Notes on Teichmuller theory

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1 Classification problems: Discrete vs. Continuous

The central problem of Teichmuller theory can be briefly stated as follows: Teichmuller theory: Let S be a smooth compact surface of genus g. Classify the complex structures on a fixed compact surface S (up to equivalence).

More precisely: Let S be a compact surface and let $\mathcal{C}$ be the set of complex structures on S. Thus if $c \in \mathcal{C}$ then c is a maximal collection $c = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ where $\phi_\alpha : U_\alpha \to \mathbf{C}$ is a coordinate chart and $\phi_\beta \phi_\alpha^{-1}$ is conformal (or, alternatively, holomorphic) and $S = \cup_{\alpha \in A} U_\alpha$.

Then moduli space and Teichmuller space are defined as follows:

$$\mathcal{R}(S) = D \backslash \mathcal{C} \ \ \textit{and} \ \ \mathcal{T}(S) = D_0 \backslash \mathcal{C}$$

where D is the group of automorphisms of S and D_0 its connected component. Teichmuller theory is the study of these spaces.

We warm up with some simpler questions:

A) Classification of finite abelian groups.

Answer: If G is a finite abelian group, then there are unique integers $m_1 | m_2 | \cdots | m_k$ such that $G \approx \mathbf{Z}/m_1\mathbf{Z} \times \cdots \times \mathbf{Z}/m_k\mathbf{Z}$.

That's the end of the story, since we finite abelian groups are discrete objects, and as such, can not be deformed. Thus it doesn't make sense to look for a natural continuous structure on the moduli space, i.e., we don't expect the moduli space to be a manifold, algebraic variety, etc.

Similarly, one can classify all finite simple groups. The answer is rather complicated, but again, it's a *discrete* answer.

B) Classification of Riemann surfaces of genus one (i.e., all elliptic curves).

Answer: Let $\mathcal{H}$ be the upper half plane. If E is a Riemann surface of genus one, then there exists a unique $\tau = \tau(E) \in SL_2(\mathbf{Z}) \backslash \mathcal{H}$ such that $E \approx \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ (biholomorphic). Thus we can say that $SL_2(\mathbf{Z}) \backslash \mathcal{H}$ is the set theoretic moduli space of elliptic curves.

This is not the end of the story however, since elliptic curves are not "discrete" - that is, they can be continuously deformed, and the moduli space respects deformations. Let's say this more precisely:

The set $SL_2(\mathbf{Z}) \backslash \mathcal{H}$ has a natural complex structure, inherited from that on $\mathcal{H}$, and as such, it is a Riemann surface. In fact, it is biholomorphic to $\mathbf{C}$. More precisely, there is a holomorphic function $j : \mathcal{H} \to \mathbf{C}$ with the following property:

$$j(\tau_1) = j(\tau_2) \iff \tau_2 = \gamma(\tau_1) \ \text{ for some } \gamma \in SL_2(Z)$$

If X is a complex manifold and if $\pi : \mathcal{E} \to X$ is a holomorphic family of elliptic curves, then the map $X \to \mathbf{C}$ given by $x \mapsto j(\tau(E_t))$ is a holomorphic map (here $E_t = \pi^{-1}(t)$). Thus we can say that $SL_2(\mathbf{Z})\backslash\mathcal{H}$ is the complex analytic moduli space of elliptic curves.

C) Classification of Complex Structures on Vector Spaces.

Let V be a vector space over $\mathbf{R}$ of dimension $2n$. We wish to classify the linear complex structures on V and show that the moduli space is a complex manifold:

A complex structure on V is a pair (ξ, T) mod equivalence where T is a complex vector space of dimension n and $\xi : V \to T$ is an isomorphisom of real vector spaces. The equivalence relation is given by $(\xi, T) \sim (\xi', T')$ if there is an isomorphism of complex vector spaces $T \to T'$ which makes the diagram commute.

This is a difficult definition to work with since the space of complex structures doesn't seem to have a natural topology. It's quite useful to introduce the following alternate definition: A complex structure on V is an element $J \in Aut(V)$ with the property $J^2 = -I$.

To see the equivalence of the two definitions, let $J : V \to V$ be such that $J^2 = -I$ and define T as follows: Then $J \otimes I$ defines and automorphism of the vector space $V \otimes \mathbf{C}$ Let T be the $+i$ eigenspace. Then $\bar{T}$ is the $-i$ eigenspace. We have

$$V \otimes \mathbf{C} \;=\; T \oplus \bar{T}$$

Now T is a complex vector space and the map $\xi : V \to T$ obtained by composing the maps $V \to V \otimes \mathbf{C} = T \oplus \bar{T} \to T$ is an isomorphism of real vector spaces. Conversely, if T is a complex vector space and if $\xi : V \to T$ is an isomorphism of real vector spaces, then $J = \xi^{-1} \circ \mathbf{i} \circ \xi$ is a complex structure. Here $\mathbf{i}$ is the map on T given by multiplication by i. Note that J depends only on the equivalence class of (ξ, T).

A slight variant is: A complex structure on V is an equivalence class of isomorphisms $f : V \to \mathbf{C}^n$, where two isomorphisms are equivalent if they differ by an element of $GL(n, \mathbf{C})$.

If we fix a basis of V, we see that a complex structure on V is a $2n \times 2n$ matrix J with the property $J^2 = -I$, where I is the $2n \times 2n$ identity matrix. Alternatively, a complex structure is an equivalence class of isomorphisms $f : \mathbf{R}^{2n} \to \mathbf{C}^n$ of real vector spaces.

Thus we see that $GL(2n, \mathbf{R})$ operates transitively on the space of complex structures, with stabilizer group $GL(n, \mathbf{C})$. So the space of complex structures on $\mathbf{R}^{2n}$ is just the space $GL(2n, \mathbf{R})/GL(n, \mathbf{C})$. Thus, if we let $\mathcal{J}(V)$ be the space of complex structures on V, we see that

$$\mathcal{J}(V) \;\approx\; GL(n, \mathbf{C})\backslash GL(2n, \mathbf{R}) \;=\; \Gamma\backslash M \tag{1.1}$$

where Γ is the group $GL(n, \mathbf{C})$ and M is the manifold $M = GL(2n, \mathbf{R})$.

Is $\Gamma\backslash M$ a manifold?

$\mathcal{J}(V)$ **as a complex manifold**.

The space $\mathcal{J}(V)$ is a smooth manifold. In fact, it has a natural structure as a complex manifold. To see this, observe that $\mathcal{J}(V)$ is the set of equivalence classes of $n\times 2n$ matrices M with entries in $\mathbf{C}$ whose columns form a basis of $\mathbf{C}^n$ viewed as a vector space over $\mathbf{R}$. In other words,

$$det\begin{pmatrix} M \\ \bar{M} \end{pmatrix} \neq 0 \tag{1.2}$$

Since such an M has maximal rank over $\mathbf{C}$, at least one of its $n\times n$ minors has non-zero determinant. Suppose that the first n columns of M form a minor of non-zero determinant. Then equivalence class of M has a unique representative of the form (I, Z) where Z, according to (1.2), is an $n\times n$ matrix such that $Im(Z)$ is non-singular. Such Z form an open subset of $M_{n\times n}(\mathbf{C})$. Since M is covered by a finite number of such open sets, with holomorphic transitions, we see that $\mathcal{J}$ is a complex manifold of dimension n^2.

D) Classify the endomorphisms of a complex vector space.

Let V be a finite dimensional complex vector space. We wish to find the moduli space of $\text{End}(V) = \{T : V \to V \ : T \text{ is a complex linear map}\}$.

Answer: Let $T \in \text{End}(V)$. Choose a basis $e_1, ..., e_n$ of V. Then, $Te_j = \sum a_{jk}e_k$. Let $A = (a_{jk}) \in M_{n\times n}(\mathbf{C})$. Choosing a different basis corresponds to conjugating A by an element $P \in GL(n, \mathbf{C})$.

Thus, if we let $\Gamma = GL(n, \mathbf{C})$ and $M = M_{n\times n}(\mathbf{C})$, then Γ is a group acting on the manifold M by conjugation:

$$\Gamma \times M \to M \ \text{ given by } (P, M) \to P^{-1}MP$$

and we see that $\Gamma\backslash M$ is the moduli space of endomorphisms of V.

When we studied the moduli space of elliptic curves, we saw that the manifold structure on $\mathcal{H}$ induced a manifold structure on $\Gamma\backslash\mathcal{H}$ and we were thus able to solve the "complex moduli problem". Similarly, when we studied the moduli space of complex structures on a vector space, we saw that the manifold structure on $M_{2n\times 2n}(\mathbf{R})$ induced a manifold structure on $\Gamma\backslash M$. Can we do the same for the moduli problem of endomorphisms?

Well, we have a problem this time since $\Gamma\backslash M$ is not a manifold. In fact, it's not even Hausdorff. This is because the action is not free (it has some bad fixed points).

One way around this is to simply throw away the endomorphisms with mulitple eigenvalues and change the problem to: Find the moduli space for $\text{End}'(V) = \{T \in \text{End}(V) : T$ has distinct eigenvalues$\}$. This is the standard method of dealing with this kind of problem in the theory of moduli - we throw away the "unstable" orbits.

E) Classify line bundles of degree zero on a fixed compact Riemann surface X.

Answer: The moduli space is $J(X)$, the Jacobi variety of X. It equals $\mathbf{C}^g/L$, where $L \subseteq \mathbf{C}^g$ is a lattice (discrete free abelian group of rank $2g$) known as the period lattice.

2 Group actions

Very often, the problem of classification leads us to the following situation: Γ is a group, M is a manifold, and $\Gamma \times M \to M$ is a group action. The quotient space $\Gamma\backslash M$ is always a topological space and $M \to \Gamma\backslash M$ is a continuous function. The basic question is: Under what conditions is $\Gamma\backslash M$ a manifold such that $M \to \Gamma\backslash M$ is a smooth map of manifolds.

Eariler we saw that the presence of fixed points was a problem. But that's not the only problem: Let $\Gamma = \mathbf{R}$ and $M = \mathbf{R}^2/\mathbf{Z}^2$. Let $h = (\alpha, \beta) \in \mathbf{R}^2$ be a vector with the property: $\frac{\beta}{\alpha} \notin \mathbf{Q}$. Consider the action $\Gamma \times M \to M$ given by $(t, p) \mapsto p + th$. Then the orbit of any point is dense so, once again, the quotient is not Hausdorff.

Even worse things can happen: Consider the action of $\mathbf{R}$ on $\mathbf{R}^2$ whose flow lines are: $\{x = b\}$ for $|b| \geq 1$ and $\{y = \frac{1}{1-x^2} + c : |x| < 1, c \in \mathbf{R}\}$. These curves are disjoint and their union is $\mathbf{R}^2$. The action is given by a point travelling along its trajectory at unit speed. It's locally proper (proper on $x > -1$ and $x < 1$) but not proper, since any small neighborhood of $(-1, 0)$ and and any small neighborhood of $(1, 0)$ have point which are equivalent under arbitrarily large values of t.

Theorem. *Let Γ be a Lie group and M a smooth manifold. Let $\Gamma \times M \to M$ be a smooth action of Γ on M. Assume*

1. *The action is free (no fixed points).*

2. *The action is proper, that is, if $K, L \subseteq M$ are compact, then*

$$\{\gamma \in \Gamma : \gamma K \subseteq L\} \subseteq \Gamma \quad \textit{is compact}$$

Then

1. *For each $x \in M$, the set $\Gamma x \subseteq M$ is a smooth submanifold.*

2. *$\Gamma\backslash M$ is a smooth manifold and $\pi : M \to \Gamma\backslash M$ is a smooth map.*

3. *If $H \subseteq T_xM$ is complementary to $T_x(\Gamma x)$, then $H \approx T_{[x]}(\Gamma\backslash M)$ via the map $d\pi$.*

Important remark: In the theorem above, we do not need to assume that Γ or M are finite dimensional manifolds. Thus they may be Hilbert manifolds.

Example: Classify all the rays in $\mathbf{R}^2$ which start at the origin.

If $\lambda \subseteq \mathbf{R}^2$ is a ray, and if $p_1, p_2 \in \Lambda$, then there is a unique positive number r such that $p_2 = rp_1$. Thus the classification space is $A = \Gamma \backslash M$ where $\Gamma = P = \mathbf{R}_+$ and $M = \mathbf{R}^2$. Inside M we have a nice submanifold $M(1) \subseteq M$ where $M(1)$ is the unit circle. More over, π gives a diffeomorphism $M(1) \to A$.

This simple example is a good one to keep in mind when we study the space of complex structures $\mathcal{C}$. For this example, M is the space of Riemannian metrics on the surface S and P is the space of positive functions; $\Gamma \backslash M = \mathcal{C}$, the space of complex structures, and $M(1)$ is the space of hyperbolic metrics (metrics of constant curvature equal to -1)

§3. Complex structures and Teichmuller space.

The basic problem of Teichmuller theory is that of classifying the complex structures on a given compact surface S.

Let S be a compact surface and let $\mathcal{C}$ be the set of complex structures on S. Thus if $c \in \mathcal{C}$ then c is a maximal collection $c = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ where $\phi_\alpha : U_\alpha \to \mathbf{C}$ is a coordinate chart and $\phi_\beta \phi_\alpha^{-1}$ is conformal (or, alternatively, holomorphic) and $S = \cup_{\alpha \in A} U_\alpha$.

Then moduli space and Teichmuller space are defined as follows:

$$\mathcal{R}(S) = \mathcal{C}/D \quad and \quad \mathcal{T}(S) = \mathcal{C}/D_0$$

where D is the group of automorphisms of S and D_0 its connected component.

Main question: Show that $\mathcal{T}(S)$ is a manifold.

Now D, D_0 are nice Lie groups, but $\mathcal{C}$ doesn't have a natural manifold structure.

Let $S = S^2 \subseteq \mathbf{R}^3$. Then stereographic projection gives us a covering of S^2 by open sets U_α, and conformal diffeomorphisms $U_\alpha \to \mathbf{R}^2$. Since the change of coordinate maps are conformal, this gives us a complex structure on S^2.

In fact, there is nothing special about S^2: if $S \subseteq \mathbf{R}^n$ is any two dimensional submanifold, then S has covering U_α with conformal diffeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbf{R}^2$. This gives us a complex structure on S.

If $S \subseteq \mathbf{R}^n$ is a submaifold and if $p \in S$, and if $v, w \in T_p(S) \subseteq \mathbf{R}^n$, let $g(v, w) = v \cdot w \in \mathbf{R}$. This function satisfies the following basic properties:

1. The function $g : T_p(S) \times T_p(S)$ is bilinear.

2. The function g is smooth in the following sense: If $V : S \to \mathbf{R}^n$ and $W : S \to \mathbf{R}^n$ are smooth functions, then $p \mapsto g(V(p), W(p))$ is a smooth function on S.

The function g allows us to measure the lengths of curves and the angles betwee curves: If $\gamma : [a,b] \to M$ is C^1, then $l(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt$.

If $\phi : U \to S$ is a local coordinate, then we write $\phi^* g(X,Y) = g(d\phi(X), d\phi(Y))$ where X, Y are tangent vectors on U. More precisely, if $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ then

$$\phi^* g(X,Y) \ = \ \delta_{\alpha\beta} d\phi(X)^\alpha d\phi(Y)^\beta = \ \delta_{\alpha\beta} X^i Y^j \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \ = \ g_{ij}(x) X^i Y^j$$

where $g_{ij}(x) = \delta_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}$. Note that $g_{ij}(x)$ is a smooth function and $(g_{ij}(x))$ is a positive defninite matrix: If X^i is any non-zero vector, then $g_{ij} X^i X^j = g(d\phi(X), d\phi(X)) > 0$.

Definition: Is S is any manifold, then a metric on S is a collection of bilinear maps $g_p : T_p(S) \times T_p(S) \to \mathbf{R}$ which vary smoothly, ie, if X, Y are smooth vector fields on M then $g(X,Y)$ is a smooth function on S. In local coordinates,

$$g(X,Y) = g_{ij}(x) X^i Y^j$$

where $g_{ij}(x)$ is a positive definite symmetric matrix. Sometimes we write $g = g_{ij} dx^i \otimes dy^j$.

Theorem(Gauss): Let (S, g) be a two dimensional riemannian manifold. Then S has an open cover $S = \cup V_\alpha$ and local coordinates $\phi : U_\alpha \to V_\alpha$ such that $\phi_\alpha^* g = e^{u_\alpha}(dx^2 + dy^2)$ for some smooth function u_α. In particular, the ϕ_α are conformal and thus S is given the structure of a Riemann surface.

Main steps.

$\mathcal{H}^s(M) = \{f : M \to \mathbf{R} : \int |D^j f|^2 \mu_g < \infty \ , \ j = 1, ..., s\}$

Let $\mathcal{A}^s = \{J \in \mathcal{H}^s(T_1^1 M) : J^2 = -1\} = \{J \in \mathcal{H}^s(T_1^1(M)) : tr(J) = 0, \det(J) = 1\}$. Thus $J \in \mathcal{A}^s \Longrightarrow J_j^i \in \mathcal{H}^s(U)$ on each coordinate neighborhood U. We write $\mathcal{A} = \cap_{s \geq 0} \mathcal{A}^s$.

$\Gamma : \mathcal{C} \to \mathcal{A}$ the natural map. It's injective: If a smooth map diffeomorphism ϕ between open subsets of $\mathbf{C}$ has the property $Jd\phi = d\phi J$, then ϕ is holomorphic. Soon we'll prove that Γ is surjective as well.

$\mathcal{A}^s \subseteq \mathcal{H}^s(T_1^1(M))$ is a smooth submanifold and $T_J \mathcal{A}^s = \{H \in \mathcal{H}^s(T_1^1(M)) : HJ + JH = 0\}$. To prove this, let $\mathrm{tr} : \mathcal{H}^s(T_1^1(M)) \to H^s(M)$ and $\det : \mathcal{H}^s(T_1^1(M)) \to H^s(M)$ be the trace and determinant maps. We show that $\mathrm{tr}^{-1}(0)$ is a smooth manifold, that $\det^{-1}(1)$ is a smooth manifold, and their intersection is transversal and equals $\mathcal{A}^s$. For example, $(D\det)_J(H) = -\mathrm{tr}(JH)$ for all $J \in \det^{-1}(1)$ and $H \in \mathcal{H}^s(T_1^1(M))$. It's easy to see that $(D\det)_J : \mathcal{H}^s(T_1^1(M)) \to \mathcal{H}^s(M)$ is surjective which proves $\det^{-1}(0)$ is a manifold. The proof for $\mathrm{tr}^{-1}(0)$ is easier.

Let $\mathcal{M}^s$ denote the space of metrics in $\mathcal{H}^s$ and $\mathcal{P}^s \subseteq \mathcal{H}^s$ the space of positive functions $(s > 1)$. Since the action of $\mathcal{P}^s$ on $\mathcal{M}^s$ is smooth, proper and free, the quotient $\mathcal{M}^s / \mathcal{P}^s$ is a C^∞ Hilbert manifold.

Define $\Phi : \mathcal{M}^s/\mathcal{P}^s \to \mathcal{A}^s$ by the formula: $\Phi(g) = -g^{ik}\mu_{kj}$ where μ is the volume form. Thus, in conformal coordinates, $g_{ij} = \lambda\delta_{ij}$, $\mu_{ij} = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ and $\Phi(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We claim: $\Phi : \mathcal{M}^s/\mathcal{P}^s \to \mathcal{A}^s$ is a smooth diffeomorphism of Hilbert manifolds. To see this, we note first that it's a bijective map between sets: If $g_1^{-1}\mu_{g_1} = g_2^{-1}\mu_{g_2}$ then, since $\mu_{g_1} = p \cdot \mu_{g_2}$ for some $p \in \mathcal{P}^s$, injectivity follows. As for surjectivity, let $J \in \mathcal{A}^s$ and let $\hat{g} \in \mathcal{M}^s$ be any metric. Let $g(u,v) = \hat{g}(u,v) + \hat{g}(Ju, Jv)$. One can check that $\Phi(g) = J$. This also shows $\mathcal{M}/\mathcal{P} \to \mathcal{A}$ is bijective and therefore, $\mathcal{C} \to \mathcal{A}$ is bijective.

To prove that Φ is a diffeomorphism, we must show that it induces an isomorphism on tangent spaces. Let $\tilde{\Phi} : \mathcal{M}^s \to \mathcal{A}^s$ be the composition of Φ with the projection. Then

$$D\Phi(g) : T_g\mathcal{M}^s = S_2^s \to T_J(\mathcal{A}^s) = \{H : HJ + JH = 0\}$$

$$= \{H \in T_1^1(M)^s : \mathrm{tr}H = 0,\ H \text{ is } g \text{ symmetric }\}$$

Here $J = \Phi(g)$ and S_2^s is the space of symmetric tensors. We say H is g symmetric if $h = gH$ is symmetric.

For every metric g, we have the canonical orthogonal splitting of S_2^s into the conformal and the traceless factor: $S_2^s = S_2^s(g)^c \oplus S_2^s(g)^T$. Note that if $h \in S_2^s$ then $h \in S_2^s(g)^T \iff H = g^{-1}h$ is a traceless, g symmetric element of $T_1^1(M)^s$.

One computes: $D\Phi(g)(h) = (\frac{1}{2}(\mathrm{tr}H)I - H)J)$ where $H = g^{-1}h \in T_1^1(M)^s$. Thus the kernel consists of conformal g symmetric matrices: $\ker D\Phi(g) = S_2^s(g)^c$ and one shows easily that $D\Phi(g)$ is surjective. In fact, $D\Phi(g) : S_2^s(g)^T \to T_J(\mathcal{A}^s)$ is an isomorphism. It's injective since we've factored out the kernel. To see that it's surjective, let $H' \in T_J(\mathcal{A}^s)$. We must show that there exists a traceless g symmetric H such that $-HJ = H'$. Well, just choose $H = H'J$. Then $H' \in T_J \Longrightarrow H'J \in T_J \Longrightarrow H'J$ is traceless and g symmetric.

3 Poincaré's Theorem.

Let (M, g) be a compact oriented Riemannian manifold of dimension two. Then $R(g)$, the curvature of g, is a smooth function on M. It's defined as follows: In isothermal coordinates, $g_{ij} = \alpha\delta_{ij}$ where α is a positive function. Then $R(g) = -\alpha^{-1}\Delta\alpha = -\alpha^{-1}(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2})$.

Theorem 1 *Suppose M is a compact oeriented surface of genus at least two. Then given any $g \in \mathcal{M}^s$ with $2 \le s \le \infty$, there is a unique $\lambda \in \mathcal{P}^s$ such that $R(\lambda g) = -1$.*

Thus, any surface of genus at least two has a smooth metric of constant negative curvature.

3.1 Review of Sobolev approach to elliptic PDE

We wish to give the main ideas in the model case of Poisson's equation. This is easier than Poincaré's theorem, since it's a linear equation, although in one respect, it's technically more difficult due to the presence of a boundary (bounded domains in $\mathbf{R}^n$ have boundaries while compact manifolds do not). Then, in the next section, we explain how these ideas can be used to prove Poincaré's theorem.

Let $\Omega \subseteq \mathbf{R}^n$ be an open bounded set with smooth boundary. We see a solution to the equation:

$$\Delta u = 0 \ \text{ in } \Omega \ \ ; \ \ u|_{\partial\Omega} = g|_{\partial\Omega} \tag{3.1}$$

Here $g : \bar{\Omega} \to \mathbf{R}$ is a given continuous function.

Here is the idea; Suppose we can find a smooth u such that $u|_{\partial\Omega} = g|_{\partial\Omega}$ for which

$$\int_\Omega |Du|^2 = \kappa \ = \ \inf\{\int_\Omega |Dv|^2 \ : \ v|_{\partial\Omega} = g|_{\partial\Omega} \ \} \tag{3.2}$$

Then, if $\phi \in \mathbf{C}_0^\infty(\Omega)$ we have

$$0 = \frac{d}{dt}\int |D(u+t\phi)|^2 \ = \ 2\int Du \cdot D\phi \ = \ -\int \phi\Delta u$$

Since this is true for all ϕ, we see that $\Delta u = 0$ and thus u solves (3.1).

How do we find a u satisfying (3.2)? We can certainly choose $u_n \in C^\infty(\Omega)$ such that $u_n|_{\partial\Omega} = g|_{\partial\Omega}$ with the property $E(u_n) \to \kappa$. If we could somehow extract a convergent subsequence, we'd be in great shape. If we knew, for example, that all the derivatives of the u_n were bounded in the sup norm, then Arzela-Ascoli would allow us to find a convergent subsequence. Unfortunately, we know very little about the u_n; essentially, all we know about them is that the L^2 norms of their first derivatives are bounded. This however points the way:

Let's consider the space of all functions whose derivatives are bounded in L^2. These form a Hilbert space. And in a Hilbert space, bounded sets are at least weakly compact.

Define $E : H^1(\Omega) \to \mathbf{R}$ by

$$E(v) \ = \ \int_\Omega |Dv|^2$$

Let $\Omega \subseteq \mathbf{R}^n$ be open and $u \in L^1_{\text{loc}}(\Omega)$. We say $v \in L^1_{\text{loc}}(\Omega)$ is the weak derivative of u in the dirction x^i if

$$\int_\Omega \phi v \ = \ -\int_\Omega u\frac{\partial\phi}{\partial x^i}\, dx$$

for all $\phi \in C_0^\infty(\Omega)$. We write $v = D_i u$.

Definition of the Sobolev space H^1:

$$H^1 \;=\; \{u \in L^2(\Omega) : D_i u \in L^2(\Omega) \;\text{ for all } 1 \leq i \leq n\}$$

The space H^1 is a Hilbert space under the norm:

$$\langle u, v\rangle \;=\; \int_\Omega u\bar{v} \;+\; \sum_j \int \partial_i u \partial_i \bar{v}$$

A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbf{C}$ is quasiconformal if $Re(f), Im(f) \in H^1$ and $|\partial_{\bar{z}} f| \leq k|\partial_z f|$ for some $k < 1$.

$$\kappa \;=\; \inf\{E(v) : v \in C^\infty(\Omega), v - g \in C_0^\infty(\Omega)\}$$

where H_0^1 is the closure of $C_0^\infty(\Omega) \subseteq H^1$. Let v_n be a minimizing sequence for $E(v)$, i.e., $E(v_n) \to \kappa$. We want to use some kind of compactness theorem to show that we can extract a convergent subsequence.

Note first that Dv_n is bounded in L^2. Also, $v_n - g \in H_0^1$ so $\|v_n - g\|_{L^2} \leq C\|D(v_n - g)\|_{L^2}$ (using the Poincare inequality) and this implies that v_n is bounded in L^2 and thus, v_n is bounded in H^1.

This means that $v_n \rightharpoonup v$ for some $v \in H^1$ (weak convergence). In particular, $v_n \rightharpoonup v$ weakly in L^2 and $Dv_n \rightharpoonup Dv$ weakly in L^2. To see this, let ϕ be a smooth function on Ω and define $H^1 \to \mathbf{R}$ by $v \to \int_\Omega Dv\phi$. This is a bounded linear functional on H^1 and thus $\int Dv_n\phi \to \int Dv\phi$ for all ϕ. Since smooth functions are dense in $L^2(\Omega)$, this proves that $Dv_n \rightharpoonup Dv$.

Since the norm on a Hilbert space is lower semi-continuous with respect to weak convergence,

$$E(v) \;\leq\; \liminf E(v_n) \;=\; \kappa$$

Also, if a sequence in H_0^1 converges weakly to an element of H^1, then the limit is in H_0^1. This is due to the fact that $H_0^1 \subseteq H^1$ is a closed Hilbert subspace. This shows that $v - g \in H_0^1$ and that $E(v) = \kappa$.

Conclusion: Equation (3.1) has a weak solution, that is, there is $u \in H^1$ with the following properties:

$$u - g \in H_0^1 \;\text{ and }\; \int_\Omega Du \cdot Dv \;=\; 0 \;\text{ for all } v \in H_0^1(\Omega) \tag{3.3}$$

Now suppose that $\partial\Omega$ is C^∞ and that $f \in H^k$ and $g \in H^{k+2}$. Suppose that $u \in H^1$ is a weak solution to $\Delta u = f$. In other words,

$$u - g \in H_0^1 \text{ and } \int_\Omega Du \cdot Dv \;=\; -\int fv \text{ for all } v \in H_0^1(\Omega)$$

Then the fundamental elliptic estimate says that $u \in H^{k+2}$ and

$$\|u\|_{H^{k+2}} \;\leq\; c \cdot (\|f\|_{H^k} + \|g\|_{H^{k+2}}) \tag{3.4}$$

Here c is a constant which depends only on Ω.

In particular, if u is a weak solution to (3.1), then (3.4) implies $u \in \cap_{k\geq 0} H^k$. On the other hand, Sobolev's theorem says that $C^\infty(\bar{\Omega}) = \cap_{k\geq 0} H^k$, and this proves u is smooth.

It's also true that u extends to a continuous function on $\partial\Omega$ and that the boundary values are given by g. I'll skip the proof of this part - it's not relevant to what we need to do, since in our case, there is no boundary.

3.2 Proof of Poincaré's theorem.

Recall the curvature formula in local coordinates:

$$R(g) = -\frac{1}{\lambda}\Delta \log \lambda$$

We write $\Delta_g = \frac{1}{\lambda}\Delta$. We must find a smooth function v on M such that $R(e^v g) = -1$, that is

$$-\frac{1}{\lambda e^v}\Delta \log (e^v \lambda) = \frac{1}{e^v}\left(-\frac{1}{\lambda}\Delta \log \lambda - \frac{1}{\lambda}\Delta v\right) \;=\; -1$$

In other words

$$-\Delta_g v + R(g) + e^v \;=\; 0 \tag{3.5}$$

This is not a linear equation, due to the presence of e^v.

Let

$$I(v) \;=\; \frac{1}{2}\int_M |\nabla_g v|^2 \, d\mu_g \;+\; \int_M (R(g)v + e^v) \, d\mu_g \tag{3.6}$$

It's easy to see that a smooth minimizer of I will solve (3.5). But, just as before, we have no way of extracting a convgent subsequence if we work with the space of smooth functions. Even worse, it's not clear that there is a minimizing sequence, since I is not a positive functional.

We need to define the right space in which to search for a solution. To do this, let's pretend that we've found the solution v. Let $x_o \in M$ be a point where v attains its maximum. Then $-\Delta_g v(x_0) \geq 0$ so $R(g)(x_0) + e^{v(x_0)} \leq 0$. This implies

$$e^{v(x_0)} \leq \; -R(g)(x_0) \;\; \leq |\min R(g)|$$

This implies that for any $x \in M$, $v(x) \;\leq\; v(x_0) \;\leq\; \log|\min R(g)| \;= \xi$.

Thus we are led to define

$$C \;=\; \{v \in H^1(M) : v(x) \;\leq\; 1+\xi \text{ a.e. }\}$$

We wish to prove that I has a minimum $v \in C$ and that v satisfies (3.5).

Again, we wish to use Poincare's inequality to show if $u \in C$ and if $I(u)$ is bounded above, then u is bounded in H^1. This will allow us to extract a weakly convergent subsequence.

Poincare's inequality says that

$$\int |\nabla_g v|^2 \; d\mu_g \;\geq\; c\int |v|^2 \; d\mu_g \;-\; c' \left(\int v\right)^2$$

We claim that the norm $\int |\nabla v|^2 + (\int v)^2$ is a norm on H^1 which is equivalent to the standard norm $\int |\nabla v|^2 + \int v^2$. Note first that $(\int v)^2 \leq \int v^2 \int 1$. Poincare's inequality gives us the reverse bound.

Let $v_0 = v - \bar{v}$ and $\bar{v} = (\int v \; d\mu / \int 1 \; d\mu) \in \mathbf{R}$, the average value of v. Now we estimate

$$I(v) \;\geq\; \frac{1}{4}\int_M |\nabla_g v|^2 \; d\mu_g \;+ \frac{c}{4}\int |v_0|^2 \; d\mu_g + \int_M R(g)v_0 \; d\mu_g \;-\; \bar{v}\cdot 4\pi|\chi(M)| \tag{3.7}$$

where we are using Gauss-Bonet, and the fact that the Euler characteristic is negative. On the other hand

$$\left|\int_M R(g)v_0 \; d\mu_g\right| \;\leq\; \int \frac{\alpha^2}{4} v_o^2 \; d\mu \;+\; \int \frac{4}{\alpha^2} R(g)^2 \; d\mu$$

for any $\alpha > 0$. Choosing $\alpha^2 = c$ we get, for $\alpha \in C$,

$$I(v) \;\geq\; \frac{1}{4}\int_M |\nabla_g v|^2 \; d\mu_g \;- \int_M \frac{4}{c} R(g)^2 \; d\mu_g \;-\; \bar{v}\cdot 4\pi|\chi(M)|$$

Case 1. If $\bar{v} > 0$ then $0 < |\bar{v}| = \bar{v} < 1+\xi$ so

$$I(v) \;\geq\; \frac{1}{4}\int_M |\nabla_g v|^2 \; d\mu_g \;- \int_M \frac{4}{c} R(g)^2 \; d\mu_g \;-\; (1+\xi)\cdot 4\pi|\chi(M)|$$

Case 1. If $\bar{v} \leq 0$ then

$$I(v) \;\geq\; \frac{1}{4}\int_M |\nabla_g v|^2 \; d\mu_g \;+\; |\bar{v}|\cdot 4\pi|\chi(M)| - \int_M \frac{4}{c} R(g)^2 \; d\mu_g$$

In either case, $I(v)$ is bounded below and, an upper bound on $I(v)$ implies an upper bound on $\int |\nabla_g v|$ and on $|\bar{v}|$. Thus there is a minimizing sequence and, since the norms are bounded, we can extract a weakly convergent subsequence. Then $\int |\nabla v_n|^2$ is lower semi-continuous and $e^{v_n} \to e^v$ in $L^1(M)$ by Taylor's theorem (Prop. 4.3, chapter 12). This gives us our minimizer.