# EXTENSION OF PSH FUNCTIONS WITH GROWTH CONTROL

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## 1. Background from complex variables

1.1. **Convex sets and domains.** We shall consider two types of convexity: relative and intrinsic. We start with the classical definition, in which the defining sheaf is the sheaf of linear functions: Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\Lambda(\Omega)$ the space of linear functions on $\Omega$. The restriction map for this sheaf $\Lambda(\mathbb{R}^n) \to \Lambda(\Omega)$ is clearly bijective.

Recall that if $K \subseteq \mathbb{R}^n$ is a subset then the (linear) convex hull is defined by

$$\hat{K} \;=\; \hat{K}_{\Lambda(\mathbb{R}^n)} \;=\; \bigcap \{\lambda \leq 0 \;:\; \lambda|_K \leq 0, \lambda \in \Lambda(\mathbb{R}^n)\;\}$$

We say that a set $K$ is convex in $\mathbb{R}^n$ if $K = \hat{K}$. If If $\Omega \subseteq \mathbb{R}^n$ is a domain and $K \subseteq \Omega$ we define the relative convex hull of $K$ with respect to $\Omega$ as

$$\hat{K}_\Omega \;= \hat{K}_{\Lambda(\Omega)} \;:=\; \bigcap\;\{\lambda \leq 0 \;:\; \lambda|_K \leq 0, \lambda \in \Lambda(\Omega)\;\} \;=\; \hat{K} \cap \Omega \;\subseteq\; \Omega$$

We say $K$ is convex relative to $\Omega$ if $K = \hat{K}_\Omega$.

Let CNV$(\Omega)$ be set of all convex functions on $\Omega$. Then we easily see

$$\hat{K}_{\Lambda(\Omega)} \;= \hat{K}_{\mathrm{CNV}(\Omega)} \;=\; \bigcap\;\{u \leq 0 \;:\; u|_K \leq 0, u \in \mathrm{CNV}(\Omega)\;\}$$

We define $\Omega$ to be (intrinsically) convex if $\hat{K}_\Omega$ is compact whenever $K \subseteq \Omega$ is compact (In fact if we wish, we may assume, in this definition, that $K$ is a set with two elements).

Another way to define convex sets is via the notion of convex exhaustion functions: We say $\Omega \subseteq \mathbb{R}^n$ is convex if there exists a convex function $f : \Omega \to \mathbb{R}$ with the property $\{f \leq c\} \subseteq \Omega$ is compact (and convex) for all $c \in \mathbb{R}$.

To see that the two definitions are equivalent, first note that if $\Omega$ has a convex exhaustion function $f$, then $\{f \leq c\}$ is clearly convex in the usual sense. On the other hand, if $\Omega$ is convex in the usual sense, then it is not hard to see that $f = -\log d_{\partial\Omega}$ is a convex exhaustion function, where $d_{\partial\Omega}(x) = \mathrm{dist}(x, \partial\Omega)$. In fact, replacing $f$ by $f + |x|^2$, we may assume the exhaustion function is strictly convex. Moreover, one can show there always exists a smooth strictly convex exhaustion function.

1.2. **Holomorphically convex sets and domains.** We next consider the complex analogue, replacing the sheaf of linear functions with the sheaf of holomorphic functions. Let $K \subseteq \mathbb{C}^n$ be a compact set. The holomorphic convex hull of $K$ is defined by

$$\hat{K}_{\mathcal{O}(\mathbb{C}^n)} \;=\; \bigcap \{|h| \leq 1 \;:\; |h||_K \leq 1,\, h \in \mathcal{O}(\mathbb{C}^n)\;\} \;\subseteq\; \mathbb{C}^n$$

We say $K$ is holomorphically convex in $\mathbb{C}^n$ if $K = \hat{K}_{\mathcal{O}(\mathbb{C}^n)}$. Note that $\hat{K}_{\mathcal{O}(\mathbb{C}^n)} = \hat{K}_{\mathbb{C}[Z]}$ so h-convex (i.e. holomorphically convex) is the same as polynomially convex.

One big source of h-convex sets are the so called h-polyhedra: Let $h_1, ...h_p$ be a collection of holomorphic functions on $\mathbb{C}^n$. Then if $\cap_{j=1}^p\{\,|h_j| \leq 1\}$ is compact, it is automatically h-convex in $\mathbb{C}^n$.

If $n = 1$ then $K$ is h-convex in $\mathbb{C}$ if and only if $K$ has no holes. The classical Runge theorem says that if $K \subseteq \mathbb{C}$ has no holes, then any holomorphic function on a neighborhood of $K$ can be uniformly approximated on $K$ by a sequence of entire (or even polynomial) functions.

If $n > 1$ then holomorphic convexity is much more subtle and has no topological characterization. But the classical Runge theorem does generalize:

**Theorem 1.** *(Oka, 1939) Let $K \subseteq \mathbb{C}^n$ be holomorphically convex in $\mathbb{C}^n$. If $f$ is holomorphic in some neighborhood of $K$ then there is a sequence of polynomials $P_j$ such that $P_j \to f$ uniformly on $K$.*

We postpone the proof - late we shall give a proof of a generalization (Oka-Weil).

We define the relative version of h-convexity mimicking the construction in $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\mathcal{O}(\Omega)$ the ring of holomorphic functions on $\Omega$. Let $K \subseteq \Omega$ be compact. Define the holomorphic convex hull of $K$ relative to $\Omega$:

$$\hat{K}_{\mathcal{O}(\Omega)} \;=\; \bigcap \{|h| \leq 1 \;:\; |h||_K \leq 1, h \in \mathcal{O}(\Omega)\;\} \;\subseteq\; \Omega$$

We say $K$ is h-convex relative to $\Omega$ (or sometimes $K$ is $\mathcal{O}(\Omega)$-convex) if $K = \hat{K}_{\mathcal{O}(\Omega)}$. Note that if $\Omega_1 \subseteq \Omega_2$ and $K \subseteq \Omega_1$ is compact, then $\hat{K}_{\mathcal{O}(\Omega_2)} \subseteq \hat{K}_{\mathcal{O}(\Omega_1)}$ but the two are not, in general, equal.

Next we define convexity for domains. We say $\Omega$ is holomorphically convex if $\hat{K}_{\mathcal{O}(\Omega)} \subseteq \Omega$ is compact whenever $K \subseteq \Omega$ is compact.

We emphasize that for compact sets, convexity is a relative notion while for open sets, convexity is an intrinsic notion. Thus a compact sub-annulus $K$ of an open annulus $\Omega \subseteq \mathbb{C}$ is h-convex in $\Omega$ (in fact, it is an h-polygon) but not h-convex in $\mathbb{C}$. On the other hand, the annulus $\Omega$ is h-convex.

**Theorem 2.** *If $\Omega$ is holomorphically convex then*

$$\hat{K}_{\mathcal{O}(\Omega)} \;=\; \hat{K}_{\mathrm{PSH}(\Omega)} \;=\; \bigcap \{\phi \le 0 \;:\; \phi|_K \le 0, \phi \in \mathrm{PSH}(\Omega)\ \} \;\subseteq\; \Omega \tag{1.1}$$

*for all compact sets $K \subseteq \Omega$.*

Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n$ be h-convex domains in $\mathbb{C}^n$. We say $\Omega_1$ is Runge in $\Omega_2$ if $\mathcal{O}(\Omega_2) \to \mathcal{O}(\Omega_1)$ has dense image.

**Corollary 1.** *If $\Omega$ is holomorphically convex then open psh-polyhedra are Runge in $\Omega$ and closed psh-polyhedra are h-convex in $\Omega$. In particular closed sublevel sets of plurisubharmonic functions are h-convex and open sublevel sets are Runge. More precisely, if $\phi \in PSH(\Omega) \cap C^0(\Omega)$ and $c \in \mathbb{R}$. Assume $\{\phi \le c\} \subseteq \Omega$ is compact. Then*

$$\{\phi \le c\} \subseteq \Omega \quad \textit{is h-convex in } \Omega \tag{1.2}$$

$$\{\phi < c\} \ \textit{is Runge in } \Omega \tag{1.3}$$

*Proof.* Statement (1.2) follows immediately from the theorem. As for (1.3), first we observe that $\Omega_c = \{\phi < c\}$ is h-convex. In fact, $-\log(c - \psi)$ is a psh exhaustion function for $\Omega_c$. To see this, recall that if $\chi(t_1, ..., t_p)$ is convex on $\mathbb{R}^p$ and increasing in each variable, then $\chi(u_1, ..., u_p)$ is psh whenever $u_1, ..., u_p$ are psh. In our setting we take $\chi(t) = -\log(c - t)$. To prove $\Omega_c$ is Runge, note that $K_\epsilon = \{\phi \le c - \epsilon\} \subseteq \Omega$ is h-convex by (1.2). Thus if $f \in \mathcal{O}(\Omega)$, Oka's theorem (more precisely, Oka-Weil) implies that for each $j$ there is a sequence $P_1, P_2, ... \in \mathcal{O}(\Omega)$ such that $P_k \to f$ uniformly on $K_{1/j}$. Now let $j \to \infty$ and take a diagonal subsequence.

Another way to define h-convex sets is via the notion of plurisubharmonic exhaustion functions: We say $\Omega \subseteq \mathbb{R}^n$ is pseudoconvex if there exists a continuous plurisubharmonic function $\psi : \Omega \to \mathbb{R}$ with the property $\{\psi \le c\} \subseteq \Omega$ is compact for all $c \in \mathbb{R}$.

**Theorem 3.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. The following are equivalent:*

1. *$\Omega$ is holomorphically convex.*
2. *$\Omega$ is pseudoconvex, i.e. there exists some continuous psh exhaustion function.*
3. *$-\log d_{\partial\Omega}$ is a continuous plurisubharmonic exhaustion function.*
4. *There exists a smooth strictly plurisubharmonic exhaustion function.*
5. *$\Omega$ is a domain of holomorphy, i.e. there exists $f \in \mathcal{O}(\Omega)$ which can not be extended to a holomorphic function on any open set which strictly contains $\Omega$.*

In particular, h-convex domains can be exhausted by compact h-convex sets and also by relatively compact Runge sub-domains: if $\psi$ is a continuous exhaustion function, then $K_c = \{\psi \le c\} \subseteq \Omega$ is compact and h-convex in $\Omega$.

The equivalence of (1) and (5) is due to Cartan-Thullen, and is not difficult. First, if $f \in \mathcal{O}(\Omega)$ can not be extended, and if $p \in K \subseteq \Omega$ with $K$ compact, then the radius of convergence of $f$ centered at $p$ is the distance from $p$ to $\partial\Omega$, which is bounded below by $r = \mathrm{dist}(K, \partial\Omega) > 0$. Thus

$$r^{|\alpha|}|D^\alpha f|(p) \ \le \ \sup_{B_{r/2}(p)} r^{|\alpha|}|D^\alpha f| \ \le \ C(n)\sup_{K'}|f|$$

where $x \in K'$ if $d_{\partial\Omega}(x) \ge r/2$. Thus, if $q \in \hat{K}_{\mathcal{O}(\Omega)}$ we have

$$r^{|\alpha|}|D^\alpha f|(q) \ \le \ C(n)\sup_{K'}|f|$$

But this means the radius of convergence of $f$ at $q$ is at least $r/c(n)$ which implies $\mathrm{dist}(\hat{K}_{\mathcal{O}(\Omega)}, \partial\Omega) \ge r/c(n)$ so $\hat{K}_{\mathcal{O}(\Omega)}$ is compact. For the converse, let $z_k \in \mathcal{O}$ be any sequence such that $\overline{\{z_k : k = 1, 2, ...\}} = \partial\Omega$. Choose an increasiong sequence $K_1 \subseteq K_2 \cdots \subseteq \Omega$ of compact sets whose union is $\Omega$. Then for each $j$, there exist $f_j \in \mathcal{O}(\Omega)$ such that $|f_j| \le 1$ on $K_j$ but $|f_j|(z_k) > 1$ for $k$ sufficiently large. After passing to a subsequence, and multiplying $f_j$ by $\tau_j f_j$ with $|\tau_j| = 1$, we may assume $|Im(f_j(z_k))| \le |Re(f_j(z_k))|$ for all $k$. Replacing $f_j$ by $f_j^N$ for large $N$, we may assume $|f_j(z_j)| \ge j2^j$ Now let $f = \sum_j 2^{-j} f_j$. We see $f|(z_j)| \to \infty$ so $\Omega$ is a domain of holomorphy.

1.3. **Characterization of Runge domains.** Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain. Recall that $\Omega$ is a Runge domain if $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\Omega)$ has dense image.

When $n = 1$, a bounded domain $\Omega \subseteq \mathbb{C}$ is Runge if and only if its complement is connected. In the general case, we don't have such a simple characterization of Runge domains, but the following is useful:

**Theorem 4.** *A pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ is Runge if and only if*

$$\hat{K}_{\mathcal{O}(\mathbb{C}^n)} \ = \ \hat{K}_{\mathcal{O}(\Omega)} \ \ \textit{for all } K \subseteq \Omega \textit{ compact} \tag{1.4}$$

To see this, let $f \in \mathcal{O}(\Omega)$ and write $\Omega = \cup K_j$ where $K_j \subseteq \Omega$ is an increasing sequence of compact h-convex subsets. Fix $j$ and let $K = K_j$. Then $K = \hat{K}_{\mathcal{O}(\mathbb{C}^n)}$ so Oka's theorem says there is a sequence of polynomial $P_1, P_2, ..$ converging uniformly to $f$ on $K_j$. Passing to a diagonal subsequence we get uniform convergence of the $P_j$ to $f$ on any compact subset.

1.4. **Stein manifolds.** Stein manifolds are generalizations of h-convex domains.

**Theorem 5.** *Let $X$ be a complex manifold of dimension $n$. The following are equivalent.*

1. *$X$ is Stein*
2. *There exists a continuous plurisubharmonic exhaustion function $\psi : X \to \mathbb{R}$.*
3. *There exists a smooth strictly plurisubharmonic exhaustion function $\psi : X \to \mathbb{R}$.*

(4) *$X$ is holomorphically convex and $\mathcal{O}(X)$ separates points.*
(5) *There exists an imbedding $X \hookrightarrow \mathbb{C}^N$ whose image is a closed submanifold. We may always take $N = 2n+1$.*
(6) *$H^p(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F} \to X$ and all $p > 0$.*

Thus a domain $\Omega \subseteq \mathbb{C}^n$ is a Stein manifold if and only if it is pseudoconvex. Also, a closed submanfold of a Stein manifold is Stein. Non-compact Riemann surfaces are Stein (this is a difficult theorem) and holomorphic covers of Stein manifolds are Stein.

Relation (1.1) generalizes to Stein manifolds: If $K \subseteq X$ is a compact subset of a Stein manifold then

$$\hat{K}_{\mathcal{O}(X)} = \hat{K}_{\mathrm{PSH}(X)} = \bigcap \{\phi \leq 0 \ : \ \phi|_K \leq 0, \phi \in \mathrm{PSH}(X) \ \} \subseteq \Omega \tag{1.5}$$

Thus every Stein manifold is exhausted by compact h-convex subsets.

Oka's theorem generalizes as well:

**Theorem 6.** *(Oka-Weil) Let $X$ be a Stein manifold and $K \subseteq X$ a compact $\mathcal{O}(X)$-convex subset. Then if $f$ is holomorphic in some neighborhood of $K$ there is a sequence $F_j \in \mathcal{O}(X)$ such that $F_j \to f$ uniformly on $K$.*

*Proof (sketch).* If $f$ is defined on $W$, an open neighborhood of $K$, then we can squeeze an analytic polyhedra between $K$ and $W$. That is, there exist $h_1, ..., h_p \in \mathcal{O}(X)$ such that

$$K \subseteq U = \bigcap_{j=1}^{p} \{|h_j| < 1\} \subseteq W$$

Thus we obtain a map $X \to \mathbb{C}^p$ whose restriction to $U$ is a proper map $U \to D^p$, where $D \subseteq \mathbb{C}$ is the unit disk. By adding more holomorphic functions $h_{p+1}, ..., h_m$, we may insure that $U \to D^m$ is a proper closed imbedding. Cartan's extension theorem says that $f$, which is holomorphic on $U \subseteq D^m$, is the restriction of a holomorphic function $F$ on $D^m$ which may be approximated by polynomials $P_1, P_2, ...$ on $\mathbb{C}^m$. Pulling back via $X \to \mathbb{C}^m$ we get a sequence of elements in $\mathcal{O}(X)$ converging uniformly to $f$ on $U$.

1.5. **Runge domains in Stein manifolds.** Let $\Omega \subseteq X$ be an Stein open subset of a Stein manfold. We say $\Omega$ is Runge in $X$ if $\mathcal{O}(X) \to \mathcal{O}(\Omega)$ has dense image.

Theorem 4 holds for Stein manifolds:

**Theorem 7.** *If $X$ is a Stein manifold and $\Omega \subseteq X$ a Stein domain then $\Omega$ is Runge in $X$ if and only if for every compact set $K \subseteq \Omega$ we have $\hat{K}_{\mathcal{O}(X)} = \hat{K}_{\mathcal{O}(\Omega)}$.*

The proof is identical to the case for domains.

If $X \subseteq Y$ is a closed submanifold of a Stein manfiold (and hence Stein) then $\mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective. This can be seen using Cartan's theorems.

In particular, if $\tilde{\Omega} \subseteq Y$ is Runge, then $\Omega = \tilde{\Omega} \cap X \subseteq X$ is Runge. Coltoiu's theorem says the converse is true.

### 1.6. **Coltoiu's theorem.**

**Theorem 8.** *Let $X \subseteq \mathbb{C}^n$ be a closed analytic set and $\Omega \subseteq X$ a Runge open subset. Then there exists a Runge open subsect $\tilde{\Omega} \subseteq \mathbb{C}^n$ such that $\tilde{\Omega} \cap X = \Omega$. Moreover, if $K \subseteq \mathbb{C}^n$ is a holomorphically convex compact subset such that $K \cap X \subseteq \Omega$ then we there exists such a $\tilde{\Omega}$ with the additional property: $K \subseteq \tilde{\Omega}$.*

## References

[CGZ] D. Coman, V. Guedj and A. Zeriahi, "Extension of PSH functions with growth control", J. reine angew. Math. 676 (2013), 33–49