# SINGULAR KÄHLER-EINSTEIN METRICS

## 1. The basic results

**1.1. Smooth varieties.** Let $X$ be a smooth projective complex manifold. A Kähler-Einstein metric on $X$ is a Kähler metric $\omega$ satisfying the equation

$$\mathrm{Ric}(\omega) \; = \; \lambda\omega$$

for some $\lambda \in \mathbb{R}$.

The existence of a Kähler-Einstein metric places a big restriction on $X$ since

(1) $\lambda < 0 \Longrightarrow K_X > 0$ (i.e. $X$ is a canonically polarized manifold)

(2) $\lambda = 0 \Longrightarrow K_X \equiv 0$ (i.e. $X$ is Calabi-Yau manifold)

(3) $\lambda > 0 \Longrightarrow K_X < 0$ (i.e. $X$ is a Fano manifold)

In cases 1) and 3) we let $L = K_X$ and $L = -K_X$. In case 2), we fix once and for all an ample line bundle $L \to X$. Thus in all three cases, $(X, L)$ is a polarized projective manifold. The existence and uniqueness results in all three cases are now well understood:

In cases 1) and 2), $X$ has a unique Kähler-Einstein metric in the class of $[L]$. Case 3) is more subtle since there are obstructions to the existence of KE. But the Fano case is now settled: a Fano manifold is Kähler-Einstein if and only if it is K-stable. Moreover, the Kähler-Einstein metric is unique up to automorphisms of $X$.

**1.2. Singular Varieties.** We are interested in generalizing this picture to the case where $\omega$ is a Kähler-Einstein metric on $X$ which is a projective variety that is not necessarily smooth, i.e. $X$ is singular. What do we mean by a Kähler-Einstein metric on a singular variety $X$? To define this notion, we again consider three types of varieties $X$ as above, those with $K_X > 0$ $K_X \equiv 0$ and $K_X < 0$ as above. Already we need to be careful since we haven't defined the canonical line bundle $K_X$ (i.e. the canonical Cartier divisor class) for a singular variety and in fact, not all projective varieties $X$ have a canonical line bundle. If $X$ is normal, it always has a canonical Weil divisor class $K_X'$ which is defined as follows. Let $\eta$ be any meromorphic section of $K_{X_{\mathrm{reg}}}$, and $D \subseteq X_{\mathrm{reg}}$ its divisor. Then we define $K_{X'} \subseteq X$ to be the closure of $D$, which is a linear combination of codimension one subvarieties. So that's easy enough to define, but $K_{X'}$ may not be Cartier, i.e. may not be locally defined by an equation of the form $f = 0$. If $X$ is not normal, then the definition of $K_X$ is a bit more involved.

Let's put these technical points aside for the moment and assume that the canonical line bundle $K_X$ exists, and further assume that one of the following conditions holds.

(1) $K_X > 0$ ($X$ is a "canonically polarized variety")

(2) $K_X \equiv 0$ ($X$ is a "Calabi-Yau variety")

(3) $K_X < 0$ ($X$ is a "Fano variety")

To explain the results, we fix and ample line bundle $L \to X$. In cases 1) and 3) we require $L = K_X$ and $L = -K_X$ respectively. Let $\omega_{FS} \in c_1(L)$ a smooth Kähler metric (the Fubini-Study metric corresponding to some high multiple of $L$ restricted to $X$). Then by the work of many people, in the first two cases there is a Kähler-Einstein metric on $X$ if an only if the singularities of $X$ are "mild" (in a sense that can be made precise). That is, there exists $\varphi_{KE} \in \mathrm{PSH}(X, \omega_{FS}) \cap C^\infty(X_{\mathrm{reg}})$ with the property $\omega_{KE} := \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi_{KE}$ is a Kähler-Einstein metric on $X_{\mathrm{reg}}$ and $\int_{X_{\mathrm{reg}}} \omega_{KE}^n = [L]^n := \int_X \omega_{FS}^n$. In other words, the singular set has Kähler-Einstein measure zero. This is also true in the Fano case provide $X$ is K-stable.

To be more precise, we say $X$ has "mild singularities" if one of the following holds.

(1) $K_X \equiv 0$ and $X$ normal and has at most canonical singularities.

(2) If $K_X > 0$ and $X$ has most semi log canonical singularities. In particular, if $X$ is normal it has at worst log canonical singularities.

(3) If $K_X < 0$ and $X$ has at worst log terminal singularities.

The main existence theorems say that a Kähler-Einstein metric exists in cases (1) and (2). And in case (3), a Kähler-Einstein metric exists if and only if $X$ is K-stable. As mentioned above, these are deep theorems and represent the work of many people over the course of many years. We will sketch the proofs of (1) and (2) later in these notes. The proof of (3) is much more difficult, and we will be addressed in a different set of notes.

So now we know when the Kähler-Einstein metric exists on a singular variety $X$. By definition, it is nice and smooth on $X_{\mathrm{reg}}$. The basic question is: what happens as we approach $X_{\mathrm{sing}}$?

If we pose this question on the level of potentials, the picture is well understood. Write $\omega_{KE} = \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi_{KE}$. Then a basic theorem says that $\varphi_{KE}$ is a bounded function if and only if $X$ has at worst log terminal singularities. And if $X$ has log canonical singularities (which can only happen when $K_X > 0$) or semi log canonical singularities (again, only if $K_X > 0$) then $\varphi_{KE}$ is still bounded above but approaches $-\infty$ very slowly near the log canonical singularities. Roughly speaking, this means that $\varphi_{KE}$ is bounded below by $\log\log|\sigma|_h$ where $\sigma = 0$ is any divisor containing the bad singularities, i.e. those which are log canonical but not log terminal. In particular, in all cases, the Lelong numbers of $\varphi_{KE}$ are all zero.

If we pose this question on the level of metrics, we are asking what happens to $\omega_{KE}$ which is smooth on $X_{\rm reg}$, as we approach $X_{\rm sing}$. This problem is more subtle. We have two metrics on $X_{\rm reg}$, that is we have $(X_{\rm reg}, \omega_{KE})$ and $(X_{\rm reg}, \omega_{FS})$. We would like to compare them. One very general result (proved using the Schwartz lemma) is that $\omega_{KE} \geq \varepsilon\omega_{FS}$. On the other hand, it may happend that $\omega_{KE}$ is much bigger than $\omega_{FS}$: If $X$ is a stable curve of genus $g \geq 2$ then $X$ has cusps which means that $\omega_{KE}$ has infinite diameter while $\omega_{FS}$ has finite diameter. In particular, there are sequences in $X$ which converge in $\omega_{FS}$ but march off to infinity in $\omega_{KE}$. So we can't expect that $\omega_{KE}$ is bounded above by $\omega_{FS}$ in any sense of the phrase "bounded above". On the other hand, we might hope

$$\omega_{KE} \ \leq \ \frac{1}{\varepsilon}\omega_{FS} \quad \text{if } \ \omega_{KE} \ \text{ has finite diameter} \tag{1.1}$$

but even this is false in general. Perhaps a more educated guess would be

$$\omega_{KE} \ \leq \ \frac{1}{|\sigma|^{2-\varepsilon}}\omega_{FS} \quad \text{if } \ \omega_{KE} \ \text{ has finite diameter} \tag{1.2}$$

for some section $\sigma$ of a line bundle $\mathcal{O}(D)$ with the property $\int \frac{1}{|\sigma|^{2-\varepsilon}}\omega_{FS}^n < \infty$. This has a chance, but it seems that such analytic estimates for $\omega_{KE}$ in terms of $\omega_{FS}$ do not appear to be known.

In the absence of analytic estimates, we can ask for a geometric comparison, which is a more coarse description of relationship between $(X_{\rm reg}, \omega_{KE})$ and $(X_{\rm reg}, \omega_{KE})$. One way to do this is by examining their completions.

Thus we let $(\bar{X}, d_{KE})$, be the metric completion of of the KE manifold $(X_{\rm reg}, \omega_{KE})$. The metric completion of $(X_{\rm reg}, \omega_{FS})$ is simply $(X, d_{FS})$. Here $d_{FS}(p, q)$ is the infimum of the lengths of all curves $\gamma : [0, 1] \to X$ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma(0, 1) \subseteq X_{\rm reg}$. Another way to say this is that $d_{FS}$ is the pullback of $\omega_{FS}$ to any smooth resolution $X'$ of $X$. Note that the pullback of $\omega_{FS}$ is only semipositive, but we can still use it to define a semi metric on $X'$ which descends to a genuine metric on $X$.

Since $\omega_{KE} \geq \varepsilon\omega_{FS}$ we see that every $\omega_{KE}$ Cauchy sequence is also and $\omega_{FS}$ Cauchy sequence. We can ask: is the converse true? In other words, we have a distance decreasing map

$$\Phi : (\bar{X}, d_{KE}) \ \to \ (X, d_{FS})$$

We can ask: Is $\Phi$ a homeomorphism? If yes, we can ask if $d_{KE}(x, y) \leq \varepsilon\, d_{FS}(\Phi(x), \Phi(y))^\alpha$ for some $\alpha > 0$.

**The main results:**

Before stating the theorems, we make one more assumption on the singularities in the Calabi-Yau case. We assume, in addition to having at worst canonical singularities, that $X$ is crepant. The difference is that if $X' \to X$ is a log resolution, and if $\Omega'$ is the pullback of $\eta \wedge \bar{\eta}$, then canonical means $\Omega'$ is a semipositive smooth volume form while creprant means it is a strictly positive smooth volume form. Here $\eta$ is a trivializing section of $K_X$.

The first result says that $(\bar{X}, d_{KE})$ has finite diameter if and only if $\varphi_{KE}$ is bounded (which is always that case if $K_X < 0$, but not always the case if $K_X > 0$).

Now we ask: What is the structure of the metric space $(\bar{X}, d_{KE})$? How is it related to the metric space $(X, d_{FS})$? Note that $\bar{X}$ and $X$ both contain copies of $X_{\text{reg}}$ as an open dense subset. The main results are as follows:

(1) The map $\Phi_{\text{reg}} : (X_{\text{reg}}, \omega_{KE}) \to (X_{\text{reg}}, \omega_{FS})$ is distance decreasing (more precisely, $\omega_{KE} \geq \varepsilon\omega_{FS}$ for som $\varepsilon > 0$).

(2) If $X$ has finite diameter, the map $\Phi_{reg}$ uniquely extends to a homeomorphism $\Phi : \bar{X} \to X$. If the diameter is infinite, then $\Phi : \bar{X} \to X^0$ is a homeomorphism where $X^0 \subseteq X$ is the non log canonical locus. This means $X \backslash X^0 \subseteq X_{\text{sing}}$ consists of the singularities which are strictly log canonical (i.e. not log terminal).

Note that 1) already implies that $\Phi_{reg}$ uniquely extends to a continuous map $\Phi : \bar{X} \to X$.

## 2. The Donaldson-Sun peak section construction

Before describing the results, we first explain the motivation. The reason we care about singular Kähler-Einstein varieties is that they arise naturally as limits of families: Suppose $(L_t, h_t) \to (X_t, \omega_t)$ is a family of polarized Kähler manifolds which has a nice Gromov-Hausdorff limit $(X_\infty, g_\infty)$. By nice, we mean that the tangent cones are good (Caution: we do not assume at this point that $\text{Ric}(h_t) = \omega_t$).

Let $p_t \in X_t$ and assume $p_t \to p \in X_\infty$. The [DS] method allows us to construct peak sections $s_t$ centered at the points $p_t$. Recall that $s_t$ is a peak section of $mL_t$ centered at $p$ if $|s|^2_{h_t^m} \sim \exp(-d^2_{m,p_t})$ (meaning the quotient approaches one as $m \to \infty$ and $t \to \infty$) where $d_m$ is the distance to $p_t$ in the metric associated to $m\omega_t$. The construction goes as follows. Let $T_p(X_\infty) = C_p(Y)$ be a tangent cone at $p$ and $\Lambda = T_p(X_\infty) \times \mathbb{C}$ the trivial line bundle equipped with the Gaussian metric $|\sigma_0|_\infty(z) = \exp(-|z|^2)$ normaliazed so that $\|\sigma|_{L^2} = 1$. Here $\sigma_0 = 1$ is the canonical trivializing section. Let $V \Subset U \Subset T_p^{\text{reg}}$ be an open subsets of the smooth locus $T_p^{\text{reg}}$ of $T_p$ satisfying the following properties. There exists $p' \in V$ very close to $p$, and there is a smooth function $\chi : T_p^{\text{reg}} \to [0, 1]$ with compact supported which equals one on $U$ whose gradient has $L^2$ norm let than $\varepsilon$. Let $\Gamma_t : U \to U_t \subseteq X_t$ be an approximating diffeomorphism and $\sigma_t = (\Gamma_t)_*(\chi \cdot \sigma_0)$. The $\sigma_t$ is a smooth section of $L_t$. Solve the equation $\bar{\partial}_t\tau_t = \bar{\partial}_t\sigma_t$ so that $\tau_t$ is very small $L^2$ norm (possible by $L^2$ estimate). Then we let $s_t = \sigma_t - \tau_t$.

To see that $s_t$ is a peak section centered at $p_t$ we verify the following:

(1) $\|\sigma_t\|_{L^2(X_t)} \sim 1$. This follows from the definition of approximating diffeomorphism.

(2) $\|\bar{\partial}_t\sigma_t\|_{L^2(X_t)} \leq \varepsilon$ . This makes use of the fact that $\partial_\infty\sigma_0 = 0$ and $\chi$ has $L^2$ norm at most $\varepsilon$ (we allow $\varepsilon$ to change from line to line, but it will always stay small).

(3) Choose a smooth section $\tau_t$ satisfying $\bar{\partial}_t\tau_t = \bar{\partial}_t\sigma_t$ and $\|\tau_t\|_{L^2(X_t)} \leq \|\bar{\partial}_t\sigma_t\|_{L^2(X_t)} \leq \varepsilon$. This follows from the Hörmander $L^2$ estimate.

(4) $\|\tau_t\|_{L^\infty(V_t)} \leq C\big(\|\tau_t\|_{L^2(U_t)} + \|\bar\partial_t\tau_t\|_{L^2(U_t)}\big) \leq \varepsilon$. This follows from the elliptic estimate together with the fact that $\partial_t\tau_t = \partial_t\sigma_t$ is very close to zero on $U_t$.

(5) The above steps show $s_t$ is a peak section centered at $p_t'$. To show that is is also a peak section centered at $p_t'$ we verify the following.

(6) $\|\nabla s_t\|_{L^\infty} \leq C\|s_t\|_{L^2} \leq C'$. This follows from Moser iteration.

(7) Since $p_t'$ is $\varepsilon$ close to $p_t$ and since $\|\nabla s_t\|_{L^\infty} \leq C$, we see that the value of $s_t$ at $p_t$ is $\varepsilon$ close to the value at $p_t'$. Thus $s_t$ peaks at $p_t$.

In practice, the only delicate points that need to be checked are the Moser iteration argument and the Hörmander $L^2$ estimate.

**2.1. The $L^2$ estimate.** We pause for a moment to explain step (3) a bit more. There are two variations of the $L^2$ esimate that we would like to explain. In each case we have a projective Kähler manifold $(X, \omega)$ and a semiample line bundle $L \to X$ equipped with a metric $h$.

**Theorem 1.** *Assume $h$ is smooth and $\mathrm{Ric}(h) = \omega > 0$ and $\mathrm{Ric}(\omega) \geq -\frac{1}{2}\omega$. Let $\eta$ be an $\bar\partial$–exact $(0,1)$ form with values in $L$. Consider the equation*

$$\bar\partial\tau \;=\; \eta \tag{2.1}$$

*Here $\tau$ is an unknown smooth section of $L$. Then the equation (2.1) has a solution satisfying*

$$\|\tau\|^2_{L^2(h,\omega^n)} \;\leq\; 2\,\|\eta\|^2_{L^2(h,\omega^n)} \quad \text{i.e.} \quad \int_X (\tau\bar\tau)\,h\,\omega^n \;\leq\; 2\int_X (\mathrm{tr}_\omega \eta\wedge\bar\eta)\,h\,\omega^n$$

**Theorem 2.** *Assume $h$ is a (possibly singular) metric such that $\mathrm{Ric}(h) \geq \delta\,\omega$. Let $\eta$ be an $\bar\partial$–closed form with values in the adjoint bundle $L + K$. Consider the equation*

$$\bar\partial\tau \;=\; \eta$$

*Here $\tau$ is an unknown smooth section of $L + K$. Then the equation has a solution $\tau$ satisfying*

$$\|\tau\|_{L^2(h,\omega^n)} \;\leq\; \frac{1}{2\pi\delta}\,\|\eta\|_{L^2(h,\omega^n)} \quad \text{i.e.} \quad \int_X (\tau\wedge\bar\tau)h \;\leq\; \frac{1}{2\pi\delta}\int_X \big(\mathrm{tr}_\omega\,(\eta\wedge\bar\eta)\big)\,h$$

*Note that $\bar\partial$–closed implies $\bar\partial$–exact by Kodaira vanishig..*

Here are the pros Theorem 2.

(1) We do not need a lower bound on $\mathrm{Ric}(\omega)$.

(2) $h$ is allowed to be singular

(3) We only need $L$ to be semiample, not ample.

The main con is that we require $\tau, \eta$ to have values in $L + K$ instead of $L$. This is not a major drawback if $X$ is Calabi-Yau, in which case $L + K = L$, or if $L = mK > 0$ in which case $L = mK = (m-1)K + K$.

In the first case, the Gromov-Hausdorff limit $X_\infty$ will be a singular metric space; in cases 1) and 3) it will contain an open dense subset $\mathcal{R} \subseteq X$ which is a smooth manifold. Not only that, in cases 1) and 3), the metric $d_\infty$ restricted to $\mathcal{R}$ is defined by $\omega_\infty$, a Kähler-Einstein metric on $U$. But $\mathcal{S}$, the complement of $\mathcal{R}$, is a set of high codimension closed subset whose structure is somewhat mysterious. For example, if $X_1, X_2, ...$ is a sequence of Riemann surfaces (i.e. elements of $\mathcal{C}$, the category of smooth projective curves) of fixed genus $g \geq 2$, each equipped with their unique hyperbolic metrics of volume one, then there always exists a convergent subsequence, but the limit $X_\infty$ may leave the category $\mathcal{C}$. On the other hand, perhaps somewhat surprisingly, the limit does not leave the category of projective curves, that is it turns out that $X_\infty$ is a singular curve. Moreover, it's singularities are very mild; they are all nodes. In fact, perhaps even more surprisingly, it turns out that $X_\infty$ is an element of $\bar{\mathcal{C}}$, the category of "stable curves". The category $\bar{\mathcal{C}}$ consists precisely of those curves which appear in the Deligne-Mumford compactification of moduli space. This will be the case in higher dimensions as well.

In the second case, $\omega_\infty$ will be a singular metric, in other words, it will be a "current"; it will be smooth on an open dense subset $U \subseteq X$, but it's behavior on the the singular set $X \backslash U$ is somewhat mysterious. For example, suppose $X$ is a smooth projective variety and $\omega_t$ is a Ricci flow on $X$, that is, $\partial_t \omega_t = -\mathrm{Ric}(\omega_t) - \omega_t$ where $\omega_0$ is fixed Kähler metric. If $K_X > 0$ then it is known that $\omega_t \to \omega_\infty$ where $\omega_\infty$ is the unique Kähler-Einstein metric on $X$ satisfying $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$. If $K_X$ is not only "nearly ample", for example if it is big and nef but not ample, then the limit $\omega_\infty$, still exists. It can't be a smooth Kähler metric however (since otherwise it would be a Kähler-Einstein metric, and that would imply $K_0$ is ample). On the other hand, it is the next best thing, it is a Kähler current, which is a smooth Kähler metric on an open subset $U \subseteq X$ and satisfying the equation $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$ on $X$. Also $\omega_\infty$ has full volume, that is, $\int_U \omega_\infty^n = \int_X \alpha^n$ where $\alpha \in [K_X]$ is any smooth $(1,1)$ form representing the class $[K_X]$.

## 3. Proving the main results

We discuss the proof of the fact that $\bar{X}$ is homeomorphic to $X$ in the case where $X$ is crepant and $K_X \equiv 0$ and $K_X > 0$. The idea in this case (as well as the $K_X > 0$ case) is to identify $\bar{X}$ as the limit of a certain family $(X_t, g_t)$ and then using Donaldson-Sun to construct $\Phi : \bar{X} \to X$ using sections of $mL$. The point here is that to show $\Phi$ is 1-1 we use the [DS] peak section method.

**3.1. The case $K_X \equiv 0$.** Assume $X$ is crepant. We start by explaining the family $(X_t, g_t)$. Let $L \to X$ be the given ample line bundle and $\chi \in c_1(L)$ a Fubini-Study metric. $\Omega$ a smooth volume form on $K_X$ with $\partial\bar{\partial} \log \Omega = 0$. Thus $\Omega = (\eta \wedge \bar{\eta})^{1/m}$ where $\eta$ is a trivializing section of $mK_X$. The Kähler-Einstein equation, which is known to have a solution by [EGZ], is

$$(\chi + \sqrt{-1}\partial\bar{\partial}\varphi_{KE})^n \;=\; \Omega$$

The construction of $(X_t, g_t) \to (X, g_{KE})$ comes from reproving the existence of $\varphi_{KE}$ using the method of a priori estimates, similar to the proof of Yau's theorem (which is slightly different than the original proof of [EGZ]). First we let $\pi : X' \to X$ be a crepant resolution, let $\Omega' = \pi^*\Omega$, and $\chi' = \pi^*\chi$. Let $\omega'$ be an arbitrary Kähler metric on $X'$ and we consider the equation

$$\omega_t^n \;=\; (\chi' + e^{-t}\omega' + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n \;=\; e^{c_t}\Omega', \quad \int_{X'} \varphi_t \, dg_t \;=\; 0 \tag{3.1}$$

and $c_t$ is chosen that that the volumes match up. These equations have smooth solutions by Aubin-Yau. Then our family of approximating spaces is $(X', g_t)$ where $g_t$ is the metric associated to $\omega_t$.

*Sketch of proof (details are in another pamphlet)*

(1) Use Kolodziej to get the $L^\infty$ bound on $\varphi_t$
(2) Apply Tosatti to get the diameter bound
(3) For second order estimates on $X \backslash E$ apply Cheng-Lu as in Yau plus Tsuji's trick
(4) Higher order estimates are then standard.

This is enough to show that $g_t \to g_\infty = g_{KE}$ on $X \backslash E$ so we can descend to $X_{\text{reg}}$ and we prove [EGZ]. □

Returning to proof of the theorem, we need to construct a homeomorphism $\Phi : \bar{X} \to X$. Note that the identity map $\Phi_{\text{reg}} : (X_{\text{reg}}, g_{KE}) \to (X_{\text{reg}}, g_{FS})$ is distance decreasing by the Schwartz lemma. Moreover, by Tosatti's result, $\bar{X}$ is compact so $\Phi$ is surjective. To complete the proof, we must show that $\Phi$ is injective.

**3.2. The Rhong-Zhang construction of $(\bar{X}, d_{KE})$.** The proof that $\Phi$ is injective makes use of the Donaldson-Sun approach. But this requires a better understanding of its tangent cones, i.e. we need to know the tangent cones are "goood". The problem we face is that we don't yet have a way of constructing $(\bar{X}, d_{KE})$. This is where the results of Rong-Zhang come into the picture. To construct $\bar{X}$, they say that instead of removing the exceptional divisor $E$ and then taking the $C^\infty$ limit on $X' \backslash E \;=\; X_{\text{reg}}$, we keep $E$ and take the Gromov-Hausdorff limit on the whole space. In other words, we let $(X_\infty, d_\infty) = \lim_{t\to\infty}(X', g_t)$. Then they prove

$$(X_\infty, d_\infty) = (\bar{X}, d_{KE}) \quad \text{and} \quad \mathcal{R} = X_{\text{reg}} \tag{3.2}$$

where $\mathcal{R} \subseteq X_\infty$ is the set of regular points. This is very useful since it allows us to apply Cheeger-Colding theory, which implies the tangent cones are good, so that the [DS] method can be applied.

Why is such (3.2) reasonable?

(1) $\mathcal{R} \supset X_{\text{reg}}$ by the a priori estimates of [EGZ].

(2) Cheeger-Colding tells us that since we have 2-sided Ricci bounds, $\mathcal{R} \subseteq X_\infty$ is an open dense subset (in particular $X_\infty$ is the completion of $\mathcal{R}$ ), and $\mathcal{R}$ is a $C^{2,\alpha}$ manifold and $d_\infty|_{\mathcal{R}} = g_\infty$ is a $C^{1,\alpha}$ metric.

(3) Moreover, since $g_t$ is a Kähler-Einstein metric, we know that $\mathcal{R}$ is $C^\infty$ and $g_\infty$ is smooth and Kähler-Einstein .

This is enough to guarantee $\bar{X} \subseteq X_\infty$ and $X_{\text{reg}} \subseteq \mathcal{R}$. Rong-Zhang prove these inequalities are actually equalities.

So this is how we get our hands on the completion. It allows us to use [DS] to study $\bar{X}$ since $(X_\infty, d_\infty)$ is a nice non-collapsing Gromov-Hausdorff limit - this means we can use Cheeger-Colding Theory to get good tangent cones on $X_\infty$, and then use those tangent cones as in [DS] to prove injectivity of $\Phi : \bar{X} \to X$.

So how do we make all this work? First of all, the Schwartz lemma implies that the identity map $\Phi_{\text{reg}} : (X_{\text{reg}}, g_{KE}) \to (X, g_{FS})$ is distance decreasing, so we get a distance decreasing map $\Phi : \bar{X} \to X$. Since $\bar{X}$ is compact, it is surjective and a cauchy sequence in $(X_{\text{reg}}, g_{FS})$ has a convergent subsequence in $(\bar{X}, g_{KE})$.

Next we use the [DS] method. The key step is proving that $\Phi$ is injective. So given distinct points $p, q \in \bar{X}$ we must show $\Phi(p) \neq \Phi(q)$. To do this we need to check the two delicate points.

Step (6) in the [DS] approach is establishing $\|s\|_{L^{\infty,\#}} + \|\nabla s\|_{L^{\infty,\#}} \leq C\|s\|_{L^{2,\#}}$ using Moser iteration. This is ok since the Sobolev constants are bounded (due to lower bounds on Ricci and volume, and upper bounds on the diameter). This step is somewhat delicate due to the fact that $X$ is singular, but can be easily handled using a cuttoff function.

We continue with the construction of the peak section. As in [DS] we pick a point $p \in X_\infty$ and use the tangent cone to construct an smooth peak section $\sigma$. Then we solve the equation $\partial\bar{\tau} = \partial\bar{\sigma}$. Since $\|\partial\sigma\|_{L^2}$ is small, we hope to find a $\tau$ with small $L^2$ norm as well.

In [DS] this is done by working on $X_i$ and using estimates for the eigenvalues of Laplacian implied by the Bochner formula. But in this case, we can't use the original method of [DS] to establish the $L^2$ estimate since $L + \varepsilon A$ is not a line bundle, even if $\varepsilon \in \mathbb{Q}$ (in which case it is just a $\mathbb{Q}$ line bundle). Thus, as $\varepsilon \to 0$ we end up constructing peak sections not of $L$, but of higher and higher powers of $L$.

To get around this difficulty, let's go back to the general strategy. We start with the smooth peak section $\sigma$ which has compact support inside of $X_{\text{reg}}$. Then we solve the $\bar{\partial}$ equation $\bar{\partial}\tau = \bar{\partial}\sigma$ where $\sigma$ is a smooth peak section of $L$. To get the $L^2$ estimate, which is step (3) of the [DS] approach, we use Demailly's $L^2$ estimate instead Bochner's formula. Of course we need to adapt Demailly to $X$ since it is singular, but this can be done using a cuttoff. So the peak sections separate points and in this way we prove injectivity.

**3.3. The case $K_X > 0$.** . The strategy is similar to the CY case. Our goal is to show $\bar{X}$ is homeomorphic to $X$ in the case where $X$ is log terminal (and homoeomorphic to $X$ with the non-klt locus removed if $X$ is log canonical). The idea in is again to identify $\bar{X}$ as the limit $X_\infty$ of a certain family $(X_t, g_t)$ using Rong-Zhang, and then applying Schwartz to construct $\Phi : \bar{X} \to X$ using sections of $mL$ and show $\Phi$ is 1-1 using the peak section method of [DS].

Again we start by describing the approximating family $(X_t, g_t)$.

Recall first that by [EGZ] the following KE equation has a solution.

$$(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n \;=\; e^{\varphi}\,\Omega$$

where $\chi$ is the Fubini-Study metric and $\Omega$ is an adapted volume form. The form $\Omega$ is smooth and $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega$.

As in the Calabi-Yau case, we let $\pi : X' \to X$ be a crepant resolution. Continuing the strategy used in the CY case, (3.1), we again add a small multiple of an a Kähler metric and invoke Aubin-Yau:

$$(\chi' + e^{-t}\omega' + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n \;=\; e^{\varphi_t}\Omega' \;\text{ i.e. }\; \mathrm{Ric}(\omega_t) \;=\; -\omega_t + e^{-t}\omega' \tag{3.3}$$

where, as before, $\chi'$ is the Fubini-Study metric (scaled so that it is in the right class) and $\sqrt{-1}\partial\bar{\partial}\log\Omega' = \chi'$. Then as before, using the [EGZ] version of Kolodziej, Yau-Tsuji, etc. as before, we show that $g_t \to g_{KE}$ on $X'\backslash E = X_{\mathrm{reg}}$.

Moreover, the $\omega_t$ have uniformly bounded diameter by Guo-Fu-Song (at the time at which Song originally proved his theorem, [GFS] was not available so he had to employ a different method. More about this later).

Continuing with the general strategy, we want to identify $\bar{X}$ and $X_\infty$ where $\bar{X}$ is the metric completion of $(X_{\mathrm{reg}}, \omega_{KE})$ and $X_\infty$ is the Gromov-Hausdorff limit of $(X, \omega_t)$. We would like to apply Rong-Zhang, but there is an obstacle that wasn't present in the Calabi-Yau case: In the Calabi-Yau case we are taking a limit of Kähler-Einstein metrics. But here the approximating metrics $\omega_t$ are not Kähler-Einstein, only approximately Kähler-Einstein. If fact, we don't even have a two-sided Ricci bound. Thus the general Cheeger-Colding results only tell us that $\mathcal{R}$ and $d_\infty|_{\mathcal{R}}$ have rather weak regularity properties (e.g without additional assumptions, we don't even know that $\mathcal{R}$ is a manifold).

To solve this regularity problem, we use a powerful result of Tian-Wang which says that from the point of view of Cheeger-Colding theory, approximate Kähler-Einstein metrics behave just like Kähler-Einstein metrics. In other words, $\mathcal{R}$ is a manifold, and $\mathcal{R} \subseteq X_\infty$ is open, dense and convex. Morever $d_\infty|_{\mathcal{R}} = \omega_\infty$ is Kähler-Einstein.

Next we apply Rong-Zhang (or rather the method of Rong-Zhang) to show that $\mathcal{R} = X_{\mathrm{reg}}$ and $\omega_\infty = \omega_{KE}$ on $\mathcal{R} = X_{\mathrm{reg}}$.

Finally, as in the CY case, the map $\Phi : (X_\infty, g_{KE}) \to (X, g_{FS})$ is distance decreasing. The proof that $\Phi$ is injective goes through as before. The fact that $X_\infty$ has compact immediately implies surjectivity. This completes the proof. □

Before moving on, we mention that in the original proof, Song employed a different method since at the time, [GFS] was not available. We now briefly sketch the idea that was used in the original paper.

The problem we face is proving the diameters of $g_t$ are uniformly bounded without using [GFS]. Since the diameters may be unbounded, we are forced to replace $(X_\infty, d_\infty)$ with a pointed Gromov-Hausdorff limit $(X_\infty, p_\infty, d_\infty)$.

The next step is showing that $\Phi : X_\infty \to X$ induces a biholomorphic map $\mathcal{R} \to X_{\rm reg}$. This use Tian-Wang combined with Rong-Zhang.

Then we prove the following:

**Lemma 1.** *Let $\Phi : X_\infty \to X$ be as above. Let $\zeta \in X$. Then the following are equivalent.*

1. *There exists a sequence $x_j \in \mathcal{R} \subseteq X$ such that $\Phi(x_j) \to \zeta$ and $d_\infty(p_\infty, x_j) \to \infty$.*
2. *For every sequence $x_j \in \mathcal{R} \subseteq X$ such that $\Phi(x_j) \to \zeta$ we have $d_\infty(p_\infty, x_j) \to \infty$.*

To prove the lemma, we assume not. Thus $x_j, y_j \in \mathcal{R}$ with $\Phi(x_j) \to \zeta$ and $\Phi(y_j) \to \zeta$ but $x_j \to \infty$ and $y_j$ is bounded. After passing to a subsequence, we may assume $y_j \to y \in X_\infty$ and since $\Phi$ is continuous, $\Phi(y) = \zeta$. Now choose a path $\gamma_j$ in $X_{reg}$ joining $\Phi(x_j)$ to $\Phi(y_j)$ such that $\gamma_j \subseteq B_{d_{FS}, r_j}(\zeta) \subseteq X$ and $r_j \to 0$. Then we can choose a point $y_j' \in \mathcal{R}$ such that $\Phi(y_j') \in \gamma_j$ and $d_\infty(y_j', y_j) = 1$. After passing to a subsequence, $y_j' \to y'$ and $\Phi(y') = \zeta$ but $d_\infty(y', y) = 1$. This contradicts the injectivity of $\Phi$.

Now the "blow-up trick" of Song-Weinkove shows that given any point $\zeta \in X$, there exists a sequence $x_j \in \mathcal{R}$ such that $\Phi(x_j) \to \zeta$ and $x_j$ is bounded. This proves $X_\infty$ has finite diameter and $\Phi$ is onto. □

## 4. The case where $X$ is smooth and $K_X$ is big and nef.

In this case, Tsuji proves the existence of a Kähler-Einstein current using the Kähler-Ricci flow:

$$\dot{\varphi} \;=\; \log \frac{(\chi + e^{-t}(\omega_0 - \chi) + i\partial\bar\partial\varphi}{\Omega} \;-\varphi\,, \quad \varphi(0) = 0. \tag{4.4}$$

Here $\chi \in [K_X]$, $\Omega$ is a smooth volume form with $\chi = \sqrt{-1}\partial\bar\partial\Omega$, and $\omega_0$ is a fixed Kähler form.

Tsuji's proof starts with an application of Kawamata's theorem which says that $K_X$ is semiample, so we can take $\chi$ to be pullback of Fubini-Study. Note that $\chi$ is only semi-positive since the Kodaira map need not be an imbedding (since $K_X$ need not be ample). Tsuji proved that the Kähler-Ricci flow converges in $C^\infty$ away from the non-Kodaira locus (i.e. the set where $\chi$ fails to be a metric). Moreover, $\varphi_{KE}$ has at worst log decay near this locus. In fact, $\varphi_{KE}$ is actually bounded by Kolodziej.

In Jian's paper "An analytic proof of Kawamata's theorem", he tries to prove Tsuji's theorem without using Kawamata's theorem, and then deduce Kawamata as a corollary.

To get started, we use the same equation (4.4) but this time $\Omega$ is any smooth volume form and $\chi = \sqrt{-1}\partial\bar{\partial}\Omega$ is a smooth $(1,1)$ form which may not have any positivity properties. The proof goes through, but with a slightly weaker conclusion.

(1) The solution $\varphi_t$ converges to a unique $\varphi_\infty \in \mathrm{PSH}(X\backslash D, \chi)$ where $D$ is any divisor such that $K_X - \varepsilon D$ is ample. Moreover, $\varphi_t$ converges to a Kähler-Einstein metric $\varphi_\infty$ on $X\backslash D$ and hence on $X\backslash B_+(K_X)$, where $B_+(K_X)$ is the augmented base locus (i.e. $X\backslash B_+(K_X)$ is the intersection of all divisors $D$ such that $K_X - \varepsilon D$ is ample for $\varepsilon$ sufficiently small and positive).

(2) $\varphi_t \geq \varepsilon \log |\sigma_D| - C_\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Actually, we can push Tsuji's proof a bit further and show that $\varphi_t \to \varphi_\infty$ smoothly on $X\backslash B(K_X)$ where $B(K_X)$ is the stable locus. The main step for this is the $C^0$ estimate which is equivalent to showing that for every fixed $\sigma \in H^0(X, mK_X)$ that $\sup_X |\sigma|^2_{h_t^m} \leq C$ for some $C$ independent of $t$. To carry out this step, we use the maximum principle to $H = \log \sum_j |\sigma_j|^{2/m}_{h_t^m}$. We easily compute $\Box H = n - \mathrm{tr}_\omega \theta$ where $\theta$ is the Fubini-Study semi-metric on $X' \to X$. More precisely, $X \dashrightarrow \mathbb{P}^N$ is a rational map so we resolve the singularity of this map $\pi : X' \to X$ so that $X' \to \mathbb{P}^N$ is a regular map. This means $\pi^*(mK_X) = L + E$ with $E$ exceptional and $L$ the pullback of $\mathcal{O}(1)$. We let $\theta$ be the Fubini-Study semi-metric on $L$.

The problem here is that when we apply the maximum principle, we get $\mathrm{tr}_\omega \theta \leq C$ which implies $\frac{\theta^n}{\omega_t^n} \leq C$, but we don't know if $H = \log \frac{\sum |\sigma_j|^{2/m}}{\omega_t^n} \leq \log \frac{\theta^n}{\omega_t^n}$ since $\theta$ is only a semi positive $(1,1)$ form. To remedy this, we introduce an auxillary bounded PSH function $\psi$ satisfying $(\theta + \sqrt{-1}\partial\bar{\partial}\psi)^n = \Omega'$ where $\Omega'$ is any fixed smooth volume form on $X'$ and apply the maximum principle to $H + \psi$. We still have a problem since although $\theta$ gets replaced by $\theta_\psi$ which is strictly positive, we can't apply the maximum principle since $\psi$ is only smooth away from $D$. So we use one more application of Tsuji to for the maximum to occur away from $D$.

Let $X$ be a compact complex manifold with $K_X$ big and nef. Let $\omega_t$ satisfy

$$\partial_t \omega \;=\; -\mathrm{Ric}(\omega) - \omega$$

Let $\Omega$ be a smooth volume form and $\chi = \sqrt{-1}\partial\bar\partial \log \Omega$. Let $h_t = \frac{1}{\omega_t^n}$ and $h_\chi = \frac{1}{\Omega}$. Then

$$\chi = \mathrm{Ric}(h_\chi), \quad -\mathrm{Ric}(\omega_t) = \mathrm{Ric}(h_t),$$

Fix $\omega_0 > 0$. Then

$$\dot\varphi = \log \frac{(\chi + e^{-t}(\omega_0 - \chi) + \sqrt{-1}\partial\bar\partial\varphi)^n}{\Omega} \;-\; \varphi \;\text{ so }\; \omega_t^n = e^{\varphi + \dot\varphi}\,\Omega$$

Choose $D \subseteq X$ such that $K_X - \varepsilon D > 0$ and $h_D$ such that $\chi - \varepsilon\,\mathrm{Ric}(h_D) \geq c\omega_0 > 0$.

**Theorem 3.** *$\varphi_t \to \varphi_\infty$ in $C^\infty(X\backslash D)$ where* $\mathrm{Ric}(\chi + i\partial\bar\partial\varphi_\infty) = \mathrm{Ric}(\omega_\infty) = -\omega_\infty$.

If we knew $K_X$ to be semi-ample, then we could identify $D$ with one of the divisors defining the base locus of $K_X$.

We sketch the proof of Theorem 4. It is easy to obtain upper bound for $\varphi$ and $\dot\varphi$. The key is proving a lower bound away from $D$.

$$\begin{aligned} \Box\,\varphi \;&=\; \dot\varphi \;-\; n \;-\; \mathrm{tr}_\omega(\chi + e^{-t}(\omega_0 - \chi)) \\ \Box\,\dot\varphi \;&=\; -\dot\varphi \;-\; e^{-t}\mathrm{tr}_\omega(\omega_0 - \chi) \end{aligned} \tag{4.5}$$

Thus

$$\Box\, Q \;=\; \Box\left(\varphi + \dot\varphi \;-\; \varepsilon \log|\sigma_D|^2_{h_D}\right) \;=\; \mathrm{tr}_\omega\,\chi \;+\varepsilon\,\partial\bar\partial \log|\sigma_D|^2_{h_D} \;-\; n \geq\; c\,\mathrm{tr}_\omega\omega_0 \;-\; n$$

Thus if $Q$ achieves its minimum at $(x_0, t_0)$ we have $\mathrm{tr}_\omega\omega_0 \leq C$ so at the min

$$\left(\frac{\omega_0^n}{\omega^n}\right)^{1/n} \;\leq\; \frac{1}{n}\mathrm{tr}_\omega\omega_0 \;\leq\; C$$

so

$$Q(x_0, t_0) \;=\; \log \frac{\omega^n}{|\sigma|^2_h\,\Omega}(x_0, t_0) \;\geq\; -C$$

Next we try to prove an analogue of Theorem with $X\backslash D$ replaced by the complement of the base locus.

Let $\pi : X' \to X$ be a resolution for the base locus of $\sigma_0, ..., \sigma_N$, a basis of $H^0(X, mK_X)$. Then

$$\pi^*(mK_X) \;=\; L' + E'$$

with $L' \to X'$ semiample and $E$ exceptional. Let $\sigma'_j$ be the section of $L$ defined so that the map $X' \to \mathbb{P}^{N_m}$ defined on $X'\backslash E$ by $(\sigma_0, ..., \sigma'_N)$ agrees with that defined by $(\sigma_0, ..., \sigma_N)$

on $X\backslash B$ where $B$ is the base locus (i.e. $B = \pi(E)$). We obtain a map $X' \to \mathbb{P}^N$ which contracts $E$. Let $\theta' = \frac{1}{m}\omega_{FS}$ where $\omega_{FS}$ is the pull back to $X'$ of the Fubini-Study metric so $\theta' \geq 0$. Since $L$ is big and semi-ample there exists an effective divisor $D'$ on $X'$ such that

$$\theta \ - \ \varepsilon \, \mathrm{Ric}(h_{D'}) \ > \ 0$$

for some choice of smooth metric on $D'$.