Approximating Minimum-Power Edge-Covers and 2, 3-Connectivity

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Abstract

Given a graph with edge costs, the power of a node is the maximum cost of an edge leaving it, and the power of a graph is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider several fundamental undirected network design problems under the power minimization criteria. The Minimum-Power Edge-Cover (MPEC) problem is: given a graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$ and a subset $S \subseteq V$ of nodes, find a minimum-power subgraph H of G containing an edge incident to every node in S. We give a 3/2-approximation algorithm for MPEC, improving over the 2-approximation by [10]. For the Min-Power k-Connected Subgraph (MPk-CS) problem we obtain the following results. For $k = 2$ and $k = 3$, we improve the best previously known ratios of 4 [3] and 7 [10] to $3\frac{2}{3}$ and $5\frac{2}{3}$, respectively. Finally, we give a $4r_{\max}$-approximation algorithm for the Minimum-Power Steiner Network (MPSN) problem: find a minimum-power subgraph that contains $r(u, v)$ pairwise edge-disjoint paths for every pair u, v of nodes.

1 Introduction

1.1 Motivation and problems considered

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the cost required at v only depends on the furthest node that is reached directly by v. This is in contrast with wired networks, in which every pair of stations that need to communicate directly incurs a cost. We study the design of symmetric wireless networks that meet some prescribed degree or connectivity properties, and such that the total power is minimized. An important network property is fault-tolerance, which is often measured by node-connectivity of the network. Node-connectivity is much more central here than edge-connectivity, as it models stations failures. Such power minimization problems were vastly studied. See [1, 3, 10, 19, 17, 13] for a small sample of papers in this area. The first problem we consider is finding a low power network that "covers" a specified set S of nodes. This is the power variant of the fundamental Edge-Cover problem, c.f., [5]. The second problem is the Min-Power k-Connected Subgraph problem which is the power variant of the classic Min-Cost k-Connected Subgraph problem. We give approximation algorithms for these problems that significantly improve the previously best known ones.

Definition 1.1 *Let $H = (V, I)$ be a graph with edge-costs $\{c(e) : e \in I\}$. For $v \in V$, the* power *$p(v) = p_H(v)$* of *v in H (w.r.t. c) is the maximum cost of an edge in I leaving v (or zero, if no such edge exists), i.e., $p(v) = p_I(v) = \max_{vu \in I} c(vu)$. The power of the graph is the sum of the powers of its nodes. (Note that in directed graphs the edges entering v do not affect its power.)*

Note that $p(H)$ differs from the ordinary cost $c(H) = \sum_{e \in I} c(e)$ of H even for unit costs; for unit costs, if H is undirected, then $c(H) = |I|$ and (if H has no isolated nodes) $p(H) = |V|$. For example, if I is a perfect matching on V then $p(H) = 2c(H)$. If H is a clique then $p(H)$ is roughly $c(H)/\sqrt{|I|/2}$. For directed graphs, the ratio of the cost over the power can be equal to the maximum outdegree, e.g., for stars with unit costs. The following statement shows that these are the extremal cases for general edge costs.

Proposition 1.1 ([10]) *$c(H)/\sqrt{|I|/2} \leq p(H) \leq 2c(H)$ for any undirected graph $H = (V, I)$, and if H is a forest then $c(H) \leq p(H) \leq 2c(H)$. For any directed graph H holds: $c(H)/\Delta(H) \leq p(H) \leq c(H)$, where $\Delta(H)$ is the maximum outdegree of a node in H.*

Minimum-power problems are usually harder than their minimum-cost versions. The Minimum-Power Spanning Tree problem is APX-hard. The problem of finding minimum-cost k pairwise edge-disjoint paths is in P (this is the Minimum-Cost k-Flow problem, c.f., [5]) while both directed and undirected minimum-power variants are unlikely to have even a polylogarithmic approximation [10, 17]. Another example is finding an arborescence rooted at s, that is, a subgraph that contains an sv-path for every node v. The minimum-cost case is in P (c.f., [5]), while the minimum-power variant is at least as hard as the Set-Cover problem.

Unless stated otherwise, graphs are assumed to be undirected and simple. Let $H = (V, I)$ be a graph. For $X \subseteq V$, $\Gamma_I(X) = \Gamma_H(X) = \{u \in V - X : v \in X, vu \in I\}$ is the set of neighbors of X, $\delta_I(X) = \delta_H(X)$ is the set of edges leaving X, and $d_I(X) = |\delta_H(X)| = |\Gamma_H(X)|$ is the degree of X in H. Given a graph $G = (V, E)$ with edge-costs, we seek to find a low power *communication network*, that is, a low power subgraph of G (that is, an edge subset of E) that satisfies some prescribed property. Two such fundamental properties are: edge-cover and fault-tolerance/connectivity.

Definition 1.2 *Given a subset $S \subseteq V$ of nodes, we say that an edge set I on V is an S-*cover *if for every $v \in S$ there is an edge in I incident to v.*

Finding a minimum-cost S-cover is a fundamental problem in Combinatorial Optimization, as this is essentially the Edge-Cover problem, c.f., [5]. The following problem is the power variant.

Minimum-Power Edge-Cover (MPEC):
Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$, and a subset $S \subseteq V$ of nodes.
Objective: Find a min-power S-cover $I \subseteq E$.

MPEC naturally arises in applications. For example, given designated sets A of "suppliers" and B of "clients" (A, B may not be disjoint), we seek to design a low power communication network in which every client can communicate with at least one supplier.

We now define our connectivity problems. A graph is k-connected if it contains k internally-disjoint uv-paths for all $u, v \in V$. We consider the min-power variant of the extensively studied Min-Cost k-Connected Subgraph (MCk-CS) problem.

Minimum-Power k-Connected Subgraph (MPk-CS):
Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E)$, and an integer k.
Objective: Find a minimum-power k-connected spanning subgraph H of G.

We also consider min-power variant of the min-cost Steiner Network problem.

Minimum-Power Steiner Network (MPSN):
Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E)$, and requirements $\{r(u, v) : u, v \in V\}$.
Objective: Find a minimum-power subgraph H of G so that H contains $r(u, v)$ pairwise edge-disjoint uv-paths for every $u, v \in V$.

1.2 Previous and related work

Results on edge-cover problems: The following generalization of MPEC was considered in [10, 13]. Given a degree requirement function r on V, an edge set I on V is an r-*cover* if $d_I(v) \geq r(v)$ for every $v \in V$. The Minimum-Power Edge-Multi-Cover problem seeks to find an r-cover of minimum power; MPEC is a particular case when r is a $0, 1$-valued function, namely, when $r(v) = 1$ if $v \in S$ and $r(v) = 0$ otherwise. In [10] the approximation ratio $\min\{r_{\max} + 1, O(\log^4 n)\}$ was derived, where $r_{\max} = \max_{v \in V} r(v)$; this is improved to $O(\log n)$ in [13]. However, for the fundamental case MPEC, the best ratio was 2.

Results on connectivity problems: The simplest connectivity problem is when we require the network to be connected. In this case, the minimum-cost variant is just the Minimum-Cost Spanning Tree problem, while the minimum-power variant is APX-hard. A 5/3-approximation algorithm for the Minimum-Power Spanning Tree problem is given by Althaus et. al [1]. Minimum-cost connectivity problems were extensively studied, see surveys in [12] and [16]. The best known approximation ratios for the Minimum-Cost k-Connected Subgraph (MCk-CS) problem are $O(\ln^2 k \cdot \min\{\frac{n}{n-k}, \frac{\sqrt{k}}{\ln k}\})$ for both directed and undirected graphs [15], and $O(\ln k)$ for undirected graphs with $n \geq 2k^2$ [4]. It turns out that (for undirected graphs) approximating MPk-CS is closely related to approximating

MCk-CS and the Min-Power k-Cover problem – a particular case of the Min-Power Edge-Multi-Cover problem when $r(v) = k$ for all $v \in V$.

Theorem 1.2 ([10])

(i) *An α-approximation for* MCk-CS *and a β-approximation for* Min-Power k-Cover *implies a $(2\alpha + \beta)$-approximation for* MPk-CS.

(ii) *A ρ-approximation for* MPk-CS *implies a $(2\rho + 1)$-approximation for* MCk-CS.

One can combine various values of α, β with Theorem 1.2 to get approximation algorithms for MPk-CS. In [10, 13] the bound $\beta = \min\{k + 1, O(\log n)\}$ was derived. The best known values for α are: $\alpha = \lceil (k + 1)/2 \rceil$ for $2 \le k \le 7$ (see [2] for $k = 2, 3$, [6] for $k = 4, 5$, and [14] for $k = 6, 7$); $\alpha = k$ for $k = O(\log n)$ [14], $\alpha = O(\ln k)$ for $n \ge k(2k - 1)$ [4], and $\alpha = O(\ln k \cdot \min\{\sqrt{k}, \frac{n}{n-k} \ln k\})$ for $n < k(2k - 1)$ [15]. Thus for undirected MPk-CS the following ratios follow: $3k$ for any k, $k + 2\lceil (k + 1)/2 \rceil$ for $2 \le k \le 7$, and $O(\log n)$ unless $k = n - o(n)$. Improvements over the above bounds are known only for $k \le 2$. As was mentioned, the Minimum-Power Spanning Tree problem (this is the case $k = 1$ of MPk-CS) admits a 5/3-approximation algorithm [1]. Calinescu and Wan [3] gave a 4-approximation algorithm for the case $k = 2$ of MPk-CS.

For further results on other minimum-power connectivity problems, among them results for problems on directed graphs, see [10, 19, 17].

1.3 Our Results

The previous best approximation ratio for MPEC was 2 [10]. We prove:

Theorem 1.3 MPEC *admits a 3/2-approximation algorithm.*

For MPk-CS we improve the best known ratios for $k = 2, 3$:

Theorem 1.4 *Undirected* MPk-CS *with $k \in \{2, 3\}$ admit a $(2k - 1/3)$-approximation algorithm.*

For $k = 2$, Theorem 1.4 improves the best previously known ratio of 4 [3] to $3\frac{2}{3}$. For $k = 3$ the improvement is from 7 to $5\frac{2}{3}$.

We also consider the MPSN problem. Williamson et. al [21] gave a $2r_{\max}$-approximation algorithm for the min-cost case, and then this was improved to $2H(r_{\max})$ in [9]. The currently best known ratio for the min-cost case is 2 [11]. We show that the algorithm of [21] for the min-cost case, has approximation ratio $4r_{\max}$ for the minimum-power variant MPSN.

Theorem 1.5 *Undirected* MPSN *admits a $4r_{\max}$-approximation algorithm.*

Theorems 1.3, 1.4, and 1.5 are proved in Sections 2, 3, and 4, respectively.

1.4 Techniques

Our approach for MPEC is inspired by the decomposition method used by Prömel and Steger [20] for the Minimum-Cost Steiner Tree problem: decomposing solutions into small parts, and then reducing the problem to the Min-Cost Spanning Tree problem in 3-uniform hypergraphs, with loss of 5/3 in the approximation ratio. A similar method was used in [1] for the Minimum-Power Spanning Tree problem. In our case, to prove Theorem 1.3, we reduce MPEC to the Min-Cost Edge-Cover problem in *graphs*, and the loss in the approximation ratio is 3/2.

For MPk-CS with $k = 2, 3$ we show a 2-approximation algorithm for the "augmentation problem" of increasing the connectivity by 1. Combining with the 5/3-approximation algorithm of [1] for the Minimum-Power Spanning Tree gives the ratio in Theorem 1.4. We note that the augmentation version admits an easy 4-approximation by combining three facts:
(i) Any minimal solution to the augmentation problems is a forest [18].
(ii) The min-cost augmentation problem admits a 2-approximation [2].
(iii) $c(F) \le p(F) \le 2c(F)$ if F is a forest, see Proposition 1.1.
These facts are also valid for an appropriate augmentation version of MPSN, c.f., [21, 9]; this is how we obtain a $4r_{\max}$-approximation for MPSN in Theorem 1.5. However, getting a ratio of 2 for the augmentation version of MPk-CS with $k = 2, 3$ is done by using a different approach. Specifically, we consider *directed* solutions to a related "k-inconnectivity problem", and show that their underlying graphs have low power.

2 A 3/2-approximation for MPEC (Proof of Theorem 1.3)

We reduce Minimum-Power S-cover to Minimum-Cost S-Cover *in graphs*; the latter is solvable in polynomial time, c.f., [5]. However, the reduction is not approximation ratio preserving, but incurs a loss of 3/2 in the approximation ratio. That is, given an instance G, c, S of Minimum-Power S-Cover, we construct in polynomial time an instance G', c', S of Minimum-Cost S-Cover such that:
(i) for any S-cover I' in G' corresponds an S-cover I in G with $p(I) \le c'(I')$.
(ii) $\mathsf{opt}' \le 3\mathsf{opt}/2$, where opt' is the minimum cost of an S-cover in G', c';
Hence if I' is an optimal (min-cost) solution to G', c', S, then $p(I) \le c'(I') = \mathsf{opt}' \le 3\mathsf{opt}/2$.

Clearly, any minimal S-cover is a union of node disjoint stars. We now define a certain decomposition of stars.

Definition 2.1 *Let I be (an edge set of) a star. A collection $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ of sub-stars of I is a t-*decomposition *of I if the following holds: the stars in $\mathcal{I}$ cover all the leaves of I, every star has at most t edges, and there is at least one star in $\mathcal{I}$ with at most $t-1$ edges. The power $p(\mathcal{I}) = \sum_{I_j \in \mathcal{I}} p(I_j)$ of $\mathcal{I}$ is the sum of the powers of its parts.*

Intuitively, every part in $\mathcal{I}$ is "in charge" to cover its leaves; any part with $t-1$ edges is also "in charge" to cover the center v_0 of I. In this way, every part is in charge to cover at most t nodes, and every node of I (including the center) is covered by some part of $\mathcal{I}$ in this way.

For the purpose of proving Theorem 1.3, we will use only 2-decompositions; we consider the case of t arbitrary, as such decompositions may have other applications.

We will be interested in establishing that for any star I there exists a t-decomposition $\mathcal{I}$ so that the ratio $p(\mathcal{I})/p(I)$ is small. Namely, we would like to bound the ratio $\max_I \min_{\mathcal{I}} p(\mathcal{I})/p(I)$, where the maximum is taken over all stars I with edge costs, and the minimum is taken over all t-decompositions $\mathcal{I}$ of I. Even for stars with unit costs, this ratio cannot be asymptotically better than $1+1/t$. Indeed, let I be a star with $\ell t+(t-1)$ leaves and with unit edge costs. Then $p(I)=(\ell+1)t$, while $p(\mathcal{I}) \geq (\ell+1)t+\ell$ for any t-decomposition $\mathcal{I}$ of I. Thus $p(\mathcal{I})/p(I) \geq (1+1/t)-1/(t(\ell+1))$, and this ratio can be arbitrary close to $1+1/t$. The following statement shows that this bound is achievable for any edge costs.

Lemma 2.1 *Any star I admits a t-decomposition $\mathcal{I}$ with $p(\mathcal{I}) \leq (1+1/t)p(I)$.*

Proof: Let I be a star with center v_0. Let $e_1,\ldots,e_d$ be the edges of I sorted by non-decreasing costs, so $c_1 \geq c_2 \geq \cdots \geq c_d \geq 0$, where $c_j=c(e_j)$ for $j=1,\ldots,d$. Let $\ell=\lceil (d+1)/t \rceil$.

We will define a specific t-decomposition $\mathcal{I}=\{I_1,\ldots,I_\ell\}$ of I. If $\ell=1$, then $\mathcal{I}=\{I\}$ and we are done. If $\ell \geq 2$, then let I_1 consist of the t most expensive edges, I_2 of the next t most expensive edges, and so on. In this way we obtain stars $I_1,\ldots,I_{\ell-1}$ with exactly t edges each. If t does not divide d, then the last star I_ℓ will consist of the remaining at most $t-1$ edges. If t divides d, $I_1,\ldots,I_{\ell-1}$ already contain all the leaves of I; in this case we set $I_\ell=\{e_d\}$ to consist of the cheapest edge. We will show that $p(\mathcal{I}) \leq (1+1/t)p(I)$.

Let $p_j(v_0)=\max_{e\in I_j} c(e)$ be the power of v_0 in I_j. Note that $p(I_j)=c(I_j)+p_j(v_0)$. The key point is that

$$p_j(v_0) \leq c(I_{j-1})/t \qquad j=2,\ldots,\ell \ .$$

This is since every edge in I_{j-1} has cost at least $p_j(v_0)$, and since there are t edges in I_{j-1}, as $j-1 \neq \ell$. Therefore,

$$\begin{aligned}
p(\mathcal{I}) &= \sum_{j=1}^{\ell}(c(I_j)+p_j(v_0)) \\
&= c(I)+c_1+\sum_{j=2}^{\ell} p_j(v_0) \\
&\leq c(I)+c_1+\sum_{j=2}^{\ell} c(I_{j-1})/t \\
&\leq c(I)+c_1+c(I)/t \\
&\leq (1+1/t)p(I) \ .
\end{aligned}$$

□

Given an instance $G=(V,E),c,S$ of MPEC, the algorithm is:

1. Construct an instance $G'=(S,E'),c'$ of Min-Cost Edge-Cover as follows.
   G' is a complete graph on S, and $c'(uv)=p(I_{uv})$ for every $u,v\in S$, where I_{uv} is some min-power $\{u,v\}$-cover that consists of one edge or of two adjacent edges.
2. Find a minimum *cost* edge-cover I' in G',c'.
3. Return $I=\cup\{I_{uv}:uv\in F'\}$

Clearly, all the steps in the algorithm can be implemented in polynomial time. The following statement is used to prove that the approximation ratio of the algorithm is 3/2.

Lemma 2.2
(i) *If I' is an edge-cover in G' then $I=\cup\{I_{uv}:uv\in I'\}$ is an S-cover in G and $p(I)\leq c'(I')$.*
(ii) $\mathsf{opt}'\leq 3\mathsf{opt}/2$*, where* opt' *is the minimum cost of an S-cover in G',c'.*

Proof: I is an S-cover since I' is an S-cover, and since I_{uv} covers $\{u,v\}$ for every $uv\in I'$. Also, $p(I)\leq c'(I')\leq 3p(I)/2$ since

$$p(I)\leq\sum_{uv\in I'}p(I_{uv})=\sum_{uv\in I'}c'(uv)=c(I')\ .$$

We now prove that $\mathsf{opt}'\leq 3\mathsf{opt}/2$. Let I be an optimal solution to MPEC in G,c,S, so $p(I)=\mathsf{opt}$. Applying Lemma 2.1 with $t=2$ implies that there exists a 2-decomposition $\mathcal{I}=\{I_1,\ldots,I_{\ell+1}\}$ of I with $p(\mathcal{I})\leq 3p(I)/2=3\mathsf{opt}/2$. To every $I_j\in\mathcal{I}$ corresponds an edge e'_j in G' and $c'(e'_j)\leq p(I_j)$. Let $I'=\{e_1,\ldots,e_{\ell+1}\}$. Then I' is an edge-cover in G', hence $\mathsf{opt}'\leq c'(I')$. Thus:

$$\mathsf{opt}'\leq c'(I')=\sum_{j=1}^{\ell+1}c'(e'_j)\leq\sum_{j=1}^{\ell+1}p(I_j)=p(\mathcal{I})\leq 3p(I)/2=3\mathsf{opt}/2\ .$$

□

Theorem 1.3 now easily follows from Lemma 2.2. Let I,I' be as in the algorithm. Then, by Lemma 2.2, we have $p(I)\leq c'(I')=\mathsf{opt}'\leq 3\mathsf{opt}/2$.

The proof of Theorem 1.3 is complete.

3 Approximating $2,3$-connectivity (Proof of Theorem 1.4)

3.1 Reduction to k-inconnectivity

A (possibly directed) graph is *k-inconnected to s* if it contains k internally-disjoint vs-paths for every $v\in V$. Note that a graph is k-connected if it is k-inconnected to every $s\in V$. We need to consider the following problem:

Minimum-Power k-Inconnectivity (MPk-IS):
Instance: A graph $G = (V, E)$ with costs $\{c(e) : e \in E\}$, an integer k, and $s \in V$.
Objective: Find a min-power k-inconnected to s subgraph $H = (V, I)$ of G.

In the next section we will prove:

Theorem 3.1 *Undirected* MPk-IS *admits a* $(2k - 1/3)$*-approximation algorithm.*

The $(2k - 1/3)$-approximation algorithm for MPk-CS with $k \in \{2, 3\}$ follows from Theorem 3.1 and the following two facts (see [2]):
(i) Any undirected minimally k-connected graph has at least $|V|/3$ nodes of degree k [18];
(ii) For $k \in \{2, 3\}$, if s is a node of degree k in an undirected graph H, then H is k-inconnected to s if, and only if, H is k-connected [2].
Hence for $k \in \{2, 3\}$ undirected MPk-CS is equivalent (via an approximation ratio preserving reduction) to the problem of finding a min-power subgraph among the k-inconnected to s subgraphs so that the degree of s is exactly k. By Theorem 3.1, the latter problem admits a $(2k - 1/3)$-approximation algorithm for any constant k, by trying $O(n^{k+1})$ possible choices of s and the k edges incident to it. In fact, using penalty methods (see [6]) the exhaustive search can be reduced to trying only $O(n)$ choices of s (details omitted). This gives a $(2k-1/3)$-approximation algorithm for MPk-CS with $k \in \{2, 3\}$.

3.2 Proof of Theorem 3.1

To prove Theorem 3.1, we need to consider the "augmentation" version of MPk-IS:

Minimum-Power k-Inconnectivity Augmentation (MPk-IA):
Instance: An integer k, a $(k - 1)$-inconnected to s graph $H_0 = (V, I_0)$, and an edge set E on V with costs $\{c(e) : e \in E\}$.
Objective: Find a min-power edge set $I \subseteq E$ so that $H_0 + I$ is k-inconnected to s.

Lemma 3.2 *If I is an inclusion minimal solution to directed* MPk-IA *then $d_I(u) \leq 1$ for every $u \in V$, and thus the power of I equals its cost. Consequently, directed* MPk-IA *is solvable in polynomial time.*

Proof: Let us say that an edge e of a (possibly directed) k-inconnected to s graph H is *critical* if $H - e$ is not k-inconnected to s. In [17] it is proved:
Let uv', uv'' be two distinct critical edges of a k-inconnected to s directed graph H. Then $d_H(u) = k$.
Now, suppose to the contrary that $d_I(v) \geq 2$, so there are distinct edges $uv', uv'' \in I$. Since I is an inclusion minimal augmenting edge set, uv', uv'' are critical edge in $H_0 + I$, hence $d_{H_0+I} = k$, by [17]. This is a contradiction, since $d_{H_0} \geq k - 1$, $d_I(v) \geq 2$, thus $d_{H_0+I} \geq k + 1$.

Thus $p(I) = c(I)$, by Proposition 1.1. Consequently, MPk-IA is equivalent to its min-cost version; the latter is solvable in polynomial time [8, 7]. □

Lemma 3.3 *Undirected* MPk-IA *admits a* 2*-approximation algorithm.*

Proof: A *bi-direction* of an undirected network H is a directed network $D(H)$ obtained by replacing every edge $e = uv$ of H by two opposite directed edges uv, vu each having the same cost as e. Note that $p(H) = p(D(H))$. The 2-approximation algorithm for MPk-IA is as follows:

1. Let $D(H_0)$ and $D(E)$ be the bi-directions of H_0 and E, respectively.
2. Compute a min-cost edge set $I_D \subseteq D(E)$ so that $D(H_0) + I_D$ is k-inconnected to s.
3. Output the underlying edge set I of I_D.

Step 2 can be implemented in polynomial time [8, 7]. We now show that the approximation ratio of the algorithm is 2. Let I^* be an optimal solution to an (undirected) MPk-IA instance (so $p(I^*) = \mathsf{opt}$) and let $D(I^*)$ be the bi-direction of I^*. W.l.o.g. assume that I_D is minimal inclusion w.r.t. the property "$D(H_0)+I_D$ is k-inconnected to s". Thus $\Delta(I_D) \leq 1$, by Lemma 3.2. This implies $p(I) \leq 2p(I_D)$, since the contribution of an edge e to $p(I_D)$ is exactly $c(e)$, while the contribution of e to $p(I)$ is at most $2c(e)$. Also note that $p(I_D) \leq p(D(I^*))$, since I_D is an optimal k-inconnected to s subgraph. Thus we have: $p(I) \leq 2p(I_D) \leq 2p(D(I^*)) = 2p(I^*) = 2\mathsf{opt}$. □

Theorem 3.1 follows by combining Lemma 3.3 with the 5/3-approximation algorithm of [1] for the Minimum-Power Spanning Tree problem. Indeed, we can apply the algorithm as in Lemma 3.3 sequentially to produce edge sets $I_1, \ldots, I_k$ so that $H_\ell = I_1 + \cdots + I_\ell$ is ℓ-inconnected (resp., ℓ-edge-inconnected) to s, and $p(I_1) \leq 5\mathsf{opt}/3$ (I_1 is a spanning tree computed by the 5/3-approximation algorithm of [1]) and $p(I_\ell) \leq 2\mathsf{opt}$ for $\ell = 2, \ldots, k$. Consequently, if $I = I_1 + \cdots + I_k$ then $H = (V, I)$ is k-inconnected (resp., k-edge-inconnected) to s, and

$$p(I) \leq p(I_1) + \sum_{\ell=2}^{k} p(I_\ell) \leq \frac{5}{3}\mathsf{opt} + \sum_{\ell=2}^{k} 2\mathsf{opt} = 2(k - 1/3)\mathsf{opt} \ .$$

The proof of Theorem 3.1, and thus also of Theorem 1.4 is complete.

Remark: Calinescu and Wan [3] also gave a $2k$-approximation algorithm for the Min-Power k-Edge-Connected Subgraph problem for arbitrary k. This ratio can be improved to $(2k - 1/3)$ as follows. Lemma 3.2 extends to the edge-connectivity version of MPk-IA, as was implicitly proved in [7]. Thus Theorem 3.1 extends to the edge-connectivity version of MPk-IS. This gives a $(2k-1/3)$-approximation for the edge-connectivity version of MPk-IS, which is equivalent (for undirected graphs) to the Min-Power k-Edge-Connected Subgraph problem.

4 Algorithm for MPSN (Proof of Theorem 1.5)

We need some definitions and a description of certain results from [21, 9]. The minimum-cost/power Steiner Network problem can be formulated as a *set-function edge-cover problem.* Let $h : 2^V \rightarrow Z_+$

be a *set-function* defined on a groundset V. An edge set I on V is an h-*cover*, if $d_I(X) \geq h(X)$ for every $X \subseteq V$. For Steiner Network problems, an appropriate choice of h is as follows. By Menger's Theorem, I is a feasible solution to minimum-cost/power Steiner network problem if, and only if, $d_I(X) \geq R(X)$ for all $\emptyset \subset X \subset V$, where $R(X) = \max\{r(u,v) : u \in X, v \in V - X\}$ (and $R(\emptyset) = R(V) = 0$). That is

$$d_I(X) \geq h(X) \equiv \max\{0, R(X)\} \quad \forall\, \emptyset \subseteq X \subseteq V. \tag{1}$$

The function h defined above is *weakly-supermodular*, that is $h(\emptyset) = 0$ and for every $X, Y \subseteq V$ with $h(X) > 0$, $h(Y) > 0$ at least one of the following holds:

$$h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \tag{2}$$

$$h(X) + h(Y) \leq h(X - Y) + h(Y - X) \tag{3}$$

Note that h is also *symmetric*, that is, $h(X) = h(V - X)$ for all $X \subseteq V$.

Several connectivity problems can be formulated as (minimum-cost/power) edge cover problems of a weakly-supermodular function, see [16]. A seminal paper of Jain [11] gives a 2-approximation algorithm for finding a minimum-cost edge-cover of an arbitrary weakly-supermodular set function h, provided certain queries related to h can be answered in polynomial time (note that h is usually not given explicitly). For h defined in (1) these queries can be realized via max-flows, which implies a 2-approximation algorithm for the Minimum-Cost Steiner Network problem. Earlier, Williamson et. al [21] gave an algorithm with approximation ratio $2h_{\max}$, which was improved later to $2H(h_{\max})$ by Goemans et. al [9].

Let h be a set function on V. For an edge set I, let $h_I(X) = \max\{h(X) - d_I(X), 0\}$. It is well known that if h is skew supermodular, so is h_I for any edge set I, see [11]. Let $\hat{h}(X) = 1$ if $h(X) = h_{\max}$ and $\hat{h}(X) = 0$ otherwise, where $h_{\max} = \max_{X \subseteq V} h(X)$. It is easy to see that any inclusion minimal edge-cover of a $\{0,1\}$-valued set function, and thus also of $\hat{h}$, is a forest. Consider the following algorithm that applies on an arbitrary set-function h, and begins with $I = \emptyset$.

While there is $X \subseteq V$ with $h_I(X) > 0$ do:

1. Find an $\hat{h}_I$-cover $F \subseteq E - I$;
2. $I \leftarrow I + F$.

EndWhile

The approximation ratio of the algorithm depends on step 1. A set function is called *uncrossable* if it is $\{0,1\}$-valued weakly-supermodular. It is easy to see that if h is weakly-supermodular, so is $\hat{h}$, that is, $\hat{h}$ is uncrossable. Williamson et. al [21] gave an algorithm that finds an edge cover of an arbitrary uncrossable function q of cost at most twice the optimum of the following LP-relaxation:

$$\min\{\sum_{e \in E} c(e) x_e : \sum_{e \in \delta_E(X)} x_e \geq q(X)\ \forall X \subseteq V, x_e \geq 0\}\ . \tag{4}$$

Williamson et. al [21] proved:

Theorem 4.1 ([21]) *For h defined by* (1) *the above algorithm can be implemented in polynomial time, so that at any iteration for $q = \hat{h}_I$ the forest F found has cost at most twice the optimal value of* (4).

Note that the number of iterations of the algorithm is at most $h_{\max}$. Thus Theorem 4.1 implies that for the Minimum-Cost Steiner Network problem the algorithm has approximation ratio $2h_{\max} \leq 2r_{\max}$. Later, Goemans et. al [9] used linear programming scaling techniques to show that the approximation ratio is in fact $2H(r_{\max})$. This scaling method does not work for the minimum-power variant.

We can show that for the minimum-power variant, the algorithm of [21] has approximation ratio $4r_{\max}$. This follows from Theorem 4.1 and the second part of Proposition 1.1. Indeed, the algorithm of [21] constructs the solution from at most $r_{\max}$ forests, where each forest has cost at most $2\mathsf{opt}_\mathsf{c}$, where opt_c is the optimal solution value to the minimum-cost variant. By Proposition 1.1, each forest has power at most $2 \cdot 2\mathsf{opt}_\mathsf{p} = 4\mathsf{opt}_\mathsf{p}$, where opt_p is the optimal solution value to the minimum-power variant. This completes the proof of Theorem 1.5.

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