The weighted Monge-Ampère energy of quasi-psh functions" by Guedj-Zeriahi (2007)

Contents

1 Introduction

We start by recalling some basic definitions.

Let $\Omega \subseteq \mathbf{C}^n$ be a bounded open set. Define

$$PSH(\Omega) \;=\; \{\varphi : \Omega \to [-\infty, \infty) \,:\, \varphi \text{ is upper semicontinuous and } dd^c\varphi \geq 0\} \tag{1.1}$$

Let $\varphi \in PSH(\Omega)$. Then Bedford-Taylor define a measure $MA(\varphi) = (dd^c\varphi)^n$ on the bounded locus $\Omega_0 = \{x \in \Omega : \varphi \text{ is bounded in an open neighborhood of } x\}$. In this section we explore different methods of extending $MA(\varphi)$ to a measure on all of Ω subject to the following key condition: $MA(\varphi)$ is continuous for decreasing sequences. We can't do this for all φ, but there is a maximal subclass $PSH(\Omega) \cap L^\infty(X) \subseteq \mathcal{D}(\Omega) \subseteq PSH(\Omega)$, called the “domain of definition for Monge-Ampère”, on for which MA has a unique extension satisfying the continuity property. Thus, if $\varphi_j \in \mathcal{D}(\Omega)$ and $\varphi_j \downarrow \varphi$ for some $\varphi \in PSH(\Omega)$, then $\varphi \in \mathcal{D}(\Omega)$ and $MA(\varphi_j) \rightharpoonup MA(\varphi)$.

Before defining $\mathcal{D}(\Omega)$, we first recall the Bedford-Taylor definition for bounded potentials. If $\varphi \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ then $MA(\varphi) = (dd^c\varphi)^n$ is a positive Borel measure on Ω which is characterized as follows: For $U \subset\subset \Omega$ we have $MA(\varphi)|_U = \lim_{j\to\infty} MA(\varphi_j)$ for *any* decreasing sequence $\varphi_j \in PSH(U) \cap C^\infty(U)$ such that $\varphi_j \downarrow \varphi$ (the limit is independent of the sequence).

For the global theory, we start with (X, ω), a compact Kähler manifold and define

$$PSH(X, \omega) \ = \ \{\varphi \in L^1(X, \omega^n) : \varphi \text{ is upper semi-continuous and } \omega + dd^c\varphi \geq 0 \ \}$$

When $\varphi \in PSH(X, \omega)$ is bounded, then we can mimic the local construction to define a Borel measure $MA(\varphi) = \omega_\varphi^n$ on X. If $\varphi \in PSH(X, \omega)$ is arbitrary, then a similar construction defines $MA(\varphi)$ as a positive Borel measure on $X\backslash X_0$.

If $\varphi \in PSH(\Omega)$ or $\varphi \in PSH(X, \omega)$ is not locally bounded, one may be tempted to proceed as follows: Let $\varphi \in PSH(X, \omega)$, let $\varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ and assume that $\varphi_j \downarrow \varphi$. Then $\int_X \omega_{\varphi_j}^n$ is bounded (in fact constant) so there is a measure μ such that, after passing to a subsequence, $\omega_{\varphi_j}^n \to \mu$. Thus we can try to define $MA(\varphi) = \mu$. as above. But this does not work since the limit may depend on the choice of decreasing sequence: Cegrell (1986) shows that if we let $\varphi = \ \log|z_1 \cdots z_n|$ and Ω is a small ball in $\mathbf{C}^n$, then there are decreasing sequences $u_j, v_j \in PSH(\Omega) \cap C^\infty(\Omega)$ such that $u_j \downarrow \varphi$ and $v_j \downarrow \varphi$ but $(dd^c u_j)^n \to 0$ and $(dd^c v_j)^n \to \delta_0$, the Dirac measure at the origin. So any theory which requires continuity for decreasing sequences must exclude φ.

Let $\varphi \in PSH(X, \omega)$. Define μ_φ to be the measure on X obtained from $MA(\varphi)$, which is a measure on $X\backslash\{\varphi = -\infty\}$, to be $MA(\varphi_j)$ on $\varphi > -j$, where $\varphi_j = \max\{\varphi, -j\}$, and extend to all of X by requiring $\{\varphi = -\infty\}$ to have measure zero. It turns out that μ_φ is a non-pluripolar measure, i.e., $\mu_\varphi(Z) = 0$ for all pluri-polar sets Z. It also turns out that $\int_X d\mu_\varphi \leq \int_X \omega^n$ for all $\varphi \in PSH(X, \omega)$. One may be tempted to define $MA(\varphi) = \mu_\varphi$, but this doesn’t work either: the Cegrell example above shows that that in the local case, this definition fails satisfy the key continuity property for decreasing sequences. There are similar counter-examples in the global case.

To define $\mathcal{D}(\Omega) \subseteq PSH(\Omega)$ we instead proceed as follows. Let $\varphi \in PSH(\Omega)$. Then $\varphi \in \mathcal{D}(\Omega)$ if there is a positive Borel measure μ on Ω such that if $U \subset\subset \Omega$ is any relatively compact domain, and $\varphi_j \downarrow \varphi$ is any decreasing sequence with $\varphi_j \in PSH(U) \cap L^\infty(U)$, then $(dd^c\varphi_j)^n \to \mu$ (weak convergence). If $\varphi \in \mathcal{D}(\Omega)$ we define $MA(\varphi) = (dd^c\varphi)^n = \mu$. The class $\mathcal{D}(X, \omega) \subseteq PSH(X, \omega)$ is defined similarly. If $\mathcal{M}(X)$ is the space of positive Borel measures on X, Then $MA : \mathcal{D}(X, \omega) \to \mathcal{M}(X)$ is well defined, and continuous (under decreasing sequences). Unfortunately, the comparison theorem fails on $\mathcal{D}(X,\omega)$, which makes the general theory rather difficult: in fact, neither the kernel nor the range of MA are known: There are examples of $\varphi,\psi \in \mathcal{D}(X,\omega)$ with $MA(\varphi) = MA(\psi)$ but $\varphi - \psi$ is not constant. Thus uniqueness for the Monge-Ampère equation does not hold. Moreover, determining the range of MA is an open problem. It is known that $MA(\varphi)$ will charge polar sets for certain φ, and some examples have been worked out, but there doesn't appear to be a complete characterization of the range.

There is an interesting class $\mathcal{E}(\Omega) \subseteq \mathcal{D}(\Omega) \subseteq PSH(\Omega)$ and, in the global settng of a Kähler manfold, $\mathcal{E}(X,\omega) \subseteq \mathcal{D}(X,\omega) \subseteq PSH(X,\omega)$, known as the Cegrell class. This is the maximal subclass for which the comparison principle holds. For this class, one has a complete theory for the operator $MA : \mathcal{E}(X,\omega) \to \mathcal{M}(X)$. The range consists of all non-pluripolar probability measures. And if $MA(\varphi) = MA(\psi)$ with $\varphi,\psi \in \mathcal{E}(X,\omega)$, then $\varphi - \psi$ is constant. The class $\mathcal{E}$ is defined as follows: Let $\mathcal{E}(X,\omega) \subseteq PSH(X,\omega)$ denote the set of all φ for for which the inequality $\int_X d\mu_\varphi \leq \int_X \omega^n$ is an equality. We have $PSH(X,\omega) \cap L^\infty(X) \subseteq \mathcal{E}(X,\omega)$. Cegrell and GZ show that $\mathcal{E} \subseteq \mathcal{D}$, i.e, they show that the key continuity property holds in $\mathcal{E}$. They prove that the comparison principle holds as well! In fact $\mathcal{E}$ is the maximal class for which the comparison property is satsified. In this theory, $MA(\varphi)$ is always non-pluripolar (again, unavoidable if we want a theory for which the comparison principle is available).

Although $\mathcal{E}(X,\omega) \subseteq \mathcal{D}(X,\omega) \subseteq PSH(X,\omega)$ are proper subsets, the gaps are not, from a certain point of view, very great. In fact, GZ show that if $\varphi \in \mathrm{PSH}(X,\omega)$ with $\varphi \leq -1$, then $-|\varphi|^p \in \mathcal{E}(X,\omega)$ for *any* $0 < p < 1$.

1.1 The Cegrell class $\mathcal{E}(X,\omega)$

This "non-pluripolar" approach goes back to Bedford-Taylor in the local case, and followed by GZ in the global case. In order to motivate the definition, we first recall that in the classical case of bounded potentials, that the Monge-Ampère measure is local with respect the the pluri-fine topology, which is the topology generated by the euclidean open sets, together with sets of the form $\{\varphi < \psi\}$, where $\varphi,\psi \in PSH(X,\omega)$. The key theorem says that if $\mathbf{O}$ is pluri-open, and if $\varphi|_{\mathbf{O}} = \psi|_{\mathbf{O}}$, then $MA(\varphi)|_{\mathbf{O}} = MA(\psi)|_{\mathbf{O}}$ for all locally bounded potentials φ,ψ. For example, we always have $\int_{\varphi<\psi} MA(\psi) = \int_{\varphi<\psi} MA(\max(\varphi,\psi))$.

Now let $\varphi \in PSH(X,\omega)$ be arbitrary. We recall the definition of $MA(\varphi)$, which a measure on $X\backslash\{\varphi = -\infty\}$. Let $B \subseteq X\backslash\{\varphi = -\infty\}$. We wish to define $MA(\varphi)(B)$ in such a way that the pluri-local property, which holds for bounded potentials, is pereserved. There is a unique way for doing this: We must have

$$MA(\varphi) = \lim_{j\to\infty} MA(\varphi)(B \cap \{\varphi > -j\}) = \lim_{j\to\infty} MA(\varphi_j)(B \cap \{\varphi > -j\})$$

where $\varphi_j = \max(\varphi, -j)$ is the *canonical approximation of* φ by bounded psh functions (the first equality comes from the definition of a measure and the second comes by imposing the plui-local property). One shows that the second sequence above is increasing and thus has a limit. This defines $MA(\varphi)$ on $X\backslash\{\varphi = -\infty\}$.

In the non-pluripolar approach, we first define a measure μ_φ by extending $MA(\varphi)$ to a measure on all of X, simply by declaring $\{\varphi = -\infty\}$ to have measure zero. Thus, for any Borel set $B \subseteq X$ we let

$$\mu_\varphi(B) \ = \ \lim_{j\to\infty} \int_{B\cap\{\varphi>-j\}} (\omega + dd^c\varphi_j)^n \tag{1.2}$$

One shows that the sequence in (1.2) is increasing and bounded above, so it has a well defined limit. One sees immediately from the definition that μ_φ vanishes on all pluri-polar sets. The measure μ_φ is called "non-pluripolar extension" associated to φ.

From the definition we see that $\int_X d\mu_\varphi \leq \int_X \omega^n$. If equality holds, then we say the φ has full mass and we define

$$\mathcal{E}(X,\omega) \ = \ \{\varphi \in PSH(X,\omega) \ : \ \int_X d\mu_\varphi = \int_X \omega^n \,\} \tag{1.3}$$

the set of all φ with full mass (this is the GZ notation: unfortunately, Cegrell uses the notation $\mathcal{E}$ to denote his class - in the local case - as well. One has to be careful, since the two classes are not analogous. For example, in the Cegrell case, $MA(\varphi)$ may charge pluri-polar sets for $\varphi \in \mathcal{E}$).

Finally, if $\varphi \in \mathcal{E}(X,\omega)$ then we define $(\omega + dd^c\varphi)^n = \mu_\varphi$. As mentioned earlier, the continuity property and the comparison principle both hold on $\mathcal{E}(X,\omega)$.

It is interesting to compare the singularities of $\mathcal{E}$ to the singularities of PSH. On the one hand, although $\mathcal{E}(X,\omega)$ is strictly smaller that $PSH(X,\omega)$, it defines the same pluripolar sets: More precisely, let $\varphi \in PSH(X,\omega)$ be arbitrary. Then GZ show that if $\varphi \leq -1$ then $-|\varphi|^p \in \mathcal{E}(X,\omega)$ for all $0 < p < 1$. In particular, if Z is a pluripolar set then there exists $\varphi \in \mathcal{E}(X,\omega)$ such that $Z = \{\varphi = -\infty\}$. On the other hand, the elements of $\mathcal{E}(X,\omega)$ have mild singularities compared to $PSH(X,\omega)$. For example, if $\varphi \in \mathcal{E}(X,\omega)$ then φ has zero Lelong numbers at every point $x \in X$. More precisely, if $\varphi \in PSH(X,\omega)$, if $z_0 \in X$, and if in some coordinate neighborhood $\varphi \leq \varepsilon \log|z - z_0| + \frac{1}{\varepsilon}$ for some $\varepsilon > 0$, then $\varphi \notin \mathcal{E}(X,\omega)$.

The fact that $-|\varphi|^p$ is in $\mathcal{E}(X,\omega)$ shows that there are many elements of $\mathcal{E}(X,\omega)$ whose gradients are non square integrable. This is in sharp contrast with the local case where, for $n = 2$, Blocki showed that a psh function on $\Omega \subseteq \mathbf{C}^2$ is in the domain of definition for MA if and only if its gradient is square integrable.

1.2 The comparison principle

One key result proved in GZ is that the comparison principle holds for $\mathcal{E}(X,\omega)$: If $\varphi,\psi \in \mathcal{E}(X,\omega)$ then

$$\int_{\varphi\le\psi}(\omega+dd^c\psi)^n \;\le\; \int_{\varphi\le\psi}(\omega+dd^c\varphi)^n \tag{1.4}$$

We observe that $\mathcal{E}$ is the maximal such family, that is, if $\mathcal{E}(X,\omega)\subseteq\mathcal{F}\subseteq PSH(X,\omega)$ and if the comparison principle holds for $\mathcal{F}$, then $\mathcal{F}=\mathcal{E}$. To see this, let $\psi\in PSH(X,\omega)$ assume we can define $(\omega+dd^c\psi)^n$ in such a way that the comparison principle (1.4) holds. We must show $\psi\in\mathcal{E}(X,\omega)$. Assume, without loss of generality, that $\psi<-1$. Now apply (1.4) to $\varphi=-|\psi|^p$ for $0<p<1$. Then $\{\varphi\le\psi\}=\{\psi=-\infty\}$. Since $MA(\varphi)(\{\psi=-\infty\})=0$ we conclude $MA(\psi)(\{\psi=-\infty\})=0$. Moreover, since $\int_{\psi<0}\omega^n\le\int_{\psi<0}(\omega+dd^c\psi)^n$, we see (using the fact that $X=\{\psi<0\}$) that $\int_X(\omega+dd^c\psi)^n=\int_X\omega^n$, so ψ has full mass. We conclude $\psi\in\mathcal{E}(X,\omega)$.

Conclusion: If we want to extend $MA(\varphi)$ in such a way that the comparison principle is still valid, then we must restrict our attention to $\varphi\in\mathcal{E}(X,\omega)$. Conversely, the comparison principle holds on $\mathcal{E}(X,\omega)$.

1.3 The general approach

Although we will mainly be interested in the nonpluripolar approach of GZ in these notes, we digress briefly to discuss the "general approach".

We would like to extend the definition of the MA operator in such a way that it is continuous under decreasing subsequences (but we no longer require the comparison principle to hold). There is a natural way to do this: Let $\varphi\in PSH(\Omega)$. Then we say $\varphi\in\mathcal{D}(\Omega)$ if there is a Borel measure μ on Ω such that if $U\subseteq\Omega$ is any open set and $\varphi_j\in PSH(U)\cap C^\infty(U)$ decreases to μ, then $(dd^c\varphi_j)^n\rightharpoonup\mu$ on U. This definition is due to Blocki (2004) who shows that this is the same as requiring $(dd^c\varphi_j)^n$ is locally weakly bounded in U for all $\varphi_j\downarrow\varphi$. We conclude that if $\varphi\in\mathcal{D}(\Omega)$ then $MA(\varphi)$ is well defined. The space $\mathcal{D}(\Omega)$ is called the domain of definition for the Monge-Ampère operator.

Blocki shows that if $\varphi\in PSH(\Omega)$ is locally bounded in a neighborhood of $\partial\Omega$, then $\varphi\in\mathcal{D}(\Omega)$ (note that $\varphi=\log|z_1\cdots z_n|$ fails this test). More generally, if $u\in\mathcal{D}(\Omega)$ and $v\in PSH(\Omega)$ with $u\le v$ outside a compact subset of Ω, then $v\in\mathcal{D}(\Omega)$.

In the case $n=2$, Blocki (2004) shows $\mathcal{D}(\Omega)=PSH(\Omega)\cap W^{1,2}_{loc}$.

In Cegrell (2004) the characterization of $\mathcal{D}(\Omega)$ is equivalent, but slightly different. He restricts his attention to the case where is hyperconvex. This means that there exists $u\in PSH^-(\Omega)$ (the negative psh functions) such that $\{u<-c\}\subset\subset\Omega$ for all $c>0$. For such domains, he defines

$$\mathcal{E}_0(\Omega) \;=\; \{\varphi\in PSH(\Omega)\cap L^\infty \;:\; \textstyle\int_\Omega(dd^c\varphi)^n<\infty\;,\;\lim_{z\to\xi}\varphi(z)=0 \text{ for all } \xi\in\partial\Omega\} \tag{1.5}$$

If $\varphi \in PSH^-(\Omega)$ then there exists $\varphi_j \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $\varphi_j \downarrow \varphi$ on Ω. The Cegrell class $\mathcal{E}(\Omega) \subseteq PSH^-(\Omega)$ is defined as follows: Let $\varphi \in PSH^-(\Omega)$. Then $\varphi \in \mathcal{E}(\Omega)$ if for every $z_0 \in \Omega$ there is an open set $z_0 \subseteq U \subseteq \Omega$ and $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $\varphi_j \downarrow \varphi$ on U and $\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty$.

Blocki shows that $\mathcal{E}(\Omega) = \mathcal{D}(\Omega) \cap PSH^-(\Omega)$.

In the global case, we define $\mathcal{D}(X,\omega) \subseteq PSH(X,\omega)$ using local coordinate charts.

Now let μ be a Borel measure on Ω. We can ask: Does there exist $\varphi \in \mathcal{D}(\Omega)$ such that $(dd^c\varphi)^n = \mu$? If μ is the Dirac measure supported on a single point in Ω, thencc Lempert and Demailly (in the convex and hyperconvex case respectively) proved the existence of a solution. Zeriahi and Xing show that there is a solution of μ is any discrete measure with compact support in a hyperconvex domain. This result is vastly generalized by Ahag, Cegrell, Czyz and Hiep (ACCH) who show that if there is a subsolution, then there is a solution.

1.4 Conjecture

We continue our digression: What is the range of the Monge-Ampère operator? This appears to be an open question (in general). Here are some known results:

1. If μ is a nonpluripolar probability measure on a Kähler manifold (X,ω), then the equation $MA(\varphi) = \mu$ has a solution $\varphi \in \mathcal{E}(X,\omega)$.

2. If μ is a positive measure on a hyperconvex domain Ω, and if there exists ψ such that $MA(\psi) \geq \mu$, then the equation $MA(\varphi) = \mu$ has a solution $\varphi \in \mathcal{D}(\Omega)$

The first theorem is due to GZ (2007) and the second to ACCH (2008).

The ACCH theorem is particularly appealing since it gives a necessary and sufficient condition for μ to be in the range of MA. On the other hand, one could argue that it doesn't really give a full answer to the original question since in a sense, it raises what appears to be an equally difficult question, namely, which positive measures μ satisfy the condition $MA(\psi) \geq \mu$ for some ψ?

The other drawback of ACCH is that its direct analogue for Kähler manifolds doesn't appear to be interesting: If μ is a probability measure and if $\psi \in \mathcal{D}(X,\omega)$ then $MA(\psi) \geq \mu$ impies $MA(\psi) = \mu$ (since both have the same mass).

Here is a (possibly very naive) guess: We say that a measure μ charges at most one pluripolar set if there exists a pluripolar set $Z \subseteq X$ such that $\mu(Z' \cap (X \backslash Z)) = 0$ for all pluripolar sets $Z' \subseteq X$.

Conjecture: Let μ be a measure on (X,ω) or Ω. Assume that μ charges at most one pluripolar set Z. Then $MA(\varphi) = \mu$ has a soluition.

2 Monge-Ampere for non-pluripolar measures

2.1 Statement of results

The main theorem of GZ says the following:

Theorem 1 *Let (X,ω) be a Kähler manifold. Let μ be a Borel measure on X such that $\int_X d\mu = \int_X \omega^n$. Then there exists $\varphi \in \mathcal{E}(X,\omega)$ such that $(\omega + dd^c\varphi)^n = \mu$ if and only if μ does not charge pluripolar sets.*

Dinew proved (later) that φ is unique up to additive constant.

The next question is the following: Suppose that μ has "mild singularities". Can we conclude that φ has mild singularities? To make this precise, GZ define the p-energy, introduced by Cegrell (1998): For $0 < p < \infty$ we define

$$\mathcal{E}^p(X,\omega) \;=\; \{\varphi \in \mathcal{E}(X,\omega) \,:\, \int_X |\varphi|^p\,\omega_\varphi^n < \infty\}$$

We say that potentials in $\mathcal{E}^p(X,\omega)$ have finite p-energy. If μ is a non-pluripolar measure, we say that μ has finite p-energy if $\mathcal{E}^p(X,\omega) \subseteq L^p(\mu)$. The following is due to GZ (proved in the local case by Cegrell (1998)):

Theorem 2 *Let $\varphi \in \mathcal{E}(X,\omega)$ and let $\mu = (\omega + dd^c\varphi)^n$. Then φ has finite p-energy if and only if μ has finite p-energy. This means: $\varphi \in \mathcal{E}^p(X,\omega)$ if and only if $\mathcal{E}^p(X,\omega) \subseteq L^p(\mu)$.*

Remark: One can show (see (4.12) that for $\varphi, \psi \in \mathcal{E}(X,\omega)$ that

$$\int_X |\psi|^p\,\omega_\varphi^n \;\le\; C_p\left(\int_X |\varphi|^p\,\omega_\varphi^n + \int_X |\psi|^p\,\omega_\psi^n\right)$$

This proves the easy direction of Theorem 1.

Remark: The map $\varphi \mapsto (\int_X |\varphi|^p\,d\mu)^{\frac{1}{p}} = \|\varphi\|_p$ is a norm but the map $\varphi \mapsto (\int_X |\varphi|^p\,\omega_\varphi^n)^{\frac{1}{p}}$ is not a norm since it doesn't scale properly. On the other hand, $\varphi \mapsto (\int_X |\varphi|^p\,\omega_\varphi^n)^{\frac{1}{p+1}} = \|\varphi\|_p'$ does scale approximately correctly: $\|\varepsilon\varphi\|_p' \le C_p + \varepsilon\|\varphi\|_p'$ for $\varepsilon < 1$. Moreover, GZ show that $\mathcal{E}^p(X,\omega) \subseteq L^p(\mu)$ if and only if $\|\varphi\|_p \le C\|\varphi\|_p'$ for some $C > 0$.

2.2 Brief sketch of the proof.

The proof of Theorem 1, which is modeled on that of Cegrell in the local case, makes uses of Yau's solution to the Calabi conjecture. The idea is to consider a sequence of smooth volume forms μ_j such that $\mu_j \rightharpoonup \mu$. The μ_j are constructed locally via convolution with symmetric bump function ρ_ε (where $\varepsilon = \frac{1}{j}$), and the patched together using a partition of unity. The importance of using this particular construction is

$$u \leq u * \rho_\varepsilon \tag{2.6}$$

whenever u is a plurisubharmonic function on a domain in $\mathbf{C}^n$. This inequality plays an key role in the proof.

Let φ_j be the unique solution (provided by Yau's theorem) to $(\omega + dd^c\varphi_j)^n = \mu_j$ with $\sup \varphi_j = 0$. Since the $\{\varphi \in PSH(X, \omega) : \sup \varphi = 0\} \subseteq L^1(X)$ is compact, we can find a subsequence such that $\varphi_j \to \varphi$ in $L^1(X)$. Then one must show, using some convergence criteria that $\varphi \in \mathcal{E}(X, \omega)$ and that

$$(\omega + dd^c\varphi_j)^n \rightharpoonup (\omega + dd^c\varphi)^n \tag{2.7}$$

(this is the hard part). The proof is then complete.

We give a very brief idea of the proof of (2.7). The key idea is to use a classical convergence criterion which says that if $\varphi_j, \varphi \in PSH(X, \omega) \cap L^\infty(X)$ and $\varphi_j \to \varphi$ in $L^1(X)$, then (2.7) holds if

$$\int_X |\varphi_j - \varphi| \omega_{\varphi_j}^n \to 0 \tag{2.8}$$

As it stands, this convergence criterion is not adequate since φ is not bounded. Thus we need to generalize the classical result to the case where $\varphi_j, \varphi \in \mathcal{E}^1(X, \omega)$ This generalization is not difficult: we rather quickly are able to reduce to the case of bounded potentials so that we can invoke the classical result. This is accomplished by replacing φ_j with $\varphi_j^{(k)} = \max(\varphi_j, -k)$, and φ with $\varphi^{(k)} = \max(\varphi, -k)$, and then showing that the error goes to zero as $k \to \infty$ (uniform in j).

Now we try to show that (2.8) holds in our setting. We work in local coordinates, and use the fact the $\mu_j = \mu * \rho_{\varepsilon_j}$. Writing $\omega_{\varphi_j} = dd^c u_j$, with u_j a psh function, we let $\tilde{u}_j = (\sup_{k \geq j} u_j)^*$ so $\tilde{u}_j \downarrow u$. Then a simple computation, which makes repeated use of (2.6), shows that on a coordinate patch $U \subseteq \mathbf{C}^n$,

$$\int_U |\varphi_j - \varphi| \omega_{\varphi_j}^n \ \leq 2 \int_U (\tilde{u}_j * \rho_{\varepsilon_j} - u) d\mu \ + \ \int_U (\varphi - \varphi_j)\, d\mu$$

The monotone convergence theorem implies the the first integral goes to zero as $j \to \infty$. As for the second integral, we wish to apply a second classical convergence result which says that if $\varphi_j, \varphi \in PSH(X, \omega) \cap L^1(X)$ are uniformly bounded in $L^\infty(X)$, and if $\varphi_j \to \varphi$ in $L^1(X)$, then

$$\int_X (\varphi - \varphi_j)\, d\mu \ \to \ 0 \tag{2.9}$$

for any non-pluripolar measure μ. Again, this criterion is not adequate since φ_j is not uniformly bounded. So again, we wish to generalize the criterion to the case where $\varphi_j \in \mathcal{E}(X, \omega)$. As before, we replace φ, φ_j by their canonical approximations $\varphi_j^{(k)}$ and $\varphi^{(k)}$. But this time, estimating the error is not so easy and the proof is rather involved. The first step is to assume that the measure μ has extremely mild singularities, that is, we assume μ is dominated by capacity. In particular, this means that μ has finite p energy for all p. It also implies, using (2.6) again, that the sequence $\int_X(-\varphi_j)MA(\varphi_j)$ is bounded (so that $\varphi \in \mathcal{E}^1(X,\omega)$). This allows us to estimate the error in passing to the canonical approximations, and proves that (2.9) holds for this class of μ. Finally, the general case is handled via the Radon-Nikodym theorem, which implies that on non-pluripolar measure is absolutely continuous with respect to some ω_ψ^n with $\psi \in PSH(X,\omega) \cap L^\infty(X)$.

Remark: The variational approach of BBGZ gives a new proof of Theorem 1 which avoids appealing to Yau's solution of the Calabi conjecture.

3 Convergence criteria

Suppose $\varphi_j, \varphi \in PSH(X,\omega)$ and $\varphi_j \to \varphi$ in $L^1(X)$. When can we conclude $\omega_{\varphi_j}^n \to \omega_\varphi^n$? And if μ is a Borel measure, when can we conclude $\int_X \varphi_j \, d\mu \to \int_X \varphi \, d\mu$?

3.1 Convergence criteria for $\omega_{\varphi_j}^n$

The proofs of the main theorems make use of some classical convergence results for bounded potentials which we now recall.

Assume $\varphi_j \in PSH(X,\omega)$ and $\varphi \in L^1(X)$ and $\varphi_j \to \varphi$ in $L^1(X)$. Then $\varphi = \lim(\sup \varphi_j)^*$ (almost everywhere), and after replacing φ by $\lim(\sup \varphi_j)^*$, which only changes φ on a set of measure zero, we have $\varphi \in PSH(X,\omega)$. Moreover, $(\omega + dd^c\varphi_j) \to (\omega + dd^c\varphi)$.

By weak compactness, $\omega_{\varphi_j}^n \to \mu$ for some measure μ (after passing to a subsequence). But even if $\varphi_j, \varphi \in PSH(X,\omega) \cap L^\infty(X)$, we still can't conclude that $\mu = \omega_\varphi^n$. In this context, a result of Lelong is particularly striking: Let $n \geq 2$, let $\Omega \subseteq \mathbf{C}^n$ be a ball, and let $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$. Then there is a uniformly bounded sequence $\varphi_j \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $\varphi_j \to \varphi$ in $L^1(\Omega)$ and $(dd^c\varphi_j)^n = 0$ for all j.

The key question then becomes: If $\varphi_j \to \varphi$ then what additional assumptions will guarantee $\omega_{\varphi_j}^n \to \omega_\varphi^n$?

Theorem 3 *Let $\varphi_j, \varphi \in PSH(X,\omega) \cap L^\infty(X)$.*

1. *If $\varphi_j \to \varphi$ in capacity and $\sup_j \|\varphi_j\|_{L^\infty} < \infty$, then $\omega_{\varphi_j}^n \to \omega_\varphi^n$.*

2. *If $\varphi_j \downarrow \varphi$ then $\varphi_j \to \varphi$ in capacity and in particular, $\omega_{\varphi_j}^n \to \omega_\varphi^n$. Moreover, $\varphi_j \to \varphi$ in $L^1(X)$ so $\omega_{\varphi_j} \to \omega_\varphi$.*

3. *If $\varphi_j \uparrow \varphi$ almost everywhere then $\omega_{\varphi_j}^n \to \omega_\varphi^n$. Moreover, $\varphi_j \to \varphi$ in $L^1(X)$ so $\omega_{\varphi_j} \to \omega_\varphi$.*

4. *Assume* $\varphi_j \to \varphi$ *in* $L^1(X)$, *i.e. assume* $\int_X |\varphi_j - \varphi|\omega^n \to 0$. *Then* $\omega_{\varphi_j} \to \omega_\varphi$. *If we further have* $\int_X |\varphi_j - \varphi|\omega^n_{\varphi_j} \to 0$ *then* $\omega^n_{\varphi_j} \to \omega^n_\varphi$.

Proof. The first three statements are well known, so will just prove statement 4. Without loss of generality, we may assume $\int_X |\varphi_j - \varphi|\omega^n_{\varphi_j} \leq \frac{1}{j^2}$.

Let $\tilde{\varphi}_j = \max(\varphi_j, \varphi - \frac{1}{j})$. Then $|\tilde{\varphi}_j - \varphi| \leq |\varphi_j - \varphi| + \frac{1}{j}$ so $\tilde{\varphi}_j \to \varphi$ in $L^1(X)$ as well. We claim that $\tilde{\varphi}_j \to \varphi$ in capacity. We must show $cap(|\tilde{\varphi}_j - \varphi| > \varepsilon) \to 0$ for all $\varepsilon > 0$. But $\{|\tilde{\varphi}_j - \varphi| > \varepsilon\} = \{\tilde{\varphi}_j - \varphi > \varepsilon\}$ for $j > \frac{1}{\varepsilon}$. Let $\psi_j = (\sup_{k \geq j} \tilde{\varphi}_j)^*$. Then $\psi_j \downarrow \varphi$. By part two of the theorem, $\psi_j \to \varphi$ in capacity. But $\psi_j \geq \tilde{\varphi}_j$. This implies $\{\tilde{\varphi}_j - \varphi > \varepsilon\} \subseteq \{\psi_j - \varphi > \varepsilon\}$ and hence $\tilde{\varphi}_j \to \varphi$ in capacity. We conclude, from part one of the theorem, that

$$(\omega + dd^c\tilde{\varphi}_j)^n \to (\omega + dd^c\varphi)^n \tag{3.10}$$

To see that $(\omega + dd^c\varphi_j)^n \to (\omega + dd^c\varphi)^n$ first observe that by compactness, we have, after passing at a subsequence, $(\omega + dd^c\varphi_j)^n \to \mu$ for some measure μ such that $\int_X \mu = \int_X \omega^n$. It suffices to show that $\mu \leq (\omega + dd^c\varphi)^n$ since both have the same total measure. We estimate, for f a continuous function,

$$\int_X f\,(\omega + dd^c\varphi_j)^n = \int_{\varphi_j > \varphi - \frac{1}{j}} f(\omega + dd^c\tilde{\varphi}_j)^n + \int_{\varphi_j \leq \varphi - \frac{1}{j}} f(\omega + dd^c\varphi_j)^n$$

But

$$\int_{\varphi_j \leq \varphi - \frac{1}{j}} \omega^n_{\varphi_j} \leq j \int_X |\varphi_j - \varphi|\omega^n_{\varphi_j} \leq \frac{1}{j}$$

Taking limits we see that $\int_X f\mu \leq \int_X f\omega^n_\varphi$ so $\mu \leq \omega^n_\varphi$. This completes the proof.

Remark: Statement 4 holds more generally if we only assume $\varphi_j, \varphi \in \mathcal{E}^1(X,\omega)$. The only place we used the fact that $\varphi_j, \varphi \in L^\infty$ was to guarantee that the $\tilde{\varphi}_j$ are uniformly bounded in L^∞ so that we can conclude (3.10) holds. If we don't make that assumption, and instead require that $\varphi_j, \varphi \in \mathcal{E}^1(X,\omega)$, then the proof can be modified so that (3.10) still holds. Indeed, let $\varphi^k = \max(\varphi, -k)$. Then for each k we see that $\tilde{\varphi}_j^{(k)}$ is uniformly bounded in L^∞, independent of j. We still have $\tilde{\varphi}_j^{(k)} \to \tilde{\varphi}^{(k)}$ in capacity, so we conclude

$$(\omega + dd^c\tilde{\varphi}_j^{(k)})^n \to (\omega + dd^c\varphi^{(k)})^n$$

Now we compare $(\omega + dd^c\tilde{\varphi}_j^{(k)})^n$ to $(\omega + dd^c\tilde{\varphi}_j)^n$ and $(\omega + dd^c\varphi^{(k)})^n$ to $(\omega + dd^c\varphi)^n$: Let h be a continuous function and estimate

$$|\int_X h\,(\omega + dd^c\tilde{\varphi}_j^{(k)})^n - \int_X h\,(\omega + dd^c\tilde{\varphi}_j)^n| = |\int_{\tilde{\varphi}_j^{(k)} \leq k} h\,(\omega + dd^c\tilde{\varphi}_j^{(k)})^n - \int_{\tilde{\varphi}_j \leq k} h\,(\omega + dd^c\tilde{\varphi}_j)^n|$$

Now

$$|\int_{\tilde{\varphi}_j \leq k} h\,(\omega + dd^c\tilde{\varphi}_j)^n| \;\leq\; \frac{\sup|h|}{k} E_1(\tilde{\varphi}_j) \leq 2^n \frac{\sup|h|}{k} E_1(\varphi - 1)$$

since $\varphi_j \geq \varphi - 1$ and $E_1(\varphi)$ gets bigger (modulo the factor of 2^n) as φ gets more singular (see (4.11). We have a similar bound for $\tilde{\varphi}_j^{(k)}$. This shows that the error goes to zero as $k \to \infty$, uniformly in j and thus (3.10) holds.

3.2 Convergence criteria for $\int_X \varphi_j \, d\mu$

The non-pluripolar property enters the theory in three important ways:

1. It allows us to obtain a stronger convergence result.

2. It allows us to to make use of energy functionals.

3. Non-pluripolar measures are, via the Radon-Nikodym theorem, absolutely continuous respect to measures of the form $MA(\varphi)$ for some bounded φ.

We first describe the convergence result. Let $\varphi_j, \varphi \in PSH(X, \omega) \cap L^\infty(X)$, and assume $\varphi_j \to \varphi$ in $L^1(X)$. If μ is another Borel measure on X which is bounded by Lebesgue measure, then we can conclude $\int_X \varphi_j \, d\mu \to \int_X \varphi \, d\mu$. In the next proposition, we show that this holds more generally, provided μ is non-pluripolar and the φ_j are uniformly bounded.

Proposition 1 *Let $\varphi_j, \varphi \in PSH(X, \omega) \cap L^\infty(X)$ and assume $\sup_X |\varphi_j| \leq C$. Let μ be a non-pluripolar measure. Assume $\varphi_j \to \varphi$ in $L^1(X)$. Then $\int_X \varphi_j \, d\mu \to \int_X \varphi \, d\mu$*

Proof. Since the φ_j are bounded we may assume, after passing to a subsequence, that $\lim_j \int_X \varphi_j \, d\mu$ exists and that $\varphi_j \to \varphi$ a.e. (with respect to Lebesgue measure). If $\varphi_j \to \varphi$ everywhere, we would be done (dominated convergence theorem).

Since the φ_j are bounded they are also bounded in $L^2(d\mu)$ so, after passing to a subsequence, there exists $\psi \in L^2(d\mu)$ so that $\psi_k \to \psi$ in $L^2(d\mu)$ where $\psi_k = \frac{1}{k}\sum_{j=1}^k \varphi_j$. In particular, after passing to another subsequence, $\psi_k \to \psi$ μ-a.e. Since $\varphi_j \to \varphi$ a.e., we also have $\psi_k \to \varphi$ a.e. Thus $\sup_{k \geq j} \psi_k \to \varphi$ a.e. which implies $(\sup_{k \geq j} \psi_k)^* \to \varphi$ *everywhere.* Putting this all together:

$$\begin{aligned}
\lim_{j\to\infty} \int \varphi_j \, d\mu &= \lim_{j\to\infty} \int \psi_j \, d\mu = \lim_{j\to\infty} \int \psi \, d\mu \\
&= \lim_{j\to\infty} \int (\sup_{k\geq j} \psi_k) \, d\mu = \lim_{j\to\infty} \int (\sup_{k\geq j} \psi_k)^* \, d\mu = \int \varphi \, d\mu
\end{aligned}$$

The first equality follows from the definition of ψ_j, the second and third from the fact that $\psi_j \to \psi$ μ-a.e., and the fourth from the fact that μ is non-pluripolar.

4 Energy functionals.

We have extended the definition of the Monge-Ampère operator from $PSH(X,\omega)\cap L^\infty(X)$ to $\mathcal{E}(X,\omega)$. In this section, we describe a filtration of $\mathcal{E}(X,\omega)$ by "Sobolev" spaces

$$PSH(X,\omega)\cap L^\infty(X) \subseteq \mathcal{E}_\chi(X,\omega) \subseteq \mathcal{E}(X,\omega)$$

of finite weighted energy, where χ ranges over all admissible weights.

4.1 The weighted energy functionals E_χ

For $\chi : \mathbf{R}^- \to \mathbf{R}^-$ an increasing function, with $\lim_{t\to-\infty}\chi(t) = -\infty$, and $\varphi \in \mathcal{E}(X,\omega)$, we define the χ-energy

$$E_\chi(\varphi) \;=\; \int_X (-\chi\circ\varphi)\,\omega_\varphi^n$$

Here we shall always assume that χ is smooth and $\chi(0) = 0$ (this is not absolutely necessary, but the more general case can always be reduced to this setting).

We define

$$\mathcal{E}_\chi(X,\omega) = \{\varphi \in \mathcal{E}(X,\omega) \,:\, E_\chi(X,\omega) < \infty\}$$

If χ is convex, we write $\chi \in \mathcal{W}^-$ and we say E_χ is a "high energy functional". If χ is concave, we write $\chi \in \mathcal{W}^+$ and we say E_χ is a "low energy functional". An important class of examples are the p-energies $\chi_p(t) = -|t|^p$. Then χ is low energy if and only if $p \le 1$ and high energy if and only if $p \ge 1$. We shall write $E_p(\chi) = E_{\chi_p}$ so

$$E_p(\varphi) \;=\; \int_X |\varphi|^p\,\omega_\varphi^n$$

The key observation is the following: Let $\varphi \in \mathcal{E}(X,\omega)$. It follows easily from the fact that $MA(\varphi)$ is non-pluripolar that there exists a weight χ such that $E_\chi(\varphi) < \infty$. In fact, we may take $\chi \in \mathcal{W}^-$. Thus for $\chi^+ \in \mathcal{W}^+$ and $\chi^- \in \mathcal{W}^-$ we have

$$PSH(X,\omega)\cap L^\infty(X) \subseteq \mathcal{E}_{\chi^+}(X,\omega) \subseteq \mathcal{E}^1(X,\omega) \subseteq \mathcal{E}_{\chi^-}(X,\omega) \subseteq \mathcal{E}(X,\omega)$$

$$PSH(X,\omega)\cap L^\infty(X) \subseteq \mathcal{E}_{\chi^+}(X,\omega) = \bigcap_{\chi\in\mathcal{W}^+} \mathcal{E}_{\chi^+}(X,\omega) \;\text{ and }\; \mathcal{E}(X,\omega) = \bigcup_{\chi\in\mathcal{W}^-} \mathcal{E}_\chi(X,\omega)$$

Let $\chi \in \mathcal{W}^-$. Although it is not obvious from the definition, it turns out that if φ is more singular than ψ, i.e., if $\varphi \le \psi$, then $E_\chi(\varphi) \ge c_\chi E_\chi(\psi)$ for some $c_\chi > 0$ (in fact, we can take $c_\chi = 2^{-n}$). Thus $E_\chi(\varphi)$ measures the strength of the singularity of φ. A similar bound holds for $\chi \in \mathcal{W}^+$, provided we assume that χ is of polynomial growth.

More precisely, we must assume that there exists $M > 0$ such that $|t\chi'(t)| \leq M|\chi(t)|$ for all $t \in \mathbf{R}^-$. If this holds, we write $\chi \in \mathcal{W}^+_M$. Then for $\chi \in \mathcal{W}^+_M$, the above inequality holds with $c_\chi = (M+1)^{-n}$. In summary, if $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$ then

$$\varphi \leq \psi \ \text{ implies } \ E_\chi(\psi) \leq c_\chi E_\chi(\varphi) \tag{4.11}$$

Clearly $E_\chi(\varphi) < \infty$ if φ is bounded. If $\varphi \in \mathcal{E}(X,\omega)$ and $\varphi_j \in PSH(X,\omega) \cap L^\infty$ is the canonical approximation, so that $\varphi_j \downarrow \varphi$, then $E_\chi(\varphi_j)$ is an increasing sequence. GZ show that $\lim_{j\to\infty} E_\chi(\varphi_j) = E_\chi(\varphi)$. Thus $\varphi \in \mathcal{E}_\chi(X,\omega)$ if and only if $E_\chi(\varphi_j)$ is bounded.

We also define, for $\varphi, \psi \in \mathcal{E}(X,\omega)$, the joint energy

$$E_\chi(\varphi,\psi) \ = \ \int_X (-\chi \circ \varphi)\, \omega_\psi^n$$

so that $E_\chi(\varphi) = E_\chi(\varphi,\varphi)$. Again, although it is not obvious from the definitions, if $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$ we have

$$E_\chi(\varphi,\psi) \ \leq \ c_\chi\,[E_\chi(\varphi) + E_\chi(\psi)] \tag{4.12}$$

In particular, if $MA(\psi) = \mu$ then $\chi \circ \varphi \in L^1(\mu)$ for all $\varphi \in \mathcal{E}_\chi(X,\omega)$, that is, μ has "finite χ-energy". This proves one direction of Theorem 2 (take $\chi(t) = -|t|^p$).

4.2 Convexity of the energy functionals

The functionals E_χ also satisfy an important convexity property. To simplify the statement, we restrict to the functionals E_p. Let $t_1, ..., t_k \geq 0$ with $\sum_{j=1}^k t_j = 1$. Then

$$E(t_1\varphi_1 + \cdots + t_k\varphi_k) \leq \ C_p(\sum_{j=1}^k \tau_j) \cdot \max_{1\leq j\leq k} E_p(\varphi_j) \tag{4.13}$$

where $\tau_j = \max(t_j, t_j^p)$. In particular, $\mathcal{E}^p = \mathcal{E}_{\chi_p}$ is convex. Moreover, if $\varphi_1, \varphi_2, ... \in \mathcal{E}_p$ with $\varphi_j \leq 0$ and $\sup_j E_p(\varphi_j) < \infty$, then $\sum_{j=1}^\infty 2^{-j}\varphi_j \in \mathcal{E}_p$.

The functionals E_p also satisfy an important scaling property. If $0 < \varepsilon < 1$ then

$$E_p(\varepsilon\varphi) \ \leq C_p[\varepsilon^p + \varepsilon^{p+1}E(\varphi)] \tag{4.14}$$

Combining the convexity and scaling properties: if $\varphi_1, \varphi_2, ... \in \mathcal{E}_p$ is an arbitrary sequence with $\varphi_j \leq -1$, and if we set $E(\varphi_j) = M_j$, then

$$\sum_{j=1}^\infty 2^{-j} M_j^{-1/(p+1)}\varphi_j \ \in \ \mathcal{E}_p \tag{4.15}$$

4.3 Capacity of sublevel sets

If $\varphi \in PSH(X,\omega)$ then for $t \geq 1$,

$$Cap_\omega(\varphi < -t) \ \leq \ \frac{C_\varphi}{t}$$

When φ has finite energy, a stronger estimate holds:

Proposition 2 *Assume that $\varphi \in \mathcal{E}_p(X,\omega)$. Then*

$$Cap_\omega(\varphi < -t) \ \leq \ \frac{C_\varphi}{t^{p+1}} \tag{4.16}$$

Moreover

$$\int_{t=1}^{\infty} t^p \, Cap_\omega(\varphi < -t) \, dt \ < \ \infty \tag{4.17}$$

Remark: There is a partial converse: If $Cap_\omega(\varphi < -t) \ \leq \ \frac{C_\varphi}{t^{p+n+\varepsilon}}$ then $\varphi \in \mathcal{E}_p(X,\omega)$.

Proof. We first prove (4.16). In fact, we prove a more general statement. Let $\varphi \in \mathcal{E}_\chi(X,\omega)$ with $\varphi \leq -1$. We claim

$$Cap_\omega(\varphi < -t) \ \leq \ \frac{C_\varphi}{t|\chi(-t)|}$$

To see this, let $u \in PSH(X,\omega)$ with $-1 \leq u \leq 0$. For $t \geq 1$ note that

$$\{\varphi < -2t\} \ \subseteq \ \{\varphi/t < u - 1\} \ \subseteq \ \{\varphi < -t\}$$

Thus

$$\begin{aligned} \int_{\varphi < -2t} \omega_u^n \ &\leq \ \int_{\frac{\varphi}{t} < u-1} \omega_u^n \ \leq \ \int_{\frac{\varphi}{t} < u-1} \omega_{\varphi/t}^n \ \leq \int_{\varphi < -t} \omega_{\varphi/t}^n \\ &\leq \ \int_{\varphi < -t} \omega^n + \frac{1}{t} \sum_{j=1}^n \binom{n}{j} \int_{\varphi < -t} \omega_\varphi^j \wedge \omega^{n-j} \end{aligned}$$

If follows from the existence of the alpha invariant that the first integral decays exponentially. As for the remaining integrals,

$$\int_{\varphi < -t} \omega_\varphi^j \wedge \omega^{n-j} \ \leq \ \frac{1}{|\chi(-t)|} \int_X (-\chi \circ \varphi) \, \omega_\varphi^j \wedge \omega^{n-j} \ \leq \ \frac{1}{|\chi(-t)|} E_\chi(\varphi)$$

This proves (4.16). To prove (4.17), we estimate

$$\begin{aligned} \int_1^\infty t^p Cap_\omega(\varphi < -2t) \, dt \ &\leq \ \int_1^\infty t^p Vol_\omega(\varphi < -t) \, dt + \sum_{j=1}^n \binom{n}{j} \int_1^\infty t^{p-1} \int_{\varphi < -t} \omega_\varphi^j \wedge \omega^{n-j} \ dt \\ &= \ \int_1^\infty t^p Vol_\omega(\varphi < -t) \, dt + \sum_{j=1}^n \binom{n}{j} \int_X |\varphi|^p \omega_\varphi^j \wedge \omega^{n-j} \end{aligned}$$

The first integral is finite since $Vol_\omega(\varphi < -t)$ decays exponentially. As for the other integrals,

$$\int_X |\varphi|^p \omega_\varphi^j \wedge \omega^{n-j} \leq \int_X |\varphi|^p \omega_\varphi^{j+1} \wedge \omega^{n-j-1} \leq \cdots \leq E_p(\varphi) \ < \ \infty$$

4.4 Criteria for finite energy

Recall (cf. the statement of Theorem 2) that a measure μ has finite p-energy if

$$\mathcal{E}_p(X,\omega) \subseteq L^p(\mu) \tag{4.18}$$

We wish to establish necessary and sufficient conditions for (4.18) to hold.

Proposition 3 *Let μ be a probability measure on X. Then $\mathcal{E}_p(X,\omega) \subseteq L^p(\mu)$ if and only if there exist $C > 0$ such that for all $\psi \in PSH(X,\omega) \cap L^\infty(X)$ the $\sup \psi = -1$, the following holds*

$$0 \leq \int_X |\psi|^p \, d\mu \ \leq \ C \left[\int_X |\psi|^p \omega_\psi^n \right]^{\frac{p}{p+1}} \tag{4.19}$$

Moreover, if (3) holds, then it also holds for all $\psi \in \mathcal{E}_p(X,\omega)$.

Proposition 4 *Let μ be a probability measure on X. Assume*

$$\mu(E) \ \leq \ A \, Cap_\omega^*(E)^\alpha \tag{4.20}$$

for some $\alpha > \frac{p}{p+1}$. Then μ has finite p-energy. We also have a partial converse: If μ has finite p-energy, then (4.20) holds for $\alpha = \frac{p}{p+1} \min(p,1)$.

Proof of Proposition 3. Suppose $\mathcal{E}_p(X,\omega) \subseteq L^p(\mu)$. Then (4.15) shows that for any sequence of $\varphi_j \in \mathcal{E}_p$, that

$$\sup_j 2^{-jp} M_j^{\frac{-p}{(p+1)}} \int |\varphi_j|^p \, d\mu < \infty$$

But this implies there exists $C > 0$ such that

$$\int |\psi|^p \, d\mu \ \leq C \left[\int |\psi|^p \omega_\psi^n \right]^{\frac{p}{p+1}} \tag{4.21}$$

for all $\psi \in PSH(X,\omega) \cap L^\infty(X)$: The proof goes by contradiction. If not, then take $C = C_j$ with $2^{-jp} C_j \to \infty$, and choose φ_j which violates the above inequality. Then $\sup 2^{-jp} C_j < \infty$, a contradiction.

Proof of Proposition 4. Recall that for a Borel set $E \subseteq X$, the extremal function is

$$h_{E,\omega}(x) \ = \ \sup\{\varphi(x) \, : \, \varphi \in PSH(X,\omega), \varphi \leq 0, \varphi|_E = 1\}$$

Then $h_{E,\omega}^* \in PSH(X,\omega)$ has the properties

1. We have $-1 \leq h_{E,\omega}^* \leq 0$ and $h_{E,\omega}^* = -1$ on $E \backslash P$ with P pluripolar.
2. $(\omega + dd^c h_{E,\omega}^*)^n = 0$ on $\Omega_E \backslash \bar{E}$ where $\Omega_E = \{h_{E,\omega}^* < 0\}$.

3. $Cap^*_\omega(E) = \int_X (-h^*_{E,\omega})(\omega + dd^c h^*_{E,\omega})^n$

Now assume $\mathcal{E}_p(X,\omega) \subseteq L^p(\mu)$. Since $-h^*_{E,\omega} = 1$ on $E\backslash P$, and since P has μ-measure zero, Proposition 3 implies

$$\mu(E) \leq \int_X (-h^*_{E,\omega})^p \, d\mu \ \leq \ A\left[\int_X (-h^*_{E,\omega})^p \omega^n_{h^*_{E,\omega}}\right]^{\frac{p}{p+1}}$$

If $p \geq 1$ then $(-h^*_{E,\omega})^p \leq -h^*_{E,\omega}$ and the right side is bounded by $ACap^*_\omega(E)^{\frac{p}{p+1}}$. If $p < 1$ then we use Hölder's inequality to see that the right side is bounded by $A'Cap^*_\omega(E)^{\frac{p^2}{p+1}}$.

For the converse, we assume $1 > \alpha > \frac{p}{p+1}$ in (4.20) and try to show μ has finite p-energy. Thus we let $\varphi \in \mathcal{E}^p(X,\omega)$ with $\sup\varphi = -1$ and estimate

$$\int (-\varphi)^p \, d\mu \ = \ 1 + p\int_1^\infty t^{p-1}\, \mu(\varphi < -t)\, dt \leq 1 + pA\int_1^\infty t^{p-1}\, [Cap_\omega(\varphi < -t)]^\alpha \, dt$$

$$= 1 + pA\int_1^\infty t^{p-\alpha-p\alpha-(1-\alpha)}\, t^{p\alpha}[Cap_\omega(\varphi < -t)]^\alpha \, dt$$

$$\leq \ 1 + pA\left[\int_1^\infty t^{\frac{p-\alpha(p+1)}{1-\alpha}-1} dt\right]^{1-\alpha} \left[\int_1^\infty t^p Cap_\omega(\varphi < -t)\, dt\right]^\alpha$$

The first integral is finite since $\alpha > \frac{p}{p+1}$ and (4.17) implies that the second is finite.

Proposition 5 *If $0 < p < \infty$ and if there exists $\alpha > \frac{p}{p+1}$ such that*

$$\mu(E) \ \leq A\, Cap(E)^\alpha \tag{4.22}$$

for all E, then μ has finite p-energy. Moreover

$$\int_1^\infty t^p Cap_\omega(\varphi < -t)\, dt \ < \ \infty$$

In particular, if $\mu(E) \leq A\, Cap(E)$, then μ has finite p-energy for all p.

Conversely, if $p > 1$ and if μ has finite p-energy, then (4.22) holds for some A and some $0 < \alpha < 1$.

5 Details of the proofs.

For $A > 0$ we let

$$\mathcal{M}_A \ = \ \{\text{probability measures } \mu \ : \ \mu(E) \leq A \cdot Cap(E) \text{ for all Borel sets } E \subseteq X\}$$

We wish to prove the following results:

Theorem 4 *Let $\mu \in \mathcal{M}_A$ for some $A > 0$. Then there exists $\varphi \in \mathcal{E}^1(X,\omega)$ such that*

$$(\omega + dd^c\varphi)^n \;=\; \mu$$

Theorem 5 *Let μ be a nonpluripolar probability measure. Let $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$. Suppose that there exists $F : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying*

1. $\lim_{t\to\infty} F(t)/t = 0$.
2. $\int_X (-\chi)\circ\psi \, d\mu \;\le\; F(E_\chi(\psi))$ *for all* $\psi \in PSH(X,\omega) \cap L^\infty(X)$

Then there exists $\varphi \in \mathcal{E}_\chi(X,\omega)$ such that $(\omega + dd^c\varphi)^n = \mu$.

Note that for $\chi = \chi_p$ we can take $F(t) = t^{\frac{p}{p+1}}$.

Proof of Theorem 4. Let μ be a probability measure on X. Cover X by a finite number of open sets U_i which are biholomorphic to the unit ball in $\mathbf{C}^n$. Let $\mu_\varepsilon^{U_i} = \mu|_{U_i} * \rho_\varepsilon$ which is a measure on the ball of radius $1-\varepsilon$. Moreover, $\mu_\varepsilon^{U_i} \to \mu|_{U_i}$ (weak convergence).

Let θ_i be a partition of unity subordinate to U_i. Then $\theta_i\mu_\varepsilon^{U_i}$ is a measure on U_i, which may also be viewed as a measure on X. Furthermore, $\theta_i\mu_\varepsilon^{U_i} \to \theta_i\mu|_{U_i} = \theta_i\mu$. This implies $\sum_i \theta_i\mu_\varepsilon^{U_i} \to \mu$. In particular, $\int_X \theta_i\mu_\varepsilon^{U_i} \to \int_X d\mu$. We may assume that $\int_X \theta_i\mu_\varepsilon^{U_i} + \varepsilon\omega^n > \int_X d\mu$ (if necessary, replace ρ_ε by $\rho_{f(\varepsilon)}$ where $f(\varepsilon) \to 0$ very rapidly as $\varepsilon \to 0$). Now let

$$\mu_j \;=\; c_j\left[\sum_i \theta_i\mu_\varepsilon^{U_i} + \varepsilon_j\omega^n\right]$$

where $\varepsilon_j = \frac{1}{j}$ and $c_j \uparrow 1$ is chosen so that $\int_X d\mu_i = \int_X d\mu$.

Then μ_j has smooth, strictly positive density so Yau's theorem provides us with a unique $\varphi_j \in PSH(X,\omega) \cap C^\infty(X)$ satisfying

$$\omega^n_{\varphi_j} = \mu_j \quad , \quad \sup_X \varphi_j = -1$$

Passing to a subsequence we may assume $\varphi_j \to \varphi$ in $L^1(X)$ with $\varphi \in PSH(X,\omega)$ and $\sup\varphi = -1$. Our goal is to prove $\varphi \in \mathcal{E}_\chi$ and $\omega^n_\varphi = \mu$.

The first step is to show that

$$\int_X (-\varphi_j)\,\omega^n_{\varphi_j} \;\le\; C\int_X (-\varphi)\,d\mu \tag{5.23}$$

This will follow easily from the fact: If u is psh in a ball in $\mathbf{C}^n$, then $(-u) * \rho_\varepsilon \le (-u)$.

Next we assume $\mu \in \mathcal{M}_A$ (this assumption will be removed later). Our goal is to show that with this assumption, that $\varphi \in \mathcal{E}^1(X,\omega)$ and that (2.8) holds. If we can show this, then this completes the proof in this case, i.e, we show that $(\omega + dd^c\varphi)^n = \mu$.

To do this, we proceed as follows.

1. Since $\mu \in \mathcal{M}_A$, Proposition 5 implies

$$\mathcal{E}^1(X,\omega) \subseteq L^1(\mu) \tag{5.24}$$

2. We combine (4.21), which says $\int_X(-\varphi_j)d\mu \leq C(\int_X(-\varphi_j)\omega_{\varphi_j}^n)^{1/2}$ if $\varphi_j \in \mathcal{E}^1(X,\omega)$ (certainly the case here since φ_j is smooth) with (5.23) to conclude $\sup_j E^1(\varphi_j) < \infty$ and thus, $\varphi \in \mathcal{E}^1(X,\omega)$. Moreover

$$\sup_j \int_X (-\varphi_j)\, d\mu < \infty \tag{5.25}$$

3. Since $\sup_j E^1(\varphi_j) < \infty$ we can apply (2) and conclude

$$Cap(\varphi_j < -t) \ \leq \ \frac{C}{t^2} \tag{5.26}$$

4. Since

$$\int_X (-\varphi_j)^{3/2}\, d\mu = 1 + \frac{3}{2}\int_1^\infty t^{1/2}\mu(\varphi_j < t)\, dt \leq 1 + \frac{3A}{2}\int_1^\infty t^{1/2} Cap(\varphi_j < -t) dt$$

we conclude

$$\int_X (-\varphi_j)^{3/2}\, d\mu \ \leq \ C \tag{5.27}$$

5. We prove

$$\int_X \varphi_j\, d\mu \to \int_X \varphi\, d\mu \tag{5.28}$$

If the φ_j were uniformly bounded in L^∞ then we could apply Proposition 1. For fixed $k > 0$, we let $\varphi_j^{(k)} = \max(\varphi_j, -k)$. Then Proposition 1 tells us

$$\int_X \varphi_j^{(k)}\, d\mu \ \to \ \int_X \varphi^{(k)}\, d\mu \tag{5.29}$$

On the other hand,

$$\int_X |\varphi_j^{(k)} - \varphi_j| \ \leq \ 2\int_{\varphi_j \leq k} (-\varphi_j)\, d\mu \leq \frac{2}{\sqrt{k}}\int_X (-\varphi_j)^{3/2}\, d\mu \ \leq \frac{C}{\sqrt{k}} \tag{5.30}$$

where C is independent of j and k. This completes step 5.

Remark: It seems that we can skip steps 3,4 and instead observe, using (5.25):

$$\int_X |\varphi_j^{(k)} - \varphi_j| \ \leq \ 2\int_{\varphi_j \leq k} (-\varphi_j)\, d\mu \leq \frac{2}{k}\int_X (-\varphi_j)\, d\mu \ \leq \frac{C}{k} \tag{5.31}$$

6. We can achieve our goal of establishing (2.8), that is, we can show

$$\int_X |\varphi_j - \varphi| \, d\mu_j \to 0 \tag{5.32}$$

It suffices to show that (5.32) holds with μ_j replaced with $\mu_j^U = \mu|_U * \rho_{\varepsilon_j}$.

To do this, write $\varphi_j = u_j - \gamma$ and $\varphi = u - \gamma$ where γ is a local potential for ω on U. Then $u_j \to u$ in $L^1(U)$ so $u = \lim_{j\to\infty}(\sup_{k\geq j} u_k)^*$ and

$$\int_U |\varphi_j - \varphi| \, d\mu_j^U \;=\; \int_U \left(\int_U |u_j(\xi) - u(\xi)| \rho_{\varepsilon_j}(z-\xi) \, d\lambda(\xi) \right) d\mu(z) \;=\; \int_U w_j(z) \, d\mu(z)$$

Define $\tilde{u}_j = (\sup_{k\geq j} u_k)^*$ so $\tilde{u}_j \geq \max(u, u_j) \geq u$ and $\tilde{u}_j \downarrow u$.