BERMAN K-POLYSTABILITY OF Q-FANO VARIETIES ADMITTING KÄHLER-EINSTEIN METRICS. INVENT. MATH. 203 (2016)

1. Main Results

Theorem 1. *(Berman) Let X be a Fano manifold. If X has a Kähler-Einstein metric then X is K-stable.*

This means the Donaldson-Futaki invariant of any test configuration $(\mathcal{X}, \mathcal{L})$ is non-negative, and moreover, if it vanishes, $\mathcal{X} \approx X \times \mathbb{C}$.

To prove the main theorem, Berman first establishes a second theorem (Theorem 2 below) which relates the Donaldson-Futaki invariant to the Ding functional. To state this theorem, we first recall the necessary notation.

Let $\omega \in c_1(-K_X)$ and $D_\omega : \mathrm{PSH}(X, \omega) \cap L^\infty(X) \to \mathbb{R}$ the Ding functional:

$$D_\omega(\phi) \;=\; -\mathcal{E}(\phi_0, \phi_1) - \log \int_X he^{-\phi}$$

where $\omega = -i\partial\bar{\partial} \log h = -dd^c \log h$. The critical points of D_ω are the KE metrics.

Let $\rho : \mathbb{C}^\times \to \mathrm{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$ be a test configuration for $(X, -K_X)$. Assume $\mathcal{X} \subseteq \mathbb{P}^N \times \mathbb{C}$ for some N. Thus $\rho(\tau) \in GL(N+1, \mathbb{C})$ for all $\tau \in \mathbb{C}^\times$ and $\rho(\tau)X = X_\tau$.

Theorem 2. *Let $h_{FS}e^{-\phi}$ be locally bounded S^1 invariant metric on $\mathcal{L}$ with $\omega_{FS} + dd^c\phi \geq 0$. Then*

$$-DF(\mathcal{X}, \mathcal{L}) \;=\; \lim_{t\to\infty} \frac{d}{dt} D_\omega(\phi_t) \;+\; q \tag{1.1}$$

where q is a non-negative rational number. Here $\phi_t \in PSH(X, -K_X)$ is defined by

$$h_{FS}e^{-\phi_t} = \rho(\tau)^*([h_{FS}e^{-\phi}]|_{L_\tau}) \;;\; e^{-t} = \log|\tau| \tag{1.2}$$

1.1. **The geodesic ray $\phi_t(\mathcal{X}, \mathcal{L})$.** We shall apply (1.1) to a particular path of metrics ϕ_t. Let $p : \mathcal{X}' \to \mathcal{X}$ be an S^1 equivariant log-resolution of singularities. Then $\phi(\mathcal{X}, \mathcal{L}) \in PSH(\mathcal{X}'_\Delta, \omega_{FS})$ is the solution to the Dirichlet problem

$$(\omega_{FS} + dd^c\phi)^{n+1} \;=\; 0 \;;\; \phi|_{\partial\Delta} = \phi_0$$

and $\phi_t \in PSH(X, \omega_{FS})$ is defined by (1.2).

2. Formula for q

After replacing $\mathcal{X}$ by a $\mathcal{X}'$, we may then $\mathcal{L}' \to \mathcal{X}'$ is semi-ample with $\mathcal{L}'_{(\mathcal{X}')^\times} \approx -K_{(\mathcal{X}')^\times/\mathbb{C}}$.

Let $A = \pi_*(\mathcal{L}' + K'_{\mathcal{X}'/\mathbb{C}})$. Then $A \to \mathbb{C}$ is a coherent sheaf, rank one, and torsion free. Thus A is a line bundle. To see this, recall that any coherent sheaf over $\mathbb{C}$ is of the form

$$A \;=\; \mathbb{C}[t]^r \oplus \mathbb{C}[t]/(f_1) \oplus \cdots \oplus \mathbb{C}[t]/(f_m)$$

Thus torsion free implies free and $r = 1$ implies free of rank one.

Now let s be a trivializing section of A over $\mathbb{C}$. Thus $s \in H^0(\mathcal{X}, \mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ and for all $\tau \in \mathbb{C}$ we have $0 \neq s|_{X_\tau} \in H^0(X_\tau, (\mathcal{L}+K_{\mathcal{X}/\mathbb{C}})|_{X_\tau})$. In particular, $s|_{X_\tau}$ is nowhere vanishing if $\tau \in \mathbb{C}^\times$. This means the divisor of s is supported on the central fiber:

$$(\pi) = \sum_i m_i E_i,\ m_i > 0$$

$$(s) = \sum_i c_i E_i,\ c_i \geq 0$$

Note that $(\pi) = (\tau)$ so we must have $m_i < c_i$ for some i (otherwise, replace s by s/τ).

There is a natural (singular) metric Φ_A on A defined by

$$\|s\|^2_{\Phi_A}(\tau) \;=\; \int_{X'_\tau} (s \wedge \bar{s}) e^{-\phi}\ ;\ \tau \neq 0$$

Before stating the formula for q we need one more construction: Let $\bar{\mathcal{X}} \to \mathbb{P}^1$ be the trivial completion of $\mathcal{X} \to \mathbb{C}$ and $\bar{\mathcal{L}} \to \bar{\mathcal{X}}$ the trivial completion of $\mathcal{L} \to \mathcal{X}$. This means that $\bar{\mathcal{X}}|_{\mathbb{P}^1\setminus\{\infty\}} = \mathcal{X}$ and $\bar{\mathcal{X}}|_{\mathbb{P}^1\setminus\{0\}} = X \times \{\mathbb{P}^1\setminus\{0\}\}$. Thus $\bar{\mathcal{X}} \subseteq \mathbb{P}^1 \times \mathbb{P}^N$ is a closed submanifold.

Proposition 1. *We have*

$$q \;=\; \max_i \frac{m_i - 1 - c_i}{m_i} \;+\; \frac{1}{c_1(L)^n} \sum_i c_i E_i \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \quad \in \quad \mathbb{Q} \tag{2.1}$$

3. Proof that $q \geq 0$

We wish to prove $q \geq 0$. To do this, we rewrite (2.1) as follows. First note that by flatness,

$$\sum_i m_i E_i \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \;=\; c_1(L)^n$$

Thus, setting

$$t_i \;=\; m_i \frac{E_i \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}'}{c_1(L)^n}$$

we see $\sum_{i \in I} t_i \;=\; 1$ where $I = \{i : t_i > 0\}$. Moreover

$$q \;=\; \max_i \left(\frac{m_i - 1}{m_i} - \frac{c_i}{m_i} \right) \;+\; \sum_{i \in I} \frac{c_i}{m_i} t_i$$

If $i_0 \in I$ is the index where the minimum of $\frac{c_i}{m_i}$ is attained, we see $q \geq \frac{m_{i_0}-1}{m_{i_0}} \geq 0$.

We have thus reduced the proof of the main theorem to the proof of Proposition 1. We first review some of the necessary tools.

4. Weight-Slope identity

Let $\eta \to \mathbb{C}$ be a holomorphic line bundle and $\rho : \mathbb{C}^\times \to \mathrm{Aut}(\eta \to \mathbb{C})$ a $\mathbb{C}^\times$ action. Let $w \in \mathbb{Z}$ be the weight of η at the origin.

Proposition 2. *Let $h = h_0 e^{-\Phi}$ a metric on η (here h_0 is a fixed smooth metric). Then*

$$\lim_{\tau \to 0} \frac{\log \|\rho(\tau) s_1)\|_h}{\log |\tau|} \; = \; w_\eta \; - \; l_\Phi$$

where

$$l_\Phi \; = \; \lim_{\tau \to 0} \frac{\Phi(\tau)}{\log |\tau|}$$

5. The weight-slope identity for the Chow bundle.

Let $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ be a test configuration, let ϕ be a locally bounded metric on $\mathcal{L}$, and $C = \langle \mathcal{L}, ..., \mathcal{L} \rangle \to \mathbb{C}$ the Chow line bundle. Since ϕ is locally bounded, one easily shows the Deligne metric $\Phi_D = \langle \phi, ..., \phi \rangle$ is locally bounded, so $l_{\Phi_D} = 0$. Let $\phi_t = \rho(\tau)^* \phi_\tau$ where $t = -\log |\tau|$. Thus ϕ_t is a family of metrics on $L \to X$ where, as usual, we make the identification $X = X_1$. Then the weight-slope identity plus S. Zhang's theorem implies

$$w_{\mathrm{C}} \; = \; \lim_{t \to \infty} \frac{d}{dt} \mathcal{E}(\phi_0, \phi_t) \tag{5.2}$$

6. The weight-slope identity for the adjoint bundle.

$$\lim_{t \to \infty} -\frac{d}{dt} \log \int_X (s_1 \wedge \bar{s}_1)^{1/r} e^{-\phi_t} \; = \lim_{t \to \infty} -\frac{d}{dt} \log \int_X e^{-\phi_t} \; = \; w_A \; - \; l_{\Phi_A} \tag{6.3}$$

In the second integral we view $e^{-\phi_t}$ as a metric on $-K_X$ using the isorphism defined by s_1.

7. The CM line bundle

We want to apply the weight-slope formula to particular line bundle $\eta \to \mathbb{C}$ whose weight is $-D(\mathcal{X}, \mathcal{L})$.

Theorem 3. *(Phong-Ross-S) Let X be Fano and $(\mathcal{X}, \mathcal{L})$ a test configuration. Assume $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein and define*

$$\eta = -\frac{1}{(n+1) c_1(L)^n} \langle \mathcal{L}, ..., \mathcal{L} \rangle + \frac{1}{c_1(L)^n} \langle \mathcal{L} + K_{\mathcal{X}/\mathbb{C}}, \mathcal{L}, ...\mathcal{L} \rangle \tag{7.4}$$

Then

$$-DF(\mathcal{X}, \mathcal{L}) \; = \; w_{\eta'} \tag{7.5}$$

If $\mathcal{X}$ is only normal, then let $\mathcal{X}' \to \mathcal{X}$ be an equivariant resolution so $K_{\mathcal{X}'/\mathbb{C}}$ is defined. Define η' as in (7.5) but with $\mathcal{X}, \mathcal{L}$ replaced by $\mathcal{X}', \mathcal{L}'$. Then

$$-DF(\mathcal{X}, \mathcal{L}) \ = \ w_{\eta'} \tag{7.6}$$

8. Decomposition of the CM line bundle

Our goal is to calculate $w_{\eta'}$. We first decompose η' as follows:

$$\eta' \ = \ F' \ + \ I' \ = \ -C' + A' + I' \tag{8.7}$$

Where

$$F' \ = \ -\frac{1}{(n+1)c_1(L)^n}\langle \mathcal{L}', ..., \mathcal{L}' \rangle + \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \ = \ -C' + A' \tag{8.8}$$

and

$$I' \ = \ \left(\frac{1}{c_1(L)^n} \langle \mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}, \mathcal{L}', ...\mathcal{L}' \rangle - \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \right)$$

To compute $w(C')$ and $w(A')$ we use the weight-asymptote identity twice:

$$w(F) \ = \ -w_{\mathrm{C}} + w_A = - \left(\lim_{t\to\infty} \frac{d}{dt} \mathcal{E}(\phi_0, \phi_t) \ \right) + \left(- \lim_{t\to\infty} \frac{d}{dt} \log \int_X e^{-\phi_t} \ + \ l_{\Phi_A} \right)$$

We obtain

$$w(F') = \lim_{t\to\infty} D_{\omega_0}(\phi_t) \ + \ l_{\Phi_A}$$

Thus

$$-DF(\mathcal{X}, \mathcal{L}) \ = \ \lim_{t\to\infty} D_{\omega_0}(\phi_t) \ + \ q$$

where

$$q \ = \ l_{\Phi_{A'}} \ + \ w(I')$$

Here $l_{\Phi_{A'}}$ is a Lelong number and, as we shall see, $w(I')$ is an intersection number.

9. Calculation of l_{Φ_A}

We wish to show

$$l_{\Phi_A} \ = \ \max_i \frac{m_i - 1 - c_i}{m_i} \tag{9.9}$$

One easily sees that the Lelong number can be characterized as follows:

$$l_{\Phi_A} \ = \ \inf \left\{ l : \int_{\Delta^*} \frac{e^{-(\Phi_A - l\log|\tau|^2)}}{|\tau|^2} d\tau \wedge d\bar{\tau} \ < \ \infty \ \right\}$$

Now

$$\int_{\Delta^*} \frac{e^{-(\Phi_A - l\log|\tau|^2)}}{|\tau|^2} d\tau \wedge d\bar\tau \;=\; \int_{\Delta^*}\int_{X_\tau} (s\wedge\bar s)^{1/r} e^{-\phi} \frac{1}{|\tau|^{2(1-l)}} d\tau\wedge d\bar\tau \;=\; \int_{\mathcal{X}'} \Omega$$

Where Ω is a positive measure on $\mathcal{X}'$.

Let $p \in \mathcal{X}'_0$ and choose $p \in U \subseteq \mathcal{X}'$ a coordinate neighborhood. Let $E_{i_1}, ..., E_{i_r}$ be the divisors passing through p, and let $(z_0, ..., z_n)$ be holomorphic coordinates on U. On U,

$$\tau \;=\; e^{\alpha(z)} z_{i_1}^{m_{i_1}} \cdots z_{i_r}^{m_{i_r}}$$

$$(s\wedge\bar s)^{1/r} e^{-\phi}\, d\tau\wedge d\bar\tau \;=\; e^{\beta(z)} z_{i_1}^{c_{i_1}} \cdots z_{i_r}^{c_{i_r}} dz\wedge d\bar z$$

where $\alpha(z), \beta(z)$ are bounded functions on U. Thus

$$\int_U \Omega \;=\; \int_U e^{\alpha+\beta} \prod_{\mu=1}^{r} |z_{i_\mu}^2|^{c_{i_\mu} - (1-l)m_{i_\mu}}\, dz\wedge d\bar z$$

which is finite if and only if $c_i - (1-l)m_i > -1$ for all i and this establishes (9.9)

10. Calculation of $w(I')$

This is inspired by Xiaowei Wang's formula for $DF(\mathcal{X}, \mathcal{L})$:

Lemma 1. *Let $\bar I \to \mathbb{P}^1$ be a line bundle with $\mathbb{C}^\times$ action. Then*

$$\deg \bar I \;=\; w_0 - w_\infty$$

We postpone the proof. Let $\bar I \to \mathbb{P}^1$ be

$$I \;=\; \left(\frac{1}{c_1(L)^n} \langle \bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}, \bar{\mathcal{L}}', ...\bar{\mathcal{L}}'\rangle - \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) \right)$$

$$w(I) \;=\; \deg \bar I \;=\; \frac{1}{c_1(L)^n} \deg\langle \bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}, \bar{\mathcal{L}}', ...\bar{\mathcal{L}}'\rangle \;-\; \deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1})$$

But the Deligne curvature formula (D1) in §6 implies

$$\deg\langle \bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}, \bar{\mathcal{L}}', ...\bar{\mathcal{L}}'\rangle \;=\; \int_{\mathbb{P}^1} \pi'_*\big[c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \wedge c_1(\bar{\mathcal{L}}') \wedge \cdots \wedge c_1(\bar{\mathcal{L}}')\big]$$

$$=\; \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \wedge c_1(\bar{\mathcal{L}}') \wedge \cdots \wedge c_1(\bar{\mathcal{L}}') \;=\; (\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}'$$

Hence

$$w(I) \;=\; \frac{1}{c_1(L)^n} (\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \;-\; \deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1})$$

We claim that $w(I)$ doesn't change if we replace $\bar{\mathcal{L}}'$ by $\bar{\mathcal{L}}' \otimes (\pi')^* O_{\mathbb{P}^1}(m)$ (see Lemma below).

This means, in the calculation of $w(I)$, we may assume

$$\deg \pi'_*(\mathcal{L}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) \;=\; 0$$

In order to proceed with the calculation, we need some notation. Write

$$\mathrm{div}(\pi^*\tau) \;=\; \sum_i m_i E_i$$

Thus $m_i \geq 1$ for all i is given by $m_i = \mathrm{ord}_{E_i}\pi^*\tau$.

Let s be a nowhere vanishing section of $\pi'_*\mathcal{L}' + K_{\bar{X}'/\mathbb{C}}$. Thus s is a section of $\mathcal{L}' + K_{\bar{X}'/\mathbb{C}}$ with the property $\tau \not| s$ so

$$\mathrm{div}(s) \;=\; \sum_i c_i E_i$$

with $c_i \geq 0$ for all i and $c_i < m_i$ for some i. We conclude

$$w(I) \;=\; \frac{1}{c_1(L)^n}(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \;=\; \frac{1}{c_1(L)^n}\sum_i c_i E_i \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \;\geq\; 0 \tag{10.10}$$

11. Weight-degree identity

Let $F \to \mathbb{P}^1$ be a line bundle with $\mathbb{C}^\times$ action. We wish to show that

$$\deg F \;=\; w_0 - w_\infty$$

Without lose of generality, we may assume $\deg F \geq 0$ (otherwise replace F by $-F$). Let s be a holomorphic section whose zeros are $\alpha_1, ..., \alpha_d \in \mathbb{C}^\times$. Then

$$\frac{\rho(\tau)(s)}{s} \;=\; c(\tau)\frac{(z-\tau\alpha_1)\cdots(z-\tau\alpha_d)}{(z-\alpha_1)\cdots(z-\alpha_d)}$$

Taking the limit as $z \to \infty$ we see that $c(\tau) = \tau^{w_\infty}$. Plugging in $z = 0$ we get

$$\tau^{w_0} = \tau^{w_\infty}\tau^{\deg F}$$

This proves the identity.

12. Twisting lemma

Lemma 2. *$w(I)$ doesn't change if we replace $\bar{\mathcal{L}}'$ by $\bar{\mathcal{L}}' \otimes (\pi')^* O_{\mathbb{P}^1}(m)$*

Proof. Note that

$$\begin{aligned} \deg \pi'_*(\mathcal{L}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1} + (\pi')^* O_{\mathbb{P}^1}(m)) \;&=\; \deg\Big([\pi'_*(\mathcal{L}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1})] \otimes O_{\mathbb{P}^1}(m))\Big) \\ &=\; \deg[\pi'_*(\mathcal{L}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1})] + m \end{aligned}$$

On the other hand, if in the integral

$$\frac{1}{c_1(L)^n}\int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) \wedge c_1(\bar{\mathcal{L}}') \wedge \cdots \wedge c_1(\bar{\mathcal{L}}')$$

we replace $c_1(\bar{\mathcal{L}}')$ with $c_1(\bar{\mathcal{L}}') + m(\pi')^*\omega_{FS}$ and $c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}})$ with $c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) + m(\pi')^*\omega_{FS}$ and expand, only terms with at most one $m(\pi')^*\omega_{FS}$ will survive. And since $c_1(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{C}}) = 0$ on a generic fiber, only

$$\frac{1}{c_1(L)^n}\int_{\bar{\mathcal{X}}} m(\pi')^*\omega_{FS}\wedge c_1(\bar{\mathcal{L}}')\wedge\cdots\wedge c_1(\bar{\mathcal{L}}') \;=\; \frac{1}{c_1(L)^n} m c_1(L)^n \;=\; m$$

survives. The two copies of m appear with opposite signs, and thus cancel.

13. Theorem 2 implies Theorem 1

We give a sketch: Let ϕ_0 be a Kähler-Einstein metric and let $\phi = \phi(\mathcal{X},\mathcal{L})$ the geodesic associated to $(\mathcal{X},\mathcal{L})$. Then $D_\omega(\phi_t)$ is convex (Berndtsson) so (1.1) implies $-D(\mathcal{X},\mathcal{L}) \geq 0$. If equality holds, we must show there is a biholomorphic map $f : X \approx X_0$.

Since $\rho(\tau) : X \to X_\tau$ we might try $f(x) = \lim_{\tau\to 0}\rho(\tau)(x)$. But such an f is usually not continuous, let alone holomorphic. If $n = 1$ the image of f is typically a finite set of points (we gave an example of this last time).

But if equality holds then $D_\omega(\phi_t)$ is linear and hence (Berndtsson) ϕ_t is a smooth geodesic associated to a non-zero holomorphic vector field V. Let $\sigma_V(\tau) : X \to X$ be the 1-parameter family of biholomorphic maps (here $\tau \in \mathbb{C}^\times$) and define $f : X \to X_0$ by $f(x) = \lim_{\tau\to 0}\rho(\tau)\sigma(\tau)(x)$. We saw that f is onto, holomorphic and generically finite. Moreover, by the result of Li-Xu, we may assume X_0 is normal. Hence, by Zariski's main theorem, f is biholomorphic.

14. Deligne pairing

We want to apply the weight-slope formula to certain hermitian line bundle $(\eta, h_\eta) \to \mathbb{C}$ whose weight is $-D(\mathcal{X},\mathcal{L})$. To explain the construction of η we first recall the concept of Deligne pairing.

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a flat projective morphism between algebraic varieties with $\mathcal{B}$ smooth. Let $\mathcal{L}_0 ..., \mathcal{L}_n \to \mathcal{X}$ be line bundles equipped with locally bounded hermitian metrics $\phi_0, ..., \phi_n$ (so the curvature of $\mathcal{L}_j$ is $dd^c\phi_j$). Then

$$\langle \mathcal{L}_0, ..., \mathcal{L}_n\rangle \to \mathcal{B}$$

is a line bundle equipped with the Deligne metric $\Phi_D = \langle \phi_0, ..., \phi_n\rangle$. It satisfies

(1) The curvature of Φ_D is given by the current

$$dd^c\Phi_D \;=\; \pi_*(dd^c\phi_0\wedge\cdots\wedge dd^c\phi_n)$$

(2) If ϕ_j, ψ_j are locally bounded metrics on $\mathcal{L}_j$ then

$$\Phi_D - \Psi_D \;=\; (n+1)\mathcal{E}(\phi,\psi) \;=\; \sum_{j=0}^{n}\int_{\mathcal{X}/B}(\phi_j - \psi_j)\bigwedge_{k<j}(dd^c\phi_k)\bigwedge_{k>j}(dd^c\psi_k)$$