SUMMARY OF SCALAR CURVARTURE AND PROJECTIVE EMBEDDINGS II

1. Introduction

In this paper Donaldson proves the following

Theorem 1. *Suppose* $\mathrm{Aut}(X, L)$ *is discrete and the there is a cscK metric* $\omega_\infty \in c_1(L)$. *Then* ω_∞ *minimizes* ν, *the Mabuchi K-energy in* $c_1(L)$.

We sketch the idea of the proof: Donaldson introduces a sequence of functionals $\mathcal{L}_k : \mathcal{H} \to \mathbb{R}$ which have the following properties:

a) If $h \in \mathcal{H}$ then $\mathcal{L}_k(h) \to \nu(h)$ as $k \to \infty$. In fact, if $h_k \in \mathcal{H}$ and $h_k \to h$ then $\mathcal{L}_k(h_k) \to \nu(h)$.
b) If h_k is k-balanced, then $\mathcal{L}_k$ reaches its minimum at h_k.

Property b) follows (indirectly) from the result of S. Zhang. And property a) follows from the Tian-Yau-Zelditch approximation theorem.

The hypothesis of the theorem implies (Donaldson's first paper) that there is a sequence of k-balanced metrics h_k such that $h_k \to h_\infty$, the cscK metric. Now assume a) and b). Let $h' \in \mathcal{H}$. Property b) says $\mathcal{L}_k(h_k) \leq \mathcal{L}_k(h')$. Now, taking the limit as $k \to \infty$, property a) implies $\nu(h_\infty) \leq \nu(h')$.

Before presenting the details, we sketch the proof of property b) which relies on another functional $Z : M \to \mathbb{R}$, where M is the space of hermitian metrics on the finite dimensional vector space $H^0(X, L)$ (this is essentially the F^0 functional, restricted to the Bergman space). It is not difficult to show that $\mathcal{L}(h') \geq Z(\mathrm{Hilb}(h'))$ for all $h' \in \mathcal{H}$, with equality if h' is balanced. On the other hand, (dropping the subscript k) if h is balanced, and if $H = \mathrm{Hilb}(h)$, Zhang's result says that $Z(H') \geq Z(H)$ for any $H' \in M$. Thus we have

$$\mathcal{L}(h') \geq Z(\mathrm{Hilb}(h')) \geq Z(\mathrm{Hilb}(h)) = \mathcal{L}(h)$$

1.1. The maps FS and $Hilb$.

Let

$$\mathcal{H} = \{h : \text{ hermitian metric on } L : \omega = -i\partial\bar{\partial}h > 0\} \tag{1}$$

$$M = \{H : \text{ hermitian metric on } H^0(X, L)\ \} \tag{2}$$

Define $FS : M \to \mathcal{H}$ by:

$$\sum_\alpha s_\alpha \bar{s}_\alpha \, FH(H) \;=\; 1 \;\text{ if }\; (s_\alpha) \text{ is } H\text{−orthonormal} \tag{3}$$

Define $Hilb : \mathcal{H} \to M$ by

$$Hilb(h)(s_\alpha, \bar{s}_\beta) \;=\; \frac{d}{V}\int_X s_\alpha \bar{s}_\beta \, h \, d\mu_h \tag{4}$$

where $d\mu_h = \omega^n/n!$.

1.2. **The Bergman approximation.** Let $h \in \mathcal{H}$. Define $h_b = FS(Hilb(h))$. Then h_b is called the Bergman approximation of h.

Proposition 1. *Let $h \in \mathcal{H}$ and define ϕ by $h = e^{-\phi} h_b$. Then*

$$\frac{1}{V}\int_X e^{-\phi} \, d\mu_h \;=\; 1 \tag{5}$$

We say that h is balanced if $h = h_b$, that is, if $\phi = 0$.

Proof of Proposition. Let (s_α) be a $Hilb(h)$-orthonormal basis so $Hilb(s_\alpha, \bar{s}_\beta) = \delta_{\alpha\beta}$. Then

$$d \;=\; \sum_\alpha Hilb(h)(s_\alpha, \bar{s}_\alpha) \;=\; \frac{d}{V}\int_X \sum_\alpha s_\alpha \bar{s}_\alpha h \, d\mu_h \;=\; \frac{d}{V}\int_X (\sum_\alpha s_\alpha \bar{s}_\alpha h_b) e^{-\phi} \, d\mu_h \;=\; \frac{d}{V}\int_X e^{-\phi} \, d\mu_h$$

1.3. **The Functional I.** Let $h_0 \in \mathcal{H}$ so $\mathcal{H} = \{\phi : \omega_0 + i\partial\bar{\partial}\phi > 0\}$. Define $I : \mathcal{H} \to \mathbb{R}$ by

$$\delta I \;=\; \int_X \delta \log h \, d\mu_h \;=\; -\int_X \delta\phi \, d\mu_h \tag{6}$$

Thus if ϕ_t is a path in $\mathcal{H}$ then

$$\frac{d}{dt} I(\phi_t) \;=\; -\int_X \dot{\phi} \, d\mu_h \;\text{ and }\; \frac{d^2}{dt^2} I(\phi_t) \;=\; -\int_X (\ddot{\phi} + \dot{\phi}\Delta\dot{\phi}) \, d\mu_h \tag{7}$$

Proposition 2. *Let $h_1, h_0 \in \mathcal{H}$ and write $h_1 = h_0 e^{-\phi}$. Then*

$$\int_X \phi \, d\mu_{h_0} \;\geq\; I(h_0) - I(h_1) \;\;\geq \int_X \phi \, d\mu_{h_1} \tag{8}$$

Proof. Let $h_t = h_0 e^{-t\phi}$ so $\phi_t = t\phi$. Then $\ddot{\phi} = 0$ so (7) implies $I(\phi_t)$ is convex. Thus

$$I(h_1) - I(h_0) \;=\; \int_0^1 \frac{d}{dt} I(\phi_t) \; dt$$

The integrand reaches its max at $t = 1$ and its min at $t = 0$. This proves the proposition.

Proposition 3. *Let $h \in \mathcal{H}$. Then*

$$I(h) \;\leq\; I(h_b) \tag{9}$$

Proof. Write $h = h_b e^{-\phi}$. Let $h_1 = h$ and $h_0 = h_b$. Then (5) implies (since log is concave)

$$\int_X \phi \, d\mu_{h_1} \ \geq \ 0$$

Thus (9) follows from (8).

1.4. **The functionals $\tilde{\mathcal{L}}$ and $\tilde{Z}$.** Define $\tilde{\mathcal{L}} : \mathcal{H} \to \mathbb{R}$ and $\tilde{Z} : M \to \mathbb{R}$ by

$$\begin{aligned} \frac{1}{d}\tilde{\mathcal{L}} &= -\frac{1}{V} I \ + \ \frac{1}{d}\log\det \circ Hilb \\ \frac{1}{V}\tilde{Z} &= -\frac{1}{V} I \circ FS \ + \ \frac{1}{d}\log\det \end{aligned}$$

To define the map det, fix a basis (s_α) of $H^0(X, L)$. Then $\det : M \to \mathbb{R}^+$ is the map $H \mapsto \det(H(s_\alpha, \bar{s}_\beta))$. Changing the basis (s_α) changes det by an additive constant.

We now have the following bounds: Let $h \in \mathcal{H}$ and $H \in M$. Then

$$\frac{1}{d}\tilde{\mathcal{L}}(h) \quad \geq \quad \frac{1}{V}\tilde{Z}(Hilb(h)) \tag{10}$$

$$\frac{1}{d}\tilde{\mathcal{L}}(FS(H)) \quad \leq \quad \frac{1}{V}\tilde{Z}(H) \tag{11}$$

Estimate (10) is a rephrasing of (9). For (11) we estimate:

$$\det\big(Hilb(FS(H))\big)^{\frac{1}{d}} \leq \frac{1}{d}\mathrm{tr}\, Hilb(FS(H)) = \frac{1}{V}\int_X \sum_\alpha s_\alpha \bar{s}_\alpha \, FS(H) \, d\mu_h \tag{12}$$

Choosing now (s_α) to be H-orthonormal, the right side of (12) is one. Thus

$$\det\big(Hilb(FS(H)(s_\alpha, \bar{s}_\beta)\big)^{\frac{1}{d}} \leq 1 = \det H(s_\alpha, \bar{s}_\beta)$$

and this proves (11).

1.5. **Balanced metrics.** Let $h \in \mathcal{H}$ and $H \in M$. We say the pair (h, H) is balanced if $h = FS(H)$ and $H = Hilb(h)$. Thus $h = FS(Hilb(h))$ (in other words, if $h = h_b$) and $Hilb(FS(H)) = H$.

An equivalent characterization is the following: Let $h \in \mathcal{H}$. Choose a basis (s_α) satisfying

$$\frac{d}{V}\int_X s_\alpha \bar{s}_\beta \, h \, d\mu_h \ = \ \delta_{\alpha\beta}$$

(In other words, (s_α) is $Hilb(h)$-orthnormal). Then h is balanced if

$$\rho_h(z) \ = \ \frac{d}{V}\sum_\alpha s_\alpha \bar{s}_\alpha \, h \ = \ \frac{d}{V}$$

Remark: $\frac{1}{V}\int_X \rho_h d\mu_h = \frac{d}{V}$.

Theorem 2. *Suppose $h \in \mathcal{H}$ is balanced and let $H = Hilb(h)$.*

(1) *H is an absolute minimum of $\tilde{Z}$.*
(2) *h is an absolute minimum of $\tilde{\mathcal{L}}$*

Proof. The first statement is the theorem of Zhang. For the second, let $h' \in \mathcal{H}$:

$$\frac{1}{d}\tilde{\mathcal{L}}(h') \;\geq\; \frac{1}{V}\tilde{Z}(Hilb(h')) \;\geq\; \frac{1}{V}\tilde{Z}(Hilb(h)) \;=\; \frac{1}{d}\tilde{\mathcal{L}}(h)$$

where the first inequality makes use of (10) and the second is a consequence of part (1).

Next we prove that the critical points of $\tilde{\mathcal{L}}$ and $\tilde{Z}$ are precisely the balanced metrics. In fact we have the following more precise statement:

Theorem 3.

$$\delta\tilde{\mathcal{L}} \;=\; \int_X (\delta\phi)(\Delta\rho_h - \rho_h + \frac{d}{V})\, d\mu_h \tag{13}$$

$$\delta\tilde{Z} \;=\; \sum_{\alpha,\beta}(\delta H)_{\alpha\bar{\beta}}\left(\int_X \bar{s}_\alpha s_\beta\, FS(H)\, d\mu_{FS(H)} \;-\; \frac{V}{d}\delta_{\alpha\beta}\right) \tag{14}$$

Proof. Let $\mathcal{L} = \log\det \circ Hilb$. Then

$$\delta\mathcal{L}(h) \;=\; \mathrm{tr}_{Hilb(h)}(\delta Hilb(h))$$

Let $h_0 \in \mathcal{H}$ and let (s_α) be a $Hilb(h_0)$-orthonormal basis. Then at the point $h = h_0$

$$\begin{aligned} \delta\mathcal{L}(h) \;&=\; \mathrm{tr}_{Hilb(h_0)}(\delta\, Hilb(h)) \;=\; \delta\,(\mathrm{tr}_{Hilb(h_0)} Hilb(h)) \\ &=\; \delta\sum_\alpha Hilb(he^{-\phi})(s_\alpha, \bar{s}_\alpha) \;=\; \frac{d}{V}\,\delta\sum_\alpha \int_X s_\alpha\bar{s}_\alpha\, he^{-\phi}\, d\mu_{he^{-\phi}} \end{aligned} \tag{15}$$

The second equality follows from: $\delta\mathrm{tr}_{H_0}H = \delta\sum_\alpha H(s_\alpha, \bar{s}_\alpha) = \sum_\alpha \delta H(s_\alpha, \bar{s}_\alpha)$. Thus

$$\delta\mathcal{L}(h) \;=\; \int_X \frac{d}{V}\sum_\alpha s_\alpha\bar{s}_\alpha\, h(-\delta\phi + \Delta\delta\phi)\, d\mu_h \;=\; \int_X (\delta\phi)(-\rho_h + \Delta\rho_h)\, d\mu_h$$

Now we prove (14). Let (s_α) be H-orthonormal. Then $\sum_\alpha s_\alpha\bar{s}_\alpha FS(H) = 1$ so

(16)
$$0 = \sum_\alpha(\delta s_\alpha\bar{s}_\alpha + s_\alpha\overline{\delta s_\alpha})FS(H) + \sum_\alpha s_\alpha\overline{s_\alpha}\,\delta FS(H) \;=\; \sum_\alpha(\delta s_\alpha\bar{s}_\alpha + s_\alpha\overline{\delta s_\alpha})FS(H) + \frac{\delta FS(H)}{FS(H)}$$

On the other hand, $H(s_\alpha, \bar{s}_\beta) = \delta_{\alpha\beta}$ so

$$0 \;=\; \delta H(s_\alpha, \bar{s}_\beta) + H(\delta s_\alpha, \bar{s}_\beta) + H(s_\alpha, \delta\bar{s}_\beta) \;=\; \delta H_{\alpha\bar{\beta}} + H(\delta s_\alpha, \bar{s}_\beta) + H(s_\alpha, \delta\bar{s}_\beta) \tag{17}$$

Next, if we write $\delta s_\alpha = \sum_\beta p_{\alpha\beta}s_\beta$ we see $p_{\alpha\beta} = H(\delta s_\alpha, \bar{s}_\beta)$ so

$$\delta s_\alpha \, \bar{s}_\alpha \;=\; \sum_{\alpha,\beta} H(\delta s_\alpha, \bar{s}_\beta)\bar{s}_\alpha s_\beta \tag{18}$$

Combining (16), (17) and (18) we conclude

$$\delta \log FS(H) \;=\; \frac{\delta FS(H)}{FS(H)} \;=\; \sum_{\alpha,\beta} (\delta H_{\alpha\bar\beta})\, \bar{s}_\alpha s_\beta \, FS(H) \tag{19}$$

Since

$$\delta I(FS(H)) \;=\; \int_X \delta \log FS(H)\, d\mu_{FS(H)} \tag{20}$$

we obtain (14).

1.6. **Raising the power of the line bundle.** We now replace L by L^k, $\mathcal{H}$ by the space $\mathcal{H}_k = \{h(k) : \text{positive hermitian metrics on } L^k\} = \{\phi(k) : \omega_0(k) + i\partial\bar\partial\phi(k) > 0\}$. Then

$$\rho_{h(k)}(z) \;=\; \frac{d_k}{V_k} \sum_\alpha s_\alpha \bar{s}_\alpha \, h(k)$$

where the s_α form an orthonormal basis of $H^0(X, L^k)$ with respect to $Hilb_k(h(k))$:

$$Hilb_k(h(k))(s_\alpha, \bar{s}_\beta) \;=\; \frac{d_k}{V_k} \int_X s_\alpha \bar{s}_\beta \, h(k)\, d\mu_{h(k)}$$

Let $I(k), \tilde{\mathcal{L}}(k)$ be the corresponding functionals on $\mathcal{H}(k)$:

$$\delta I(k) \;=\; -\int \delta\phi(k)\, d\mu_{h(k)}\ , \quad \tilde{\mathcal{L}}(k) \;=\; -\frac{d_k}{V_k} I(k) + \log\det \circ Hilb_k$$

Then (13) implies

$$\delta\tilde{\mathcal{L}}(k) \;=\; \int_X (\delta\phi(k))(\Delta_k \rho_{h(k)} - \rho_{h(k)} + \frac{d_k}{V_k})\, d\mu_{h(k)} \tag{21}$$

Now apply this to $h(k) = h^k$ for some $h \in \mathcal{H}$. Then $\phi(k) = k\phi$, $\Delta_k = \frac{1}{k}\Delta$, $d\mu_{h(k)} = k^n d\mu_h$

$$I(k)(\phi(k)) \;=\; k \cdot k^n I(\phi) \tag{22}$$

$$\delta\tilde{\mathcal{L}}(k) \;=\; \int_X (\delta\phi)(\Delta\rho_{h(k)} - k\rho_{h(k)} + k\frac{d_k}{V_k})\, d\mu_{h(k)} \tag{23}$$

If f is a function on X write $[f]_h = f - \hat{f}$ where $\hat{f}$ is the average of f defined using the measure $d\mu_h$. Then we can write this last identity as

$$\delta\tilde{\mathcal{L}}(k) \;=\; \int_X (\delta\phi)[\Delta\rho_{h(k)} - k\rho_{h(k)}]_h \, d\mu_{h(k)} \tag{24}$$

We define $\rho_{k,h} = \sum_\alpha s_\alpha \bar{s}_\alpha h^k$ where $\int_X s_\alpha \bar{s}_\alpha d\mu_h = \delta_{\alpha\beta}$. Thus $\rho_{h(k)} = k^{-n}\rho_{h,k}$ so

$$\delta\tilde{\mathcal{L}}_k \;=\; \int_X (\delta\phi)[\Delta\rho_{k,h} - k\rho_{k,h}]_h \, d\mu_h \tag{25}$$

Where $\tilde{\mathcal{L}}_k(h) = \tilde{\mathcal{L}}(k)(h^k)$. Now TYZ tells us $\rho_{k,h} = k^n + \frac{s}{2\pi}k^{n-1} + \cdots$ so

$$[\Delta\rho_{h(k)} - k\rho_{h(k)}]_h = -k^n \frac{1}{2\pi}[s]_h + \cdots$$

On the other hand, $\delta\nu = -\int_X [s]_h \, d\mu_h$. Thus $\delta \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k \to \delta\nu$ so

$$\frac{2\pi}{k^n}\tilde{\mathcal{L}}_k + \lambda_k \to \nu \tag{26}$$

for some $\lambda_k \in \mathbb{C}$. The convergence is uniform over bounded subsets of $\mathcal{H}$.

1.7. **Proof of Theorem.** Let $h_0 \in \mathcal{H}$ be a base point and let us normalize I, ν and $\log\det \circ Hilb_k$ by requiring

$$I(h_0) = 0, \;\; \nu(h_0) = 0 \;\; \text{and} \;\; \log\det \circ Hilb_k(h_0(k)) = 0$$

Then $\tilde{\mathcal{L}}_k(h_0) = \nu(h_0) = 0$ for all k so we may take $\lambda_k = 0$.

Note that

$$\left| \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(\phi_1) - \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(\phi_0) \right| \;\leq\; \sup|\phi_1 - \phi_0| \cdot \sup\left| \frac{1}{2\pi}[s]_h + O(\frac{1}{k}) \right|$$

The second factor is uniformly bounded in k when ϕ_1 is in a bounded neighborhood of ϕ_0 in $\mathcal{H}$.

Assume now that there exists $\omega_\infty \in c_1(L)$ with constant scalar curvature, corresponding to some $h_\infty \in \mathcal{H}$. The Donaldson's theorem shows that there exists $h(k) \in \mathcal{H}_k$, a balanced metric such that $h_k = h(k)^{1/k} \to h_\infty$.

$$\left| \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(h_k) - \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(h_\infty) \right| \to 0 \tag{27}$$

Thus if $h \in \mathcal{H}$ we have, for k large,

$$\nu(h_\infty) \leq \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(h_\infty) + \epsilon \;\leq \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(h_k) + 2\epsilon \leq \frac{2\pi}{k^n}\tilde{\mathcal{L}}_k(h) + 2\epsilon$$

The first inequality follows from (26), the second from (27) and the third from the fact that h_k is the absolute minimum of $\tilde{\mathcal{L}}_k$. Taking the limit as $k \to \infty$ we prove our theorem.

2. Appendix

Let $f : X \to Y$ be a smooth maps between manifolds (which may be infinite dimensional). Let $x \in X$ and let $y = f(x)$. Then $(\delta f)(x) : T_xX \to T_yY$ is a linear map. If $\delta x \in T_xX$. Then we define

$$\delta(f(x)) \;=\; (\delta f)(x)(\delta x) \in T_yY$$

For example, $\delta(x^2) = 2x \cdot \delta x$. If $x(t)$ is a curve in X, then $\frac{d}{dt} f(x(t)) \;=\; (\delta f)(x)(\dot{x})$.

The chain rule: Let $f : X \to Y$ and $g : U \to X$. Then

$$\delta(f(g(u)) \;=\; (\delta f)(g(u))(\delta g(u))$$

Now let $H_0 \in M$ and let $\mathrm{tr}_{H_0} : M \to \mathbb{R}$ be the map $\mathrm{tr}_{H_0}(H) = \sum_\alpha H(s_\alpha, \bar{s}_\alpha)$. Moreover, if we differentiate both sides of this last equation with respect to t (so $\delta H = \dot{H}$) we obtain

$$\delta(\mathrm{tr}_{H_0} H) \;=\; \mathrm{tr}_{H_0} \delta H$$

Consider the map $\log\det : GL(n, \mathbb{C}) \to \mathbb{C}^*$. Then

$$\delta \log\det(A) \;=\; \mathrm{tr}_A(\delta A)$$

Let $\mathcal{L} = \log\det \circ Hilb$. Then

$$\delta\mathcal{L}(h) \;=\; \mathrm{tr}_{Hilb(h)}(\delta Hilb(h))$$

so

$$\begin{aligned} \delta\mathcal{L}(h)(h_0) &= \mathrm{tr}_{Hilb(h_0)}(\delta\, Hilb(h)(h_0)) \;=\; (\delta\, \mathrm{tr}_{Hilb(h_0)}(Hilb(h))(h_0) \\ &= \delta \sum_\alpha Hilb(he^{-\phi})(s_\alpha, \bar{s}_\alpha) \;=\; \frac{d}{V}\, \delta \sum_\alpha \int_X s_\alpha \bar{s}_\alpha\, he^{-\phi}\, d\mu_{he^{-\phi}} \end{aligned} \tag{28}$$

Let $h_0 \in \mathcal{H}$ and let (s_α) be a $Hilb(h_0)$-orthonormal basis. Then at the point $h = h_0$

$$\begin{aligned} \delta\mathcal{L}(h) &= \mathrm{tr}_{Hilb(h_0)}(\delta\, Hilb(h)) \;=\; \delta\, (\mathrm{tr}_{Hilb(h_0)} Hilb(h)) \\ &= \delta \sum_\alpha Hilb(he^{-\phi})(s_\alpha, \bar{s}_\alpha) \;=\; \frac{d}{V}\, \delta \sum_\alpha \int_X s_\alpha \bar{s}_\alpha\, he^{-\phi}\, d\mu_{he^{-\phi}} \end{aligned} \tag{29}$$