NOTES ON THE OPENNESS CONJECTURE FOR PSH FUNCTIONS

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Theorem 1. *Let $B \subseteq \mathbb{C}^n$ be the open unit ball and $u \in PSH(B)$ with $u \leq 0$. Assume*

$$\int_B e^{-u} \ < \ \infty$$

Then there is a $p > 1$ such that

$$\int_B e^{-pu} \ < \ \infty$$

Let

$$H \ = \ \{h : B \to \mathbb{C} \,|\, \|h\|_0^2 = \int_B |h|^2 < \infty, \quad h \text{ holomorphic } \}$$

For $s \geq 0$ define

$$\|h\|_s^2 \ = \ \int_B |h|^2 e^{-2u_s}$$

where

$$u_s \ = \ \max(u + s, 0)$$

The following Lemma is elementary:

Lemma 1. *For $0 < p < 2$ we have*

$$\int_B |h|^2 \, e^{-pu} \ = \ a_p \int_0^\infty \ e^{ps} \|h\|_s^2 \, ds \ + \ \|h\|_0^2 \tag{0.1}$$

Proof of Theorem. Given the Lemma, our assumption becomes

$$\int_0^\infty \ e^s \|h\|_s^2 \, ds \ < \ \infty$$

where $h = 1$. Let $E = B \times H$, which we view as an infinite dimensional vector bundle. Berndtsson's (2009) theorem says that $\log \|h^*\|_s$ is convex, where h^* is any holomorphic section of E^*, the dual bundle. Suppose we knew that $\log \|h\|_s$ were *concave*. Then (0.1) would imply that

$$\|h\|_s^2 \ \leq \ ce^{-(1+\epsilon)s} \tag{0.2}$$

for some $\epsilon > 0$, and Lemma 1 would allow us to conclude the proof.

It turns out that (0.2) is almost true (and holds in great generality):

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Theorem 2. *Let H be a Hilbert space equipped with a decreasing family of norms $\|\cdot\|_s$ with positive curvature. Suppose $h \in H$ has the property*

$$N(h) \ = \ \int_0^\infty e^s \|h\|_s^2 \, ds \ < \ \infty$$

Then for all sufficiently small $\epsilon > 0$ there exists h_ϵ such that

$$\|h_\epsilon\|_s^2 \ \leq \ e^{-(1+\epsilon)s} \|h\|_0^2 \quad \text{for} \quad s > 1/\epsilon$$

and

$$\|h_\epsilon - h\|_0^2 \ \leq \ 2\epsilon N(h)$$

Theorem 1 now follows: Since $h = 1$ we see, for ϵ small enough, that $\|h_\epsilon - h\|_{L^\infty(B/2)} < 1/2$, so $|h|_\epsilon > 1/2$ on $B/2$. Since $1/2 = h/2$ we obtain

$$\|h/2\|_s^2 \leq \|h_\epsilon\|_s^2 \ \leq \ e^{-(1+\epsilon)s} \|h\|_0^2 \quad \text{for} \quad s > 1/\epsilon$$

In particular, $e^{ps}\|h\|_s^2$ is integrable for some $p > 1$ and the result follows from Lemma 1.