Spectral questions in endoscopic transfer for real reductive groups

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May 20, 2013

- two related transfer principles introduced by Langlands 1970$\pm$, 2010$\pm$
- archimedean local case and its relation to broader picture

- **endoscopic transfer** relates invariant harmonic analysis on given group $G(\mathbb{R})$ to stable harmonic analysis on the generally lower dimensional endoscopic groups $H_1(\mathbb{R})$

- part of broader themes involving stable conjugacy, packets of representations and stabilization of the Arthur-Selberg trace formula

- second principle, **stable transfer**, concerns stable harmonic analysis on any two groups $G(\mathbb{R})$, $H(\mathbb{R})$ related by a morphism of L-groups, part of *Beyond Endoscopy*, not discussed here

- stable conjugacy in $G(\mathbb{R})$: $G(\mathbb{C})$-conjugacy with small refinement
- start with **geometric transfer**: unstable combinations of orbital integrals on given group $G(\mathbb{R})$ match stable combinations on an endoscopic group $H_1(\mathbb{R})$
- matching: based on norm correspondence for very regular stable conjugacy classes in $H_1(\mathbb{R})$ and (twisted) classes in $G(\mathbb{R})$
- matching provides a transfer of test functions from $G(\mathbb{R})$ to $H_1(\mathbb{R})$, then a dual map from $\mathfrak{Z}$-finite stable distributions on $H_1(\mathbb{R})$ to $\mathfrak{Z}$-finite invariant distributions on $G(\mathbb{R})$
- **spectral transfer:** interpret this dual map in terms of traces of irreducible admissible representations

- geometric side: transfer for orbital integrals has been proved using *transfer factors*
- transfer factors = coefficients for unstable combinations: are defined *a priori* and have various properties useful for descent arguments, comparison among inner forms, global questions *etc.* [Langlands-Shelstad, Kottwitz-Shelstad]
- introduce spectral transfer factors with same basic structure (incomplete) and prove similar properties
- show that they are the only possible coefficients for spectral interpretation of dual transfer
- apply this to various known identities to get (partial) spectral transfer
- the spectral factors contain precise information needed about packets

Endoscopic transfer: geometric side

a. general twisted setting

- G connected, reductive algebraic group defined over $\mathbb{R}$
  θ an $\mathbb{R}$-automorphism of G, ϖ a quasi-character on $G(\mathbb{R})$
  study representations π for which $\pi \circ \theta \simeq \varpi \otimes \pi$

- **quasi-split data (G^*, θ^*) :**
  G^* quasi-split over $\mathbb{R}$, has an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
  [ultimately choice of spl^* will not matter]
  θ^* an $\mathbb{R}$-automorphism of G^* preserving spl^*

- **inner form (G, θ, η) of (G^*, θ^*) :**
  (G, θ) as above, and $\eta : G \longrightarrow G^*$ an inner twist such that
  η transports θ to θ^* up to inner automorphism:
  $\theta = Int(h_\theta) \circ \eta^{-1} \circ \theta^* \circ \eta$, where $h_\theta \in G$

- up to **isomorphism** of inner forms, can arrange that transport
  $\eta^{-1} \circ \theta^* \circ \eta$ is defined over $\mathbb{R}$, so $Int(h_\theta) \in G_{ad}(\mathbb{R})$
  [use **fundamental splittings** – exist for all G]

- **dual complex group** $G^\vee$ with splitting $spl^\vee$ dual to spl^*, action of Weil group $W_\mathbb{R}$ through $W_\mathbb{R} \to \Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ action preserves $spl^\vee$, **and** L**-group** ${}^LG = G^\vee \rtimes W_\mathbb{R}$

- **automorphism** $\theta^\vee$ **of** $G^\vee$: preserves $spl^\vee$ and dual to θ^* quasi-character ϖ comes from $a : W_\mathbb{R} \to {}^LZ = Center(G^\vee) \rtimes W_\mathbb{R}$

- **automorphism** ${}^L\theta_a$ **of** ${}^LG$ extends $\theta^\vee$ with twist by a: ${}^L\theta_a(g \times w) = \theta^\vee(g)a(w)$, for $g \in G^\vee, w \in W_\mathbb{R}$

- in talk: assume $G^\vee$-component of a is **bounded**, so ϖ unitary [otherwise, insert *essentially* in various statements ...]

Endoscopic transfer: geometric side

bb. endoscopic data

- **(bounded) supplemented endoscopic data $\mathfrak{e}_z$:**
  endoscopic data $\mathfrak{e} = (H, \mathcal{H}, s)$, together with
  z-extension data (H_1, ξ_1) [Kaletha refinement ...]

- **basic picture:**

$$
\begin{array}{ccccccc}
 & & & & {}^L H_1 & & \\
 & & & \overset{\xi_1}{\nearrow} & & & \\
1 \longrightarrow Cent_{\theta^\vee}(s, G^\vee)^0 & \longrightarrow & \mathcal{H} & \rightleftarrows & W_{\mathbb{R}} & \longrightarrow 1 & \qquad (1) \\
 & & & \underset{incl}{\searrow} & & & \\
 & & & & {}^L G & &
\end{array}
$$

where $W_{\mathbb{R}}$ acts on $Cent_{\theta^\vee}(s, G^\vee)^0 = H^\vee$ by conjugation by elements of $Cent_{{}^L\theta_a}(s, {}^L G)$

- in talk: assume θ **preserves a fundamental splitting** [at each step should note effect of extra twist by elt of $G_{ad}(\mathbb{R})$]

- there is Γ-map $\mathcal{A}$ from the set $Cl_{ss}(H_1)$ of semisimple conjugacy classes in $H_1(\mathbb{C})$ to the set $Cl_{\theta\text{-}ss}(G,\theta)$ of θ-semisimple θ-conjugacy classes in $G(\mathbb{C})$:

$$\begin{array}{ccccc} Cl_{ss}(H_1) & & & & \\ \downarrow & & & & \\ Cl_{ss}(H) & \overset{endo}{\longrightarrow} & Cl_{\theta^*\text{-}ss}(G^*,\theta) & \overset{inner}{\longrightarrow} & Cl_{\theta\text{-}ss}(G,\theta) \end{array} \qquad (2)$$

- γ_1 is **strongly** G**-regular** if and only if $\mathcal{A}$ maps its class to a class of strongly θ-regular elements in G
- strongly G-regular γ_1 **is a norm of** strongly θ-regular δ, *i.e.* (γ_1, δ) is a **norm pair**, if and only if δ is in image of class of γ_1

- sufficient to specify geometric transfer on **very regular set**: all pairs $(\gamma_1, \delta) \in H_1(\mathbb{R}) \times G(\mathbb{R})$, where γ_1 is strongly G-regular and δ is strongly θ-regular
- **transfer factor** Δ is complex-valued function on very regular set
- define $\Delta(\gamma_1, \delta) = 0$ if (γ_1, δ) is not a norm pair

- now assume $(\gamma_1, \delta), (\gamma_1', \delta')$ are norm pairs
- our transfer statement will not fix normalization for $\Delta(\gamma_1, \delta)$ instead define canonical relative factor $\Delta(\gamma_1, \delta; \gamma_1', \delta')$ and use any factor $\Delta(\gamma_1, \delta)$ satisfying

$$\Delta(\gamma_1, \delta)/\Delta(\gamma_1', \delta') = \Delta(\gamma_1, \delta; \gamma_1', \delta') \tag{3}$$

- two versions of transfer: here use factors for classical version other version: (turns out to be) complex conjugate

- definitions allow simultaneously treatment of inner forms extended group = K-group: fills out stable conjugacy classes
- particular normalizations , esp. Whittaker normalization for several inner forms of quasi-split data (G^*, θ^*)

- relative Δ is product $\Delta_I \Delta_{II} \Delta_{III}$; only Δ_{III} is genuinely relative
- Δ_I, Δ_{III} have Galois-cohomological definitions, spectral versions in same groups [sample at end of talk]

- $\Delta_{II}(\gamma_1, \delta)$ comes from analysis of jumps in orbital integrals spectral version: different form, involves character formula

- **toral data** associated with norm pair (γ_1, δ) : there is θ^*-stable pair (B, T) in G^*, with T defined over $\mathbb{R}$, and various maps yielding

$$\begin{array}{ccccccc} & & \delta & \overset{inner}{\rightsquigarrow} & \delta^* \in T(\mathbb{C}) \\ & & & & \downarrow \\ \gamma_1 & \overset{z}{\longrightarrow} & \gamma_H & \overset{endo}{\longleftrightarrow} & \gamma^* \in T_{\theta^*}(\mathbb{R}) \end{array} \qquad (4)$$

- $R_{res} = \theta^*$-restricted root system for T in G^*,Galois orbits $\mathcal{O}_{res}$
  $R_1 =$ root system for T_1 in H_1,Galois orbits $\mathcal{O}_1$
  to each indivisible $\mathcal{O}_{res}$ attach well-defined $\chi_\alpha(\frac{N\alpha(\delta^*)^{r_\alpha} - 1}{a_\alpha})$
  to each $\mathcal{O}_1$ attach well-defined $\chi_{\alpha_1}(\frac{\alpha_1(\gamma_1)-1}{a_{\alpha_1}})$ [notation]
  $\Delta_{II}(\gamma_1, \delta)$ is quotient over all indivisible $\mathcal{O}_{res}$ by all $\mathcal{O}_1$
- χ-**data**, a-**data:** $\{\chi_\alpha\}, \{a_\alpha\}$ etc. as above
- same data used in Δ_I, Δ_{III} ; two of the three affect each of relative $\Delta_I, \Delta_{II}, \Delta_{III}$ but product Δ is independent of all choices

Theorem

For each θ-Schwartz fdg on $G(\mathbb{R})$ there exists λ_1-Schwartz f_1dh_1 on $H_1(\mathbb{R})$ such that

$$SO(\gamma_1, f_1dh_1) = \sum_{\delta,\ \theta\text{-}conj} \Delta(\gamma_1, \delta)\ O^{\theta,\varpi}(\delta, fdg) \tag{5}$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

Corollary

If f has compact support then we may take f_1 of compact support mod $Z_1(\mathbb{R})$.

Endoscopic transfer: geometric side

f. remarks on statement

- corollary follows immediately from a theorem of Bouaziz
- notation: $Z_1 = Ker(H_1 \rightarrow H)$, $\mathfrak{e}_z$ determines character λ_1 on $Z_1(\mathbb{R})$, require $f_1(z_1 h_1) = \lambda_1(z_1)^{-1} f_1(h_1)$ for $z_1 \in Z_1(\mathbb{R})$, $h_1 \in H_1(\mathbb{R})$
- $\Delta(\gamma_1, \delta)$ is invariant under stable conjugacy in first variable, also has correct behavior under translation by $Z_1(\mathbb{R})$
- $SO(\gamma_1, f_1 dh_1)$ is usual normalized stable orbital integral
- left and right: compatible Haar measures in denominators of quotients
- (θ, ϖ)-twisted orbital integral

$$O^{\theta,\varpi}(\delta, fdg) := \int_{Cent_\theta(\delta,G)(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}\delta\theta(g))\varpi(g)\frac{dg}{dt_\delta} \qquad (6)$$

- $\Delta(\gamma_1, \delta)$ has correct behavior under θ-conjugacy to make right side of (5) well-defined

- **For proof of theorem:**

- (old) characterization of stable orbital integrals via Harish-Chandra Plancherel theory in terms of jump behavior
- introduce form better adapted to canonical transfer factors

- Harish-Chandra descent for twisted orbital integrals and *semi-regular is sufficient* principle, along with descent properties of the norm correspondence, reduce problem to simple wall-crossing properties for transfer factors

- (long) calculations with transfer factors to check these properties ■

Endoscopic transfer: spectral side

a. dual transfer: summary

- for each test fdg on $G(\mathbb{R})$ attach test $f_1 dh_1$ on $H_1(\mathbb{R})$ with matching orbital integrals in the sense of (5) of main theorem

- Θ_1 : stable distribution on $H_1(\mathbb{R})$, correct $Z_1(\mathbb{R})$ behavior and eigendistribution for center $\mathfrak{Z}_1$ of universal enveloping algebra

- then $\Theta : fdg \to \Theta_1(f_1 dh_1)$ well-defined θ-twisted invariant distribution on $G(\mathbb{R})$ and eigendistribution for $\mathfrak{Z}$

- Θ_1 tempered $\Longrightarrow$ Θ tempered
- endo $\mathfrak{e}_z$ determines shift in infinitesimal character
- formula for Θ_1 as smooth function on regular set $\Longrightarrow$ formula for Θ as smooth function on regular set

- **goal:** for a stable character $\Theta_1 = St\text{-}Trace\ \pi_1$, where π_1 irreducible admissible representation of $H_1(\mathbb{R})$ with correct $Z_1(\mathbb{R})$ behavior, **to describe** Θ explicitly as a combination of (θ, ϖ)-twisted traces

$$f \longrightarrow Trace\ [\pi(f) \circ \pi(\theta, \varpi)] \qquad (7)$$

notation: $\pi(\theta, \varpi)$ intertwines $\pi \circ \theta$ and $\varpi \otimes \pi$ [also drop dg, dh]

- **thus to establish** dual transfer in the form

$$St\text{-}Trace\ \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi)\ [Trace\ \pi(f) \circ \pi(\theta, \varpi)] \qquad (8)$$

- term on right side will be independent of normalization of $\pi(\theta, \varpi)$ [Δ_{II} involves twisted character formula and effects cancel]

- in place of very regular norm pairs $(\gamma_1, \delta), (\gamma_1', \delta')$, consider very regular related pairs $(\pi_1, \pi), (\pi_1', \pi')$: define (almost) canonical $\Delta(\pi_1, \pi; \pi_1', \pi')$

- again Δ has same form $\Delta_I \Delta_{II} \Delta_{III}$; may also define $\Delta(\pi_1, \pi; \gamma_1, \delta)$
- in transfer theorems use geom-spec compatible factors: $\Delta(\pi_1, \pi)/\Delta(\gamma_1, \delta) = \Delta(\pi_1, \pi; \gamma_1, \delta)$

- **standard setting:** $\theta =$ *identity*, $\varpi =$ *trivial character* results $\Longrightarrow$ structure on packets of representations ... then twisted setting $\Longrightarrow$ compatible additional structure on packets preserved by $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$

- prescribe very regular pairs via Arthur parameters, start with G^*
- Arthur parameter: $G^\vee$-conjugacy class of an admissible hom
  $$\psi = (\varphi, \rho) : W_\mathbb{R} \times SL(2, \mathbb{C}) \to {}^L G$$
  here φ [in general, essentially] bounded Langlands parameter
- let $S = S_\psi = Cent(\psi(W_\mathbb{R} \times SL(2, \mathbb{C})), G^\vee)$: ψ is **elliptic** if S^0 central
- $\rho(SL(2, \mathbb{C})) \subset M^\vee = M^\vee_\varphi =$ Levi group $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ in $G^\vee$
- call ψ *u*-**regular** if $\rho(SL(2, \mathbb{C}))$ contains regular unipotent elts of $M^\vee$
- define group $\mathcal{M} = \mathcal{M}_\varphi$ in ${}^L G$ as subgp gen by $M^\vee$ and $\varphi(W_\mathbb{R})$
  $$1 \longrightarrow M^\vee \longrightarrow \mathcal{M} \rightleftarrows W_\mathbb{R} \longrightarrow 1$$
  extract L-action same way as endo, $M^* =$ dual, quasi-split over $\mathbb{R}$

Endoscopic transfer: spectral side

cc. very regular pairs (cont.)

- u-regular ψ is elliptic $\Longleftrightarrow$ $T \hookrightarrow M^* \hookrightarrow G^*$ all over $\mathbb{R}$, with T anisotropic modulo the center of G
- u-regular $\psi = (\varphi, triv)$ is elliptic $\Longleftrightarrow$ φ discrete series parameter

- attach packet Π to u-regular ψ : L-packet if $\rho = triv$, or Arthur packet otherwise [see Adams-Johnson, just elliptic here]
- do same for endo group: use only those u-regular ψ_1 such that $\psi_1(W_\mathbb{R} \times SL(2,\mathbb{C}))$ lies in the image of endo $\mathcal{H}$, up to conjugacy [$\Longleftrightarrow$ members of attached Π_1 have correct $Z_1(\mathbb{R})$ behavior]

- such ψ_1 determines parameter ψ_{ψ_1} for G^*, Levi group $\mathcal{M}_1$ for ψ_1 determines subgroup $\mathcal{M}_H$ of $\mathcal{H}$ contained in Levi $\mathcal{M}$ for ψ_{ψ_1} : call ψ_1 G-regular if $\mathcal{M}_H = \mathcal{M}$
- (ψ_1, ψ) **very regular pair**: ψ_1, ψ are u-regular and ψ_1 is G-regular
- very regular **related** pair: also $\psi = \psi_{\psi_1}$

- **same defs for** pairs (π_1, π) in packets (Π_1, Π) attached to (ψ_1, ψ)

- **start with standard setting, tempered** $(\rho = triv)$ **and elliptic:**
  (8) says: $St\text{-}Trace\ \pi_1(f_1) = \sum_{\pi} \Delta(\pi_1, \pi)\ Trace\ \pi(f)$
  (π_1, π), (π_1', π') related pairs discrete series representations with Langlands parameters (φ_1, φ), (φ_1', φ')
- define relative factor $\Delta(\pi_1, \pi; \pi_1', \pi')$
- toral data $T_1 \to T$, with T anisotropic mod center of G, a-data, χ-data for $\Delta_I, \Delta_{II}, \Delta_{III}$
- Δ_{II} involves local formula for $Trace\ \pi(f)$ as smooth function ... [fourth root of unity if rewrite usual Harish-Chandra formula]

- via parabolic induction extend defns to $\Delta(\pi_1, \pi; \pi_1', \pi')$, $\Delta(\pi_1, \pi; \gamma_1, \delta)$, for all very regular norm pairs (γ_1, δ) and all tempered very regular related pairs $(\pi_1, \pi), (\pi_1', \pi')$ [set $\Delta(\pi_1, \pi) = 0$ if pair not related]

- **proof of (8) for tempered very regular pairs:** reduce quickly to elliptic case, discrete series both sides, and then apply Harish-Chandra characterization theorem: transfer Θ is tempered invariant eigendistribution with correct infinitesimal character and agrees with $\sum_{\pi} \Delta(\pi_1, \pi)$ *Trace* $\pi(f)$ on regular elliptic set

- now **theorem for all tempered pairs?** for example, need this for converse: spec transfer for $(f_1, f) \Longrightarrow$ geom transfer for (f_1, f)

- main case = elliptic on left: transfer discrete series to limits of discrete series, limits which arise have Levi $\mathcal{M}$ of type $(A_1)^n$ then Hecht-Schmid character identities + analysis in $G^\vee$ identifies transfer Θ as right side of (8), where factor $\Delta(\pi_1, \pi)$ is defined via analog of Zuckerman translation for parameters

- conclude the following continuation of geom transfer thm, std setting:

Theorem

Suppose geom, spec factors Δ are compatible. Then

$$St\text{-}Trace\ \pi_1(f_1 dh_1) = \sum_{\pi} \Delta(\pi_1, \pi)\ \ Trace\ \pi(fdg) \tag{9}$$

for all tempered irreducible admissible representations π_1 such that $Z_1(\mathbb{R})$ acts by λ_1.

- **Conversely:** *if* fdg, $f_1 dh_1$ *are test measures satisfying (9) then*

$$SO(\gamma_1, f_1 dh_1) = \sum_{\delta\ conj} \Delta(\gamma_1, \delta)\ O(\delta, fdg) \qquad (10)$$

for all strongly G*-regular* γ_1 *in* $Z_1(\mathbb{R})$.
Proof: Use both transfer thms plus *same* $SO's \Longrightarrow$ *same* St*-Traces*

- **alternate argument** to prove tempered spectral transfer:
- **(i)** in the elliptic case the chosen $\Delta(\pi_1, \pi)$ are the only possible coefficients for a spectral version of dual transfer ..., plus they have correct properties re translation principle and parabolic induction ... again this depends also on properties of the geometric factors and compatibility factors
- **(ii)** theorem is true for some choice of coefficients [old result] and so it is true with the factors $\Delta(\pi_1, \pi)$ we have defined

- still in standard setting, nontempered examples?
  define $\Delta(\pi_1, \pi)$ for very regular pairs in general:
  enough to define $\Delta(\pi_1, \pi; \pi_1', \pi')$ for some tempered (π_1', π'),
  then $\Delta(\pi_1, \pi) := \Delta(\pi_1, \pi; \pi_1', \pi').\Delta(\pi_1', \pi')$

- start with elliptic case: construct (π_1', π') tempered elliptic or just π_1' tempered elliptic in some cases
- for transfer statement (9): apply alternate argument to character identities of Adams-Johnson [see Arthur, Kottwitz]
- or check directly that these factors $\Delta(\pi_1, \pi)$ work in A-J arguments: use familiar formula for relative factor
  $\Delta(\pi_1, \pi; \pi_1', \pi') := \Delta(\pi_1, \pi)/\Delta(\pi_1', \pi')$ when π, π' lie in same Arthur packet
- [remove elliptic assumption]

- return to twist by (θ, ϖ) and start with tempered setting
- now concerned only with (θ, ϖ)-stable packets Π, *i.e.* those Π preserved by $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$, along with attached twist-packet $\Pi^{\theta,\varpi}$ consisting of those $\pi \in \Pi$ fixed by this map
- enough: θ preserves fundamental splitting [earlier comment]
- essentially harmonic analysis on group $G(\mathbb{R}) \rtimes \langle \theta \rangle$ outside Harish-Chandra class [some results not yet written in sufficient generality to claim transfer results in general]
- approach to defining tempered spectral factors: again elliptic setting first, translation, and then parabolic descent [Mezo 2013: use results of Duflo for parabolic induction]

- spectral factors in tempered elliptic case: now constructions parallel those for twisted geometric factors of Kottwitz-Shelstad, again compatibility factors, parallel properties, *etc.*

- Proof of transfer: apply alternate argument again, here to character identities of Mezo
- Mezo 2012: identities for elliptic (π_1, π), also when only π_1 elliptic, with coefficients written in terms of data from Duflo's method rather than directly from Harish-Chandra character formula

- again similar approach to standard case to define twisted factors $\Delta(\pi_1, \pi)$ for nontempered very regular pairs (π_1, π) ...

- summary: along with geometric transfer factors come spectral factors, in both standard and twisted settings; these express dual transfer as a spectral transfer [incomplete ...]

- now we use the relative factors $\Delta(\pi_1, \pi; \pi_1', \pi')$ to establish pairings of a packet Π with a finite group defined on dual side

- then twisted relative factors $\Delta(\pi_1, \pi; \pi_1', \pi')$ provide compatible pairings for twist-packets $\Pi^{\theta,\varpi}$ within (θ, ϖ)-stable Π

- various (Galois-cohomological) properties of pairings have consequences for harmonic analysis, *e.g.* inversion of spectral transfer in tempered setting

  [Whittaker normalizations $\Longrightarrow$ simplest spectral pairings]

- start with tempered packet Π and use relative factors $\Delta(\pi_1, \pi; \pi_1, \pi')$, with $\pi, \pi' \in \Pi$, to put structure on Π
- π_1 determined by spectral construction of endo data:

- $\varphi : W_{\mathbb{R}} \to {}^L G$ Langlands parameter for Π
  $S = Cent(\varphi(W_{\mathbb{R}}), G^{\vee})^0$, $\quad S^{ad}$ = image of S in $(G^{\vee})_{ad}$,
  S^{sc} = preimage of S^{ad} in $(G^{\vee})_{sc}$, $\quad s_{sc}$ = semisimple element in S^{sc}
- s = image of s_{sc} in $G^{\vee}$
  $\mathcal{H}(s)$ = subgroup of ${}^L G$ generated by $Cent(s, G^{\vee})^0$ and $\varphi(W_{\mathbb{R}})$
  $\mathfrak{e}_z(s_{sc}) = \mathfrak{e}_z(s)$ = attached suppl. endo data
- by construction, φ factors through *well-positioned* $\varphi^s : W_{\mathbb{R}} \to {}^L H_1$
- now for π_1 take any $\pi^s \in \Pi^s$ = packet attached to φ^s

- **Theorem:** $s_{sc} \to \Delta(\pi^s, \pi; \pi^s, \pi')$ depends only on the image of s_{sc} under $S^{sc} \to S^{ad} \to \pi_0(S^{ad}) = \mathbb{S}^{ad}$ = sum of $\mathbb{Z}/2$'s
- and defines character on $\mathbb{S}^{ad}$, trivial iff $\pi = \pi'$, all ...
  [in general this requires a dual, uniform by packet, version of Knapp-Zuckerman decomposition of unitary principal series]
- elliptic case: just Tate-Nakayama duality $\mathbb{C}/\mathbb{R}$
- in general, don't use duality with $\mathbb{S}^{ad}$ but with extension, e.g. $\mathbb{S}^{sc}$
  so will write $\Delta(\pi^s, \pi; \pi^s, \pi') = \langle \pi, s_{sc} \rangle \ / \ \langle \pi', s_{sc} \rangle$:
  pick base point π_0 for Π and specify character $s_{sc} \to \langle \pi_0, s_{sc} \rangle$,
  then $\langle \pi, s_{sc} \rangle := \Delta(\pi^s, \pi; \pi^s, \pi_0) \ \langle \pi_0, s_{sc} \rangle$
  ... pairing of type proposed by Arthur for global picture [2007]
  [better, new approach of Kaletha]
- simpler case... **Theorem:** G of quasi-split type, Whittaker norm of absolute Δ, π_0 generic, trivial character $s_{sc} \to \langle \pi_0, s_{sc} \rangle$:
  $\langle \pi, s \rangle := \Delta(\pi^s, \pi)$ gives perfect pairing ... Π as dual of $\mathbb{S}^{ad}$

- **Corollary:** invert transfer in Whittaker setting simply as

$$Trace\ \pi(f) = \left|\mathbb{S}^{ad}\right|^{-1} \sum_{s \in \mathbb{S}^{ad}} \langle \pi, s \rangle\ St\text{-}Trace\ \pi^s(f_1^s) \qquad (11)$$

for all tempered π, test f and corresponding test f_1^s

- now review some constructions, focus on Whittaker case, and move to twisted setting ...
- elliptic case, Whittaker setting: calculate $\langle \pi, s \rangle$?
 G^* cuspidal, T anisotropic mod center, also $T_G \subseteq G$
 $\pi =$ discrete series, π_0 determines Weyl chamber(s) $\mathcal{C}_0$ yielding toral data for T in G^* and then well-defined character κ on $H^1(\Gamma, T^{sc})$; π determines chamber for T_G; inner twist carries this chamber to $\mathcal{C}_0$ up to inner automorphism; make a well defined element ω in $H^1(\Gamma, T^{sc})$; finally, $\langle \pi, s \rangle = \kappa(\omega)$

- remarks on last calculation:
- (i) $\langle \pi, s \rangle$ is the absolute version of Δ_{III} available in this setting
  in general setting, there is a central obstruction to defining ω in H^1
  which is handled by going to relative version using a trick from
  the original definition of geometric factors in [L-S]
  [trick works for any pair T, T' of maximal tori over local field F ...]
  nonabelian variant for general elliptic u-regular case
- (ii) it is easy to extend this type of calculation (for discrete series)
  to the twisted setting using fundamental splittings
  (Weyl chambers $\rightsquigarrow$ fnd. splittings):
- assume θ preserves fnd. splitting spl_f; may assume inner twist η
  transports spl_f to fnd. splitting spl_f^* of G^* preserved by θ^*,
  spl_f^* provides toral data to transport objects from $G^\vee$...

- $\Pi = (\theta, \varpi)$-stable packet of discrete series
  fnd. splitting spl_π for π in twist-packet $\Pi^{\theta,\varpi}$ is preserved by θ
  up to inner automorphism η transports spl_π to spl_f^*
  make Galois cocycle in this setting (relative in general)
- cocycle almost takes values in θ^*-invariants; instead, satisfies hypercocycle condition, so back to setting of Kottwitz-Shelstad for geometric transfer factors

- compatibility statement: introduce twisted version of S,
- work in $G^\vee \rtimes \langle \theta^\vee \rangle$ ...
- for nontrivial twisting character ϖ, analysis exploits map on endo data: $\mathfrak{e}_z \to (\mathfrak{e}_z)_{ad}$ dual to $G_{sc} \to G$
  ■