ON THE STRUCTURE OF ENDOSCOPIC TRANSFER FACTORS

D. SHELSTAD

ABSTRACT. There are two versions of endoscopic transfer factors to accommodate the different versions, classical or renormalized, of the local Langlands correspondence. An examination of the structure of these complex-valued factors shows that the versions are related in a simple manner. We gather various consequences and indicate ways in which these are useful for the harmonic analysis associated with endoscopic transfer.

1. Introduction

The purpose of this paper is to examine certain properties of endoscopic transfer factors. We are guided by the discussion in Sections 4 and 5 of [KS12] on the expected two versions for the local Langlands correspondence and some of the implications for transfer.

We begin with transfer factors for twisted endoscopy, specifically the factors Δ' and Δ_D of [KS12] for the transfer of orbital integrals. These are assembled in [KS12], primarily from terms in [KS99] and [LS87]. Our first goal here is a brief review and then to establish a simple relation between Δ' and Δ_D; see Corollary 6.1.2 and then Lemmas 6.3.1, 6.3.2 for the particular case of Whittaker normalization. This allows for quick passage back and forth between the associated transfers; see (6.4) where we make this explicit. In the nonarchimedean case the celebrated theorem of Waldspurger proves the existence of transfer for Δ' [Wa08]. In the archimedean case, [Sh12] also uses Δ'.

We continue in the archimedean case to the attached spectral transfer, confining our attention here to tempered representations and standard endoscopy. The spectral transfer factors of [Sh10] provide coefficients for the transfer of traces dual to the transfer of orbital integrals. There are relations that parallel our earlier observations; see Lemmas 8.2.1, 8.3.1. These are applied to the dual transfer in (8.4) and (8.5).

Our main interest is then the structure on tempered packets provided by the spectral transfer factors; see (9.1). We also relate the two versions of the *strong base point property* for tempered packets in the Whittaker normalization, thereby confirming that the renormalized version is also true; see Lemma 9.2.1. Our results will be applied to the twisted setting in [Sh] where we construct twisted spectral factors that provide an additional structure on packets preserved by the twisting. Compatibility of the additional structure with the standard one is of significance globally [Ar13, Chapter 2]. This provides additional motivation for our approach in Sections 2 - 4.

Note: A preliminary version of this paper was posted on the author's website andromeda.rutgers.edu/~shelstad in July, 2012.

2. **Setting**

We consider twisted endoscopy over a local field F of characteristic zero. Thus G is a connected reductive algebraic group defined over F, θ is an F-automorphism of G and a is a 1-cocycle of the Weil group W_F in the center $Z_{G^\vee}$ of the complex dual group $G^\vee$. There are twisting characters on $G(F)$ attached to a which we label as ϖ, ϖ_D ([KS99], [KS12] use the notation ω, ω_D); they are discussed in (4.2). Although we do not use it explicitly here, there is also an underlying inner twist (G, θ, ψ) of a quasi-split pair (G^*, θ^*), where θ^* preserves an F-splitting of G^*; see Appendix B of [KS99] for definitions. In the transfer theorems we cite, it is assumed that θ^* has finite order.

For a set $\mathfrak{e} = (H, \mathcal{H}, s)$ of endoscopic data we follow the definitions of Chapter 2 of [KS99], except that we assume harmlessly that $\mathcal{H}$ is a subgroup of $^L G$ and $\xi : \mathcal{H} \to {}^L G$ is inclusion. We then usually ignore ξ in notation. We also fix a z-pair (H_1, ξ_1), again as in [KS99], and write $\mathfrak{e}_z$ for the endoscopic data $\mathfrak{e}$ so supplemented. Then H_1 is the *endoscopic group* attached to $\mathfrak{e}_z$ by which we mean the group that appears in the associated endoscopic transfer to G.

We will call the supplemented data $\mathfrak{e}_z$ *bounded* if the section $c : W_F \to \mathcal{H}$ of p. 18 of [KS99] and the embedding $\xi_1 : \mathcal{H} \to {}^L H_1$ are such that $\xi \circ c$, $\xi_1 \circ c$ are bounded Langlands parameters for G, H_1 respectively. It is not difficult to check that given $\mathfrak{e}$, we may always choose c and ξ_1 so that this is true. Boundedness of $\mathfrak{e}_z$ will be used in Lemma 5.0.2 and in the notion of a related pair of bounded Langlands parameters for H_1, G in (7.2).

There is a map $\mathfrak{e} \to \mathfrak{e}'$ on endoscopic data which we will use just in the case of standard endoscopy ($\theta = id$, $a = 1$) where it is particularly simple to describe. Thus suppose that $\mathfrak{e} = (H, \mathcal{H}, s)$ is standard. This returns us to the setting of [LS87], although z-pairs are used a little differently there. We will follow [KS99] regarding z-pairs; see (3.1) below. From $\mathfrak{e}$ we obtain another standard set of endoscopic data $\mathfrak{e}'$ by inverting the datum s: $\mathfrak{e}' = (H, \mathcal{H}, s^{-1})$. The same z-pair (H_1, ξ_{H_1}) serves $\mathfrak{e}'$ and we denote the supplemented data by $\mathfrak{e}'_z$. The endoscopic group and the point correspondence for geometric transfer are unchanged. Write Δ for the relative transfer factor attached to $\mathfrak{e}_z$ in [LS87]; see (3.1). Then we attach to $\mathfrak{e}_z$ another transfer factor Δ' by taking the factor Δ for $\mathfrak{e}'_z$. This construction has the same effect as inverting the Tate-Nakayama pairing in the definition of Δ; see (3.1) again. We may also apply it in other settings; see (6.2), (6.3).

2.1. **Organizing terms in relative transfer factors.** Terms to be used in the definition of (relative) geometric transfer factors are described in Section 3 of [LS87] and Section 4 of [KS99]. One is a quotient labelled Δ_{IV} made from discriminants. It may be used instead in the definition of orbital integrals themselves, specifically in the normalization of invariant measures on orbits. Lemma 2.12 of [La83] (on measures for orbital integrals as fiber integrals) provides additional geometric motivation for doing so. Thus, throughout this paper:

- *we omit Δ_{IV} from all formulas for transfer factors.*

Of the other terms, Δ_{II} is defined concretely and the properties we will need are read off quickly from lemmas in [LS87] and [KS99]. There remain Δ_I and Δ_{III}, and in the case of standard endoscopy Δ_{III} splits into a product $\Delta_{III_1} \Delta_{III_2}$. These various terms can be gathered into a single term quite naturally (as remarked in [KS99]) but that is not our concern here. As defined in [LS87] and [KS99], each term

Δ_I, Δ_{III}, Δ_{III_1}, Δ_{III_2} is given by a pairing in Galois (hyper)cohomology. It will be helpful to review briefly the pairings involved and their various compatibilities.

2.2. **Pairings in Galois cohomology.** We return to Appendix A of [KS99]. In (A.3.12) a pairing $\langle -, - \rangle$ is defined between two groups. The first is the Galois hypercohomology group $H^1(F, T \xrightarrow{f} U)$ attached there to the complex $T \xrightarrow{f} U$ of tori over F, concentrated in degrees 0 and 1. The second is the group $H^1(W_F, \widehat{U} \xrightarrow{\widehat{f}} \widehat{T})$ attached to the complex $\widehat{U} \xrightarrow{\widehat{f}} \widehat{T}$, concentrated in the same degrees and made from the corresponding $\mathbb{C}$-duals.

In the case that f is trivial, the pairing $\langle -, - \rangle$ is the product of the classical Langlands pairing (or of the *inverse* of the renormalized Langlands pairing $\langle -, - \rangle_D$) and the Tate-Nakayama pairing as on p. 138 of [KS99] which we will denote temporarily by $\langle -, - \rangle_{tn}$.

In general, the compatibilities for $\langle -, - \rangle$ are as follows:

$$(2.2.1) \qquad \langle j(u), \widehat{z} \rangle = \left\langle u, \widehat{i}(\widehat{z}) \right\rangle_D^{-1}$$

and

$$(2.2.2) \qquad \left\langle z, \widehat{j}(\widehat{t}) \right\rangle = \left\langle i(z), \widehat{t} \right\rangle_{tn}.$$

We refer to pp. 137, 138 of [KS99] for unexplained notation. We also record the compatibility statements for the inverted pairing $\langle -, - \rangle_*$ as

$$(2.2.3) \qquad \langle j(u), \widehat{z} \rangle_* = \left\langle u, \widehat{i}(\widehat{z}) \right\rangle^{-1},$$

where on the right we have the classical Langlands pairing for tori, and

$$(2.2.4) \qquad \left\langle z, \widehat{j}(\widehat{t}) \right\rangle_* = \left\langle i(z), \widehat{t} \right\rangle_{tn}^{-1}.$$

3. **Standard endoscopy**

3.1. **Factors Δ and Δ'.** As remarked already, we will use the convention of [KS99] for z-pairs. Thus the standard *relative* transfer factor $\Delta = \Delta(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)$ is attached to two very regular related pairs (γ_1, δ), $(\gamma_1^\dagger, \delta^\dagger)$ of points in $H_1(F) \times G(F)$. In brief, (γ_1, δ) is a very regular related pair if γ_1 is strongly G-regular, δ is strongly regular and the image γ of γ_1 in $H(F)$ coincides with the image of δ under the inverse $T \to T_H$ of an admissible isomorphism of maximal tori $T_H \to T$, where $T_H = Cent(\gamma, H)$ and $T = Cent(\delta, G)$; see [LS87]. When we come to the twisted case in the next section we will use the generalization of this formalism to the norm correspondence in Section 3 of [KS99]; there $T \to T_H$ becomes an *abstract norm map*.

Following our removal of Δ_{IV}, the factor Δ is a product

$$(3.1.1) \qquad \Delta := \Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_2}.$$

Only the term $\Delta_{III_1} = \Delta_{III_1}(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)$ is genuinely relative in the sense that every other term is defined as a quotient of absolute terms, $\Delta_I(\gamma_1, \delta) \big/ \Delta_I(\gamma_1^\dagger, \delta^\dagger)$ and so on.

With the exception of Δ_{III_2}, each term depends on the images γ, $\gamma^\dagger$ of γ_1, $\gamma_1^\dagger$ in $H(F) = H_1(F)/Z_1(F)$ rather than on γ_1, $\gamma_1^\dagger$ themselves and so, with this exception, we may use the definitions of Section 3 of [LS87] as written.

Now consider the definition of $\Delta_{III_2}(\gamma_1, \delta)$. If $H_1 = H$, so that $\gamma_1 = \gamma$, then we again follow [LS87]. Namely, fix χ-data $\chi = \{\chi_\alpha\}$ for the tori T_H, T as in (3.1) of [LS87]. In fact, we will always make in advance all three choices of that section, namely χ-data, a-data and toral data $T_H \to T$, whenever we write the terms of transfer factors. Write $a(\chi)$ for the cocycle a defined on p. 247. It defines an element of $H^1(W_F, T^\vee)$ which we also denote by $a(\chi)$. Then

$$(3.1.2) \qquad \Delta_{III_2}(\gamma, \delta) := \langle a(\chi), \delta \rangle,$$

where $\langle -, - \rangle$ denotes the classical Langlands pairing for tori.

Now to drop the assumption $H_1 = H$, we replace T by the pullback T_2 of the isomorphism $T \to T_H$ given by toral data and the projection $T_1 \to T_H = T_1 / Z_1$; T_2 is a torus and is labelled T_1 on p. 42 of [KS99] where our present T_1 is written T_{H_1}. Define $\delta_2 = (\gamma_1, \delta) \in T_2(F)$ and $a_2(\chi)$ to be the class in $H^1(W_F, T_2^\vee)$ of the cocycle $a_T(w)$ on p. 45 of [KS99]. Then

$$(3.1.3) \qquad \Delta_{III_2}(\gamma, \delta) := \langle a_2(\chi), \delta_2 \rangle.$$

The terms Δ_I and Δ_{III_1} are defined in terms of the Tate-Nakayama pairing for tori. We see immediately from the definitions in [LS87] that replacing the endoscopic datum s by s^{-1} inverts these terms, and that no others are affected. Thus the factor Δ' can be written as

$$(3.1.4) \qquad \Delta' = (\Delta_I)^{-1} \Delta_{II} \, (\Delta_{III_1})^{-1} \Delta_{III_2}.$$

In the archimedean case, both Δ_I and Δ_{III_1} are signs, and so Δ' coincides with Δ.

3.2. **Factor Δ_D.** It will be helpful to introduce the factor Δ_D of (5.1) in [KS12] slightly differently, and we also alter notation. Namely, we now define the term $\Delta_{III_2,D}$ by just replacing the pairing in the definition of Δ_{III_2} with the renormalized one. In other words,

$$(3.2.1) \qquad \Delta_{III_2,D} := (\Delta_{III_2})^{-1}.$$

The factor Δ_D is then given as

$$(3.2.2) \qquad \Delta_D := \Delta_I \, (\Delta_{II})^{-1} \Delta_{III_1} \Delta_{III_2,D}.$$

Thus it is immediate that

$$(3.2.3) \qquad \Delta_D = (\Delta')^{-1}.$$

The formula (5.3.1) of [KS12] shows that we have not changed the factor Δ_D itself, at least if $H_1 = H$. We review the argument in our present notation. The terms Δ_{II} and Δ_{III_2} each depend on the choice of χ-data but their product does not, and no other term depends on χ-data; see Lemmas 3.3.D, 3.5.A and the proof of Theorem 3.7.A of [LS87], with minor adjustments for the case $H_1 \neq H$ (or we could just cite Theorem 4.6.A of [KS99]). Write χ^{-1} for the χ-data $\{\chi_\alpha^{-1}\}$ and set $\Delta_{II} = \Delta_{II}(\chi)$ when χ is used in the definition of Δ_{II}. Clearly,

$$(3.2.4) \qquad (\Delta_{II}(\chi))^{-1} = \Delta_{II}(\chi^{-1}).$$

The term Δ_{II} also depends on the choice of a-data; we assume the same choice on both sides of (3.2.4).

The use of χ-data in the definition of Δ_{III_2} is more subtle but we do have the simple relation

$$(3.2.5) \qquad \Delta_{III_2,D}(\gamma, \delta) = \langle a_2(\chi), \delta_2 \rangle_D = \langle a_2(\chi), \delta_2 \rangle^{-1}.$$

Because we may replace χ by χ^{-1} without affecting $\Delta_{II}\Delta_{III_2}$ we may rewrite Δ_D in a form where only Δ_{III_2} is changed:

$$(3.2.6) \qquad \Delta_D = \Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_{2,D}}(\chi^{-1}).$$

Since

$$(3.2.7) \qquad \Delta_{III_{2,D}}(\chi^{-1})(\gamma, \delta) = \left\langle a_2(\chi^{-1}), \delta_2 \right\rangle_D = \left\langle a_2(\chi^{-1}), \delta_2 \right\rangle^{-1},$$

this agrees with the definition of Δ_D in (5.1) of [KS12], which completes our review.

4. Twisted transfer factors

We will be working with the terms Δ_I, Δ_{II}, Δ_{III} as defined in [KS99], with the following caveat.

- Δ_I *is replaced by the term* Δ_I^{new} *of (3.4) in* [KS12] *without change in notation,*
- Δ_{II} *is exactly as in (4.3) of* [KS99],
- Δ_{III} *is exactly as in (4.4) of* [KS99]; *in particular, it is defined in terms of the pairing (A.3.12).*

Again we are concerned with relative transfer factors, and as in the standard case above, only Δ_{III} is genuinely relative. We will also assume for convenience that we are in the setting of Chapter 4 of [KS99], where no twisting is needed on the endoscopic group. The modifications for the general case in Sections 5.3 and 5.4 of [KS99] are simple but require a more detailed look at the norm correspondence.

It is convenient to introduce also the term Δ_{III}^* defined via the inverted pairing $\langle -, - \rangle_*$ from (2.2) above. Of course we have $\Delta_{III}^* = (\Delta_{III})^{-1}$. We will also sometimes write Δ_I^* for $(\Delta_I)^{-1}$.

Recall from Section 1 of [KS12] that the product $\Delta_I \Delta_{II} \Delta_{III}$, called Δ in [KS99], is *not* a transfer factor. In other words, the constructions of Section 4 of [KS99] and the pairing of (A.3.12) do not provide an extension of the factor Δ of [LS87] to the twisted case. Following [KS12] we may extend Δ' by these methods (that this is possible was pointed out by Waldspurger), and also Δ_D. In (4.1) and (4.2) we give a brief review.

4.1. **Factor Δ'.** First, for compatibility with the classical local Langlands correspondence, we follow [KS12] and introduce

$$(4.1.1) \qquad \Delta' := (\Delta_I \Delta_{III})^{-1} \Delta_{II} = \Delta_I^* \Delta_{III}^* \Delta_{II}.$$

In the case of standard endoscopy, the definitions of Section 4.4 of [KS99] imply that

$$(4.1.2) \qquad \Delta_{III} = \Delta_{III_1}(\Delta_{III_2})^{-1} = \Delta_{III_1} \Delta_{III_{2,D}}$$

(see also (2.2) above), and so

$$(4.1.3) \qquad \Delta' = (\Delta_I \Delta_{III_1})^{-1} \Delta_{III_2} \Delta_{II}$$

which coincides with our definition of Δ' using [LS87].

Remark: In [KS12] we corrected an error in [KS99] by replacing Δ with Δ'. Here we highlight changes for key arguments in Chapters 4 and 5. First, we replace $\langle -, - \rangle$ by $\langle -, - \rangle_*$ throughout. Theorem 4.6.A now states the independence of Δ' from the choices for χ-data, a-data and toral data. It follows because (i) Δ_{III}^* has the properties we wanted for Δ_{III} in regard to χ-data and the classical Langlands pairing for tori, (ii) use of $\langle -, - \rangle_*$ does not affect cancellation for Δ_I^*, Δ_{III}^* in regard

to toral data, and (iii) a change in a-data causes only sign changes in each of Δ_I, Δ_{II} and so Δ_I^*, Δ_{II} works just as well as Δ_I, Δ_{II}. Now consider the results of (5.1) in [KS99]. If we replace Δ by Δ' in Lemmas 5.1.A, 5.1.B then the statements remain correct and the same proofs apply. Lemma 5.1.C and Theorem 5.1.D may be rewritten as statements about the values of relative transfer factors (we write an analogue for Δ_D explicitly in (4.2.8) below). Lemma 5.1.C and its generalization at the bottom of p. 53 also concern characters on certain central subgroups. For the proofs we follow those in [KS99] and continue to use the classical Langlands pairing for tori. Next, the formula (1) of Theorem 5.1.D links our factor with the notion of κ-orbital integral. The statement now involves $\langle -, - \rangle_*$, and the proof is the same. The formula (2) in Theorem 5.1.D concerns twisting by a nontrivial quasi-character and is critical for a well-defined transfer statement for twisted orbital integrals in that case. The proof is delicate, but we may again follow line by line that in [KS99] with the caveats we have made. In particular, the parenthetical remark on p. 56, l. -9 is replaced by *use the pairing* $\langle -, - \rangle_*$, and on the next line we invoke (2.2.3) above in place of (A.3.13).

4.2. **Factor Δ_D.** Next, for compatibility with the renormalized Langlands correspondence, we have

$$(4.2.1) \qquad \Delta_D := \Delta_I \Delta_{III} (\Delta_{II})^{-1}.$$

It of course extends Δ_D in the standard case, and again

$$(4.2.2) \qquad \Delta_D = (\Delta')^{-1}.$$

As in the standard case, we can replace χ-data χ by χ^{-1} without harm. Thus we get the alternative formula

$$(4.2.3) \qquad \Delta_D = \Delta_I \Delta_{II} \Delta_{III}(\chi^{-1}) = \Delta_I \Delta_{II} \Delta_{III}^*(\chi^{-1})^{-1}.$$

For key properties of Δ_D we can appeal to (4.2.2) and the corresponding properties for Δ' reviewed in the last section (see Remark). Recall from [KS12] that in place of the two twisting characters ϖ, λ_{H_1} we now use their companions ϖ_D, $\lambda_{H_1,D}$ in the various statements. More precisely, the quasi-character ϖ on $G(F)$ attached on p.17 of [KS99] to the class $\mathbf{a}$ in $H^1(W_F, Z_{G^\vee})$ of our twisting datum a is replaced by ϖ_D given by

$$(4.2.4) \qquad \varpi_D(g) := \langle g, \mathbf{a} \rangle_D = \varpi(g)^{-1}.$$

Also the quasi-character λ_{H_1} on $Z_1(F)$ is replaced by $\lambda_{H_1,D}$, where

$$(4.2.5) \qquad \lambda_{H_1,D}(h_1) := \langle h_1, \mathbf{b} \rangle_D = \lambda_{H_1}(h_1)^{-1}$$

and $\mathbf{b}$ is the Langlands parameter defined by the homomorphism

$$(4.2.6) \qquad W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\xi_1} {}^L H_1 \longrightarrow {}^L Z_1$$

considered on p. 23. For example, we restate Lemma 5.1.C as

$$(4.2.7) \qquad \Delta_D(z_1 \gamma_1, \delta) = \lambda_{H_1,D}(z_1)^{-1} \Delta_D(\gamma_1, \delta),$$

or, in terms of relative factors,

$$(4.2.8) \qquad \Delta_D(z_1 \gamma_1, \delta; \gamma_1, \delta) = \lambda_{H_1,D}(z_1)^{-1},$$

and (2) of Theorem 5.1.D as

$$(4.2.9) \qquad \Delta_D(\gamma_1, \delta') = \varpi_D(h) \Delta_D(\gamma_1, \delta),$$

where $\delta' = h^{-1}\delta\theta(h)$ and $h \in G(F)$.

5. Two lemmas for relative factors

We repeat (4.2.2) as:

Lemma 5.0.1. *For all very regular related pairs* (γ_1, δ), $(\gamma_1^\dagger, \delta^\dagger)$ *we have*

$$\Delta_D(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger) = \Delta'(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)^{-1}.$$

Recall from Section 2 that $\mathfrak{e}_z$ is our notation for a set of endoscopic data supplemented by the choice of a z-pair.

Lemma 5.0.2. *For all very regular related pairs* (γ_1, δ), $(\gamma_1^\dagger, \delta^\dagger)$ *we have*

$$\left|\Delta_D(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)\right| = \left|\Delta'(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)\right| = 1$$

provided $\mathfrak{e}_z$ *is bounded.*

Proof. Consider first the case of standard endoscopy since the argument here points the way to the general case. Since Δ_I and Δ_{III_1} are defined by Tate-Nakayama pairings we have $|\Delta_I| = |\Delta_{III_1}| = 1$. A check of definitions [LS87] shows that we may choose the quasi-characters of χ-data to be unitary. Then $|\Delta_{II}| = 1$. It remains to consider Δ_{III_2}. Now because the χ-data are unitary and $\mathfrak{e}_z$ is assumed bounded, the cocycle $a_2(\chi)$ of (3.1) is bounded; recall this is the cocycle $a_T(w)$ constructed on p. 45 of [KS99]. Then we apply the second part of Theorem 1 of [La97] to conclude that $|\Delta_{III_2}| = 1$, and so we are done in the standard case.

Turning to the general case, we have then only to check that $|\Delta_{III}| = 1$ given that $\mathfrak{e}_z$ is bounded and the χ-data are unitary. We could verify an analogue of Langlands' result for the hypercohomology pairing of (2.2) from [KS99]. This is straightforward since the arguments of [KS99] are based on those of Langlands: we replace $\mathbb{C}^\times$ by the unit circle $\mathbb{C}_u^\times$ (notation of [La97]) at certain steps in the argument. Alternatively, we may return directly to the standard case by observing that it is enough to prove this for a sufficiently high power of the pairing. Then it is easy to check in our application that we may assume that, in the notation of Section 2.2, the first term in the pairing is in the image of j. We have already noted that the cocycle $a_T(w)$ is bounded. Now the compatibility rules (2.2.1) and (2.2.3) complete the argument. $\square$

Corollary 5.0.3. *For all very regular related pairs* (γ_1, δ), $(\gamma_1^\dagger, \delta^\dagger)$ *we have*

$$\Delta_D(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger) = \overline{\Delta'(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)}$$

provided $\mathfrak{e}_z$ *is bounded.*

6. Absolute transfer factors

6.1. Normalization in general.
To reduce the notational burden, we assume the following for the rest of this paper.

- *The supplemented endoscopic data* $\mathfrak{e}_z$ *are bounded.*

Adjustments for the general case are minor.

Just for this subsection, we write the relative factors Δ' and Δ_D as Δ'_{rel} and $\Delta_{D,rel}$. By an *absolute* transfer factor Δ' we mean a complex-valued function Δ' on all very regular pairs (γ_1, δ) with the following two properties: $\Delta'(\gamma_1, \delta) \neq 0$ if and only if the pair (γ_1, δ) is related, and

$$(6.1.1) \qquad \Delta'(\gamma_1, \delta) \diagup \Delta'(\gamma_1^\dagger, \delta^\dagger) = \Delta'_{rel}(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger)$$

whenever each pair $(\gamma_1, \delta), (\gamma_1^\dagger, \delta^\dagger)$ is related. We define absolute Δ_D analogously.

Lemma 6.1.1. *Suppose that Δ' is an absolute transfer factor for Δ'_{rel}. Then $(\Delta')^{-1}$, $\overline{\Delta'}$ are absolute transfer factors for $\Delta_{D,rel}$.*

Here by $(\Delta')^{-1}$ we mean the factor defined by $(\Delta')^{-1}(\gamma_1, \delta) = \Delta'(\gamma_1, \delta)^{-1}$ for a very regular *related* pair (γ_1, δ) and by $(\Delta')^{-1}(\gamma_1, \delta) = 0$ for an unrelated pair.

Proof. This is immediate from Lemma 5.0.1 and Corollary 5.0.3. $\qquad\qquad$ □

Corollary 6.1.2. *Let Δ', Δ_D be absolute transfer factors for Δ'_{rel}, $\Delta_{D,rel}$ respectively. Then there exists $c \in \mathbb{C}^\times$ such that*

$$(6.1.2) \qquad\qquad \Delta_D(\gamma_1, \delta) = c \, \overline{\Delta'(\gamma_1, \delta)}$$

for all very regular pairs (γ_1, δ).

There are significant cases where $c \neq 1$. Examples are provided by the Whittaker normalization when we define absolute Δ', Δ_D using the *same* Whittaker data for each; see Lemma 6.3.1 below.

6.2. **Quasi-split setting.** Assume that G is quasi-split over F and choose an F-splitting spl. We start with the standard case. There are, of course, absolute versions of Δ_I, Δ_{II} and Δ_{III_2}. This is true for any G, and absolute Δ_I depends additionally on the choice of spl (see 3.2 in [LS87]). Just in our present quasi-split setting, an absolute version of Δ_{III_1} is defined on p. 248 of [LS87], and

$$(6.2.1) \qquad\qquad \Delta_0 := \Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_2}$$

is a transfer factor in the usual sense:

$$(6.2.2) \qquad\qquad \Delta_0(\gamma_1, \delta) \diagup \Delta_0(\gamma_1^\dagger, \delta^\dagger) = \Delta(\gamma_1, \delta; \gamma_1^\dagger, \delta^\dagger),$$

for all very regular related pairs $(\gamma_1, \delta), (\gamma_1^\dagger, \delta^\dagger)$. We may apply the $'$-operation to Δ_0 to obtain another factor

$$(6.2.3) \qquad\qquad \Delta_0' := (\Delta_I)^{-1} \Delta_{II} (\Delta_{III_1})^{-1} \Delta_{III_2}.$$

Similarly, we define

$$(6.2.4) \qquad\qquad \Delta_{0,D} := \Delta_I (\Delta_{II})^{-1} \Delta_{III_1} \Delta_{III_{2,D}}.$$

This extends to twisted factors [KS12]. Assume that the twisting automorphism θ preserves spl. Then we have the twisted factors

$$(6.2.5) \qquad\qquad \Delta_0' := (\Delta_I)^{-1} \Delta_{II} (\Delta_{III})^{-1}$$

and

$$(6.2.6) \qquad\qquad \Delta_{0,D} := \Delta_I (\Delta_{II})^{-1} \Delta_{III},$$

where the absolute twisted factor Δ_{III} is that defined on p. 63 of [KS99]. It is clear then that

$$(6.2.7) \qquad\qquad (\Delta_0')^{-1} = \Delta_{0,D},$$

provided we define Δ_I in terms of the same *spl* in each case.

6.3. **Whittaker normalization.** We continue in the setting of (6.2). The Whittaker normalization was introduced in Sections 5.2 - 5.4 of [KS99]; see also (5.5) in [KS12]. It shifts dependence from the F-splitting *spl* preserved by θ to a set λ of Whittaker data preserved by θ. This is done by inserting a certain epsilon factor $\varepsilon_L(V, \psi)$ [KS99, p. 65] in the factors Δ_0' and $\Delta_{0,D}$. Thus

$$(6.3.1) \qquad \Delta_\lambda' := \varepsilon_L(V, \psi)\Delta_0' = \varepsilon_L(V, \psi)(\Delta_I)^{-1}\Delta_{II}(\Delta_{III})^{-1}$$

and

$$(6.3.2) \qquad \Delta_{\lambda,D} := \varepsilon_L(V, \psi)\Delta_{0,D} = \varepsilon_L(V, \psi)\Delta_I(\Delta_{II})^{-1}\Delta_{III}.$$

A check of the arguments on pp. 65-66 of [KS99] shows that Δ_λ' as well as $\Delta_{\lambda,D}$ depends only on the choice of λ.

Lemma 6.3.1.

$$(6.3.3) \qquad \Delta_{\lambda,D}(\gamma_1, \delta) = c\,\overline{\Delta_\lambda'(\gamma_1, \delta)}$$

for all very regular pairs (γ_1, δ), *where*

$$(6.3.4) \qquad c = (det(V))(-1).$$

Here we regard $det(V)$ as a character on $F^\times$; the choice of local classfield theory isomorphism for this does not matter.

Proof. Comparing the right sides of (6.3.1) and (6.3.2), we see that this amounts to a property of $\varepsilon_L(V, \psi)$ given on p. 65 of [KS99], namely $(\varepsilon_L(V, \psi))^2 = (det(V))(-1)$. The property comes from the fact that the chosen V is defined over $\mathbb{R}$. $\square$

Next we replace λ by $\overline{\lambda} = \lambda^{-1}$ on the left side of (6.3.3). Here we are using λ as notation both for Whittaker data, namely a $G(F)$-conjugacy class of Whittaker characters, and for a representative character.

Lemma 6.3.2.

$$(6.3.5) \qquad \Delta_{\overline{\lambda},D}(\gamma_1, \delta) = \overline{\Delta_\lambda'(\gamma_1, \delta)}$$

for all very regular pairs (γ_1, δ).

Proof. Again the needed property is found on p. 65 of [KS99]. With representative λ defined using the F-splitting *spl* and character ψ, replacing λ by $\overline{\lambda}$ amounts to replacing ψ by ψ_{-1}, and then $\Delta_{\overline{\lambda},D}(\gamma_1, \delta) = c\,\Delta_{\lambda,D}(\gamma_1, \delta)$, with c as in (6.3.4). The lemma now follows. $\square$

6.4. **Consequences for transfer of orbital integrals.** Suppose Δ' and Δ_D are arbitrary normalizations, *i.e.* are absolute transfer factors for Δ_{rel}', $\Delta_{D,rel}$ respectively. Assume Δ'-transfer exists. Thus if f is a test function on $G(F)$, *i.e.* $f \in C_c^\infty(G(F))$, then there exists a test function f_1 on $H_1(F)$, *i.e.* $f_1 \in C_c^\infty(H_1(F), \lambda_{H_1})$, with Δ'-matching orbital integrals:

$$(6.4.1) \qquad SO(\gamma_1, f_1) = \sum_{\{\delta\}} \Delta'(\gamma_1, \delta)\, O^{\theta,\varpi}(\delta, f)$$

for all strongly G-regular elements γ_1 in $H_1(F)$. Here we follow the usual conventions for orbital integrals [Wa08], and the sum is over θ-conjugacy classes $\{\delta\}$ of strongly θ-regular elements δ in $G(F)$.

Recall Corollary 6.1.2 above and notice that the function

$$(6.4.2) \qquad\qquad f_{1,D} := c\,\overline{(\overline{f})_1}$$

lies in $C_c^\infty(H_1(F), \lambda_{H_1,D})$. Then, applying complex conjugation to (6.4.1) with $\overline{f}$ in place of f, we see that f and $f_{1,D}$ have Δ_D-matching orbital integrals:

$$(6.4.3) \qquad SO(\gamma_1, f_{1,D}) = \sum_{\{\delta\}} \Delta_D(\gamma_1, \delta)\, O^{\theta,\varpi_D}(\delta, f)$$

for all strongly G-regular elements γ_1 in $H_1(F)$. Thus Δ_D-transfer exists. Conversely, assume that Δ_D-transfer exists. Then we argue similarly to recover the existence of Δ'-transfer.

Remark: See [Ka13] for an alternate approach to the case of Whittaker normalization for quasi-split groups. There is a further comment on this in (7.4) when we come to the dual spectral transfer for real groups. We will see that our approach with complex conjugation is sufficient for the tempered spectrum, both in the Whittaker case and in general, because of unitarity and the simple relations of Section 5. We expect other applications also.

Remark: The arguments of [Sh12] can also be used to prove *directly* a Δ_D-transfer of orbital integrals in the archimedean case. First, notice that [Sh12] proves transfer for the factor $\Delta' = \Delta_I^* \Delta_{II} \Delta_{III}^*$. Since $\Delta_I = \Delta_I^*$ in the archimedean setting, proving transfer for Δ_D then amounts to checking that Δ_{III}^* may be replaced by $\Delta_{III}^*(\chi^{-1})^{-1}$ in the limit formula of the main lemma (Lemma 9.3). We may certainly invert both sides of that limit formula without harm, and further it is clear from the definitions in Section 3 of the paper that we may also invert the two sets of χ-data simultaneously. Thus we are done.

7. Archimedean case: spectral setting

From now on, $F = \mathbb{R}$ and we consider only standard endoscopy. In this setting, the relative transfer factors Δ'_{rel} and Δ_{rel} are the same (recall (3.1) above) and so we may as well normalize the absolute factors to be the same also. Thus we will use the notation Δ from [Sh10] interchangeably with Δ'.

7.1. **Statement of dual tempered transfer.** Our interest is in the tempered spectral transfer dual to (6.4.1), the tempered spectral transfer dual to (6.4.3), and the precise relation between the two.

We state the two dual transfer formulas as

$$(7.1.1) \qquad St\text{-}Trace\ \pi_1(f_1) = \sum_{\pi} \Delta'(\pi_1, \pi)\ Trace\ \pi(f)$$

and

$$(7.1.2) \qquad St\text{-}Trace\ \pi_{1,D}(f_{1,D}) = \sum_{\pi} \Delta_D(\pi_{1,D}, \pi)\ Trace\ \pi(f).$$

Here π_1, $\pi_{1,D}$ denote tempered irreducible admissible representations of $H_1(\mathbb{R})$ for which the central subgroup $Z_1(\mathbb{R})$ acts by the character λ_{H_1}, $\lambda_{H_1,D}$ respectively, and π denotes an tempered irreducible admissible representation of $G(\mathbb{R})$. Notationally we identify a representation with its isomorphism class. Also, we may as well limit our attention to unitary representations on Hilbert spaces (and unitary isomorphism). For each π_1 there will be only finitely many (isomorphism classes)

π such that $\Delta'(\pi_1, \pi)$ is nonzero, and similarly for each $\pi_{1,D}$ there will be only finitely many π such that $\Delta_D(\pi_{1,D}, \pi)$ is nonzero. This is by construction, although it may be argued directly that this has to be the case. Further, we will see that such π form the packet, classical in the case of Δ', renormalized in the case of Δ_D, predicted by Langlands' functoriality principle.

The terms $St\text{-}Trace\ \pi_1$, $St\text{-}Trace\ \pi_{1,D}$ on the left sides denote the stable trace over the packet of π_1, $\pi_{1,D}$ respectively. Each of these is a stable tempered distribution on $H_1(\mathbb{R})$.

The transfer theorems state that for f, f_1 as in (6.4.1) and $f, f_{1,D}$ as in (6.4.3), the formulas (7.1.1) and (7.1.2) are true when $\Delta'(\pi_1, \pi)$, $\Delta_D(\pi_{1,D}, \pi)$ are as we specify. The factor $\Delta'(\pi_1, \pi)$ is the factor $\Delta(\pi_1, \pi)$ from [Sh10] and the associated transfer theorem has been proved there following the method of [Sh82]. We review briefly and build on this.

As on the geometric side, we first isolate and examine a *very regular* setting. To this end, we introduce the notion of *very regular (related) pair* for Langlands parameters.

7.2. **Very regular related pairs.** Suppose that $\varphi : W_\mathbb{R} \to {}^L G = G^\vee \rtimes W_\mathbb{R}$ is an admissible homomorphism relevant to G, so that the orbit of φ under the conjugation action of $G^\vee$ is a Langlands parameter for G. We use the notation $\boldsymbol{\varphi}$ (boldface) for the parameter when it is helpful to distinguish it from a representative. We call φ or $\boldsymbol{\varphi}$ *regular* if the subgroup $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ of $G^\vee$ of elements fixing (pointwise) the image of $\mathbb{C}^\times$ is abelian. For the endoscopic group H_1 we consider only parameters $\boldsymbol{\varphi}_1$ for which composition with ${}^L H_1 \to {}^L Z_1$ produces the parameter $\mathbf{b}$ of (4.2.5). Equivalently, $\boldsymbol{\varphi}_1$ has a (*good*) representative φ_1 with image in $\xi_1(\mathcal{H})$, as described in Section 2 of [Sh10]. Recall that we have:

$$
\begin{array}{ccc}
 & & {}^L H_1 \\
 & \overset{\xi_1}{\nearrow} & \\
\mathcal{H} & & \\
 & \underset{\xi = incl}{\searrow} & \\
 & & {}^L G
\end{array}
$$

(7.2.1)

Since $H_1 \to H$ is a z-extension, good φ_1 is unique up to conjugation by elements of the subgroup $H^\vee$ of $\mathcal{H}$. The $G^\vee$-orbit of $\varphi^* = \xi \circ \xi_1^{-1} \circ \varphi_1$ defines a parameter $\boldsymbol{\varphi}^*$ for the quasi-split form G^* of G (it need not be relevant to G). We say $\boldsymbol{\varphi}_1$ is $G^\vee$-*regular* if $Cent(\varphi^*(\mathbb{C}^\times), G^\vee)$ is abelian.

Assume $\boldsymbol{\varphi}_1$ is $G^\vee$-regular and $\boldsymbol{\varphi}$ is a regular parameter for G. We call the pair $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$ a *very regular pair*. We call $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi})$, or representatives (φ_1, φ), *related* if $\boldsymbol{\varphi}^*$ is relevant to G and coincides with $\boldsymbol{\varphi}$.

7.3. **Parameters in explicit form.** Recall from Section 3 of [La89] a description of essentially bounded Langlands parameters. Throughout we assume a choice of splitting $spl^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$ preserved by the action of the Galois group $\Gamma = \{1, \sigma\}$; the final results are independent of this choice.

We start with those parameters that factor through no proper parabolic subgroup of ${}^L G$, namely the parameters for essentially discrete series representations, provided they exist. Each such parameter has an almost canonical representative $\varphi = \varphi(\mu, \lambda)$ attached to $spl^\vee$. Here μ is a uniquely determined element of

$X_*(\mathcal{T}) \otimes \mathbb{C}$ that is integral and regular dominant for $(\mathcal{B}, \mathcal{T})$ and λ is a representative for a uniquely determined coset in $X_*(\mathcal{T}) \otimes \mathbb{C}$ of the submodule $\mathcal{K}$ generated by $X_*(\mathcal{T})$ and the (-1)-eigenspace of the automorphism $\overline{\sigma}$ to be defined shortly. The *almost canonical* refers to the fact that this representative is unique only up to the action of $\mathcal{T}$ (see [Sh10] for a slight extension of the discussion in [La89]).

In particular, for $\varphi = \varphi(\mu, \lambda)$, we have that $\varphi(\mathbb{C}^\times)$ is contained in $\mathcal{T} \times \mathbb{C}^\times$. If w has image σ under $W_\mathbb{R} \to \Gamma$ then $\varphi(w)$ normalizes $\mathcal{T}$ and its action on $\mathcal{T}$ is independent of choice for w. We write it as $\overline{\sigma}$. Then $\overline{\sigma}$ acts on $\mathcal{T} \cap G_{der}^\vee$ as inversion $t \to t^{-1}$. By construction,

$$(7.3.1) \qquad \varphi(z) = z^\mu \overline{z}^{\overline{\sigma}\mu} \times z$$

for $z \in \mathbb{C}^\times$. Suppose $w^2 = -1$. Then w has image σ under $W_\mathbb{R} \to \Gamma$ and, by construction again,

$$(7.3.2) \qquad \varphi(w) \in e^{2\pi i \lambda} G_{der}^\vee \times w.$$

Here $e^{2\pi i \lambda}$ denotes the element of $\mathcal{T}$ satisfying

$$(7.3.3) \qquad \lambda^\vee(e^{2\pi i \lambda}) = e^{2\pi i \langle \lambda^\vee, \lambda \rangle}$$

for all $\lambda^\vee \in X^*(\mathcal{T})$, where $\langle -, - \rangle$ is the canonical pairing between $X^*(\mathcal{T})$ and $X_*(\mathcal{T})$. Also, $G_{der}^\vee$ denotes the derived group of $G^\vee$.

Boundedness of φ is a condition on $\mu + \overline{\sigma}\mu$ which we will impose to allow us to drop the adjective *essentially* from our discussion; it states that $\langle \lambda^\vee, \mu + \overline{\sigma}\mu \rangle$ is purely imaginary for all $\lambda^\vee \in X^*(\mathcal{T}) \otimes \mathbb{R}$ and forces unitarity of the character data of the next subsection. Here we use *essentially* as in Section 3 of [La89]. Thus a discrete series representation has unitary central character and has matrix coefficients that are square-integrable on $G(\mathbb{R})/Z_G(\mathbb{R})$, where Z_G denotes the center of G. An essentially discrete series representation π has no condition on its central character, but π then has the property that $\zeta \otimes \pi$ is a discrete series representation, for some quasicharacter ζ on $G(\mathbb{R})$. The *essentially* case is thus no harder to handle, for discrete series or more generally for tempered representations, but it does add to the notational burden.

The boundedness condition is of course vacuous if Z_G is anisotropic, and $\overline{\sigma}$ then acts on the entire torus $\mathcal{T}$ as inversion. In that case we may take $\lambda = 0$ and then ignore it. More precisely, in general we have that

$$(7.3.4) \qquad \frac{1}{2}(\mu - \overline{\sigma}\mu) - \iota \equiv \lambda + \overline{\sigma}\lambda \bmod X_*(\mathcal{T}),$$

where ι is one-half the sum of the coroots of $(\mathcal{B}, \mathcal{T})$. This is the critical Lemma 3.2 of [La89]. In the case that the center of G is anisotropic we may replace (7.3.4) by the simpler rationality condition

$$(7.3.5) \qquad \mu - \iota \in X_*(\mathcal{T}).$$

Now to drop the assumption that bounded $\boldsymbol{\varphi}$ factors through no proper parabolic subgroup of ${}^L G$, we follow the argument of [La89] to attach a parabolic subgroup of ${}^L G$ standard relative to $(\mathcal{B}, \mathcal{T})$ through which a representative φ factors minimally. Then if ${}^L M$ is the corresponding standard Levi subgroup we may assume that φ has image in ${}^L M$ but factors through no proper parabolic subgroup of ${}^L M$. Using the subsplitting $spl_M^\vee$ of $spl^\vee$ (defined in the obvious way), we have a description as above, with $\overline{\sigma} = \overline{\sigma}_G$ replaced by $\overline{\sigma}_M$ and $\iota = \iota_G$ replaced by ι_M.

7.4. **Attached packets (discrete series).** The packet attached to a parameter φ in the local Langlands correspondence depends on the choice of an inner class of inner twists from G to the fixed quasi-split form G^*. Thus we choose an inner twist $\psi : G \to G^*$ and are free to adjust it by an inner automorphism when convenient.

Throughout this subsection we assume that φ is bounded and factors through no proper parabolic subgroup of ${}^L G$, and write Π for the discrete series packet attached via the classical local Langlands correspondence of [La89]. We recall how this is done. The existence of $\overline{\sigma}$ ensures (more precisely, is equivalent to) the existence of a maximal torus T over $\mathbb{R}$ in G that is anisotropic modulo the center of G. Then we adjust ψ so that its restriction to T is defined over $\mathbb{R}$, and use ψ together with toral data to transport μ, λ to $X^*(T) \otimes \mathbb{C}$; $\overline{\sigma}$ becomes σ_T. Let Ω be the Weyl group of T in G. We have then a well-defined collection $\{(\omega\mu - \iota, \lambda); \omega \in \Omega\}$ of *character data*, *i.e.* one that is independent of the choice of splittings and toral data, and that depends only on the inner class of ψ. Informally, these character data provide well-defined character formulas on the regular points of $T(\mathbb{R})$ along the lines of the familiar Weyl character formula. Harish-Chandra's theorems on the discrete series allow us to use this information to characterize the distributions $Trace\, \pi$ and $St\text{-}Trace\, \pi$ for $\pi \in \Pi$ (see Section 3 of [La89]).

Assuming a choice of toral data, we may now identify $\pi \in \Pi$ by an element ω of Ω, or more precisely by a left coset of the subgroup $\Omega_{\mathbb{R}}$ of elements realized in $G(\mathbb{R})$. We then write $\pi = \pi(\omega)$; see Section 7b in [Sh10] and below. As in the definition of the geometric factors, for spectral transfer factors we choose toral data, a-data and χ-data to assemble terms and then observe that factor itself is independent of these choices.

Before turning to Π_D, we review a critical step in the definition of the classical packet Π, namely forming characters on $T(\mathbb{R})$, to be denoted $\Lambda_{(\omega\mu - \iota, \lambda)}$, from the character data $\{(\omega\mu - \iota, \lambda); \omega \in \Omega\}$. This is done in [La89] via an explicit form of the classical local Langlands correspondence for real tori, as follows. The congruence

$$(7.4.1) \qquad \frac{1}{2}[(\omega\mu - \iota) - \sigma_T(\omega\mu - \iota)] \equiv \lambda + \sigma_T \lambda \bmod X^*(T),$$

is a consequence of (7.3.4). Moreover, it is a necessary and sufficient condition that the formula

$$(7.4.2) \qquad \Lambda_{(\omega\mu - \iota, \lambda)}(\exp H \,\exp i\pi\lambda^\vee) = e^{\langle \omega\mu - \iota, H \rangle}\, e^{2\pi i \langle \lambda, \lambda^\vee \rangle}.$$

yields a well-defined character $\Lambda_{(\omega\mu - \iota, \lambda)}$ on $T(\mathbb{R})$. Here we have identified the Lie algebra $\mathfrak{t}$ of T as $X_*(T) \otimes \mathbb{C}$ and the Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of $T(\mathbb{R})$ as the real subspace of invariants for the twisted action of σ_T (meaning σ acts on both components). The element H lies in $\mathfrak{t}_{\mathbb{R}}$ and $\lambda^\vee \in X_*(T)$ is σ_T-invariant. These elements $\exp i\pi\lambda^\vee$ form a finite subgroup F of $T(\mathbb{R})$ such that $T(\mathbb{R}) = T(\mathbb{R})^0\, F$.

Notice that if T itself is anisotropic over $\mathbb{R}$, *i.e.* the center Z_G of G is anisotropic, then the group $T(\mathbb{R})$ is connected and the character $\Lambda_{\omega\mu - \iota}$ is just the restriction of the rational character $\omega\mu - \iota$ to $T(\mathbb{R})$.

In general, consider restriction to $Z_G(\mathbb{R})$ of the character $\Lambda_{(\omega\mu - \iota, \lambda)}$. Since the group F is central we may write $z \in Z_G(\mathbb{R})$ as $z = \exp H \,\exp i\pi\lambda^\vee$, with $H \in \mathfrak{z}_{G,\mathbb{R}}$ and $\lambda^\vee$, as above, a σ_T-invariant cocharacter of T. Then for each $\omega \in \Omega$,

$$(7.4.3) \qquad \Lambda_{(\omega\mu - \iota, \lambda)}(z) = e^{\langle \mu, H \rangle}\, e^{2\pi i \langle \lambda, \lambda^\vee \rangle}.$$

We rewrite (7.4.3) as $\Lambda_{(\mu,\lambda)}(z)$, keeping in mind that in general the character $\Lambda_{(\mu,\lambda)}$ is well-defined only on $Z_G(\mathbb{R})$. Then $\Lambda_{(\mu,\lambda)}$ is the central character of each of $\pi \in \Pi$; this is evident from the character formula for π on $T(\mathbb{R})_{reg}$.

For consistency with the abelian case of the local Langlands correspondence, *i.e.* the case $G = T$, we require that the renormalized packet Π_D consists of the discrete series representations attached to the characters $(\Lambda_{(\omega\mu-\iota,\lambda)})_D$, $\omega \in \Omega$. Recall from (4.2) of [KS12], if χ is a character on $T(\mathbb{R})$ then χ_D is defined to be χ^{-1}. In other words, we replace the characters $\Lambda_{(\omega\mu-\iota,\lambda)}$ by $\Lambda_{(-\omega\mu+\iota,-\lambda)}$ and the central character $\Lambda_{(\mu,\lambda)}$ by $\Lambda_{(-\mu,-\lambda)}$. Notice that this applies also to the parameters that we have been ignoring, namely those that are essentially bounded but not bounded.

First we check that this prescription of Π_D agrees with that in (4.4.) of [KS12]. Recall the automorphism of $^LG = G^\vee \rtimes W_\mathbb{R}$ given by $^L\theta_0 = \theta_0^\vee \times id$, where $\theta_0^\vee$ (written in [KS12] as θ_0) denotes the automorphism of $G^\vee$ that preserves $spl^\vee$ and acts on $\mathcal{T}$ as $t \to \omega_0(t)^{-1}$. Here ω_0 is the longest element of the Weyl group of $\mathcal{T}$. In particular, if Z_G is anisotropic then $\theta_0^\vee$ is given by the action of the Galois element σ on $G^\vee$. According to [KS12], the packet Π_D attached to φ is the classical packet for $^L\theta_0 \circ \varphi$. Since

$$(7.4.4) \qquad\qquad \theta_0^\vee(\mu) = -\omega_0\mu, \;\; \theta_0^\vee(\lambda) \equiv -\lambda \bmod \mathcal{K},$$

we see that Π_D consists of the representations attached to $\varphi(-\omega_0\mu, -\lambda)$ in the classical correspondence. The character data for these coincide with that for the representations we have prescribed above, and so we are done.

There is also another more convenient way to describe the parameter $^L\theta_0 \circ \varphi$. Namely we replace $spl^\vee$ by its opposite $(spl^\vee)^{opp}$, *i.e.* we replace $\mathcal{B}$ by its opposite relative to $\mathcal{T}$ and replace the root vector $X_{\alpha^\vee}$ by the root vector $X_{-\alpha^\vee}$ that forms a simple triple with $X_{\alpha^\vee}$ and the coroot $H_{\alpha^\vee}$ for $\alpha^\vee$. Relative to this new splitting, $\varphi(-\mu, -\lambda)$ is the almost canonical representative for $^L\theta_0 \circ \varphi$.

Now fix a parameter φ and consider both the classical packet Π and the renormalized packet Π_D attached as we have described. There are various ways to relate Π_D directly to Π; see also [AV12], Section 4 of [Ka13]. Since our representations are unitary we will use complex conjugation. For each such π we consider its complex conjugate $\overline{\pi}$ on the conjugate Hilbert space; $\overline{\pi}$ is the contragredient or dual of π.

Relative to given toral data, we have written a member π of Π as $\pi(\omega)$, where ω represents a *left* coset of $\Omega_\mathbb{R}$ in Ω. By definition [Sh10], the character formula for π on $T(\mathbb{R})_{reg}$ involves only the characters $\Lambda_{(\omega^{-1}\mu-\iota,\lambda)}$, for ω in this coset. Relative to the same toral data and $(spl^\vee)^{opp}$, we write a member π_D of Π_D as $\pi_D(\omega)$ if its character formula on $T(\mathbb{R})_{reg}$ involves only the characters $(\Lambda_{(\omega^{-1}\mu-\iota,\lambda)})_D$.

Lemma 7.4.1. *(i) For each* $\omega \in \Omega$, *if* $\pi \in \Pi$ *and* $\pi = \pi(\omega)$ *then* $\overline{\pi} \in \Pi_D$ *and* $\overline{\pi} = \pi_D(\omega)$. *(ii) The map* $\pi \to \pi_D = \overline{\pi}$ *defines a bijection* $\Pi \to \Pi_D$.

Proof. From Harish-Chandra's character formula for $\overline{\pi}$ on $T(\mathbb{R})_{reg}$ and his characterization of discrete series characters (see [HC75, Section 27]), we see that $\overline{\pi} = \pi_D(\omega)$, and the result follows. $\square$

Remark 7.4.2. *We could also argue Lemma 7.4.1 by applying Theorem 1.3 of* [AV12].

Finally, recall the stable characters $St\text{-}Trace\,\pi$, $St\text{-}Trace\,\pi_D$ attached to Π, Π_D. Here each is simply the sum of the characters of the members of the packet. Then

the last lemma implies that

$$(7.4.5) \qquad St\text{-}Trace \ \pi_D = \overline{St\text{-}Trace \ \pi}.$$

Recall that for a distribution Θ and test function f we have $\overline{\Theta}(f) = \overline{\Theta(\overline{f})})$. Thus if $St\text{-}Char(\pi, -)$, $St\text{-}Char(\pi_D, -)$ denote the analytic functions on the regular semisimple set of $G(\mathbb{R})$ that represent these two distributions then

$$(7.4.6) \qquad St\text{-}Char(\pi_D, \delta) = \overline{St\text{-}Char(\pi, \delta)},$$

for all regular semisimple $\delta \in G(\mathbb{R})$.

7.5. Very regular related pairs and attached packets.
Consider a very regular related pair of bounded parameters (φ_1, φ) as in (7.2). We first attach packets Π_1, Π in the classical correspondence. For any $\pi_1 \in \Pi_1$ and $\pi \in \Pi$, we call (π_1, π) a very regular related pair (of tempered irreducible representations). Recall that the $G^\vee$-regularity of φ_1 ensures that if Π_1 consists of discrete series representations of $H_1(\mathbb{R})$ then Π consists of discrete series representations of $G(\mathbb{R})$; see also [Sh10]. More generally, we invoke parabolic descent in standard endoscopy. Namely, to (φ_1, φ) we may attach an endoscopic pair of Levi subgroups M_1, M and discrete series packets for both $M_1(\mathbb{R})$, $M(\mathbb{R})$ respectively. The members of the corresponding packets for $H_1(\mathbb{R})$, $G(\mathbb{R})$ are representations parabolically induced from representations in the packets for $M_1(\mathbb{R})$, $M(\mathbb{R})$ respectively.

Observe that Lemma 7.4.1 may be applied to all bounded parameters because complex conjugation commutes in the obvious sense with unitary parabolic induction in our setting (or again we could argue via [AV12]). The assertions following Lemma 7.4.1 on stable characters apply also.

Continuing with a very regular related pair of bounded parameters (φ_1, φ), we also use the renormalized correspondence to attach packets $\Pi_{1,D}, \Pi_D$ to (φ_1, φ). Let $\pi_{1,D} \in \Pi_{1,D}$ and $\pi_D \in \Pi_D$. Then we call $(\pi_{1,D}, \pi_D)$ a very regular related pair also. Thus if (π_1, π) is a very regular related pair in the first (classical) sense then $(\pi_{1,D}, \pi_D) = (\overline{\pi_1}, \overline{\pi})$ is a very regular related pair in the second (renormalized) sense, with the converse also true. We will use this throughout Sections 8 and 9 without further comment.

8. Archimedean case: transfer theorems

8.1. Spectral relative factors Δ' and Δ_D.
We work in the setting of very regular related pairs of bounded parameters; we take two such pairs and consider the various attached packets. The relative terms $\Delta_I, \Delta_{II}, \Delta_{III}$ were defined in [Sh10] for related pairs in the classical sense.

The term Δ_I is a sign given by Tate-Nakayama pairing. Thus we use the same definition of Δ_I for the renormalized related pairs (see Section 8 of [Sh10]), so that

$$(8.1.1) \qquad \Delta_I(\pi_{1,D}, \pi_D) = \Delta_I(\pi_1, \pi)$$

for any $(\pi_1, \pi), (\pi_{1,D}, \pi_D)$ attached to given very regular related pair of parameters.

In the case of Δ_{III}, another Tate-Nakayama sign, we follow Section 10 of [Sh10]. Then we conclude that

$$(8.1.2) \qquad \Delta_{III}(\pi_1, \pi; \pi_1^\dagger, \pi^\dagger) = \Delta_{III}(\overline{\pi_1}, \overline{\pi}; \overline{\pi_1^\dagger}, \overline{\pi^\dagger})$$

for all very regular related pairs (π_1, π), $(\pi_1^\dagger, \pi^\dagger)$.

In Section 9 of [Sh10], Δ_{II} is defined as a quotient of absolute terms that are fourth roots of unity. We now define the term $\Delta_{II,D}$ in such a way that

$$(8.1.3) \qquad \Delta_{II,D}(\overline{\pi_1}, \overline{\pi}) = \Delta_{II}(\pi_1, \pi)^{-1} = \overline{\Delta_{II}(\pi_1, \pi)},$$

for all very regular related pairs (π_1, π). Namely we proceed as in Section 9 of [Sh10], but use $-\mu_1, -\iota_1, -\lambda_1$ in place of $\mu_1, \iota_1, \lambda_1$ in character formulas. We also replace μ^*, λ^* described there by $-\mu^*, -\lambda^*$. Notice that this is done also when we define explicitly the geometric $\Delta_{III_2,D}$ of (3.2.1). Then the pair denoted μ, λ in [Sh10], namely $\mu = \mu_1 + \mu^*, \lambda = \lambda_1 + \lambda^*$, is replaced by $-\mu, -\lambda$. There μ is not necessarily dominant. In the case that μ is not dominant there is an additional sign contribution to Δ_{II}; we use the same one in $\Delta_{II,D}$. If μ is dominant then we call φ_1 *well-positioned* relative to φ; the discussion of Section 7c of [Sh10] shows that this is a well-defined notion. Notice also that $\Delta_{II}, \Delta_{II,D}$ depend only on the pair of parameters and not on the choice among attached representations since only the stable characters are used in the definition. Recall that the stable character of $\Pi_{1,D}, \Pi_D$ respectively is the complex conjugate of that of Π_1, Π respectively. So far, we have arranged to replace everything in each character formula in Section 9 of [Sh10] by its complex conjugate except for the constant, a fourth root of unity, that contributes to $\Delta_{II}(\pi_1, \pi)$. We thus also replace $\Delta_{II}(\pi_1, \pi)$ by its complex conjugate, as desired. Recall that the *relative* version of Δ_{II} is a sign [Sh10].

As usual for the real case, we identify Δ' with $\Delta = \Delta_I \Delta_{II} \Delta_{III}$. Next, set

$$(8.1.4) \qquad \Delta_D = \Delta_I \Delta_{II,D} \Delta_{III}.$$

The following is now immediate.

Lemma 8.1.1. *Suppose that (π_1, π) and $(\pi_1^\dagger, \pi^\dagger)$ are very regular related pairs. Then we have an equality of signs:*

$$(8.1.5) \qquad \Delta(\pi_1, \pi; \pi_1^\dagger, \pi^\dagger) = \Delta_D(\overline{\pi_1}, \overline{\pi}; \overline{\pi_1^\dagger}, \overline{\pi^\dagger}).$$

We emphasize that (8.1.5) takes the form given because all the critical contributions are signs.

8.2. **Spectral absolute Δ' and Δ_D.** We define absolute spectral factors $\Delta'(\pi_1, \pi)$, $\Delta_D(\pi_{1,D}, \pi_D)$ following the same method as for the geometric factors $\Delta'(\gamma_1, \delta)$, $\Delta_D(\gamma_1, \delta)$. In particular, if the very regular pair (φ_1, φ) is not related then Δ' and Δ_D vanish on the attached packets; more precisely,

$$(8.2.1) \qquad \Delta'(\pi_1, \pi) = \Delta_D(\pi_{1,D}, \pi_D) = 0$$

for all $\pi_1 \in \Pi_1$, $\pi \in \Pi$, $\pi_{1,D} \in \Pi_{1,D}$ and $\pi_D \in \Pi_D$.

Now consider just the factors Δ' for the classical local Langlands correspondence. It is clear that for a given choice of geometric factor Δ' there is at most one choice of spectral factor Δ' for which the dual spectral transfer statement (7.1.1) is true, and conversely the choice of spectral factor fixes the choice of geometric factor. To describe which normalizations are compatible in this sense we introduce yet another canonical factor, the compatibility factor $\Delta'(\pi_1, \pi; \gamma_1, \delta)$ (see Section 12 of [Sh10]). Here we again identify Δ' with $\Delta = \Delta_I \Delta_{II} \Delta_{III}$, and observe that only the term $\Delta_{III}(\pi_1, \pi; \gamma_1, \delta)$ is genuinely relative. Then we call $\Delta'(\gamma_1, \delta)$ and $\Delta'(\pi_1, \pi)$ compatible if and only if

$$(8.2.2) \qquad \Delta'(\pi_1, \pi) = \Delta'(\pi_1, \pi; \gamma_1, \delta) \, \Delta'(\gamma_1, \delta)$$

for some choice of very regular related pairs $(\pi_1, \pi), (\gamma_1, \delta)$. Then it is true for all such pairs by the transitivity properties (ii) and (iii) of Lemma 12.1 in [Sh10].

To consider compatibility of the factors $\Delta_D(\gamma_1, \delta)$ and $\Delta_D(\pi_{1,D}, \pi_D)$ we introduce $\Delta_D(\pi_{1,D}, \pi_D; \gamma_1, \delta)$, using once again the paradigm $\Delta_D = \Delta_I \Delta_{II,D} \Delta_{III}$, and then we proceed in the same way as for Δ'. Following the arguments for Corollary 5.0.3 and Lemma 8.1.1 we observe that

$$(8.2.3) \qquad \Delta_D(\overline{\pi_1}, \overline{\pi}; \gamma_1, \delta) = \overline{\Delta'(\pi_1, \pi; \gamma_1, \delta)},$$

for all very regular related pairs $(\pi_1, \pi), (\gamma_1, \delta)$.

Lemma 8.2.1. *Let $\Delta'(\gamma_1, \delta)$, $\Delta_D(\gamma_1, \delta)$ be absolute transfer factors and let $c \in \mathbb{C}^\times$ be such that*

$$(8.2.4) \qquad \Delta_D(\gamma_1, \delta) = c \; \overline{\Delta'(\gamma_1, \delta)}$$

for all very regular pairs (γ_1, δ). Suppose that $\Delta'(\pi_1, \pi)$ is compatible with $\Delta'(\gamma_1, \delta)$ and that $\Delta_D(\pi_{1,D}, \pi_D)$ is compatible with $\Delta_D(\gamma_1, \delta)$. Then

$$(8.2.5) \qquad \Delta_D(\overline{\pi_1}, \overline{\pi}) = c\overline{\Delta'(\pi_1, \pi)}$$

for all very regular pairs (π_1, π). Conversely, (8.2.4) and (8.2.5) imply that either both or neither of Δ', Δ_D have geometric-spectral compatibility.

Recall that Corollary 6.1.2 proves the existence of c for given factors $\Delta'(\gamma_1, \delta)$, $\Delta_D(\gamma_1, \delta)$.

Proof. This is a consequence of (8.2.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

8.3. Quasi-split case: spectral $\Delta_0, \Delta_{0,D}$ and $\Delta_\lambda, \Delta_{\lambda,D}$.

If G is quasi-split over $\mathbb{R}$ we may as well take $G = G^*$ and the inner twist ψ to be the identity automorphism of G. We assemble spectral factors $\Delta_0, \Delta_{0,D}$ following the paradigm for the geometric case in (6.2). These spectral factors are compatible with the corresponding geometric ones; see Lemma 12.3 in [Sh10] for Δ_0 and argue similarly for $\Delta_{0,D}$ or use (8.2.3) above. We do the same for the Whittaker factors $\Delta_\lambda, \Delta_{\lambda,D}$.

Lemma 8.3.1. *For all very regular pairs (π_1, π) we have*

$$(8.3.1) \qquad \Delta_{\lambda,D}(\overline{\pi_1}, \overline{\pi}) = c \; \overline{\Delta_\lambda(\pi_1, \pi)},$$

where $c = (det(V))(-1)$ as in Lemma 6.3.1, and

$$(8.3.2) \qquad \Delta_{\overline{\lambda},D}(\overline{\pi_1}, \overline{\pi}) = \overline{\Delta_\lambda(\pi_1, \pi)}.$$

Proof. This follows from Lemmas 6.3.1 and 6.3.2, geometric-spectral compatibility for each of the factors $\Delta_\lambda, \Delta_{\lambda,D}$ and (8.2.3). $\qquad\qquad\qquad\qquad \square$

See (8.4.3) below to extend this to all bounded related pairs.

8.4. Consequences for spectral transfer.

Suppose we have compatible geometric and spectral factors Δ' and that the test functions $f \in C_c^\infty(G(\mathbb{R}))$ and $f_1 \in C_c^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ have Δ'-matching orbital integrals as in (6.4.1). Here we may replace $C_c^\infty(-)$ by the corresponding Harish-Chandra-Schwartz space $\mathcal{C}(-)$ if we wish. Then f and f_1 also have Δ'-matching traces on the very regular pairs of irreducible tempered representations (π_1, π) as in (7.1.1). A proof of this, which comes from [Sh82] and [LS90], is reviewed in detail in [Sh10].

Suppose instead we work with compatible geometric and spectral factors Δ_D. Returning to (6.4), suppose as above that f is a test function on $G(\mathbb{R})$. Use geometric Δ'-transfer for $\overline{f}$ to define $f_{1,D}$ as in (6.4.2) such that f and $f_{1,D}$ have Δ_D-matching orbital integrals. We claim that f and $f_{1,D}$ also have Δ_D-matching traces on the very regular pairs of irreducible tempered representations $(\pi_{1,D}, \pi)$ as in (7.1.2). For this, we will change notation on the right side of (7.1.2), writing π_D in place of π and then setting $\pi = \overline{\pi_D}$, $\pi_1 = \overline{\pi_{1,D}}$. We observe that

$$(8.4.1) \qquad \Delta_D(\pi_{1,D}, \pi_D) \; Trace \; \pi_D(f) = c \; \overline{\Delta(\pi_1, \pi) \; Trace \; \pi(\overline{f})}$$

and

$$(8.4.2) \qquad St\text{-}Trace \; \pi_{1,D}(f_{1,D}) = c \; \overline{St\text{-}Trace \; \pi_1((\overline{f})_1)}.$$

Thus the claim follows. Conversely if we start with Δ_D-transfer then we may deduce Δ'-transfer.

We now drop the *very regular* assumption on the spectral side. The transfer proved on the geometric side provides a correspondence on test functions that is full in the sense that it is sufficient to establish the dual transfer to a tempered invariant eigendistribution on $G(\mathbb{R})$ of *any* stable tempered trace $St\text{-}Trace \; \pi_{1,D}$ on $H_1(\mathbb{R})$ for which $Z_1(\mathbb{R})$ acts correctly. It remains then to define the relevant spectral transfer factors and identify the distribution as the right side of (7.1.2). We could proceed directly, copying the approach for Δ'-transfer described in [Sh10]. That involves long and delicate arguments with limits of discrete series representations, although the adjustments needed are minor. Alternatively, we may invoke the results for Δ'-transfer and use the formula (8.2.5) to extend the definition of $\Delta_D(\pi_{1,D}, \pi_D)$ by

$$(8.4.3) \qquad \Delta_D(\pi_{1,D}, \pi_D) = c \; \overline{\Delta'(\overline{\pi_{1,D}}, \overline{\pi_D})},$$

where the constant c is given by (8.2.4). Notice that Lemma 8.3.1 remains true. Now we deduce (7.1.2) from (7.1.1) as in the very regular case. In conclusion:

Lemma 8.4.1. *Suppose test functions f on $G(\mathbb{R})$ and $f_{1,D}$ on $H_1(\mathbb{R})$ have Δ_D-matching orbital integrals. Then*

$$(8.4.4) \qquad St\text{-}Trace \; \pi_{1,D}(f_{1,D}) = \sum_{\pi_D} \Delta_D(\pi_{1,D}, \pi_D) \; Trace \; \pi(f),$$

for each tempered irreducible admissible representation $\pi_{1,D}$ of $H_1(\mathbb{R})$ such that the central subgroup $Z_1(\mathbb{R})$ acts by the character $\lambda_{H_1,D}$, where the summation is over tempered irreducible admissible representations π_D of $G(\mathbb{R})$.

8.5. **Extended groups and extended packets.** Turning to structure on stable classes and packets, we recall that if the simply-connected covering G_{sc} of the derived group of G has nontrivial first Galois cohomology set $H^1(\Gamma, G_{sc})$ then stable conjugacy classes of regular elliptic elements have insufficient conjugacy classes, and discrete series packets have insufficient members, for an exact duality with an evident candidate. In the notation of (7.3), this candidate is the group of $\overline{\sigma}$-invariants in the quotient of $\mathcal{T}$ by the center of $G^\vee$, and the evidence is Tate-Nakayama duality.

As Vogan first pointed out, such problems may be resolved by using several inner forms simultaneously in place of the single group G. We follow Kottwitz's formulation of an extended group, also called a K-group [Ar99]. See Section 4 of [Sh08] for a review with examples; unitary groups are discussed at length in

the reference [S7] of that paper, available at the author's website. This solves our problem minimally and is useful in stabilization of the Arthur-Selberg trace formula [Ar06, Ar13].

Both geometric and spectral factors extend naturally to this setting. See Section 5 of [Sh08] for a discussion of $\Delta = \Delta'$. The discussion for Δ_D parallels that for Δ' in the same manner as for a single group. We will assume from now on, *without changing notation*, that we are in the extended group setting.

9. Archimedean case: applications

9.1. Structure on tempered packets. Here we recall that our factors $\Delta = \Delta'$ give structure on tempered packets in the form proposed by Arthur in [Ar06], remark that there is an analog for Δ_D, and comment on the relation between the two.

Before starting, we should note that for extended groups without a quasi-split component, recent work of Kaletha [Ka13a] allows us to replace $\mathcal{S}_\varphi^{sc}$ and ζ with better behaved objects, but only after modifying the notion of extended group. Our present elementary approach will be sufficient for the applications in [Sh] and will simplify some already delicate computations.

Let $\boldsymbol{\varphi}$ be a bounded Langlands parameter for the extended group G. As usual, φ will denote a representative for $\boldsymbol{\varphi}$. We consider the group S_φ^{ad} consisting of the images in $G_{ad}^\vee = (G^\vee)_{ad}$ of those elements in $G^\vee$ which fix the image of φ under the conjugation action. Write $\mathcal{S}_\varphi$ for the component group of S_φ^{ad}. Then $\mathcal{S}_\varphi$ is a finite abelian group, in fact a sum of groups of order two.

A change to representative φ' determines a unique isomorphism $\mathcal{S}_\varphi \to \mathcal{S}_{\varphi'}$. In the quasi-split case the group $\mathcal{S}_\varphi$ is sufficient for our purposes, but in general we introduce also S_φ^{sc}, the inverse image of S_φ^{ad} in $G_{sc}^\vee = (G^\vee)_{sc}$. Then $\mathcal{S}_\varphi^{sc}$ will be the component group of S_φ^{sc}. In Section 7 of [Sh08] we attach to $s \in \mathcal{S}_\varphi^{sc}$ an endoscopic group and parameter $\varphi^{(s)}$ for that group such that $(\varphi^{(s)}, \varphi)$ is a related pair. Moreover, $\varphi^{(s)}$ is well-positioned relative to φ (see (8.1) above). Let $\Pi^{(s)}, \Pi$ denote the corresponding classical packets. Let $\pi^{(s)} \in \Pi^{(s)}$; the choice does not matter. We also pick a basepoint π^{base} for Π and character ζ on $\mathcal{S}_\varphi^{sc}$ as specified below. Then

$$(9.1.1) \qquad\qquad \left\langle s, \pi^{base} \right\rangle = \zeta(s)$$

and

$$(9.1.2) \qquad\qquad \left\langle s, \pi \right\rangle \diagup \left\langle s, \pi^\dagger \right\rangle = \Delta(\pi^{(s)}, \pi; \pi^{(s)}, \pi^\dagger),$$

for all $s \in \mathcal{S}_\varphi^{sc}$ and $\pi, \pi^\dagger \in \Pi$, together define a pairing

$$(9.1.3) \qquad\qquad \left\langle -, - \right\rangle : \mathcal{S}_\varphi^{sc} \times \Pi \to \mathbb{C}^\times$$

that identifies Π as the set of characters $\zeta \otimes \chi$, where χ is a character on $\mathcal{S}_\varphi^{sc}$ trivial on the kernel of $\mathcal{S}_\varphi^{sc} \to \mathcal{S}_\varphi$. For all this, as well as the requirements on ζ, see Sections 7 and 11 of [Sh08]. The main result there is Theorem 7.5 and its reinterpretation in Corollary 11.1.

Thus Π is identified as the dual of $\mathcal{S}_\varphi$ only up to a twist. If we turn now to Δ_D and the renormalized packets $\Pi_D^{(s)}, \Pi_D$ attached to the related pair $(\varphi^{(s)}, \varphi)$, we may take $\pi_D^{base} = \overline{\pi^{base}}$ and reuse ζ if we wish because the character ζ_G of

Corollary 11.1 of [Sh08] has order one or two and so coincides with its renormalized version. Then, by Lemma 8.1.1 above, we get the pairing

(9.1.4) $$\langle -, - \rangle_D : \mathcal{S}_\varphi^{sc} \times \Pi_D \to \mathbb{C}^\times$$

given by

(9.1.5) $$\langle s, \pi_D \rangle_D = \langle s, \pi \rangle,$$

for all $s \in \mathcal{S}_\varphi^{sc}$ and $\pi_D \in \Pi_D$, where $\pi = \overline{\pi_D}$. We could also have used $\overline{\zeta}$ or a different basepoint, etc.. In short, Lemma 8.1.1 allows us to apply the results in Sections 7 and 11 of [Sh08] to renormalized packets.

In the quasi-split case, *i.e.* the case that the extended group has a quasi-split component which is then unique [Sh08], we may return to $\mathcal{S}_\varphi$ as described separately in the next subsection.

9.2. **Whittaker normalizations and strong base-point property.** We first observe the strong base-point property for the renormalized Whittaker factors $\Delta_{\lambda, D}$.

Lemma 9.2.1. *Let φ be a bounded Langlands parameter for the extended group G of quasi-split type and denote by Π_D the renormalized extended packet attached to φ. Let λ be a set of Whittaker data for G and suppose π_D is the unique member of Π_D that is λ-generic. Then*

(9.2.1) $$\Delta_{\lambda, D}(\pi_D^{(s)}, \pi_D) = 1$$

for all $s \in \mathcal{S}_\varphi$.

Proof. The parallel statement for the factors $\Delta'_\lambda = \Delta_\lambda$ and the classical extended packet Π attached to φ is proved as Theorem 11.5 in [Sh08]. Then (8.3.2) finishes the proof. $\square$

Continuing with Π, the classical extended packet attached to φ, we recall from Lemma 11.4 and Corollary 14.1 of [Sh10] that

(9.2.2) $$\Delta_\lambda(\pi^{(s)}, \pi) = \pm 1,$$

for all $s \in \mathcal{S}_\varphi, \pi \in \Pi$. Then, by (8.3.2) again,

(9.2.3) $$\Delta_{\lambda, D}(\pi_D^{(s)}, \pi_D) = \pm 1,$$

for all $s \in \mathcal{S}_\varphi, \pi_D \in \Pi_D$.

We have further from Theorem 7.5 and Corollary 11.1 of [Sh08] that the pairing

(9.2.4) $$\langle -, - \rangle_\lambda : \mathcal{S}_\varphi \times \Pi \to \{\pm 1\},$$

given by

(9.2.5) $$(s, \pi) \to \Delta_\lambda(\pi^{(s)}, \pi)$$

is perfect in the sense that it identifies Π as the dual of the finite abelian group $\mathcal{S}_\varphi$. From (8.3.2) yet again, we conclude:

Lemma 9.2.2. *The map*

(9.2.6) $$\langle -, - \rangle_{\lambda, D} : \mathcal{S}_\varphi \times \Pi_D \to \{\pm 1\}$$

given by

(9.2.7) $$(s, \pi_D) \to \Delta_{\lambda, D}(\pi_D^{(s)}, \pi_D)$$

is a perfect pairing of Π_D with $\mathcal{S}_\varphi$, and moreover,

$$(9.2.8) \qquad \langle s, \pi_D \rangle_{\lambda, D} = \langle s, \pi \rangle_{\overline{\lambda}},$$

for all $s \in \mathcal{S}_\varphi, \pi_D \in \Pi_D$, *where* $\pi = \overline{\pi_D} \in \Pi$.

The sign $\langle s, \pi_D \rangle_{\lambda, D}$ is easily computed explicitly via the Tate-Nakayama pairing when φ is regular or, more generally, when Π is presented with nondegenerate data. On the other hand, see [Sh08, Sh14] for the *totally degenerate* limits of discrete series that exist, for example, when the derived group of G is simply-connected. This case, which includes the representations studied by Carayol and Knapp [CK07], is also easy to calculate since each member of the packet is generic and the sets of Whittaker data index the packet. We take this up in [Sh14].

References

[Ar99] Arthur, J. On the transfer of distributions: weighted orbital integrals, Duke Math. J., 99 (1999), 209-283.

[Ar06] Arthur, J. A note on L-packets Pure & Appl. Math. Quart., 2 (2006), 199-217.

[Ar13] Arthur, J. *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*, AMS Colloq. Publ., vol. 61, 2013

[AV12] Adams, J. and Vogan, D. The contragredient, preprint (2012), *arXiv*:1201.0496v2

[CK07] Carayol, H. and Knapp, A. Limits of discrete series with infinitesimal character zero, Trans. Amer. Math Soc., 359 (2007), 5611-5651.

[HC75] Harish-Chandra, Harmonic analysis on real reductive groups I, J. Funct. Anal., 19 (1975), 104-204.

[Ka13] Kaletha, T. Genericity and contragredience in the local Langlands correspondence, Alg. Number Thy., 7 (2013), 2447 - 2474.

[Ka13a] Kaletha, T. Rigid inner forms of real and p-adic groups, preprint (2013), *arXiv:*1304.3292v3

[KS99] Kottwitz, R. and Shelstad, D. *Foundations of Twisted Endoscopy*, Astérisque, 255, 1999.

[KS12] Kottwitz, R. and Shelstad, D. On splitting invariants and sign conventions in endoscopic transfer, preprint (2012), *arXiv:*1201.5658

[La83] Langlands, R. Orbital integrals on forms of SL(3) I, Amer. Jour. Math. 105 (1983), 465-506.

[La89] Langlands, R. On the classification of irreducible representations of real algebraic groups, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, AMS Math Surveys and Monographs, 31, 1989, 101-170.

[La97] Langlands, R. Representations of abelian algebraic groups, Pacific J. Math., 181 (1997), 231-250.

[LS87] Langlands, R. and Shelstad, D. On the definition of transfer factors, Math. Ann., 278 (1987), 219-271.

[LS90] Langlands, R. and Shelstad, D. Descent for transfer factors, in *The Grothendieck Festschift* II, Birkhauser, Boston, 1990, 485-563.

[Sh82] Shelstad, D. L-indistinguishability for real groups, Math. Ann. 259 (1982), 385-430.

[Sh08] Shelstad, D. Tempered endoscopy for real groups III: inversion of transfer and L-packet structure, Represent. Theory, 12 (2008), 369-402.

[Sh10] Shelstad, D. Tempered endoscopy for real groups II: spectral transfer factors, in *Automorphic Forms and the Langlands Program,* Higher Ed. Press/Int'l Press, 2009/2010, 236-276.

[Sh12] Shelstad, D. On geometric transfer in real twisted endoscopy, Annals of Math, Vol. 176 (2012), 1919-1985.

[Sh14] Shelstad, D. On elliptic spectral factors in real endoscopic transfer I, preprint (2014), arXiv:1412.4647

[Sh] Shelstad, D. On elliptic spectral factors in real endoscopic transfer II, in preparation.

[Wa08] Waldspurger, J-L. *L'endoscopie tordue n'est pas si tordue*, Memoirs AMS, Nr. 908, 2008.

MATHEMATICS DEPARTMENT, RUTGERS UNIVERSITY-NEWARK, NEWARK NJ 07102
E-mail address: shelstad@rutgers.edu