Title:	Fixed parameter approximability and hardness
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Problem Definition

NP-hard problems are believed to be intractable. This is the widely believe assumption that $P \neq NP$. For all our problems, the size of their input is denoted by n. In parameterized complexity the input is refined to (I, k) with k a parameter related to the input and the goal is to find an exact algorithm for the problem, that runs in time $f(k) \cdot n^{O(1)}$, for some function f. In this survey we parameterize by the optimum value of the instance unless stated otherwise. In addition, the optimum is always integral. In approximation algorithms a ρ approximation for a minimization (maximization) problem P, is a polynomial time algorithm A, such that for any instance I, A returns a solution of value $A(I)$ and $A(I)/\text{OPT}(I) \leq \rho$ ($\text{OPT}(I)/A(I) \leq \rho$) with $\text{OPT}(I)$ the optimum value for the instance. In both subjects, there are intractability results. The

class FPT are the problems that admit an $f(k)n^{O(1)}$ time, exact solution for some function f. The classes $W[i]$ for every integer $i \geq 1$ satisfy $\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \ldots$. It is widely believed that all inclusions are strict. Consider the CLIQUE problem. Given a graph $G(V, E)$, a subset $U \subseteq V$, forms a *Clique*, if for every $u, v \in U$, $(u, v) \in E$. The problem is:
Input: A graph G and a parameter k
Question: Is there in G a clique U of size $|U| \geq k$?
In [Hastad et al(1996)Hastad] it is proved that CLIQUE admits no $n^{1-\epsilon}$ approximation unless $P = NP$. It is known that CLIQUE is $W[1]$-complete. Thus it is considered highly unlikely that CLIQUE $\in$ FPT. The SETCOVER problem is defined as follows:
Input: A universe U and a collection $\mathcal{S} = \{S_i\}$ of subsets of $\mathcal{U}$ and a parameter k.
Question: Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ containing at most k sets so that $\bigcup_{S_i \in \mathcal{S}'} S_i = \mathcal{U}$?
SETCOVER is $W[2]$-complete. In addition Raz and Safra (STOC 1997) show that unless $P = NP$, SETCOVER admits no $c \ln n$ algorithm for some constant c, almost matching the simple greedy $\ln n + 1$ ratio approximation algorithm.
Our subject: Formally we deal with the following subject: An algorithm for a minimization (resp., maximization) problem P, is called an (r, t)-FPT-approximation algorithm for P with input parameter k, if the algorithm takes as input an instance I with value OPT and an integer parameter k and either computes a feasible solution to I with value at most $k \cdot r(k)$ (resp., at least $k/r(k)$ and $k/r(h) = o(k)$) or computes a certificate that $k <$ OPT (resp., $k >$ OPT) in time $t(k) \cdot |I|^{O(1)}$. The requirement that $k/r(k) = o(k)$ avoids returning a single vertex in the clique problem, claiming OPT approximation.

A problem is called (r, t)-FPT-inapproximable (or, (r, t)-FPT-hard) if it does not admit an (r, t)-FPT-approximation algorithm. An FPT-approximation is mainly interesting if the problem is $W[1]$-hard and allowing running time $f(k) \cdot n^{O(1)}$ gives improved approximation. We restrict our attention to this scenario. Thus, we do not discuss many subjects such as approximations in OPT that run in *polynomial time* in n and upper and lower bounds, on algorithms with sub exponential time in n, for problems.
Our complexity assumption:
We assume the following conjecture throughout. Impagliazzo et al. [Calabro et al(1995)Calabro, Impagliazzo, and Paturi] conjectured the following:

Exponential Time Hypothesis (ETH)
3-SAT cannot be solved in $2^{o(q)}(q + m)^{O(1)}$ time where q is the number of variables and m is the number of clauses.

The following is due to [Calabro et al(1995)Calabro, Impagliazzo, and Paturi].

Lemma 1. *Assuming* ETH, *3-*SAT *cannot be solved in* $2^{o(m)}(q + m)^{O(1)}$ *time where q is the number of variables and m is the number of clauses.*

It is known that the ETH implies that $W[1] \neq \text{FPT}$. This implies that $W[2] \neq \text{FPT}$ as well.

Key Results

We survey some FPT-approximability and inapproximability results. Our starting point is a survey by Marx [Marx et al(2005)Marx], and we also discuss recent results. The simplest example we are aware of in which combining FPT running time

and approximation algorithm, gives an improved result is for the Strongly Connected Directed Subgraph (SCDS) problem.
Input: A directed graph $G(V, E)$, a set $T = \{t_1, t_2, \ldots, t_p\}$ of terminals and an integer k
Question: Is there a subgraph $G'(V, E')$ so that $|E'| \leq k$ and for every $t_i, t_j \in T$, there is a directed path in G' from t_i to t_j and vise-versa?
The problem is in $W[1]$-hard. The best approximation algorithm known for this problem is n^ϵ for any constant ϵ. See Charikar et al Journal of Algorithms 1999.

The following is due to [Chitnis et al(2013)Chitnis, Hajiaghayi, and Kortsarz].

Theorem 1. *The SCDS problem admits an* FPT *time* 2 *approximation ratio.*

Proof. The directed Steiner Tree problem is given a directed edge weighted graph and a root r and a set $T = \{t_1, t_2, \ldots, t_p\}$ of terminals, find a minimum cost directed tree rooted by r containing T. This problem belongs to FPT. See Dreyfus and Wagner, Networks 1971. Note that for every terminal t_i any feasible solution contains a directed tree from t_i to T, and a a reverse directed Steiner tree from T to t_i. These two problems can be solved optimally in FPT time. In the second application, we reverse the direction of edges before we find the directed Steiner tree. Moreover, two such trees give a feasible solution for the SCDS problem as every two terminals t_j, t_k have a path via t_i. Clearly, the solution has value at most $2 \cdot \text{OPT}$ with OPT the optimum value for the SCDS instance. The claim follows.

Definition 1. *A* Polynomial time approximation scheme *(PTAS) for a problem P is a $1+\epsilon$ approximation for any constant ϵ that runs in time $n^{f(1/\epsilon)}$. An EPTAS is such an algorithm that runs in time $f(1/\epsilon)n^{O(1)}$*

The *Vertex Cover* problem, is to select the smallest possible subset U of V so that for every edge (u, v), either $u \in U$ or $v \in U$ (or both). In the *Partial Vertex Cover problem* a graph $G(V, E)$ and a an integer k are given. The goal is to find a set U of k vertices, that is touched by the largest number of edges. An edge (u, v) is touched by a set U if $u \in U$ or $v \in U$, or both. It is known that this problem admits no *PTAS* unless $P = NP$ (see Dinur and Safra, STOC 2002). The corresponding Minimum Partial Vertex Cover problem requires a set of k vertices touched by the *least number of edges*. This problem admits no better than 2-ratio, under the *Small Set Expansion Conjecture*. See [Gandhi et al(2014)Gandhi and Kortsarz]. Both problems are $W[1]$-hard. The following theorem of [Marx et al(2005)Marx] relies on a technique called *color coding* [Alon et al(1995)Alon, Yuster Zwick].

Theorem 2. [Marx et al(2005)Marx] *For every constant ϵ, the Partial Vertex Cover problem (and in in a similar proof the Minimum Partial Vertex Cover problem) admits an EPTAS that runs in time $f(k, 1/\epsilon) \cdot n^{O(1)}$ with n the number of vertices in the graph.*

Proof. Let $D = \binom{k}{2}/\epsilon$. Sort the vertices $v_1, v_2, ..v_n$ by non-increasing degrees. If for the largest degree, $d(v_1)$ satisfies $d(v_1) \geq D$, the algorithm outputs the set $\{v_1, v_2, \ldots, v_k\}$. These k vertices cover at least $\sum_{i=1}^{k} deg(v_i) - \binom{k}{2}$ edges. Clearly, $\text{OPT} \leq \sum_{i=1}^{k} deg(v_i)$. Hence the value of the constructed solution is at least:

$$\frac{\sum_{i=1}^{k} deg(v_i) - \binom{k}{2}}{\sum_{i=1}^{k} deg(v_i)} \geq 1 - \frac{\binom{k}{2}}{D} \geq 1 - \frac{\epsilon}{2} \geq \frac{1}{1+\epsilon}$$

times the optimum for a $1+\epsilon$ approximation. In the other case, the optimum $\text{OPT} \leq k \cdot D$. We guess the correct value of the optimum by trying all values between $1, \ldots, k \cdot D$.

Fix the run with the correct OPT. Let E^* be the set of OPT edges that are touched by the optimum. An OPT-labeling is an assignment of a label in $\{1, \ldots, \text{OPT}\}$ to the edges of E. We show that if the labels of E^* are pairwise distinct, we can solve the problem in time $h(k, 1/\epsilon)$. Let $\{u_1, u_2, \ldots, u_k\}$ be the optimum set. Let L_i be the labels of the edges of u_i. As all labels of E^* are pairwise distinct, $\{L(u_i)\}$ is a disjoint partition of all labels (as otherwise there is a labeling with less than OPT labels). The number of possible partitions of the labels into k sets, is at most k^{OPT}. Given the *correct* partition $\{L_i\}$, we need to match every L_i with a vertex u_i so that the labels of u_i are L_i. This can be done in polynomial time by matching computation. To get a labeling with different pairwise labels on E^* we draw for every edges a label between 1 and OPT, randomly and independently. The probability that the labels of E^* are disjoint is more than $1/\text{OPT}^{\text{OPT}}$. Repeating the random experiment for OPT^{OPT} times implies that with probability at least $1 - 1/e$, one of the labeling has different pairwise labels for E^*. This result can be derandomized [Alon et al(1995)Alon, Yuster Zwick].

We consider one example in which OPT is not the parameter [Marx et al(2005)Marx]. Consider a graph that contains a set $X = \{x_1, \ldots, x_k\}$. so that $G \setminus X$ is a planar graph. Thus the parameter here is the number of vertices that need to be removed to make the graph. Consider the minimum coloring problem on G. We can determine the best coloring of X in time k^k. Then we can color $G \setminus X$ by 4 (different) colors. An simple calculation shows that this algorithm has approximation ratio at most 7/3.

The following is a simple relation exist between $EPTAS$ and FPT theory.

Proposition 1. *If an optimization problem P admits an $EPTAS$, then $P \in$ FPT.*

Proof. We prove the theorem for minimization problems. For maximization problems the proof is similar. Assume that P has a $1 + \epsilon$ approximation that runs in time $f(1/\epsilon) \cdot n^{O(1)}$. Set $\epsilon = 1/(2k)$. Using the EPTAS algorithm gives an $f(2k)n^{O(1)}$ time $(1 + \epsilon)$ approximation. If the optimum is at most k, we get a solution of size at most $(1 + \epsilon)k = k + 1/2 < k + 1$. As the solution is integral, the cost is at most k. If the minimum is $k + 1$ the approximation will not return a better than $k + 1$ size solution. Thus the approximation returns cost at most k if and only if there is a solution of size at most k.

Thus we can rule out the possibility of an $EPTAS$ if a problems is $W[1]$-hard. For example this shows that the Maximum independent set for unit disks graphs admits no $EPTAS$ as it belongs to $W[1]$. See many more examples in [Marx et al(2005)Marx]. Chen Grohe and Gruber (IWPEC 2006) provide an early discussion of our topic. Lange wrote a PDF presentation for recent FPT approximation. The following theorem is due to Grohe and Gruber (see ICALP 2007).

Theorem 3. *If a maximization problem admits an FPT-approximation algorithm with performance ratio $\rho(k)$ then for some function ρ' there exists an $\rho'(k)$* polynomial time *approximation algorithm for the problem.*

In the Traveling Sales person with deadline the input is a metric on n points and a set $D \subseteq V$ with each $v \in D$ having a deadline t_v. A feasible solution is a simple path containing all vertices, so that *for every* $v \in D$, the length of the tour until v is at most t_v. The problem admits no constant approximation and is not in FPT when parameterized by $|D|$. See Bockenhauer, Hromkovic, Kneis, and Kupke (see Theory of Computing Systems 2007). In this paper the authors give a 2.5 approximation that runs in time $n^{O(1)} + |D|! \cdot |D|$. The parameterized undirected Multicut problem is given an undirected graph and a collection $\{s_i, t_i\}_{i=1}^m$ of pairs, and a parameter k, is possible to remove at most k edges and disconnect all pairs? Garg Vazirani and Yannakakis give an

$O(\log n)$ approximation for the problem (SIAM J. Comput. 1996). In 2009 it was given a ratio 2 Fixed-parameter approximation (Marx and Razgon) algorithm. However, Marx and Razgon (STOC'11) and Bousquet (STOC'11) show that this problem is in fact in FPT. Fellows, Kulik, Rosamond and Shachnai give the following tradeoff (see ICALP 2012). The best known exact time algorithm for the Vertex Cover problem has running time 1.273^k. The authors show that if we settle for an approximation result, then the running time can be improved. Specifically, they gave $\alpha \geq 1$ approximation for Vertex Cover that runs in time $1.237^{(2-\alpha)k}$. The minimum edge dominating set problem is given a graph and a parameter k, is there a subset $E' \subset E$ of size at most k so that every edge in $E \setminus E'$ is adjacent to at least one edge in E'. Escoffier, Monnot, Paschos, and Mingyu Xiao (see IPEC 2012) prove that the problem admits a $1+\epsilon$ ratio for any $0 \leq \epsilon \leq 1$ that runs in time $2^{(2-\epsilon)\cdot k}$. A kernel for a problem P is a reduction from an instance I to an instance I' whose size is $g(k)$ namely, a function of k, so that a yes answer for I implies a yes answer for I' and a no answer for I implies a no answer for I'. If a kernel exists it is clear that $P \in$ FPT. However, the size of the kernel may determine what is the function of k in the $f(k) \cdot n^{O(1)}$ exact solution. The following result seems interesting because it may not be intuitive. In the *Tree Deletion* problem, we are given a graph $G(V, E)$ and a number k and the question is if we can delete up to k vertices and get a tree. Archontia Giannopoulou Lokshtanov Saket and Suchy prove (see arXiv:1309.7891 2013) that the tree deletion problem admits a kernel of size $O(k^4)$. However, the problem does not admit an approximation ratio of OPT^c for any constant c.

Other parameters: An *independent set* is a set vertices so that no two vertices in the set share an edge. In parameterized version given k, the question if if there is an independent set of size at least k. Clearly the problem is $W[1]$-complete. Grohe (Combinatorica 2003) show that the Maximum Independent Set admits a FPT-Approximation scheme if the parameter is the genus of the graph. E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi (FOCS 2005) showed that Vertex Coloring has a ratio 2 approximation when parameterized by the *genus* of a graph. The tree augmentation problem is given an edge weighted graph and a spanning tree whose edges have cost 0, find a minimum cost collection of edges to add to the tree, so that the resulting graph is 2-edge connected. The problem admits several polynomial time, ratio 2, approximation algorithms. Breaking the 2 ratio for the problem is an important challenge in approximation algorithms. Cohen and Nutov parameterized the problem by the diameter D of the tree and gave an $f(D) \cdot n^{O(1)}$ time, $1 + \ln 2 < 1.7$ approximation algorithm for the problem (Theor. Comput. Sci. 2013)

Fixed parameter Inapproximability The following inapproximability is from [Downey et al(1995)Downey, Fellows, and McCartib]. The additive Maximum independent set problem is given a graph and a parameter k and a constant c, and the question is if the problem admit an independent set of size at least $k - c$, or no independent set of size k exists.

It turns out the problem equivalent to the independent set problem.

Theorem 4. *Unless* $W[1] = \text{FPT}$ *(hence, under the* ETH*) the Independent Set problem admits no additive c approximation*

Proof. Let I be the instance. Find the smallest d so that

$$\left\lceil \frac{dk - c}{d} \right\rceil \geq k.$$

Output d copies of G and let $k \cdot d$ be the parameter of the new instance I'. We show that the new graph has Independent set of size $dk - c$ if and only if the original instance has an independent set of size k. If the original instance has an independent set of size k, taking union of d independent sets we get an independent set of size $k \cdot d$.

Now say that I' has an independent set of size $dk - c$. The average size of an independent set in a graph in I' is then $(dk - c)/d$. Since the size of the independent set is integral, there is a copy that admits an independent sets of size

$$\left\lceil \frac{dk - c}{d} \right\rceil \geq k.$$

An independent set I is *maximal* if for every $v \notin I$, $v + I$ is not an independent set. The problem of Minimum size maximal independent set (MSDIS) is shown to be completely inapproximability in [Downey et al(1995)Downey, Fellows, and McCartib] Namely, this problem is $(r(k), t(k))$-FPThard for any r, t, unless $\mathrm{FPT} = W[2]$ (hence, under the ETH). The problem admits no $n^{1-\epsilon}$ approximation. (see Halldórsson Inf. Process. Letters 1993).

In the Min-WSAT problem a Boolean circuit is given and the task is to find a satisfying assignment of minimum weight. The weight of an assignment is the number of true variables. Min-WSAT was given a complete inapproximability by Chen, Grohe, Grber (see IWPEC 2006) 2006.

The above two problems are not monotone. This implies that the above result are non-surprising. The most meaningful complete inapproximability is given by Marx (JCSS 2013) who shows that the weighted circuit satisfiability for monotone or antimonotone circuits is completely FPT inapproximable.

Of course, if the problem has almost no gap, namely the instance can have value k or $k - 1$ its hard to get strong hardness.

A natural question is can we use gap reductions from Approximation Algorithms Theory to get some strong lower bounds, in particular for Clique and Set-Cover. It turns out that this is very difficult even under the ETH conjecture. This subject is related to almost linear PCP (see Moshkovitz APPROX-RANDOM 2012). In this paper Moshkovitz poses a conjecture called *the Projection game conjecture* (PGC). M. Hajiaghayi, R. Khandekar, G, Kortsarz show the following theorem:

Theorem 5. *Under the* ETH *and* PGC *conjectures,* SETCOVER *is* (r, t)-FPT-*hard for* $r(k) = (\log k)^{\gamma}$ *and* $t(k) = \exp(\exp((\log k)^{\gamma})) \cdot \mathrm{poly}(n) = \exp\left(k^{(\log^f k)}\right) \cdot \mathrm{poly}(n)$ *for some constant* $\gamma > 1$ *and* $f = \gamma - 1$.

Cross-References

Recommended Reading

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