Notes on Lempert's solution of the Dirichlet problem for Monge-Ampére

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1 Introduction

In these notes we summarize Lempert's papers:

1. "La métrique de Kobayashi et la représentation des domaines sur la boule", Bulletin de la S.M.F., tome 109, (1981), 427-474
2. "Intrinsic distances and holomorphic retracts", Complex analysis and applications, Sofia, 1984
3. "Solving the degenerate complex Monge-Ampère equation with one concentrated singularity", Math. Ann. 263, (1983) 515-532

One of the main problems considered in these papers is solving the homogeneous Monge-Ampère (HMA) equation:

$$\det(\partial_i \partial_{\bar{j}} u) = 0 \text{ in } D \ ; \quad u = 0 \ \text{ on } \partial D, \quad \partial\bar{\partial} u \geq 0$$

where $D \subseteq \mathbf{C}^n$ is a smoothly bounded convex domain (later we will allow more general boundary values). Thus the Hessian of u has non-negative eienvalues, and at each point $z \in D$, at least one of the eigenvalues vanishes. The maximum principle tells that the only smooth solution to the HMA equation is $u = 0$, so, in order to get a non-trivial solution, we must allow u to be singular.

For example, if $D = U = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$, then $u = \log|\zeta|$ is the solution to the HMA with logarithmic singularity at the origin. More generally, if $\zeta_0 \in U$ is arbitrary, then

$$u(\zeta) \ = \ \log \frac{|\zeta - \zeta_0|}{|1 - \zeta\bar{\zeta}_0|} \tag{1.1}$$

is the solution to the HMA with a log singularity at ζ_0, that is, $u(\zeta) - \log|\zeta - \zeta_0|$ is bounded.

Now let $D \subseteq \mathbf{C}^n$ be an arbitrary convex domain. Lempert shows that for any point $z_0 \in D$, there is a unique solution u, smooth on $D \backslash \{z_0\}$, with the property: $u(z) - \log|z - z_0|$ is bounded on D. In fact, Lempert's solution has the following properties: $\text{rank}(\partial\bar{\partial}u) = n-1$ on $D\backslash\{z_0\}$ and u vanishes on ∂D to first order. In otherwords, $u = pr$ where p is a positive function in a neighborhood of ∂D and r is a defining function for D, that is, $D = \{r < 0\}$ and $\partial D = \{r = 0\}$. Moreover, for Lempert's solution, the function $\epsilon(z) = u(z) - \log|z - z_0|$ is not only bounded, but it becomes smooth on B, the blow up of D at the point z_0. In otherwords, if $v \in \mathbf{C}^n$ is a non-zero, vector then the limit $\epsilon(z, v) = \lim_{t\to 0, t\in\mathbf{C}} \epsilon(z + tv)$ always exists, and if if $z = z_0$, it depends (in general) on the direction $[v] \in \mathbf{P}^{n-1}$. Moreover, $\epsilon(z, v)$ is a smooth function of $(z, v) \in B = \{(z, [v]) : z \in D, [v] \in \mathbf{P}^{n-1}, (z - z_0) \in [v]\}$.

To do describe Lempert's solution, we first rewrite the explicit formula given by (1.1) in the case $D = U$ in a slightly different fashion: Let $\delta(\zeta_1, \zeta_2)$ be the hyperbolic distance in U. Thus, $\delta(0, \zeta) = \log \frac{1+|\zeta|}{1-|\zeta|}$. Then we can solve for $|\zeta|$ and we get $|\zeta| = \frac{e^\delta - 1}{e^\delta + 1}$ so

$$u(\zeta) \;=\; \log\left(\frac{e^{\delta(\zeta_0,\zeta)} - 1}{e^{\delta(\zeta_0,\zeta)} + 1}\right) \tag{1.2}$$

Note that u vanishes on the boundary since $\delta(\zeta_0, \zeta) \to \infty$ as $\zeta \to \partial U$.

To write down Lempert's solution, we first need to generalize δ in (1.2): Let $D \subseteq \mathbf{C}^n$ be an arbitrary bounded domain. Then D carries a natural metric, called the Kobayashi metric, which generalizes δ on U: Let $z_1, z_2 \in D$ and define

$$K_D(z_1, z_2) \;=\; \inf\{\delta(\zeta_1, \zeta_2) : f(\zeta_1) = z_1, f(\zeta_2) = z_2 \;\text{ for some holomorphic } f : U \to D\} \tag{1.3}$$

and let

$$\tilde{K}_D(z_1, z_2) \;=\; \inf\{\sum_{j=1}^{k} K(w_{j-1}, w_j) \;:\; w_j \in D, w_0 = z_1, w_k = z_2\}$$

Then $\tilde{K}_D$ is a metric on D, called the Kobayashi metric. Lempert shows that $\tilde{K}_D = K_D$ if D is convex. Moreover, the inf which appears in (1.3) is realized by some unique $f : U \to D$, depending only on z_1, z_2 (such an f is called "extremal").

Lempert's solution to the HMA is then given by the following formula:

$$u(z) \;=\; \log\left(\frac{e^{K_D(z_0,z)} - 1}{e^{K_D(z_0,z)} + 1}\right)$$

Why does this work, that is, why is the Kobayashi metric the right generalization of δ? To understand this, let's start from the beginning, and try to find a function u satisfying the HMA. The first observation is that $\text{rank}(\partial\bar{\partial}u) = n - 1$ implies that $\ker(\partial\bar{\partial}u)$ is one dimensional. Thus, $\ker(\partial\bar{\partial}u)$ gives us a distribution in the tangent bundle which is of real rank two. A theorem of Bedford-Kalka (1977) says that the Frobenius condition is satisfied for this distribution, and hence, there exists a foliatation $\mathcal{F}$ of D, whose leaves are one-dimensional complex maniolds, and with the property that u is harmonic on each leaf of $\mathcal{F}$. If u is plurisubharmonic, this implies that $\partial u = (\partial_1 u, ..., \partial_n u)$ is holomorphic on each leaf (if a hermitian matrix H is positive semi-definite, and if $H[X] = 0$ for some vector X, then $HX = 0$). In other words, if $U \subseteq \mathbf{C}$ is the unit disk and if $f : U \to D$ is holomorphic function, parametrizing a leaf of $\mathcal{F}$, then $\partial u \circ f : U \to \mathbf{C}^n$ is holomorphic. Let's write this condition down a little more explicitly:

Choose a smooth function $r : \mathbf{C}^n \to \mathbf{R}$ so that $D = \{r < 0\}$. Then $u|_{\partial D} = 0 \Longrightarrow u = pr$ for some positive (replacing u by $-u$ if necessary) function $p : \partial D \to \mathbf{R}$. Let $U \subseteq \mathbf{C}$ be the unit disk, and $f : U \to D$ be a leaf of the foliation $\mathcal{F}$, chosen so that $f(0) = z_0$. Thus f is holomorphic and $\partial u \circ f$ is also holomorphic, that is, $q(\partial r \circ f) : U \to D$ is holomorphic where $q = p \circ f$ is a positive function on U. Note that $\bar{\partial} r(z)$ is an outward pointing normal vector. Thus, in order to find u, we are led to the search for smooth functions $f : (\bar{U}, \partial U) \to (\bar{D}, \partial D)$ such that $f : U \to D$ is holomorphic, and such that the function $\partial U \to \mathbf{C}^n$, given by $\zeta \mapsto q_1(\zeta)(\bar{\nu}_1(f(\zeta)), ..., \bar{\nu}_n(f(\zeta))$ extends to a holomorphic function $\tilde{f} : U \to \mathbf{C}^n$. Here $\nu(z)$ is the outward pointing unit normal vector at the point $z \in \partial D$ and $q_1 : \partial U \to \mathbf{R}$ is some unknown positive function. Lempert calls such f "extremal". The bulk of Lempert's paper is the proof of the following theorem: for each unit vector $v \in \mathbf{C}^n$ there is a unique extremal f such that $f'(0)/|f'(0)| = v$.

A second (slightly different) way to think about the results in these papers is as an attempt to generalize the Riemann mapping theorem and the construction of the Green's function for the Laplacian to higher dimensions:

Let $U = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$. Let $D \subseteq \mathbf{C}$ be a simply connected bounded domain with smooth boundary, $z \in D$ and $v \in \mathbf{C}$ with $|v| = 1$. Then the Riemann mapping theorem says that there is a unique diffeomorphism $\phi = \phi_{z,v} : \bar{D} \to \bar{U}$ such that $\phi|_D : D \to U$ is biholomorphic, $\phi(z) = 0$ and $\phi'(0) = \lambda v$ from some $\lambda > 0$.

Let $u_z(w) = \log |\phi_z(w)|$ and let $G(z, w) = u_z(w)$ (which is independent of the choice of v). Then G is the Green's function for D, that is,

1. $\partial_w \bar{\partial}_w G(z_0, w) = 0$
2. $G(z, w) = 0$ if $w \in \partial D$
3. c. $G(z, w) - \log |w|$ is bounded in a neighborhood of z, and $\partial_w \bar{\partial}_w G(z, w) = 0$.

Now suppose $D \subseteq \mathbf{C}^n$ is a bounded convex domain with smooth boundary and let $B^{(n)}$ be the unit ball in $\mathbf{C}^n$ centered at the origin. Let $z \in D$. We want to find a homeomorphism $\Phi : \bar{D} \to \bar{B}^{(n)}$ which generalizes ϕ above. We can't expect Φ to be biholomorphic. But Lempert's theorem provides us with the next best thing: It says that there exists a , Lipshitz homeomorphism $\Phi_{z_0} = \Phi : \bar{D} \to \bar{B}^{(n)}$ with $\Phi(z_0) = 0$ and $\Phi : \bar{D} \backslash \{z_0\} \to B^{(n)} \backslash \{0\}$ a diffeomorphism, satisfying the following:

1. $\det(\partial_i \partial_{\bar{j}}\ u(w))\ = 0$ where $u = \log |\Phi_{z_0}(w)|$
2. $u(w) = 0$ if $w \in \partial D$
3. $u(w) - \log |w|$ is bounded in a neighborhood of z_0.

In other words, u is a smooth solution to the Dirichlet problem on D with an isolated logarithmic singularity at z.

How does Lempert construct such a Φ? To explain this, we first construct the canonical foliation of $B^{(n)}$ by holomorphic disks: Let $[v] \in \mathbf{P}^{n-1}$. Choose a representative $v \in \mathbf{C}^n$ of $[v]$ such that $|v| = 1$ (so that v is uniquely determined up to an element of S^1). Now consider the map $f = f_{0,v} : U \to B^{(n)}$ given by $f(\zeta) = \zeta v$. The image, $U_{[v]}$, is a disk in $B^{(n)}$ passing through the origin. Moreover, if $[v] \neq [v']$, then $U_{[v]} \cap U_{[v']} = \{0\}$ and the intersection is transversal.

Now let $D \subseteq \mathbf{C}^n$ be any convex bounded domain with smooth boundary, and fix $z \in D$. Lempert constructs a canonical foliation $\mathcal{F}$ of D by holomorphic disks $\tilde{f}_{z,v} : U \to \tilde{U}_{[v]} \subseteq D$ which all pass through z: In fact, The $\tilde{f}$ each extend to smooth maps $\tilde{f}_{z,v} : \bar{U} \to \bar{D}$ with the following properties: $\tilde{f}(\partial U) \subseteq \partial D$, $\tilde{f}_{z,v}(0) = z$ and $\tilde{f}'_{z,v}(0) = \lambda v$ for some $\lambda > 0$. These disks have the property that $[v] \neq [v']$ implies $\tilde{U}_{[v]} \cap \tilde{U}_{[v']} = \{z\}$ and the intersection is transversal.

Now we can define $\Phi : D \to B^{(n)}$. It will not, in general, be holomorphic (although it turns out that $\Phi : D \to B^{(n)}$ is biholomorphic if and only if there exists *some* biholomorphic map $D \to B^{(n)}$). But it will restrict to a biholomorphic map from $\tilde{U}_{[v]}$ to $U_{[v]}$ for each $[v] \in \mathbf{P}^{n-1}$: the restriction is given by the obvious formula: $\Phi|_{\tilde{U}_{[v]}} = \tilde{f}_{z,v}^{-1} \circ f_{0,v} : \tilde{U}_{[v]} \to U_{[v]}$. In particular, $|\Phi(\tilde{f}(\zeta))| = |\zeta|$ for any extremal map $f : U \to D$.

Lempert shows as well that u is pluri-subharmonic, that is, the hessian $H = (\partial_i \partial_{\bar{j}} u)$ is non-negative. On the other hand, if $p \in D$ and if $X \in T_p$ is tangent to the foliation, then $H[X] = 0$, that is, X is a null vector of H. This shows that $(\partial\bar{\partial} u)^n = 0$.

In order to completely specify Φ, we must explain how the maps $f_{z,v}$ are defined. One way to specify these maps is via the following extremal characterization: Fix z, v as above. Assume $g : \bar{U} \to \bar{B}^{(n)}$ is a smooth imbedding with such that $g|_U : U \to B^{(n)}$ holomorphic, $g(\partial U) \subseteq \partial B^{(n)}$, $g(0) = z$ and $g'(0) = \eta v$ for some $\eta > 0$. Then $|g'(0)| \leq |\tilde{f}'_{z,v}(0)|$ with equality if and only if $g = \tilde{f}_{z,v}$.

Thus, to prove Lempert's theorem, one must first prove that for a given z, v, that there exists a unique extremal $\tilde{f}_{z,v}$. This is done using the method of continuity: After scaling and translating, we may assume $0 \in D \subseteq D_0 = B^{(n)}$. Let $D_t = (1-t)B^{(n)} + tD$. When $t = 0$, we have already constructed the canonical foliation, given by the maps $f_{0,v}(\zeta) = \zeta v$. To use the method of continuity, we must show that the set of $t \in [0, 1]$ for which there exists an extremal map $\tilde{f}_{0,v}$ is both open and closed. As usual, the openness is proved via the implicit function theorem and the closedness via apriori estimates on the $\tilde{f}_{0,z}$. The apriori estimates are proved by a simple application of the Schwartz lemma. Openess is more difficult, and requires solving the "Riemann-Hilbert" problem.

One nice feature of Lempert's approach is that one stays within the category of holomorphic functions. Thus, for example, one only needs rather weak apriori estimates on f_t: It suffices to have C^α estimates for any $\alpha > 0$ in order to extract a subsequence which converges uniformly in the C^∞ topology to a holomorphic limit. In Lempert's paper, he obtains C^α apriori estimates with $\alpha = \frac{1}{2}$. On the other hand, these estimates rely quite heavily on the convexity of D.

It turns out that the $\tilde{f}_{z,v}$ which Lempert constructs have a number of amazing properties which allow one to derive all sorts of interesting consequences in complex analysis. For example, Lempert proves that the Kobayashi metric equals the Caratheodory metric for convex domains. He also uses these techniques to give a new proof of Fefferman's theorem on the existence of a C^∞ extension for biholomorphic maps between strictly pseudoconvex domains.

2 The class E

Let $D \subseteq \mathbf{C}^n$ be bounded and convex with smooth boundary.

Suppose $f : U \to D$ is such that $f(0) = z$ and $f'(0) = \lambda v$ for some $\lambda > 0$. Then, for any $\eta \leq \lambda$ we can define $g : U \to D$ via the formula $g(\zeta) = f(a\zeta)$ where $a = \eta/\lambda$. Then $g(0) = z$ and $g'(0) = \eta v$. Thus, we can always make λ smaller by this simple procedure.

Similarly, if $f : U \to D$ is such that $f(0) = z_1$ and $f(\xi) = z_2$, with $\xi \in (0, 1)$, and and if $0 < \xi \leq \xi' < 1$, then we can define $g : U \to D$ via the formula $g(\zeta) = f(a\zeta)$ with $a = \xi/\xi'$. Then $g(0) = z_1$ and $g(\xi') = z_2$. Thus we can always make ξ bigger by this simple procedure.

Thus we are led to two (aparently different, but, as we shall soon see, if fact equivalent) existence problems for extremal maps:

Definition: Let $\zeta \in U$, let $z \in D$ and $0 \neq v \in \mathbf{C}^n$. We that $f : U \to D$ is extremal with respect to (ζ, z, v) if $f(\zeta) = z$, $f'(\zeta) = \lambda v$ for some $\lambda > 0$, and if for any other holomorphic map $g : U \to D$ such that $g(\zeta) = z$, $g'(\zeta) = \eta v$ for some $\eta > 0$ we have $\eta \leq \lambda$.

Definition: Suppose $z_1, z_2 \in D$ with $z_1 \neq z_2$. We say $f : U \to D$ is extremal with respect to (z_1, z_2) if there exists $\zeta_1, \zeta_2 \in U$ such that $f(\zeta_j) = z_j$ and if for any other holomorphic map $g : U \to D$ such that $g(\zeta_j') = z_j$ for some $\zeta_1', \zeta_2' \in U$, we have $\delta(\zeta_1', \zeta_2') \geq \delta(\zeta_1, \zeta_2)$, where δ is the hyperbolic distance function on the unit disk (normalized so that $\delta(0, \xi) = \log \frac{1+\xi}{1-\xi}$ for $\xi \in (0, 1)$).

The following theorem shows that the extremal problems associated with the two definitons are, in fact, the same, as the following uniqueness theorem shows:

Theorem 1 . *Fix $z_1, z_2 \in D$ distinct. Then there exists a unique $f : U \to D$ which is extremal with respect to z_1, z_2. Moreover, f is extremal with respect to any distinct pair $z_1', z_2' \in f(U)$. Moreover, f is extremal with respect to any triple (ζ, z, v) where $z \in f(D)$*

is arbitrary, and $f(\zeta) = z$ and $f'(\zeta) = v$. Similarly, if we fix ζ, z, v, then there exists a unique $f : U \to D$ which is extremal with respect to ζ, z, v, and this f is also extremal with respect to any distinct pair $z_1, z_2 \in f(U)$.

In order to prove this uniqueness theorem, as well as the existence theorem, we give a characterization of extremal maps which is more useful than those provided by the definitions. To motivate this characterization, we begin with a simple discussion of the geometry of convex domains in $\mathbf{C}^n$:

Let $H \subseteq \mathbf{C}^n$ be a smooth real hypersurface (so $\dim_{\mathbf{R}}(H) = 2n-1$). Let $p = (p_1, ..., p_n) \in H$ and let $\nu = (\nu_1, ..., \nu_n)$ be a vector in $\mathbf{C}^n$ which is normal to H at the point p. Then the tangent plane of H at the point p is given as follows:

$$T_p(H) \;=\; \{(z_1, ..., z_n) \in \mathbf{C}^n : \mathrm{Re}\Big(\sum_{j=0}^{n} \bar{\nu}_j (z_j - p_j)\Big) = 0\}$$

This is a vector space over $\mathbf{R}$ of dimension $2n-1$. We define $T_p^{\mathbf{C}}(H) = T_p(H) \cap \sqrt{-1}T_p(H)$. This is a vector space over $\mathbf{C}$ of dimension $n-1$:

$$T_p^{\mathbf{C}}(H) \;=\; \{(z_1, ..., z_n) \in \mathbf{C}^n : \sum_{j=0}^{n} \bar{\nu}_j (z_j - p_j) = 0\}$$

Thus $T_p^{\mathbf{C}}(H) \in \mathbf{P}^{n-1}$ has homogeneous coordinates $[\bar{\nu}_1 : \cdots : \bar{\nu}_n]$.

Now let $D \subseteq \mathbf{C}^n$ be a bounded convex domain with smooth boundary $H = \partial D$. Consider the map

$$\Psi : \partial D \to \; \mathbf{C}^n \times \mathbf{P}^{n-1} \quad \text{given by} \quad p \mapsto (p, T_p^{\mathbf{C}}(\partial D))$$

Explicitly:

$$\Psi(p) \;=\; (p, [\bar{\nu}_1(p) : \cdots \bar{\nu}_n(p)])$$

For example, if $D = B^{(n)}$ then $\Psi(p) = (p_1, ..., p_n, [\bar{p}_1 : \cdots : \bar{p}_n])$. Note that Ψ does not extend to a holomorphic map $D \to \mathbf{C}^n \times \mathbf{P}^{n-1}$, but that its restriction to $\partial U_{[v]}$ does extend to a holomorphic map on U. To see this, fix v. Then for $p \in \partial U_{[v]}$, we have $\Psi(p) = (p, [\bar{v}])$ and this formula clearly gives the extension to all of $U_{[v]}$ (in fact, this extension is essentially the identity map, since the second term, $[\bar{v}]$, is a constant, independent of p). Thus, if Ψ did extend to D the extension would be given by the formula $\Psi(p) = (p, [\bar{p}])$. But this is not holomorphic in p. Moreover, it's not even defined when $p = 0$.

This simple example shows that if the plan outlined in the introduction is to succeed, that is, if we will be able to construct $\Phi : D \to B^{(n)}$ taking the foliation by extremal disks to the canonical foliation in $B^{(n)}$, then the restriction of Ψ to the boundary of the extremal disks $\tilde{U}_{[v]}$ must extend to a holomorphic function on all of $\tilde{U}_{[v]}$. This extension property gives a necessary condition for a holomorphic disk $g : U \to D$ (sending $\partial U \to \partial D$) to be extremal. It turns out that this condition is essentially sufficient as well. To make this precice, we introduce the notion of an E-disk:

Let $D \subseteq \mathbf{C}^n$ be a smoothly bounded convex domain, and let $\nu : \partial D \to \mathbf{C}^n$ be the outward pointing normal vector. Suppose $f : U \to D$ is a holomorphic map. We say that $f \in E$ if

1. f extends to a $C^{1/2}$ map $f : \bar{U} \to \bar{D}$ such that $f(\partial U) \subseteq \partial D$.
2. There exists a positive $C^{1/2}$ function $p : \partial U \to \mathbf{R}$ such that the mapping $\partial U \to \mathbf{C}^n$

$$\zeta \;\mapsto\; \zeta p(\zeta)\Big(\bar{\nu}_1(f(\zeta)), ..., \bar{\nu}_n(f(\zeta))\Big)$$

   extends to a holomorphic function $\tilde{f} : \bar{U} \to \bar{D}$.
3. The winding number of the function $\phi(\zeta) = \bar{\nu}(f(\zeta)) \cdot (z - f(\zeta))$ is zero for some (and hence all) $z \in D$ (here $z \cdot w = \sum z_j w_j$ and the function ϕ maps $\partial U \to \mathbf{C}$).

The second condition implies (but is slightly stronger than) requiring that Ψ extends to a holomorphic function on $f(\bar{U})$, and it is easily seen to be necessary if the plan outlined in the introduction is to work. Similarly for the third condition.

3 Elementary properties of E.

The elements of E enjoy some remarkable elementary properties, all of which are quite easy to prove (just a few lines per property):

Property 1 (Regularity). If $f \in E$ then f satisfies conditions 1,2,3 with $C^{1/2}$ replaced by C^∞. The proof is a simple application of the reflection principle. We omit the details.

Property 2 (Retract). If $f \in E$ then f has a holomorphic retract: There exists $F : \bar{D} \to \bar{U}$, holomorphic on D, such that $F \circ f = \mathrm{id}_{\bar{U}}$ and $F(z) \in U$ for all $z \notin f(\partial U)$.

Proof. Let $z \in D$. Consider the equation

$$\tilde{f}(\zeta) \cdot (z - f(\zeta)) = 0, \quad \zeta \in \bar{U}$$

We claim this equation has a unique solution $\zeta = F(z) \in U$. To see this, note that the right side is a holomorphic function of ζ. Thus, we need only show that the winding number of the right side (on ∂U) equals one:

$$\mathrm{wind}(\tilde{f}(\zeta) \cdot (z - f(\zeta))) \;=\; \mathrm{wind}(\zeta) \;+\; \mathrm{wind}(\bar{\nu}(f(\zeta)) \cdot (z - f(\zeta))) \;=\; 1 + 0 = 1$$

This proves the claim. Clearly, if $z = f(\zeta)$ for some $\zeta \in U$, then $F(z) = z$.

Property 3 (Extremal). Let $f \in E$. Then f is the unique extremal with respect to $z = f(0)$ and $v = f'(0)$ and also with respect to any pair of distinct points $z_1, z_2 \in f(U)$.

Proof. If $g : U \to D$ is such that $g(0) = f(0)$ and $g'(0) = \lambda f'(0)$ then $F \circ g : U \to U$ is a holomorphic self-mapping which fixes 0, so by Schwartz's lemma, $|(F \circ g)'(0)| \leq 1$, that is, $\lambda |F'(f(0)) f'(0)| = \lambda |(F \circ f)'(0)| \leq 1$. But $F \circ f = \mathrm{id}$ so $\lambda \leq 1$ with equality if and only if $F \circ g$ is the identity map (which implies, by a simple argument, that $g = f$).

Property 4 (Constancy). If $f \in E$ then $f'(\zeta) \cdot \tilde{f}(\zeta)$ is a positive constant.

Proof. Let $f_t(\zeta) : U \to D$ be any smooth family of holomorphic functions with the property: $f_t(\partial U) \subseteq \partial D$, $f_t(0) = z_0$, and $f = f_0 \in E$. Let $g : U \to \mathbf{C}^n$ be the map: $g(\zeta) = \frac{d}{dt}|_{t=0}\ f_t(\zeta)$. Then we claim

$$\mathrm{Re}[\zeta^{-1} g(\zeta) \cdot \tilde{f}(\zeta)] \;=\; 0 \;\text{ if }\; \zeta \in U$$

To see this, note that for $\zeta \in \partial U$, that $g(\zeta)$ is tangent to ∂D at the point $f(\zeta)$ and hence $\mathrm{Re}(g(\zeta) \cdot \bar{\nu}(f(\zeta)) = \mathrm{Re}(\zeta^{-1} g(\zeta) \cdot \tilde{f}(\zeta)) = 0$ for all $\zeta \in \partial U$ and hence, by the open mapping theorem, for all $\zeta \in U$.

If we apply this to $f_t(\zeta) = f(e^{it}\zeta)$ then $\zeta^{-1} g(\zeta) = i f'(\zeta)$ so we obtain

$$0 \;= \mathrm{Im}(f'(\zeta) \cdot \tilde{f}(\zeta))$$

Thus the holomorphic function $f'(\zeta) \cdot \tilde{f}(\zeta)$ is real valued on ∂U. The Fourier expansion of this function shows that it must be constant on U. A simple argument shows that this constant, which is certainly real, is in fact positive.

Property 5 (Kobayashi). If $D \subseteq \mathbf{C}^n$ is an open set, and $z_1, z_2 \in D$, let

$$c_D(z, w) \;=\; \sup\{\ \delta(F(z_1), F(z_2)) : F \in \mathrm{Hol}(D, U)\ \}$$

$$k_D(z, w) = \;\inf\{\delta(\zeta_1, \zeta_2)) \;:\; f \in \mathrm{Hol}(U, D), f(\zeta_j) = z_j\ \}$$

The function c_D is called the Carathéodory distance. It clearly decreases under holomorphic maps $D \to D'$. Clearly $c_U = k_U = \delta$, the hyperbolic distance.

The function k_D is not necessarily a metric, since it may not satisfy the triangle inequality. To remedy this we define

$$k'_D(z_1, z_2) \;=\; \inf\{\ \sum_{j=1}^{m} k(w_{j-1}, w_j) \;:\; w_0 = z_1, w_m = z_2\ \}$$

Then k' is a distance, and the Schwartz lemma implies $k' \geq c$.

Let $f \in E$ and let $\zeta_1, \zeta_2 \in U$. Let $z_j = f(\zeta_j)$. Then

$$c_D(z_1, z_2) = k'_D(z_1, z_2) = k_D(z_1, z_2) = \delta(\zeta_1, \zeta_2)$$

Proof. Let F be the retraction of f. Since the Caratheodory distance decreases under holomorphic maps:

$$c(z_1, z_2) \;\geq\; \delta(F(z_1), F(z_2)) \;=\; \delta(\zeta_1, \zeta_2) \;\geq k(z_1, z_2) \;\geq k'(z_1, z_2) \;\geq\; c(z_1, z_2)$$

4 Apriori estimates

Main Theorem.

1. *Let $f : U \to D$ be a holomorphic map. Then $f \in E$ if and only if f is extremal with respect to $z = f(0)$ and $v = f'(0)$.*

2. *Let $z \in D$ and $0 \neq v \in \mathbf{C}^n$. Then there exists a unique $f : U \to \mathbf{C}$ extremal with respect to z, v.*

Outine of proof: The first part has already been proved in the previous section (this is the "extremal property"). As for the second part, we must show that for any $z \in D$ and $0 \neq v \in \mathbf{C}^n$, there is an $f \in E = E(D)$ such that $f(0) = z$ and $f'(0) = \lambda v$ with $\lambda > 0$.

We may assume that $D \subseteq B^{(n)}$. Let $D_t = tD + (1-t)B^{(n)}$. Then $z \in D_t$ for all $t \in [0, 1]$. Let $T \subseteq [0, 1]$ be the set of all $t \in [0, 1]$ for which there is an $f_t \in E(D_t)$ such that $f_t(0) = z$ and $f'_t(0) = \lambda_t v$ with $\lambda_t > 0$. We must show that T is open and closed. In this section, we show that T is closed.

Lemma. *Fix $D \subseteq \mathbf{C}^n$ convex and let $z \in D$. Let $f \in E$ be such that $f(0) = z$. Then*

1. $|f(\zeta_1) - f(\zeta_2)| \;\leq\; C|\zeta_1 - \zeta_2|^{1/2}$ *for all* $\zeta_1, \zeta_2 \in U$.

2. $|\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2)| \;\leq\; C|\zeta_1 - \zeta_2|^{1/2}$ *for all* $\zeta_1, \zeta_2 \in U$.

Here C is a constant, independent of ζ_1, ζ_2. In fact, C depends only on the geometry of (D, z): To be precise, let $\epsilon > 0$. We say that $(D, z) \in \mathcal{C}(\epsilon)$ if

1. $\operatorname{diam}(D) < \frac{1}{\epsilon}$ *and* $\epsilon < |\operatorname{curvature}(\partial D)| < \frac{1}{\epsilon}$

2. $\operatorname{dist}(z, \partial D) > \epsilon$

3. For every pair $z_1, z_2 \in D$ there exist balls $B_1, ..., B_m$ of radius $\frac{\epsilon}{2}$,f centered at $p_1, ..., p_m$, such that $z_1 \in B_1$, $z_2 \in B_m$, $\operatorname{dist}(p_{j-1}, p_j) < \frac{\epsilon}{4}$ and $m < \frac{1}{\epsilon^2}$.

If $(D, z) \in \mathcal{C}(\epsilon)$ then C depends only on ϵ.

Proof. We first prove that there exists C, depending only on ϵ, such that

$$\operatorname{dist}(f(\zeta), \partial D) \;\leq\; C(1 - |\zeta|) \qquad (*)$$

To prove this, note that by the Kobayashi property,

$$k'(f(0), f(\zeta)) \;=\; \delta(0, \zeta) \;=\; \log \frac{1 + |\zeta|}{1 - |\zeta|} \;\geq\; -\log(1 - |\zeta|) \qquad (**)$$

Assume, for the moment, that $\operatorname{dist}(f(\zeta), \partial D) > \frac{1}{\epsilon}$. Then it follows from property c) that $C \geq k'(z, f(\zeta))$. But this, combined with (**) shows that $1 - |\zeta|$ is bounded below by a positive constant. On the other hand, the $\operatorname{diam}(D)$ is bounded above, and this gives (*).

Now suppose $\operatorname{dist}(f(\zeta), \partial D) \leq \frac{1}{\epsilon}$. Again property c) implies:

$$k'(z, f(\zeta)) \;\leq\; k'(z, p_m) \;+\; k'(p_m, f(\zeta)) \;\leq\; C \;+\; \log \frac{1}{\operatorname{dist}(f(\zeta), \partial D)}$$

Combining this with (**) gives (*).

To prove the lemma, it suffices to show that for each $\zeta_0 \in U$,

$$|f'(\zeta_0)| \;\leq C(1 - |\zeta_0|)^{-1/2} \qquad (***)$$

This will follow from (*) and the Schwartz lemma which says the following: Suppose $g : U \to B_R(0)$ is holomorphic. Then the Schwartz lemma says that

$$|g(0)|^2 + |g'(0)|^2 \;\leq\; R^2$$

Fix $\zeta_0 \in U$ and apply the Schwartz lemma to $g(\zeta) = f(\frac{\zeta_0 - \zeta}{1 - \bar{\zeta}_0 \zeta})$ where B_R is chosen as follows: Let $w \in \partial D$ be chosen so that $\operatorname{dist}(f(\zeta_0), \partial D) = |f(\zeta_0) - w|$. Let B_R be a ball tangent to ∂D at the point w, with radius $R = R(\epsilon)$ chosen such that $D \subseteq B_R$. Without loss of generality, we may assume that B_R is centered at 0, so that $R = |w|$. Then

$$|g'(0)|^2 \;\leq\; |w|^2 - |f(\zeta_0)|^2 \;\leq\; C(|w| - |f(\zeta_0|) \;\leq C(|w - f(\zeta_0)|) \;=\; C(\operatorname{dist}(f(\zeta_0), \partial D)$$

We get:

$$|f'(\zeta_0)| \;=\; \frac{|g'(0)|}{1 - |\zeta_0|^2} \;\leq\; \frac{|g'(0)|}{1 - |\zeta_0|} \;\leq\; C \frac{\operatorname{dist}(f(\zeta_0), \partial D)^{1/2}}{1 - |\zeta_0|} \;\leq\; C(1 - |\zeta_0|)^{-1/2}$$

where the last inequality comes from (*). This proves the first part of the lemma. The second part is proved in a similar manner.

Closedness of T now follows; we need only observe that $D_t \in \mathcal{C}(\epsilon)$ for some $\epsilon > 0$, independent of t.

5 Implicit function theorem.

We want to show that if $f : U \to D_0$ is extremal with $f(0) = z$ and $f'(0) = \lambda_0 v_0$, then if D_t and v_t are small pertubations of D_0 and v_0, there is an extremal disk $f_t : U \to D_t$ such that $f_t(0) = z$ and $f_t'(0) = \lambda_t v_t$ with $\lambda_t > 0$.

To prove this, we first must make precise the meaning of "perturbation": How do we perturb a convex domain D_0? This can be done in a very down to earth way, via the defining function of the boundary of D_0. To simplify matters (although Lempert treats the case of smooth boundary as well), we shall assume that $D_0 = \{r_0 < 0\}$ where $r = r_0 : \mathbf{C}^n \to \mathbf{R}$ is a real analytic function such that $dr \neq 0$ on ∂D. For example, if D_0 is the unit ball, we can take $r(z) = |z|^2 - 1$.

An outward pointing normal vector is given by $\bar{\nu}(z) = r_z = (\partial_{z_1} r, ..., \partial_{z_n} r)$. Thus, in the example of the unit ball, $\nu(z) = (z_1, ..., z_n)$.

Associated to $r = r(z, \bar{z})$ is its complexification $r(z, w)$, which is a holomorphic function in $\mathbf{C}^{2n}$. Thus, in the example of the ball, $r = z\bar{z} - 1$ and $r(z, w) = zw$. Now we fix V_0, a neighborhood of ∂D in $\mathbf{C}^n = \mathbf{R}^{2n}$, and we let $V \subseteq \mathbf{C}^{2n}$ be its complexification. Let X be the Banach space of bounded holomorphic functions on V which are real valued on V_0 (with respect to the sup norm). The theorem we need to prove is the following:

Theorem 2 *Let $f_0 : U \to D_0$ be a mapping in $E(D_0)$ with $f_0(0) = z$ and $f_0'(0) = v_0$. Then there is an open set $M \subseteq X \times \mathbf{C}^n$ with $(r_0, v_0) \in M$, and an analytic map $F : M \to C^{1/2}(\bar{U})$ such that $F(r_0, v_0) = f_0$ such that for $(r, v) \in M$ the map $f = F(r, v)$ is an E mapping $f : U \to D_r = \{r < 0\}$ with $f(0) = z$ and $f'(0) = \lambda v$, $\lambda > 0$.*

Outline of Proof.

We want to use the implicit function theorem, which says the following: Suppose that $\Phi : E_1 \times E_2 \to F$ is a smooth map between Banach spaces, and suppose $(e_1, e_2) \in E_1 \times E_2$ is such that $\Phi(e_1, e_2) = 0$. Assume $D_2\Phi(e_1, e_2) : E_2 \to F$ is invertible. Then there exists a smooth function $F : U_1 \to E_2$, where $e_1 \in U_1 \subseteq E_1$ is open, such that $F(e_1) = e_2$ and $\Phi(e, F(e)) = 0$ for all $e \in U_1$.

Let us fix $(r, v) \in X \times \mathbf{C}^n$, close to (r_0, v_0). Here $X \times \mathbf{C}^n$ will play the role of E_1. Suppose we are given (f, p, λ) where $f : U \to \mathbf{C}^n$ is a holomorphic function, p is a positive real valued function (normalized so that $p(1) = 1$), and λ is a positive real number. It's not hard to give the space of such triples the structure of a Banach space (this will be done in detail below), which will play the role of E_2. What does it mean to say that the triple (f, p, λ) is an E mapping with respect to the given pair (r, v)? Well, by definition, it means that

1. $f(\partial U) \subseteq D_r$,
2. $\zeta p(\zeta)(r_z \circ f)$, which is a map $\partial U \to \mathbf{C}^n$, extends to a holomorphic map $U \to \mathbf{C}^n$.
3. $f'(0) = \lambda v$.

We want to use the implicit function theorem to show that for a given $(r, v) \in E_1$, close to (r_0, v_0), there is a unique $(f, p, \lambda) \in E_2$ satisfying conditions 1,2,3. In order to do this, we must rewrite these conditions as an equation $\Phi(r, v; f, p, v) = 0$ where $\Phi : E_1 \to E_2 \to F$ is a smooth map between Banach spaces (which we will be defined below). In other words, we must translate conditions 1,2,3 into equations.

We haven't yet defined E_2 and F, but nevertheless, it's easy to see how to formally proceed: The third condition is already an equation, namely, $f'(0) - \lambda v = 0$. The first is also an equation, namely $r \circ f = 0$. As for the second condition, this is also an equation, namely $\pi[\zeta p(\zeta)(r_z \circ f)] = 0$ where π is orthogonal complement of the projection onto the Hardy space. Thus, on a formal level,

$$\Phi(r, v; f, q, \lambda) = (r \circ f, \pi[\zeta(p_0 + q)(\zeta)(r_z \circ f)], f'(0) - \lambda v)$$

where we have rewritten $p = p_0 + q$ and q is normalized so that $q(1) = 0$.

Step 1. We need to defne the Banach spaces E_2 and F: Let

$$A = \{a : \partial U \to \mathbf{C}^n : a \in L^2(\partial U), \quad \text{and} \quad a', a'' \in L^2(\partial U)\}$$

In other words, A is the Sobolev space $L^2_2(\partial U, \mathbf{C}^n)$. Note that $A \subseteq C^{1/2} \subseteq C^0$. If $a \in A$ then we can write $a = \sum_{n=-\infty}^{\infty} a_n e(nt)$ where $a_n \in \mathbf{C}^n$, and $t \in \mathbf{R}/\mathbf{Z}$.

Let

$$B = \{a \in A : a_n = 0 \quad \text{if} \quad n \leq 0\}$$

Thus $b \in B$ defines a holomorphic function $b : U \to \mathbf{C}^n$ with the property $b(0) = 0$.

Let

$$Q = L^2_2(\partial U, \mathbf{R}) \quad \text{and} \quad Q_0 = \{q \in Q : q(1) = 1\}$$

Let $\pi : A \to \bar{B}$ be the map

$$\pi(a) = \sum_{n=-\infty}^{-1} a_n e(nt)$$

Thus $\ker(\pi) \subseteq A$ is the subspace of functions with holomorphic extensions to U.

Finally we let

$$E_1 \;=\; X \times \mathbf{C}^n \;,\; E_2 \;=\; B \times Q_0 \times \mathbf{R} \quad \text{and} \quad F \;=\; Q \times \bar{B} \times \mathbf{C}^n$$

and we define $\Phi : E_1 \times E_2 \to F$ by the formula above:

$$\Phi(r, v; f, q, \lambda) \;=\; (r \circ f, \pi[(p_0 + q)\zeta(r_z \circ f)], f'(0) - \lambda v)$$

Then $\Phi(r, v; f, q, \lambda) = 0$ if and only if (f, q, λ) is an E mapping for (r, v).

Step 2. After making a change of variables, we may assume that

$$f_0(\zeta) = (\zeta, 0, ..., 0), \text{ if } \zeta \in U \quad \text{and} \quad \bar{\nu}(\zeta, 0, ..., 0) = (r_0)_z(f_0(\zeta)) = (\bar{\zeta}, 0, ..., 0)$$

That is, we may assume that D behaves like the unit ball, at least as far as f_0 is concerned. In particular, $p_0 = 1$. The proof is a short but tricky winding number argument, which I will omit. The condition that D is convex at the points $f_0(\zeta)$ becomes

$$\sum_{i,j=2}^{n} (r_0)_{z_i\bar{z}_j}(f_0(\zeta))v_i\bar{v}_j \quad > \quad \left| \sum_{i,j=2}^{n} (r_0)_{z_i z_j}(f_0(\zeta))v_i v_j \right| \tag{1}$$

for all $0 \neq v \in \mathbf{C}^{n-1}$

Step 3. Computation of the derivative. We must show that $D_2\Phi(r_0, v_0; , f_0, 0, 1) : E_2 \to F$ is invertible, that is, we must establish the invertibility of the operator

$$L \;=\; \Phi_{(f,q,\lambda)}(r_0, v_0; f_0, 0, 1) : B \times Q_0 \times \mathbf{R} \;\to\; Q \times \bar{B} \times \mathbf{C}^n$$

Thus we let $(\dot{f}, \dot{q}, \dot{\lambda}) \in B \times Q_0 \times \mathbf{R}$. For example, $\dot{f}$ is a column vector whose entries are $\dot{f}_j = \sum_{k \geq 1} a_{jk}\zeta^k$. Let us compute $L(\dot{f}, \dot{q}, \dot{\lambda})$:

$$L(\dot{f}, \dot{q}, \dot{\lambda}) \;=\; \frac{d}{dt}\bigg|_{t=0} (r_0 \circ (f_0 + t\dot{f}), \pi[(1 + t\dot{q})\zeta(r_{0z} \circ (f_0 + t\dot{f})], \dot{f}_0'(0) + t\dot{f}'(0) - (1 + t\dot{\lambda})v_0)$$

$$= \; ((r_{0z} \circ f_0)\dot{f} \;+\; (r_{0\bar{z}} \circ f_0)\overline{\dot{f}},\; \pi[\dot{q}\zeta(r_{0z} \circ f_0) \;+\; \zeta(r_{0zz} \circ f_0)\dot{f} \;+\; \zeta((r_{0z\bar{z}} \circ f_0)\overline{\dot{f}}],\; \dot{f}'(0) - \dot{\lambda}v_0)$$

In this last epression, we view $r_{0z} \circ f_0$ and $r_{0\bar{z}} \circ f_0$ as a row vectors, $\dot{f}$ and $\overline{\dot{f}}$ as a column vectors, and $r_{0zz} \circ f_0$ and $r_{0z\bar{z}} \circ f_0$ as an $n \times n$ matrices. Thus $(r_{0z} \circ f_0)\dot{f}$ is a scalar function, $(r_{0zz} \circ f_0)\dot{f}$ is a column vector of functions, and $\dot{f}'(0) - \dot{\lambda}v_0$ is a column vector of scalars.

Now, to show that L is invertible, we must give ourselves an arbitrary element $(\rho, \phi, \nu) \in Q \times \bar{B} \times \mathbf{C}^n$ and show that the system of linear equations

$$(r_{0z} \circ f_0)\dot{f} \;+\; (r_{0\bar{z}} \circ f_0)\overline{\dot{f}} \;=\; \rho \in Q \tag{i}$$

$$\pi[\dot{q}\zeta(r_{0z}\circ f_0) \ + \ \zeta(r_{0zz}\circ f_0)\dot{f} \ + \ \zeta((r_{0z\bar{z}}\circ f_0)\overline{\dot{f}}] \ = \ \phi \ \in \bar{B} \tag{ii}$$

$$\dot{f}'(0) - \dot{\lambda}v_0 \ = \ \nu \ \in \ \mathbf{C}^n \tag{iii}$$

has a unique solution $(\dot{f}, \dot{q}, \dot{\lambda}) \in B \times Q_0 \times \mathbf{R}$.

Since $r_{0z} \circ f_0 = (\bar{\zeta}, 0, ..., 0)$, the equation (i) becomes $\bar{\zeta}\dot{f}_1 + \zeta\overline{\dot{f}_1} = \rho \in Q = L^2_2(\partial U, \mathbf{R})$. Here $\dot{f}_1 : U \to \mathbf{C}$ is an unkown holomorphic function such that $\dot{f}_1(0) = 0$. Thus we can write (i) as follows: $\mathrm{Re}(\zeta^{-1}\dot{f}_1) = \rho$. This determines all the coefficients of $\dot{f}_1 = \sum_{k\geq 1} a_{1k}\zeta^k$, except for a_{11}, which is only determined up to the addition of an arbitrary purely imaginary constant (so $\mathrm{Re}(a_{11})$ is determined, and a_{1k} is determined if $k \geq 2$).

Now the equation (iii) implies that $\mathrm{Re}(a_{11}) - \mathrm{Re}(\nu_1) = \dot{\lambda}$ (since $v_0 = (1, 0, ..., 0)$). This determines $\dot{\lambda}$. Similarly, $\mathrm{Im}(a_{11}) - \mathrm{Im}(\nu_1) = 0$. This determines $\mathrm{Im}(a_{11})$.

Conclusion: (i) and (iii) uniquely determine $\dot{f}_1$, $\dot{\lambda}$, and $\dot{f}'_j(0)$ for all j.

Now we turn our attention to equation (ii), which is system of n linear equations. We focus on the last $n-1$ equations: For $z = (z_1, ..., z_n) \in \mathbf{C}^n$, let $z_* = (z_2, ..., z_n) \in \mathbf{C}^{n-1}$. Then the last $n-1$ equations become:

$$\pi\left[\zeta(r_{0z_*z_*}\circ f_0)\dot{f}_* \ + \ \zeta((r_{0z_*\bar{z}_*}\circ f_0)\overline{\dot{f}}_* \ - \ \psi \ \right] \ = \ \pi\left[\alpha g \ + \ \beta\bar{g} \ - \ \psi\right] \ = \ 0 \tag{2}$$

where $\psi : \partial U \to \mathbf{C}^{n-1}$ is a known L^2_2 mapping, $\alpha = \zeta^2(r_{0z_*z_*}\circ f_0)$, $\beta = ((r_{0z_*\bar{z}_*}\circ f_0)$, and and $g = \zeta^{-1}\dot{f}_*$.

To solve (2), we establish the following:

Lemma *Let $\alpha, \beta : \partial U \to M_{(n-1)\times(n-1)}(\mathbf{C})$ be real analytic functions with α symmetric and β hermitian. Let $\psi \in L^2_2(\partial U, \mathbf{C}^{n-1})$, and $b \in \mathbf{C}^{n-1}$. Then there exists a unique holomorphic function $g : U \to \mathbf{C}^{n-1}$ satisfying:*

$$\pi\left[\alpha g \ + \ \beta\bar{g} \ - \ \psi\right] \ = \ 0, \quad g(0) = b \ . \tag{3}$$

Proof. Since β is self-adjoint, by the "Riemann-Hilbert" theorem (proved by Lempert in his paper), there is a holomoprhic $H : \bar{U} \to GL(n-1, \mathbf{C})$ such that $HH^* = \beta$. Now (3) becomes:

$$\pi\left[H^{-1}\alpha g \ + \ H^*\bar{g} - H^{-1}\psi\right] \ = \ 0$$

Putting $h = {}^tHg$ and $T = (H^{-1})\alpha({}^tH^{-1})$, we write this as:

$$\pi\left[\bar{h}\right] \ = \ \pi\left[H^{-1}\psi - Th\right], \quad h(0) = a$$

Now let's take the complex conjugate of both sides, and note that $\bar{\pi}\bar{h} = \overline{\pi(\bar{h})} = h - a$ if h is holomorphic, and $h(0) = a$, so we can rewrite (3) as:

$$h \;=\; \bar{\pi}\left[H^{-1}\psi - Th\right] \;+\; a$$

Thus, if we define $K : L^2_1(\partial U, \mathbf{C}^{n-1}) \to L^2_1(\partial U, \mathbf{C}^{n-1})$ by $K(h) = \bar{\pi}\left[H^{-1}\psi - Th\right] \;+\; a$, then equation (3) can be expressed as: $K(h) = h$.

To show that K has a fixed point, we would like to show that it's a contraction mapping, that is,

$$\|\bar{\pi}Th\| \;\leq\; \mu\|h\|$$

for some $\mu < 1$. To do this, we first observe that (1) implies $\|T\|_{op} < 1$. Now it's certainly true that K has a fixed point in L^2 since $\|\bar{\pi}Th\|_{L^2} \;\leq\; \|Th\|_{L^2} \;\leq \|T\|_{op}\|h\|$. But if we use the L^2_2 norm, then the derivatives of T appear, and all we know about these derivative is that they are bounded, the the sup norm. On the other hand, the following simple trick allows us to avoid this problem: Define the norm $\|h\|_\epsilon$ on L^2_2 as follows:

$$\|h\|_\epsilon \;=\; \|h\|_{L^2} \;+\; \epsilon\|h'\|_{L^2} \;+\; \epsilon^2\|h''\|_{L^2}$$

Then the $\epsilon T'h$ and $\epsilon^2 T''h$ terms can be asborbed in $\|Th\|_{L^2}$ and the $\epsilon^2 T'h'$ term can be absorbed in $\epsilon\|h'\|_{L^2}$. This proves K has a fixed point and the lemma is established.

Finally we observe that we can now choose $\dot{q}$ to uniquely solve the first equation in (ii).

6 Solving the HMA equation.

We wish to prove that $u = \log|\Phi|$ is plurisubharmonic, that it solves the homogeneous Monge-Ampere equation, and that it has a logarithmic singularity at $z_0 \in D$.

6.1 Logarithmic singularity.

First, why is the singularity logarithmic? To see this, recall the definition of k_D:

$$k_D(z_1, z_2) \;=\; \inf\{\delta(\zeta_1, \zeta_2) : f(\zeta_j) = z_j \;\text{ for some holomorphic } f : U \to D\}$$

Let's assume $z_0 = 0 \in D$. Then we must show that $u(z) - \log|z|$ is bounded in a neighborhood of $0 \in D$. To see this, choose two balls, $B_1 \subseteq D \subseteq B_2$, with radii r_1, r_2, and centered at the origin. Then $k_{B_1}(0, z) \geq k_D(0, z) \geq k_{B_2}(0, z)$ for all $z \in B_1$, that is, if $f(0) = 0$ and $f(\zeta) = z$ with f extremal, then

$$\log \frac{1+\frac{|z|}{r_1}}{1-\frac{|z|}{r_1}} \;\geq\; \log \frac{1+|\zeta|}{1-|\zeta|} \;\geq\; \log \frac{1+\frac{|z|}{r_2}}{1-\frac{|z|}{r_2}}$$

Here we have used the Kobayashi property. Thus we obtain:

$$\log \frac{|z|}{r_1} \;\geq\; \log|\zeta| \;\geq\; \log \frac{|z|}{r_2}$$

Recall that $u(z) = \log|\zeta|$ (by the definition of u), we see $u(z) - \log|z|$ is bounded on D.

6.2 The family of Kobayashi balls.

Next we show that u is plurisubharmonic. To do this, we study the Kobayashi balls in D: If $z \in D$ and $z \neq z_0$ then z determines a unique extremal map $f : U \to D$. Let $\zeta \in U$ be such that $f(\zeta) = z$ and let $v = f'(0)/|f'(0)|$. Recall that $z \mapsto \zeta v$ defines a homeomorphism $\Phi : D \to B^{(n)}$. Let $0 < r \leq 1$ and let $D_r = \{z \in D : \Phi(z) < r\}$. Then $D_r \subseteq D$ is a ball, in the Kobayashi metric, centered at z_0. When $r = 1$, then $D_r = D$.

We claim that D_r is convex: Let $z_1, z_2 \in D_r$ and f_1, f_2 extremal such that $f_1(r) = z_1$ and $f_2(r) = z_2$. Let $h = \lambda f_1 + (1-\lambda) f_2$ with $0 \leq \lambda \leq 1$. Then $h : U \to D$ is holomorphic, $h(0) = z_0$ and $h(r) = \lambda z_1 + (1-\lambda) z_2$. By virtue of the extremal characterization of the Kobayashi metric, we see that $\lambda z_1 + (1-\lambda) z_2 \in D_r$ so D_r is convex.

Recall that if $f : U \to D$ is extremal, then $\tilde{f} : U \to \mathbf{C}^n$ is holomorphic and if $|\zeta| = 1$, then $\tilde{f}(\zeta)$ is the "slope" of H_z, the holomorphic tangent plane to $z = f(\zeta) \in \partial D$. Moreover, $f(\partial U) \subseteq \partial D$ is transversal to H (note that $\dim_{\mathbf{R}}(H) = 2n-2$ and $\dim_{\mathbf{R}}(\partial D) = 2n-1$).

Now choose $0 < r \leq 1$ and consider the maps $f_r : U \to D_r$ and $\tilde{f}_r : U \to \mathbf{C}^n$ defined $f_r(\zeta) = f(\zeta r)$ and $\tilde{f}_r(\zeta) = \tilde{f}(\zeta r)$. Then it is clear from the definitions that f_r is extremal for D_r. We claim as well that that $\tilde{f}_r$ is the corresponding holomorphic normal. To see this, we need the "constancy property" which we now recall:

Let $f_t(\zeta) : U \to D$ be any smooth family of holomorphic functions with the property: $f_t(\partial U) \subseteq \partial D$, $f_t(0) = z_0$, and $f = f_0 \in E$. Let $g : U \to \mathbf{C}^n$ be the map: $g(\zeta) = \frac{d}{dt}|_{t=0}\, f_t(\zeta)$. Then we proved

$$\mathrm{Re}[\zeta^{-1} g(\zeta) \cdot \tilde{f}(\zeta)] \;=\; 0 \;\text{ if }\; \zeta \in U$$

We apply this as follows: Let $z \in \partial D_r$ and let $z(t)$ be any smooth curve in ∂D_r such that $z(0) = z$. Let f_t be the extremal map determined by z_t. Then f_t is a smooth family satisfying the hypothesis of the constancy property. Thus we conclude: $\mathrm{Re}[g(r) \cdot \tilde{f}(r)] = 0$. But $g(r) = \frac{d}{dt}|_{t=0} f_t(r) = \frac{d}{dt}|_{t=0}\; z(t) = z'(0)$. Since $z'(0)$ is an arbitrary tangent vector to ∂D_r at the point z, we see that $\tilde{f}(r)$ is normal to ∂D_r.

6.3 Plurisubharmonicity of u

We wish to show that u is plurisubharmonic and that it satisfies the HMA. So let's fix $w \in D$ and choose r such that $w \in \partial D_r$. Let f be the extremal map corresponding to w. We can choose coordinates so that the vector field $\frac{\partial}{\partial z_1}$ is tangent to $f(U)$ and such that $\frac{\partial}{\partial z_2}, ..., \frac{\partial}{\partial z_n}$ are tangent to ∂D_r at w. The convexity of D_r implies that $(u_{z_i, \bar{z}_j}(w))_{2 \le i,j \le n}$ is positive semi-definite. We claim that $(u_{z_1, \bar{z}_j}(w))_{1 \le j \le n} = 0$. This will prove that u is pluri-subharmonic and that it satisfies the HMA equation.

To see this, let X and Y be two holomorphic vector fields along $f(U) \subseteq D$ with Y tangent to $f(U)$ and Y nowhere vanishing. Then $Y(f(\zeta)) \cdot \tilde{f}(\zeta)$ is a nowhere vanishing holomorphic function on U and

$$\alpha(\zeta) \; = \; \frac{X(f(\zeta)) \cdot \tilde{f}(\zeta)}{Y(f(\zeta)) \cdot \tilde{f}(\zeta)}$$

is a holomorphic function on U. Let

$$V(\zeta) \; = \; X(f(\zeta)) \; - \; \alpha(\zeta) Y(f(\zeta))$$

Then $V(\zeta)$ is holomorphic along $f(U)$ and $V(\zeta) \cdot \tilde{f}(\zeta) = 0$. Hence $V(\zeta)$ is tangent to ∂D_r at $f(\zeta)$ (where $r = |\zeta|$). Since u is constant on ∂D_r, we conclude $du(V(\zeta)) = 0$, that is,

$$du(X(\zeta)) \; = \; \alpha(\zeta) du(Y(\zeta))$$

On the other hand, $du(Y(f(\zeta)) = \partial(u \circ f) = \partial \log |\zeta|$ which is holomorphic in ζ. Thus $du(X(\zeta))$ is holomorphic. Applying this to $X = \frac{\partial}{\partial z_j}$ we get:

$$0 \; = \frac{\partial}{\partial \bar{\zeta}} \; u_{z_j}(f(\zeta))|_{\zeta = r} \; = \; u_{z_i \bar{z}_k}(f(\zeta)) \cdot \overline{f_j'(r)}$$

But we have normalized so that $f(\zeta) \; = \; (f_1(\zeta), 0, ..., 0)$ so the last equality becomes the following: $u_{z_i \bar{z}_1}(f(\zeta)) = 0$ for $1 \le i \le n$. Thus the complex hessian of u is a matrix whose first row and column are all zero, and whose lower right hand $(n-1) \times (n-1)$ corner is positive semi-defninite. This proves that u is plurisubharmonic, and that it satsifies the HMA equation.