Elliptic partial differential equations

Contents

1. Introduction

The theory of linear elliptic partial differential equations is formally analogous to the theory of linear maps between finite dimensional vector spaces. To describe the analogy, we need some notation: Let $x, x' \in \mathbb{R}^n$. We view x, x' as $n \times 1$ matrices. Define $\langle x, x' \rangle = {}^t x x'$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then $L(x) = Ax$ for some $m \times n$ matrix A. Note that

$$|L(x)| \;\leq\; \|A\|_{\mathrm{HS}} \cdot |x|$$

where $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}({}^t AA)$. In particular, this shows that linear maps are (Lipshitz) continuous.

Let $L^* : \mathbb{R}^m \to \mathbb{R}^n$ be the adjoint map: $Ly = {}^t Ay$. Then we have the basic formula:

$$\langle Lx, y \rangle \;=\; \langle x, L^* y \rangle$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. This implies $\mathrm{Im}(L) \subseteq \ker(L^*)^\perp$.

Theorem 1. $\mathrm{Im}(L) = \ker(L^*)^\perp$.

Discussion: If we fix $y_0 \in \mathbb{R}^m$, the equation $Lx = y_0$ may not have a solution: in order for a solution to exist, y_0 must satisfy the following obvious necessary condition: $y_0 \in \ker(L^*)^\perp$. The theorem says that the obvious necessary condition is also sufficient.

Discussion: Even if y_0 does not satisfy the obvious necessary condition, $\mathrm{Im}(L) \subseteq \mathbb{R}^m$ is a linear subspace so there is a unique $L(x_0) \in \mathrm{Im}(L)$ such that

$$|L(x_0) - y_0| \;\leq\; |L(x) - y_0| \;\text{ for all } x \in \mathbb{R}^n \tag{1.1}$$

Proof of Theorem. Let $y_0 \in \ker(L^*)^\perp$. Let $E : \mathbb{R}^n \to \mathbb{R}$ be the function $E(x) = |Lx - y_0|^2$. Then (1.1) implies that E achieves its minimum at some point $x_0 \in \mathbb{R}^n$. In particular, if $v \in \mathbb{R}^n$ is arbitrary, and $f(t) = E(x_0 + tv)$ for $t \in \mathbb{R}$, then f achieves its minimum at $t = 0$ so $f'(0) = 0$. Thus

$$f(t) = \langle L(x_0 + tv) - y_0, L(x_0 + tv) - y_0 \rangle = f(0) \;+\; 2t\langle L(v), L(x_0) - y_0 \rangle + O(t^2)$$

so

$$0 \;=\; f'(0) \;=\; 2\langle L(v), L(x_0) - y_0 \rangle \;=\; 2\langle v, L^*(L(x_0) - y_0) \rangle$$

for all $v \in \mathbb{R}^n$. We conclude $L(x_0) - y_0 \in \ker(L^*)$. But $L(x_0) \in \mathrm{Im}(L) \subseteq \ker(L^*)^\perp$ and $y_0 \in \ker(L^*)^\perp$ (by assumption) so

$$L(x_0) - y_0 \;\in\; \ker(L^*) \cap \ker(L^*)^\perp \;=\; 0\ .$$

An linear elliptic PDE is an equation of the form

$$Lu = f \tag{1.2}$$

Here L is a linear "elliptic operator" (e.g., the Laplacian), f is given, and u is the unknown. The domain and range of L will be vector spaces, but unlike the linear algebra theory described above, these vector spaces will be infinite dimensional, so the very simple techniques that work in $\mathbb{R}^n$ do not directly apply. Part of the PDE "art" is choosing well adapted domains and ranges (preferably Hilbert or Banach spaces).

The fundamental questions are:

(1) Existence: Under what conditions does a solution u to (1.2) exist?
(2) Regularity/Estimates: Suppose $Lu = f$ and assume f is smooth. Is u smooth? Is the inverse of L (viewed as a map from $\text{Im}(L) \to \ker(L)^{\perp}$) continuous?

The rough answers to these questions are as follows.

(1) Existence: equation (1.2) has a solution if f satisfies the "obvious necessary conditions". For example, if M is a compact Riemannian manifold, and f is a C^0 function on M, and $L = \Delta$, the Laplacian, then (1.2) has a solution u if and only if $\int_M f = 0$. The solution is unique if we require $\int_M u = 0$. In general, $\text{Im}(L) = \ker(L^*)^{\perp}$.
(2) Regularity: The regularity is "the best one could hope for". For example, if $f \in C^{k,\alpha}$ and $\Delta u = f$ then $u \in C^{k+2,\alpha}$. Moreover, if we normalize so that $\int_M u = 0$, then u satisfies the apriori estimate $\|u\|_{C^{k+2,\alpha}} \leq C\|f\|_{C^{k,\alpha}}$, where $C > 0$ is a constant, independent of f. If $f \in H^k$ (the k^{th} Sobolev space) then $u \in H^{k+2}$ and u satisfies the apriori estimate $\|u\|_{H^{k+2}} \leq C\|f\|_{H^k}$. In general, L^{-1} is a continuous functional $\ker(L^*)^{\perp} \cap H^k \to H^{k+l}$ where

Equation (1.2) will be studied in two basic settings. The first is that of compact manifolds M without boundary (i.e., $\partial M = \emptyset$) and the second is compact manifolds with boundary. The second setting includes, as a very important special case, bounded domains in $\mathbb{R}^n$ with smooth boundary.

We now make precise the "obvious necessary conditions" mentioned above: First assume $\partial M = \emptyset$. If $Lu = f$ then for every $\phi \in C^{\infty}(M)$ integration by parts implies

$$\langle f, \phi \rangle = \langle Lu, \phi \rangle = \langle u, L^*\phi \rangle \tag{1.3}$$

where $\langle g, h \rangle$ is the L^2 inner product and L^* is the dual of L (which will be another elliptic operator). In the case $L = \Delta$ we have $\Delta^* = \Delta$). Thus $Lu = f$ implies f is orthogonal to $\ker L^*$ (which, as we shall see, is a finite dimensional vector space). It turns out that this necessary condition is also sufficient: we will see that if f is orthogonal to $\ker L^*$, then $Lu = f$ has a unique solution u with the property: u is orthogonal to $\ker L$.

If $\partial M \neq \emptyset$, then the situation is similar (although the proofs are more complicated). Equation (1.3) still holds if we require that ϕ vanishes on the boundary (since, when one integrates by parts, the boundary terms will then vanish). The main theorem says that if we fix a smooth function g on ∂M, then the there is a unique solution to $Lu = f$ on M satisfying $u|_{\partial M} = g$ provided f satisfies the necessary condtion: f is orthogonal to the elements of $\ker L^*$ which vanish on the boundary. In the case $L = \Delta$, this last condition is vacuous (there are no non-zero harmonic functions which vanish on the boundary).

The goal of these notes is to prove the existence and regularity/estimates results described above. Our treatment will be as follows: First we will prove the regularity theorems in the case $M = \mathbb{R}^n/\mathbb{Z}^n$ (the so called "periodic case"). Second, we will apply the results from the periodic case to treat the "local case", that is, the case where $M = U \subseteq \mathbb{R}^n$ is a bounded open subset. In particular, we will prove the local apriori estimates for bounded open subsets of $\mathbb{R}^n$. Finally, we will prove the regularity theorems for arbitrary compact manifolds without boundary.

After proving the regularity results, we turn our attention to the existence results. The main theorem gives necessary and sufficient conditions for the existence of a solution to the equation $Lu = f$ where L is an elliptic operator between smooth vector bundles over a fixed compact Riemannian manifold M.

2.1. L^2 and C^0.

We define the two basic spaces $L^2(M)$ and $C^0(M)$. The first is a Hilbert space. The second is a Banach space. Each of these spaces fits into a natural family: We will introduce the Banach cases C^k for $k \in \mathbb{N}$ (and C^0 will be the special case $k = 0$). We shall also introduce Hilbert spaces H^s, known as Sobolev spaces. The space L^2 will then be (canonically isomorphic to) the special case H^0. First some notation.

For $\xi = (\xi_1, ..., \xi_n) \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ we let $e_\xi(x) = e^{i\xi\cdot x}$. If $\phi : \mathbb{R}^n \to \mathbb{C}^m$ is a smooth function and $\alpha \in \mathbb{N}^n$ we let $D^\alpha\phi = \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ so that $D^\alpha e_\xi = \xi^\alpha e_\xi$.

Let $M = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and $dV = \frac{1}{(2\pi)^n} dx_1 \cdots dx_n$. Let

$$L^2(M, \mathbb{C}^m) = \{\phi : M \to \mathbb{C}^m \mid \phi \text{ measurable and } \|\phi\|_{L^2}^2 = \int_M |\phi|^2\, dV < \infty\}$$

Then L^2 is a Hilbert space. The Hilbert space inner product in L^2 is given by the formula: $\langle \phi, \psi\rangle = \int_M {}^t\phi\bar{\psi} dV$ and the topology induced by $\|\cdot\|_{L^2}$ is called the L^2-topology. Let

$$C^0(M, \mathbb{C}^m) = \{\phi : M \to \mathbb{C}^m \mid \phi \text{ is continuous }\}$$

Then C^0 is a Banach space with norm $\|\phi\|_{C^0} = \sup_M |\phi|$. Note that the inclusion

$$C^0(M, \mathbb{C}^m) \hookrightarrow\ L^2(M, \mathbb{C}^m)$$

is continuous. In fact, for $\phi \in C^0$ we have $\|\phi\|_{L^2} \leq\ \|\phi\|_{C^0}$.

For $k > 0$ we define, inductively,

$$C^k(M, \mathbb{C}^m) = \{\phi \in C^{k-1}(M, \mathbb{C}^m) : D^\alpha\phi \text{ exists and is continuous for all } \alpha \text{ with } |\alpha| = 1\}$$

Then C^k is a Banach space with norm $\|\phi\|_{C^k} = \sum_{|\alpha|\leq k} \sup_M |D^\alpha\phi|$.

2.2. **Fourier series.** Let $\phi \in L^2(M, \mathbb{C})$. Define the Fourier transform $\hat{\phi} : \mathbb{Z}^n \to \mathbb{C}$ as follows:

$$\hat{\phi}(\xi) \ = \ \langle \phi, e_\xi\rangle_{L^2} \ = \ \int_M \phi\bar{e}_\xi\, dV$$

More generally, let $\phi \in L^2(M, \mathbb{C}^m)$. Then $\phi = {}^t(\phi^1, ..., \phi^m)$ where $\phi^\mu \in L^2(M, \mathbb{C})$. Define the Fourier transform $u = \hat{\phi} : \mathbb{Z}^n \to \mathbb{C}^m$ as follows: $u^\mu(\xi) = \widehat{\phi^\mu}(\xi)$ for $1 \leq \mu \leq m$.

Theorem 2. *Let $\phi \in L^2(M, \mathbb{C}^m)$ and let $u : \mathbb{Z}^n \to \mathbb{C}^m$ be its Fourier transform. Then*

(1) We have $u \in \ell^2(\mathbb{Z}^n, \mathbb{C}^m)$ that is, $\sum_{\xi\in\mathbb{Z}^n} |u(\xi)|^2 < \infty$.
(2) The map $\phi \mapsto \hat{\phi}$ defines an isomorphism of Hilbert spaces $L^2(M, \mathbb{C}^m) \mapsto \ell^2(\mathbb{Z}^n, \mathbb{C}^m)$.
(3) We have $\phi = \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(\xi)e_\xi$ where the convergence is in L^2.
(4) If $\phi \in C^\infty(M, \mathbb{C}^m)$ then $\phi = \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(\xi)e_\xi$ where the convergence is in C^k for all $k \geq 0$.

2.3. Elementary estimates for Sobolev norms.

Lemma 1. *If $\phi \in C^k(M)$ then $D^\alpha \phi \in L^2$ if $|\alpha| \le k$ so $\sum_\xi (1+|\xi|^2)^s |u_\xi|^2 < \infty$ if $s \le k$. In particular, if $\phi \in C^\infty(M, \mathbb{C}^m)$ then $\sum_\xi (1+|\xi|^2)^s |u_\xi|^2 < \infty$ for all $s \in \mathbb{R}$.*

Problem 1. *Prove Lemma 1. Then prove part (4) of Theorem 2. Hint: For part (4), first show that if $f_k : (a,b) \to \mathbb{R}$ is in C^1, and if $f_k' \to g$ uniformly for some function $g : (a,b) \to \mathbb{R}$, then $g = f'$.*

Hint: To show that

(2.4) $$\phi \in C^k(M) \Longrightarrow \sum_\xi (1+|\xi|^2)^s |u_\xi|^2 < \infty \text{ if } s \le k$$

we must first show

(2.5) $$\widehat{D^\alpha \phi}(\xi) \;=\; \xi^\alpha \hat{\phi}(\xi)$$

Let's try this when $n = 1$ so $\phi : [0, 2\pi] \to \mathbb{C}$ is periodic. Then

$$\frac{1}{i}\widehat{f'}(\xi) \;=\; \frac{1}{2\pi i}\int_0^{2\pi} f'(x)e^{-i\xi x}\,dx \;=\; -\frac{1}{2\pi i}\int_0^{2\pi} f(x)(-i\xi)e^{-i\xi x} = \xi \hat{f}(\xi)$$

The general case of (2.5) follows in a similar fashion (please supply the details). Once we know (2.5) holds for all $|\alpha| = \alpha_1 + \cdots \alpha_n \le k$ then, using the fact that $D^\alpha\phi \in C^0 \Longrightarrow D^\alpha \phi \in L^2$ we get

$$\sum_\xi |\xi^\alpha \hat{\phi}(\xi)|^2 \;=\; \sum_\xi |\widehat{D^\alpha \phi}(\xi)|^2 \;<\; \infty$$

for all $|\alpha| \le k$. Now sum over all such α and try to use this to extablish (2.4). To complete the proof of (4) you must show

(2.6) $$\text{If } \phi \in C^\infty(M) \text{ then } D^\alpha \phi = \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(\xi) D^\alpha e_\xi$$

We know that the derivative of a finite sum is the sum of the derivatives, but here we are dealing with an infinite sum. Let f_N be the N^{th} partial sum and $f = \lim_{N\to\infty} f_N$ the infinite sum. Then by the previous step in this problem, we know that $\lim_{N\to\infty} D^\alpha f_N$ converges to some function g. What we must show is $g = D^\alpha f$. Try this first when $n = 1$. In fact, try to prove the following more general lemma:

(2.7) $$f_k : (a,b) \to \mathbb{R} \text{ is in } C^1, \text{ and } f_k' \to g \text{ uniformly then } g = f'.$$

For this, you can use the fundamental theorem of calculus:

$$\lim_{k\to\infty} f_k(x) \;=\; \lim_{k\to\infty} \int_0^x f_k'(t)\,dt$$

The result follows if you can justify switching the limit and the integral.

This motivates the definition the Sobolev spaces: Let $s \in \mathbb{R}$. Define

(2.8) $$H_s(\mathbb{C}^m) \;=\; \{u : \mathbb{Z}^n \to \mathbb{C}^m : \|u\|_{H_s}^2 = \sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^s |u_\xi|^2 \;<\; \infty\} \;=\; L^2(\mathbb{Z}^n, \mathbb{C}^m; d\mu_s)$$

Here μ_s is the discrete measure on $\mathbb{Z}^n$ defined by $\mu_s(\xi) = (1+|\xi|^2)^s$.

We have a natural hermitian pairing $H_s \times H_t \to \mathbb{C}$ defined as follows. Let $u \in H_s, v \in H_t$. Then

$$\langle u, v\rangle = \sum_\xi (1+|\xi|^2)^{\frac{s+t}{2}} u(\xi)\cdot \overline{v(\xi)} \quad \text{so that} \quad |\langle u,v\rangle| \;\leq\; \|u\|_{H_s}\cdot \|v\|_{H_t}$$

Then H_s is a Hilbert space with respect to this inner product (in the case $s=t$). The collection $\{e_\xi\}_{\xi\in\mathbb{Z}^n}$ is an orthogonal basis of H_s. The map $f \mapsto \hat{f}$ gives a Hilbert space isomorphism $L^2(M) \to H_0$ and, for all $k \geq 0$, continuous dense imbeddings $C^k(M) \hookrightarrow H_k$. Moreover, H^s and H^{-s} are dual with respect to this pairing.

If $t \leq s$ then $\|u\|_t \leq \|u\|_s$ (why?) and

$$C^\infty(M, \mathbb{C}^m) \;=\; H_\infty \subseteq H_s \subseteq H_t \subseteq H_{-\infty}$$

where $H_\infty = \cap_s H_s$ and $H_{-\infty} = \cup_s H_s$. Moreover $H_s \to H_t$ is continuous (why?) H_s and H_{-s} are dual.

For $|\alpha| = k$ we define the map $D^\alpha : H_s \to H_{s-k}$ by $(D^\alpha u)(\xi) = \xi^\alpha u(\xi)$.

Proposition 1. *(continuity of the derivative) If $u \in H_s$ and $|\alpha| = k$ then $D^\alpha u \in H_{s-k}$ Moreover the map $D^\alpha : H_s \to H_{s-k}$ is continuous.*

Let k be a non-negative integer and define

$$\ell^{2,k} \;=\; \{u \in H_0 \,:\, D^\alpha u \in H_0 \quad \text{for all } |\alpha| \leq k\}$$

If $u \in \ell^{2,k}$ then we define

$$\|u\|_{\ell^{2,k}} \;=\; \sum_{|\alpha|\leq k} \|D^\alpha u\|_0$$

Proposition 2. *(H_k equivalence with $\ell^{2,k}$) We have $H_k = \ell^{2,k}$. Moreover, there exist $c(n,k) > 0$ such that*

$$c(n,k)\|u\|_{H_k} \;\leq\; \|u\|_{L^2} + \sum_{|\alpha|=k} \|D^\alpha u\|_0 \;\leq\; \|u\|_{\ell^{2,k}} \;\leq c(n,k)^{-1}\|u\|_{H_k} \tag{2.9}$$

Problem 2. *Prove Proposition 2*

Proposition 3. *(Peter-Paul estimate). Let $t \in \mathbb{R}$, $\epsilon, a, b > 0$ with $\epsilon \in (0,1)$ and $u \in H_{t+a}$. Then*

$$\|u\|_{t-b} \leq \|u\|_t \leq \|u\|_{t+a}$$

Moreover

$$\|u\|_t^2 \;\leq\; \epsilon^a |u\|_{t+a}^2 + \frac{1}{\epsilon^b}\|u\|_{t-b}^2 \tag{2.10}$$

Problem 3. *Show that*

$$1 \leq \; \epsilon^a(1+|\xi|^2)^a + \frac{1}{\epsilon^b}(1+|\xi|^2)^{-b}$$

and use this to prove Proposition 3.

If $u, v \in H_{-\infty}$ define

$$u * v(\xi) \ = \ \sum_{\eta\in\mathbb{Z}^n} u(\xi-\eta)v(\eta)$$

where $u(\xi)v(\eta) = (u^1(\xi)v^1(\eta), ..., u^m(\xi)v^n(\eta))$.

Proposition 4. *(Convolution estimate) Let $s, p, q \in \mathbb{R}$. Then there exists $C = C(n, s) > 0$ such that*

$$\|u * v\|_s \ \le \ C\Bigg[\|u\|_{s+|p-s|}\|v\|_{s-p} \ + \ \|v\|_{s+|q-s|}\|u\|_{s-q} \Bigg]$$

Taking $q = s$ we get:

$$\|u * v\|_s \le C(n, s)\|u\|_0\|v\|_s \ + \ C(n, s)\|u\|_{s+|p-s|}\|v\|_{s-p}$$

If $f \in C^\infty(M)$ and $p \ge 1$ we get the "top sup bound":

$$\|fv\|_s \le C(n, s)\|f\|_{L^\infty}\|v\|_s \ + \ C(n, s)\|f\|_{s+|p-s|}\|v\|_{s-p}$$

Plugging in $p = 0$ we get the two-absolute bound:

$$\|u * v\|_s \le C(n, s)\|u\|_{2|s|}\|v\|_s \tag{2.11}$$

Problem 4. *Assume $x, y, s, p, q \in \mathbb{R}$ with $x, y \ge 0$. Show that*

$$(1 + x + y)^s \ \le \ C(n, s)\Bigg[(1 + x)^{s+|p-s|}(1 + y)^{s-p} \ + \ (1 + y)^{s+|q-s|}(1 + y)^{s-q} \Bigg]$$

Then use this inequality to prove Proposition 4.

Remark: If $u \in H_0$ then u is an L^2 function on M. If $u \in H_{-k}$ with $k > 0$ then u is no longer a function on M - it is, in the old language, a "generalized function" and, in modern language, a "distribution". More precisely, it is a distribution of order k meaning that if $\phi_j, \phi \in C^\infty(M)$ and $D^\alpha\phi_j \to D^\alpha\phi$ uniformly for all $|\alpha| \le k$, then $\langle u, \phi_j\rangle_{H_0} \to \langle u, \phi\rangle_{H_0}$.

2.4. **Sobolev's lemma.** We have already noted the continuous dense inclusion $C^k(M) \hookrightarrow H_k$. Sobolev's theorem is a kind of converse:

Theorem 3. *Let $s > k + \frac{n}{2}$. Then $H_s \hookrightarrow C^k(M)$ is a continuous inclusion.*

Proof. We first treat the case $k = 0$. Let $u \in H_s$. Note that

$$\sum_\xi |u_\xi| \ = \ \sum_\xi (1 + |\xi|^2)^{s/2}|u_\xi| \cdot (1 + |\xi|^2)^{-s/2} \ \le \ \|u_\xi\|_s \cdot \left(\sum_\xi \frac{1}{(1 + |\xi|^2)^s} \right)^{1/2} \ = \ C(s)\|u\|_{H_s}$$

and the last sum is finite since $s > \frac{n}{2}$.

Problem 5. *a) Why can we conclude $u \in C^0$ and that $H_s \to C^0(M)$ is a continuous map?*
b) Prove Theorem for arbitrary $k \ge 0$.

Hint: We must show that if $s > k + \frac{n}{2}$ then

$$\|D^\alpha \phi\|_{L^\infty} \;\leq\; c(n,s,k)\|u\|_s \;\text{ for all } |\alpha| \leq k$$

To so this, start as follows:

$$\|D^\alpha \phi\| \;=\; \left|\sum_\xi \widehat{D^\alpha \phi}(\xi) e^{i\xi\cdot x}\right| \;=\; \left|\sum_\xi \xi^\alpha \widehat{\phi}(\xi) e^{i\xi\cdot x}\right| \;\leq\; \sum_\xi |\xi^\alpha u(\xi)|$$

and now use the same trick as in the case $k = 0$.

In the previous proof we made use of the following convergence criterion:

$$\text{(2.12)} \qquad \sum_{\xi\in\mathbb{Z}^n} \frac{1}{(1+|\xi|^2)^p} \;<\; \infty \qquad \Longleftrightarrow \qquad p > \frac{n}{2}$$

To see this, we compare the sum to the integral $\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^p}\, dx \;=\; c_n \int_0^\infty \frac{1}{(1+r^2)^p}\, r^{n-1}\, dr$.

2.5. **Newton quotients.** Let $h \in \mathbb{R}^n$ with $h \neq 0$. Define the translation map $T_h : H_s \to H_s$ by $(T_h u)(\xi)\xi = e^{ih\cdot\xi} u(\xi)$. Define the Newton quotient map $N_h : H_s \to H_s$ by

$$N_h(u) = \frac{1}{i}\cdot\frac{T_h u - u}{|h|} \;=\; \frac{1}{i}\cdot\frac{e^{ih\cdot\xi}-1}{|h|} u(\xi), \text{ so } \;|N_h(u)(\xi)| \;=\; \left|\frac{2\sin(\frac{h\cdot\xi}{2})}{|h|} u_\xi\right|$$

If $u : \mathbb{Z}^n \to \mathbb{C}$ we define the derivative in the direction of h as follows: $(\nabla_h u)_\xi = \frac{h\cdot\xi}{|h|} u_\xi$. Thus we have, for each $\xi \in \mathbb{Z}^n$, the following:

$$\text{(2.13)} \qquad \lim_{t\searrow 0} N_{th}(u)(\xi) \;=\; \nabla_h u(\xi)$$

Remark: If $\phi \in C^\infty(M)$ then

$$\phi(x) \;=\; \sum_\xi u_\xi e^{i\xi\cdot x} \quad\Longrightarrow\quad \frac{1}{i}\cdot\frac{\phi(x+h)-\phi(x)}{|h|} \;=\; \sum_\xi N_h(u)(\xi) e^{i\xi\cdot x}$$

and $\nabla_h \phi = \frac{d}{dt}\big|_{t=0} \phi(x + t\frac{h}{|h|})$.

Theorem 4. *(Newton quotient theorem).*

a) Let $u \in H_s$. Then

$$u \in H_{s+1} \quad\Longleftrightarrow\quad \sup_{0\neq h\in\mathbb{R}^n} \|N_h(u)\|_s \;<\; \infty$$

b) If $u \in H_{s+1}$ then

$$\sup_h \|N_h u\|_s^2 \;\leq\; \|Du\|_s^2 \;\leq n \sup_h \|N_h u\|_s^2$$

where

$$\|Du\|_s^2 \;:=\; \sum_{j=1}^n \|D_j u\|_s^2 \;=\; \sum_\xi (1+|\xi|^2)^s |\xi|^2 |u(\xi)|^2$$

Here $D_j u = D^\alpha u$ where $\alpha = (0,0,...,1,0,...0)$ where the 1 is the j^{th} entry.

c) If $u \in H_{s+1}$ then the following limit holds in the H_s topology.

$$\lim_{t \searrow 0} N_{th}(u) \ = \ \nabla_h u$$

Proof. Let $u \in H_{s+1}$. Then $|\sin(x)| \leq |x|$ implies

$$\|N_h(u)\|_2^s \ = \ \sum_\xi (1+|\xi|^2)^s \left| \frac{2\sin(h\cdot\xi/2)}{|h|} u(\xi)\right|^2 \ \leq \ \sum_\xi (1+|\xi|^2)^s \left|\frac{h\cdot\xi}{|h|}u(\xi)\right|^2$$
$$\leq \ \sum_\xi (1+|\xi|^2)^s |\xi|^2 |u(\xi)|^2 \ \leq \ \|u\|_{s+1}^2 \ < \ \infty$$

Conversely, if $\sup_{0\neq h\in\mathbb{R}^n} \|N_h(u)\|_s = C < \infty$ then, letting $h_1 = (1,0,...,0)$ we have, for all $M > 0$,

$$\sum_{|\xi|\leq M} (1+|\xi|^2)^s \left|\frac{2\sin(th_1\cdot\xi/2)}{|th_1|}u(\xi)\right|^2 \ \leq \ C$$

Taking the limit as $t \to 0$ and then taking the limit as $M \to \infty$ we obtain

$$\sum_{|\xi|\leq M} (1+|\xi|^2)^s |\xi_1|^2 |u(\xi)|^2 \ \leq \ C$$

Replacing ξ_1 by ξ_i and summing over i we get

$$\sum_{|\xi|\leq M} (1+|\xi|^2)^s |\xi|^2 |u(\xi)|^2 \ \leq \ nC$$

so $\|u\|_{s+1}^2 \ \leq \ \|u\|_s^2 \ + \ n\sup\|N_h u\|_s^2$. This proves a) and b).

Problem 6. *Supply a proof for part c).*

Hint: Note that the convergence is in the H_s topology. In other words, you must show

$$\lim_{t\searrow 0} \|N_{th}(u) - \nabla_h u\|_s = 0$$

You will need (2.13) but you will also need some additional argument.
We summarize this discussion as follows:

Corollary 1. *Let $s \in \mathbb{R}$. Then $\|u\|_{s+1} \sim \|u\|_s + \sup_h \|N_h u\|_s$.*

2.6. **Rellich Compactness.**

Theorem 5. *Let $s > t$. Then the map $H_s \to H_t$ is super-compact, that is, if $u_j \in H_s$ is a bounded sequence, then there exists $u_\infty \in H_s$ such that, after passing to a subsequence, $u_j \to u_\infty$ in H_t.*

Proof. By assumption there is a $C > 0$ such that for all $j > 0$ we have

$$\sum_\xi (1+|\xi|^2)^s |u_j(\xi)|^2 \ \leq \ C$$

Fix $\xi \in \mathbb{Z}^n$. Then we have $(1+|\xi|^2)^s|u_j(\xi)|^2 \ \leq \ C$ so there exists $u_\infty(\xi)$ such that, after passing to a subsequence, $u_j(\xi) \to u_\infty(\xi)$. Using the diagonalization procedure, we may assume that the subsequence does not depend on ξ, that is, $u_j(\xi) \to u_\infty(\xi)$ for all ξ. From now on, we replace the original sequence by this subsequence.

We claim that $u_\infty \in H_s$. To see this, fix $N > 0$. Then we have $\sum_{|\xi| \le N}(1+|\xi|^2)^s |u_j(\xi)|^2 \ \le\ C$. Passing to the limit we see $\sum_{|\xi|\le N}(1+|\xi|^2)^s|u_\infty(\xi)|^2 \ \le\ C$. Taking $N \to \infty$ we conclude

$$\sum_\xi (1+|\xi|^2)^s|u_\infty(\xi)|^2 \ \le\ C$$

and the claim is proved. Fix $N > 0$. We estimate

$$\|u_j - u_\infty\|^2_{H_t} \ = \ \sum_{|\xi|<N}(1+|\xi|^2)^t|u_j(\xi)-u_\infty(\xi)|^2 + \sum_{|\xi|\ge N}(1+|\xi|^2)^{t-s}(1+|\xi|^2)^s|u_j(\xi)-u_\infty(\xi)|^2$$

The second term is bounded by $N^{-(s-t)}\|u_j - u_\infty\|^2_{H_s} \le N^{-(s-t)}(2C)^2$. The first term is less than $\frac{1}{N}$ for j sufficiently large (its a finite sum of terms which approach zero pointwise). Thus, for j sufficiently large, we have

$$\|u_j - u_\infty\|^2_{H_t} \ \le\ \frac{1}{N} + \frac{(2C)^2}{N^{s-t}}$$

This shows that $\|u^j - u^\infty\|_{H_t} \to 0$ as $j \to \infty$.

Problem 7. *Let $A > 0$, let H be a hilbert space with countable orthonormal basis $e_1, e_2, \ldots$ and let $\lambda_1, \lambda_2, \ldots \in [0, A]$.*

a) Show that there is a unique bounded linear transformation $T : H \to H$ with the property $T(e_j) = \lambda_j e_j$.

b) Assume that $\lambda_j \to 0$. Show that T is compact, that is, if $x_1, x_2, \ldots \in H$ is a bounded sequence then $T(x_1), T(x_2), \ldots$ has a convergent subsequence.

c) Use part b) to show that if $s > t$, then the map $H_s \to H_t$ is compact, that is if $u_j \in H_s$ is a bounded sequence then $T(u_j)$ has a bounded subsequence. Why isn't this a second proof of Rellich compactness.

Hint: In problem 7, the linear transformation T maps a Hilbert space H to itself. But the map $H_s \to H_t$ is a map between two different Hilbert spaces. Thus, in order to apply problem 7, you must first compose $H_s \to H_t$ with a Hilbert space isomorphism $K : H_t \to H_s$ and obtain $H_s \to H_t \to H_s$. Then apply Problem 7 to the composition $H_s \to H_s$. So the question becomes: can we write down an explicit isomorphism K?

2.7. **Elliptic estimates.** A homogeneous differential operator $P(D)$ on $M = \mathbb{R}^n/\mathbb{Z}^n$ of order l and rank one is a formal sum $P(D) = \sum_{|\alpha|=l} a_\alpha(x) D^\alpha$ where $a_\alpha(x) \in C^\infty(M, \mathbb{C})$. Then we have $P(D) : C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$: If $\phi \in C^\infty(M,\mathbb{C})$ then $P(D)\phi = \sum_{|\alpha|=l} a_\alpha(x) D^\alpha \phi(x)$. Similarly we can define an elliptic operator $P(D)$ on an open subset of $\mathbb{R}^n$.

For example, the following is an elliptic operator on $\mathbb{R}^3$ of rank one with and order 4 and.

$$\begin{aligned} P(D)\phi &= \frac{1}{x_3^2x_2^8+1}\frac{\partial^4\phi}{\partial^3 x_1\partial^1 x_3} - (x_1x_2+\sin(x_1e^{x_3}))\frac{\partial^4\phi}{\partial^2 x_1\partial^1 x_2\partial^1 x_3} + 7\frac{\partial^3\phi}{\partial^2 x_1\partial^2 x_3} \\ &= a_{(3,0,1)}\frac{\partial^4\phi}{\partial^3 x_1\partial^1 x_3} + a_{(2,1,1)}\frac{\partial^4\phi}{\partial^2 x_1\partial^1 x_2\partial^1 x_3} + a_{(2,0,2)}\frac{\partial^3\phi}{\partial^2 x_1\partial^2 x_3} \end{aligned}$$

More generally, if $u \in H_s$ then $L = P(D)u \in H_{s-l}$ is defined by $P(D)u = \sum_{|\alpha|=l} a_\alpha(x)D^\alpha u$. More precisely,

$$[P(D)u](\xi) \ := \ a_\alpha D^\alpha u(\xi) := \ [a_\alpha \xi^\alpha u(\xi)] := \xi^\alpha[\widehat{a_\alpha} * u](\xi)$$

Here, as always, if $f \in C^\infty(M, \mathbb{C})$ and $u \in H_s(\mathbb{C})$ then $fu := \hat{f} * u$.

The map $P(D) : H_s \to H_{s-l}$ is continuous (by (3) of Proposition 1). In fact,

$$\|Lu\|_{s-l} \ \leq \ c(L, s)\cdot\|u\|_s \tag{2.14}$$

where $c(L, s)$ depends only on L and s. To see this we estimate

$$\|Lu\|_{s-l} \ \leq \ \sum_{|\alpha|=l}\|a_\alpha(x)D^\alpha u\|_{s-l} \ \leq \ \sum_{|\alpha|=l}\|a_\alpha(x)\|_{2|s-l|}\|D^\alpha u\|_{s-l} \leq \ \left[\sum_{|\alpha|=l}\|a_\alpha(x)\|_{2|s-l|}\right]\|u\|_s$$

If L is a homogeneous elliptic operator we define $P_L : \mathbb{Z}^n \to C^\infty(M)$, the symbol of L, as follows. If $\xi \in \mathbb{R}^n$ then $P_L(\xi)(x) = \sum_\alpha a_\alpha(x)\xi^\alpha$. Thus $P_L(\xi)$ is a smooth function on M which is homogeneous of degree l. That is, $P(\lambda\xi) = |\lambda|^l P(\xi)$ for $\lambda \in \mathbb{C}$. We sometimes write $P_L(\xi, x) := P_L(\xi)(x)$.

A homogeneous differential operator $P(D)$ on $M = \mathbb{R}^n/\mathbb{Z}^n$ of order l and rank m is a matrix $P(D) = (P^\mu_\nu(D))_{1\leq\mu,\nu\leq m}$ where $P^\nu_\mu(D) = \sum_\alpha a^\mu_{\alpha,\nu}(x)D^\alpha$ is a homogeneous differential operator of order l and rank one. Thus we may write $P(D) = \sum_\alpha A_\alpha(x)D^\alpha$ where $A_\alpha(x) = (a^\mu_{\alpha,\nu})_{1\leq\mu,\nu\leq m}$ is a matrix valued smooth function on M. The operator $P(D)$ defines a map $P(D) : C^\infty(M, \mathbb{C}^m) \to C^\infty(M, \mathbb{C}^m)$ in the usual way: $(P(D)\phi)^\mu = \ P^\mu_\nu(D)\phi^\nu$.

More generally, we can extend $P(D)$ to a linear map $P(D) : H_s(\mathbb{C}^m) \to H_{s-l}(\mathbb{C}^m)$ where $H_s(\mathbb{C}^m) = H_s(\mathbb{C})^m$. More precisely

$$[P(D)u](\xi)^\mu \ := \ a^\mu_{\nu,\alpha}\xi^\alpha u^\nu(\xi) \ := \ \xi^\alpha[\widehat{a^\mu_{\nu,\alpha}} * u^\nu](\xi)$$

If L is a homogeneous operator of order l and rank m, then for $\xi \in \mathbb{R}^n$ we define

$$[P_L(\xi)]^\mu_\nu \ = \ a^\mu_{\alpha,\nu}(x)\xi^\alpha$$

Thus $P_L(\xi) : M \to \mathrm{End}(\mathbb{C}^m)$ is a smooth function on M with values in $\mathrm{End}(\mathbb{C}^m)$. We sometimes write $P_L(\xi)(x) := P_L(\xi, x)$. We say that that P is elliptic if

$$P_L(\xi) : M \to \mathrm{Aut}(\mathbb{C}^m)$$

For example, if $n = 2$ and $a, b, c \in \mathbb{R}$ then consider

$$P(D)\phi \;=\; a\frac{\partial^2\phi}{\partial x_1^2} \;+\; 2b\frac{\partial^2\phi}{\partial x_1 \partial x_2} \;+\; c\frac{\partial^2\phi}{\partial x_2^2}$$

This is the most general homogeneous operator of rank one degree two with constant coefficients. Assume that $a \geq 0$. The symbol is the 1×1 matrix

$$P(\xi) \;=\; a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 \;=\; \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_1\end{pmatrix} \quad \text{with } (\xi_1, \xi_2) \in \mathbb{R}^2$$

To say that P is elliptic is the same as saying that $a > 0$ and $ac - b^2 > 0$ which is the same as saying that the two eigenvalues of $\begin{pmatrix}a & b\\ b & c\end{pmatrix}$ are both positive (note the eigenvalues of a symmetric real matrix are always real). In general, if a, b, c are smooth functions of $x \in M$, and if $a(x_0) \geq 0$ at some point $x_0 \in M$, then P is elliptic if the two eigenvalues of $\begin{pmatrix}a(x) & b(x)\\ b(x) & c(x)\end{pmatrix}$ are both positive for every $x \in M$. Alternatively, we can require that there exist $\lambda, \Lambda > 0$ such that for all $x \in M$ we have

$$\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \;\leq\; \begin{pmatrix}a(x) & b(x)\\ b(x) & c(x)\end{pmatrix} \;\leq\; \Lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

A differential operator $L(D)$ on M of order l and rank m is a sum $L(D) = \sum_{0 \leq j \leq l} P_j(D)$ where $P_j(D)$ is homogeneous of order j. We say that $L(D)$ is an elliptic if $P_l(D)$ is elliptic.

Problem 8. *Let L_1 and L_2 be differential operators of rank 1 and orders l_1 and l_2. Show that $L_1 \circ L_2$ is a differential operator of rank m and order $l_1 + l_2$. Show that if $m = 1$ then $L_1 \circ L_2 - L_2 \circ L_1$ is a differential operator of rank 1 and order at most $l_1 + l_2 - 1$.*

Fix $f \in C^\infty(M)$ and fix L a homogeneous operator of order l. Then we can define a new operator $L_f(u) = L(fu) - fL(u)$.

Lemma 2. *L_f is a differential operator of order $l - 1$:*

$$L_f(u) = \sum_\alpha a_\alpha(x)[D^\alpha(fu) - fD^\alpha u]$$

*Proof.*The product rule says that if $\alpha = l$ then

$$D^\alpha(fu) \;=\; \sum_{\beta+\gamma=\alpha} c(\beta,\gamma)(D^\beta f)(D^\gamma u) \;=\; fD^\alpha u + \sum_{\beta+\gamma=\alpha, |\gamma|<l} c(\beta,\gamma)(D^\beta f)(D^\gamma u)$$

where $c(\beta, \gamma)$ are binomial coefficients. Thus

$$L(fu) = \sum_{|\alpha|=l} a_\alpha(x)D^\alpha(fu) \;=\; f\sum_\alpha a_\alpha(x)D^\alpha u \;+\; \sum_{|\gamma|<l}\left(\sum_{\{\beta:|\beta+\gamma|=l\}} c(\beta,\gamma)a_{\beta+\gamma}(x)D^\beta f\right)D^\gamma u$$

Theorem 6. *Let L be an elliptic order of order l and rank m on M. Let $s, t \in \mathbb{R}$ with $s + l \geq t$. There exists $C = C(L) > 0$ with the following property. If $u \in H_{s+l}$ then $Lu \in H_s$ and we have the apriori estimate*

$$\|u\|_{s+l} \;\leq\; C(L,s,t)(\|Lu\|_{H_s} + \|u\|_{H_t}) \quad \textit{for all } u \in H_{s+l} \tag{2.15}$$

Conversely, if $u \in H_{-\infty}$ and if $Lu \in H_s$ then $u \in H_{s+l}$.

Remark 1. The reverse inequality holds trivially: $\|u\|_{s+l} \;\geq\; C'(\|Lu\|_{H_s} + \|u\|_{H_t})$.

Remark 2. We shall see that if $u \in \ker(L^*)^{\perp}$ than we can drop the $\|u\|_t$ term in (2.15).

Proof of theorem.
First assume that L is homogeneous and that $L = P_l$ has constant coefficients. Thus, for all $\xi \in \mathbb{R}^n$ we have $P_L(\xi)(x) \in GL_n(\mathbb{C})$ is independent of $x \in M$. Moreover, for all $u \in H_s(\mathbb{C}^m)$ and $\xi \in \mathbb{R}^n$ we have

$$[P(D)u](\xi) \;=\; [P_L(\xi)u](\xi) \;\in\; \mathbb{C}^m \tag{2.16}$$

To see this, recall that $(D^{\alpha}u)(\xi) = \xi^{\alpha}u(\xi)$ for $u \in H_s(\mathbb{C})$. Thus if $u = (u^{\mu})_{1\leq\mu\leq m}$ we have

$$[(P(D)u)(\xi)]^{\nu}_{\mu} \;=\; (a^{\nu}_{\mu,\alpha}D^{\alpha}u^{\mu})(\xi) \;=\; a^{\nu}_{\mu,\alpha}\xi^{\alpha}u^{\mu}(\xi) \;=\; P_L(\xi)^{\nu}_{\mu}u^{\mu}(\xi) = \; P(\xi)u(\xi)$$

We claim that there exists $c_1 > 0$ satisfying the following. Let $\xi \in \mathbb{R}^n$ and $u \in \mathbb{C}^m$. Then

$$|P(\xi)u|^2 \geq c_1|\xi|^{2l}|u|^2 \tag{2.17}$$

for some $0 < c_1 < 1$. Indeed, this follows by from the compactness of the unit sphere when $|\xi| = |u| = 1$, and by homogeneity in general.

Now we estimate:

$$\|L\phi\|_s^2 \;=\; \sum_{\xi}|P(\xi)u(\xi)|^2(1+|\xi|^2)^s \;\geq\; c_1\sum_{\xi}|u(\xi)|^2|\xi|^{2l}(1+|\xi|^2)^s$$
$$\|\phi\|_t^2 \;\geq\; c_1\sum_{\xi}|u(\xi)|^2(1+|\xi|^2)^t$$
$$|\xi|^{2l}(1+|\xi|^2)^s + (1+|\xi|^2)^t \;\;\geq\; c_2(1+|\xi|^2)^{s+l}$$

We thus obtain (2.15) with $C = \frac{1}{c_1c_2} = C(s,t,l)$

Problem 9. *Let $L = \sum_{1\leq j\leq l} P_l$ be an elliptic operator on M with constant coefficients.*
a) Show that the map $L : C^{\infty}(M,\mathbb{C}) \to C^{\infty}(M,\mathbb{C})$ has a finite dimensional kernel.
b) Show that there is a complex vector subspace $S \subseteq C^{\infty}(M,\mathbb{C})$ of finite codimension (i.e. $C^{\infty}(M,\mathbb{C})/S$ is a finite dimensional vector space) such that for every $f \in S$ there exists $\phi \in C^{\infty}(M,\mathbb{C})$ such that $L\phi = f$.

Hint: Use (2.16) and (2.17). First do the case where L is the laplace operator (which is homogeneous of order 2 and rank 1) and then do the general case.

Now we treat the general case. Let $L = \sum_{1\leq j\leq l} P_l$ be an elliptic operator on M and let $p \in M$. Let $L_0 = P_l(p)$. Then L_0 is a homogeneous elliptic operator of order l and degree m with constant coefficients. Let U be an open neighborhood of p and let ϕ be a smooth function on M with support in U. Then

$$\|\phi\|_{s+l} \leq C(s,t,l)(\|L_0\phi\|_s + \|\phi\|_t) \;\leq\; C(s,t,l)(\|L\phi\|_s + \|(L-L_0)(\phi)\|_s + \|\phi\|_t)$$

The leading term in $L - L_0$ vanishes at p. The leading term is

$$[P_l(x) - P_l(p)](\phi) = \sum_{|\alpha|=l} (a_\alpha(x) - a_\alpha(p)) D^\alpha \phi \;=\; \sum_{|\alpha|=l} f_\alpha(x) D^\alpha \phi$$

Thus, shrinking U if necessary, we can guarantee that $|f_\alpha| \leq \epsilon$ on the open set U.

Now we apply the convolution estimate

$$\begin{aligned} \|f_\alpha D^\alpha \phi\|_s &\leq C(n,s)\|f_\alpha\|_{L^\infty}\|D^\alpha\phi\|_s \;+\; C(n,s)\|f_\alpha\|_{s+|p-s|}\|D^\alpha\phi\|_{s-p} \\ &\leq C(n,s)\epsilon\|\phi\|_{s+l} \;+\; C(n,s)\|f_\alpha\|_{s+|p-s|}\|D^\alpha\phi\|_{s-p} \end{aligned}$$

Let $N_l \;=\; \#\{\alpha : |\alpha| = l\}$. Then, choosing $p = s + l - 1$, we obtain

$$\|[P_l(x) - P_l(p)](\phi)\| \;\leq\; C(n,s)\epsilon N_l \|\phi\|_{s+l} + C(n,s)C_1(L)\|D^\alpha\phi\|_{s+l-1}$$

Choose ϵ so that $C(n,s)\epsilon N_l \leq \frac{1}{4}$. Then

$$C(n,s)\|(L - L_0)(\phi)\|_s \;\leq \frac{1}{4}\|\phi\|_{s+l} + C(n,s,L)\|\phi\|_{s+l-1}$$

Now $t \leq s + l - 1 \leq s + l$ so Peter-Paul implies

$$C\|(L - L_0)(\phi)\|_s \;\leq \frac{1}{2}\|\phi\|_{s+l} + C'(n, sL)\|\phi\|_t$$

This proves (2.15) in the case where ϕ has support in U.

For the general case, we cover M by a finite collection open sets U_j as above and let θ_j be a partition of unity subordinate to the U_j. Then

$$\|\phi\|_{s+l} = \|\sum_j \theta_j\phi\|_{s+l} \;\leq\; \sum_j \|\theta_j\phi\|_{s+l} \;\leq\; \sum_j C(\|L(\theta_j\phi)\|_s + \|\theta_j\phi\|_t)$$

Let $L_j(\phi) = L(\theta_j\phi) - \theta_j L(\phi)$. Then L_j is an operator of order $l - 1$ and

$$\begin{aligned} \|L(\theta_j\phi)\|_s \;\leq\; \|\theta_j L(\phi)\|_s + \|L_j\phi\|_s \;&\leq\; C(n,s)\|\theta_j\|_{2|s|}\|L(\phi)\|_s + C_1\|\phi\|_{s+l-1} \\ \|\phi\|_{s+l-1} \;&\leq \epsilon\|\phi\|_{s+l} + \frac{1}{\epsilon}\|\phi\|_t \\ \|\theta_j\phi\|_t \;&\leq\; C\|\theta_j\|_{2|t|}\|\phi\|_t \end{aligned}$$

This completes the proof.

2.8. **Elliptic Regularity.**

Theorem 7. *Let L be an elliptic operator on M of order l. Assume that $u \in H_{-\infty}$ and the $Lu \in H_t$ for some t. Then $u \in H_{t+l}$.*

Before proving the theorem we recall some previous results that will be needed.

Continuity of PDO's. Let $L = a_\alpha D^\alpha$ be a PDO of order l and rank m. Here $a_\alpha \in C^\infty(M, \mathrm{End}(\mathbb{C}^m))$ for all $|\alpha| \leq l$. Then $L : H_s \to H_{s-l}$ is continuous: $\|L(u)\|_{s-l} \;\leq\; C(L)\|u\|_s$ This follows from the convolution estimate. More precisely, if $\max_\alpha \|a_\alpha\|_{2|s|} \leq A$ then

$$\|L(u)\|_{s-l} \;\leq\; C(A)\|u\|_s$$

Translation operator. Let $h \in \mathbb{R}^n$. The $T_h : H_s \to H_s$ is the continuous map

$$(T_h u)(\xi) \; = \; e^{i\xi\cdot x} u(\xi)$$

We have the estimate

$$\|T_h u\|_s \; = \; \|u\|_s$$

Newton quotient estimate. Let $0 \neq h \in \mathbb{R}^n$. Then $N_h : H_s \to H_s$ is the continuous map $N_h u = \frac{T_h u - u}{|h|}$. More precisely,

$$(N_h u)(\xi) \; = \; \left(\frac{e^{i\xi\cdot x} - 1}{h} \right) u(\xi)$$

To see that N_h is continuous we estimate

$$\|N_h u\|_s \; = \; \frac{1}{|h|} \|T_h u - u\|_s \; \leq \; \frac{1}{|h|} 2\|u\|_s$$

Since $H_{s+1} \subseteq H_s$ we also have $N_h : H_{s+1} \to H_s$ is continuous but this time

$$\|N_h(u)\|_s \quad = \; \|u^h\|_s \; \leq \; \|u\|_{s+1}$$

where $u^h \; := \; N_h(u)$. The estimate we need says the following. Let $u \in H_s$ and assume $\sup_h \|u^h\|_s < \infty$. Then $u \in H_{s+1}$. In fact

$$\|u\|_s + \sup_h \|u^h\|_s \leq \|u\|_{s+1} \; \leq \; C(n,s)(\|u\|_s + \sup_h \|u^h\|_s)$$

Elliptic estimate. Let L be an elliptic PDO of order l. Let $u \in H_s$. Then

$$\|u\|_s \; \leq \; C(L)(\|Lu\|_{s-l} + \|u\|_{s-l})$$

Proof. Claim: Let $u \in H_s$. Assume $Lu \in H_{s-l+1}$. Then $u \in H_{s+1}$.

Assume the claim for the moment and let's try to prove the theorem. Choose $k \leq l \in \mathbb{Z}$ maximal with the property $u \in H_{t+k}$. If $k < l$ than $k \leq l-1$ so $k - l + 1 \leq 0$. Let $s = t + k$ so

$$u \in H_s \quad \text{and } Lu \in H_t \subseteq H_{t+k-l+1} = H_{s-l+1}$$

The claim implies that $u \in H_{s+1} = H_{t+k+1}$, a contradiction.

Now we prove the claim. Assume

- $u \in H_s$
- $Lu \in H_{s-l+1}$.

We want to show $u \in H_{s+1}$. To do this we must prove

$$\|u^h\|_s \leq B < \infty$$

for some constant B which is independent of h. But

$$\begin{aligned} \|u^h\|_s \; &\leq \; C(L)(\|L(u^h)\|_{s-l} + \|u^h\|_{s-1}) \\ &\leq C(L)(\|L(u)^h\|_{s-l} + \|u^h\|_{s-1}) + C(L)(\|L(u^h) - L(u)^h\|_{s-l}) \\ &\leq C(L)(\|L(u)\|_{s-l+1} + \|u\|_s) + C(L)(\|L(u^h) - L(u)^h\|_{s-l}) \end{aligned}$$

Lemma 3. *Let $u \in H_{-\infty}$ and let L be a periodic operator with smooth coefficients. Then*

$$(Lu)^h - L(u^h) \; = \; L^h(T_h u) \tag{2.18}$$

where $L^h = a^h_\alpha D^\alpha$.

Proof. First we treat the case $u = \phi \in C^\infty(M)$. Then

$$\begin{aligned}(L\phi)^h = \frac{1}{|h|}[(L\phi)(x+h) - (L\phi)(x)] &= \frac{1}{|h|}[a_\alpha(x+h)D^\alpha\phi(x+h) - a_\alpha(x)D^\alpha\phi(x)] \\ = \left[\frac{a_\alpha(x+h) - a_\alpha(x)}{|h|}\right] D^\alpha[\phi(x+h)] &+ a_\alpha(x)D^\alpha\left(\frac{\phi(x+h)-\phi(x)}{|h|}\right) \\ &= \; L^h(T_h\phi) \; + \; L(u^h)\end{aligned}$$

This proves (2.18) in the case where u is smooth. Now let $u \in H_r$ and choose $\phi_j \to u$ in H_r with ϕ_j smooth. Apply (2.18) to ϕ_j. Since $u \mapsto u^h$ and $u \mapsto T_h u$ is continuous in H_s for all s, we can take the limit as $j \to \infty$ in H_{r-l} and the lemma is proved.

Let $A = \max \|a_\alpha\|_{2|s|+1}$. Then

$$\max \|a^h_\alpha\|_{2|s|} \; \leq \; A$$

Now we can estimate:

$$\|L(u^h) - L(u)^h\|_{s-l} \; = \; \|L^h(T_h u)\|_{s-l} \leq C(A)\|T_h u\|_s = C(A)\|u\|_s$$

Proposition 1 implies that the coefficients of L^h are uniformly bounded in H_s for $h \neq 0$. Thus are constants c_1, c_2, independent of h such that

$$\|L^h(T_h u)\|_{s-l} \leq c_1\|T_h u\|_s \leq c_1 c_2 \|u\|_s$$

3. Estimates and Interior Regularity for domains in $\mathbb{R}^n$

3.1. **Apriori estimates.** Let $U \subseteq M$ be open. Define

$$C_o^\infty(U) = \{u \in C^\infty(M) \,:\, \text{u has compact support in } U\}$$
$$C_o^\infty(\bar{U}) = \{u \in C^\infty(M) \,:\, \text{u vanishes outside } \bar{U}\}$$

Thus $\phi \in C_o^\infty(\bar{U})$ if and only if $\chi\phi = 0$ for all $\chi \in C^\infty(M)$ such that $\chi = 0$ on U. Define

$$H_o^s(U) = \overline{C_o^\infty(U)} \subseteq H^s(M)$$
$$H_o^s(\bar{U}) = \overline{C_o^\infty(\bar{U})} \subseteq H^s(M)$$

Lemma 4. *Let $u \in H^s(M)$. Then $u \in H_o^s(\bar{U})$ if and only if $\chi u = 0$ for all $\chi \in C^\infty(M)$ such that $\chi = 0$ on U.*

Proof. Let $u \in H_o^s(\bar{U})$ and let $\chi \in C^\infty(M)$ such that $\chi = 0$ on U. Then $\phi_j \to u$ for some $\phi_j \in C_o^\infty(\bar{U})$ so $\chi\phi_j \to \chi u$. But $\chi\phi_j = 0 \Longrightarrow \chi u = 0$.

Conversely, let $u \in H^s(M)$ and assume $\chi u = 0$ for all χ. Write $M \backslash \bar{U} = \cup_{p\in\mathbb{N}}\Omega_\mu$, a locally finite countable union of open sets. Let χ_p be a partition of unity subordinate to Ω_μ. Let $\phi_j \to u$ with $\phi_j \in C^\infty(M)$. Then $\chi_1\phi_j \to \chi_m u = u$ and $\chi_1\phi_j$ vanishes on the suport of χ_1. Similarly, $\chi_1\chi_2\phi_j \to u$ and $\chi_1\chi_2$ vanishes on $\mathrm{supp}(\chi_1) \cup \mathrm{supp}(\chi_2)$. Using the diagonalization process, we obtain a sequence $\phi_j \to u$ such that $\phi_j \in C_c^\infty(\bar{U})$.

Proposition 5. *Let $\bar{U}_1 \subseteq U_2$. Then*

(1) $H_o^s(\bar{U}_1) \subseteq H_o^s(U_2) \subseteq H_o^s(\bar{U}_2)$.
(2) $H_o^s(\bar{U}) \cap C^k(M) \subseteq C_o^k(\bar{U})$.
(3) Let $t \leq s$. Then $H_o^t(\bar{U}) \cap H^s(M) \subseteq H_o^s(\bar{U})$.

Proof. Part (1) follows from $C_o^\infty(\bar{U}_1) \subseteq C_o^\infty(U_2) \subseteq C_o^\infty(\bar{U}_2)$.

For (2), let $u \in H_o^s(\bar{U}) \cap C^k(M)$ and let $\chi \in C^\infty(M)$ with $\chi = 0$ on U. We must show $\chi u = 0$. To see this, let $\phi_j \to u$ in $H^s(M)$ with $\phi_j \in C_o^\infty(\bar{U})$. Then $\chi\phi_j \to \chi u$. But $\chi\phi_j = 0$ and thus $\chi u = 0$.

For (3), let $u \in H_o^t(\bar{U}) \cap H^s(M)$ and $\chi = 0$ on U. The $\chi u = 0$ so $u \in H_o^s(\bar{U})$.

The basic lemmas in §2.3 hold word for word, with $H^s(M)$ replaced by $H_o^s(U)$.

For example, since $D^\alpha : C_o^\infty(U) \to C_o^\infty(U)$ is continuous, taking closures we conclude that $D^\alpha : H_o^s(U) \to H_o^{s-l}(U)$. Moreover, $\|D^\alpha u\|_{s-l} \leq \|u\|_s$ when $|\alpha| = l$ and $u \in H_o^s(U)$.

Also, if $\psi \in C^\infty(M)$ let $M_\psi : C^\infty(M) \to C^\infty(M)$ be the map $\phi \mapsto \psi\phi$. Then we have the estimate $\|M_\psi(\phi)\|_s \leq c(s,\psi)\|\phi\|_s$ so M_ψ extends to a continuous map $M_\psi : H^s(M) \to H^s(M)$. Since $M_\psi : C_c^\infty(U) \to C_c^\infty(U)$ we conclude that $M_\psi : H_o^s(U) \to H_o^s(U)$.

In particular, $M_\psi : H_o^s(U) \to H_o^s(U)$ is a well defined continuous linear map for $\psi \in C^\infty(\bar{U})$. To see this, let $\psi \in H_o^s(\bar{U})$, that is $\psi \in C^\infty(U')$ for some $\bar{U} \subseteq U' \subseteq M$. Choose $\chi \in C_c^\infty(U')$ such that $\chi = 1$ on $\bar{U}$. Then $\tilde{\psi} = \chi\psi \in C^\infty(M)$ and it agrees with ψ on $\bar{U}$ (that is, $\tilde{\psi}$ is an extension of ψ from $\bar{U}$ to M). Now $M_{\tilde{\psi}} : H_o^s(U) \to H_o^s(U)$ is continuous and is the unique extension of $M_\psi : C_o^\infty(U) \to C_o^\infty(U)$.

Combining these remarks, let $L = (P_\nu^\mu)$ satisfy the following: For each $1 \leq \mu, \nu \leq m$ we have $P_\nu^\mu = \sum_{|\alpha| \leq l} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^\infty(\bar{U})$. Then $L : H_o^s(U) \to H_o^{s-l}(U)$ is continuous, that is, $\|Lu\|_{s-l} \leq c\|u\|_s$ for some $c = c(s, L)$. Such an operator is elliptic if it satisfies the usual condition: $(P_\nu^\mu(x,\xi))_{1 \leq \mu,\nu \leq m}$ is an invertible $m \times m$ matrix for all $x \in \bar{U}$ and all $0 \neq \xi \in \mathbb{R}^n$.

Theorem 5, when applied to U, takes the following form: $H_o^s(\bar{U}) \subseteq C_o^k(\bar{U})$ if $s > k + \frac{n}{2}$, where $C_0^k(U)$ is the set of smooth functions on $\mathbb{R}^n$ which vanish on the complement on U.

To see this, note that for $s > k + \frac{n}{2}$ that $H_o^s(\bar{U}) \subseteq H^s(M) \subseteq C^k(M)$. But now we can apply the proposition which says $H_o^s(\bar{U}) \cap C^k(M) \subseteq C_o^k(\bar{U})$.

Theorem 8. *If $s < t$ and $u_j \in H_o^s(\bar{U})$ is bounded, then there exists $u_\infty \in H^s(\bar{U})$ such that $u_j \to u_\infty$ in $H_o^t(\bar{U})$.*

Proof. We apply Theorem 5 to conclude that, after passing to a subsequence, there exist $u_\infty \in H^s(M)$ such that $u_j \to u_\infty$ in $H^t(M)$. Choose χ vanishing on U. Then $0 = \chi u_j \to \chi u_\infty$ so $\chi u_j = 0 \Longrightarrow u_j \in H_o^s(\bar{U})$. □

Theorem 9. *Let L be an elliptic operator of order l and rank m on $\bar{U}$. Let $s, t \in \mathbb{R}$ with $s \geq t$. Then there exists $C > 0$ with the following property. If $u \in H_o^{s+l}(U)$ then $Lu \in H_o^s(U)$ and*

$$\|u\|_{H_o^{s+l}(U)} \leq C(\|Lu\|_{H_o^s(U)} + \|u\|_{H_o^t(U)}) \tag{3.19}$$

Proof. For every $p \in \bar{U}$ choose small open sets V_p, U_p containing p with $\bar{V}_p \subseteq U_p$. Let $\phi_p \in C_c^\infty(U_p)$ be a cut-off function (i.e., $0 \leq \phi_p \leq 1$) such that $\phi_p = 1$ on V_p. Let $\tilde{L}_p = \phi_p L + (1 - \phi_p) L(p)$.

Lemma 5. *For U_p sufficiently small, $\tilde{L}_p$ is an elliptic operator on M which agrees with L on V_p.*

Proof. We write $L = a_\alpha D^\alpha$ and $L_p = a_\alpha(p) D^\alpha$ so

$$\begin{aligned} \tilde{L}_p &= [a_\alpha(p) + \phi_p(x)(a_\alpha(x) - a_\alpha(p))] D^\alpha \\ P_{\tilde{L}_p}(\xi) &= a_\alpha(p)\xi^\alpha \; + \; f_\alpha(x)\xi^\alpha : C^\infty(M) \to \mathrm{End}(\mathbb{C}^m) \end{aligned}$$

We wish to show $P_{\tilde{L}_p}(\xi) : C^\infty(M) \to \mathrm{Aut}(\mathbb{C}^m)$. By homogeneity, we may assume $|\xi| = 1$. Then for U_p sufficiently small, $|f_\alpha(x)| \leq \epsilon$ and $\det(a_\alpha(p)\xi^\alpha) > 0$ on M for all $\xi \in M \times S^{n-1}$. But $M \times S^{n-1}$ is a compact set so $\det(a_\alpha(p)\xi^\alpha) \geq \delta > 0$ for all $(x,\xi) \in S^{n-1}$. Since det is a continuous function, we have $\det(a_\alpha(p)\xi^\alpha + f_\alpha(x)\xi^\alpha) \geq \delta/2 > 0$ for all $(x,\xi) \in M \times S^{n-1}$ for ϵ sufficiently small. □

Choose an open cover $U \subseteq V_{p_1} \cup \cdots \cup V_{p_N}$ and let ψ_j be a partition of unity subordinate to the V_j so $\psi_j \tilde{L}_{p_j} = \psi_j L$. Then

$$\|u\|_{H_o^{s+l}(U)} = \|\sum_j \psi_j u\|_{H^{s+l}(M)} \leq \sum_j \|\psi_j u\|_{H^{s+l}(M)}$$

On the other hand,

$$\|\psi_j u\|_{H^{s+l}(M)} \le C(\|\tilde{L}_j\psi_j u\|_{H^s(M)}+\|\psi_j u\|_{H^t(M)}) \le C(\|\psi_j\tilde{L}_j u\|_{H^s(M)}+\|M_j u\|_{H^s(M)}+\|\psi_j u\|_{H^t(M)})$$

where $M_j = \tilde{L}_j\psi_j - \psi_j\tilde{L}_j$ is an operator on M of order $l-1$.

Since $\psi_j\tilde{L}_j = \psi_j L$ we see $\psi_j\tilde{L}_j u = \psi_j Lu \in H^s_o(U)$. Thus

$$\|\psi_j\tilde{L}_j u\|_{H^s(M)} = \|\psi_j Lu\|_{H^s(M)} \le c_j\|Lu\|_{H^s(M)} = c_j\|Lu\|_{H^s_o(U)}$$

We also have

$$\|\psi_j u\|_{H^t(M)} \le c_j\|u\|_{H^t(M)} = c_j\|u\|_{H^t_o(U)}$$

Now we treat the M_j term: Since M_j is of order $l-1$, the continuity estimate give us

$$\|M_j u\|_{H^s(M)} \ \le c_j\|u\|_{H^{s+l-1}(M)} \le \epsilon\|u\|_{H^{s+l}(M)} + C_j\|u\|_{H^t(M)} = \epsilon\|u\|_{H^{s+l}_o(U)} + C_j\|u\|_{H^t_o(U)}$$

Combining we obtain (3.19).

3.2. **Interior regularity.** We prove the following:

Theorem 10. *Let $u \in H^t_o(U)$ and L and elliptic operator of order l on $\bar{U}$ and let $t < s+l$. Assume that $Lu \in H^s_o(U)$. Then $\phi u \in H^{s+l}_o(U)$ for every $\phi \in C^\infty_c(U)$.*

Remark: The theorem shows that $\phi u \in H^{s+l}_o(U)$. One might guess that in fact we have (the stronger conclusion) $u \in H^{s+l}_o(U)$, but this is only true if the boundary of U has some regularity (C^∞ would suffice). This boundary regularity theorem will not be proved in these notes (the proof is quite intricate).

Proof. As in the proof of Theorem 7, it suffices to prove the theorem in the case $t = s+l-1$. We shall show that for every $p \in U$ there is an open set V_p with $p \in V_p \subseteq U$ such that the theorem holds for all $\phi \in C^\infty_c(V_p)$. This implies the general case: Let $\phi \in C^\infty_c(U)$ and let K be the support of ϕ. Choose $p_1, ..., p_N$ such that $K \subseteq \cup_j V_{p_j}$. Let $\psi_j \in C^\infty_c(V_j)$ be such that $\sum_j \psi_j = 1$ on K. Then $\psi_j\phi$ has support in V_j so $\psi_j\phi u \in H^{s+l}_o(U)$. On the other hand, $\sum_j \psi_j\phi = \phi$ so we conclude $\phi u \in H^{s+l}_o(U)$.

Fix $p \in U$ and choose V_0 open, $p \in V_0 \subseteq U$ so that there exists a periodic elliptic operator $\tilde{L}$ on M which agrees with L on V_0. Thus $\phi\tilde{L} = \phi L$ if $\phi \in C^\infty_c(V_0)$. Let V be open such that $p \in V \subseteq \bar{V} \subseteq V_0$ and fix $\phi \in C^\infty_c(V)$. Let $u \in H^{s+l-1}_o(V)$ and assume $Lu \in H^s_o(V)$. Then $Lu \in H^s_o(V)$ implies

$$\phi\tilde{L}(u) = \phi L(u) = \phi Lu \in H^s_o(V)$$

But $\phi\tilde{L}(u) = \tilde{L}(\phi u) + M(u)$ where $M = \phi\tilde{L} - \tilde{L}\phi$ has order $l-1$. Since $u \in H^{s+l-1}_o(U)$ we have $M(u) \in H^s_o(U)$. We conclude $\tilde{L}(\phi u) \in H^s_o(U)$ and hence, by the periodic regularity theorem, we see $\phi u \in H^{s+l}(M)$. But ϕu has compact support in U so $\phi u \in H^{s+l}_o(U)$.

Corollary 2. *Let $u \in H^t_o(U)$ and $f \in C^\infty_o(U)$. Assume that $Lu = f$. Then $u \in C^\infty(U)$.*

Remark: The boundary regularity theorem would imply $u \in C_o^\infty(U)$, but again, this conclusion is only true with some additional regularity assumption on the boundary of U.

4. Estimates and Regularity for compact manifolds

Let M be a smooth manifold of dimension n. This means that we can write $M = \cup_\alpha V_\alpha$ with $V_\alpha \subseteq M$ open, and $\phi_\alpha : V_\alpha \to U_\alpha \subseteq \mathbb{R}^n$ a homeomorphism with the following property. The map $\phi_{\beta\alpha} = \phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(V_\alpha \cap V_\beta) \to \phi_\alpha^{-1}(V_\alpha \cap V_\beta)$ is a diffeomorphism. The V_α are called coordinate neighborhoods on M.

Problem 10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, let $M = \{x \in \mathbb{R}^n : f(x) = 0\}$ and assume ∇f, the gradient of f, has the property $\nabla f(x) \neq 0$ for all $x \in M$. Show that M is a smooth manifold.*

Let $p \in M$ and define

$$T_p(M) \;=\; \{\gamma : (-\epsilon, \epsilon) \to M \;:\; \gamma \text{ smooth and } \gamma(0) = p\}/\sim$$

Here we say that $\gamma_1 \sim \gamma_2$ if there exists α such that $p \in V_\alpha$ and $\sigma_1'(0) = \sigma_2'(0)$ where

$$\sigma_j = \phi_\alpha \circ \gamma_j : (-\epsilon, \epsilon) \to \mathbb{R}^n$$

Problem 11. *Show that the equivalence relationship is independent of the chosen α. Show as well that $T_p(M)$ has a well defined vector space structure (Hints were given in class). If $F : M \to N$ is a smooth manifold, $p \in M$ and $q = f(p)$, define $DF : T_p(M) \to T_q(N)$ by $DF([\gamma]) = [F \circ \gamma]$. Show that DF is a well defined linear map. Show that if $q \in \mathbb{R}^n$ that $T_q(\mathbb{R}^n) = \mathbb{R}^n$ (a canonical isomorphism).*

Problem 12. *Let M be a smooth manifold and let $T = \{(p, v) : p \in M, v \in T_p(M)\}$. Let $\pi : T \to M$ be the map $(p, v) \mapsto p$. Suppose $M = \cup_\alpha V_\alpha$ and $\phi_\alpha : V_\alpha \to U_\alpha$ are the coordinate maps. Let $\tilde{V}_\alpha = \pi^{-1}(V_\alpha)$ and define $\tilde{\phi}_\alpha : \tilde{V}_\alpha \to U_\alpha \times \mathbb{R}^n$ by $(p, v) \mapsto (\phi_\alpha(p), D\phi_\alpha(p))$. Show that $T = \cup_\alpha \tilde{V}_\alpha$ and $\tilde{\phi}_\alpha$ give T the structure of a smooth manifold.*

If V_α is a coordinate neighborhood, and $p \in V_\alpha$, and $\phi_\alpha(p) = (x^1(p), ..., x^m(p))$, then $x^j : V_\alpha \to \mathbb{R}$ are called the local coordinates on V_α. If $y^j : U_\beta \to \mathbb{R}$ are local coordinates on V_β, then $\phi_{\beta\alpha}(x^1(p), ..., x^m(p)) = (y^1(p), ..., y^m(p))$. Sometimes we simply write $y = \phi_{\beta\alpha}(x)$.

If $\pi : N \to M$ is a smooth map between manifolds, and $V \subseteq M$ an open set. We say that $s : V \to N$ is a section of π over U if $\pi \circ s(p) = p$ for all $p \in U$. If $V = M$ we say s is a global section.

Let $\pi : E \to M$ a smooth complex vector bundle of rank m. This means that E is a smooth manifold and π is a smooth map with the following properties.

1. For every $p \in M$, $E_p := \pi^{-1}(p)$ is an m dimension vector space over $\mathbb{C}$.
2. E is locally trivial. This means that there is an open cover $M = \cup_\alpha V_\alpha$ and diffeomeorphisms $F_\alpha : E_{V_\alpha} = \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^m$ satisfying the following. If we write $F_\alpha(x) = (p, v)$ then $p = \pi_\alpha(x)$ and F_α is linear on the fibers. This means $v = \lambda_\alpha(p)(x)$ where $\lambda_\alpha(p) : E_p \to \mathbb{R}^m$ is a linear isomorphism.
   (a) $\pi_\alpha \circ F_\alpha = \pi$
   (b) If we write $F_\alpha(x) = (p, \lambda_\alpha(p)(x))$

3. some other

1. First item
2. Second item
   2.1. First subitem
   2.2. Second subitem
   2.3. Third subitem
3. Third item

(1) For every $p \in M$, $E_p := \pi^{-1}(p)$ is an m dimension vector space over $\mathbb{C}$.
(2) There is an open cover $M = \cup_\alpha V_\alpha$ and diffeomeorphisms $F_\alpha : E_{V_\alpha} = \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^m$ satisfying

(1) For every $p_0 \in M$, $E_{p_0} = \pi^{-1}(p_0)$ is an m dimension vector space over $\mathbb{C}$.
(2) For every $p_0 \in M$ there exists and open set $p_0 \in V \subseteq M$ and smooth sections $e_1, ..., e_m$ of π over V such that $\{e_1(p), ..., e_m(p)\} \subseteq E_p$ is a basis of E_p for all $p \in V$.

The set $e_1, ..., e_m$ is called a local frame of E. Let $\Gamma(V)$ be the space of local sections over V and $u \in \Gamma(V)$. Then we can write $u(p) = \sum u^j(p)e_j(p)$. If $\phi(p) = x$ then $u(\phi^{-1}(x)) = \sum_j u^j(\phi^{-1}(x)e_j(\phi^{-1}(x))$. Let $u^j(x) = u^j(\phi^{-1}(x))$. Then $u = (u^1, ..., u^m) \in C^\infty(U, \mathbb{C}^m)$ and we have an isomorphism $\Gamma(V) \approx C^\infty(U, \mathbb{C}^m)$ and $\Gamma_c(V) \approx C_c^\infty(U, \mathbb{C}^m)$, where $\Gamma_c(V) \subseteq \Gamma(V)$ consists of those sections of E over V which vanish outside a compact subset of V.

Let $\Gamma(M) = \{s : M \to E : \pi \circ s = \mathrm{Id}\}$, the space of global sections on M. If $V \subseteq M$ is and open subset, then $s|_V \in \Gamma(V)$. Let M be compact and $M = \cup_\alpha V_\alpha$ be a covering by small coordinate neighborhoods and choose a frame for E over each V_α. Let ψ_α be a partition of unity subordinate to $\{V_\alpha\}$. If $u \in \Gamma(E)$ then $u = \sum_\alpha \psi_\alpha u$ and $\psi_\alpha u \in \Gamma_c(V_\alpha) \approx C_c^\infty(U_\alpha, \mathbb{C}^m)$. Thus we have an imbedding

$$\Gamma(M, E) \hookrightarrow \bigoplus_\alpha C_c^\infty(U_\alpha, \mathbb{C}^m) \subseteq \bigoplus H_o^s(U_\alpha, \mathbb{C}^m)$$

If $u \in \Gamma(M, E)$ we define

$$\|u\|_{H^s(M,E)} = \sum_\alpha \|\psi_\alpha u\|_{H_o^s(U_\alpha, \mathbb{C}^m)}$$

and define $H^s(M, E)$ to be the completion of $\Gamma(M, E)$ with respect to this norm. In particular, we have

$$H^s(M, E) \subseteq \bigoplus_\alpha H_o^s(U_\alpha, \mathbb{C}^m)$$

Note that if $u \in \Gamma_c(U_\alpha, \mathbb{C}^m) \approx \Gamma_c(V_\alpha, E)$ then we extend by zero to all of M and view $u \in \Gamma(M, E)$. Thus $\Gamma_c(U_\alpha, \mathbb{C}^m) \hookrightarrow \Gamma(M, E)$, which implies, upon taking completions,

$$H_o^s(U_\alpha, \mathbb{C}^m) \hookrightarrow H^s(M, E)$$

Thus, if $u \in H^s_o(U_\alpha, \mathbb{C}^m)$ then

$$\|u\|_{H^s} \;\leq\; C(U_\alpha, s)\|u\|_{H^s_o(U_\alpha)}$$

Proposition 6. *Let $E \to M$ be a smooth vector bundle and $s \in \mathbb{R}$.*

(1) The norms $\|u\|_{H^s(M,E)}$ defined by different open coverings and local frames are equivalent.
(2) The inclusion map $H^s_o(U_\alpha) \to H^s(M,E)$ is continuous.
(3) The inclusion

$$H^s(M,E) \hookrightarrow \bigoplus_\alpha H^s_o(U_\alpha, \mathbb{C}^m)$$

is a homeomorphism onto its image.

4.1. **Sobolev Theory for $E \to M$.** In this section we shall prove all the basic theorems for the space $H^s(M,E)$. Let's recall the set-up.

Let $M = \cup_\alpha V_\alpha$ and $\phi_\alpha : V_\alpha \to U_\alpha$.

Let $\pi : E \to M$ be a vector bundle of rank m such that

$$E_{V_\alpha} := \pi^{-1}(V_\alpha) \;\approx\; V_\alpha \times \mathbb{C}^m \;\approx\; U_\alpha \times \mathbb{C}^m$$

where the diffeormorphisms are linear on fibers.

If $V \subseteq M$ is open then

$$\Gamma(V,E) \;=\; \{s : V \to E \,:\, s \text{ smooth and } \pi \circ s = Id\}$$

is the "space of sections over V".

We now have

$$\Gamma(V_\alpha, E) \;=\; C^\infty(U_\alpha, \mathbb{C}^m)$$

Let ψ_α be a partition of unity with $\text{supp}(\psi_\alpha) \subseteq V_\alpha$. If $u \in \Gamma(M)$ then $\psi_\alpha u \in \Gamma_c(V_\alpha) = C^\infty_c(U_\alpha, \mathbb{C}^m)$ and

$$\Gamma(M) \hookrightarrow \bigoplus_\alpha C^\infty_c(U_\alpha, \mathbb{C}^m)$$

Taking completions, we get

$$H^s(M,E) \;\subseteq\; \bigoplus_\alpha H^s_o(U_\alpha, \mathbb{C}^m)$$

and if $u \in H^s(M,E)$ then

$$\|u\|_s \;=\; \sum_\alpha \|\psi_\alpha u\|_{H^s_o(U_\alpha, \mathbb{C}^m)}$$

If $M = \cup_\alpha V'_\alpha$ and ψ'_α is a partition of unity, then for $u \in \Gamma(M)$ we have

$$\frac{1}{A}\|u\|_s \;\leq\; \|u\|'_s \;\leq\; A\|u\|_s$$

Thus $u_k \in \Gamma(M)$ is a cauchy sequence with respect to the $\|\cdot\|_s$ norm $\iff$ it is a cauchy sequence with respect to the $\|\cdot\|'_s$ norm. Hence $H^s(M,E)$ is independent of the choice of $M = \cup_\alpha V_\alpha$.

Proposition 7. *Let $\psi \in C^\infty(M, \mathbb{C})$. Then the map $M_\psi : H^s(M,E) \to H^s(M,E)$ defined by $M_\psi(u) = \psi u$ is continuous.*

To see this, let $\psi \in C^\infty(M,E)$ and $u \in \Gamma(M,E)$. Then

$$\|\psi u\|_{H^s(M,E)} = \sum_j \|\psi_j \psi u\|_{H^s_o(U_j,\mathbb{C}^m)} \leq \sum_j c(\psi_j,s)\|u\|_{H^s_o(U_j,\mathbb{C}^m)} \leq (\sum_j c(\psi_j,s)c(U_j,s))\|u\|_{H^s(M,E)}$$

Similarly, one shows that Peter-Paul holds for the space $H^s(M,E)$.

Let $u \in H^s(M,E)$. Then we have $\psi_\alpha u \in H^s_o(U_\alpha,\mathbb{C}^m)$ for all α. We say $u \in C^k(M,E)$ if $\psi_\alpha u \in C^k_o(U_\alpha,\mathbb{C}^m)$ for all α.

Next we observe that the Sobolev lemma holds for $H^s(M,E)$.

Proposition 8. *Assume $s > k + \frac{n}{2}$. Then $H^s(M,E) \subseteq C^k(M,E)$.*

Proof. Let $u \in H^s(M,E)$. Then $u = \sum_\alpha \psi_\alpha u$ and $\psi_\alpha u \in H^s_o(U_\alpha,\mathbb{C}^m) \subseteq C^k_o(U_\alpha,\mathbb{C}^m)$ and hence $u \in C^k(M,E)$ (by definition).

Next we prove the Rellich compactness theorem.

Proposition 9. *Let $u_k \in H^s(M,E)$ with $\|u_k\|_{H^s(M,E)} \leq C$. Then there exists $u \in H^s(M,E)$ such that after passing to a subsequence, we have $u_k \to u$ in $H^t(M,E)$ for all $t < s$.*

Proof. First we recall the local version of Rellich compactness. Let $U' \subseteq \mathbb{R}^n$ and $v_k \in H^s_o(U')$. Assume $\|v_k\|_{H^s_o(U')} \leq C$. Then there exists $v_\infty \in H^s_0(\bar{U}')$ such that $v_k \to v_\infty$ in $H^t_0(\bar{U}')$.

If $\bar{U}' \subseteq U$ then $H^s_0(\bar{U}') \subseteq H^s_o(U)$. Thus we conclude: Let $v_k \in H^s_o(U')$. Assume $\|v_k\|_{H^s_o(U')} \leq C$. Thenthere exists $v_\infty \in H^s_0(U)$ such that $v_k \to v_\infty$ in $H^t_0(U)$.

To prove Proposition 9, choose an open subset $U'_\alpha \subseteq U_\alpha$ with the property $\text{supp}(\psi_\alpha) \subseteq U'_\alpha \subseteq \bar{U}'_\alpha \subseteq U_\alpha$, and use U'_α instead of U_α to define the open cover of M and the norm $H^s(M,E)$.

Thus

$$\|u\|'_{H^s(M,E)} = \sum_\alpha \|\psi_\alpha u\|_{H^s_o(U'_\alpha,\mathbb{C}^m)}$$

We have $\frac{1}{A}\|u\|_{H^s(M,E)} \leq \|u\|'_{H^s(M,E)} \leq A\|u\|_{H^s(M,E)}$ for some A independent of u.

Let $u_k \in H^s(M,E)$ with $\|u_k\|_{H^s(M,E)} \leq C$. Then $\psi_\alpha u_k \in H^s_o(U'_\alpha,\mathbb{C}^m) \subseteq H^s_o(M,E)$ and

$$\|\psi_\alpha u_k\|_{H^s_o(U'_\alpha,\mathbb{C}^m)} \leq C_1\|\psi_\alpha u_k\|'_{H^s(M,E)} \leq C_2\|u_k\|'_{H^s(M,E)} \leq C_2 A \leq C_3$$

where C_i is independent of k. Thus, by the local version of Rellich compactness, there exists $u^\alpha_\infty \in H^s_o(U_\alpha) \subseteq H^t(M,E)$ such that $\psi_\alpha u_k$ converges in $H^t_o(U_\alpha) \subseteq H^t(M,E)$ to u^α_∞ for all α (after passing to a subsequence). In other words, $\psi_\alpha u_k \in H^s(M,E)$ and $u^\alpha_\infty \in H^s_o(M)$ and $\psi_\alpha u_k \to u^\alpha_\infty$ in $H^t_o(M,E)$. This implies $u_\infty := \sum_\alpha u^\alpha_\infty \in H^s(M,E)$ and $\sum_\alpha \psi_\alpha u_k = u_k$ converges to u_∞ in $H^t(M,E)$ for all $t < s$. Let u be the limit of u_k so that $u \in H^t(M,E)$. Th $u \in H^s(M,E)$ since $\psi_\alpha u_k \to \psi_\alpha u$ in $H^t_o(U_\alpha,\mathbb{C}^m)$ and the local version of Rellich says that $\psi_\alpha u \in H^s_o(U_\alpha,\mathbb{C}^m)$ for all α.

Theorem 11. *Let E_1, E_2 be vector bundles of rank m, equipped with smooth metrics, and let $L : \Gamma(M, E_1) \to \Gamma(M, E_2)$ be an elliptic operator of order l. Let $u \in H^t(M, E_1)$ and assume $Lu \in H^s(M, E_2)$. Then $u \in H^{s+l}(M, E_1)$ and*

$$\|u\|_{H^{s+l}} \;\leq\; C \cdot (\|Lu\|_{H^s} + \|u\|_{H^t}) \tag{4.20}$$

Moreover, the kernel of $L : H^{s+l}(M, E_1) \to H^s(M, E_2)$ is finite dimensional, independent of s, and contained in $\Gamma(M, E_1)$. Finally, if $u \in H^{s+l} \cap (\ker L)^\perp$, then

$$\|u\|_{H^{s+l}} \;\leq\; C \cdot \|Lu\|_{H^s} \tag{4.21}$$

Corollary 3. *Let $u \in H^t(M, E)$ and suppose $Lu \in C^\infty(M, E)$. Then $u \in C^\infty(M, E)$.*

Proof. The estimate (4.20) follows from the analysis on $\mathbb{R}^n$: Let $u \in H^{s+l}$. Then

$$\|u\|_{H^{s+l}(M,E)} = \sum_\alpha \|\psi_\alpha u\|_{H_o^{s+l}(U_\alpha, \mathbb{C}^m)} \leq C \sum_\alpha (\|L(\psi_\alpha u)\|_{H_o^s(U_\alpha,\mathbb{C}^m)} + \|\psi_\alpha u\|_{H_o^t(U_\alpha,\mathbb{C}^m)})$$

Now the definition of the $H^t(M, E)$ norm tells us that $\sum_\alpha \|\psi_\alpha u\|_{H_o^t(U_\alpha,\mathbb{C}^m)} = \|u\|_{H^t(M,E)}$. Moreover,

$$\|L(\psi_\alpha u)\|_{H_o^s(U_\alpha,\mathbb{C}^m)} \leq C_\alpha \|L(\psi_\alpha u)\|_{H^s(M,E)} \leq C_\alpha \|\psi_\alpha L(u)\|_{H^s(M,E)} + C_\alpha \|M(u)\|_{H^s(M,E)}$$

where $M = \psi_\alpha L - L\psi_\alpha$ has order $l - 1$. Since M is also continuous we obtain

$$\|M(u)\|_{H^s(M,E)} \leq C\|u\|_{H^{s+l-1}(M,E)} \leq \epsilon \|u\|_{H^{s+l}(M,E)} + C'\|u\|_{H^t(M,E)}$$

Since

$$\sum_\alpha \|\psi_\alpha L(u)\|_{H^s(M,E)} \leq C \sum_\alpha \|\psi_\alpha L(u)\|_{H_o^s(U_\alpha,\mathbb{C}^m)} = C\|L(u)\|_{H^s(M,E)}$$

we obtain (4.20).

Now let $u \in H^t$. If $Lu = 0$ then u is smooth, since it is in H^k for every k. If $\ker(L)$ is infinite dimensional, then there exists a sequence $u_p \in \ker(L)$ which is orthonormal in H^{s+l}. Applying (4.20), with s replaced by $s + 1$ and $t = s + l$, the sequence u_p is bounded in H^{s+l+1}. Rellich's theorem implies that, after passing to a subsequence, u_p converges in H^{s+l}. But this contradicts the fact that u_p is orthonormal in H^k. Thus $\ker(L)$ is finite dimensional.

Assume now that (4.21) fails. Then there exists $u_p \in H^{s+l} \cap (\ker L)^\perp$ with $\|u_p\|_{H^{s+l}} = 1$ and $\|Lu_p\|_{H^s} \to 0$. Rellich implies that after passing to a subsequence, there exists $u_\infty \in H^{s+l}$ such that $u_p \to u_\infty$ in H^{s+l-1}. Then $Lu_p \to Lu_\infty$ in H^{s-1}. On the other hand, $\|Lu_p\|_{H^{s-1}} \to 0$ so $Lu_\infty = 0$, that is, $u_\infty \in \ker L$.

Now let $v_p = u_p - u_\infty \in H^{s+l}$. Then $\|Lv_p\|_{H^s} \to 0$ and $\|v_p\|_{H^{s-1}} \to 0$. This implies $\|v_p\|_{H^{s+l}} \to 0$, that is, $u_p \to u_\infty$ in H^{s+l}. But $u_p \perp u_\infty$. Thus $u_\infty = 0$. So $u_p \to 0$ in H^{s+l} which contradicts $\|u_p\|_{H^{s+l}} = 1$. This proves (4.21).

5.1. **Norms and inner products.** Let V be a finite dimensional vector space over $\mathbb{R}$. A norm on V is a function

$$N : V \to \mathbb{R} \qquad v \mapsto N(v) := \|v\|_N$$

with the following properties:

(1) $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0 \iff v = 0$.
(2) $\|\lambda v\| = |v| \cdot \|v\|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.
(3) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

A map $h : V \times V \to \mathbb{R}$ is called an inner product (or a metric) if

(1) $h(v, w) = h(w, v)$ for all $v, w \in V$.
(2) $h(\lambda v, w) = \lambda h(v, w)$ for all $\lambda \in \mathbb{R}$ and $v, w \in V$.
(3) $h(v_1 + v_2, w) = h(v_1, w) + h(v_2, w)$ for all $v_1, v_2, w \in V$.
(4) $h(v, v) \geq 0$ with equality if and only if $v = 0$

We shall often write $h(v, w) = \langle v, w\rangle_h$.

Problem 13. *a) Let h be an inner product on V and define $N_h(v) = \sqrt{\langle v, v\rangle_h}$. Show that N_h is a norm and the map $N_h^2 : V \to \mathbb{R}$ is smooth.*

b) Let N be a norm on V. Assume $N^2 : V \to \mathbb{R}$ is smooth. Show that there is a metric $h = h_N$ on V such that $N = N_h$.

Hint: Let $\langle v, w\rangle_N = \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$ and consider the Taylor expansion of the function $F(v, w) = \langle v, w\rangle$.

Now let $\pi : E \to M$ be a vector bundle. A function $N : E \to \mathbb{R}$ is called a norm on E if $N^2 : E \to \mathbb{R}$ is smooth and if $N_p = N|_{E_p}$ is a norm for each $p \in M$. If $s, t \in \Gamma(M, E)$ are smooth sections, define $\langle s, t\rangle : M \to \mathbb{R}$ to be the funtion $\langle s, t\rangle(p) = \langle s(p), t(p)\rangle_{h_{N_p}}$.

Problem 14. *Let $\pi : E \to M$ be a vector bundle of rank m.*
a) Let $s, t \in \Gamma(M, E)$. Show that $\langle s, t\rangle \in C^\infty(M, \mathbb{R})$.

b) Show that for every point $p \in M$ there is an open set $p \in U \subseteq M$ and $e_1, ..., e_m \in \Gamma(U, E)$ such that $e_1(x), ..., e_m(x)$ is a basis of E_x for every $x \in U$. (we call $e_1, ..., e_m$ a local frame for E).

c) Let $s \in \Gamma(U, E)$ where U be as in part b). Show that $s = s^j e_j$ where $s^j \in C^\infty(U)$.

d) Let $s, t \in \Gamma(M, E)$ and N a metric on E and $s = s^j e_j, t = t^k e_k \in \Gamma(U, M)$. Let $h_{jk} = \langle e_j, e_k\rangle_N$. Show that $\langle s, t\rangle = h_{jk} s^j t^k$.

5.2. **Linear Algebra.** . Let V be a finite dimensional vector space over $\mathbb{R}$ of dimension m and let $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Thus if $\lambda \in V^*$ then $\lambda : V \to \mathbb{R}$ is a linear map. We often write $\langle v, \lambda\rangle = \lambda(v)$. If $e_1, ..., e_m$ is a basis of V then $e^1, ..., e^m$ is a basis of V^* where e^k is defined by the equation $\langle e_j, e^k\rangle = \delta_j^k$, the Kronecker δ. Thus $\delta_j^k = 1$ if $j = k$ and is zero otherwise.

Let $k \geq 0$. Then $\Lambda^k(V)$ is a vector space of dimension $\binom{n}{k}$ if $k \leq n$ and is zero otherwise. If $v \in \Lambda^k(V)$ and $w \in \Lambda^l(V)$ then $v \wedge w \in \Lambda^{k+l}(V)$ and $v \wedge w = (-1)^{kl}\, w \wedge v$. If $e_1, ..., e_m$ is a basis of V then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \;:\; i_1 < i_2 < \cdots < i_k\} \tag{5.22}$$

is a basis of $\Lambda^k(V)$.

We have $(\Lambda^k V)^* = \Lambda^k V^*$. In fact, $\{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k} \;:\; i_1 < i_2 < \cdots < i_k\}$ is the dual basis of the basis (5.22).

Let $f : V \to W$ be a linear map between two vector spaces. Then $f^* : W^* \to W^*$ is the dual linear map and is defined as follows: $f^*(\lambda) = \lambda \circ f$. Similarly $\Lambda^k f : \Lambda^k V \to \Lambda^k W$ is the linear map defined by $v_1 \wedge \cdots v_k \mapsto f(v_1) \wedge \cdots f(v_k)$. If $\dim(V) = \dim(W) = m$, then $\Lambda^m f : \Lambda^m V \to \Lambda^m W$ is a map between 1-dimensional vector spaces and is called $\det(f)$, the determinant of f.

Problem 15. *Show that if $e_1, ..., e_m$ is a basis of V and $u_1, ..., u_m$ a basis of W then*

$$f(e_1 \wedge \cdots \wedge e_m) = \det(A) u_1 \wedge \cdots \wedge u_m$$

where A is the matrix associated to the linear transformation that is, $f(e_j) = A^k_j u_k$.

Hint: Note that left side of the above equation are invariant if we replace e_i by $e_i + \lambda e_j$ and similarly for the right side. Here $i \neq j$ and $\lambda \in \mathbb{R}$.

5.3. **Operations on vector bundles.** . Let $\pi : E \to M$ be a vector bundle of rank m.

Let $E^* = \cup_{p \in M} E^*_p$ and $\Lambda^k E = \cup_{p \in M} \Lambda^k E_p$.

Problem 16. *a) Show that E^* can be given the structure of a vector bundle of rank m so that the following property holds for all open sets $U \subseteq M$. If $e_1, ..., e_m$ is a local frame for $\Gamma(U, M)$ let $\{e^1(p), ..., e^m(p)\}$ be the dual basis of $\{e_1(p), ..., e_m(p)$ for all $p \in U$. Then $e^1, ..., e^m$ is a local frame for E^*.*

b) Show that $\Lambda^k E$ can be given the structure of a vector bundle of rank $\binom{m}{k}$ over M so that the following property holds for all open sets $U \subseteq M$. If $e_1, ..., e_m$ is a local frame for $\Gamma(U, M)$ Then $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \;:\; i_1 < i_2 < \cdots < i_k\}$ a local frame for $\Lambda^k E$ (here $(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})(p) := e_{i_1}(p) \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}(p))$.

Let M be a manifold of dimension n and $p \in M$. Let f be a real valued smooth function defined in some neighborhood U_1 of p and let g be a real valued smooth function defined in some neighborhood U_2 of p. We say $f \sim g$ if $f = g$ on some open neighborhood $p \in U_3 \subseteq U_1 \cap U_2$. This is an equivalence relation and we call an equivalence class a germ at p. Let $\mathcal{G}_p$ be the set of all germs at p. Then $\mathcal{G}_p$ is a ring.

Let $\delta : \mathcal{G}_p \to \mathbb{R}$ be a derivation. This means $\delta(f) = 0$ if f is constant, that $\delta(f+g) = \delta(f) + \delta(g)$ and $\delta(fg) = f(p)\delta(g) + g(p)\delta(f)$. Let $Der_p(M)$ be the set of derivations of $\mathcal{G}_p$.

Problem 17. *a) Show that the map $T_pM \to Der_p(M)$ given by $[\gamma] \mapsto \delta_\gamma$ where $\delta_\gamma(f) = (f\circ\gamma)'(0)$ is an isomorphism.*

b) Define $T'M = \cup_p Der_p(M)$, let $U \subseteq M$ be open and let $\delta : U \to T'M$ be a function with the property $\delta(p) \in Der_p(M)$ for all $p \in M$. We say δ is a smooth section over U if for every $f \in C^\infty(U)$ the map $p \mapsto \delta(p)(f)$ is a smooth function on U. Show that $T'M$ is the structure of a vector bundle with the following property: $\Gamma(U, T'M)$ is the set of all smooth sections over U.

c) Show that if $\phi : U \to V$ is a coordinate map that $\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n}$ is a local frame for $T'M$.

This problem shows TM is isomorphic to $T'M$ so we shall often write TM instead of $T'M$.

5.4. **Existence of solutions.** Let E, F be vector bundles over M of rank m and let $L : \Gamma(E) \to \Gamma(F)$ be a linear map. We say that L is a partial differential operator if there is an open cover $M = \cup_{V\in A} V$ by coordinate neighborhoods $\phi_V : V \to U$ such that $E_V = U \times \mathbb{C}^m$ and $F|_V = U \times \mathbb{C}^m$ for all V, and there is a partial differential operator $L_V : C^\infty(U, \mathbb{C}^m) \to C^\infty(U, \mathbb{C}^m)$ with the following properties. For every $s \in \Gamma(E)$ we have $L(s)|_V = L_U(s|_V)$. We say L is elliptic if L_V is elliptic for all V.

Now let $L : \Gamma(E) \to \Gamma(F)$ be a differential operator of order l between two vector bundles of rank m and let $h, \tilde{h}$ be metrics on E and F. Let dV be a fixed volume form on M. Then the adjoint of L is the differential operator $L^* : \Gamma(F) \to \Gamma(E)$ which is characterized by

$$\int_M \langle L\sigma, \tau\rangle_{h_2}\, dV \;=\; \int_M \langle \sigma, L^*\tau\rangle_{h_1}\, dV$$

for all $\sigma \in \Gamma(E_1)$ and $\tau \in \Gamma(E_2)$. Thus, $\mathrm{Im}(L) \subseteq \ker(L^*)^\perp$. If L is elliptic one easily sees that L^* is elliptic.

Let $\sigma \in C^\infty(M)$ and let

$$\|\sigma\|_h^2 \;=\; \int_M \langle \sigma, \sigma\rangle_h\, dV$$

Then $\|\sigma\|_h^2 \sim \|\sigma\|_{H^o(M,E)}^2$. If $\sigma, \tau \in \Gamma(V)$ we let we let

$$(\sigma, \tau)_h \;=\; \int_M \langle \sigma, \tau\rangle_h\, dV$$

Then for every $s \in \mathbb{R}$ we have $(\sigma, \tau)_h \leq C\|\sigma\|_{H^s(M,E)}\|\tau\|_{H^{-s}(M,E)}$. In particular, we can extend to a pairing $H^s(M, E) \times H^{-s}(M, E) \to \mathbb{C}$ with the property

$$(u, v)_h \leq C\|u\|_{H^s(M,E)} \cdot \|v\|_{H^{-s}(M,E)}$$

for all $u \in H^s(M, E)$ and $v \in H^{-s}(M, E)$. Moreover, if $\sigma_k \to u$ in H^s and $\tau_k \to v$ in H^{-s} then $(\sigma_k, \tau_k)_h \to (u, v)_h$.

Let $\sigma \in \Gamma(M, E_1)$ and $\tau \in \Gamma(M, E_2)$. Then

$$(L\sigma, \tau)_{h_2} \;=\; (\sigma, L^*\tau)_{h_1}$$

If $u \in H^l(M,E_1)$ and $v \in H^0(M,E_2)$ we let $\sigma_j \to u$ in $H^s(M,E_1)$ and and $\tau_k \to v$ in $H^0(M,E_2)$. Taking limits we conclude

$$(Lu,v)_{h_2} \;=\; (u,L^*v)_{h_1}$$

Theorem 12. *Let L be an elliptic operator from V_1 to V_2 and L and elliptic operator. Assume that V_1, V_2 are equipped with metrics. Then* $\ker L, \ker L^*$ *are finite dimensional and* $\operatorname{Im}(L) = (\ker L^*)^\perp$

Proof. Let $f \in (\ker L^*)^\perp \cap \Gamma(V_2)$ and choose $u_j \in H^l(V_1) \cap \ker(L)^\perp$ such that

$$\|Lu_j - f\|_{H^0(M,E;h_2)} \to \mu \;=\; \inf_{u \in H^l(V_1)} \|Lu - f\|_{H^0(M,E;h_2)}$$

We claim that u_j is a cauchy sequence in H^l. To see this, we use the parallelogram identity:

$$\|Lu_j - Lu_k\|^2_{H^0(M,E;h_2)} \;+\; 4\,\|L(\frac{u_j+u_k}{2}) - f\|^2_{H^0} \;=\; 2\,\|Lu_j - f\|^2_{H^0} + 2\,\|Lu_k - f\|^2_{H^0}$$

so

$$\|Lu_j - Lu_k\|^2_{H^0} \;\;\le\;\; 2\,\|Lu_j - f\|^2_{H^0} + 2\,\|Lu_k - f\|^2_{H^0} \;-\; 4\mu$$

Taking the limit we see that Lu_j is cauchy in $H^0(M,E_2)$. Thus the u_j are cauchy in H^l so there exist $u_\infty \in H^l(V_1) \cap \ker(L)^\perp$ such that $u_j \to u_\infty$ in H^l. In particular, $\|Lu_\infty - f\|_{H^0} = \mu$.

Now let $\phi \in \Gamma(V_1)$ be arbitrary. Then

$$0 \;=\; \frac{d}{dt}\bigg|_{t=0} \|L(u_\infty + t\phi) - f\|^2_{h_2} = \langle L\phi, Lu_\infty - f\rangle_{h_2} = (\phi, L^*(Lu_\infty - f))_{h_1}$$

Since this holds for all ϕ, we see that $Lu_\infty - f \in \ker(L^*)$. On the other hand, $Lu_\infty - f \in \ker(L^*)^\perp$. This shows $Lu_\infty = f$. Since f is smooth, then elliptic regularity imples u_∞ is smooth as well.

6. The Hodge Theorem

6.1. **Vector Bundles.** Let E, M be smooth manifolds and $\pi : E \to M$ a smooth map. We say E is a vector bundle of rank m over M if the following conditions hold.

(1) For every $p \in M$, the set E_p is vector space of dimension m.
(2) There is an open cover $M = \cup_\alpha V_\alpha$ and smooth sections $e_1^\alpha, ..., e_\alpha^m \in \Gamma(V_\alpha)$ such that $e_1^\alpha(p), ..., e_\alpha^m(p)$ is a basis of E_p for every $p \in V_\alpha$

In this section we give an alternate definition of a vector bundle which will be useful in practice.

First, let us recall the definition of a manifold. A manifold of dimension n is an equivalence class of triples $(M, \{V_\alpha\}, \{\phi_\alpha\})$ where M is a set, $\{V_\alpha\}$ is a family of subsets of M, $\phi_\alpha : V_\alpha \to \mathbb{R}^n$ is a family of functions satisfying the following.

(1) $M = \cup_\alpha V_\alpha$
(2) Let $U_\alpha = \phi_\alpha(V_\alpha) \subseteq \mathbb{R}^n$, let $U_{\alpha\beta} = \phi_\alpha(V_\alpha \cap V_\beta)$, and let $\phi_{\beta\alpha} : U_{\alpha\beta} \to U_{\beta\alpha}$ be the function $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$. Then $\phi_{\beta\alpha}$ is a smooth map for all α, β.
(3) M is Hausdorff with respect to the topology generated by the ϕ_α (i.e. the smallest topology for which the ϕ_α are homeomorphisms).

We say that $(M, \{V_\alpha\}, \{\phi_\alpha\})$ and $(M, \{V'_\alpha\}, \{\phi'_\alpha\})$ are equivalent if there is a diffeomorphism between them.

Now we give an alternate definition of a vector bundle. A smooth vector bundle of rank m is an equivalence class of six-tuples $(M, \{V_\alpha\}, \{\phi_\alpha\}, E, \pi, \{e_1^\alpha, ..., e_m^\alpha\})$ where $(M, \{V_\alpha\}, \{\phi_\alpha\})$ is a smooth manifold, E is a set, $\pi : E \to M$ is a function and $e_j^\alpha : V_\alpha \to \pi^{-1}(V_\alpha)$ are functions satisfying the following.

(1) For every $p \in M$, the set $E_p = \pi^{-1}\{p\}$ is a vector space over $\mathbb{R}$ of dimension m.
(2) For every $p \in V_\alpha$, the set $\{e_1^\alpha(p), ..., e_m^\alpha(p)\}$ is a basis of E_p.
(3) If $p \in V_\alpha \cap V_\beta$ we write $e_k^\beta(p) = a_k^j(p) e_j^\alpha$. Then $a_k^j \in C^\infty(V_\alpha \cap V_\beta)$.

Remark: If we let $A_{\beta\alpha} = (a_k^j)_{1 \le j,k \le m}$ then $A_{\beta\alpha} \in GL(m, C^\infty(V_\alpha \cap V_\beta))$. Moreover, the following cocycle relation is satisfied: $A_{\gamma\alpha} = A_{\gamma\beta} A_{\beta\alpha}$. We say that A is a cocycle on M with values in $GL(m)$. If $C_\alpha \in GL(m, C^\infty(V_\alpha))$ is any collection, then $\tilde{A}_{\beta\alpha} = C_\beta A_{\beta\alpha} C_\alpha^{-1}$ satisfies the cocycle relation. We say that $\tilde{A}$ is equivalent to A. Let $H^1(M, GL(m))$ be the equivalence classes of cocycles. Then there is a one-to-one correspondence between isomorphism classes of vector bundles of rank m over M and the set $H^1(M, GL(m))$.

6.2. **The Tangent Bundle.** Let M be a smooth manifold and let $p \in M$. If $f, g \in C^\infty(M)$ we say $f \sim g$ if there is an open set $p \in V \subseteq M$ such that $f|_V = g|_V$. Let $C_p^\infty = C^\infty(M)/\sim$. We call $C_p^\infty(M)$ the ring of smooth germs at p. Note that $C_p^\infty(V) = C_p^\infty(M)$ for every open set $p \in V \subseteq M$.

If $F : M \to N$ is a smooth map and $F(p) = q$, then the map $F^* : C_q^\infty(N) \to C_p^\infty(M)$ is a ring homomorphism.

A derivation is a map $v : C_p^\infty(M) \to \mathbb{R}$ satisfying: $v(f+g) = vf + vg$ for all $f, g \in C_p^\infty$ and $v(fg) = f(p)vg + g(p)vf$. The set $T_p(M)$ of derivations of $C_p^\infty(M)$ is a vector space over $\mathbb{R}$.

If $F : M \to N$ is a smooth map and $F(p) = q$ then $DF : T_p(M) \to T_q(N)$ is a linear map, where

$$DF(v)(g) = v(F^* g)$$

If F is a diffeomorphism then DF is an vector space isomorphism.

Proposition 10. *Let U be an open set in $\mathbb{R}^n$ and let $a \in U$. Let $\partial_{j,a} : C_a^\infty \to \mathbb{R}$ be the map $\partial_{j,a} f = \frac{\partial f}{\partial x^j}(a)$. Then $\partial_{j,a} \in T_p(U)$. Moreover, $\partial_{1,a}, ..., \partial_{n,a}$ is a basis of $C_a^\infty(U)$ over $\mathbb{R}$.*

Proof. Let $f \in C_a^\infty$. Then $f = f(a) + \sum_j \partial_{j,p} f \cdot (x^j - a^j) + \sum_{j,k} c_{jk}(x)(x^i - a^i)(x^j - a^k)$ for some $c_{jk} \in C_a^\infty$. Let $v \in T_a U$ and let $v^i = v(x^i)$. Then $vf = v^i \partial_{i,a} f$ so $v = v^i \partial_{i,a} = v^i \frac{\partial}{\partial x^i}$.

Let $U \subseteq \mathbb{R}^n$ and $U' \subseteq \mathbb{R}^m$ be open sets and Let $F : U \to U'$ be a smooth map. Write $y = F(x) = (F^1(x), ..., F^m(x))$ where $x = (x^1, ..., x^n) \in U$ and $y = (y^1, ..., y^n) \in U'$. Fix $a \in U$ and let $b = F(a) \in U'$. Then the linear map $DF : T_a U \to T_b(U')$ is the map

$$v^i \frac{\partial}{\partial x^i} \mapsto F_i^\alpha v^i \frac{\partial}{\partial y^\alpha}$$

where $F_i^\alpha = \frac{\partial F^\alpha}{\partial x^i}$.

Example: Let $\gamma : (-\epsilon, \epsilon) \to M$ be smooth with $\gamma(0) = p$. Define $X = \gamma'(0) \in T_p(M)$ as follows: $Xf = (f \circ \gamma)'(0)$. Every element $X \in T_p M$ is of the form $X = \gamma'(0)$ for some γ.

Let $TM = \{(p, v) | p \in M, v \in T_p(M)\}$ and $\pi : TM \to M$ the map $(p, v) \mapsto p$. We give TM the structure of a smooth vector bundle as follows. Let $M = \cup_\alpha V_\alpha$ be a covering by coordinate neighborhoods equipped with coordinate maps $\phi_\alpha : V_\alpha \to U_\alpha$. Define $e_1, ..., e_n : V_\alpha \to \pi^{-1}(V_\alpha)$ as follows: Let $p \in V_\alpha$ and $f \in C_p^\infty(M)$. Then $e_j^\alpha(p)(f) = \partial_j(f \circ \phi_\alpha^{-1}) = \frac{\partial f \circ \phi_\alpha^{-1}}{\partial x^j}$ where $x = \phi_\alpha(p)$. To see that this defines the structure of a smooth vector bundle, we need only check that the cocycle is smooth. To do this, let's fix α and β and let $x = \phi_\alpha(p)$ and $y = \phi_\beta(p)$. Then $y = \phi_{\beta\alpha}(x)$ and

$$e_j^\alpha(p)(f) = \partial_j(f \circ \phi_\alpha^{-1}) = \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x^j} = \frac{\partial(f \circ \phi_\beta^{-1})}{\partial y^k} \cdot \frac{\partial y^k}{\partial x^j} = a_j^k(p) e_k^\beta(p)(f)$$

and $a_j^k = \frac{\partial y^k}{\partial x^j}$ where $y = \phi_{\beta\alpha}(x)$ is a smooth function. Since the derivative of a smooth function is again smooth, we see that the cocycle $A = (a_j^k)$ is smooth.

Remark on notation: If V_α is as above and $X \in \Gamma(V, TM)$ then $X(p) = X^j(p) e_j^\alpha(p)$ for some smooth functions $X^j \in C^\infty(V)$. If $\phi_\alpha : V_\alpha \to U_\alpha$ is the local coordinate mapping, then let us write $D\phi_\alpha(X)(x) = X(x)$ (this is an abuse of notation). Then we have $X(x) = X^j(x)\partial_j$ where $x = \phi_\alpha(p)$ and $X^j \in C^\infty(U_\alpha)$.

6.3. **Tensor bundles.** Recall that if V and W are vector spaces of dimensions m and n, then $V \otimes W$ is a vector space spanned by $\{v \otimes w | v \in V, w \in W\}$. The elements $v \otimes w$ satisfy the relations $(a_1 v_+ a_2 v_2) \otimes w = a_1 v_1 \otimes w + a_2 v_2 \otimes w$ and $v \otimes (b_1 w_1 + b_2 w_2) = b_1 v \otimes w_1 + b_2 v \otimes w_2$. Let $e_1, ..., e_m$ and $f_1, ..., f_n$ be bases of V and W. Then $e_i \otimes f_j$ is a basis of $V \otimes W$.

We let $V^* = \text{Hom}(V, \mathbb{R})$. If $e_1, ..., e_m$ is a basis of V, let $e^1, ..., e^m$ be the dual basis: $e^i(e_j) = \delta^i_j$. The map $V^* \otimes V \to \mathbb{R}$ defined by $v^* \otimes v \mapsto v^*(v) = (v^*, v)$ is called the contraction map.

For $0 \le r \le m$, the vector space $\Lambda^r V$ has dimension $\binom{m}{r}$. It is spanned by $\{v_1 \wedge \cdots \wedge v_r : v_i \in V\}$. The wedge product is multilinear and alternating: If we fix $v_2, ..., v_m$ then $(av + bw) \wedge v_2 \wedge \cdots \wedge v_m = av \wedge v_2 \wedge \cdots \wedge v_r + bw \wedge v_2 \wedge \cdots \wedge v_r$. Moreover, if σ is a permutation of $\{1, ..., r\}$ then $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(r)} = \text{sign}(\sigma) v_1 \wedge \cdots \wedge v_r$. If $e_1, ..., e_m$ is a basis for V then $\{e_{i_1} \wedge \cdots e_{i_r} : i_1 < i_2 < \cdots i_r\}$ is a basis for $\Lambda^r V$. If $I = \{i_1, ..., i_r\}$ then we write $e_I = e_{i_1} \wedge \cdots e_{i_r}$.

Note that $(E \otimes F)^* = E^* \otimes F^*$ and $(\Lambda^r V)^* = \Lambda^r(V^*)$. To see that last identity, we need to define the pairing $\Lambda^r V^* \times \Lambda^r V \to \mathbb{R}$ by

$$(v_1^* \wedge \cdots \wedge v_r^*, v_1 \wedge \cdots \wedge v_r) \mapsto \sum_{\sigma \in S_r} \text{sign}(\sigma)\, (v_j^*, v_{\sigma(j)})$$

In particular $(e^I, e_J) = \delta^I_J$ so $\{e^I\}$ is the dual basis of $\{e_I\}$.

Now let $h : V \times V \to \mathbb{R}$ be a metric on V and let $e_1, ..., e_m$ be an orthonormal basis of V. Let $e^1, ..., e^m$ be the dual basis of V^*. Then we define a metric h^* on V^* as follows: If $e_1, ..., e_m$ is an orthonormal basis of V with respect to h, then the dual basis $e^1, ..., e^m$ is an orthonormal basis of V^* with respect to the metric h^*. We also define $\Lambda^r h$ on $\Lambda^r V$ by requiring that $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : i_1 < \cdots < i_r\}$ be orthonormal for $\Lambda^r h$.

Now let $E \to M$ and $F \to M$ be vector bundles of ranks m and n. Then we define $E \otimes F \to M$ as follows: If $p \in M$ then $(E \otimes F)_p = E_p \otimes F_p$. Moreover, if $e_1, ..., e_m$ and $f_1, ..., f_n$ are local frames for E and F, then $\{e_i \otimes f_j : 1 \le i \le m, 1 \le j \le n\}$ is a local frames for for $E \otimes F$. We need only check that the transition functions are smooth. Let $e_1', ..., e_m'$ and $f_1', ..., f_n'$ be frames over V' and write $e_i' = a^k_i e_k$ and $f_j' = b^l_j f_l$ with a^k_i and b^l_j smooth. Then $e_i' \otimes f_j' = a^k_i b^l_j e_k \otimes f_l$ and $a^k_i b^l_j$ is smooth (since the product of smooth functions is smooth).

Similarly, we define E^* by defining $(E_p^*) = (E_p)^*$ and requiring $e^1, ..., e^m$ to be a smooth frame for E^*. We define $\Lambda^r E$ by defining $(\Lambda^r E)_p = \Lambda^r(E_p)$ and requiring $\{e_{i_1} \wedge \cdots e_{i_r} : i_1 < \cdot < i_r\}$ to be a smooth frame for $\Lambda^r E$.

Example. The bundle M has local frame $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$. We call the dual frame $dx^1, ..., dx^n$. Thus $dx^i(\partial_j) = \delta^i_j$. Then a local frame for $\Lambda^p T^*M$ is given by $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Sections of $\Lambda^p T^*M \to M$ are called differential p-forms. If η is a differential p-form, then locally on $V_\alpha \subseteq M$ we can write

$$\eta = \sum_{i_1 < \cdots < i_p} \eta_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

for some unique smooth $\eta_{i_1 \cdots i_p} \in C^\infty(U_\alpha)$.

6.4. **Ellipticity.** Let $E \to M$ and $F \to M$ be smooth vector bundles and $L : \Gamma(E) \to \Gamma(F)$ a differential operator of order l. Let $p \in M$ and $\xi \in T_p^*(M)$. Then $\sigma(\xi) : E_p \to F_p$ is a linear map, called the symbol of L. It is defined as follows: Choose local frames $e_1, ..., e_m$ for E and $f_1, ..., f_m$ for F over $V \subseteq M$. Let $V \to U$ be a coordinate map. Then $\Gamma(V, E) = C^\infty(U, \mathbb{R}^m)$,

$\Gamma(V,F) = C^\infty(U,\mathbb{R}^m)$ and $L : C^\infty(U,\mathbb{R}^m) \to C^\infty(U,\mathbb{R}^m)$ is given by $L = (P_i^j(x,D))_{1\le i,j\le m}$. Here $P(x,D) = \sum_{|\alpha|\le l} a_\alpha(x)D^\alpha$. For $\xi \in \mathbb{R}^n$ we let $P(x,\xi) = \sum_{|\alpha|=l} a_\alpha(x)\xi^\alpha$ so $P(x,\xi)$ is a homogeneous polynomial of degree l in the variable ξ. Then $\sigma_L(x,\xi) = (P_i^j(x,\xi))$. We say L is elliptic if for every $x \in M$ and every $0 \neq \xi \in \mathbb{R}^m$ and $0 \neq v \in \mathbb{R}^n$, we have $\sigma(x,\xi)v \neq 0$.

Alternatively, we can define the symbol as follows: Let $s_p \in E_p$ and let $s \in \Gamma(E)$ be such that $s(p) = s_p$. Let $\phi \in C^\infty(M)$ be such that $\phi(p) = 0$ and $d\phi(p) = \xi$. Then

$$L(\phi^l s)(p) = \frac{1}{l!}\sigma_L(\xi)(s_p)$$

We say L is elliptic if $\sigma_L(\xi) : E_p \to F_p$ is invertible for all p and all ξ.

Note that if h_E and h_F are metrics on E and F, then $\sigma_L(\xi)^* = \sigma_{L^*}(\xi)$.

Let $E_0, ..., E_N$ be smooth vector bundles over M equipped with smooth metrics $h_0, ..., h_N$. Let $L_k : \Gamma(E_k) \to \Gamma(E_{k+1})$ be a sequence of differential operators. Let $V = \oplus_{k\ge 0}E_{2k}$ and let $W = \oplus_{k\ge 0}E_{2k+1}$

Proposition 11. *Assume that for each $p \in M$ and each $\xi \in T_p^*M$ the sequence*

$$0 \to (E_0)_p \to (E_1)_p \to \cdots \to (E_N)_p \to 0$$

is exact, where the maps are $\sigma_{L_k}(\xi) : (E_k)_p \to (E_{k+1})_p$. Then

$$L + L^* : V \to W \quad \text{and} \quad L + L^* : W \to V$$

are elliptic operators.

Proof. Assume that $(\sigma + \sigma^*)(\sum_k x_{2k}) = 0$ where $x_{2k} \in (E_{2k})_p$. We must show $x_{2k} = 0$ for all k. Since $\sigma x_{2k} + \sigma^* x_{2k+2} = 0$ for all k we conclude $\sigma^*\sigma x_{2k} = 0$ so

$$\sigma x_{2k} \in \ker(\sigma^*) \cap \mathrm{Im}(\sigma) \subseteq \ker(\sigma^*) \cap \ker(\sigma^*)^\perp = 0$$

Similarly $\sigma^* x_{2k+2} = 0$ and, replacing k by $k-1$, $\sigma^* x_{2k} = 0$.

Now $x_{2k} \in \ker(\sigma)$ implies, by exactness, that $x_{2k} = \sigma(y_{2k-1})$ for some y_{2k-1}. Then $0 = \sigma^* x_{2k} = \sigma^*\sigma(y_{2k-1})$ so $\sigma(y_{2k-1}) \in \ker(\sigma^*) \cap \mathrm{Im}(\sigma) = 0$ as before. Thus $x_{2k} = 0$ and this proves $(\sigma + \sigma^*)$ an injective map from V_p to W_p. Since these two spaces have the same dimension, we see that $(\sigma + \sigma^*)$ is also surjective, that is, $L + L^*$ is elliptic.

Corollary 4. *Assume $L \circ L = 0$. Then $LL^* + L^*L : E_k \to E_k$ is elliptic for all $0 \le k \le N$.*

6.5. **The d operator.** Let $A^k = \Lambda^k T^*M$ and let $A^k(M)$ be the space of differential k-forms We define a first order differential operator $d : \Gamma(A^k) \to \Gamma(A^{k+1})$ as follows:

$$d\eta = \sum_{i_1<\cdots<i_k} \partial_k \eta_{i_1\cdots i_k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

If $\xi \in \mathbb{R}^n$ then the symbol map $\sigma(\xi) : A_p^k \to A_p^{k+1}$ is the map $\eta \mapsto \xi \wedge \eta$ where $\xi = \xi_j dx^j$.

Proposition 12. *Let $p \in M$. The sequence of vector spaces $0 \to A_p^0 \to A_p^1 \to \cdots \to A_p^n \to 0$, with maps given by the symbols $\sigma(\xi)$, is exact.*

Let $Z^p = \ker\{d : A^p \to A^{p+1}\}$ and $B^p = \mathrm{Im}\{d : A^{p-1} \to A^p\}$. Then $B^p \subseteq Z^p$ and we define

$$H^p_{\mathrm{dR}}(M) = Z^p/B^p$$

We define the cup product

$$H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \;\to\; H^{p+q}_{\mathrm{dR}}(M)$$

by the formula: $(\eta + B^p, \omega + B^q) \mapsto \eta \wedge \omega + B^{p+q}$.

Let M be a compact manifold and g a metric on TM. We say that (M, g) is a Riemannian manifold and call g a Riemannian metric. Let $\Delta = dd^* + d^*d$. Then $\Delta : A^p \to A^p$ is elliptic.

Theorem 13.

(1) Let $H^p = \ker(\Delta)$. Then H^p is a finite dimensional vector space over $\mathbb{R}$.
(2) $H^p = \ker(d) \cap \ker(d^*)$
(3) $A^p = H^p \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*) = Z^p \oplus \mathrm{Im}(d^*)$
(4) $Z^p = H^p \oplus B^p = H^p \oplus \mathrm{Im}(d)$. In particular, $H^p_{\mathrm{dR}} \approx H^p$.

Proof. The first statement just says that the kernel of an elliptic operator is finite dimensional. For the second, assume $\eta \in H^p$. Then

$$0 \;=\; ((dd^* + d^*d)\eta, \eta) = (d\eta, d\eta) + (d^*\eta, d^*\eta)$$

For the third statement, observe that H^p, $\mathrm{Im}(d)$ and $\mathrm{Im}(d^*)$ are mutually orthogonal, so we have $H^p \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*) \subseteq A^p$. On the other hand, $A^p = H^p \oplus \mathrm{Im}(\Delta)$ and $\mathrm{Im}(\Delta) \subseteq \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$. Finally, $\ker(d) = \mathrm{Im}(d^*)^\perp$.

6.6. **Poincare duality.** We wish to show that $\dim(H^p_{\mathrm{dR}}(M)) = \dim(H^{n-p}_{\mathrm{dR}}(M))$. Moreover, the cup product pairing is perfect.

To do this, we need to define the Hodge $*$ operator. We first review some linear algebra. Let V be a vector space of dimension m over $\mathbb{R}$. Then $\Lambda^m V\backslash\{0\}$ has two components which are called orientations of V. An oriented vectors space is a pair (V, C) where V is a vector space and C is a connected component of $\Lambda^m V\backslash\{0\}$. Let V be an oriented vector space over $\mathbb{R}$ and h a metric on V. Then there is a unique element dV_h of Λ^m which is positively oriented and of norm one. This element is called the volume form of V. Thus, is $e_1, ..., e_n$ is an oriented orthornomal basis of V, then $dV_h = e_1 \wedge \cdots \wedge e_n$.

We define $* : \Lambda^k V \to \Lambda^{n-k} V$ as follows: Let $e_1, ..., e_m$ be an orthonormal basis of V and let $l = n - k$. Then

$$*(e_{i_1} \wedge \cdots e_{i_k}) = \epsilon \cdot (e_{j_1} \wedge \cdots \wedge e_{j_l})$$

where $\{i_1, ..., i_k, j_1, ..., j_l\} = \{1, 2, ..., n\}$ and $\epsilon \in \{1, -1\}$ is chosen so that

$$\epsilon \cdot e_{j_1} \wedge \cdots \wedge e_{j_l} \wedge e_{i_1} \wedge \cdots e_{i_k} = e_1 \wedge \cdots \wedge e_n \;=\; dV_h$$

A coordinate free characterization of $*$ is given as follows: If $w \in \Lambda^k V$ then $*w$ is the unique element of $\Lambda^{n-k}V$ satsifying

$$v \wedge *w \;=\; (v,w)_h \cdot dV_h$$

One easily checks that $** = (-1)^{p(n-p)}$ and $(-1)^n * \Delta = \Delta *$.

Proposition 13. *We have the following formula for d^*:*

$$d^* \;=\; (-1)^{(p+1)(n-p)+1}(*d*)$$

Proof.

$$\int (d\eta,\omega)\, dV_h \;=\; \int d\eta \wedge (*\omega) \;=\; \int d(\eta\wedge(*\omega)) - (-1)^p \eta \wedge d(*\omega) \;=\; \int (-1)^{p+1}(-1)^{p(n-p)}\eta \wedge (*d*)\omega$$

Corollary 5. *(Poincaré duality)*

(1) *If $\omega \in H^p$ then $*\omega \in H^{n-p}$. Moreover, the map $* : H^p \to H^{n-p}$ is an isomorphism.*

(2) *The map*

$$H^p \times H^{n-p} \;\to\; \mathbb{R}$$

given by $(\eta,\omega) \mapsto \int \eta \wedge \omega$ is a perfect pairing. In particular, the cup product

$$H^p_{\mathrm{dR}}(M) \times H^p_{\mathrm{dR}}(M) \;\to\; \mathbb{R}$$

is a perfect pairing.

7. The spectral theorem

Let M be a compact manifold, $E \to M$ a smooth vector bundle, and $L : \Gamma(E) \to \Gamma(E)$ an elliptic operator. Fix h a metric on E and dV a volume form on M.

Theorem 14. *Assume $L = L^*$. Let $A = \{\lambda \in \mathbb{R} : Ls = \lambda s$ for some non-zero $s \in \Gamma(E)\}$. Then all the elements of A are non-negative. Furthermore, $V_\lambda \;=\; \{s \in \Gamma(E) : Ls = \lambda s\ \}$ is finite dimensional and*

$$L^2(M,E) \;=\; \bigoplus_{\lambda \in A} V_\lambda$$

is an orthogonal direct sum.

Proof. Consider the map $L : H^{s+l}(M,E) \to H^s(M,E)$. Let

$$H^s_0 = \{u \in H^s : (u,f) = 0\}$$

for all $f \in \ker(L)\}$. Then $L_o : H^{s+l}_0 \to H^s_0$ and then there exists $C > 0$ such that

$$\frac{1}{C}\|Lu\|_s \leq \|u\|_{s+l} \leq C\|Lu\|_s$$

so L_0 is an isomorphism of Hilbert spaces.

Consider $L_0^{-1} : H^0 \to H^l$. Then L_0^{-1} is also an isomorphism. On the other hand, $\iota : H^l \to H^0$ is a compact operator. Thus $G = \iota \circ L_0^{-1} : L^2_0 \to L^2_0$ is a compact operator (known as the Green's operator). If follows from functional analysis that the set $\{\mu\}$ of eigenvalues of G form a sequence which converges to zero and $L_0^2 = \oplus W_\mu$ where W_μ is the μ eigenspace of G. But $W_\mu = V_\lambda$ where $\lambda = \mu^{-1}$.

8. The Peter-Weyl Theorem.

Now let $M = G$, a compact Lie group. A representation of G is a continuous homomorphism

$$\pi : G \to GL(n, \mathbb{C})$$

We write $\pi(\gamma) = (a_{ij}^\pi(\gamma))$ so the matrix coeficients a_{ij}^π are continuous functions on G. Let

$$R \subseteq L^2(G) \text{ be the vector space spanned by } \{a_{ij}^\pi : \pi \text{ irreducible}, \ 1 \leq i, j \leq d(\pi)\}$$

Theorem 15. *The space $R \subseteq L^2(G)$ is dense.*

Proof. Let g be a riemannian metric on G which is G invariant and let $\Delta : C^\infty(G) \to C^\infty(G)$ be the Laplacian. Then for every $\gamma \in G$ and every $f \in C^\infty(G)$ we have

$$\Delta(f \circ \gamma) \ = \ (\Delta f) \circ \gamma$$

where $(f \circ \gamma)(g) = f(g\gamma)$. In particular, if $f \in V_\lambda$ then $f \circ \gamma \in V_\lambda$ for all γ since

$$\Delta(f \circ \gamma) \ = \ (\Delta f) \circ \gamma \ = \ (\lambda f) \circ \gamma \ = \ \lambda(f \circ \gamma)$$

This means that G acts on V_λ so $V_\lambda = \oplus V_\pi$ where the V_π are irreducible representations of G. Let $\phi_1, ..., \phi_d$ be a basis of V_π. Then

$$\phi_j(g\gamma) = \sum_i a_{ij}(\gamma)\phi_i(g)$$

Taking $g = 1 \in G$ we see that $\phi_j \in R$.

Remark: In fact, one can show that the a_{ij}^π form an orthogonal basis of $L^2(G)$. Thus, if Σ is the set of all irreducible representation of G then

$$L^2(G) \ = \ \bigoplus_{\pi \in \Sigma} V_\pi^{d(\pi)}$$