before); and $H = T$.*

To indicate that H is defined by (T, κ) we write $H = H(T, \kappa)$. Note that the choice (of diagonalization) made in defining the isomorphism between $X_*(T)$ *and* $X^*({}^LT^\circ)$ does not affect $H(T, \kappa)$.

4. Embedding L-groups. For later use we specify an embedding ι_H of ${}^LH$ in ${}^LG$. We refer to the proposition above. In (a) ι_H is to be the identity, in (b) ι_H is the inclusion and in (c) ι_H extends the inclusion of ${}^LT^\circ$ in ${}^LG^\circ$ by:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w \longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w$$

if $w \in W_{K/F}$ maps to 1 under $W_{K/F} \to \mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$, and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w \longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_* \times w$$

if w maps to σ°.

As remarked earlier there are analogues of these groups "H" for any connected reductive group G over F. In general, it need not be that ${}^LH$ embeds in ${}^LG$; [7] indicates how to accommodate this (see Lemma 1 and the last several paragraphs).

5. Orbital integrals (normalization). We will transfer certain integrals from G to H. For this, normalizations are required. Fix a pair (T, κ).

We choose Haar measures dt and dg on $T(F)$ and $G(F)$ respectively. If $f \in C_c^\infty(G(F))$, $\delta \in \mathfrak{D}(T)$, and $\gamma \in T(F)$ is regular in G then we set

$$\Phi^\delta(\gamma, f) = \int_{h^{-1}T(F)h\backslash G(F)} f(g^{-1}h^{-1}\gamma hg) \frac{dg}{(dt)^h}$$

where $h \in \mathfrak{A}(T)$ represents δ and $(dt)^h$ is the measure on $h^{-1}T(F)h$ obtained from dt by means of ad h. For $\delta \in \mathscr{E}(T) - \mathfrak{D}(T)$ we set $\Phi^\delta(\ \ , f) \equiv 0$. Recall that $\mathfrak{A}(T)$, $\mathfrak{D}(T)$, $\mathscr{E}(T)$ were defined in §1.

In the case that κ is trivial (whatever T) we define

$$\Phi^{T/\kappa}(\ \ , f) = \Phi^{T/1}(\ \ , f) = \varepsilon(G) \sum_{\delta \in \mathscr{E}(T)} \Phi^\delta(\ \ , f)$$

where $\varepsilon(G) = 1$ if $G = \mathrm{SL}_2$ and $\varepsilon(G) = -1$ if G is anisotropic. We call $\Phi^{T/1}(\ ,\ )$ a "stable" orbital integral.

If T is split and κ nontrivial then we define

$$\Phi^{T/\kappa}(\gamma, f) = \frac{|\gamma_1 - \gamma_2|}{|\gamma_1\gamma_2|^{1/2}} \sum_{\delta \in \mathscr{E}(T)} \kappa(\delta)\Phi^\delta(\gamma, f)$$

where γ_1, γ_2 are the eigenvalues of γ.

Suppose that T is anisotropic and that κ is nontrivial. Then our normalization depends on two choices. First choose a nontrivial additive character ψ_F on F. If E is the quadratic extension of F attached to T then we define $\lambda(E/F, \psi_F)$ as in [5]. Also choose a regular element γ° of $T(F)$. Let κ' denote the quadratic character of $F^\times$ attached to E. Then we define

$$\Phi^{T/\kappa}(\gamma,f) = \lambda(E/F,\psi_F)\kappa'\left(\frac{\gamma_1-\gamma_2}{\gamma_1^\circ-\gamma_2^\circ}\right)\frac{|\gamma_1-\gamma_2|}{|\gamma_1\gamma_2|^{1/2}}\sum_{\delta\in\mathscr{E}(T)}\kappa(\delta)\Phi^\delta(\gamma,f)$$

where the order on the eigenvalues γ_1, γ_2 of γ and γ_1^0, γ_2^0 of γ° is prescribed by fixing a diagonalization of T. A different choice of ψ_F or γ° causes at most a sign change in the normalizing factor.

6. Transferring orbital integrals. The integrals $\Phi^{T/\kappa}(\ ,f)$ can be transferred to stable orbital integrals on $H = H(T,\kappa)$ in the following sense:

LEMMA. *If $f \in C_c^\infty(G(F))$ then there exists $f_H \in C_c^\infty(H(F))$ such that*

$$\Phi^{T/1}(\gamma, f_H) = \Phi^{T/\kappa}(\gamma, f)$$

and

(i) *if G is anisotropic and κ trivial (so that H is SL_2) then*

$$\Phi^{T'/1}(\ ,f_H) \equiv 0 \quad \text{for } T' \text{ split};$$

(ii) *if G is SL_2 and κ trivial (so that H is SL_2 again) then*

$$\Phi^{T'/1}(\ ,f_H) \equiv \Phi^{T'/1}(\ ,f)$$

for all T', provided that the Haar measures dt' are chosen consistently.

This is proved (in **[4]**) by a case-by-case argument. Note that if H is T then $\Phi^{T/1}(\ ,f_H)$ is just f_H itself.

For general (real) groups a formalism for transferring the $\Phi^{T/\kappa}(\ ,f)$ to H is developed in **[10]**. Some progress towards obtaining a result as above for any real group is made in **[9]** and **[10]**.

7. Stable distributions and a map. We define the space of stable distributions on $G(F)$ to be the closed subspace, with respect to simple convergence, of the space of all distributions on $G(F)$ generated by stable orbital integrals, that is, by the distributions $f \to \Phi^{T/1}(\gamma, f)$, where γ is a regular semisimple element of $G(F)$ and T denotes the torus containing γ.

The transfer of the $\Phi^{T/\kappa}(\ ,f)$ to H establishes a correspondence (f, f_H) between $C_c^\infty(G(F))$ and $C_c^\infty(H(F))$. Dual to this correspondence there is a well-defined map from the space of stable distributions on $H(F)$ to the space of invariant distributions on $G(F)$; that is, if Θ_H is a stable distribution on $H(F)$ then we may define a (conjugation-) invariant distribution Θ on $G(F)$ by the formula $\Theta(f) = \Theta_H(f_H)$, $f \in C_c^\infty(G(F))$. This map, $\Theta_H \to \Theta$, will be central to our study of L-indistinguishability.

We denote by χ_π the character of (the infinitesimal equivalence class of) an irreducible admissible representation of $G(F)$; we regard χ_π as a function on the regular semisimple elements of $G(F)$, using the same normalization of Haar measure as in §5. There is a simple way to determine whether χ_π, as distribution, is stable. Let $\tilde{G}$ be GL_2 in the case that G is SL_2, or the full multiplicative group of the underlying quaternion algebra in the case that G is anisotropic. Then a linear combination χ of characters is stable if and only if χ is invariant under $\tilde{G}(F)$ ... or (of relevance for generalizations (cf. **[9]**)) if and only if χ is invariant under $\mathfrak{A}(T)$, $T \subset G$.

8. Local L-packets and some stable characters. We assume that the Langlands correspondence has been proved for G; in fact enough has been proved in each of the cases being considered. In the following definitions we also allow G to be a torus. We denote by $\Phi(G)$ the set of equivalence classes of admissible homomorphisms of W'_F into ${}^LG^\circ \rtimes W'_F$ (notation and definitions as in [1]). To each $\{\varphi\} \in \Phi(G)$ there is attached a finite collection $\Pi_{\{\varphi\}}$ of irreducible admissible representations of $G(F)$, an L-packet. Two irreducible admissible representations of $G(F)$ are said to be L-indistinguishable if they belong to the same L-packet. In the case that G is an inner from of SL_2 there is a simpler definition: π_1 and π_2 are L-indistinguishable if and only if there exists $g \in \tilde{G}(F)$ such that π_2 is equivalent to $\pi_1 \circ \operatorname{ad} g$. We set $\chi_{\{\varphi\}} = \sum_{\pi \in \Pi_{\{\varphi\}}} \chi_\pi$. Clearly:

PROPOSITION. *$\chi_{\{\varphi\}}$ is stable.*

Since the Langlands correspondence has been proved for any (connected reductive) real group [6] we can define characters $\chi_{\{\varphi\}}$ in that case. In general $\chi_{\{\varphi\}}$ need not be stable; however if the constituents of $\Pi_{\{\varphi\}}$ are tempered (cf. [3]) then $\chi_{\{\varphi\}}$ is stable [9].

From §12 on we will not need to distinguish in notation between an admissible homomorphism $\varphi\colon W'_F \to {}^LG^\circ \rtimes W'_F$ and its equivalence class; we will then denote both by φ and write Π_φ, χ_φ, etc.

9. Character identities (introduction). We return to the map of stable distributions on $H(F)$ to invariant distributions on $G(F)$. If $\{\varphi_H\} \in \Phi(H)$ then, as we have observed, $\chi_{\{\varphi_H\}}$ is stable. Its image in invariant distributions on $G(F)$ is represented by some function χ on the regular semisimple elements of $G(F)$. This function χ is computed by the Weyl integration formula. We write: $\chi_{\{\varphi_H\}} \approx \chi$.

Recall the embedding ι_H of ${}^LH$ in ${}^LG$ (§4). We could have used W'_F in place of $W_{K/F}$ in defining ${}^LG$, ${}^LH$, and ι_H. With these modifications, suppose that φ_H is an admissible homomorphism of W'_F into ${}^LH$. Then $\varphi = \iota_H \circ \varphi_H$ maps W'_F to ${}^LG$. Suppose that φ is admissible; recall that (local) admissibility imposes the condition that the image of φ lie only in parabolic subgroups of ${}^LG$ which are "relevant to G" (cf. [1]). Then we say that $\{\varphi\} \in \Phi(G)$ factors through $\{\varphi_H\} \in \Phi(H)$.

Suppose that $\{\varphi\}$ factors through $\{\varphi_H\}$. Then linear combinations of the characters of the representations in $\Pi_{\{\varphi\}}$ make natural candidates for χ ($\approx \chi_{\{\varphi_H\}}$).

10. "$S^\circ \backslash S$". We introduce a useful group. It is easier to work with a homomorphism $\varphi\colon W'_F \to {}^LG$ rather than an equivalence class $\{\varphi\}$. We exclude the case of sp(2) [1] and corresponding special representation of $G(F)$, and consider just an admissible homomorphism $\varphi\colon W_{K/F} \to {}^LG$ where K, ${}^LG$ (and ${}^LH$, ι_H) are as earlier.

We define S_φ to be the centralizer in ${}^LG^\circ$ of the image of φ; S°_φ will be the connected component of the identity in S_φ. If φ' is equivalent to φ then there exists $g \in {}^LG^\circ$ such that $S_{\varphi'} = gS_\varphi g^{-1}$ and $S^\circ_{\varphi'} = gS^\circ_\varphi g^{-1}$.

Suppose that $\varphi = \iota_H \circ \varphi_H$ where φ_H is an admissible homomorphism of $W_{K/F}$ into ${}^LH$ and $H = H(T, \kappa)$. We regard κ as an element of ${}^LT^\circ$. Recall that ${}^LH^\circ$ is the connected component of the identity in the centralizer of κ in ${}^LG^\circ$. By definition (§3), σ_H fixes κ, $\sigma \in \mathfrak{G}_{K/F}$. It follows then that κ lies in the center of the image of ${}^LH$ in ${}^LG$. Therefore κ centralizes the image of $\varphi_H(W_{K/F})$ in ${}^LG$; that is, κ centralizes

$\varphi(W_{K/F})$. We have then that $\kappa \in S_\varphi$. We define $s_\varphi(\kappa)$, or just $s(\kappa)$ when φ is understood, to be the coset of κ in $S_\varphi^\circ \backslash S_\varphi$. Recall that this quotient appeared in [3].

Suppose that φ' is equivalent to φ and that $\varphi' = \iota_H \circ \varphi'_H$ where $\varphi'_H : W_{K/F} \to {}^LH$ and $H = H(T', \kappa')$. Then $\kappa' \in S_{\varphi'}$. Write φ' as $\operatorname{ad} g \circ \varphi$, $g \in {}^LG^\circ$. Then $g^{-1}\kappa' g \in S_\varphi$. We define $s_\varphi(\kappa')$ to be the coset of $g^{-1}\kappa' g$ in $S_\varphi^\circ \backslash S_\varphi$. We will see that $S_\varphi^\circ \backslash S_\varphi$ is abelian. Therefore $s_\varphi(\kappa')$ is independent of the choice for g.

We continue with the same φ, φ'. The conjugation ad g induces an isomorphism between $S_\varphi^\circ \backslash S_\varphi$ and $S_{\varphi'}^\circ \backslash S_{\varphi'}$ which carries $s_\varphi(\kappa)$ to $s_{\varphi'}(\kappa)$. Because both groups are abelian this isomorphism is independent of the choice of g in the equivalence $\varphi' = \operatorname{ad} g \circ \varphi$. We may therefore regard $S_\varphi^\circ \backslash S_\varphi$ and the elements $s_\varphi(\kappa)$ as attached to $\{\varphi\}$.

11. Calculations. We compute explicitly $S_\varphi^\circ \backslash S_\varphi$ and the elements $s(\kappa) = s_\varphi(\kappa)$. We need take just one homomorphism $\varphi \colon W_{K/F} \to {}^LG$ from each equivalence class. If we write $\varphi(w)$ as $\varphi_1(w) \times w$, $w \in W_{K/F}$, then the homomorphism $\varphi_1 \colon W_{K/F} \to {}^LG^\circ = \mathrm{PGL}_2(\boldsymbol{C})$ lifts to a two dimensional representation $\tilde\varphi_1$ of $W_{K/F}$ [7, Lemma 3].

(i) *Suppose that $\tilde\varphi_1$ is reducible.*

Then $\tilde\varphi_1$ factors through $\tau \colon W_{K/F} \to F^\times$ and is defined by a pair (μ, ν) of quasi-characters on $F^\times$. We take

$$\tilde\varphi_1(w) = \begin{pmatrix} \mu(\tau(w)) & 0 \\ 0 & \nu(\tau(w)) \end{pmatrix}, \qquad w \in W_{K/F}.$$

If $(\mu/\nu)^2 \neq 1$ then $S_\varphi = S_\varphi^\circ = {}^LT^\circ$ and if $\mu = \nu$ then $S_\varphi = S_\varphi^\circ = {}^LG^\circ$. However, if μ/ν has order two then $S_\varphi = {}^LT^\circ \rtimes \langle 1, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})_* \rangle$; recall that $(\ )_*$ denotes the image of $(\ )$ in $\mathrm{PGL}_2(\boldsymbol{C})$. Hence $S_\varphi^\circ \backslash S_\varphi = \boldsymbol{Z}/(2)$.

Whatever μ/ν, φ factors through ${}^LT$, T a split torus. The corresponding κ $((\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix})_*$, some $z \neq 1)$ lies in S_φ° and $s(\kappa)$ is trivial. Note that (this or any) φ factors through H when $H = H(\ , 1)$; $s(1)$ is trivial also.

Suppose that μ/ν has order two. In (ii) we will show that a homomorphism equivalent to φ factors through ${}^LT$, where T is a torus defined by the quadratic extension E of F attached to μ/ν, and that the associated $s(\kappa) = s_\varphi(\kappa)$ is the coset of $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})_*$.

(ii) *Suppose that $\tilde\varphi_1$ factors through a representation $\tilde\varphi_2 = \mathrm{Ind}(W_{E/F}, E^\times, \theta)$ of $W_{E/F}$ where $E \subset K$ is a quadratic extension of F and θ is a quasi-character on $E^\times$.*

Then φ_1 factors through φ_2, the projective representation defined by $\tilde\varphi_2$. Let σ° be the nontrivial element of $\mathfrak{G}_{E/F}$ and define $\bar\theta(x) = \theta(x^{\sigma^\circ})$, $x \in E^\times$.

We realize $W_{E/F}$ explicitly as $\{x \times \rho; x \in E^\times, \rho \in \{1, \sigma^\circ\}\}$, with multiplication rule

$$(x \times \rho)(x' \times \rho') = x(x')^\rho a_{\rho,\rho'} \times \rho\rho'$$

where $a_{\rho,\rho'} = 1$ unless $\rho = \rho' = \sigma^\circ$ and $a_{\sigma^\circ,\sigma^\circ}$ is some (chosen) element α of $F^\times - Nm_{E/F}E^\times$. Then we may assume that

$$\tilde\varphi_2(x \times 1) = \begin{pmatrix} \theta(x) & 0 \\ 0 & \bar\theta(x) \end{pmatrix}, \qquad x \in E^\times,$$

and

$$\tilde\varphi_2(1 \times \sigma^\circ) = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$$

where z is a chosen square root of $\theta(\alpha)$. Thus

$$\tilde{\varphi}_2(x \times 1) = \begin{pmatrix} \theta(x) & 0 \\ 0 & \bar{\theta}(x) \end{pmatrix}_*, \qquad x \in E^\times,$$

and

$$\varphi_2(1 \times \sigma^\circ) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*.$$

Let T be a torus attached to E. Then φ factors through ${}^LT$; indeed we chose the embedding ι_T of ${}^LT$ in ${}^LG$ (§4) so as to insure this. Recall that $T = H(T, \kappa)$ where κ, as element of ${}^LT^\circ$, is $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$; $s(\kappa)$ is the coset of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$ in $S_\varphi^\circ / S_\varphi$.

To compute S_φ, suppose first that $(\theta/\bar{\theta})^2 \neq 1$. Then

$$S_\varphi = \left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*\right\}$$

so that $S_\varphi^\circ \backslash S_\varphi = \mathbf{Z}/(2)$; we have already recovered the nontrivial element of $S_\varphi^\circ \backslash S_\varphi$ as an $s(\kappa)$.

Suppose now that $\theta/\bar{\theta}$ has order two. Then S_φ coincides with $\varphi_1(W_{K/F})$; that is,

$$S_\varphi = \left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_*\right\}.$$

Therefore $S_\varphi^\circ \backslash S_\varphi = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$. We have recovered only $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$ (and 1) as an $s(\kappa)$. To recover the remaining elements of $S_\varphi^\circ \backslash S_\varphi$ we recall homomorphisms φ', φ'': $W_{K/F} \to {}^LG$ which are equivalent to φ and factor through other ${}^LH$.

Let E_0 be the quadratic extension of E defined by $\theta/\bar{\theta}$. Then $F^\times / Nm_{E_0/F} E_0^\times = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ so that there are two distinct quadratic extensions E' and E'' of F which are distinct from E and contained in E_0. We pick a quasi-character θ' on $(E')^\times$ such that $\theta' \circ Nm_{E_0/E'} = \theta \circ Nm_{E_0/E}$ and $\theta'/\bar{\theta}'$ is the quadratic character attached to E_0/E' ($\bar{\theta}'(x) = \theta'(x^{\sigma'})$, $1 \neq \sigma' \in \mathfrak{G}_{E'/F}$); we pick θ'' similarly. Then $\tilde{\varphi}_2' = \mathrm{Ind}(W_{E'/F}, (E')^\times, \theta')$ and $\tilde{\varphi}_2'' = \mathrm{Ind}(W_{E''/F}, (E'')^\times, \theta'')$ define representations $\tilde{\varphi}_1'$, $\tilde{\varphi}_1''$ of $W_{K/F}$, each equivalent to $\tilde{\varphi}_1$, and hence homomorphisms $\varphi', \varphi'' : W_{K/F} \to {}^LG$, each equivalent to φ. We define φ_1', φ_2' and φ_1'', φ_2'' as we did φ_1, φ_2; we realize φ_2' and φ_2'' as we did φ_2. Then $\varphi_1(W_{K/F})$, $\varphi_1'(W_{K/F})$, $\varphi_1''(W_{K/F})$, S_φ, $S_{\varphi'}$ and $S_{\varphi''}$ all coincide and equal

$$\left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_*\right\}.$$

As with φ earlier, φ' (... φ'') factors through ${}^L(T')$ (... ${}^L(T'')$), where T' (...T'') is some torus attached to E' (... E''). Suppose $T' = H(T', \kappa')$ and $T'' = H(T'', \kappa'')$. Then, writing $\varphi = \mathrm{ad}\, g' \circ \varphi' = \mathrm{ad}\, g'' \circ \varphi''$, $g', g'' \in {}^LG^\circ$, we have

$$s(\kappa') = s_\varphi(\kappa') = g' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_* g'^{-1}$$

and

$$s(\kappa'') = s_\varphi(\kappa'') = g'' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_* g''^{-1}.$$

An elementary argument shows that none of the conjugations ad g', ad g'', ad$(g')^{-1}g''$ may fix $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)_*$ and so we conclude that

$$\{s(\kappa'), s(\kappa'')\} = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_*, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}_*\right\}.$$

Thus we have recovered each element of $S_\varphi^\circ\backslash S_\varphi$ as an "$s(\kappa)$".

We have one further case to consider: $\theta = \bar{\theta}$. We pick a quasi-character μ on $F^\times$ such that $\theta = \mu \circ Nm_{E/F}$. Then φ is equivalent to the homomorphism of type (i) above attached to the pair $(\mu, \zeta\mu)$ where ζ is the quadratic character of $F^\times$ defined by the extension E/F. We now call that homomorphsim φ'. We had that $S_{\varphi'}^\circ\backslash S_{\varphi'}$ has two elements and that $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)_*$ is a representative for the nontrivial coset. We now recover this coset as an $s_{\varphi'}(\kappa)$. Let T be a torus attached to E. Then φ factors through ${}^LT$ and the associated κ is $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)_*$. If $\varphi' = \text{ad}\, g \circ \varphi$, $g \in {}^LG^\circ$, then

$$g\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}_* g^{-1} \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_* \qquad (\text{mod}\ {}^LT^\circ),$$

and we are done.

(iii) *Suppose that $\tilde{\varphi}_1$ is of tetrahedral or octahedral type.*

Then $S_\varphi = S_\varphi^\circ = 1$. This is a straightforward exercise.

In summary:

PROPOSITION. (1) *$S_\varphi^\circ\backslash S_\varphi$ is one of* 1, $\boldsymbol{Z}/(2)$ *or* $\boldsymbol{Z}/(2) \oplus \boldsymbol{Z}/(2)$.

(2) *For each element s of $S_\varphi^\circ\backslash S_\varphi$ there is a pair (T, κ) such that $s = s(\kappa)$.*

12. Character identities (continued). From now on we will not distinguish in notation between an admissible homomorphism φ: $W_{K/F} \to {}^LG$ and its equivalence class; that is, we denote both by φ ... as we have noted, $S_\varphi^\circ\backslash S_\varphi$ can be regarded as attached to the equivalence class.

Suppose that φ factors through φ_H, where $H = H(T, \kappa)$. Let $s = s(\kappa)$. Then, in the notation of §§8, 9:

LEMMA. *There exist integers $\langle s, \pi\rangle$, $\pi \in \Pi_\varphi$, such that*

$$\chi_{\varphi_H} \approx \sum_{\pi \in \Pi_\varphi} \langle s, \pi\rangle\, \chi_\pi.$$

The proof [4] is again a case-by-case argument. Here is a summary of the explicit results.

(A) $G = \mathrm{SL}_2$. We consider φ as in (i), (ii), (iii) of the last section.

(i) If $(\mu/\nu)^2 \neq 1$ or if $\mu = \nu$ then Π_φ contains one element which we denote by π; $S_\varphi^\circ\backslash S_\varphi = 1$ and $\langle 1, \pi\rangle = 1$. If μ/ν has order two then Π_φ has two elements which we denote by π_1 and π_2. Recall that $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2)$; $\langle s, \pi_1\rangle$ and $\langle s, \pi_2\rangle$ are the two characters in s. These characters depend on the choices we made in normalizing orbital integrals (§5); that is, for some choices $\langle\ , \pi_1\rangle$ is the trivial character and for others $\langle\ , \pi_2\rangle$ is the trivial one.

(ii) We have already considered the case $\theta = \bar{\theta}$. If $\theta/\bar{\theta}$ is not of order two then Π_φ has two elements and $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2)$; the result is as in (i).

Suppose that $\theta/\bar{\theta}$ has order two. Then we had that $S_\varphi^\circ\backslash S_\varphi$ is $\boldsymbol{Z}/(2) \oplus \boldsymbol{Z}(2)$; Π_φ has four elements, too. Suppose $\Pi_\varphi = \{\pi_i, i = 1, \cdots, 4\}$. Then we may (and do) normalize orbital integrals so that the $\langle\ , \pi_i\rangle$ are the four characters on $S_\varphi^\circ\backslash S_\varphi$.

(iii) Here Π_φ has one element, say π; $S_\varphi^\circ \backslash S_\varphi = 1$ and $\langle 1, \pi \rangle = 1$.

(B) G *anisotropic*. We have only to consider φ as in (ii) with $\theta \neq \bar{\theta}$, and (iii).

Suppose $F = \boldsymbol{R}$. Then $S_\varphi^\circ \backslash S_\varphi = \boldsymbol{Z}/(2)$ since we exclude (iii) also. However Π_φ has only one element, say π; $\langle s, \pi \rangle$ is a character in s, trivial or nontrivial according to our choice of normalizing factors for orbital integrals.

Suppose that F is nonarchimedean. Then Π_φ has the same number of elements as the corresponding L-packet in SL_2, and the result is as there, *except* when φ is of type (ii) with $\theta/\bar{\theta}$ of order two. Then the corresponding packet in SL_2 has four elements; Π_φ has only one element, say π. We must set $\langle 1, \pi \rangle = 2$ and $\langle s, \pi \rangle = 0$, $s \neq 1$; in particular, $\langle \ , \pi \rangle$ is not a character.

As for generalizing these identities we will report some progress for real groups in a forthcoming paper; in the case that κ is trivial, so that H is a quasi-split inner form of G, an appropriate identity is known provided that the constituents of Π_φ are tempered [9].

13. Structure of local L-packets. From the results of the last section we conclude:

PROPOSITION. *If G is the quasi-split form* (SL_2) *then the pairing* $\langle \ , \ \rangle$ *identifies* Π_φ (*noncanonically*) *as the dual group of* $S_\varphi^\circ \backslash S_\varphi$.

If G is anisotropic we remark only the existence of the functions $\langle \ , \pi \rangle$, $\pi \in \Pi_\varphi$. With some qualifications, we can expect an analogue of the proposition for a general quasi-split real group (cf. [3]).

14. A global application. Suppose now that F is a global field and that G is SL_2. There are analogues for the groups H of the local case; again H may be a maximal torus in G, defined over F, or G itself.

We assume that K is some large but finite Galois extension of F and consider just homomorphisms φ: $W_{K/F} \to {}^L G$ defined by representations of the form $\mathrm{Ind}(W_{K/F}, W_{K/E}, \theta)$, where E is a quadratic extension of F contained in K and θ is a grossencharacter for E not factoring through $Nm_{E/F}$. If T is a torus attached to E then φ factors through ${}^L T$ (as earlier, provided that ${}^L T$ is correctly embedded in ${}^L G$). We define Π_φ to be the set of representations $\pi = \bigotimes_v \pi_v$ of $G(\boldsymbol{A})$ which are irreducible, admissible (as in [1]) and such that $\pi_v \in \Pi_{\varphi_v}$ for each place v. The set Π_φ is an L-packet *in the following sense.*

We define two irreducible admissible representations $\pi^1 = \bigotimes_v \pi_v^1$ and $\pi^2 = \bigotimes_v \pi_v^2$ of $G(\boldsymbol{A})$ to be L-indistinguishable if for all places v, π_v^1 and π_v^2 are L-indistinguishable and for almost all v, π_v^1 and π_v^2 are equivalent. An L-packet is an equivalence class for this relation.

We have singled out the packets Π_φ for the following reason. As is proved in [4], two representations from such a packet may appear with different multiplicities in the space of cusp forms for G. One may appear, with multiplicity one, and the other not appear. For the remaining L-packets the members do appear with equal multiplicity (conjectured to be either zero or one).

In [4] there is a formula for the multiplicity with which a representation from Π_φ appears in cusp forms; it is this we wish to discuss.

As in the local case, we define S_φ to be the centralizer of $\varphi(W_{K/F})$ in ${}^L G^\circ$ and S_φ° to be the connected component of the identity in S_φ. As in (ii) of §11 we have that either $S_\varphi^\circ \backslash S_\varphi = \boldsymbol{Z}/(2)$ or $S_\varphi^\circ \backslash S_\varphi = \boldsymbol{Z}/(2) \oplus \boldsymbol{Z}/(2)$. For each place v, S_φ embeds naturally in S_{φ_v} and S_φ° in $S_{\varphi_v}^\circ$. There is then a natural map of $S_\varphi^\circ \backslash S_\varphi$ into $S_{\varphi_v}^\circ \backslash S_{\varphi_v}$. In notation we will not distinguish between an element of $S_\varphi^\circ \backslash S_\varphi$ and its image in $S_{\varphi_v}^\circ \backslash S_{\varphi_v}$.

Recall that the local characters $\langle\ ,\ \pi_v\rangle$ depend on the choices we made when normalizing orbital integrals (§§5, 12). We chose a nontrivial additive character ψ_{F_v} on F_v and regular element $\gamma^\circ = \gamma_v^\circ$ of $T(F_v)$ for each torus T anisotropic over F_v. Instead, we now choose a nontrivial additive character ψ on $F\backslash A$ and for each torus T anisotropic over F a regular element γ° in $T(F)$. At each place v where T does not split we use ψ_v and γ° to specify $\Phi^{T/\kappa}(\ ,\ )$, κ nontrivial.

Fix $s \in S_\varphi^\circ \backslash S_\varphi$ and $\pi \in \Pi_\varphi$. Then we have $\langle s, \pi_v\rangle = 1$ for almost all v. Therefore we may define $\langle s, \pi\rangle = \Pi_v \langle s, \pi_v\rangle$. Then [4]:

PROPOSITION. (i) *$\langle s, \pi\rangle$ is independent of the choices made for ψ and γ°.*
(ii) *$\langle\ ,\ \rangle$ induces a (canonical) surjection of Π_φ onto the dual group of $S_\varphi^\circ \backslash S_\varphi$.*

Also:

THEOREM. *The multiplicity with which $\pi \in \Pi_\varphi$ appears in the space of cusp forms is*:

$$\frac{1}{[S_\varphi^\circ \backslash S_\varphi]} \sum_{s \in S_\varphi^\circ \backslash S_\varphi} \langle s, \pi\rangle .$$

That is, those π for which $\langle\ ,\ \pi\rangle$ is the trivial character appear with multiplicity one and those π for which $\langle\ ,\ \pi\rangle$ is nontrivial do not appear.

15. Afterword. More generally, and of relevance also for Shimura varieties (cf. [8]), we can consider inner forms of a group G such that $\mathrm{Res}_{E/F}\mathrm{SL}_2 \subset G \subset \mathrm{Res}_{E/F}\mathrm{GL}_2$ with E some finite Galois extension of F, and develop a local and a global theory in the same way. The analogous multiplicity formula is [4]:

$$\frac{d_\varphi}{[S_\varphi^\circ \backslash S_\varphi]} \sum_{s \in S_\varphi^\circ \backslash S_\varphi} \langle s, \pi\rangle$$

where d_φ is defined as follows. If $\varphi, \varphi'\colon W_F \to {}^LG$ are admissible homomorphisms (cf. [1, §16]) call φ and φ' locally equivalent everywhere if φ_v is equivalent to φ_v' for each place v. Call φ and φ' weakly globally equivalent if φ' is equivalent to $\omega\varphi$ where ω is a continuous 1-cocycle of W_F with values in the center of ${}^LG^\circ$ such that the restriction of ω to each local group is trivial. Then d_φ is the number of weak global equivalence classes in the everywhere local equivalence class of φ.

In [8] L-indistinguishability plays a role in relating the zeta-functions of certain Shimura varieties to automorphic L-functions. Briefly, in the case discussed in [2], where G is the full multiplicative group of some quaternion algebra over a totally real field and there is no L-indistinguishability, functions $L(s, \pi, \rho)$ appear (π an automorphic representation of $G(A)$, ρ a certain representation of ${}^LG$, fixed as in that lecture). In the general case of [8], where G is a subgroup of the full multiplicative group, we must consider at least some of the L-packets Π_φ where different multiplicities in cusp (...automorphic) forms occur. Then $\varphi\colon W_F \to {}^LG$ factors through $\iota_T\colon {}^LT \to {}^LG$, with T a nonsplit torus of G. If $\varphi = \iota_T \circ \varphi_T$, $\pi_T \in \Pi_{\varphi_T}$ and $\rho_T = \rho \cdot \iota_T$ then $L(s, \pi, \rho) = L(s, \pi_T, \rho_T)$. There is a natural decomposition $\rho_T = \rho_T^1 \oplus \rho_T^2$, where ρ_T^1, ρ_T^2 may be reducible, and it is the functions $L(s, \pi_T, \rho_T^i)$ which are relevant.

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