Donaldson's theorems on scalar curvature

§1. Linear Algebra.

Complex Structures on Vector Spaces.

Let V be a vector space over $\mathbb{R}$ of dimension $2n$. Then a complex structure on V is an element $J \in Aut(V)$ with the property $J^2 = -I$.

Alternatively, a complex structure on V is a pair (ξ, T) mod equivalence where T is a complex vector space of dimension n and $\xi : V \to T$ is an isomorphisom of real vector spaces. The equivalence relation is given by $(\xi, T) \sim (\xi', T')$ if there is an isomorphism of complex vector spaces $T \to T'$ which makes the diagram commute.

To see the equivalence of the two definitions, let $J : V \to V$ be such that $J^2 = -I$ and define T as follows: Then $J \otimes I$ defines and automorphism of the vector space $V \otimes \mathbb{C}$ Let T be the $+i$ eigenspace. Then $\bar{T}$ is the $-i$ eigenspace. We have

$$V \otimes \mathbb{C} \; = \; T \oplus \bar{T}$$

Now T is a complex vector space and the map $\xi : V \to T$ obtained by composing the maps $V \to V \otimes \mathbb{C} = T \oplus \bar{T} \to T$ is an isomorphism of real vector spaces. Conversely, if T is a complex vector space and if $\xi : V \to T$ is an isomorphism of real vector spaces, then $J = \xi^{-1} \circ \mathbf{i} \circ \xi$ is a complex structure. Here $\mathbf{i}$ is the map on T given by multiplication by i. Note that J depends only on the equivalence class of (ξ, T).

A slight variant is: A complex structure on V is an equivalence class of isomorphisms $f : V \to \mathbb{C}^n$, where two isomorphisms are equivalent if they differ by an element of $GL(n, \mathbb{C})$.

If we fix a basis of V, we see that a complex structure on V is a $2n \times 2n$ matrix J with the property $J^2 = -I$, where I is the $2n \times 2n$ identity matrix. Alternatively, a complex structure is an equivalence class of isomorphisms $f : \mathbb{R}^{2n} \to \mathbb{C}^n$ of real vector spaces.

Thus we see that $GL(2n, \mathbb{R})$ operates transitively on the space of complex structures, with stabilizer group $GL(n, \mathbb{C})$. So the space of complex structures on $\mathbb{R}^{2n}$ is just the space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. Thus, if we let $\mathcal{J}(V)$ be the space of complex structures on V, we see that

$$\mathcal{J}(V) \; \approx \; GL(2n, \mathbb{R})/GL(n, \mathbb{C}) \tag{1.1}$$

$\mathcal{J}(V)$ as a complex manifold.

The space $\mathcal{J}(V)$ is a smooth manifold. In fact, it has a natural structure as a complex manifold. To see this, observe that $\mathcal{J}(V)$ is the set of equivalence classes of $n \times 2n$ matrices M with entries in $\mathbb{C}$ whose columns form a basis of $\mathbb{C}^n$ viewed as a vector space over $\mathbb{R}$. In other words,

$$det \begin{pmatrix} M \\ \bar{M} \end{pmatrix} \neq 0 \tag{1.2}$$

Since such an M has maximal rank over $\mathbb{C}$, at least one of its $n \times n$ minors has non-zero determinant. Suppose that the first n columns of M form a minor of non-zero determinant. Then equivalence class of M has a unique representative of the form (I, Z) where Z, according to (1.2), is an $n \times n$ matrix such that $Im(Z)$ is non-singular. Such Z form an open subset of $M_{n\times n}(\mathbb{C})$. Since M is covered by a finite number of such open sets, with holomorphic transitions, we see that $\mathcal{J}$ is a complex manifold of dimension n^2.

Symplectic structures on vector spaces.

A symplectic form on V is an non-degenerate alternating form $\omega : V \times V \to \mathbb{R}$. In other words, a symplectic form is an element $\omega \in \Lambda^2 V^*$ which is non-degenerate.

If $V = \mathbb{R}^{2n}$ then a symplectic form is a non-singular $2n \times 2n$ matrix ω such that ${}^t\omega = -\omega$. The standard symplectic form on $\mathbb{R}^{2n}$ is $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $GL(2n, \mathbb{R})$ acts transitively on the set of symplectic froms with stabilizer $Sp(2n, \mathbb{R})$, we see that the set of symplectic structures on $\mathbb{R}^{2n}$ is $GL(2n, \mathbb{R})/Sp(2n, \mathbb{R})$. Thus, if we let $\mathcal{S}(V)$ be the space of symplectic structures on V, we see that

$$\mathcal{S}(V) \;\approx\; GL(2n, \mathbb{R})/Sp(2n, \mathbb{R}) \tag{1.3}$$

$\mathcal{S}(V)$ as a symplectic manifold.

The space $\mathcal{S}(V)$ is a smooth manifold with a natural symplectic structure. If $\omega \in \mathcal{S}(V)$ then $\omega + \eta \in \mathcal{S}(V)$ for sufficiently small $\eta \in T_\omega(\mathcal{S}(V)) = \Lambda^2(V^*)$. This shows that $S(V)$ is a manifold of dimension $\binom{n}{2}$.

If $\eta_1, \eta_2 \in T_\omega(\mathcal{S}(V))$ then define

$$\Omega(\eta_1, \eta_2) \;=\; Tr(\eta_1 \omega \eta_2)$$

Then Ω is a non-degenerate closed 2-form on $\mathcal{S}(V)$.

Complex structures compatible with symplectic forms.

Let V be a finite dimensional vector space over $\mathbb{R}$ and fix ω, a symplectic form on V. We say that a complex structure J is compatible with ω if $\omega(Ju, Jv) = \omega(u, v)$ and if $\omega(u, Ju) > 0$ if $u \neq 0$. Let $\mathcal{J}_\omega(V) \subseteq \mathcal{J}(V)$ be the set of complex structures on V compatible with ω. We have seen that $\mathcal{J}(V)$ is a complex manifold, which is covered by coordinate neighborhoods $\mathcal{J}(V)_i$, with $1 \leq i \leq \binom{2n}{n}$ where

$$\mathcal{J}(V)_i \;=\; \{Z \in M_{n\times n}(\mathbb{C}) : \det(Im(Z)) \neq 0\}$$

Let $\mathcal{J}_{\omega,i} = \mathcal{J}_i \cap \mathcal{J}_\omega$.

Claim. *$\mathcal{J}_\omega$ is a complex submanifold of $\mathcal{J}$. Moreover, for every i,*

$$\mathcal{J}_\omega = \mathcal{J}_{\omega,i} \;=\; \{Z \in M_{n\times n}(\mathbb{C}) : \quad Z \;=\; {}^tZ,\; Im(Z) > 0 \;\} \;=\; Sp(2n,\mathbb{R})/U(n)$$

In other words, $\mathcal{J}_\omega$ is the Siegel upper half plane of genus n, and the natural action of $Sp(2n;\mathbb{R})$ on $\mathcal{J}_\omega$ is the standard Möbius action: If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R})$ and if Z is a point in the upper half plane then

$$\gamma(Z) \;=\; (AZ+B)(CZ+D)^{-1}$$

Proof. A complex structure $\xi : V \to \mathbb{C}^n$ is compatible with ω if and only if $\nu = \xi_*\omega$ (which is a symplectic form on the real vector space $\mathbb{C}^n$) has the property: $\nu(iz, iw) \;=\; \nu(z,w)$ for all $z, w \in \mathbb{C}^n$. In other words, $\nu(z,w) = Im\big(\langle z,w\rangle_\xi\big)$ where $\langle z,w\rangle_\xi$ is the hermitian metric on $\mathbb{C}^n$ given by the formula:

$$\langle z,w\rangle_\xi \;=\; \nu(z,iw) \;+\; i\nu(z,w) \tag{1.4}$$

(Our convention for the definition of a a hermitian pairing is: $\langle z, aw\rangle \;=\; a\langle z,w\rangle$ and $\langle az, w\rangle \;=\; \bar{a}\langle z,w\rangle$ for all $a \in \mathbb{C}$). Since any two hermitian metrics on $\mathbb{C}^n$ are equivalent under the action of $GL(n,\mathbb{C})$, we see that inside the equivalence class of ξ there is a representative (which we also call ξ) with the property $\langle,\rangle_\xi = \langle,\rangle$ where $\langle,\rangle$ is the standard hermitian pairing on $\mathbb{C}^n$: $\langle z,w\rangle \;=\; {}^t\bar{z}w$. This representative is unique up to $U(n)$, the symmetry group of the form $\langle,\rangle$. Thus the set of complex structures compatible with ω is the set of $U(n)$ equivalence classes of isomorphisms $\xi : V \to \mathbb{C}^n$ with the property $\omega = \xi^*\nu$ where $\nu = Im\big(\langle\;,\;\rangle\big)$. Fix one such ξ. Then any other ξ must be of the form $\eta = \xi \circ f$ where $f \in GL(V)$. But the condition $\eta^*\nu = \omega$ implies, $f^*\omega = \omega$ which means that $f \in Sp(V,\omega)$. This shows that $\mathcal{J}_\omega \;= Sp(2n,\mathbb{R})/U(n)$. In other words the inclusion $\mathcal{J}_\omega(V) \hookrightarrow \mathcal{J}(V)$ is equivalent to the inclusion

$$Sp(2n,\mathbb{R})/U(n) \;\hookrightarrow\; GL(2n,\mathbb{R})/GL(n,\mathbb{C}) \tag{1.5}$$

(thus we've proved that $U(n) = Sp(2n,\mathbb{R}) \cap GL(n,\mathbb{C})$). The rest of the claim follows by simple calculation.

§2. The groups, the manifolds, and the actions.

We digress for a moment to discuss the definition of an infinite dimensional manifold. We start with the two basic examples:

Let M be a smooth manifold, and let N be a smooth manifold. Let $C^\infty(M,N)$ be the set of smooth maps from M to N. Then $C^\infty(M,N)$ is an example of an infinite dimensional manifold (which is actually finite dimensional when M is a finite collection of points). If $f \in C^\infty(M,N)$ then the tangent space at f is defined as $T_f(C^\infty(M,N)) = \Gamma(f^*TN))$. Thus an element in the tangent space assigns, in a smooth fashion, to each point $x \in M$ a tangent vector at the point $f(x) \in N$.

To give an example of an infinite dimensional complex manifold, we again let M be a smooth manifold and N a complex manifold. Then $C^\infty(M,N)$ is a complex manifold. The complex structure on the tangent space T_f is defined to be f^*J, were $J : TN \to TN$ is the complex structure on N.

Now we describe a more general class of examples: The category of infinite dimensional manifolds based on a given finite dimensional manifold M is contains the category of fiber bundles over M. Recall that a fiber bundle over M is a manifold F and a map $F \to M$ with the property that locally, $F = U_\alpha \times F_o$ where F_o is a fixed smooth manifold. We require that on the overlaps, the transition functions $\phi_{\alpha\beta}(x)$ are diffeomorphisms of F_o which vary smoothly with x. Then the infinite dimensional manifold associated to $F \to M$ is the space $\Gamma(F/M)$ of smooth sections $s : M \to F$. The tangent space at s is defined to be $T_s(\Gamma(F/M)) = \Gamma(s^*(TF^v))$, where $TF^v \subseteq TF$ is the subsheaf consisting of "vertical vectors", that is, TF^v is the kernel of the map $TF \to TM$. Thus a tangent vector at s assigns to every $x \in M$ a vector tangent to the fiber F_x at the point $s(x) \in F_x \subseteq F$.

Similarly we can define the complex manifold assoicated to a fiber bundle, is similar, but we require that F_o be a complex manifold and that the transition functions $\phi_{\alpha\beta}(x)$ be biholomorphic maps which vary smoothly with x.

Now maps between fiber bundles over M give rise to maps between corresponding manifolds in the obvious way. Embeddings $F \hookrightarrow F'$ correspond to submanifolds.

The simplest examples come from the case where F_o is a vector space, in other words, if F is a vector bundle. The assoicated manifold is the space of smooths sections. These manifolds are affine: This means that all the tangent spaces are canonically identified with the manifold itself.

The general definition of a manifold based on a given M is similar to the usual definition of a finite dimensional manifold: It's a topological space $\mathcal{F}$ which is covered by "Euclidean balls over M": A Euclidean ball is a set of the form $\{s \in \Gamma(E) : ||s - s_0||_{C^\infty} < r\}$, where E is a smooth vector vector bundle on M endowed with a connection ∇, $s_0 \in \Gamma(E)$ is a fixed smooth section, r is a positive number, and the norm is the C^∞ norm defined by ∇.

Next we define the relevant infinite dimensional Lie groups and the infinite dimensional manifolds upon which they act, and we calculate the infintesimal actions of the Lie algebras.

The groups Diff and Sym.

Let M be a smooth manifold. Then $Diff(M)$ is the group of diffeomorphisms of M. We have $Lie(Diff(M)) = Vect(M)$, the space of smooth vector fields on M.

Now let ω be a symplectic form on M. We define $Sym(M,\omega) \subseteq Diff(M)$ to be the group of exact symplectomorphisms of (M,ω). Then

$$Lie(Sym(M,\omega)) \;=\; C^\infty(M)/\mathbb{R}$$

The Lie algebra imbedding $C^\infty(M)/\mathbb{R} = Lie(Sym(M,\omega)) \hookrightarrow Lie(Diff(M)) = Vect(M)$ is given by $f \mapsto X_f$, where X_f is the symplectic gradient of f. In other words,

$$X_f^j = f_i \omega^{ij} \tag{2.1}$$

In other words, $df = i_{X_f}\omega$. The Lie algebra structure on $C^\infty(M)/\mathbb{R}$ is given by the Poisson bracket: If $f, g \in C^\infty(M)/\mathbb{R}$ then $\{f,g\} \ = \ \omega(X_f, X_g)$

The manifolds $\mathrm{Aut}, \mathcal{J}, \mathcal{S}, \mathcal{J}_\omega, \mathcal{C}$ **and** $\mathcal{K}$**.**

Define

$$\begin{aligned}
\mathrm{Aut} \ &= \ \{J : TM \to TM \mid J \ \text{ is a bundle automorphism } \} \\
\mathcal{J} \ &= \ \{J : \mathrm{Aut} \mid J^2 = -I\} \\
\mathcal{S} \ &= \ \{\omega : TM \otimes TM \to C^\infty \mid \omega \text{ is a symplectic form}\} \\
\mathcal{C} \ &= \ \{(J,\omega) \in \mathcal{J} \times \mathcal{S} : \ \omega(Ju, Jv) = \omega(u,v), \ \omega(u, Ju) > 0 \text{ for } u \neq 0\}
\end{aligned}$$

and for a fixed symplectic form ω,

$$\mathcal{J}_\omega \ = \ \{J \in \mathcal{J} : (J,\omega) \in \mathcal{C} \ \}$$

For $(J,\omega) \in \mathcal{C}$ we define $g_{(J,\omega)}(u,v) = \omega(u, Jv)$ which is a Riemannian metric on M, and we let $\nabla_{(J,\omega)}$ be the corresponding Levi-Civita connection. Then define the space of Kahler structures on M as follows:

$$\mathcal{K} \ = \ \{(J,\omega) \in \mathcal{C} : \nabla J \ = \ 0\}$$

Finally we define $\mathcal{J}_{int} \subseteq \mathcal{J}$ as follows:

$$\mathcal{J}_{int} \ = \ \{J \in \mathcal{J} : N(J) = 0 \ \}$$

where N is the Nijenhuis tensor. Recall that if $(J,\omega) \in \chi C$ with $J \in \mathcal{J}_{int}$ then $(J,\omega) \in \mathcal{K}$.

The discussion in §1 shows that $\mathcal{J}, \mathcal{S}$ and $\mathcal{J}_\omega$ are all infinite dimensional manifolds which are associated to various fiber bundles. For example, $\mathcal{J}$ is the set of all sections of a certain $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ bundle over M. In a similar way, $\mathcal{S}$ is associated to a $GL(2n,\mathbb{R})/Sp(2n,\mathbb{R})$ bundle and $\mathcal{J}_\omega$ to a $Sp(2n,\mathbb{R})/U(n))$ bundle.

Since $Sp(2n,\mathbb{R})/U(n))$ is the Siegel uper half plane, which has a complex structure, we see that $\mathcal{J}$ is a complex manifold. The submanifolds $\mathcal{J}_{int}$ and $\mathcal{K}$ are defined by smooth first order (non-linear) differential equations. They are not associated to a fiber bundle and are thus more difficult to understand.

The tangent spaces.

The tangent spaces can be described as follows:

$$\begin{aligned}
T_J(\mathcal{A}ut) \ &= \ \{A : TM \to TM \ : A \text{ is a bundle map}\} \ = \ \mathrm{End}(TM) \\
T_J(\mathcal{J}) \ &= \ \{A \in \mathrm{End}(TM) : \ JA + AJ = 0\} \\
T_\omega(\mathcal{S}) \ &= \ \{\eta : TM^* \otimes TM^* \to C^\infty : \eta(u,v) = -\eta(v,u)\} \\
T_{(J,\omega)}(\mathcal{C}) \ &= \ \{(A,\eta) \ : \ \omega(JAu, v) + \omega(u, JAv) \ = \ \eta(Ju, Jv) - \eta(u,v)
\end{aligned}$$

$$T_J(\mathcal{J}_\omega) \;=\; \{A \in T_J(\mathcal{J}) \;:\; \omega(JAu, v) + \omega(u, JAv) = 0\}$$

Complex Coordinates

If $J \in \mathcal{J}$ then we can decompose $TM \otimes \mathbb{C} \ = \ T \oplus \bar{T}$, where T is the i eigenspace of J and $\bar{T}$ the $-i$ eigenspace. Using this decomposition, one can give a simpler description of the tangent spaces above. For example, if $A \in T_J(\mathcal{A}ut)$ then we can write

$$A \ = \ \begin{pmatrix} A^i_j & A^i_{\bar{j}} \\ A^{\bar{i}}_j & A^{\bar{i}}_{\bar{j}} \end{pmatrix}, \ \overline{A^i_j} = A^{\bar{i}}_{\bar{j}}, \ \overline{A^i_{\bar{j}}} = A^{\bar{i}}_j \tag{2.2}$$

with $A^i_j \in Hom(T,T), A^i_{\bar{j}} \in Hom(\bar{T},T)$, etc. An element $A \in T_J(\mathcal{A}ut)$ is in $T_J(\mathcal{J})$ if an only if $A^i_j = A^{\bar{i}}_{\bar{j}} = 0$. Thus we have an isomorphism $\mu : T_J(\mathcal{J}) \to \Gamma(T \otimes \bar{T}^*)$ which we normalize as follows:

$$\mu(A) \ = \ \frac{\frac{\sqrt{-1}}{2}}{2} \cdot A^i_{\bar{j}} \tag{2.3}$$

Since $\Gamma(T \otimes \bar{T}^*)$ is a complex vector space, we see that $\mathcal{J}$ has a complex structure.

If $(J,\omega) \in \mathcal{C}$ then, with respect to the eigenspace decomposition $TM \otimes \mathbb{C} \ = \ T \oplus \bar{T}$,

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \text{ and } \omega = \begin{pmatrix} 0 & \omega_{i\bar{j}} \\ \omega_{\bar{i}j} & 0 \end{pmatrix}, \overline{\omega_{i\bar{j}}} \ = \ \omega_{\bar{i}j}, \ \ \omega_{i\bar{j}} = -\omega_{\bar{j}i}$$

Thus if $J \in \mathcal{J}_\omega$ we see that

$$T_J(\mathcal{J}_\omega) \ = \ \{\mu^i_{\bar{j}} \in T_J(\mathcal{J}) : \omega_{\bar{i}k}\mu^k_{\bar{j}} \ = \ \omega_{\bar{j}k}\mu^k_{\bar{i}}\}$$

Thus, if we let $S^2(\bar{T}^*)$ be the set of symmetric tensors in $\Gamma(\bar{T}^* \otimes \bar{T}^*)$ we have an isomorphism

$$s : T_J(\mathcal{J}_\omega) \to S^2(\bar{T}^*)$$

given by $\mu^i_{\bar{j}} \mapsto s_{\bar{i}\bar{j}} \ = \ \omega_{\bar{i}k}\mu^k_{\bar{j}}$. Again, since $S^2(\bar{T}^*)$ is a complex vector space, $\mathcal{J}\omega$ has a complex structure.

Finally, the symplectic gradient (2.1) can also we written in complex coordinates: For a smooth function $f \in C^\infty(M)/\mathbb{R}$ we have $X_f = \xi_f + \overline{\xi_f}$ where $\xi_f \in \Gamma(T)$ is given by

$$\xi^i_f \ = \ f_{\bar{j}}\omega^{\bar{j}i} \tag{2.4}$$

The actions.

Now we define the actions of $Diff(M)$ on $\mathcal{J}$ and on $\mathcal{S}$: If $\phi \in Diff(M)$ and $J \in \mathcal{J}$, then $\phi \cdot J = D\phi \circ J \circ D\phi^{-1}$, where $D\phi : TM \to TM$ is the derivative of ϕ. If $\omega \in \mathcal{S}$ then we define $\phi \cdot \omega = \phi^*\omega$. These actions induce actions on $\mathcal{C}$ and $\mathcal{K}$ and $\mathcal{J}_{int}$. Note however that the manifold $\mathcal{J}_\omega$ is invariant under $Sym(M,\omega)$ but not under $Diff(M)$.

Now we calculate the infinitestimal action of $Diff(M)$ on $\mathcal{A}ut$ and on $\mathcal{S}$. If $v \in Vect(M) = Lie(Diff(M))$, and if $J \in \mathcal{A}ut$, we have

$$v \cdot J \ = \ \mathcal{L}_v J \ \in \ T_J(\mathcal{J}) \tag{2.5}$$

where $\mathcal{L}_v$ is the Lie derivative (this is essentially the definition of the Lie derivative). Note that if $J \in \mathcal{J}$ then $J^2 = -1$ so we have $AJ + JA = 0$ where $A = \mathcal{L}_v J$. In other words, $\mathcal{L}_v J \in T_J(\mathcal{J})$ as expected.

Next we re-write (2.5) in terms of coordinates: Let v be a smooth vector field let ϕ_{tv} be the 1-parameter family of diffeomorphisms associtated to v. Then by definition,

$$v \cdot J \;=\; \frac{d}{dt}\Big|_{t=0} D\phi_{tv} \circ J(\phi_{tv}(x)) \circ D\phi_{tv}^{-1} \;=$$

$$\frac{d}{dt}\Big|_{t=0} \left(\delta^i_p + tv^i_p\right)\left(J^p_k \;+\; tJ^p_{k;l}v^l\right)\left(\delta^k_j - tv^k_j\right) \;=\; \left(v^i_p J^p_j - J^i_p v^p_j\right) \;+\; J^i_{j;l}v^l$$

Now suppose that ∇ is a connection on TM with the property: $\nabla J = 0$ (this is the case, for example, when M is Kähler). Choosing normal coordinates:

$$(v \cdot J)^i_j \;=\; v^i_p J^p_j - J^i_p v^p_j \tag{2.6}$$

Since the right side of (2.6) is a tensor, we see that (2.6) holds in any coordinate system, where, as usual, v^i_j denotes covariant differentiation.

Equation (2.6) takes a particularly simple form if $J \in \mathcal{J}$:

$$\mu(v \cdot J) \;=\; \bar{\partial} v \tag{2.7}$$

In other words, if we identify $T_J(\mathcal{J})$ with $\Gamma(T \otimes \bar{T}^*)$, we obtain:

$$(v \cdot J)^i_{\bar{j}} \;=\; v^i_{\bar{k}} \tag{2.8}$$

Again (2.7) and (2.8) hold when $J \in \mathcal{J}$ and when $\nabla J = 0$. For example, they hold when $J \in \mathcal{J}_{int}$ is compatible with a symplectic form ω.

To prove (2.6) and (2.7), we simply calculate the right side of (2.6):

$$\begin{pmatrix} v^i_j & v^i_{\bar{j}} \\ v^{\bar{i}}_j & v^{\bar{i}}_{\bar{j}} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \;-\; \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} v^i_j & v^i_{\bar{j}} \\ v^{\bar{i}}_j & v^{\bar{i}}_{\bar{j}} \end{pmatrix} \;=\; \begin{pmatrix} 0 & -2\frac{\sqrt{-1}}{2}v^i_{\bar{j}} \\ 2\frac{\sqrt{-1}}{2}v^{\bar{i}}_j & 0 \end{pmatrix}$$

Applying (2.3) we get (2.8).

§3. The lifted groups, manifolds and actions.

Let M be a compact manifold and let $\pi : L \to M$ be a complex line bundle on M. Let $\mathcal{A}ut(M)$ be the diffeomorphism group of M. Define $\mathcal{A}ut(M, L)$ to be the group of bundle automorphisms of L: Thus $\mathcal{A}ut(M, L)$ consists of all pairs $\mathcal{F} = (F, \tilde{F})$ where $F : M \to M$ and $\tilde{F} : L \to L$ are diffeomorphisms with the properties: $\pi\tilde{F} = F\pi$ and $\tilde{F}(\lambda x) = \lambda F(x)$ for all $\lambda \in \mathbb{C}$ and all $x \in L$. Clearly the map $\mathcal{A}ut(M, L) \to \mathcal{A}ut(M)$ mapping $(F, \tilde{F}) \mapsto F$ is a homomorphism. It's image is the group $\mathcal{A}ut_0(M)$ consisting off all diffeomorphisms such that $F^*c_1(L) = c_1(L)$.

The Lie algebra of $\mathrm{Aut}(M)$ (and of $\mathcal{A}ut_0(M)$) is the space $Vect(M)$ consisting of all smooth vector fields on M. The Lie algebra of $\mathrm{Aut}(M, L)$ is given by the space of vector fields on $L^\times$ (the complement of the zero section in L) which are $\mathbb{C}^\times$ invariant. If we choose on L a hermitian metric h and a unitary connection A, then we can describe the Lie algebra in a very convenient fashion:

$$Lie(\mathcal{A}ut(M,L)) \;=\; \{(\tilde{X} + \psi\mathbf{t}) : X \in Vect(M), \psi \in C^\infty(M,\mathbb{C})\}$$

where $\tilde{X}$ is the horizontal lift of the vector field X, $\mathbf{t}$ is the vector field on $L^\times$ generated by the infinitesimal action of $U(1)$ and $C^\infty(M,\mathbb{C})$ is the space of complex valued smooth functions on M.

Now fix (M,ω), a compact symplectic manifold, and (L,h,A), a complex vector bundle with hermitian metric and unitary connection, satisfying the following curvature property:

$$F_A \;=\; -i\omega \tag{3.1}$$

We let $\mathcal{G}_0 = \mathcal{A}ut(M,\omega)$ be the group of exact symplectomorphisms of M. Recall that the Hamiltonian construction gives an isomorphism $Lie(\mathcal{A}ut(M,\omega) = \mathbb{C}^\infty(M)/\mathbb{R}$. We now define $\mathcal{G} = \mathcal{A}ut((M,\omega),(L,A,\omega))$ to be the subroup of $\mathcal{A}ut(M,L)$ which preserve h and A. Thus and element $\mathcal{F} = (F,\tilde{F}) \in \mathcal{A}ut(M,L)$ is in $\mathcal{A}ut((M,\omega),(L,A,\omega))$ if $|F(x)|_h \;=\; |x|_h$ for all $x \in L$ and if $\tilde{F}^*A \;=\; A$: Recall that if $F : N \to M$ is a smooth map, and $E \to M$ a vector bundle with connection A, the F^*E is a vector bundle on N with connection F^*A. There are various (equivalent) ways of defining F^*A: in local coordinates, $A = (A_{ij})$ is a matrix of one forms on M. Then $F^*A = (F^*A)_{ij}$ is a matrix of one forms on N defined by $(F^*A)_{ij} = F^*(A_{ij})$. Thus F^*A is characterized by the formula:

$$\nabla_{(F^*A)}(F^*s) \;=\; F^*(\nabla_A s)$$

for every section s of E. More geometrically, we can view the pullback of a connection via holonomy maps as follows: The connection A assigns to each path γ in M an isomorphism: $A_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$. If σ is a path on N, define $(F^*A)_\sigma : (F^*L)_{\sigma(0)} \to (F^*L)_{\sigma(1)}$ by the formula $(F^*A)_\sigma = A_{F\sigma}$, where we make the canonical identification $(F^*L)_{\sigma(t)} = L_{F\sigma(t)}$. These holonomy isomorphisms determine F^*A.

Now we calculate the lie algebra $Lie(\mathcal{G})$: Since $\mathcal{G} \subseteq \mathcal{A}ut(M,L)$, every element in $\mathcal{G}$ must be of the form $V = \tilde{X} + \psi\mathbf{t}$ for some $X \in Vect(M)$ and some $\psi \in \mathbb{C}^\infty(M,\mathbb{C})$. Then $V \in Lie(\mathcal{G})$ if and only if $\mathcal{L}_V(|x|_h) = 0$ and $\mathcal{L}_V A = 0$, where $\mathcal{L}$ is the Lie derivative. The infintesimal action of V on the metric is given by

$$\mathcal{L}_V(|x|_h) \;=\; \frac{d}{dt}\bigg|_{t=0} |\exp(it\psi)x|_h \;=\; -Im(\psi)|x|_h$$

Since the elements in $\mathcal{G}$ are required to preserve the metric, we must have $Im(\psi) = 0$, that is, $\psi = f \in C^\infty(M,\mathbb{R})$, a real valued function.

Next we calculate the infinitesimal action of V on the connection: We view A as a one-form on $L(1)$ (the elements of L of norm one) with values in $i\mathbb{R}$ (the lie algebra of $U(1)$). Then

$$(\mathcal{L}_V A)(Z) \;=\; V(A(Z)) \;-\; A([V,Z]) \;=\; 0 \tag{3.2}$$

for all $Z = \tilde{Y} + g\mathbf{t}$, where $Y \in Vect(M)$ and $g \in \mathbb{C}^\infty(M)$, where $\mathcal{L}$ is the lie derivative acting on one-forms. Now $A(Z) = A(\tilde{Y} + g\mathbf{t}) = g$ since A kills horizonal vectors and $A(\mathbf{t}) = 1$. Thus $V(A(Z)) = V(g) = (\tilde{X} + f\mathbf{t})(g) = X(g)$ since g is constant on fibers and is thus killed by $\mathbf{t}$. Thus (3.2) becomes

$$X(g) \;=\; A([V,Z])$$

for all g and all Y. Now

$$[V,Z] \;=\; [\tilde{X} + f\mathbf{t}, \tilde{Y} + g\mathbf{t}] \;=\; [\tilde{X},\tilde{Y}] + [f\mathbf{t},\tilde{Y}] + [\tilde{X}, g\mathbf{t}] + [f\mathbf{t}, g\mathbf{t}]$$

Now

$$[\tilde{X},\tilde{Y}] \;=\widetilde{[X,Y]} + iF_A(X,Y)\mathbf{t}$$

using the definition of curvature. But we are assuming that $F_A = -i\omega$. Thus we obtain $A([\tilde{X},\tilde{Y}]) = \omega(X,Y)$. Now

$$[\tilde{X}, g\mathbf{t}] \;=\; \mathcal{L}_{\tilde{X}} g\mathbf{t} \;=\; X(g)\mathbf{t} + \mathcal{L}_{\tilde{X}}\mathbf{t} \;=\; X(g)\mathbf{t}$$

and

$$[f\mathbf{t}, g\mathbf{t}] \;=\; \mathcal{L}_{f\mathbf{t}}(g\mathbf{t}) \;=\; f(\mathcal{L}_{\mathbf{t}} g)\cdot\mathbf{t} \;+ fg\mathcal{L}_{\mathbf{t}}\mathbf{t} \;=\; 0+0 \;=\; 0$$

Thus

$$X(g) \;=\; A([V,Z]) \;=\; \omega(X,Y) \;+\; X(g) - Y(f)$$

which implies $\omega(X,Y) = Y(f) = df(Y)$ for all Y, in other words, $X = X_f$. Thus, we see that

$$Lie(\mathcal{G}) \;=\; \{\tilde{X}_f + f\mathbf{t} : f \in C^\infty(M)\} \;\approx\; C^\infty(M)$$

where the isomorphism is an isomorphism of Lie algebras (where $C^\infty(M)$ has the lie algebra structure given by the Poisson bracket). The map $\mathcal{G} \to \mathcal{G}_0$ induces a map on Lie algebras $C^\infty(M) \to C^\infty(M)/\mathbb{R}$ which is just the canonical quotient map.

§4. The mirror principle.

As before, we let $\mathcal{H} = \{(\underline{s}, I) : \underline{s}$ is a basis of $H^0(\mathcal{L}_I^k)$ $\}$ and we define, for $a > 0$,

$$\mu_a \;=\; \{(\underline{s}, I) \in \mathcal{H} \;:\; \sum_{\alpha=0}^{N} |s_\alpha|_{h_0}^2 \;\; = a \;\}$$

We wish to show:

$$\mu_a/\mathcal{G} \;=\; \mathcal{H}/\mathcal{G}^c \tag{4.1}$$

To do this, we first prove the following:

Lemma (mirror principle). *Fix $(\underline{s}, I) \in \mathcal{H}$. There is a natural diffeomorphism of infinite dimensonal manifolds:*

$$\mathcal{G}^c(\underline{s}, I)/\mathcal{G} \;\;\approx\; \mathrm{Herm}(\mathcal{L}_I) \tag{4.2}$$

where $\mathrm{Herm}(\mathcal{L}_I) \;=\; \{h : \mathcal{L}_I \to \mathbb{R} :\; h \;$ *is a hermitian metric with positive curvature* $\}$

Proof. We first define a map $\mathcal{G}^c(\underline{s}, I) \to \mathrm{Herm}(\mathcal{L}_I)$ as follows: Let $(\underline{s}', I') \in \mathcal{G}^c(\underline{s}, I)$ and choose $\mathcal{F} \in \mathcal{A}ut(M, L)$ such that $\mathcal{F}(\underline{s}', I') = (\underline{s}, I)$. Thus $\mathcal{F} = (F, \tilde{F})$, where the map $F : (M, I) \to (M, I')$ is biholomorphic, $\tilde{F} : \mathcal{L}_I \to \mathcal{L}_{I'}$ is a holomorphic isomorphism of line bundles, and $\underline{s} = \tilde{F}^{-1}\underline{s}'F$.

The choice of $\mathcal{F}$ is unique, since if $(\underline{s}, I) = (\underline{s}', I')$ then $\mathcal{F}(F, \tilde{F})$ has the property: F is a holomorphic automorphism of (M, I) such that $\tilde{F}\underline{s} = sF$. Evaluating at any $x \in M$: $\tilde{F}_x(\underline{s}(x)) \;=\; \underline{s}(F(x))$. But $\tilde{F}_x$ is a non-zero complex number, and thus $\underline{s}(x)$ and $\underline{s}(F(x))$ define the same point in projective space. But we are assuming that $\underline{s}$ provides and embedding of M into projective space. Thus $x = F(x)$ and F is the identity. Thus $\underline{s}(x) = \tilde{F}_x \cdot \underline{s}(x)$. Since at least one of the elements in the basis $\underline{s}$ is non-zero, we see that $\tilde{F}_x = 1$. This shows that $\mathcal{F}$ is unique.

Now define $h = \mathcal{F}(h_0)$, that is, $h = h_0 \circ \tilde{F}$. Then $R(h) = F^*(R(h_0)) = F^*(\omega)$. Since I' is compatible with ω, we have that $I = \mathcal{F}(I')$ is compatible with $F^*(\omega)$. In other words, $F^*(\omega)$ is a positive $(1,1)$ form, and thus $R(h)$ is positive, that is, $h = h(\underline{s}', I') \in \mathrm{Herm}(\mathcal{L}_I)$.

Note that $h(s', I') = h(s'', I'')$ if $(s', I') = \mathcal{F}(s'', I'')$ for some $\mathcal{F} \in \mathcal{G}$. We claim the converse is true as well: Assume $h(s', I') = h(s'', I'')$. Then there is an $\mathcal{F} \in \mathcal{A}ut(M, L)$ such that $\mathcal{F}(s', I') = (s'', I'')$ and such that $\mathcal{F}(h_0) = h_0$. Now consider the two connections: A and $\mathcal{F}^*A$. The complex structure on $\mathcal{L}_{I'}$ is compatible with both, and the metric h_0 is compatible with both. Since the connection compatible with the metric and the complex structure is uniquely determined, we conclude that $A = \mathcal{F}^*(A)$, and thus $\mathcal{F} \in \mathcal{G}$.

We now see that the map $h : \mathcal{G}^c(\underline{s}, I)/\mathcal{G} \;\;\to\;\; \mathrm{Herm}(\mathcal{L}_I)$ is well defined and injective. It remains to prove that it is surjective. So let $h \in \mathrm{Herm}(\mathcal{L}_I)$. Then $R(h) = \omega + i\partial\bar{\partial}\phi > 0$. By Moser's lemma, there is $F : M \to M$ such that $F^*(\omega) \;=\; \omega + i\partial\bar{\partial}\phi$. This shows that $c_1(F^*L) = c_1(L)$, so there is a diffeomorphism $(G, \tilde{G}) : (M, F^*L) \to (M, L)$ where G is the identity. Thus there exists $\tilde{F}$ such that $\mathcal{F} = (F, \tilde{F}) \in Aut(M, L)$. Now the curvature of $\mathcal{F}(h_0)$ is $\omega + i\partial\bar{\partial}\phi$ and thus $\mathcal{F}(h_0) = ah$ for some $a > 0$. Replacing $\tilde{F}$ by $a^{-1}\tilde{F}$, we conclude $\mathcal{F}(h_0) = h$, and thus our map is surjective.

Now we extablish (4.1): Note that for every $(s', I') \in \mathcal{G}^c(s, I)$ we have

$$\sum_\alpha |s'_\alpha|^2_{h_0} \;=\; \sum_\alpha |s_\alpha|^2_h$$

where $h = h(s', I')$. Thus there exists $(s', I') \in \mathcal{G}^c(s, I)$ with the property $\sum_\alpha |s'_\alpha|^2_{h_0} = a$ if and only if there exists $h \in \mathrm{Herm}(\mathcal{L}_I)$ with the property

$$\sum_\alpha |s_\alpha|^2_h \;=\; a \tag{4.3}$$

Now for a fixed $(s, I) \in \mathcal{H}$ there is clearly a unique h satisfying (4.3), namely:

$$h \;=\; a\cdot\frac{h_0}{\sum_\alpha |s_\alpha|^2_{h_0}}$$

Thus, by the mirror lemma, for a fixed $(s,I)\in\mathcal{H}$ there is, up to the action of $\mathcal{G}$, a unique $(s',I')\in\mathcal{G}^c(s,I)$ such that $(s',I')\in\mu_a$.

§5. Moment maps: Uniqueness and existence of zeros.

Let G be a compact Lie group acting on a Kahler manifold (Z,ω,I). Let h be a invariant inner product on $Lie(G)$ (which always exists when G is compact) and let $\nu : Z\to Lie(G)$ be a moment map for the action of G, where we identify $Lie(G)$ with $Lie(G)^*$ using h.

For $z\in Z$, let $\sigma_z : Lie(G)\to TZ_z$ be the infinitesimal action: $\sigma_z(\xi)=\frac{d}{dt}|_{t=0}\exp(t\xi)\cdot z$. Let $I: TZ\to TZ$ be the complex structure. Then to say that ν is a moment map is to say:

$$\langle d\nu(w),\xi\rangle_{Lie(G)} \;=\; \langle w, I\sigma_z(\xi)\rangle_{TZ_z} \tag{5.1}$$

for all $w\in TZ_z$ and all $\xi\in Lie(G)$. Replacing w by Iw, and using the fact that the metric is invariant under I, we can restate (5.1) as follows:

$$d\nu\circ I \;=\; \sigma^* \tag{5.2}$$

Using the relation $g_{ij}=\omega_{ik}I^k_j$, we can rewrite (5.1) using indices as follows:

$$\nu^\alpha_j h_{\alpha\beta} \;=\; \sigma^i_\beta\omega_{ij} \tag{5.3}$$

Uniqueness of moment map zeros.

Let $Lie(G)^c = Lie(G)\otimes\mathbb{C}$ be the complexified Lie algebra. Let G^c be the associated complexified group. Then $Lie(G^c)=Lie(G)^{\mathbb{C}}$ and $G\subseteq G^c$ is a maximal compact subgroup.

If G is the set of real points of a linear algebraic group over $\mathbb{R}$ (i.e., a subgroup of $GL(n)$ defined by polynomial equations with coefficients in $\mathbb{R}$), then G^c is just the set of complex points. For example,

$$U(n) \;=\; \{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in M_n(\mathbb{R}),\; a=d, b=-c,\; a^ta+b^tb=I, b^ta=a^tb\}$$

where we identify $u=a+bi\in U(n)$ with the $2n\times 2n$ matrix $\begin{pmatrix} a & b\\ -b & a\end{pmatrix}$. Thus

$$U(n)^c \;=\; \{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in M_n(\mathbb{C}),\; a=d, b=-c,\; a^ta+b^tb=I, b^ta=a^tb\}$$

where $M_n(\mathbb{R})$ (*resp.* $M_n(\mathbb{R})$) is the set of $n\times n$ matrices with entries in $\mathbb{R}$ (*resp.* $\mathbb{C}$). Note that $U(n)^c\approx GL(n,\mathbb{C})$ via the map $\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\mapsto a+bi$. Similarly $SU(n)^c=SL(n,\mathbb{C})$.

The action of G on (Z, ω) extends to an action of G^c on Z (which no longer preserves ω). To see this, we first extend infinitsimally by defining

$$\sigma_z(\xi_1 + i\xi_2) \ = \ \sigma_z(\xi_1) \ + \ I\sigma_z(\xi_2) \tag{5.4}$$

and then the action of $g = \exp(\xi_1 + i\xi_2)$ is obtained by integrating (5.4).

Let $z \in Z$. We are interested in the solutions to the moment map equation:

$$\nu(gz) = 0$$

Note that by the equivariance of ν, if $\nu(z) = 0$ then $\nu(gz) = 0$ for all $g \in G$.

The solution to the moment map equation may not always exist, but if it does, it is unique modulo the action of G:

Lemma (uniqueness of moment map zero) *Let $z \in Z$, and assume $\nu(z) = 0$.*

1. Assume that $\nu(gz) = 0$, for some $g \in G^c$. Then we can factor g as follows:

$$g = g_0 \exp(i\xi)$$

were $g_0 \in G$, $\xi \in Lie(G)$ and $\exp(it\xi) \cdot z = z$ for all $t \in \mathbb{R}$.

2. If $\nu(gz) = 0$, for some $g \in G^c$, then $z = g_o z$ for some $g_o \in G$.

3. We have the following isomorphism of discrete groups:

$$\frac{z^G}{(z^G)^0} \ \approx \ \frac{z^{G^c}}{(z^{G^c})^0}$$

where H^0 denotes the connected component of the identity of a topological group H, and $z^G = \{g \in G : gz = z\}$.

Proof. We start with the proof of statement 1. Define Q_z, and endomorphism of $Lie(G)$, as follows:

$$Q_z \ = \ \sigma_z^* \sigma_z \ = \ d\nu \circ I \circ \sigma_z$$

Assume $\nu(z) = \nu(gz) = 0$ for some $g \in G^c$. Write $g = g_o \exp(i\xi)$ for some $g_o \in G$ and some $\xi \in Lie(G)$. Then we have $\nu(z) = \nu(g_o \exp(i\xi)z) = \nu(\exp(i\xi)z) = 0$. Let $z(t) = \exp(it\xi) \cdot z$. Then

$$z'(t) \ = \ I\sigma_{z(t)}(\xi) \ = \ \sigma_{z(t)}(i\xi) \tag{5.5}$$

Define

$$f(t) \ = \ \langle \xi, \nu(z(t)) \rangle$$

Then

$$f'(t) \ = \langle \xi, d\nu(I\sigma_{z(t)}(\xi)\rangle \ = \ \langle \xi, Q_{z(t)}\xi \rangle \ = \ \langle \sigma_{z(t)}\xi, \sigma_{z(t)}\xi \rangle \tag{5.6}$$

where in the first equality we've made use of (5.2).

Now we are assuming $f(0) = f(1) = 0$. Equation (5.6) implies $\sigma_{z(t)}\xi = 0$ for all t. Thus, by (5.5), we have $z'(t) = 0$ for all t, which shows that $z(t)$ is constant, and this proves part one. Parts two and three are immediate consequences of part one.

We can rephrase part two as follows: Let $Z^s \subseteq Z$ denote the set of stable elements, that is, those elements whose complex orbits meet the set $\nu(z) = 0$. Then

$$Z//G \;=\; \frac{\{z \in Z : \nu(z) = 0\}}{G} \;=\; \frac{Z^s}{G^c} \tag{5.7}$$

where, for the moment, we view (5.7) is a bijections of sets.

The gradient flow and existence of moment map zero.

The solutions to the equation $\nu(g \cdot z) = 0$, are the same as the solutons to $\phi(g \cdot z) = 0$, where $\phi(z) = |\nu(z)|^2$. We shall try to solve this equation by flowing along the descending gradient lines of ϕ.

First we compute the gradient of ϕ: Since $\phi : Z \to \mathbb{R}$, we have $\mathrm{grad}_z\phi \in T_zZ$. We claim:

$$\mathrm{grad}_z\phi \;=\; 2I\sigma_z(\nu(z)) \tag{5.8}$$

To see this, we start with the defintion: $\phi(z) = \langle \nu(z), \nu(z)\rangle_{Lie(G)}$. Thus, for $w \in T_zZ$,

$$\langle \mathrm{grad}_z\phi, w\rangle_{TZ} \;=\; d\phi(w) \;=\; 2\langle d\nu, \nu\rangle_{Lie(G)}(w) \;=\; 2\langle d\nu(w), \nu\rangle_{Lie(G)} \;=\; 2\langle w, I\sigma_z(\nu(z))\rangle_{TZ}$$

The first equality is the definiton of $\mathrm{grad}_z\phi$. Since this holds for all w, we obtain (5.8).

Remark: Equation (5.8) shows that if $\mathrm{grad}_z\phi = 0$, then either $\nu(z) = 0$ or z has a non-discrete stablizer group (that is, there is a one paramenter subgroup of G fixing z). Thus, if we assume that all stabilizers are discrete, the critical points of ϕ are in 1-1 correspondence with the zeros of the moment map.

Using equation (5.8), we see that the descending gradient flow equation is:

$$\frac{dz}{dt} \;=\; -I\sigma_z(\nu(z));\;\; z(0) = z_0\ . \tag{5.9}$$

We also consider the lifted equation

$$\frac{d\xi}{dt} \;=\; \nu(\exp(i\xi(t)) \cdot z_0);\quad \xi(0) = 0\ . \tag{5.10}$$

where $\xi : \mathbb{R} \to Lie(G)$ is an unknown fucntion.

Lemma on the gradient flow.

1. *Equation (5.9) preserves the G^c orbits, that is, $z(t) \in \Gamma = G^c z_0$ for all t.*
2. *Equations (5.9) and (5.10) have solutions $z(t)$ and $\xi(t)$ which exists for all $t \in \mathbb{R}$.*
3. *a) If the flow $\xi(t)$ has an accumulation point $\xi_1 \in Lie(G)$, then $z_1 = \exp(i\xi_1) \cdot z_0$ is a critical point of ϕ.*
   b) Moreover, if the points of Γ have discrete stabilizers, then z_1 is the unique (modulo G action) zero of the moment map and $\lim_{t\to\infty} \xi(t) = \xi_1$, $\lim_{t\to\infty} z(t) = z_1$.
4. *If V is compact, then the flow lines converge to the critical set of ϕ.*

Proof. Part 1. follows from the simple equation: $-I\sigma_z(\nu(z)) = \sigma_z(-i\nu(z))$ which says that the tangent line of the flow stays inside the tangent space of the G^c orbit.

More explicitly, let's consider the lifted equation

$$\frac{d\xi}{dt} \;=\; \nu(\exp(i\xi(t))\cdot z_0); \quad \xi(0) = 0 \;. \tag{5.10}$$

where $\xi : \mathbb{R} \to Lie(G)$ is an unknown fucntion. Since $\nu(\exp(i\xi)\cdot z_o)$ is a smooth vector field on $Lie(G)$, equation (5.10) has a solution ξ which exists for t is some interval $[0, T_\infty)$. Choose T_∞ to be maximal. Thus $T_\infty \in (0, \infty]$.

Let $z(t) = \exp(\xi(t))\cdot z_0$. Then we clearly have $z(t) \in G^c z_0$ for all t. We claim that $z(t)$ is a solution (and thus the unique solution) to (5.9). To see this, we differentiate:

$$\frac{dz}{dt} \;=\; \sigma_{z(t)}(\xi'(t)) \;=\; \sigma_{z(t)}\big(-i\nu(z(t))\big) \;=\; -I\sigma_{z(t)}(\nu(z(t))$$

which shows that $z(t)$ is a solution to (5.9).

To finish the proof of part one and two of the lemma, we must show that $T_\infty = \infty$: Since $|\nu(z(t)|$ is a decreasing function, we see that $|\xi'(t)|$ is a bounded function. Thus, if $T_\infty < \infty$, the curve $\xi : [0, T_\infty)$ has bounded lenth. Thus $\lim_{t\to T_\infty} \xi(t) = \xi_1$ exists. But now the gradient flow (5.10) with initial condition $\xi(T_\infty) = \xi_1$ has a smooth solution on some interval $(T_\infty - \epsilon, T_\infty + \epsilon)$ which patches together with the solution $\xi : [0, T_\infty) \to V$ to give a solution on $[0, T_\infty + \epsilon)$, contradicting the maximality of T_∞.

Now we prove part three: If $\xi(t_n)$ converges to $\xi_1 \in Lie(G)$, then we wish to show that $\text{grad}_{z_1}\phi = 0$, where $z_1 = \exp(i\xi_1)$. First observe that $|\nu(t)|$ is decreasing, and thus has a limit $c \geq 0$. If $c = 0$, then $\nu(z_1) = 0$, and thus z_1 is a critical point of ϕ. Assume therefore that $c > 0$, and fix $\epsilon > 0$. Then $|\text{grad}_{t_n}| > \epsilon$ implies that $|\text{grad}_t| = |\sigma_{z(t)}(\nu(z(t)))| > \epsilon/2$ on some interval $[t_n, t_n + \delta]$, where δ depends only on ϵ (Reason: $|\text{grad}\phi| = |\sigma_z(\nu(z))|$ is a uniformly conintuous function of z on compact sets. Since $|\nu(z(t))|$ is decreasing, it's bounded above, which means that the velocity of $\xi(t)$ is bounded above. So $\xi(t)$ stays close to $\xi(t_n)$ for a bounded amount of time. For that time interval, $\sigma_{z(t)}$ is bounded above which means that $z'(t)$ is bounded above. Thus $z(t)$ stays close to $z(t_n)$ for a bounded amount of time depending on ϵ).

Now observe that if $z(t)$ is a solution to (5.9), then

$$\int_0^T \left|\frac{dz}{dt}\right|^2 \, dt \;=\; \int_0^T |\text{grad}\phi|^2 \, dt \;=\; -\int_0^T \frac{d}{dt}|\nu(z(t))|^2 \, dt \;=\; \nu(z(0)) - \nu(z(T)) \;\leq \nu(z(0))$$

The first equality follows from (5.9). The second from the fact that $-|\text{grad}\phi| = \frac{d\phi}{ds}$, where s is the arc length parameter, and the fact that $\frac{ds}{dt} = |z'(t)| = |\text{grad}\phi|$.

Thus $\left|\frac{dz}{dt}\right|$ is in $L^2\big([0, T_\infty)\big)$. But it might not be in L^1 (that is, $z(t)$ need not have bounded length).

Returning now to the proof of part three, the fact that $|\text{grad}_t \phi|$ is bounded in L^2 says that the set of n for which $|\text{grad}_t| = |\sigma_{z(t)}(\nu(z(t)))| > \epsilon/2$ on some interval $[t_n, t_n + \delta]$ is a finite set. But this shows that $\text{grad}_{t_n} \phi \to 0$, which completes the proof of part 3a).

To prove 3b): We have already seen that if the stabilizers are discrete, then every critical point of ϕ is a moment map zero. And the uniqueness of z_1 modulo G action has also been proved. To prove $\lim \xi(t) = \xi_1$, let A be an annulus cenetered at ξ with finite inner and outer radius. Then $\xi^{-1}(A)$ is a disjoint union of bounded intervals. If this union is infinite for every A, then every A contains an accumulation point ξ_A which, by uniqueness, must satisfy $\exp(i\xi_A) \cdot z_0 = g_A \exp(i\xi_1) \cdot z_0$ for some $g_A \in G$. But the $\xi_A \to \xi_1$ as we take smaller and smaller annuli. Thus, for every pair A, B we have

$$g_A^{-1} g_B \exp(i(\xi_B - \xi_1)) \cdot z_0 \;=\; \exp(i(\xi_A - \xi_1)) \cdot z_0 \qquad (*)$$

Since G is compact, we can choose a family of pairs such that $g_A^{-1} g_B \to 1$. But then (*) together with the assumption that the stabilizer of z_0 is discrete, implies that $g_A = g_B$ and $\xi_A = \xi_B = \xi_1$. But the ξ_A are in different annuli, as A varies, so this is a contradiction. Thus we conclude that for some A, $\xi^{-1}(A)$ is a finite union of intervals. This shows that $\xi(t) \to \xi_1$ and the completes the proof of part 3.

The proof of part 4 is similar, and we omit it.

Next we prove Proposition 17, which guarantees the existence of a solution to the moment map equation under the assumption that the operator Q_z is bounded below.

Assume that the stabilizers of all points under the G action are discrete. This implies that σ_z is injective for all z and thus Q_z in invertible for all $z \in Z$. In otherwords, the eigenvalues of Q, which are all non-negative real numbers, are strictly positive. Let Λ_z be the operator norm of $Q_z^{-1} : Lie(G) \to Lie(G)$, defined using the metric h on Lie(G). Thus Λ_z, the inverse of the smallest eigenvalue of Q_z, is a positive continuous function on Z.

Proposition 17. *Let $z_0 \in Z$ and let $\delta > 0$. Assume that $\Lambda_z \leq 1$ for all $z = \exp(i\xi)$ with $|\xi| \leq \delta$. Suppose that $|\nu(z_0)| < \delta$. Then there is a point $w = \exp(i\eta) \cdot z_0$ with $|\eta| < \delta$ and $\nu(w) = 0$.*

Proof. Let $\xi(t)$ be the solution to (5.10). Let $s : [0, \infty) \to \mathbb{R}$ be the arc length function: Thus $s(T) = \int_0^T |\xi'(t)| dt$. Since s is an increasing function of t, there are two possibilities: A) $\lim_{t\to\infty} s(t) < \delta$ or B) $\lim_{t\to\infty} s(t) \geq \delta$.

If A) holds, let $\eta = \lim_{t\to\infty} \xi(t)$ and let $w = \exp(\eta) \cdot z_0$. Then we must have $\nu(w) = 0$, for otherwise, $|\xi'(t)| = |\nu(\exp(\xi(t) \cdot z_0)| \to |\nu(w)| > 0$ as $t \to \infty$ which implies that $\lim_{t\to\infty} s(t) = \int_0^\infty |\xi'(t)| dt = \infty$. This contradicts A), so we conclude that $\nu(w) = 0$. Moreover, $|\eta| < \delta$, so the proposition is proved if A occurs.

Thus we may assume that B) holds. Then, letting $z(t) = \exp(\xi(t)) \cdot z_0$ and $\nu(t) = \nu(z(t))$,

$$\frac{d}{ds}|\nu(t)| \;=\; \frac{d}{ds}\langle \nu(t), \nu(t)\rangle^{\frac{1}{2}} \;=\; \frac{1}{2}\frac{2\langle d\nu(\sigma(-i\nu(t), \nu(t)\rangle}{|\nu(t)|} \cdot \frac{dt}{ds}$$

where we use (5.9) to establish the second equality. Now $\frac{ds}{dt} = |\nu(t)|$. Equation (5.2) yields:

$$-\frac{d}{ds}|\nu(t)| \;=\; \frac{\langle Q_z(\xi)\nu, \nu\rangle}{\langle \nu, \nu\rangle} \geq 1 \qquad (5.11)$$

provided $s \leq \delta$. Here we are using the assumption that the smallest eigenvalue of Q_z is at least one, inside the closed ball of radius δ. But we are assuming that $|\nu(0)| < \delta$. Hence (5.11) shows that $|\nu(t)| = 0$ for some $s = s(t) < \delta$, and this proves the proposition.

§6. The symplectic quotients.

Whenever a Lie group H acts on a symplectic manifold (W, ω), one can ask for the existence of an equivariant moment map μ. If such a μ exists, and if the action is discrete, then one can construct the symplectic quotient, $W//H$, which is defined by $W//H = \mu^{-1}(0)/H$. It turns out that $W//H$ has a natural structure of symplectic manifold. If W has additional structure (eg a line bundle compatible with ω, or a complex structure compatible with ω), then the symplectic quotient also has the same additional sturcture. We consturct the symplectic quotient in this section, and we show how the additional structure descends to the symplectic quotient.

The symplectic quotient of (W, ω).

We start with the simplest setting: Let H act on a symplectic manifold (W, ω). There may not be a moment map for the action of H on W, but if one exists, it is essentially unique:

Theorem (uniqueness of moment maps)

1. If μ_H is a moment map for the action of H, then the set of all moment maps is the set $\mu_H + c$, where c ranges over all elements of $[Lie(H), Lie(H)]^0$ the set of elements in $Lie(H)^*$ which kills $[Lie(H), Lie(H)]$.

2. If H is semi-simple (by defintion, this means that we have $[Lie(H), Lie(H)] = Lie(H)$) then there exists a unique moment map μ_H for the action of H.

Examples of semi-simple Lie groups are $SU(n), SO(n), Sp(n), SL(n)$, for $n \geq 2$, and any product of such groups.

We are interested in the sets $\mu^{-1}(0)$, where μ ranges over the moment maps for H. By the theorem above, this is equivalent to fixing a moment map μ, and considering the sets $\mu^{-1}(c)$ where $c \in [Lie(H), Lie(H)]^0$. Let $W//H = \mu^{-1}(c)/H$.

If H acts freely, then it turns out that $W//H$ has the structure of a smooth symplectic manifold, known as the "symplectic quotient" of W:

Theorem (Marsden-Weinstein-Meyer). *Assume that H is compact and that it acts freely on $\mu^{-1}(c)$.*

1. The set $\mu^{-1}(c) \subseteq W$ is a smooth manifold.

2. The set $W//H$ has the structure of a smooth manifold and the map $\pi : \mu^{-1}(c) \to W//H$ is a smooth principal H bundle.

3. There is a symplectic form ω_{red} on $W//H$ with the property $\pi^*\omega_{red} = \iota^*\omega$ where ι is the inclusion map: $\iota : \mu^{-1}(c) \hookrightarrow W$.

Proof of parts one and three. Let $w \in \mu^{-1}(c)$. Then $d\mu : T_wW \to Lie(H)^*$.

Claim: $\text{codim}(Ker(d\mu)) = dim(H)$.

To see this, recall the moment map conditon:

$$d\mu(\xi)(Y) \;=\; \omega(X_\xi, Y) \tag{6.2}$$

for all $\xi \in Lie(H)$ and all $Y \in T_wW$. Thus $Y \in Ker(d\mu)$ if and only if $0 = d\mu(\xi)(Y) = \omega(X_\xi, Y)$ for all $\xi \in Lie(H)$. Since H acts freely the X_ξ span a subspace of T_wW of dimenson $\dim H$. The claim now follows from the assumption that ω is non-degenerate. We now see that $d\mu$ has maximal rank at all points of $\mu^{-1}(c)$ which implies that $\mu^{-1}(c)$ is a smooth manifold, and completes the proof of part one.

Next we prove part three: Consider the linear maps

$$Lie(H) \;\to \sigma \; T_sW \;\to d\mu \; Lie(H)^*$$

where σ is the map $\xi \mapsto X_\xi$. Since μ is constant on the orbit of w, we have $im(\sigma) \subseteq ker(d\mu)$.

Claim: $ker(d\mu) \;=\; ann(im(\sigma))$ where

$$ann(im(\sigma)) \;=\; \{Y \in T_wW : \omega(X_\xi, Y) \;\text{ for all } \xi \in Lie(H)\;\} \tag{6.2a}$$

is the annihilator of the image of σ.

To prove the claim, observe first that (6.2) implies that $ker(d\mu) \subseteq ann(im(\sigma))$. Furthermore, we have seen that $\text{codim}(ker(d\mu)) = \dim(H)$. On the other hand, since H acts freely, $\dim(im(\sigma)) = \dim(H)$. Since ω is non-degenerate, $\text{codim}\,[ann(im(\sigma)] = \dim(H)$.

Now the claim implies that we have a non-degenerate pairing induced by ω:

$$\frac{ker(d\mu)}{im(\sigma)} \times \frac{ker(d\mu)}{im(\sigma)} \;\;\to \omega_{red} \;\; \mathbb{R}$$

On the other hand, we have a canonical isomorphism

$$T_{[w]}\,(W//H) \;=\; T_{[w]}\left(\mu^{-1}(c)/H\right) \;=\; \frac{ker(d\mu)}{im(\sigma)} \tag{6.3}$$

where $[w] = wH \subseteq \mu^{-1}(c)/H$. This proves part three.

Remark: We are assuming in the above that H acts freely. If we only assume that the stabilizers are finite, then the above proof shows that $\mu^{-1}(c)$ is still a smooth manifold and that $W//H$ is a symplectic *orbifold.*

Example 1. The simplest example is $H = U(1)$, $W = \mathbb{C}^{N+1}$, $\omega = -i\sum dw_j \wedge d\bar{w}_j$. Then $Lie(U(1)) = i\mathbb{R}$ so $Lie(U(1))^* = i\mathbb{R}$ where the pairing sends $(ix, iy) \mapsto xy$. Define $\mu : \mathbb{C}^{N+1} \to i\mathbb{R}$ to be the map $\mu(w) = i\sum |w_j|^2$. Then we claim μ is a moment map. To see this, let $\xi = ix \in Lie(U(1))$. We must show

$$d\mu(\xi)(Y) \ = \ \omega(X_\xi, Y) \ = \ (\iota_{X_\xi}\omega)(Y)$$

for every tangent vector Y, where X_ξ is the infinitesimal action of ξ, that is, identifying the complexification of the tangent space of $\mathbb{C}^{N+1}$ with $\mathbb{C}^{N+1} \oplus \mathbb{C}^{N+1}$: $X_\xi(w) = ix \cdot w - ix \cdot \bar{w}$. Thus $\iota_{X_\xi}\omega = x\sum(w_j d\bar{w}_j + \bar{w}_j dw_j)$. On the other hand, $\mu(\xi) = x\sum |w_j|^2$. Thus we see $d\mu(\xi) = \iota_{X_\xi}\omega$.

Now let $c = i$, so that

$$\mu^{-1}(c) = \{(w_0, ..., w_N) \in \mathbb{C}^{N+1}\backslash\{0\} : \sum |w_i|^2 = 1\}$$

and $W//H = \mathbb{C}P^N$, where ω_{red} is the Fubini-Study form.

Example 2. Let (V, h) be a hermitian vector space and let

$$W \ = \ \{ \ \underline{s} = (s_0, ..., s_N) : \text{ the } s_i \text{ form a basis of } V\}$$

Then $T_{\underline{s}}(W) = \{\underline{\sigma} = (\sigma_0, ..., \sigma_N) : \sigma \in V \ \}$. Define $\omega(\underline{\sigma}, \underline{\sigma}') \ = \ \sum_j Im\langle\sigma_j, \sigma_j'\rangle$ Then ω is a symplectic form on W, and $U(h)$, the unitary group of h, acts on (W, ω).

Define $\mu : W \to u(N+1)$ by

$$\mu(\underline{s}) \ = \ \frac{1}{2}i\langle s_\alpha, s_\beta\rangle_h$$

where we identify $u(h) = u(N+1)$ and $u(h)^* = u(N+1)^*$. The first identification is via the basis $\underline{s}$, and the second via the invariant pairing on $u(N+1)$ given by the formula $\langle A, B\rangle_{u(N+1)} = Tr(AB^*) = -Tr(AB)$.

Then we claim μ is a moment map. To see this, let $\xi = iA \in u(N+1)$. Then

$$d\mu(\xi)_{\underline{s}}(\underline{\sigma}) \ = \ \lim_{t\to 0} \frac{\mu(\xi)(\underline{s} + t\underline{\sigma}) - \mu(\xi)(\underline{s})}{t} \ = \ \frac{1}{2}\sum \left(\langle\underline{\sigma}_\alpha, \underline{s}_\beta\rangle + \langle\underline{s}_\alpha, \underline{\sigma}_\beta\rangle\right) \bar{A}_{\alpha\beta} \ = \ \sum_{\alpha,\beta} Re\langle\sigma_\alpha, A_{\alpha\beta}s_\beta\rangle$$

On the other hand, $X_\xi = i\sum A_{\alpha\beta}s_\beta$ so

$$\omega(X_\sigma, \sigma) \ = \sum_\alpha Im\langle iA_{\alpha\beta}s_\beta, \sigma_\alpha\rangle \ = \ \sum_{\alpha,\beta} Re\langle\sigma_\alpha, A_{\alpha\beta}s_\beta\rangle$$

Now the annihilator of $[Lie(H), Lie(H)]$ is the set of scalar diagonal elements. Let $c = i\lambda$ where $\lambda \in \mathbb{R}$. Now $\mu^{-1}(0) = 0$, so the action of H is not free. So let's consider the case

$c \neq 0$: $\mu^{-1}(2i)$ is the set of bases which are orthonormal. Since any two orthonormal bases are in the same H orbit, we see that $W//H$ is a single point in this case.

Symplectic quotient of $(W, \omega; L, h, A)$

Assume, as above, that we are given an action of a compact Lie group H on a symplectic manifold (W, ω) and an equivariant moment map μ , such that H acts freely on W.

Assume as well that we are given a hermitian complex line bundle on W, with unitary connection (L, h, A), compatible with ω, that is, assume that the property

$$F_A \ = \ -i\omega$$

Then the action of H on (W, ω) extends to an action of H on (L, h, A), covering the action on W (and preserving h and A).

To see this, let $\mathcal{G}_0 = Aut(W, \omega)$ and let $\mathcal{G} = Aut(L, h, \omega)$. Then we proved in §3 that $Lie(\mathcal{G}_0) = C^\infty(W)/\mathbb{R}$ and $Lie(\mathcal{G}) = C^\infty(W)$ We are given a homomorphism $H \to \mathcal{G}_0$, which, on the level of Lie algebras is the map $\xi \mapsto H_\xi$, where H_ξ is the unique element of $C^\infty(W)/\mathbb{R}$ whose symplectic gradient is the vector field $\sigma_w(\xi)$ (the inifinitesimal action).

Now define $Lie(H) \to C^\infty(W)$ by the rule: $\xi \to \mu(w)(\xi)$, where $\mu : W \to Lie(H)^*$ is the moment map. This is clearly a lift of the map $Lie(H) \to C^\infty(W)/\mathbb{R}$, and thus defines a lifted homorphism $H \to \mathcal{G}$.

Now $L|_{\mu^{-1}(0)}$ is a line bundle on a smooth manifold together with an H action. Now let $U \subseteq W//H$ be an open subset and let $\pi : \mu^{-1}(0) \to W//H$ be the canonical quotient map. Then we define a line bundle L_{red} on $W//H$ as follows: Let $s : U \to \mu^{-1}(0)$ be any section of π, viewed as a principal H bundle (which exists, provided U is sufficiently small). Then $L_{red}|_U = s^*L$. Since there is a canonical isomorphism $s^*L = s'^*L$ for any sections s, s', this is well defined and patches together to define L_{red}. Thus we have

$$L_{red}(U) = \ \{s \in L(\pi^{-1}(U)) : s \text{ is } H \text{ invariant } \}$$

in other words, $s \in L_{red}(U)$ is a section $s : \pi^{-1}(U) \to L$ satisfying $s(hw) = \rho(h)s(w)$ where ρ is the canonical isomorphism $\rho : L \to h^*L$. Sometimes we refer to ρ as a "factor of automorphy".

The metric and the connection clearly descend as well. Thus $(L_{red}, h_{red}, A_{red})$ is a hermitian line bundle with connection on $(W//H, \omega_{red})$.

In example 1 of the previous section, $H = U(1)$, $W = \mathbb{C}^{N+1}$, and $\omega = -i \sum dz_j \wedge d\bar{z}_j$. Let L be the trivial line bundle $\mathbb{C}^{N+1} \times \mathbb{C}$, with the metric: $|(w, z)| = e^{-|w|^2}|z|$. The curvature of the metric is ω. Now $U(1)$ acts on L: If $\exp(ix) \in U(1)$ then $\exp(ix) \cdot (w, z) = (\exp(ix)w, \exp(ix)z)$. The action clearly preserves the metric and the connection (which we take to be the trivial connection).

Now we compute the quotient: $\mu^{-1}(c) = S^{2n+1} = \{w : |w| = 1\}$. The sections of L_{red} are the functions f on S^{2n+1} which transform by the rule: $f(\zeta w) = \zeta f(w)$ for $\zeta \in U(1)$. The principal $U(1)$ bundle $S^{2n+1} \to \mathbb{C}P^n = S^{2n}$ is the standard Hopf bundle, and the line bundle L_{red} is the Hopf line bundle on complex projective space.

The linear maps $f : \mathbb{C}^{N+1} \to \mathbb{C}$ are all global sections of L_{red}. If f is such a linear map, then $||f([z])|| = \frac{|f(z)|}{|z|}$ where $z \in \mathbb{C}^{N+1}$ is any representative of $[z] \in \mathbb{C}P^N$.

Symplectic Quotient of a Kahler Manifold.

Let H be a compact Lie group acting freely (*resp.* with finite stabilizers) on a symplectic manifold (W, ω) and fix an equivariant moment map $\mu : W \to Lie(H)^*$. Then we saw in the previous section that

$$(W//H, \omega_{red})$$

is a symplectic manifold (*resp.* orbifold) where $W//H = \mu^{-1}(0)/H$. Moreover, the map $\mu^{-1}(0) \to W//H$ is a principal H bundle.

Lemma on the Kahler quotient. *Assume that (W, ω, I) is a Kahler manifold and that H acts freely on W.*

1. Then $Z = W//H$ has a natural almost complex structure I_{red}, compatible with ω_{red}.

2. If $z = Hw \in Z$ for some $w \in \mu^{-1}(c)$, we have a canonical isomorphism of complex vector spaces:

$$T_wW/T_w(H^cw) \;=\; T_zZ \tag{6.5}$$

(in fact, this is how I_{red} is defined). This gives us a canonical isomorphism

$$T_zZ \;=\; T_w(H^cw)^{\perp} \;\subseteq\; T_wW \tag{6.6}$$

where the hermitian inner product on T_wW is the one defined by ω and I: For $u, v \in T_wW$,

$$\langle u, v\rangle \;=\; \omega(u, Iv) \;+\; i\omega(u, v) \;=\; g(u, v) \;+\; i\omega(u, v) \tag{6.7}$$

(where g is the Riemannian metric). Moreover, the restriction of $\langle , \rangle$ to T_zZ gives a hermitian structure on T_zZ whose imaginary part is ω_{red}.

3. Assume the action of H extends to an action of H^c on W^s via biholomorphic maps. Then $(Z, \omega_{red}, I_{red})$ is a Kahler manifold.

Proof. We start with the proof of statements one and two.

Claim:

$$T_{[w]}(W//H) \;=\; \frac{ker(d\mu)}{im(\sigma)} \;=\; \{u \in T_wW : \langle u, \sigma(\xi)\rangle = 0 \text{ for all } \xi \in Lie(H^c)\ \} \tag{6.8}$$

Proof of claim. The first equality is (6.3). We thus have a canonical identification:

$$T_{[w]}(W//H) \;=\; \{u \in ker(d\mu) : g(u, \sigma(\xi)) = 0 \text{ for all } \xi \in Lie(H)\ \}$$

This follows from the fact that g is a metric on the real vector space $ker(d\mu)$. On the other hand, (6.2a) implies

$$ker(d\mu) \;=\; \{u \in T_wW : \omega(u, \sigma(\xi)) = 0 \text{ for all } \xi \in Lie(H)\ \}$$

Thus (6.7) implies (6.8), and the claim is proved. Since the right side of (6.8) is clearly a complex subspace of T_wW, we have defined an almost complex structure I_{red} on T_wW, which is clearly compatible with ω_{red}. This proves parts one and two.

Now we sketch the proof of part three: (which I don't fully understand): First observe that H^c has no continuous isotropy groups (since $\sigma(i\xi) = I\sigma(\xi)$). Thus (5.4a) implies that H^c acts freely on W. In particular, the orbits H^cw are smooth complex manifolds (isomorphic to H^c).

Now for $w \in \mu^{-1}(0)$ relation (6.5) implies

$$T_{[w]}(W//H) \;=\; (T_wW)/T_w(H^cw) \tag{6.9}$$

One way to prove statement 3 is to show directly (by showing that the Nijenhuis tensor vanishes) that the complex structure induced by (6.9) is integrable.

Another way to prove 3 is to use the fact that was proved earlier:

$$W//H \;=\; W^s/H^c \tag{6.10}$$

where $W^s \subseteq W$ is the subset of points $w \in W$ for which there is a zero of the moment map in the orbit H^cw. It turns out that W^s is always open (I'm not sure why). Thus W^s/H^c has a natural complex structure (I'm not sure why. I guess: If $U \subseteq W//H = W^s/H^c$, then a smooth function $f : U \to \mathbb{C}$ is holomorphic if and only if $\pi \circ f$ is a holomorphic function on U, where $\pi : W^s \to W^s/H^c$ is the canonical quotient map. But it's not clear that there exist non-constant holomorphic functions...).

Finally we prove 3b): Let $\xi \in Lie(H)$, and $z = H^cw \in Z$ where $\mu(w) = 0$. The $\sigma(i\xi) = I\sigma(\xi) \in T_wW$ is the infinitesimal action of $i\xi$ on w. Statement 3b) says that

$$\pi(\sigma(i\xi)) \;=\; \pi(I\sigma(\xi)) \;=\; I_{red}\sigma(\xi) \tag{$*$}$$

where

$$\pi : T_wW \to T_zZ$$

is the projection map. But the projection map is a map of complex vector spaces (by definition of I_{red}). Thus $\pi I = I_{red}\pi$. This proves (*).

In the example above, we give $W = \mathbb{C}^{N+1}$ the usual complex structure. Then W^s is the set of non-zero points, $H^c = \mathbb{C}^\times$ and every H^c orbit meets S^{2n+1} uniquely up to the action of H. If $z \in \mathbb{C}^{N+1}\backslash 0$, the tangent space at a point $[z] \in \mathbb{C}P^N$ is $z^\perp$ and the Fubini-Study metric is the euclidean metric on $z^\perp$.

The action of $G \times H$

Let G and H be Lie groups acting on a symplectic manifold (W, ω) which commute, in other words, assume that we are given an action of $G \times H$ on W. Let μ_G and μ_H be equivariant moment maps for the actions of G and H.

Observe that if $g \in G$ is a fixed element, then then $\mu_H(gw)$ is a moment map for the action of H. Thus $\mu_H(gw) - \mu_H(w)$ is a constant, depending on $g \in G$. We shall make the following assumptions:

$$\mu_H(gw) = \mu_H(w) \text{ and } \mu_G(hw) = \mu_G(w) \text{ for all } g \in G, h \in H \text{ and } w \in W \qquad (A)$$

For example, if G and H are semi-simple, then assumption (A) is automatic.

Let $Z_H = W//H$ and $Z_G = W//G$. Then (A) implies that G acts symplectically on Z_H and H act symplectically on Z_G. More precisely, if $w \in \mu_H^{-1}(0)$ so that $z = Hw \in Z_H$, and if $g_o \in G$, then

$$g_o z \;=\; g_o(Hw) \;=\; H(g_o w)$$

In other words:

$$\pi_o(g_o w) \;=\; g_o(\pi_o w)$$

where $\pi_o : \mu^{-1}(0) \to Z_H$ is the canonical quotient map.

Moreover, μ_G defines an equivariant moment map for the action of G on Z_H, and similarly for μ_H.

Now assume that (W, ω) is Kähler. In order to simplify the discussion, assume as well that every point in W is H stable and G stable. Then $Z_H = W/H^c$ and $Z_G = W/G^c$.

Now G acts on W and $Z = Z_H$, so G^c also acts on W and $Z = Z_H$ (whenever a compact group acts on a Kahler manifold, then the action extends to an action of G^c which is given, infinitesimally by the formula $\sigma(i\xi) \;=\; I\sigma(\xi)$). We wish to show that these two actions are compatible:

Lemma on the action of G^c on Z_H. *Let $\pi : W \to Z_H$ be the canoncial quotient map. Then*

$$\pi(gw) = g\pi(w)$$

for all $w \in \mu^{-1}(0)$ and $g \in G^c$.

Proof. It suffices to prove this infinitesimally: Thus, we must show that for $\xi \in Lie(G)$, the following formula holds:

$$\pi_*(\sigma_w(i\xi)) \;=\; I_{red}\sigma_z(\xi)$$

where $\pi_* : T_w W \to T_z Z$, $\sigma_w : Lie(G) \to T_w W$ is the infinitesimal action of G on $T_w W$ and $\sigma_z : Lie(G) \to T_z Z$ is the infinitesimal action of G on $T_z W$, where $z = \pi w$.

The uniqueness of moment map zeros lemma says that $\pi|_{\mu^{-1}(0)} \;=\; \pi_o$. Thus () implies

$$\pi_*(\sigma_w(\xi)) \;=\; \sigma_z(\xi) \qquad ()$$

for all $\xi \in Lie(G)$. On the other hand, $\sigma_w(i\xi) = I\sigma(\xi)$ (by definition). Thus () follows from () and the fact that $\pi_* I = I_{red}\pi$ (this is the defintion of I_{red}). This proves the lemma, and shows as well that () holds for all $\xi \in Lie(G^c)$.

§7. Zeros of the moment map: The line bundle point of view.

Let K be a compact Lie group acting on a Kahler manifold (V, ω, I). Then the complexified group satisfies $Lie(K^c) = Lie(K) \otimes \mathbb{C}$, and K^c acts on (V, I), preserving the complex structure, but not the Kahler form or the metric.

Let $\nu : V \to Lie(H)$ be an equivariant moment map, where we identify $Lie(H)$ with its dual via an invariant metric on $Lie(H)$. Let $\phi(x) = |\nu(x)|^2$. Then $\text{grad}_x \phi = 2\sigma(I\nu(x))$ and the gradient flow is:

$$\frac{dz}{dt} \;=\; -\sigma(I(\nu(x))\;;\; z(0) = z_0 \tag{7.1}$$

If $\xi(t)$ is a solution to

$$\frac{d\xi}{dt} \;=\; -\nu(\exp(i\xi(t)) \cdot z_0)\;;\; \xi(0) = 0 \tag{7.2}$$

then $z(t) = \exp(i\xi(t)) \cdot z_0$ is a solution to (7.1).

Let $\Gamma = H^c z_0$. We have seen that the flow $z(t)$ stays inside Γ. If the stabilizers of points in Γ are discrete, then either $z(t)$ has a limit in Γ, and the limit is the unique (up to G action) moment map zero, or $z(t)$ has no accumulation point in Γ.

A very useful way of interpreting the zeros of the moment map is via the norm on the line bundle $\mathcal{L}$: Let (L, h, A) be a complex hermitian line bundle with unitary connection on a Kahler manifold (V, ω, I). Let $\mathcal{L} = (L, I)$ be the holomorphic line bundle determined by I. Let $\nu : V \to Lie(K)$ be an equivariant moment map. Then ν allows us to lift the action of K^c to $\mathcal{L}$ as follows: If $\xi \in Lie(K)$ then the infinitesimal action of K on $\mathcal{L}$ is given by

$$\hat{\sigma}(\xi) \;=\; \widetilde{\sigma(\xi)} + \nu(\xi)\mathbf{t}$$

where $\mathbf{t}$ is the infinitesimal action of $U(1)$. This gives the infinitesimal action of K on $\mathcal{L}$ which integrates to an action of K. We thus get as well an action of K^c on $\mathcal{L}$:

$$\hat{\sigma}(\Xi) \;=\; [\sigma(\xi_1) + I\sigma(\xi_2)]^{\tilde{}} \;+\; [\nu(\xi_1) + i\nu(\xi_2)] \tag{7.3}$$

$\Xi = \xi_1 + i\xi_2 \in Lie(K^c)$ (where $\xi_1, \xi_2 \in Lie(K)$).

Now let $\tilde{\Gamma} \subseteq \mathcal{L}$ be a fixed orbit for K^c acting on $\mathcal{L}$. Then $\tilde{\Gamma}$ is a smooth manifold which lies over an orbit $\Gamma \subseteq V$ (also a smooth manifold). Define

$$h : \tilde{\Gamma} \to \mathbb{R}$$

by

$$h(\gamma) = \; -\log|\gamma|^2$$

Let $Q = K^c/K$ and fix $\gamma_o \in \tilde{\Gamma}$. Define

$$H : Q \to \mathbb{R}$$

by

$$H(g) \; = \; h(g \cdot \gamma_o)$$

Thus $\tilde{\Gamma} = H(Q)$.

Theorem.

1. If $\gamma \in \tilde{\Gamma}$, then γ is a critical point of h if and only if $\nu(\pi(\gamma)) = 0$.

2. For $\xi \in Lie(K)$ let $H_\xi(t) = H(\exp(it\xi))$. Then

$$H'_\xi(t) \; = \; 2\langle \nu(\exp(it\xi) \cdot x_o), \xi\rangle \tag{7.4}$$

and

$$H''_\xi(t) \; = \; 2\langle \sigma_x(\xi), \sigma_x(\xi)\rangle \tag{7.5}$$

where $x = x(t) = \exp(it\xi) \cdot \gamma_o$.

3. The gradient flow lines of H on Q map to the gradient flow lines of Φ on Γ.

Proof. If $\Xi \in Lie(K^c)$ then $\hat{\sigma}(\Xi)$ is a smooth vector field on $\tilde{\Gamma}$. We wish to compute the lie derivative $\mathcal{L}_{\hat{\sigma}(\Xi)} h$.

Claim:

$$(\mathcal{L}_{\hat{\sigma}(\Xi)} h)(\gamma) \; = \; 2\langle \nu(x), \xi_2\rangle \tag{7.6}$$

where $x = \pi(\gamma) \in V$ (here $\pi : L \to V$).

Proof. Clearly $\mathcal{L}_{\tilde{X}} h = 0$ for any vector field X on V (since $|\gamma|^2$ is infinitesimally constant in the horizontal direction). Thus, (7.3) implies

$$(\mathcal{L}_{\hat{\sigma}(\Xi)} h)(\gamma) \; = -\frac{d}{dt} \log \left| \exp(it[\nu(\xi_1) + i\nu(\xi_2)]\gamma \right|^2 \; = \; -\frac{d}{dt} \exp(-2t\nu(\xi_2))$$

which yields (7.6). Taking $\Xi = i\xi$ in (7.6) we get (7.4). Differentiating one more time we get

$$H''_\xi(t) \; = \; 2\langle d\nu(\sigma_x(i\xi)), \xi\rangle \; = \; 2\langle \sigma_x^* \sigma_x(\xi), \xi\rangle \; = \; 2\langle \sigma_x(\xi), \sigma(\xi)\rangle$$

and this proves (7.5). Now (7.4) says that $\text{grad}_g H = \nu(g \cdot x_o)$. Thus the gradient flow equation of H is just given by (7.2), whose solutions, as we have seen, map to the gradient flow of Φ. To prove statement one, observe that $\mathcal{L}_{\hat{\sigma}(\Xi)} h)(\gamma) \; = 0$ for all Ξ if and only if $\nu(x) = 0$.

Now g is a critical point of H, if and only if $g\gamma_o$ is a critical point of point of h. Thus a moment map zero exists if and only if H has a critical point. Moreover, (7.5) shows that the critical points of H are all global minima, and that the global minima are all in the isotropy group of the moment map zero. Thus, if the isotropy group is discrete, H can have at most one critical point.