Approximability status of Survivable Network problems

by Zeev Nutov

In Survivable Network problems (a.k.a. Survivable Network Design Problem – SNDP) we are given a graph with costs/weights on edges and/or nodes and prescribed connectivity requirements/demands. Among the subgraphs of G that satisfy the requirements, we seek to find one of minimum cost. Formally, the problem is defined as follows. Given a graph $G=(V,E)$ and $Q \subseteq V$, the *Q-connectivity* $\lambda_G^Q(uv)$ of uv in G is the maximum number of edge-disjoint uv-paths such that no two of them have a node in $Q-\{u,v\}$ in common. The case $S=\emptyset$ is just the *edge-connectivity* when the paths should be edge-disjoint, and the case $S=V$ is just the *node-connectivity* when the paths should be internally node-disjoint.

Survivable Network
Instance: A (possibly directed) graph $G=(V,E)$ with edge/node-costs, a node subset $Q \subseteq V$, and a nonnegative integer requirements $\{r_{uv} : uv \in D\}$ on a set D of *demand pairs* on a set $S \subseteq V$ of *terminals*.
Objective: Find a minimum cost subgraph G' of G such that $\lambda_{G'}^Q(uv) \geqslant r_{uv}$ for all $uv \in D$.

Extensively studied particular choices of Q are *edge-connectivity* ($Q=\emptyset$), *node-connectivity* ($Q=V$), and *element-connectivity* ($r_{uv}=0$ whenever $u \in Q$ or $v \in Q$).

Given an instance of Survivable Network let $k=\max_{uv \in D} r_{uv}$ denote the maximum connectivity requirement, and let k-Survivable Network be the restriction of Survivable Network to instances with $\max_{uv \in D} r_{uv}=k$.

Survivable Network has received considerable attention in the past, c.f. surveys in [17, 28, 34]. The edge-connectivity version admits an elegant 2-approximation algorithm via the seminal iterative rounding method by Jain [27] (see also [39] for an elegant and short proof). On the other hand, the only known nontrivial ratio for the node-connectivity version is $O(k^3 \log |D|)$ [12] due to Chuzhoy and Khanna; the problem also admits a folklore ratio $|D|$.

The following classification of Survivable Network problems is widely used, c.f. [34]. We may assume that the input graph G is complete (edges that do not appear in G can be added to G and assigned infinite costs). Under this assumption, the edge costs are categorized as follows:

- $\{0,1\}$-*costs* (known also as "augmentation problems"): here we are given an initial graph G_0 (formed by the edges of cost 0), and the goal is to find a min-size augmenting edge set F of new edges (any edge is allowed and has cost 1) such that the graph $G'=G_0+F$ satisfies the requirements.
- $\{1,\infty\}$-*costs* (known also as "min-size subgraph problems" or "uniform costs"): given a graph H (formed by the edges of cost 1 of G, while edges not in H have cost ∞) find a min-size spanning subgraph G' of H that satisfies the requirements.
- *Metric Costs:* here we assume that the edge costs satisfy the triangle inequality.
- *General (non-negative) costs.*

For each type of costs, the following four types of requirements were studied extensively:

- *Uniform requirements:* $r_{uv} = k$ for every pair $u, v \in V$.
  The corresponding edge-connectivity and node-connectivity versions are the k-Edge-Connected Subgraph and the k-Connected Subgraph problems, respectively.
- *Rooted (single source) requirements:* there is $s \in V$ such that $r_{uv} > 0$ implies $u = s$; this gives the Rooted Survivable Network problem.
- *Subset uniform requirements:* $r_{uv} = k$ for every pair $u, v \in U \subseteq V$.
  The corresponding edge-connectivity and node-connectivity versions are the Subset k-Edge-Connected Subgraph and the Subset k-Connected Subgraph problems.
- *Arbitrary requirements.*

Many fundamental problems are particular cases of Survivable Network. When there is only one pair uv with $r_{uv} > 0$ (namely, when $|D| = 1$) we get the (uncapacitated) Min-Cost k-Flow problem, which is solvable in polynomial time (cf., [50]). The undirected 1-Connected Subgraph is just the MST problem; however, the directed 1-Connected Subgraph is NP-hard. The 1-Survivable Network problem (the case $r_{uv} \in \{0, 1\}$) is the Steiner Forest problem which admits ratio 2 for undirected graphs [1, 23] and ratio $O(n^{2/3+\epsilon})$ for directed graphs [2]. Rooted 1-Survivable Network is the extensively studied Steiner Tree problem; c.f. [49, 3] for the undirected case and [4] for the directed case; the undirected Steiner Tree problem can also be casted as the undirected Subset 1-Connected Subgraph problem. Several other fundamental problems are also particular cases of the Survivable Network problem.

For directed graphs, many Survivable Network problems with node-costs are equivalent to those with edge-costs, but for undirected graphs the node-costs problems are usually harder to approximate. For example, Steiner Tree with edge-costs admits a constant ratio, while the version with node-costs is Set-Cover hard [31]. We will consider mainly Survivable Network problems with edge-costs. For Survivable Network problems and some other Network Design problems with node-costs see, for example, [31, 24, 42, 43, 47, 51, 5, 20].

In *low connectivity* Survivable Network problems, $k = 1, 2$, among them: Directed Steiner Tree, Directed Steiner Forest, Tree Augmentation, Directed Rooted 2-Survivable Network, and others. Examples of high connectivity Survivable Network problems are k-Connected Subgraph and the general Survivable Network with edge/node costs. Table 1 summarizes the current approximability status for high edge/node-connectivity Survivable Network problems. See also surveys in [16, 28, 34]. We mention some additional results not appearing in the table.

Element connectivity: Element-Connectivity Survivable Network admits ratio 2 [15, 10]. For $\{0, 1\}$-costs the problem is NP-hard even for $r(u, v) \in \{0, 2\}$ [30]. For $\{0, 1\}$-costs the best known ratio is 7/4 [40].

Rooted requirements: A graph $G = (V, E)$ is *k-edge-outconnected from s* (*k-outconnected from s*) if it contains k edge disjoint (k internally disjoint) sv-paths for every $v \in V \setminus \{s\}$. In the corresponding k-Edge-Outconnected Subgraph and the k-Outconnected Subgraph problems, $r_{sv} = k$ for every $v \in V$. For *directed* graphs, both problems can be solved in polynomial time, see [14]

c,r	Edge-Connectivity		Node-Connectivity	
	Undirected	*Directed*	*Undirected*	*Directed*
$\{0,1\}$,U	in P [52]	in P [16]	$\min\{2, 1+\frac{k^2}{2\mathsf{opt}}\}$ [18, 26]	in P [18]
$\{0,1\}$,R	in P [16]	$O(\ln n)$ [35] $\Omega(\ln n)$ [16]	$O(\min\{\ln^2 k, \ln n\})$ [44, 35] $\Omega(\ln n)$ [40]	$O(\ln n)$ [35] $\Omega(\ln n)$ [16]
$\{0,1\}$,S	in P [16]	$O(\ln n)$ [35] $\Omega(\ln n)$ [16]	$\frac{\lvert S\rvert}{\lvert S\rvert-k}\cdot O(\min\{\ln^2 k, \ln n\})$ [45] $\Omega(2^{\ln^{1-\varepsilon} n})$ [38]	$\frac{\lvert S\rvert}{\lvert S\rvert-k}\cdot O(\ln n)$ [45] $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]
$\{0,1\}$,G	in P [16]	$O(\ln n)$ [35] $\Omega(\ln n)$ [16]	$k\cdot O(\min\{\ln^2 k, \ln n\})$ [44, 35] $\Omega(2^{\ln^{1-\varepsilon} n})$ [41]	$O(k\ln n)$ [35] $\Omega(2^{\ln^{1-\varepsilon} n})$ [41]
$\{1,\infty\}$,U	$1+\frac{2}{k}$ [22, 8]	$1+\frac{1}{k}$ [37]	$1-\frac{1}{k}+\frac{n}{\mathsf{opt}}$ [8] ([46])	$1-\frac{1}{k}+\frac{2n}{\mathsf{opt}}$ [8] ([46])
$\{1,\infty\}$,R	2 [27] ([39])	$\lvert D\rvert$ $\Omega(\ln^{2-\varepsilon} n)$ [25]	$O(k\ln k)$ [43] $\Omega(\ln^{2-\varepsilon} n)$ [38]	$\lvert D\rvert$ $\Omega(\ln^{2-\varepsilon} n)$ [25]
$\{1,\infty\}$,S	2 [27]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]	$\frac{\lvert S\rvert}{\lvert S\rvert-k}\cdot O(k\ln k)$ [45] $\Omega(2^{\ln^{1-\varepsilon} n})$ [32]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]
$\{1,\infty\}$,G	2 [27]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]	$O(k^3\ln\lvert S\rvert)$ [12] $\Omega(2^{\ln^{1-\varepsilon} n})$ [32]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]
MC,U	2 [29]	2 [29]	$2+(k-1)/n$ [33]	$2+k/n$ [33]
MC,R	2 [27]	$\lvert D\rvert$ $\Omega(\ln^{2-\varepsilon} n)$ [25]	$O(\ln k)$ [11]	$\lvert D\rvert$ $\Omega(\ln^{2-\varepsilon} n)$ [25]
MC,S	2 [27]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]	24 [11]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]
MC,G	2 [27]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]	$O(\ln k)$ [11]	$\lvert D\rvert$ $\Omega(2^{\ln^{1-\varepsilon} n})$ [13]
GC,U	2 [29]	2 [29]	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [48] 6 if $n\geqslant k^3$ [9] ([21]) $\lceil(k+1)/2\rceil$ if $k\leqslant 8$ [33]	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [48] $k+1$ if $k\leqslant 6$ [33]
GC,R	2 [27]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]	$O(k\ln k)$ [43] $\Omega(\max\{k^{1/10}, \lvert D\rvert^{1/4}\})$ [36]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]
GC,S	2 [27]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]	$\frac{\lvert S\rvert}{\lvert S\rvert-k}\cdot O(k\ln k)$ [45] $\Omega(\max\{k^{1/10}, \lvert D\rvert^{1/4}\})$ [36]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]
GC,G	2 [27]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]	$O(k^3\ln\lvert S\rvert)$ [12] $\Omega(\max\{k^{1/6}, \lvert D\rvert^{1/4}\})$ [36]	$\lvert D\rvert$ $\Omega(\max\{k^{1/2}, \lvert D\rvert^{1/4}\})$ [36]

TABLE 1. Known approximability status of Survivable Network problems. MC and GC stand for metric and general costs, U, R, S, and G stand for uniform, rooted, subset uniform, and general requirements, respectively. $k = \max_{uv\in D} r_{uv}$ is the maximum requirement and S is the set of terminals; $|D| = |S|-1$ in the case of rooted requirements and $|D| = \Theta(|S|^2)$ in the case of subset-uniform requirements. Ratio $|D|$ is obtained by computing a min-cost r_{uv}-flow for every $uv \in D$. References in brackets either contain a simplified proof, or a slight improvement of the main result needed to achieve the approximation ratio or threshold stated.

and [19], respectively. This implies a 2-approximation algorithm for undirected graphs. This fact is widely used for designing approximation algorithms for k-Edge-Connected Subgraph and k-Connected Subgraph problems. For additional literature see [4, 14, 17, 19, 40, 6, 12, 43, 7].

Relation between directed and undirected Node-Connectivity Survivable Network *problems:* In [38] it is shown that for $k = n/2 + k'$ the approximability of the undirected Node-Connectivity Survivable Network variant is the same (up to factor of 2) as that of the directed one with maximum requirement k'.

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