

BASE CHANGE AND A MATCHING THEOREM FOR
REAL GROUPS

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§1. INTRODUCTION.

In the problem of "base change" for automorphic representations (cf. [1]) of (even low-dimensional) groups other than GL_n , a new difficulty is introduced by the instability of conjugacy and twisted-conjugacy. Let G be a simply-connected semisimple quasi-split linear algebraic group over a number field F and \tilde{G} be the group obtained from G by restriction of scalars from K to F , K being some cyclic extension of F , of prime degree over F , contained in the algebraic closure \bar{F} of F . Then two elements of $G(F)$ may be conjugate in $G(\bar{F})$ (i.e. "stably" conjugate) but not conjugate in $G(F)$, and two elements of $\tilde{G}(F)$ may be twisted-conjugate in $\tilde{G}(\bar{F})$ (i.e. "stably" twisted-conjugate) but not twisted-conjugate in $\tilde{G}(F)$. If F is replaced by a local field (of characteristic zero) then a similar situation applies. In order to get some indication, at least for some low-dimensional groups, of how the trace formula for G and the twisted trace formula for

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\tilde{G} might be applied to "base-change", we consider a local problem, that of matching linear combinations of the invariant integrals of a function in $C_c^\infty(\tilde{G}(F))$ along the twisted-conjugacy classes in certain stable twisted-conjugacy classes with integrals which arise in the ordinary trace formula for G . Our results concern only the case $F = \mathbb{R}$ and $K = \mathbb{C}$ (one apparent difficulty here is not present for F nonarchimedean (cf. Section 3)), and we are less restrictive in our assumptions on G .

To be precise about the matching we need to review basic facts about \tilde{G} , (stable) twisted-conjugacy and the norm map from stable twisted-conjugacy classes in $\tilde{G}(F)$ to the stable conjugacy classes in $G(F)$; this will be done in Section 2, where we also note results special to the case $F = \mathbb{R}$. In Section 3, we will return to the matching theorem to be proved, and continue our introductory discussion.

From the point of view of analysis on real groups also, matching theorems are useful. In the case of L-indistinguishability, for example, a matching theorem for unstable orbital integrals, in conjunction with the dual character liftings, reduces the problem of existence and explicit computation of the Fourier transform of Harish-Chandra's transform F_f to the analogous problem for the stabilized F_f transform. The

results of L-indistinguishability, in fact, provide a guide for what we shall prove. We expect also an analogue (yet to be made precise) of the functoriality principle for L-indistinguishability. In the present paper, however, our concern will be just a "generic" matching theorem, as described in Section 3.

§2. THE NORM MAP.

(2.1) The group \tilde{G} , twisted-conjugacy, etc.

Let F be a field of characteristic zero, with algebraic closure \bar{F} . Let $K \subset \bar{F}$ be a cyclic extension of F , of degree l over F . Fix a generator σ of $\text{Gal}(K/F)$. Let G be a connected reductive algebraic group defined over F , and \tilde{G} denote the product of G with itself l times, regarded as group over F . Let α be the automorphism

$$(x_1, x_2, \dots, x_l) \rightarrow (x_l, x_1, x_2, \dots, x_{l-1})$$

of $\tilde{G}(\bar{F})$; α is defined over F .

The group $\text{Gal}(K/F)$ acts on $\tilde{G}(\bar{F})$ by

$$\sigma^r \cdot x = \alpha^r(x), \quad x \in \tilde{G}(\bar{F}), \quad r = 0, 1, \dots, l-1,$$

and $\text{Gal}(\bar{F}/F)$ acts also, through the projection

$\text{Gal}(\bar{F}/F) \rightarrow \text{Gal}(K/F)$. This action of $\text{Gal}(\bar{F}/F)$ defines an

element of $H^1(\text{Gal}(\bar{F}/F), \text{Aut}(\tilde{G}(\bar{F})))$, and hence a group

$\tilde{G} = \text{Res}_F^K G$ defined over F . We realize $\tilde{G}(\bar{F})$ as $G(\bar{F}) \times \dots \times G(\bar{F})$

(l factors), i. e. as $\tilde{G}(\bar{F})$, with Galois action:

$$\tau(x_1, x_2, \dots, x_l) = \alpha^{-r}((\tau x_1, \tau x_2, \dots, \tau x_l))$$

if $\tau \in \text{Gal}(\bar{F}/F)$ and $\tau|_K = \sigma^r$. Then $\tilde{G}(K) = G(K) \times \dots \times G(K)$

(l factors) and $\tilde{G}(F) = \{\delta = (x, \sigma^{-1}x, \sigma^{-2}x, \dots, \sigma^{(l-1)}x); x \in G(K)\}$
 (when x and δ are so related, we write $x = x(\delta)$ and
 $\delta = \delta(x)$).

Both $\tilde{G}(\bar{F})$ and $\tilde{G}(\bar{F})$ contain $D(\bar{F}) = \{(g, g, \dots, g) : g \in G(\bar{F})\}$
 as subgroup defined over F . Also the map $g \rightarrow (g, g, \dots, g)$
 of $G(\bar{F})$ into $D(\bar{F})$ is an isomorphism over F .

In the following, we will identify $G(L)$ with $D(L)$,
 $F \subseteq L \subseteq \bar{F}$. Thus we have $\tilde{G}(F) \cap \tilde{G}(F) = G(F)$ and
 $G(L) = \{\delta \in \tilde{G}(L) : \alpha(\delta) = \delta\} = \{\delta \in \tilde{G}(L) : \alpha(\delta) = \delta\}$.

In the case that G is abelian, there are norm maps as
 follows:

$\tilde{G}(F)$	$\tilde{G}(F)$
$\delta = \delta(x), x \in G(K)$	$\delta = (x_1, x_2, \dots, x_l), x_j \in G(F)$
$\downarrow \text{Nm}_{\tilde{G}}$	$\downarrow \text{Nm}_{\tilde{G}}$
$\begin{aligned} &\delta \alpha(\delta) \alpha^2(\delta) \dots \alpha^{l-1}(\delta) \\ &= x \sigma(x) \sigma^2(x) \dots \sigma^{l-1}(x) \end{aligned}$	$\begin{aligned} &\delta \alpha(\delta) \alpha^2(\delta) \dots \alpha^{l-1}(\delta) \\ &= x_1 x_2 \dots x_l \end{aligned}$
$\tilde{G}(F) \cap \tilde{G}(F) = G(F)$	

In this paper we will ignore the group \tilde{G} . We now describe a
 norm for \tilde{G} under the assumption that G is simply connected,
semisimple and quasi-split over F .

Definition 2.1.1: For $\delta \in \tilde{G}(\bar{F})$, set $N(\delta) = \delta \alpha(\delta) \alpha^2(\delta) \dots \alpha^{l-1}(\delta)$;
for $x \in G(K)$, set $N(x) = x \sigma(x) \sigma^2(x) \dots \sigma^{l-1}(x)$.

It is immediate that if $\delta \in \tilde{G}(F)$ then $N(\delta) \in \tilde{G}(F)$, that $N(x) \in G(K)$ and that $x(N\delta) = N(x(\delta))$, $\delta \in \tilde{G}(F)$.

Lemma 2.1.2: Let $\delta = \delta(x) \in \tilde{G}(F)$. Then δ is a regular semi-simple element of \tilde{G} if and only if x is a regular semi-simple element of G .

Proof: It is clear that δ is semisimple if and only if x is semisimple. Both G and \tilde{G} are simply-connected, so that G_x , the centralizer of x in G , and \tilde{G}_δ , the centralizer of δ in \tilde{G} , are connected reductive groups. We have only to show that either both are abelian or neither are. Since G_x is defined over K , we can form $\text{Res}_F^K G_x = G_1$, a group over F . It is easily verified that $\tilde{G}_\delta(F) = G_1(F)$, if G_1 is identified as a subgroup of \tilde{G} in the obvious way. The lemma now follows easily.

Definition 2.1.3: The elements δ_1, δ_2 of $\tilde{G}(F)$ are twisted-conjugate if $\delta_2 = g^{-1}\delta_1\alpha(g)$, for some $g \in \tilde{G}(F)$, and are stably twisted-conjugate if $\delta_2 = g^{-1}\delta_1\alpha(g)$, for some $g \in \tilde{G}(\bar{F})$.

We will investigate the effect of N on those elements δ of $\tilde{G}(F)$ for which $N\delta$ is regular semisimple. Let \mathfrak{S} be the set of all such elements δ ; since $N(g^{-1}\delta\alpha(g)) = g^{-1}N\delta g$, \mathfrak{S} is closed under stable twisted-conjugacy. Another description of \mathfrak{S} will be given in Lemma 2.3.1.

(2.2) Lemmas.

Lemma 2.2.1: Let $\delta \in \mathcal{J}$. Then there exists $g \in \tilde{G}(\bar{F})$ such that $g^{-1}N\delta g$ belongs to $G(F)$.

Proof: Let $x = x(\delta)$. Then the conjugacy class of $N(x)$ in $G(\bar{F})$ is defined over F , and so contains an F -rational point, say $g_0^{-1}N_x g_0$ ([15]). Let $g =$

$$(g_0, \sigma^{-1}(x)g_0, \sigma^{-2}(x)\sigma^{-1}(x)g_0, \dots, \sigma^{-(l-1)}(x) \dots \sigma^{-1}(x)g_0).$$

Then $g^{-1}N\delta g = g_0^{-1}N_x g_0$, and the lemma is proved.

Lemma 2.2.2: Let $\delta \in \mathcal{J}$. Suppose that both $\gamma_1 = g_1^{-1}N\delta g_1$ and $\gamma_2 = g_2^{-1}N\delta g_2$ belong to $G(F)$. Then γ_1 and γ_2 are stably conjugate in G .

Proof: Since G is simply-connected and semi-simple we have only to show that there exists $h_0 \in G(\bar{F})$ such that $\gamma_2 = h_0^{-1}\gamma_1 h_0$. Let $h = g_1^{-1}g_2$. Then $\gamma_2 = h^{-1}\gamma_1 h$, so that $h^{-1}\gamma_1 h = \alpha(h^{-1}\gamma_1 h) = \alpha(h^{-1})\gamma_1\alpha(h)$. Thus $h\alpha(h^{-1})$ lies in $\tilde{T}_1 = \text{Res}_F^K T_1$, where T_1 is that maximal torus in G which contains γ_1 . Let $t = h\alpha(h^{-1})$. Then $N(t) = 1$. A straightforward calculation shows that $t = u\alpha(u^{-1})$, for some $u \in \tilde{T}$. Then $u^{-1}h = \alpha(u^{-1}h)$, so that $u^{-1}h = h_0$ belongs to $G(\bar{F})$. Since $\gamma_2 = h_0^{-1}\gamma_1 h_0$, the lemma is proved.

Lemma 2.2.3: Let $\delta_1, \delta_2 \in \mathcal{J}$, and suppose that $g_1^{-1}N\delta_1g_1$ and $g_2^{-1}N\delta_2g_2$ belong to $G(F)$ and are stably conjugate in G . Then δ_1, δ_2 are stably twisted-conjugate in \tilde{G} .

Proof: Let $\gamma_1 = g_1^{-1}N\delta_1g_1$, $\gamma_2 = g_2^{-1}N\delta_2g_2$ and $h_0 \in G(\bar{F})$ be such that $\gamma_2 = h_0^{-1}\gamma_1h_0$. Then $\gamma_2 = N(g_2^{-1}\delta_2\alpha(g_2)) = N(h_0^{-1}g_1^{-1}\delta_1\alpha(g_1h_0))$. It is sufficient to prove that there exists $g_3 \in \tilde{G}(\bar{F})$ such that $g_2^{-1}\delta_2\alpha(g_2) = g_3^{-1}(h_0^{-1}g_1^{-1}\delta_1\alpha(g_1h_0))\alpha(g_3)$. Let T_2 be the maximal torus in G containing γ_2 , and $\tilde{T}_2 = \text{Res}_F^K T_2$. Then since $\alpha(\gamma_2) = \gamma_2$, both $g_2^{-1}\delta_2\alpha(g_2)$ and $h_0^{-1}g_1^{-1}\delta_1\alpha(g_1h_0)$ belong to $\tilde{T}_2(\bar{F})$. A straightforward calculation in \tilde{T}_2 shows the existence of the desired g_3 , and so the lemma is proved.

Lemma 2.2.4: Let $\delta \in \mathcal{J}$. Assume that δ has the following property:

(2.2.5) if $g^{-1}N\delta g \in G(F)$ then the maximal torus in G containing $g^{-1}N\delta g$ splits over K .

Then there exists $\delta' \in \tilde{G}(F)$ such that:

- 1) δ' is twisted-conjugate to δ and
- 2) $N\delta' \in G(F)$.

Proof: Let $x = x(\delta)$. Choose $g_0 \in G(\bar{F})$ and $g \in \tilde{G}(\bar{F})$ as in

the proof of Lemma 2.2.1; $g_0^{-1}Nx g_0 \in G(F)$ and $g_0^{-1}Nx g_0 = g^{-1}N\delta g$.
 Let T be the maximal torus in G containing $g_0^{-1}Nx g_0$. Then,
 by (2.2.5), T splits over K . Since $g_0 \in \mathcal{O}(T, K)$ we may
 assume that $g_0 \in G(K)$. Then set
 $g_1 = (g_0, \sigma^{-1}(g_0), \sigma^{-2}(g_0), \dots, \sigma^{-(l-1)}(g_0))$; $g_1 \in \tilde{G}(F)$ and
 $g_1^{-1}N\delta g_1 = g^{-1}Nx g$. Set $\delta' = g_1^{-1}\delta \alpha(g_1)$. Then $N(\delta') = g_1^{-1}N\delta g_1$,
 and so δ' has the desired properties.

Note: (a) If $F = \mathbb{R}$ then every $\delta \in \mathcal{L}$ satisfies (2.2.5), and
 (b) in Lemma 2.2.4, if T is the maximal torus
 containing $N\delta'$ then $\delta' \in \tilde{T} = \text{Res}_F^K T$.

(2.3) The norm map η .

The results of (2.2) allow us to define an injection η
 of the set of stable twisted-conjugacy classes in \mathcal{L} into the
 set of stable conjugacy classes of regular semisimple elements
 in $G(F)$:

if \mathcal{O}_{tw} is a stable twisted-conjugacy class in \mathcal{L} ,
choose $\delta \in \mathcal{O}_{tw}$ and $g_* \in \tilde{G}(\bar{F})$ such that $g_*^{-1}N\delta g_* \in G(F)$;
 η maps \mathcal{O}_{tw} to \mathcal{O} , the stable conjugacy class of
 $g_*^{-1}N\delta g_*$ in $G(F)$.

For the rest of this section, fix $\delta \in \mathcal{L}$ and $g_* \in \tilde{G}(\bar{F})$

such that $\gamma_* = g_*^{-1} N \delta g_*$ lies in $G(F)$; let T denote the maximal torus in G containing γ_* and $\tilde{T} = \text{Res}_F^K T$.

Recall that the $G(F)$ -conjugacy classes in the stable conjugacy class of γ_* are parametrized by $\mathfrak{D}(T, F) = T(\bar{F}) \backslash \Omega(T, F) / G(F)$, where $\Omega(T, F) = \{g \in G(\bar{F}) : t \rightarrow g^{-1} t g : T \hookrightarrow G \text{ is defined over } F\}$, and that the map $g \rightarrow (\rho \rightarrow \rho(g) g^{-1}, \rho \in \text{Gal}(\bar{F}/F))$ of $\Omega(T, F)$ into the 1-cocycles in T induces an embedding of $\mathfrak{D}(T, F)$ into $H^1(T, F) = H^1(\text{Gal}(\bar{F}/F), T(\bar{F}))$. This embedding is surjective if $H^1(G, F) = 1$.

Suppose now that $\delta' \in \mathfrak{L}$, $\delta' = h^{-1} \delta \alpha(h)$, $h \in \tilde{G}(\bar{F})$. Let $\rho \in \text{Gal}(\bar{F}/F)$. Then $\rho(h^{-1} \delta \alpha(h)) = h^{-1} \delta \alpha(h)$, so that $(\rho(h) h^{-1}) \delta \alpha((\rho(h) h^{-1})^{-1}) = \delta$. Let $\delta_* = g_*^{-1} \delta \alpha(g_*)$. Then $\delta_* \in \tilde{T}(\bar{F})$ since $N(\delta_*) = \gamma_* \in T(F)$; moreover, $\{x \in \tilde{G}(\bar{F}) : x \delta_* \alpha(x^{-1}) = \delta_*\} = T(\bar{F})$. Thus $h_\rho = g_*^{-1} (\rho(h) h^{-1}) g_* \in T(\bar{F})$. Since $g_*^{-1} \in \Omega(\tilde{T}, F)$ (clearly), $\rho \rightarrow h_\rho$ is a 1-cocycle of $\text{Gal}(\bar{F}/F)$ in $T(\bar{F})$. If $h_1^{-1} \delta \alpha(h_1) \in \tilde{G}(F)$ is twisted-conjugate to $h^{-1} \delta \alpha(h)$ then $\rho \rightarrow (h_1)_\rho$ is cohomologous to $\rho \rightarrow h_\rho$ and conversely, if the corresponding cocycles are cohomologous then $h_1^{-1} \delta \alpha(h_1)$ and $h^{-1} \delta \alpha(h)$ are twisted-conjugate.

Our approach makes the definition of "twisted $\Omega(\quad)$ " and "twisted $\mathfrak{D}(\quad)$ " rather untidy, so we will just remark that the arguments of the last paragraph give an embedding of the set of twisted-conjugacy classes in the stable twisted-

conjugacy class of δ into $H^1(T, F)$.

In the case that δ satisfies (2.2.5) we may as well assume that $g_* = 1$, i.e. that $\delta \in \tilde{T}(F)$. Then "twisted $\alpha(\)$ " etc. are easily defined (see (2.5) for the case $F = \mathbb{R}$).

Another characterization of \mathcal{L} will be useful, and it is convenient to note it here. If $\delta \in \tilde{G}(F)$, set $G_\delta^\alpha = \{g \in G: g^{-1} \delta \alpha(g) = \delta\}$. Then G_δ^α is a subgroup of \tilde{G} , defined over F . Also, calculation shows that if $K \subseteq L \subseteq \bar{F}$ then $G_\delta^\alpha(L) = G_\delta^\alpha \cap \tilde{G}(L)$ consists exactly of the elements

$$(g_1, \sigma^{-1}(x)g_1\sigma^{-1}(x^{-1}), \sigma^{-2}(x)\sigma^{-1}(x)g_1\sigma^{-1}(x^{-1})\sigma^{-2}(x^{-1}), \dots)$$

where $x = x(\delta)$ and g_1 lies in the centralizer of Nx in $G(L)$... and that, in fact, G_δ^α is an F -form of the centralizer of Nx in G . If $\delta \in \mathcal{L}$, and g_* and T are as usual, then the map $\gamma \rightarrow g_*^{-1} \gamma g_*$ of G_δ^α to T is an isomorphism over F . The following is immediate.

Lemma 2.3.1: \mathcal{L} consists of those elements δ of $\tilde{G}(F)$ for which G_δ^α is a torus.

(2.4) The case that F is a local field.

We continue with $\delta \in \mathcal{L}$, and g_* and T as before. We

have an injection of the set of twisted-conjugacy classes in the stable twisted-conjugacy class of δ into $H^1(T, F)$. Since $\gamma \rightarrow g_*^{-1} \gamma g_*$ defines an isomorphism over F between G_δ^α and T , we could have used $H^1(G_\delta^\alpha, F)$ in place of $H^1(T, F)$. Then the image of the injection is exactly the set of those elements of $H^1(G_\delta^\alpha, F)$ which are trivial in $H^1(\tilde{G}, F)$. If F is a nonarchimedean local field then $H^1(\tilde{G}, F) = 1$ since \tilde{G} is simply-connected, and if $F = \mathbb{R}$ then an easy calculation shows that $H^1(\tilde{G}, F) = 1$ also. Thus we have the following:

Lemma 2.4.1: If F is a local field then there are $[H^1(G_\delta^\alpha, F)]$ twisted-conjugacy classes in the stable twisted-conjugacy class of δ .

To further investigate the case $F = \mathbb{R}$, we have the following result, valid for any field F of characteristic zero.

Lemma 2.4.2: Suppose that T splits over K and δ is chosen in $\tilde{T}(F)$. Then:

(i) each twisted-conjugacy class in the stable twisted-conjugacy class of δ has a representative $t^{-1}_\delta \alpha(t)$ with t in $\tilde{T}(K)$,

(ii) a complete set of representatives for these twisted-

conjugacy classes is provided by $\{u\delta : u \in \tilde{T}(F), N(u) = 1\}$;

if $u, u' \in \tilde{T}(F)$ and $Nu = Nu' = 1$, then $u\delta$ is twisted-conjugate
to $u'\delta$ if and only if $x(u) = x(u')t_0 \sigma(t_0^{-1})$, for some
 $t_0 \in T(K)$.

Proof: Each such twisted-conjugacy class has a representative in $\tilde{T}(F)$, say $g^{-1}\delta \alpha(g)$. Then g centralizes $N\delta$ and so can be chosen in $\tilde{T}(K)$, and (i) follows; (ii) is obtained from a straightforward calculation.

Suppose now that $F = \mathbb{R}$. Let δ, g_* and T be as before; we will assume that $g_* = 1$, so that $\delta \in \tilde{T}(\mathbb{R})$. We may even assume that δ belongs to

$$T(\mathbb{R})_* = \{t \in T(\mathbb{R})^0 : t^2 \text{ is regular in } G\},$$

where $T(\mathbb{R})^0$ denotes the connected component of the identity in $T(\mathbb{R})$, since if $\delta = (\exp X, \sigma(\exp X))$, with $X \in \mathfrak{t}_{\mathbb{R}}$, the Lie algebra of $T(\mathbb{R})$, then we may rewrite δ as $\tilde{t}^{-1}t \alpha(\tilde{t})$, where $t = \exp(\frac{1}{2}(X + \sigma X)) \in T(\mathbb{R})^0$ and $\tilde{t} = (\exp(\frac{\sigma X}{2}), \sigma(\exp(\frac{\sigma X}{2})))$. The following is then immediate:

Lemma 2.4.3:

$$(i) \quad \eta(\delta) = U_T(T(\mathbb{R})^0 \cap G_{\text{reg}}).$$

(ii) $U_T T(\mathbb{R})_*$ provides a complete set of representatives for the twisted-conjugacy classes in δ , and

(iii) if $\delta \in T(\mathbb{R})_*$ then a complete set of representatives for the twisted-conjugacy classes in the stable twisted-conjugacy class of δ is provided by $\{u\delta : u \in T(\mathbb{R})^0, u^2 = 1\}$; two such representatives $u\delta$ and $u'\delta$ are twisted-conjugate if and only if $u' = u \exp i\pi(\mu^\vee + \sigma\mu^\vee)$, some $\mu^\vee \in X_*(T)$.

In (iii) we have identified $\dagger_{\mathbb{C}}$ with $X_*(T) \otimes \mathbb{C}$ in the usual way, and in (i) we have regarded η as a correspondence between $\tilde{G}(\mathbb{R})$ and $G(\mathbb{R})$.

(2.5) A more general norm map for real groups.

Let G be any connected reductive quasisplit group over \mathbb{R} . Let G_{der} be the derived group of G , G_{sc} be the simply-connected covering of G_{der} , and Z be the connected center of G . Then because (1) $\tilde{G}(\mathbb{R}) = \tilde{Z}(\mathbb{R})\tilde{G}_{\text{der}}(\mathbb{R})$ and (2) the natural map $\tilde{G}_{\text{sc}}(\mathbb{R}) \rightarrow \tilde{G}_{\text{der}}(\mathbb{R})$ is surjective, we can extend the definitions of (2.1) and (2.3.) to G without difficulty. Recall that the stable conjugacy class of $\gamma \in G(\mathbb{R})_{\text{reg}}$ is $\{w\gamma w^{-1} : w \in \mathcal{G}(T_\gamma), T_\gamma \text{ being the maximal torus containing } \gamma\}$; in general, this set is properly contained in the intersection of $G(\mathbb{R})$ with the $G(\mathbb{C})$ -conjugacy class of γ . For groups not quasi-split over \mathbb{R} we could use an inner twist to a quasi-split group to define a partial norm map; this however, will

not be done here.

Thus, for the rest of this section, G will be a connected reductive quasi-split group over \mathbb{R} . We define twisted-conjugacy, N and \mathcal{L} as for simply-connected groups.

Also, for T a maximal torus in G , define

$$T(\mathbb{R})_* = \{t \in T(\mathbb{R})^0 = t^2 \text{ is regular in } G\} \text{ and set}$$

$$G(\mathbb{R})_* = \bigcup_T T(\mathbb{R})_*.$$

Lemma 2.5.1: Let $\delta \in \mathcal{L}$. Then the twisted-conjugacy class of δ meets $G(\mathbb{R})_*$.

Proof: Via passage to G_{sc} we find $g \in \tilde{G}(\mathbb{R})$ such that $g^{-1}N\delta g$ lies in $G(\mathbb{R})$. An argument as in the last paragraph of (2.4) completes the proof.

Let T be a maximal torus over \mathbb{R} in G . Set

$\mathcal{A}_{\text{tw}}(T) = \{g \in \tilde{G}(\mathbb{R}) : t \mapsto g^{-1}tg : \tilde{T} \hookrightarrow \tilde{G} \text{ is defined over } \mathbb{R}\},$
 and $\mathfrak{D}_{\text{tw}}(T) = T(\mathbb{R}) \setminus \mathcal{A}_{\text{tw}}(T) / \tilde{G}(\mathbb{R})$. Call $\delta' \in \tilde{G}(\mathbb{R})$ stably twisted-conjugate to $\delta \in T(\mathbb{R})_*$ if $\delta' = g^{-1}\delta\alpha(g)$ for some $g \in \mathcal{A}_{\text{tw}}(T)$, and extend this relation transitively to all pairs of elements of \mathcal{L} .

The set $\mathfrak{D}_{\text{tw}}(T)$ parametrizes the twisted-conjugacy classes in the stable twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$. There is a bijection from $\mathfrak{D}_{\text{tw}}(T)$ to $H^1(T)$; this is established easily, following our earlier arguments. We have further:

Lemma 2.5.2: A complete set of representatives for the twisted-conjugacy classes in the stable twisted conjugacy class of $\delta \in T(\mathbb{R})_*$ is provided by $\{u\delta: u \in T(\mathbb{R})^0, u^2 = 1\}$; two such elements $u\delta$ and $u'\delta$ are twisted-conjugate if and only if $u' = u \exp i\pi(\mu^\vee + \sigma\mu^\vee)$, some $\mu^\vee \in X_*(T)$, and moreover every element $u\delta, u^2 = 1, u \in T(\mathbb{R})^0$, is stably twisted-conjugate to δ .

Proof: Let $g \in \mathcal{O}_{tw}(T)$. Then $\rho(g)g^{-1} \in T(\mathbb{C})$ for $\rho \in \text{Gal}(\mathbb{C}/\mathbb{R})$. Since $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \tilde{T}(\mathbb{C})) = 1$, we may write g as tg_0 , where $t \in \tilde{T}(\mathbb{C}), g_0 \in \tilde{G}(\mathbb{R})$. Thus we may as well assume that $g \in \tilde{T}(\mathbb{C})$. But then $g^{-1}\delta\alpha(g) = g^{-1}\alpha(g)\delta$ is easily seen to be twisted-conjugate to an element $u\delta, u^2 = 1, u \in T(\mathbb{R})^0$. The second and third assertions of the lemma follow from similar calculations. We omit the details.

To define the norm map η we may use passage to G_{sc} , or we may simply define $\eta(\mathcal{O}_{tw})$, where \mathcal{O}_{tw} denotes the stable twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$, to be the stable conjugacy class of δ^2 . It follows from the last lemma that η is an injection.

Regarding η as a correspondence between $\tilde{G}(\mathbb{R})$ and $G(\mathbb{R})$ we have finally,

$$\eta(\mathcal{O}_{tw}) = U_T(T(\mathbb{R})^0 \cap G_{reg}).$$

§3. THE MATCHING PROBLEM.

Let G be a connected reductive quasi-split group over \mathbb{R} . We will assume that for each maximal torus over \mathbb{R} in G , the natural map $H^1(T_{sc}) \rightarrow H^1(T)$ is surjective (i.e. $\mathfrak{E}(T) = H^1(T)$ (cf. [4])). This condition is mainly for convenience; it is satisfied by the groups in which we are mainly interested, the simply-connected semisimple groups. Our arguments in Sections 4 and 5 can be strengthened slightly to provide matching theorems for any connected reductive quasi-split group over \mathbb{R} (see Remark (8) of (5.4)).

Let $\delta \in \mathfrak{g}$. Replacing δ by a twisted-conjugate element, we assume that $\delta \in T(\mathbb{R})_*$, $T(\mathbb{R})$ some Cartan subgroup of $G(\mathbb{R})$. The set of twisted-conjugacy classes in the stable twisted-conjugacy class of δ is parametrized by $\mathfrak{D}_{tw}(T) \cong H^1(T)$. Thus if κ is a character on $H^1(T)$ then we may form

$$\sum_{\mathfrak{g}} \kappa(\mathfrak{g}) \int_{T(\mathbb{R}) \backslash G(\mathbb{C})} f(x^{-1} \mathfrak{g}^{-1} \delta \alpha(\mathfrak{g}) \alpha(x)) \frac{dx}{dt},$$

for $f \in C_c^\infty(G(\mathbb{C}))$... such integrals are shown to be finite in Section 4. Here $\Sigma_{\mathfrak{g}}$ denotes summation of a set of representatives in $\tilde{G}(\bar{\mathbb{F}})$ for the members of $\mathfrak{D}_{tw}(T)$; dx and dt are Haar measures on $G(\mathbb{C})$ and $T(\mathbb{R})$ respectively. Since

$(G_\delta^\alpha)^0 = T$, this expression is a linear combination of the (invariant) integrals of f along the twisted conjugacy classes in the stable twisted-conjugacy class of δ . We write the expression as $\oint_f^{(T, \sigma, \kappa)}(\delta, dt, dx)$, although we will usually drop dx and dt from the notation.

If now $\sqrt{\cdot}$ is a cross-section for the map $\delta \rightarrow \delta^2$ of $G(\mathbb{R})_* = \cup_T T(\mathbb{R})_*$ to $G(\mathbb{R})_0 = \cup_T T(\mathbb{R})_{\text{reg}}^0$ then we may form

$$\oint^{(T, \kappa)}(\gamma) = \begin{cases} \oint_f^{(T, \sigma, \kappa)}(\sqrt{\gamma}) & \text{if } \gamma \in G(\mathbb{R})_0 \\ 0 & \text{otherwise,} \end{cases}$$

for each $\gamma \in G(\mathbb{R})_{\text{reg}}$.

On the other hand, for the given pair (T, κ) we may form, for $f_0 \in C_c^\infty(G(\mathbb{R}))$, $\gamma \in T(\mathbb{R})_{\text{reg}}$, the expression

$$\sum_w \kappa(w) \int_{T(\mathbb{R}) \setminus G(\mathbb{R})} f_0(x^{-1} w^{-1} \gamma w x) \frac{dx}{dt} = \oint_{f_0}^{(T, \kappa)}(\gamma)$$

where \sum_w denotes summation over a set of representatives for elements of $\mathfrak{D}(T) \subseteq \mathfrak{E}(T)$. This expression is a linear combination of the (invariant) integrals of f_0 along the conjugacy classes in the stable conjugacy class of γ ... we call it a " κ -orbital integral of f " (cf. [11]).

In Sections 4 and 5 we will see that the functions $\oint^{(T, \kappa)}$ behave very much like the functions $\oint_{f_0}^{(T, \kappa)}$, provided

a suitable $\sqrt{\quad}$ is chosen (cf. (5.1)). This is somewhat surprising since, in general, there are more twisted-conjugacy classes in the stable twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$ than conjugacy classes in the stable conjugacy class of δ^2 .

Attached to G is a finite collection of connected reductive quasisplit groups over \mathbb{R} ([4]), the endoscopic groups of G (see also [13]). Each pair (T, κ) is attached to one of the groups ([13, Lemma 2.4.2]). Let H be an endoscopic group for G . In [13] we showed that, provided ${}^L H$, the L -group of H , embeds admissibly in ${}^L G$ (..this happens most of the time [cf. [12]]), the κ -orbital integrals of f_0 , for the (T, κ) 's associated to H , match the stable (i.e. $\kappa = 1$) orbital integrals of some function on $H(\mathbb{R})$. These stable orbital integrals must be carefully normalized; the normalization factors are determined by, or attached to, an admissible embedding of ${}^L H$ in ${}^L G$.

Recall that in the stabilization of the trace formula for G (cf. [3] for SL_2) the κ -orbital integrals appear and are replaced by the stable orbital integrals on H . Here we shall show that provided H does not satisfy a certain rather special property (cf. (5.3.1)) and ${}^L H$ embeds in ${}^L G$, the κ -twisted orbital integrals $\oint_f^{(T, \sigma, \kappa)}$, or more precisely the $\oint^{(T, \kappa)}$ defined earlier, match the stable orbital integrals of

some function on $H(\mathbb{R})$. The integrals must be normalized and the normalization factors are those found for the matching of κ -orbital integrals; in particular, a matching is attached to an embedding of ${}^L H$ in ${}^L G$. Our result depends on the choice (and existence of) a suitable crosssection $\sqrt{}$; also we find a function in the Schwartz space of $H(\mathbb{R})$, rather than a compactly supported function, as needed for trace formula applications. The "certain rather special property" assumed of H excludes few endoscopic groups; it does not exclude G itself, nor any of the endoscopic groups if G is $SU(p, p+1)$ or $SU(p, p)$. However, the case $G = SL_2$ and H an anisotropic torus in G is excluded. We handle this case separately after the main theorem (cf. (5.4)). The other excluded cases bear striking similarities to this SL_2 one. We expect that matching theorems can be found in all cases; in the excluded ones it seems that there is a more important embedding of ${}^L H$ in ${}^L \tilde{G}$ than that provided by an embedding of ${}^L H$ in ${}^L G$ and the natural inclusion of ${}^L G$ in ${}^L \tilde{G}$ (cf. (5.4) for the case of SL_2).

The proof of the main theorem occupies Sections 4 and 5. Section 4 is concerned with the convergence of twisted orbital integrals (for compactly supported functions) and the behavior of the "twisted F_f transform"; the results hinge on a Compactness Theorem (cf. (4.2)). The case $G = GL_2(\mathbb{R})$

has been studied in great detail in [14].

After the proof of the main theorem, in Section 5 we include some remarks about the dual lifting of stable characters and some special cases of the theorem.

§4. TWISTED ORBITAL INTEGRALS FOR REAL GROUPS.

(4.1) Notation.

Suppose now that G is a connected reductive group over \mathbb{R} . Until (4.3) we will make no further assumptions on G .

Let $f \in C_c^\infty(G(\mathbb{C}))$. For the integral of f along the twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$, $T(\mathbb{R})$ a Cartan subgroup of $G(\mathbb{R})$, we will use $\int_{\tilde{G}(\mathbb{R})/T(\mathbb{R})} f(\alpha(g)\delta g^{-1}) \frac{dg}{dt}$. This will enable us to follow the notation of [10], [11], etc.

Also it will be convenient to identify $\tilde{G}(\mathbb{R})$ with $G(\mathbb{C})$. This will cause few problems because we will choose representatives for the twisted-conjugacy classes in the stable twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$ as in Lemma 2.5.2. Recall that if $g \in \tilde{G}(\mathbb{R})$ corresponds to $x \in G(\mathbb{C})$ then Ng corresponds to $Nx = x\sigma(x)$ and if $\delta \in T(\mathbb{R})_*$, then $\alpha(g)\delta g^{-1}$ corresponds to $\sigma(x)\delta x^{-1}$ (... the embedding of $G(\mathbb{R})$ in $\tilde{G}(\mathbb{R})$ becomes the natural inclusion of $G(\mathbb{R})$ in $G(\mathbb{C})$). Also for $\delta \in T(\mathbb{R})^0$, we replace G_δ^α by G_δ^σ , where $G_\delta^\sigma(\mathbb{C})$ is the centralizer of δ^2 in $G(\mathbb{C})$ and $G_\delta^\sigma(\mathbb{R}) = \{y \in G(\mathbb{R}) : \sigma(y)\delta y^{-1} = \delta \text{ i.e. } \delta^{-1}\sigma(y)\delta = y\} = \{y \in G_\delta^\sigma(\mathbb{C}) : \tilde{\sigma}(y) = y, \text{ where } \tilde{\sigma} = \text{ad } \delta^{-1} \cdot \sigma\}$.

(4.2) Twisted F_f .

Let $T(\mathbb{R})$ be a Cartan subgroup of $G(\mathbb{R})$. We introduce

next a "twisted" analogue of Harish-Chandra's transform F_f^T , although we shall omit a normalizing factor in the definition. This twisted transform will be a function on $T(\mathbb{R})_* = \{\delta \in T(\mathbb{R})^0 : \delta^2 \text{ is regular in } G\}$. As with F_f^T , the first step is a Compactness Theorem.

Theorem 4.2.1: Let $\delta_0 \in T(\mathbb{R})^0$ and $H = (G_{\delta_0}^\sigma)^0$. Then there exists a neighborhood $\mathcal{N}(\delta_0)$ of δ_0 in $T(\mathbb{R})^0$ with the following property: if C is a compact subset of $G(\mathbb{C})$ then there exists a compact subset \bar{C} of $G(\mathbb{C})/H(\mathbb{R})$ such that if $\delta \in \mathcal{N}(\delta_0)$, $y \in G(\mathbb{C})$ and $\sigma(y)\delta y^{-1} \in C$ then $yH(\mathbb{R}) \in \bar{C}$.

Proof: We may assume G simply-connected, semisimple. Fix C as in the statement of the theorem. If $\sigma(y)\delta y^{-1} \in C$ then $y\delta^2 y^{-1} = \sigma(\sigma(y)\delta y^{-1})\sigma(y)\delta y^{-1} \in \sigma(C)C = C_1$. According to [16, Lemma 8.1.4.2] we may choose a neighborhood $\mathcal{N}(\delta_0^2)$ of δ_0^2 in $T(\mathbb{R})^0$ and compact set \bar{C}_1 in $G(\mathbb{C})/H(\mathbb{C})$ such that if $y \in G(\mathbb{C})$, $\delta \in \mathcal{N}(\delta_0^2)$ and $y\delta y^{-1} \in C_1$ then $yH(\mathbb{C}) \in \bar{C}_1$. Thus it will be sufficient to prove the following:

(1) there is a neighborhood $\mathcal{N}(\delta_0)$ of δ_0 in $T(\mathbb{R})^0$ with the following property: if C_2 is a compact subset of $H(\mathbb{C})$ then there exists a compact subset \bar{C}_2 of $H(\mathbb{C})/H(\mathbb{R})$ such that if $y \in H(\mathbb{C})$, $\delta \in \mathcal{N}(\delta_0)$ and $\sigma(y)\delta y^{-1} \in C_2$ then $yH(\mathbb{R}) \in \bar{C}_2$.

Recall that for $y \in H(\mathbb{C})$, the Galois action σ_H is given

by $\sigma_H(y) = \delta_0^{-1} \sigma(y) \delta_0$. Thus $\sigma(y) \delta y^{-1} = \delta_0 \sigma_H(y) (\delta_0^{-1} \delta) y^{-1}$, and so it follows that (1) is a restatement of the theorem to be proved for the case $\delta_0 = 1$.

We therefore return to the original statement of the theorem and assume $\delta_0 = 1$. Again it is enough to consider the case G simply-connected, semisimple. Let C be as in the statement of theorem and set $C_1 = \sigma(C)C$. We claim:

(2) there is a neighborhood \mathbb{N}_1 of 1 in $T(\mathbb{R})^0$ and a compact subset C_2 of $G(\mathbb{C})$ such that if $\delta \in \mathbb{N}_1$, $y \in G(\mathbb{C})$ and $y \delta^2 y^{-1} \in C_1$ then $y \delta y^{-1} \in C_2$.

To prove (2) regard $G(\mathbb{C})$ as a real Lie group with (real) Lie algebra $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{\mathbb{R}}$. As usual [cf. [16]], define $\tilde{\mathfrak{g}}(\pi) = \{X \in \tilde{\mathfrak{g}} : |\operatorname{Im} \lambda| < \pi, \text{ for each eigenvalue } \lambda \text{ of } \operatorname{ad} X\}$. Let \mathbb{N}_1 be a neighborhood of 1 in $T(\mathbb{R})^0$ such that if $\delta \in \mathbb{N}_1$ then $\delta = \exp H$ where $2H \in \tilde{\mathfrak{g}}(\pi)$. We may find a compact subset C_2 of $G(\mathbb{C})$ such that (a) C_2 contains $C_1 \cap \exp \tilde{\mathfrak{g}}(\pi)$ and (b) if $\exp H \in C_2$, $H \in \tilde{\mathfrak{g}}(\pi)$, then $\exp(\frac{H}{2}) \in C_2$ (cf. Lemma 8.1.6.5 of [16]). If $\delta \in \mathbb{N}_1$, write δ as $\exp H$, where $2H \in \tilde{\mathfrak{g}}(\pi)$. Then $2 \operatorname{Ad} y(H) \in \tilde{\mathfrak{g}}(\pi)$, so that $y \delta^2 y^{-1} \in C_1 \cap \exp \tilde{\mathfrak{g}}(\pi)$, if y is as in the statement of (2). Thus $y \delta y^{-1} = \exp(\operatorname{Ad} y(H)) \in C_2$ and (2) is proved.

Applying (2), we have that if $\delta \in \mathbb{N}_1$, $y \in G(\mathbb{C})$ and $\sigma(y) \delta y^{-1} \in C$ then $\sigma(y) y^{-1} \in C C_2^{-1} = C_3$. Thus the following lemma will complete the proof of Theorem 4.2.1.

Lemma 4.2.2: Let G be a connected reductive group defined
over \mathbb{R} and C a compact subset of $G(\mathbb{C})$. Then there exists
a compact subset \bar{C} of $G(\mathbb{C})/G(\mathbb{R})$ such that if $y \in G(\mathbb{C})$ and
 $\sigma(y)y^{-1} \in C$ then $yG(\mathbb{R}) \in \bar{C}$.

Proof: First assume that G is anisotropic modulo its center.
 We will replace the conclusion of the lemma by:

(4.2.3) there is a compact subset C_1 of $G(\mathbb{C})$ such that
 $y \in C_1 S(\mathbb{R})$, where S is the maximal split
 torus in the center of M .

To prove (4.2.3) we may reduce immediately to the case $S = 1$.
 Regard $G(\mathbb{C})$ as a real Lie group; σ is a Cartan involution on
 $G(\mathbb{C})$ (cf. [16]). Let $G(\mathbb{C}) = G(\mathbb{R})P$ be the corresponding
 Cartan decomposition and write y as up , $u \in G(\mathbb{R})$, $p \in P$.
 Then $\sigma(y)y^{-1} = u(p^{-2})u^{-1}$, so that $p^{-2} \in G(\mathbb{R})C G(\mathbb{R})$. Then p
 belongs to a compact set also, and (4.2.3) is proved.

For arbitrary G (connected, reductive, over \mathbb{R}) let
 P be a minimal parabolic subgroup over \mathbb{R} . Let U be a
 maximal compact subgroup of $G(\mathbb{C})$. We may assume that $G(\mathbb{C}) =$
 $UM(\mathbb{C})N(\mathbb{C})$, where $P = MN$ is a Levi decomposition of P over
 \mathbb{R} ; over \mathbb{R} , M is anisotropic modulo its center. Let y be
 such that $\sigma(y)y^{-1} \in C$. Write y as umn , $u \in U$, $m \in M(\mathbb{C})$ and
 $n \in N(\mathbb{C})$. Then $\sigma(m)m^{-1} m(\sigma(n)n^{-1})m^{-1} \in \sigma(U)CU = C_1$, a compact

set. Since $m(\sigma(n)n^{-1})m^{-1} \in N(\mathbb{C})$, we have that there are compact subsets C_2 in $M(\mathbb{C})$ and C_3 in $N(\mathbb{C})$ such that $\sigma(m)m^{-1} \in C_2$ and $m(\sigma(n)n^{-1})m^{-1} \in C_3$. Applying (4.2.3) we may as well assume that $m \in S(\mathbb{R})$. Then $m(\sigma(n)n^{-1})m^{-1} = \sigma(n')(n')^{-1}$, where $n' = mn m^{-1} \in N(\mathbb{C})$. Thus we have now only to prove the assertion of Lemma 4.2.2 when G is a simply-connected nilpotent group. Under this hypothesis on G , we may write $y \in G(\mathbb{C})$ as $\exp X \exp Y$, where $X \in \mathfrak{g}_{\mathbb{R}}$ and $Y \in i\mathfrak{g}_{\mathbb{R}}$ ([16, Lemma 1.1.4.1]), and the result is immediate. This completes the proof of Lemma 4.2.2 and hence the proof of Theorem 4.2.1.

Let $f \in C_c^\infty(G(\mathbb{C}))$. Then for $\delta \in T(\mathbb{R})_*$ we set

$$(4.2.4) \quad \Phi_f^{(T, \sigma)}(\delta, dt, dg) = \int_{G(\mathbb{C})/T(\mathbb{R})} f(\sigma(g)\delta g^{-1}) \frac{dg}{dt}$$

given Haar measures dg on $G(\mathbb{C})$ and dt on $T(\mathbb{R})$. Much of the time, the choice of dg and dt plays no real role, so we will often omit these measures from notation.

Theorem 4.2.1, applied with $\delta_0 \in T(\mathbb{R})_*$, shows that $\delta \rightarrow \Phi_f^{(T, \sigma)}(\delta)$ is a well-defined C^∞ function on $T(\mathbb{R})_*$.

Fix some system of positive roots for (G, T) . For $\delta \in T(\mathbb{R})^0$, we set

$$\Delta_\sigma^T(\delta) = 2^{\ell} \prod_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha}} (\alpha(\delta) - \alpha(\delta^{-1})) \prod_{\substack{\alpha > 0 \\ \sigma\alpha \neq -\alpha}} |\alpha(\delta) - \alpha(\delta^{-1})|,$$

where ℓ is the dimension of T . Sometimes it will be more convenient to consider

$$\Delta_{\sigma}^T(\delta) = \Delta_{\sigma}^T(\delta) \prod_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha}} \alpha(\delta^{-1}).$$

Also, set

$$\Psi_f^{(T, \sigma)}(\delta) = \Delta_{\sigma}^T(\delta) \Phi_f^{(T, \sigma)}(\delta)$$

for $\delta \in T(\mathbb{R})_*$, and

$$\Psi_f^{(T, \sigma)}(\delta) = \Delta_{\sigma}^T(\delta) \Phi_f^{(T, \sigma)}(\delta).$$

In the following, \mathcal{J} will denote the set of invariant differential operators on T .

Lemma 4.2.5:

(i) There is a compact subset C of $T(\mathbb{R})^0$ such that
 $\Phi_f^{(T, \sigma)}(\delta) = 0$ for $\delta \in T(\mathbb{R})_* \setminus C$.

(ii) $\sup_{\delta \in T(\mathbb{R})_*} |D\Psi_f^{(T, \sigma)}(\delta)| < \infty$

for each $D \in \mathcal{J}$.

Proof: (i) Regard $G(\mathbb{C})$ as a real Lie group. For $x \in G(\mathbb{C})$ and indeterminate λ write $\det((\lambda+1) - \sigma \text{Ad } x)$ as $\sum_{i=0}^n D_i^{\sigma}(x) \lambda^i$. Consider D_{ℓ}^{σ} , where ℓ is the rank of the derived group of $G(\mathbb{R})$. For $\delta \in T(\mathbb{R})^0$, $|D_{\ell}^{\sigma}(\delta)| = |\Delta_{\sigma}^T(\delta)|^2$, as direct calculation shows. Moreover D_{ℓ}^{σ} is an analytic function on $G(\mathbb{C})$ and $D_{\ell}^{\sigma}(\sigma(y)xy^{-1}) = D_{\ell}^{\sigma}(x)$ for all $x, y \in G(\mathbb{C})$.

To prove (i) we may assume that G is semi-simple. The discussion of the last paragraph shows that if $\delta \in T(\mathbb{R})_*$ and $\sigma(y)\delta y^{-1}$ lies in the support of f then $\alpha(\delta)$ is bounded, for all roots α of (G, T) . Hence δ lies in a compact subset of $T(\mathbb{R})^0$.

(ii) By (1) we have only to show that around each element δ_0 of $T(\mathbb{R})^0$ there is a neighborhood $\mathcal{N}(\delta_0)$ such that

$$\sup_{\delta \in \mathcal{N}(\delta_0) \cap T(\mathbb{R})_*} |D\psi_f^{(T, \sigma)}(\delta)| < \infty.$$

Let $\delta_0 \in T(\mathbb{R})^0$, and write H for $(G_{\delta_0}^\sigma)^0$. Choose a neighborhood $\mathcal{N}(\delta_0)$ of δ_0 in $T(\mathbb{R})_0$ and compact set \bar{C} in $G(\mathbb{C})/H(\mathbb{C})$ so that if $\delta \in \mathcal{N}(\delta_0)$, $y \in G(\mathbb{C})$ and $\sigma(y)\delta y^{-1}$ lies in the support of f then $yH(\mathbb{C}) \in \bar{C}$ (cf. Theorem 4.2.1). Choose $F \in C_c^\infty(G(\mathbb{C}))$ so that $\int_{H(\mathbb{R})} F(xh)dh = 1$ for $xH(\mathbb{C}) \in \bar{C}$, dh being some fixed Haar measure on $H(\mathbb{R})$, and set

$$\tilde{f}(h) = \int_{G(\mathbb{C})} F(x) f(\sigma(x)\delta_0 hx^{-1}) dx, \quad h \in H(\mathbb{R}).$$

Then $\tilde{f} \in C_c^\infty(H(\mathbb{R}))$. For $t \in T(\mathbb{R}) \cap H_{\text{reg}}$, set

$$\Phi_{\tilde{f}}(t) = \int_{H(\mathbb{R})/T(\mathbb{R})} \tilde{f}(hth^{-1}) \frac{dh}{dt}.$$

Then for $\delta \in T(\mathbb{R})_* \cap \mathcal{N}(\delta_0)$ we have

$$\Phi_f^{(T, \sigma)}(\delta) = \Phi_{\tilde{f}}(\delta_0^{-1}\delta).$$

Let α be a root of (G, T) such that $\alpha(\delta_0^2) = 1$, i.e. such that α is a root of (H, T) . Then if $\alpha(\delta_0) = -1$ then $1 - \alpha((\delta_0^{-1}\delta)^{-1}) = 1 + \alpha(\delta^{-1})$ and if $\alpha(\delta_0) = 1$ then $1 - \alpha((\delta_0^{-1}\delta)^{-1}) = 1 - \alpha(\delta^{-1})$. By a well known result of Harish-Chandra (cf. [16, Cor. 8.5.5.3]), each derivative of

$$\delta \rightarrow \prod_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha}} (1 - \alpha((\delta_0^{-1}\delta)^{-1})) \prod_{\substack{\alpha > 0 \\ \sigma\alpha \neq -\alpha}} |\alpha^{\frac{1}{2}}(\delta_0^{-1}\delta) - \alpha^{-\frac{1}{2}}(\delta_0^{-1}\delta)| \phi_{\tilde{f}}(\delta_0^{-1}\delta)$$

(where α runs over the roots of (H, T)), is bounded on the set of those elements δ in $\mathbb{N}(\delta_0)$ for which $\delta_0^{-1}\delta$ is regular in H . The assertion (ii) now follows easily, and Lemma 4.2.5 is proved.

Lemma 4.2.6: $\psi_f^{(T, \sigma)}$ extends to a Schwartz function on
 $T(\mathbb{R})_{**} = \{\delta \in T(\mathbb{R})^0 : \alpha(\delta^2) \neq 1 \text{ for all imaginary roots } \alpha$
of (G, T)].

Proof: In view of Lemma 4.2.5 we have only to show that $\psi_f^{(T, \sigma)}$ extends to a C^∞ function on some neighborhood of any $\delta_0 \in T(\mathbb{R})_{**} \setminus T(\mathbb{R})_*$.

Fix $\delta_0 \in T(\mathbb{R})_{**} \setminus T(\mathbb{R})_*$ and set $H = (G_{\delta_0}^\sigma)^0$. Then each root of (H, T) is either real or complex. Define \tilde{f} as in the proof of the last lemma. Then the extension property of $\phi_{\tilde{f}}$, with suitable normalization factor (cf. [16, Theorem 8.5.1.4]), implies the extension of $\psi_f^{(T, \sigma)}$ to a C^∞ function around δ_0 ,

and the lemma is proved.

Finally we come to the behavior of $\phi_f^{(T,\sigma)}$ near points δ_0 in $T(\mathbb{R})^0$ for which $\alpha(\delta_0^2) = 1$ for some imaginary root α of (G,T) . We will also suppose that δ_0^2 is semi-regular, i.e. if $\alpha(\delta_0^2) = 1$ and $\alpha > 0$ then α is unique among roots of (G,T) . Moreover we will assume α noncompact. The case α compact is handled similarly, but we will have no use for it in this paper.

Lemma 4.2.7: If δ_0^2 is semiregular and $\alpha(\delta_0) = -1$ for a noncompact root α of (G,T) , then $\psi_f^{(T,\sigma)}$ extends to a C^∞ function around δ_0 .

Proof: Define H , $\tilde{N}(\delta_0)$ and \tilde{f} as in the proof of Lemma 4.2.5. Since $\alpha(\delta_0) = -1$, we must have that H , a group of type A_1 , is an-isotropic modulo its center. Then the lemma follows from the fact that $t \rightarrow \phi_{\tilde{f}}(t)$ is smooth on $T(\mathbb{R})$.

The case that δ_0^2 is semiregular and $\alpha(\delta_0) = 1$, α noncompact, requires some preparation.

Let $s: T \rightarrow T_s$ be a standard Cayley transform with respect to α (cf. [10, Section 2]). Then s fixes δ_0 . While $\delta_0 \in T(\mathbb{R})^0$, it need not happen that $\delta_0 \in T_s(\mathbb{R})^0$. However, it is easily checked that δ_0 is σ -conjugate to an element of

of $T_s(\mathbb{R})^0$. Write $\delta_0^{(s)}$ for any such element. Let β denote the root α of T_s and regard a coroot as element of the Lie algebra. Set $\delta_v = \delta_0 \exp iv\alpha^v$ and $\delta_v^{(s)} = \delta_0^{(s)} \exp v\beta^v$.

To define $\Psi^{(T,\sigma)}$, use any ordering on the roots of (G,T) for which $\alpha > 0$ and if $\beta > 0$ is imaginary then $\langle \alpha, \beta \rangle \geq 0$. Also, use any Haar measure dt on $T(\mathbb{R})$. For $\Psi^{(T_s,\sigma)}$, use the ordering induced by s and the ordering for the roots of (G,T) (cf. [10, Section 4]) and the measure $(dt)^s$ determined by s (cf. [10, Section 4]). For a differential operator D in \mathcal{J} define D^s as in [10, Section 4]. Set $d(\alpha) = 2$ if the Weyl reflection w_α with respect to α can be realized in $G(\mathbb{R})$ and $d(\alpha) = 1$ otherwise.

Lemma 4.2.8: If δ_0^2 is semiregular and $\alpha(\delta_0) = 1$, for a noncompact root α of (G,T) then

$$\begin{aligned} \lim_{v \downarrow 0} D \Psi_f^{(T,\sigma)}(\delta_v) &= \lim_{v \uparrow 0} D \Psi_f^{(T,\sigma)}(\delta_v) \\ &= id(\alpha) \lim_{v \rightarrow 0} D^s \Psi_f^{(T_s,\sigma)}(\delta_v^{(s)}) \end{aligned}$$

where s is a standard Cayley transform with respect to α and the remaining notation is as described in the last paragraph.

Proof: In this case $H = (G_{\delta_0}^\sigma)^0$ coincides with the connected centralizer of δ_0^2 in G . The argument now follows a well-

known procedure (see, for example, [10, Proposition 4.5]).

We omit the details.

(4.3) κ -twisted orbital integrals.

We assume now that G is connected, reductive and

(4.3.1) G is quasi-split over \mathbb{R}

and

(4.3.2) for each maximal torus over \mathbb{R} in G ,
the map $H^1(T_{sc}) \rightarrow H^1(T)$ is surjective,
i.e. $\mathcal{E}(T) = H^1(T)$ (cf. (2.5)).

We remark on the assumption (4.3.1) after the proof of Theorem 5.4.1. The assumption (4.3.2), in place of the assumption that G is simply-connected and semisimple, allows us to consider groups such as GL_n and PGL_n .

We form κ -twisted orbital integrals as follows. Let $\delta \in T(\mathbb{R})_*$, $T(\mathbb{R})$ being some Cartan subgroup of $G(\mathbb{R})$. We choose representatives for the twisted-conjugacy classes in the stable twisted-conjugacy class of δ in $G(\mathbb{R})$ as in 2.5, i.e. we choose representatives $u\delta$, where $u \in T(\mathbb{R})^0$ and $u^2 = 1$; each such u generates a 1-cocycle for T (i.e. $1 \rightarrow 1$,

$\sigma \rightarrow u$ is a 1-cocycle of $\text{Gal}(\mathbb{E}/\mathbb{R})$ in $T(\mathbb{E})$... for distinct representatives $u\delta$ and $u'\delta$, u and u' are not cohomologous and moreover every class in $H^1(T)$ is generated by some u for which $u\delta$ is a representative. For $f \in C_c^\infty(G(\mathbb{E}))$, we set

$$\Phi_f^{(T, \sigma, \kappa)}(\delta) = \sum_u \kappa(u) \Phi_f^{(T, \sigma)}(u\delta), \quad \delta \in T(\mathbb{R})_*.$$

For technical reasons we assume that κ is a quasicharacter on $X_*(T_{\text{sc}})/X_*(T_{\text{sc}}) \cap \{\mu^\vee - \sigma\mu^\vee : \mu^\vee \in X_*(T)\}$ which contains $H^1(T)$ as a submodule (cf. [4]). Note that $\Phi_f^{(T, \sigma, \kappa)}$ coincides with the expression introduced in Section 3.

From the results of (4.2) we have immediately:

Lemma 4.3.1: (1) $\delta \rightarrow \Phi_f^{(T, \sigma, \kappa)}(\delta)$ defines a C^∞ function on $T(\mathbb{R})_*$ which vanishes off a relatively compact subset of $T(\mathbb{R})$.

(2) $\delta \rightarrow \Delta_\sigma^T(\delta) \Phi_f^{(T, \sigma, \kappa)}(\delta)$ extends to a Schwartz function on $T(\mathbb{R})_{**} = \{\delta \in T(\mathbb{R})^0 : \alpha(\delta^2) \neq 1 \text{ for each imaginary root } \alpha \text{ of } (G, T)\}$.

Let $w \in \mathcal{Q}(T)$. Then w induces a map from $X_*(T_{\text{sc}})$ to $X_*(T_{\text{sc}}^W)$. Hence κ^W is well-defined (cf. [11, Section 3]). We write $\kappa(w)$ for $\kappa(w_\sigma)$ where w_σ is the cohomology class of the cocycle $\sigma \rightarrow \sigma(w^{-1})w$.

Lemma 4.3.2:

$$\phi_f^{(T, \sigma, \kappa)}(\delta) = \kappa(w) \phi_f^{(T^w, \sigma, \kappa^w)}(w\delta w^{-1}), \quad \delta \in T(\mathbb{R})_*,$$

for each $w \in \mathcal{O}(T)$.

Here, if dt is the measure on $T(\mathbb{R})$ used to define $\phi_f^{(T, \sigma, \kappa)}$ then $(dt)^w$ must be used to define $\phi_f^{(T^w, \sigma, \kappa^w)}$.

Proof: $uw\delta w^{-1} = \sigma(w)((\sigma(w^{-1})w)w^{-1}uw\delta)w^{-1}$, so that

$$\begin{aligned} \kappa(w) \sum_u \kappa^w(u) \phi_f^{(T^w, \sigma)}(uw\delta w^{-1}) \\ &= \sum_u \kappa((\sigma(w^{-1})w)w^{-1}uw) \phi_f^{(T, \sigma)}(\sigma(w^{-1})ww^{-1}uw\delta) \\ &= \sum_{u'} \kappa(u') \phi_f^{(T, \sigma)}(u'\delta), \end{aligned}$$

and the lemma is proved.

In the next result, the assumption that G is quasi-split is used in an essential way.

Lemma 4.3.3: Let $\delta_0 \in T(\mathbb{R})^0$ be such that δ_0^2 is semiregular and $\alpha(\delta_0^2) = 1$ where α is imaginary. Suppose also that $\kappa(\alpha^\vee) = -1$. Then

$$\lim_{v \downarrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_v) = \lim_{v \uparrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_v),$$

$D \in \mathcal{J}$, where $\Psi_f^{(T, \sigma, \kappa)} = \Delta_\sigma^T \phi_f^{(T, \sigma, \kappa)}$.

The notation has been explained in (4.2).

Proof: Because G is quasi-split, we may assume that α is noncompact (... for any imaginary root α of (G, T) , there is a noncompact root in $\{\omega\alpha: \omega \text{ in the imaginary Weyl group of } (G, T)\}$ ([11, Lemma 9.2])); now apply Lemma 4.3.2 ...).

To compute

$$\lim_{\nu \downarrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu) - \lim_{\nu \uparrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu)$$

we may replace $\Psi_f^{(T, \sigma, \kappa)}(\delta)$ by

$$\Psi(\delta) = \Delta_\sigma^T(\delta) \sum_u' \kappa(u) \Phi_f^{(T, \sigma)}(u\delta),$$

where Σ' denotes summation over those u for which $\alpha(u\delta_0) = 1$ (cf. Lemma 4.2.7). Note that if u' is cohomologous to u then $\alpha(u') = \alpha(u)$ (cf. Lemma 2.4.3). If u appears in Σ' then so does (a generator for a 1-cocycle cohomologous to) $\sigma(w_\alpha^{-1})w_\alpha u$, where $w_\alpha \in G(\mathbb{C})$ realizes the Weyl reflection with respect to α . Since α is noncompact we have $\kappa(\alpha^\vee) = \kappa(w_\alpha) = -1$, so that $\sigma(w_\alpha^{-1})w_\alpha u$ is not cohomologous to u . Also $\kappa(w_\alpha \sigma(w_\alpha^{-1})u) = -\kappa(u)$ and $\sigma(w_\alpha^{-1})w_\alpha u\delta = \sigma(w_\alpha^{-1})(w_\alpha u\delta w_\alpha^{-1})w_\alpha$. Hence $\Psi(\delta_0 \delta^\alpha) = \Psi(\delta_0 \delta)$. Thus if $D \in \mathcal{J}$ is invariant under w_α , we have the desired result.

If $D^\alpha = -D$ then Lemma 4.2.8 shows that

$$\lim_{\nu \downarrow 0} D \Psi_f^{(T, \sigma)}(u\delta_\nu) = \lim_{\nu \uparrow 0} D \Psi_f^{(T, \sigma)}(u\delta_\nu) = 0$$

for each u such that $\alpha(u\delta_0) = 1$. The lemma now follows for arbitrary $D \in \mathcal{J}$.

Suppose that δ_0^2 is semiregular and $\alpha(\delta_0^2) = 1$ where α is imaginary and $\kappa(\alpha^\vee) = 1$. Let $s: T \rightarrow T_s$ be a Cayley transform with respect to α (in the sense of [12, Section 6]); since G is quasi-split, such a transform exists ([10]). Define $\delta_0^{(s)}$ as for the case s standard (cf. (4.2)); we will use the other notation from (4.2) as well.

Since $\kappa(\alpha^\vee) = 1$, the quasicharacter κ^s is well-defined ([11, Section 3]); the κ -signature of s , $\epsilon_\kappa(s)$, has been defined in [11, Section 4].

Lemma 4.3.4: If $\alpha(\delta_0) = 1$ then

$$\begin{aligned} \lim_{\nu \downarrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu) &= \lim_{\nu \uparrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu) \\ &= 2i\epsilon_\kappa(s) \lim_{\nu \rightarrow 0} D^s \Psi_f^{(T_s, \sigma, \kappa^s)}(\delta_\nu^{(s)}). \end{aligned}$$

Here

$$\Psi_f^{(T, \sigma, \kappa)} = \Delta_\sigma^T \Phi_f^{(T, \sigma, \kappa)}.$$

Proof: Because of Lemma 4.3.2 we may assume α noncompact and s standard. Then $\epsilon_\kappa(s) = 1$ (cf. [11, Section 4]).

To compute the left-hand side of the formula we may replace $\Psi_f^{(T, \sigma, \kappa)}$ by $\delta \rightarrow \Delta_\sigma^T(\delta) \Sigma'_u \Phi_f^{(T, \sigma)}(u\delta)$, where Σ' denotes summation over those u for which $\alpha(u) = 1$.

Suppose then $\alpha(u) = 1$. We have, from Lemma 4.2.8,

$$\lim_{\nu \downarrow 0} D \Psi_f^{(T, \sigma)}(u\delta_\nu) - \lim_{\nu \uparrow 0} D \Psi_f^{(T, \sigma)}(u\delta_\nu) = \text{id}(\alpha) \lim_{\nu \rightarrow 0} D^s \Psi_f^{(T_s, \sigma)}(u\delta_\nu^{(s)}).$$

Here $u\delta_\nu^{(s)} = (u\delta_0)' \exp \nu(s\alpha^\vee)$, where $(u\delta_0)'$ is an element of $T_s(\mathbb{R})^0$ twisted-conjugate to $u\delta_0$.

Since $\alpha(u) = 1$, $u^s = u$ lies in $T_s(\mathbb{R})$, and so generates a 1-cocycle in T_s . Denote a generator in $T_s(\mathbb{R})^0$ for this new cocycle by $u^{(s)}$. Then $u\delta_\nu^{(s)} = (u^{(s)})(\delta_\nu^{(s)})$. Clearly,
 $\kappa(u) = \kappa^s(u^{(s)})$. The following is easily verified:

(i) the map $u \rightarrow u^{(s)}$ induces a surjective map from the classes in $H^1(T)$ with representative u , where $u \in T(\mathbb{R})^0$ and $\alpha(u) = 1$, onto $H^1(T_s)$,

(ii) if $d(\alpha) = 2$ then the map is a bijection and if $d(\alpha) = 1$ then the fiber over the class of $u^{(s)}$ consists of the classes generated by the (non-cohomologous) elements u and $\sigma(w_\alpha^{-1})w_\alpha u$. We omit the details of the proof except to note that $\sigma(w_\alpha^{-1})w_\alpha$ is cohomologous to $\exp i\pi\alpha^\vee$ (cf. [11, Proposition 2.1]).

Since $\kappa(\sigma(w_\alpha^{-1})w_\alpha u) = \kappa(w_\alpha)\kappa(u) = \kappa(\alpha^\vee)\kappa(u) = \kappa(u)$, the lemma now follows.

§5. MATCHING THEOREM FOR REAL GROUPS.

(5.1) Smooth cross-section for η .

We continue with G connected, reductive, defined over \mathbb{R} and satisfying (4.3.1) and (4.3.2) (... G quasi-split over \mathbb{R} , and $H^1(T) = \mathcal{E}(T)$, for each Cartan subgroup $T(\mathbb{R})$). We also continue with the conventions of (4.1), so that $\tilde{G}(\mathbb{R})$ is identified with $G(\mathbb{C})$, etc..

As noted already in Section 3, if we are to transfer $\mathfrak{g}_{\mathbb{F}}^{(T, \sigma, \kappa)}$ to $G(\mathbb{R})$ by means of η , then we will need a way of choosing for each $\gamma \in G(\mathbb{R}) \cap \eta(\delta)$ a twisted-conjugacy class in that stable twisted-conjugacy class in $G(\mathbb{C})$ which maps to the stable conjugacy class of γ under η . This and smoothness requirements of the matching theorem we have in mind motivate the following:

Definition 5.1.1: $\sqrt{}$ is a smooth cross-section for η if

- (1) $\sqrt{}$ is an injective C^∞ map of $G(\mathbb{R})_0 = \bigcup_T T(\mathbb{R})_{\text{reg}}^0$ into $G(\mathbb{R})_* = \bigcup_T T(\mathbb{R})_*$,
- (2) $(\sqrt{\gamma})^2 = \gamma$, $\gamma \in G(\mathbb{R})_0$,
- (3) $\sqrt{w\gamma w^{-1}} = w\sqrt{\gamma}w^{-1}$, $w \in \mathcal{G}_\gamma(T_\gamma)$, and
- (4) if $T(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$ then:
 - (a) $\sqrt{}$ extends smoothly to $\{\gamma \in T(\mathbb{R})^0 : \alpha(\gamma) \neq 1$

for all imaginary roots α of (G, T) and

(b) if $\gamma_0 \in T(\mathbb{R})^0$ is semiregular, $\alpha(\gamma_0) = 1$ for some α noncompact imaginary, and $\gamma_0^s \in T_s(\mathbb{R})^0$ for some (and hence any) Cayley transform s with respect to α then

$$\lim_{\nu \downarrow 0} (\sqrt{\gamma_\nu})^s = \lim_{\nu \uparrow 0} (\sqrt{\gamma_\nu})^s = \lim_{\nu \rightarrow 0} \sqrt{\gamma_\nu^{(s)}}.$$

In (3), T_γ denotes the maximal torus containing γ , and in

$$(5), \gamma_\nu = \gamma_0 \exp i\nu\alpha^\vee, \gamma_\nu^{(s)} = \gamma_0^s \exp \nu(s\alpha^\vee).$$

Remark: In cases such as $G = GL_n, SL_n$ or $SU(p, q)$ the construction of a smooth cross-section for η is easy. We have yet to complete a suitably general construction.

(5.2) Transfer of twisted orbital integrals.

Let $f \in C_c^\infty(G(\mathbb{R}))$. With (T, κ) as usual, we define

$$\phi^{(T, \kappa)}(\gamma) = \begin{cases} \phi_f^{(T, \sigma, \kappa)}(\sqrt{\gamma}) & \text{if } \gamma \in T(\mathbb{R})_{\text{reg}}^0, \\ 0 & \text{if } \gamma \in T(\mathbb{R})_{\text{reg}} \setminus T(\mathbb{R})_{\text{reg}}^0, \end{cases}$$

using a smooth cross-section $\sqrt{\quad}$ of η .

Lemma 5.2.1:

- (1) $\phi^{(T, \kappa)}(\gamma) = \kappa(w) \phi^{(T^w, \kappa^w)}(\gamma^w), \quad w \in \mathcal{O}(T).$
- (2) $\gamma \rightarrow \Delta_T(\gamma) \phi^{(T, \kappa)}(\gamma)$ is smooth on $T(\mathbb{R})_{\text{reg}}$, vanishes

off a subset relative compact in $T(\mathbb{R})$, and extends to a Schwartz function on $\{\gamma \in T(\mathbb{R}) : \alpha(\gamma) \neq 1, \alpha \text{ any imaginary root of } (G, T)\}$.

The proof is straightforward. We omit the details.

Lemma 5.2.2: If α is an imaginary root of (G, T) , $\kappa(\alpha^\vee) = 1$, $\alpha(\gamma_0) = 1$ and γ_0 is semiregular then

$$\begin{aligned} \lim_{\nu \downarrow 0} \hat{D}(\Delta_{T\hat{\phi}}^{(T, \kappa)})(\gamma_\nu) - \lim_{\nu \uparrow 0} \hat{D}(\Delta_{T\hat{\phi}}^{(T, \kappa)})(\gamma_\nu) \\ = 2i\epsilon_\kappa(s) \lim_{\nu \rightarrow 0} \hat{D}^s(\Delta_{T_s\hat{\phi}}^{(T_s, \kappa^s)})(\gamma_\nu^{(s)}), \end{aligned}$$

for each Cayley transform s with respect to α .

We have used the following new notation:

$$\Delta_T(\gamma) = \prod_{\substack{\alpha > 0 \\ \sigma\alpha = -\alpha}} (1 - \alpha(\gamma^{-1})) \prod_{\substack{\alpha > 0 \\ \sigma\alpha \neq -\alpha}} |\alpha^{\frac{1}{2}}(\gamma) - \alpha^{-\frac{1}{2}}(\gamma)|$$

for $\gamma \in T(\mathbb{R})$. Note that $\Delta_T(\gamma) = 2^{-l} \Delta_\sigma^T(\overline{\gamma})$ for $\gamma \in T(\mathbb{R})_*$,

provided the same positive system for the imaginary roots

of (G, T) is used in the definitions of Δ_σ^T and Δ_T .

The differential operators \hat{D} and \hat{D}^s are as in [10, Section 4].

The restrictions of (4.2) apply to the choice of Haar measure on $T(\mathbb{R})$ and $T_s(\mathbb{R})$ (as used in defining $\hat{\phi}^{(T, \kappa)}$ and $\hat{\phi}^{(T_s, \kappa^s)}$) and positive systems for the imaginary roots of

(G, T) and (G, T_s) respectively (as used in defining Δ_T and Δ_{T_s}).

Proof of Lemma 5.2.2: In view of Lemma 5.2.1 we may assume α noncompact. If $\gamma_0 \in T(\mathbb{R}) \setminus T(\mathbb{R})^0$ then $\gamma_0^s \in T_s(\mathbb{R}) \setminus T_s(\mathbb{R})^0$ and there is nothing to prove. If $\gamma_0 \in T(\mathbb{R})^0$ and $\gamma_0^s \in T_s(\mathbb{R})^0$ then we apply the property (4b) of $\sqrt{\quad}$ (see (5.1.1)) to conclude that $\delta_0 = \lim_{\nu \downarrow 0} \sqrt{\gamma_\nu} = \lim_{\nu \uparrow 0} \sqrt{\gamma_\nu}$ exists and that, for suitable $c \neq 0$,

$$c \lim_{\nu \downarrow 0} \hat{D}(\Delta_T^{\otimes} (T, \kappa))(\gamma_\nu) = \lim_{\nu \downarrow 0} D_* \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu),$$

$$c \lim_{\nu \uparrow 0} \hat{D}(\Delta_T^{\otimes} (T, \kappa))(\gamma_\nu) = \lim_{\nu \uparrow 0} D_* \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu)$$

and

$$c \lim_{\nu \rightarrow 0} \hat{D}^s(\Delta_{T_s}^{\otimes} (T_s, \kappa^s))(\gamma_\nu^{(s)}) = \lim_{\nu \rightarrow 0} D_*^s \Psi_f^{(T_s, \sigma, \kappa^s)}(\delta_\nu^{(s)}),$$

where D_* is the image of D under the automorphism of \mathfrak{J} induced by the map $X \rightarrow 1/2X$ of the Lie algebra of T to itself.

Since $\alpha(\delta_0) = 1$ (by property (4b) of $\sqrt{\quad}$) we can apply Lemma 43.4 to obtain the desired formula.

If now $\gamma_0 \in T(\mathbb{R})^0$ and $\gamma_0^s \notin T_s(\mathbb{R})^0$ then the right-hand side of the formula to be proved is zero. To show that the left-hand side is zero also, we need some additional analysis.

It is easily checked that if $\delta^2 = \gamma_0$ then $\alpha(\delta) = -1$.

Clearly $\delta_0 = \lim_{\nu \downarrow 0} \sqrt{\gamma_\nu}$ and $\delta'_0 = \lim_{\nu \uparrow 0} \sqrt{\gamma_\nu}$ are well-defined and

$\delta'_0 = \delta_0^{w_\alpha}$. Moreover, for suitable $c \neq 0$,

$$(a) \quad c \lim_{\nu \downarrow 0} \hat{D}(\Delta_T^\phi(T, \kappa))(\gamma_\nu) = \lim_{\nu \rightarrow 0} (D_* \Psi_f^{(T, \sigma, \kappa)})(\delta_\nu)$$

and

$$(b) \quad c \lim_{\nu \uparrow 0} \hat{D}(\Delta_T^\phi(T, \kappa))(\gamma_\nu) = \lim_{\nu \rightarrow 0} (D_* \Psi_f^{(T, \sigma, \kappa)})(\delta'_\nu).$$

Note that we have used Lemma 4.2.7 to write "lim" on the right
 $\nu \rightarrow 0$

hand side. It will be sufficient to show

$$(c) \quad \lim_{\nu \rightarrow 0} D \Psi_f^{(T, \sigma, \kappa)}(\delta_\nu) = \lim_{\nu \rightarrow 0} D \Psi_f^{(T, \sigma, \kappa)}(w_\alpha \delta_\nu w_\alpha^{-1})$$

for all $D \in \mathcal{J}$.

Suppose that $D^{w_\alpha} = D$. Then both sides of (c) are zero.

Indeed, for each u generating a 1-cocycle of T we have $\alpha(u\delta) = -1$. We carry the argument of Lemma 4.2.7 further to conclude that $D(\delta \rightarrow \Delta_\sigma^T(\delta) \phi_f^{(T, \sigma)}(u\delta))$ vanishes at δ_0 and, similarly, at $\delta'_0 = w_\alpha \delta_0 w_\alpha^{-1}$, since δ'_0 satisfies the same hypothesis as δ_0 . Hence both sides of (c) are zero.

Suppose now that $D^{w_\alpha} = -D$. Since we must have $\alpha(u) = 1$ for each generator u of 1-cocycle in T it follows from Lemma 4.3.2 that

$$\Psi_f^{(T, \sigma, \kappa)}(w_\alpha \delta w_\alpha^{-1}) = -\kappa(\alpha^\nu) \Psi_f^{(T, \sigma, \kappa)}(\delta) = -\Psi_f^{(T, \sigma, \kappa)}(\delta),$$

and so (c) follows. This completes the proof of the lemma.

Consider now the following possibility:

(5.2.3) $\gamma_0 \in T(\mathbb{R})^0$ is semiregular, $\alpha(\gamma_0) = 1$ for a noncompact root α , and γ_0^s , the image of γ_0 under some Cayley transform s with respect to α , does not lie in $T_s(\mathbb{R})^0$.

In case that (5.2.3) holds and $\kappa(\alpha^\vee) = 1$ was discussed in the last lemma. Now assume (5.2.3) and $\kappa(\alpha^\vee) = -1$. Then arguing as in the last lemma we obtain

$$(5.2.4) \quad \lim_{\nu \downarrow 0} \hat{D}^{\Psi(T, \kappa)}(\gamma_\nu) = - \lim_{\nu \uparrow 0} \hat{D}^{\Psi(T, \kappa)}(\gamma_\nu)$$

for all $D \in \mathcal{J}$ (...if $D^{\alpha^\vee} = D$ then both sides of (5.2.4) are zero). Thus we have found a way in which the $\hat{\phi}_f^{(T, \sigma, \kappa), s}$ behave differently from κ -orbital integrals. For the present we will simply exclude the case (5.2.3) with $\kappa(\alpha^\vee) = -1$ from consideration.

Lemma (5.2.5): Assume that if α is a noncompact root for which (5.2.3) occurs then $\kappa(\alpha^\vee) = 1$.

Suppose that γ_0 is semiregular and $\alpha(\gamma_0) = 1$ where α is imaginary and $\kappa(\alpha^\vee) = -1$. Then

$$\lim_{\nu \downarrow 0} D(\Delta_T^{\hat{\phi}}(T, \kappa))(\gamma_\nu) = \lim_{\nu \uparrow 0} D(\Delta_T^{\hat{\phi}}(T, \kappa))(\gamma_\nu)$$

for all $D \in \mathcal{J}$.

Proof: We may assume α noncompact. Then either $\gamma_0 \notin T(\mathbb{R})^0$ and $\gamma_0^s \notin T_s(\mathbb{R})^0$, in which case there is nothing to prove, or $\gamma_0 \in T(\mathbb{R})^0$ and $\gamma_0^s \in T_s(\mathbb{R})^0$. In the latter case we argue as we did in the proof of Lemma 5.2.2 and apply Lemma 4.3.3. Thus the lemma is proved.

(5.3) Endoscopic groups, etc..

An endoscopic group H for G is prescribed by L-group data. Since we will not use that data explicitly, we refer to [13, Section 2] for details. Let T be a maximal torus over \mathbb{R} in G . A "pseudo-diagonalization" (p-d) η of T is, essentially, a conjugation of T to the "most split" torus in G (see [13, §1.3] for a more precise definition...recall that we are assuming G quasi-split and so may take the pair (G^*, ψ) of [13, §1.3] to be $(G, \text{identity map})$). For each such pair (T, η) attached to H , i.e. belonging to the set $\mathcal{J}_H(G)$ of [13, §2.4], we have an embedding of T in H defined over \mathbb{R} . We can therefore introduce the terminology of " $\gamma' \in H(\mathbb{R})$ originating from $\gamma \in G(\mathbb{R})_{\text{reg}}$ via (T, η) " and of "a maximal torus T' in H originating in G (via some $(T, \eta) \in \mathcal{J}_H(G)$ ". Further we can associate to a Haar measure dt on T , a Haar measure dt' on any T' originating from T via (T, η) (cf.

[13, Section 2.4]); we denote by $d_* t'$ the measure $2^l d' t$, l being the dimension of T (or T').

To each $(T, \eta) \in \mathcal{J}_H(G)$ there is attached a quasicharacter κ of the kind we have been considering since Section 4.3. Concerning the characters on $\mathfrak{E}(T) = H^1(T)$, note Lemma 2.4.2 of [13].

Let $\xi: {}^L H \hookrightarrow {}^L G$ be an admissible embedding. Then to ξ there is associated two families of normalization factors $\{\Delta_{(T, \eta)}, (T, \eta) \in \mathcal{J}_H(G)\}$ and $\{-\Delta_{(T, \eta)}\}$ (see [13, Section 3] for details).

We now fix H , assume that $\xi: {}^L H \hookrightarrow {}^L G$ is an admissible embedding and fix one of the two attached families of normalization factors, say $\{\Delta_{(T, \eta)}\}$. We say that $\gamma' \in H(\mathbb{R})$ originates from $\delta \in G(\mathbb{R})_*$ via (T, η) and $\sqrt{\quad}$ (a smooth cross-section for η) if $\delta = \sqrt{\gamma}$, where γ' originates from γ via (T, η) .

In line with (5.2), we assume that:

(5.3.1) if κ is the quasicharacter attached to some
 $(T, \eta) \in \mathcal{J}_H(G)$ and α is a root of T
satisfying (5.2.3) for some $\gamma_0 \in T(\mathbb{R})^0$ then
 $\kappa(\alpha^\vee) = 1$.

Finally, we choose a Haar measure dh on $H(\mathbb{R})$. The space of

Schwartz functions on $H(\mathbb{R})$ will be denoted $\mathcal{C}(H(\mathbb{R}))$.

(5.4) The main theorem.

We assume that G satisfies (4.3.1) and (4.3.2), and that $\sqrt{\quad}$ is a smooth cross-section for η .

Let H be an endoscopic group for G satisfying (5.3.1); choose data as in (5.3).

Theorem 5.4.1: For each $f \in C_c^\infty(G(\mathbb{R}))$ there exists $f_H \in \mathcal{C}(G(\mathbb{R}))$ such that

$$\Phi_{f_H}^{(T,1)}(\gamma', d_* t', dh) = \begin{cases} \Delta_{(T,\eta)}^{(\delta^2)} \Phi_f^{(T,\sigma,\kappa)}(\delta, dt, dg) & \text{if} \\ \gamma' \text{ originates from } \delta \in G(\mathbb{R})_* \text{ via} \\ (T,\eta) \text{ and } \sqrt{\quad}, \\ 0 & \text{if } \gamma' \text{ originates in } G(\mathbb{R})_{\text{reg}} \setminus G(\mathbb{R})_{\text{reg}}^0 \\ \text{via } (T,\eta) \end{cases}$$

for all Cartan subgroups $T'(\mathbb{R})$ of $H(\mathbb{R})$ and pairs (T,η) in $\mathcal{J}_H(G)$ with attached quasicharacter κ .

Proof: In the arguments of [11, Sections 9, 10] and [13, Section 3] we replace the " $\Phi_f^{(T,\kappa)}(\gamma, dt, dg)$ " by $\Phi_f^{(T,\kappa)}(\gamma, dt, dg)$ as defined in (5.2), and use the results of Lemmas 5.2.1, 5.2.2 and 5.2.5 in place of the properties of " $\Phi_f^{(T,\kappa)}(\gamma, dt, dg)$ ".

Remarks: (1) For application to the trace formula we seek to replace f_H by a (smooth) function of compact support. That requires a characterization of the stable orbital integrals of C_c^∞ -functions but, apparently, no further analysis of the type we have just carried out.

(2) Dual to the correspondence (f, f_H) there is a map $\Theta_H \rightarrow \Theta$ (i.e. $\Theta(f) = \Theta_H(f_H)$) from stable tempered characters on $H(\mathbb{R})$ to distributions on $G(\mathbb{C})$ invariant under twisted-conjugacy. Further study of twisted orbital integrals will reveal more about these lifts Θ . It is clear from the definition that Θ is represented by a locally L^1 function, and that this function is analytic on \mathcal{S} . The function is readily computed in terms of Θ_H . See (6) below.

(3) We may replace G by an inner form (i.e. consider groups not quasi-split) provided we make some assumptions about the behavior of $\mathfrak{p}_f^{(T, \sigma)}$ in the neighborhood of a point δ in $T(\mathbb{R})^0$ for which δ^2 is annihilated by a totally compact (cf. [13, (4.3.6)]) imaginary root.

(4) Let $G = GL_n$. For $\gamma \in G(\mathbb{R})_{\text{reg}}$ we say that $\gamma = Nm \delta$ has a solution if the image of the twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$ under η is the conjugacy class of γ . Then in view of the existence of a smooth cross-section for η , Theorem 5.4.1, applied with the inclusion ${}^L G \hookrightarrow {}^L \tilde{G}$, provides the following:

Corollary 5.4.2: For each $f \in C_c^\infty(\mathrm{GL}_n(\mathbb{C}))$ there exists
 $f_0 \in C(\mathrm{GL}_n(\mathbb{R}))$ such that

$$\Phi_{f_0}^T(\gamma, d_*t, dg_0) = \begin{cases} \Phi_f^{(T, \sigma)}(\delta, dt, dg) \\ \text{if } \gamma = Nm \delta \text{ has a solution,} \\ 0 \text{ otherwise,} \end{cases}$$

for all $\gamma \in G(\mathbb{R})_{\mathrm{reg}}$, T denoting the maximal torus containing γ .

Here dg_0 , dg are fixed Haar measures on $G(\mathbb{R})$ and $G(\mathbb{C})$ respectively, dt is an arbitrary Haar measure on $T(\mathbb{R})$ and $d_*t = 2^n dt$. If $n = 2$ then we can apply Lemma 4.1 of [5] (cf. [8]) to obtain $f_0 \in C_c^\infty(G(\mathbb{R}))$; thus we recover the well known matching result for GL_2 (cf. [5, Lemma 4.2]).

(5) The lifting dual to (f, f_0) follows the principle of functoriality in the L-group. This is seen most easily by combining the explicit formula for a lift with the computations of [7] (for $n = 2$ the result is well known (cf. [5], [14])). See also [2].

(6) We may generalize (4) and (5). Let G be a connected reductive quasi-split group over \mathbb{R} for which $\mathcal{E}(T) = H^1(T)$ for all Cartain subgroups $T(\mathbb{R})$ of $G(\mathbb{R})$. Assume that a smooth cross-section for \mathcal{N} does exist. We say that for $\gamma \in G(\mathbb{R})_{\mathrm{reg}}$, " $\gamma = Nm \delta$ has a solution" if the stable conjugacy class

of γ is the image of the stable twisted-conjugacy class of $\delta \in T(\mathbb{R})_*$ under η . Then:

Corollary 5.4.3: For each $f \in C_c^\infty(G(\mathbb{C}))$ there exists
 $f_0 \in C(G(\mathbb{R}))$ such that

$$\Phi_{f_0}^{(T,1)}(\gamma, d_*t, dg_0) = \begin{cases} \Phi_f^{(T,\sigma,1)}(\delta, dt, dg) \\ \text{if } \gamma = Nm \delta \text{ has a solution,} \\ 0 \text{ otherwise,} \end{cases}$$

for all $\gamma \in G(\mathbb{R})_{reg}$.

The dt , dg_0 and T are as usual.

If Θ_0 is a stable tempered character on $G(\mathbb{R})$, denote by Θ its lift to $G(\mathbb{C})$ (i.e. $\Theta(f) = \Theta_0(f_0)$). If $\delta \in \mathcal{L}$ (i.e. $\delta \in G(\mathbb{C})$ and $\delta\sigma(\delta)$ is regular semisimple in G) denote by $Nm(\delta)$ any element in the stable conjugacy class in $G(\mathbb{R})$ corresponding to the stable twisted conjugacy class of δ under η . Then we also have:

Corollary 5.4.4:

$$\Theta(\delta) = \Theta_0(Nm \delta), \quad \delta \in \mathcal{L}.$$

Proof: This is a straightforward computation once we note the following analogue of the Weyl Integration Formula:

$$\int_{G(\mathbb{C})} f(g) dg = \sum_T \frac{1}{n(T)} \int_{T(\mathbb{R})_*} \mathfrak{F}_f^{(T, \sigma)}(\delta) |\Delta_\sigma^T(\delta)|^2 d\delta$$

for $f \in C_c^\infty(G(\mathbb{C}))$. Here Σ_T denotes summation over a set of representatives for the conjugacy classes of Cartan subgroups $T(\mathbb{R})$ of $G(\mathbb{R})$ and $n(T)$ is the order of $\mathcal{G}_*(T)/T(\mathbb{R})$, where $\mathcal{G}_*(T) = \{w \in G(\mathbb{C}) : w \text{ normalizes } T \text{ and } \sigma(w^{-1})w \in T(\mathbb{R})^0\}$; $n(T)$ is easily checked to be finite (cf. Lemma 2.4.3 iii). The formula follows from the fact that the map

$$T(\mathbb{R})_* \times G(\mathbb{C})/T(\mathbb{R}) \rightarrow G(\mathbb{C})$$

defined by $(\delta, gT(\mathbb{R})) \rightarrow \sigma(g)\delta g^{-1}$ is a local diffeomorphism with Jacobian $|\Delta_\sigma^T(\delta)|^2$, and the fact that $\sigma(g)\delta g^{-1} = \sigma(g')\delta'g'^{-1}$ if and only if $g' = gw$ and $\delta' = \sigma(w^{-1})\delta w$ for some $w \in \mathcal{G}_*(T)$.

To calculate $\Theta(f)$ we identify Θ_0 with an analytic function on $G(\mathbb{R})_{\text{reg}}$. Then $\Theta(f) = \Theta_0(f_0)$

$$= \sum_T \frac{1}{m(T)} \int_{T(\mathbb{R})_{\text{reg}}} \Theta_0(\gamma) \mathfrak{F}_{f_0}^{(T, 1)}(\gamma) |\Delta_T(\gamma)|^2 d\gamma$$

where $m(T)$ is the order of the group $\Omega_0(T)$ of elements in the Weyl group of T which commute with the Galois action on T ,

$$= \sum_T \frac{1}{m(T)} \int_{T(\mathbb{R})_{\text{reg}}} \Theta_0(\gamma) \mathfrak{F}_{f_0}^{(T, 1)}(\gamma) |\Delta_T(\gamma)|^2 d_* t$$

$$= \sum_{\mathbb{T}} \frac{1}{m(\mathbb{T})} \int_{\mathbb{T}(\mathbb{R})_{\text{reg}}^0} \tilde{\Theta}(\sqrt{\gamma}) \Phi_f^{(\mathbb{T}, \sigma, 1)}(\sqrt{\gamma}) |\Delta_{\sigma}^{\mathbb{T}}(\sqrt{\gamma})|^2 dt$$

where $\tilde{\Theta}$ is defined by $\tilde{\Theta}(\delta) = \Theta_0(\text{Nm } \delta)$, $\delta \in \mathcal{A}$,

$$= \sum_{\mathbb{T}} \frac{1}{m(\mathbb{T})} \int_{\Lambda} \tilde{\Theta}(\delta) \Phi_f^{(\mathbb{T}, \sigma, 1)}(\delta) |\Delta_{\sigma}^{\mathbb{T}}(\delta)|^2 dt$$

where $\Lambda = \{\sqrt{\gamma} : \gamma \in \mathbb{T}(\mathbb{R})_{\text{reg}}^0\}$,

$$= \sum_{\mathbb{T}} \frac{1}{m(\mathbb{T})} \sum_u \int_{u\Lambda} \tilde{\Theta}(\delta) \Phi_f^{(\mathbb{T}, \sigma)}(\delta) |\Delta_{\sigma}^{\mathbb{T}}(\delta)|^2 dt$$

where Σ_u indicates summation over representatives in $\mathbb{T}(\mathbb{R})^0$

for a maximal set of non-cohomologous 1-cocycles in \mathbb{T} ,

$$= \sum_{\mathbb{T}} \frac{1}{n(\mathbb{T})} \sum'_u \int_{u\Lambda} \tilde{\Theta}(\delta) \Phi_f^{(\mathbb{T}, \sigma)}(\delta) |\Delta_{\sigma}^{\mathbb{T}}(\delta)|^2 dt$$

where Σ'_u indicates summation over all $u \in \mathbb{T}(\mathbb{R})^0$ such that $u^2 = 1$,

$$= \int \tilde{\Theta}(g) f(g) dg.$$

Thus $\tilde{\Theta} = \Theta$ and the lemma is proved.

(7) For the case $G = \text{SL}_2$ and H an anisotropic torus in G (...excluded by (5.3.1)), the conclusion of Theorem 5.4.1 is valid provided we make a simple change in the factor $\Delta_{(\mathbb{T}, \eta)}$. More explicitly, we have the following. Set

$$r(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad T(\mathbb{R}) = \{r(\theta) : -\pi \leq \theta \leq \pi\}, \quad H = T,$$

$$dt = d\theta, \quad dh = 2 d\theta$$

and $f_H(r(\theta)) = e^{i\theta/2} (e^{i\theta} - e^{-i\theta}) (\phi_f^{(T,\sigma)}(r(\frac{\theta}{2})) - \phi_f^{(T,\sigma)}(r(-\frac{\theta}{2})))$
 for $-\pi < \theta < \pi$, $\theta \neq 0$. Then f_H extends to a smooth function on $T(\mathbb{R})$. This follows easily from the discussion of (5.2)

(cf. (5.2.4)). We investigate this example further. First $L_{\tilde{G}} = L_G^0 \times L_G^0 \rtimes W$ with W acting through $\text{Gal}(\mathbb{T}/\mathbb{R})$, and the nontrivial element of $\text{Gal}(\mathbb{T}/\mathbb{R})$ acting by permutation of factors (we follow the notation of [12] here and below). We embed $L_T = L_H$ in $L_{\tilde{G}}$ as follows:

$$\xi(t \times 1 \times 1) = (t, t^{-1}) \times 1 \times 1, \quad t \in L_T^0$$

$$\xi(1 \times z \times 1) = \left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}_*, \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{bmatrix}_* \right) \times z \times 1,$$

$$\text{for } z = re^{i\theta} \in \mathbb{C}^\times$$

and

$$\xi(1 \times 1 \times \sigma) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_*, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_* \right) \times 1 \times \sigma.$$

This is, clearly, not the embedding provided by the usual

embedding of ${}^L T$ in ${}^L G$ (cf. [9]). Let $\varphi: W \rightarrow {}^L T$ be the homomorphism

$$z \times 1 \rightarrow \begin{bmatrix} e^{2i\theta} & 0 \\ 0 & 1 \end{bmatrix}_* \times z \times 1, \quad 1 \times \sigma \rightarrow 1 \times 1 \times \sigma.$$

Then as pointed out in [6, Example 2], no homomorphism in the equivalence class of $\tilde{\varphi} = \xi \circ \varphi$ factors through the natural embedding ${}^L G \hookrightarrow {}^L \tilde{G}$. However $\tilde{\varphi}$ is " σ -invariant" (cf. [6]) and so the attached representation $\tilde{\pi}$ of $SL_2(\mathbb{C}) \cong \tilde{G}(\mathbb{R})$ has a well-defined (up to sign) twisted character. Fix a sign and denote the twisted character by $\chi_{\tilde{\pi}}^{\sigma}$. By a result (unpublished) of Harish-Chandra, we can regard $\chi_{\tilde{\pi}}^{\sigma}$ as a function on \mathcal{L} . We will not attempt to compute $\chi_{\tilde{\pi}}^{\sigma}$, but just note that because

$\tilde{\pi}(-1) = -1$ (as is easily verified), we have

$$\chi_{\tilde{\pi}}^{\sigma}(w g w^{-1}) = \chi_{\tilde{\pi}}^{\sigma}(\sigma(w^{-1}) w g) = -\chi_{\tilde{\pi}}^{\sigma}(g), \quad \text{for } g \in \mathcal{L} \text{ and } w = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Hence

$$\chi_{\tilde{\pi}}^{\sigma}(r(-\theta)) = -\chi_{\tilde{\pi}}^{\sigma}(r(\theta)), \quad r(\theta) \in T(\mathbb{R})_*$$

and

$$\chi_{\tilde{\pi}}^{\sigma}(a) = 0 \quad \text{for } a \in A(\mathbb{R})_*,$$

A denoting the diagonal subgroup of SL_2 (...the differential equation for $\chi_{\tilde{\pi}}^{\sigma}$ would allow us to say more...).

On the other hand, we can compute explicitly the lift

of the character on $T(\mathbb{R})$ attached to the homomorphism φ above. The character is $r(\theta) \rightarrow e^{in\theta}$ and its lift Θ is given by

$$\begin{aligned}\Theta(f) &= 2 \int_{-\pi}^{\pi} e^{in\theta} f_H(\theta) d\theta \\ &= 2 \int_{-\pi}^{\pi} e^{in\theta} e^{i\theta/2} (e^{i\theta} - e^{-i\theta}) (\Phi_f^{(T,\sigma)}(r(\frac{\theta}{2})) - \Phi_f^{(T,\sigma)}(r(-\frac{\theta}{2}))) d\theta \\ &= 2 \int_{-\pi/2}^{\pi/2} e^{i(2n+1)\theta} (e^{2i\theta} - e^{-2i\theta}) (\Phi_f^{(T,\sigma)}(r(\theta)) - \Phi_f^{(T,\sigma)}(r(-\theta))) d\theta,\end{aligned}$$

where we have adjusted the measure defining $\Phi_f^{(T,\sigma)}$,

$$\begin{aligned}&= 2 \int_{-\pi/2}^{\pi/2} (e^{i(2n+1)\theta} + e^{-i(2n+1)\theta}) (e^{2i\theta} - e^{-2i\theta}) \Phi_f^{(T,\sigma)}(r(\theta)) d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{-(e^{i(2n+1)\theta} + e^{-i(2n+1)\theta})}{e^{2i\theta} - e^{-2i\theta}} |\Delta_{\sigma}^T(r(\theta))|^2 \Phi_f^{(T,\sigma)}(r(\theta)) d\theta \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{-(e^{i(2n+1)\theta} + e^{-i(2n+1)\theta})}{e^{2i\theta} - e^{-2i\theta}} |\Delta_{\sigma}^T(r(\theta))|^2 \Phi_f^{(T,\sigma)}(r(\theta)) d\theta\end{aligned}$$

since $\Phi_f^{(T,\sigma)}(r(-\theta)) = \Phi_f^{(T,\sigma)}(r(\theta+\pi))$. Thus Θ is represented by the function

$$\Theta(\delta) = \begin{cases} \frac{-(e^{i(2n+1)\theta} + e^{-i(2n+1)\theta})}{e^{2i\theta} - e^{-2i\theta}} & \text{if } \delta = r(\theta), \\ & -\pi < \theta < \pi, \theta \neq 0, \pm\pi/2, \\ 0 & \text{if } \delta \in A(\mathbb{R})_*. \end{cases}$$

This gives some suggestion for a functoriality principle.

Note that for the case $G = H = \mathrm{SL}_2$, Corollary 5.4.3 applies

(f_0 can be found in $C_c^\infty(G(\mathbb{R}))$), and so does Corollary 5.4.4.

Using the results for $\mathrm{GL}_2(\mathbb{R})$, the lift of a stable tempered

character on $\mathrm{SL}_2(\mathbb{R})$ is easily shown to be the twisted character

predicted by functoriality. Unlike the lifts in the case

$H = \mathrm{T}$, this twisted character is, of course, invariant under

stable twisted-conjugation.

(8) Let G be a connected reductive quasi-split group over \mathbb{R} . We do not assume (4.3.2). For each maximal torus T over \mathbb{R} in G there is a natural projection of $H^1(T)^\vee$ onto $\mathcal{E}(T)^\vee$, where $^\vee$ indicates the dual group. For each T fix some cross-section C_T for this projection. Assume that $(C_T(\kappa))^w = C_{T^w}(\kappa^w)$ for each $w \in \mathcal{G}(T)$ and that if α is an imaginary root of T , s is a Cayley transform with respect to α and $\kappa(\alpha^\vee) = 1$, then $(C_T(\kappa))^s = C_{T_s}(\kappa^s)$. If H is an endoscopic group for G , $(T, \eta) \in \mathcal{J}_H(G)$ (cf. (5.3)) and κ is a quasicharacter attached to (T, η) , we will write

$$\Phi_f^{(T, \sigma, \kappa)}(\delta, dt, dg) \quad \text{for} \quad \Phi_f^{(T, \sigma, C_T(\kappa))}(\delta, dt, dg)$$

where $C_T(\kappa)$ denotes C_T applied to the restriction of κ to $\mathcal{E}(T)$. Suppose, as before, that $\sqrt{\quad}$ is a smooth cross-section for η and that H satisfies (5.3.1). Then the statement of Theorem 5.4.1 remains true with our new definition of $\Phi_f^{(T, \sigma, \kappa)}$. This is verified by an examination of our earlier arguments (... the assumption (4.3.2) played no real role until we came to consider endoscopic groups).

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