Proposition 1. *Let X be a compact Kähler manifold (or* Kähler *variety). Let T be a positive* $(1,1)$ *current which has locally bounded potentials on $X\backslash S$ where $S \subseteq X$ is an analytic subvariety (or more generally, a closed pluripolar set). Then for any $\varepsilon > 0$ and $K \subset\subset X\backslash S$ there exists $\rho_\varepsilon \in C^\infty(X)$ such that*

(1) $0 \leq \rho_\varepsilon \leq 1$
(2) $\operatorname{supp}(\rho_\varepsilon) \subseteq X\backslash S$
(3) $\rho_\varepsilon = 1$ *on K.*
(4) $\int_{X\backslash S} \sqrt{-1}\partial\rho_\varepsilon \wedge \bar{\partial}\rho_\varepsilon \wedge\ T^{n-1} < \varepsilon$

Here $\sqrt{-1}\partial\rho_\varepsilon \wedge \bar{\partial}\rho_\varepsilon \wedge\ T^{n-1}$ is the usual Bedford-Taylor measure on $X\backslash S$ (well defined since T has locally bounded potentials).

Proof. After passing to a resolution of singularities we may assume X is smooth and S is a divisor. It is also Kähler by Kollar's theorem above. Let s be the canonical section of $O(S)$ so that $S = \{s = 0\}$ and h be a smooth metric on $O(S)$ such that $|s|_h \leq 1$. Let θ be a smooth representative of the class $[T]$ and write $T = \theta + \sqrt{-1}\partial\bar{\partial}\psi$ for some $\psi \in \mathrm{PSH}(X,\theta) \cap L^\infty_{\mathrm{loc}}(X\backslash S)$ and let $\omega\ =\ T|_{X\backslash S}$.

Choose ω_X a Kähler metric on X such that $\omega_X > \mathrm{Ric}(h)$ and $\omega_X > -\theta$. Let $F \in C^\infty([0,\infty))$ have the properties $0 \leq F \leq 1$, $F = 1$ on $[0, 1/2]$ and $F = 0$ on $[1,\infty)$.

Let

$$\eta_\varepsilon\ =\ \max(\log |s|_h^2, \log\varepsilon)\ \in\ \mathrm{PSH}(X,\omega_X) \cap C^0(X)$$

so $\log\varepsilon \leq \eta_\varepsilon \leq 0$, and let

$$\rho_\varepsilon\ =\ F\left(\frac{\eta_\varepsilon}{\log\varepsilon}\right)$$

Then $\rho_\varepsilon = 1$ on K if ε is sufficiently small and if we let $N_\delta(S)$ be a closed tubular neighborhood of S with radius $0 < \delta \ll \varepsilon$ then

$$\int_{X\backslash S} \sqrt{-1}\partial\rho_\varepsilon \wedge \bar{\partial}\rho_\varepsilon \wedge \omega^{n-1}\ =\ \frac{1}{(\log\varepsilon)^2}\int_{X\backslash S} (F')^2\sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \wedge \omega^{n-1}$$
$$\leq\ \frac{C}{(\log\varepsilon)^2}\int_{X\backslash S} \sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \wedge \omega^{n-1}\ =\ \frac{C}{(\log\varepsilon)^2}\int_{X\backslash N_\delta(S)} \sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \wedge \omega^{n-1}$$
$$=\ \frac{C}{(\log\varepsilon)^2}\int_{X\backslash N_\delta(S)} (-\eta_\varepsilon)\sqrt{-1}\partial\bar{\partial}\eta_\varepsilon \wedge \omega^{n-1}\ \leq\ \frac{C}{(\log\varepsilon)^2}\int_{X\backslash N_\delta(S)} (-\eta_\varepsilon)(\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge \omega^{n-1}$$
$$\leq\ \frac{C}{|\log\varepsilon|}\int_{X\backslash N_\delta(S)} (\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge \omega^{n-1}\ =\ \frac{C}{|\log\varepsilon|}\int_{X\backslash N_\delta(S)} (\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge (\theta + \sqrt{-1}\partial\bar{\partial}\psi)^{n-1}$$
$$\leq\ \frac{C}{|\log\varepsilon|}\int_{X\backslash N_\delta(S)} (\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge (\theta + \omega_X + \sqrt{-1}\partial\bar{\partial}\psi)^{n-1}$$

$$= \frac{C}{|\log \varepsilon|} \int_{X \setminus N_\delta(S)} (\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge (\theta + \omega_X + \sqrt{-1}\partial\bar{\partial}\psi_k)^{n-1}$$

$$\leq \frac{C}{|\log \varepsilon|} \int_X (\omega_X + \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon) \wedge (\theta + \omega_X + \sqrt{-1}\partial\bar{\partial}\psi_k)^{n-1} = \frac{C}{|\log \varepsilon|} \int_X \omega_X \wedge (\theta + \omega_X)^{n-1}$$

where $k > 0$ is chosen such that $\psi = \psi_k := \max\{\psi, -k\}$ on $X \setminus N_\delta(S)$. Note that we have $\psi_k \in \mathrm{PSH}(X, \theta + \omega_X)$ since $\theta + \omega_X$ is a Kähler metric and $\psi \in \mathrm{PSH}(X, \theta) \subseteq \mathrm{PSH}(X, \theta + \omega_X)$.

If S is a closed pluripolar set, then the same proof works if we replace $\log |s|_h$ by a function $\Psi \in \mathrm{PSH}(X, \omega_X)$ such that $S = \{\Psi = -\infty\}$ and $\sup \Psi = 0$.