Supplementary Material to Paper "Analysis of Markovian Competitive Situations using Nonatomic Games"—an Example involving Dynamic Pricing

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Abstract

Results of the paper "Analysis of Markovian Competitive Situations using Nonatomic Games" [7] work well with a competitive situation involving dynamic pricing. First, Theorem 2 is applied to a transient case where inventories of all firms are gradually depleted. Then, Theorem 3 is applied to a stationary case where production under a given policy helps maintain the whole game in a steady state. In both, equilibria derived in nonatomic (NG) counterparts of the respective games help firms in finite-rival situations to combat uncertainty and competition.

Keywords: Nonatomic Game; Markov Equilibrium; Dynamic Pricing

1 Introduction

When a nonatomic-game (NG) equilibrium is handy, the paper "Analysis of Markovian Competitive Situations using Nonatomic Games" (Yang [7]) shows that it can be used on the original finite Markovian game to serve our intended purpose. Its Theorem 2 states that players can use the observation-blind NG equilibrium in the finite-player situation and gain an average performance that is ever harder to beat as the number of players grows. The "average" on players' states is assessed on the state distribution prevailing in either the NG or the finite game. The paper's Theorem 3 applies to the stationary situation involving time-invariant payoff and transition functions, fixed discountings over time, and an infinite time horizon. It answers affirmatively to the question of whether stationary equilibria (SE) studied in past literature can be useful to large finite games.

The above theory will be most useful when the NG counterpart is relatively easy to deal with in comparison to the corresponding finite games. Besides evidence in literature, this point is further buttressed by a dynamic pricing game we study here. The game participated by heterogeneous firms. Since the random demand arrival process is influenced by prices charged by all firms and leftover items are stored for future sales, the finite-player version of this problem is virtually intractable. The usefulness of the transient result Theorem 2 is thus at full display. To the stationary case also involving production, the stationary result Theorem 3 can further be applied. Moreover, depending on which portion of the outside environment, whether it be merely the other firms' prices or both their prices and inventory levels, are observable, there can be different versions of the finite game. The NG approximation renders these differences irrelevant.

We added to the revenue management literature a new way to tackle the notoriously difficult dynamic pricing game in which firms are capable of reacting in real time to their changing inventory levels and are yet blind to or only partially aware of the other firms' inventory levels. In contrast, many existing results on dynamic price competition were achieved on the condition that all firms' inventory levels are on full display. In addition, no concavity- or monotonicity-related assumption which could reduce the appeal of price-competition results is needed here. We conduct literature survey in Section 2, introduce the dynamic pricing game in Section 3, examine its transient case in Section 4, and go on to its stationary case in Section 5. The paper is concluded in Section 6.

2 Literature Survey

In today's marketplace, companies are often obliged to take the competition element into their revenue management considerations. The permeation of Internet access among virtually all homes and businesses brings price competition to an even more intense, global level. Sites that pool dynamically changing prices from tens or even hundreds of providers and let customers choose the most suitable ones for themselves include Expedia, Hotwire, Orbitz, and Travelocity, just to name a few. The competition is heated in several industries and is most manifest in the hospitality industry and the secondary market of ticket selling.

Yet, research on competitive dynamic pricing has only been sporadic. Perakis and Sood [5] studied such a problem in which each firm reacts most conservatively to the inventory-independent price schedules of its competitors. Xu and Hopp [6] considered a multi-seller dynamic pricing game, where demand arrival is governed by a geometric Brownian motion and only the lowest-priced seller sells. Lin and Sibdari [3] and Levin, McGill, and Nediak [2] resolved equilibrium existence issues for specific dynamic pricing situations; neither revealed structural properties of their equilibria. Martinez-de-Albeniz and Talluri [4] studied a dynamic pricing game in which demand arrives one unit a time, sellers decide prices upon demand arrivals, the lowest-priced seller sells, and all inventory levels are publicly known. They showed the existence of a unique subgame-perfect equilibrium.

Gallego and Hu [1] modeled deterministic transient price competition as differential games and applied heavy-traffic approximation to the corresponding stochastic situations. Regarding precisions of the respective approximations, we note that the just-mentioned work relied on there being sufficiently high supply and demand, while we count on there being enough number of competing firms. The earlier work also assumed that opponents' inventories are visible, a point rendered irrelevant by our NG approach. Yang and Xia [8] provided computational support to the current approach. They tackled a continuous-time dynamic pricing game, and found that an NG equilibrium would fare well in an actual competitive situation with as few as 30 firms.

3 Model Primitives

Here, players are heterogeneous firms engaged in price competition whilst items unsold in a period are put aside for future sales. For strictly positive integers $\bar{s}_T$ and $\bar{s}_I$, let $S_T \equiv \{1, ..., \bar{s}_T\}$ be the set of firm types and $S_I \equiv \{0\} \cup \{1, ..., \bar{s}_I\}$ be the set of potential inventory

levels. The product $S \equiv \bar{S}_T \times \bar{S}_I$ containing $\bar{s} \equiv \bar{s}_T \times (\bar{s}_I + 1)$ elements constitutes the current problem's state space. We let action space be the set $X \equiv \{1, ..., \bar{x}\} \cup \{\infty\}$ of price choices, whereas the price associated with each choice x is denoted by $\bar{p}_x$. Suppose that

$$0 < \bar{p}_1 < \cdots < \bar{p}_{\bar{x}} < \bar{p}_\infty < +\infty. \tag{1}$$

Firms with no stock left and only those firms will charge the demand-stopping price $\bar{p}_\infty$.

For any natural number n, let $\Delta(n)$ be the simplex in the Euclidean space $\Re^n$:

$$\Delta(n) = \{(y_1, ..., y_n) \in [0,1]^n | y_1 + \cdots + y_n \leq 1\}, \tag{2}$$

whose boundary

$$\partial\Delta(n) = \{(y_1, ..., y_n) \in [0,1]^n | y_1 + \cdots + y_n = 1\}. \tag{3}$$

There are homeomorphisms between $\mathcal{P}(\{1, ..., n\})$ equipped with the Prohorov metric and $\Delta(n-1)$ or $\partial\Delta(n)$ equipped with any metric that is equivalent to the Euclidean one; hence, we can equate $\mathcal{P}(S) \equiv \mathcal{P}(S_T \times S_I)$ with $\partial\Delta(\bar{s}) \equiv \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$. A state-variable profile, namely, a joint type-inventory distribution $\sigma \in \mathcal{P}(S_T \times S_I)$ can be represented by a vector $(\sigma_{s_T s_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$. Within the vector, each $\sigma_{s_T s_I}$ represents the fraction of firms that are of type s_T and with s_I items left.

Besides locking the price choice ∞ with inventory level 0, we suppose that a firm's type will not affect demands going to the other firms once it has run out of stock. Thus, the space of in-action environments need only be the $(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x} + 1)$-member set $(S_T \times \{1, ..., \bar{s}_I\} \times \{1, ..., \bar{x}\}) \cup \{(0, \infty)\}$. In it, the element $(0, \infty)$ stands for the situation where the firm is of an arbitrary type, with inventory level 0 and hence price choice ∞. We can equate $\mathcal{P}((S_T \times \{1, ..., \bar{s}_I\} \times \{1, ..., \bar{x}\}) \cup \{(0, \infty)\})$ with $\Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x})$. Every joint type-inventory-price distribution $\mu \in \mathcal{P}((S_T \times \{1, ..., \bar{s}_I\} \times \{1, ..., \bar{x}\}) \cup \{(0, \infty)\})$ can be understood as a vector $(\mu_{s_T s_I x})_{s_T=1,...,\bar{s}_T, s_I=1,...,\bar{s}_I, x=1,...,\bar{x}} \in \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x})$, where each $\mu_{s_T s_I x}$ stands for the fraction of firms with type s_T, inventory level s_I, and price choice x. The number $\mu_{0\infty} \equiv 1 - \sum_{s_T=1}^{\bar{s}_T} \sum_{s_I=1}^{\bar{s}_I} \sum_{x=1}^{\bar{x}} \mu_{s_T s_I x}$ will be the fraction of firms without any stock.

We further suppose that demand coming to a firm, though potentially affected by the other firms' types and price choices, is unperturbed by the other firms' inventory levels. It is thus necessary to deal with the space $(S_T \times \{1, ..., \bar{x}\}) \cup \{\infty\}$. The element ∞ stands for the situation where the firm is of an arbitrary type and with price choice ∞. We can equate $\mathcal{P}((S_T \times \{1, ..., \bar{x}\}) \cup \{\infty\})$ with $\Delta(\bar{s}_T \cdot \bar{x})$. A joint type-price distribution $\xi \in \mathcal{P}((S_T \times \{1, ..., \bar{x}\}) \cup \{\infty\})$ can be represented by a vector $(\xi_{s_T x})_{s_T=1,...,\bar{s}_T, x=1,...,\bar{x}} \in \Delta(\bar{s}_T \cdot \bar{x})$, such that

each $\xi_{s_T x}$ represents the fraction of firms that are of type s_T and charging price $\bar{p}_x$. Now, the value $\xi_\infty \equiv 1 - \sum_{s_T=1}^{\bar{s}_T} \sum_{x=1}^{\bar{x}} \xi_{s_T x}$ will represent the fraction of firms without any stock. The assumptions placed so far on demand arrivals are to make them closer to reality. They do not make the analysis easier. A less restrictive model where demand depends on elements of $\mathcal{P}(S \times X) \equiv \mathcal{P}(S_T \times (\{0\} \cup \{1, ..., \bar{s}_I\}) \times (\{1, ..., \bar{x}\} \cup \{\infty\}))$ can be similarly handled.

For post-action shocks, we let $I = [0, 1]$ and the distribution $\bar{\iota}$ be uniform. For each $s_T = 1, ..., \bar{s}_T$ and $x = 1, ..., \bar{x}$, let $\tilde{\lambda}_{s_T x}(\cdot)$ be a function from $\Delta(\bar{x})$ to $[0, 1]$. In a period, suppose a particular type-s_T firm charges price $x = 1, ..., \bar{x}$, faces an outside type-price distribution $\xi \in \Delta(\bar{s}_T \cdot \bar{x})$, and receives a shock $i \in I$. Then, it will experience

$$\text{a unit demand when } i \leq \tilde{\lambda}_{s_T x}(\xi) \text{ and no demand otherwise.} \tag{4}$$

Therefore, demand is effectively distributed as a Bernoulli random variable with parameter $\tilde{\lambda}_{s_T x}(\xi)$ when the firm is of type s_T and charges price $\bar{p}_x$, while the outside environment induces $\xi \in \Delta(\bar{s}_T \cdot \bar{x})$. When periods are short, this setup is versatile enough to accommodate a variety of situations involving random price-demand relationships and independent demand arrivals. We make the following continuity assumption:

(P1) Every $\tilde{\lambda}_{s_T x}(\cdot)$ is continuous from $\Delta(\bar{s}_T \cdot \bar{x})$ to $[0, 1]$.

This is a mild condition. For instance, for positive constants $\bar{\alpha}_{11}, ..., \bar{\alpha}_{\bar{s}_T \bar{x}}$, $\bar{\beta}_{11}, ..., \bar{\beta}_{\bar{s}_T \bar{x}}, \bar{\beta}_\infty$, and $\bar{\gamma}_{11}, ..., \bar{\gamma}_{\bar{s}_T \bar{x}}, \bar{\gamma}_\infty$ that satisfy, for every $s_T = 1, ..., \bar{s}_T$ and $x = 1, ..., \bar{x}$,

$$\bar{\alpha}_{s_T x} \leq \frac{(\min_{s'_T=1}^{\bar{s}_T} \min_{x'=1}^{\bar{x}} \bar{\beta}_{s'_T x'} \cdot \exp(-\bar{\gamma}_{s'_T x'} \cdot \bar{p}_{x'})) \wedge (\bar{\beta}_\infty \cdot \exp(-\bar{\gamma}_\infty \cdot \bar{p}_\infty))}{\bar{\beta}_{s_T x} \cdot \exp(-\bar{\gamma}_{s_T x} \cdot \bar{p}_x)}, \tag{5}$$

it allows each $\tilde{\lambda}_{s_T x}(\xi)$ to be equal to

$$\frac{\bar{\alpha}_{s_T x} \cdot \bar{\beta}_{s_T x} \cdot \exp(-\bar{\gamma}_{s_T x} \cdot \bar{p}_x)}{\sum_{s'_T=1}^{\bar{s}_T} \sum_{x'=1}^{\bar{x}} \xi_{s'_T x'} \cdot \bar{\beta}_{s'_T x'} \cdot e^{-\bar{\gamma}_{s'_T x'} \cdot \bar{p}_{x'}} + (1 - \sum_{s'_T=1}^{\bar{s}_T} \sum_{x'=1}^{\bar{x}} \xi_{s'_T x'}) \cdot \bar{\beta}_\infty \cdot e^{-\bar{\gamma}_\infty \cdot \bar{p}_\infty}}. \tag{6}$$

But the above is consistent with a multinomial logit price-demand model. Let us also suppose that unit holding cost per period for a type-s_T firm is $\bar{h}_{s_T}$.

4 The Transient Game

Given a joint type-inventory-price distribution $\mu \equiv (\mu_{s_T s_I x})_{s_T=1,...,\bar{s}_T, s_I=1,...,\bar{s}_I, x=1,...,\bar{x}} \in \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x})$, its type-price marginal $\mu|_{S_T \times X} \equiv (\mu|_{S_T \times X, s_T x})_{s_T=1,...,\bar{s}_T, x=1,...,\bar{x}} \in \Delta(\bar{s}_T \cdot \bar{x})$ satisfies

$$\mu|_{S_T \times X, s_T x} = \sum_{s_I=1}^{\bar{s}_I} \mu_{s_T s_I x}, \qquad \forall s_T = 1, ..., \bar{s}_T, x = 1, ..., \bar{x}. \tag{7}$$

This corresponds to a single-period payoff $\tilde{f}_t$ that satisfies

$$\tilde{f}_t(s_T, s_I, x, \mu) = \mathbf{1}(s_I \geq 1) \cdot \bar{p}_x \cdot \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X}) - \bar{h}_{s_T} \cdot s_I. \tag{8}$$

The above (8) reflects that a type-s_T firm can earn $\bar{p}_x$ with a $\tilde{\lambda}_{s_T x}(\mu|_{S_T \times X})$ chance when it charges the price $\bar{p}_x$ at a nonzero inventory level and when the outside environment is μ. The term $\bar{h}_{s_T} \cdot s_I$ reflects the holding cost, which we have assessed on the pre-selling inventory level. For the state-transition map $\tilde{s}_t \equiv (\tilde{s}_{T,t}, \tilde{s}_{I,t})$,

$$\begin{cases} \tilde{s}_{T,t}(s_T, s_I, x, \mu, i) = s_T, \\ \tilde{s}_{I,t}(s_T, s_I, x, \mu, i) = \mathbf{1}(s_I \geq 1) \cdot [s_I - \mathbf{1}(i \leq \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X}))]. \end{cases} \tag{9}$$

The T-portion of (9) confirms that a firm's type is its innate characteristics that is unmovable by time. The I-portion reflects that, if a type-s_T firm starts a period with inventory level $s_I \geq 1$ and outside environment μ, charges price $\bar{p}_x$, and experiences shock i in the period, then it will secure a unit demand as long as $i \leq \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X})$; also, the firm's inventory will remain empty once it reaches 0.

For pre-action shocks, we let $G = [0, 1]$ and $\bar{\gamma}$ be the uniform distribution. This is as though every firm is equipped with a random number generator and use it to assist in decision making. We focus only on action plans $x_t \in \mathcal{M}(S \times G, X)$ that correspond to members of the space $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$. Given any $\chi_t \equiv (\chi_{t s_T s_I})_{s_T=1,...,\bar{s}_T, s_I=1,...,\bar{s}_I} \in (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ where each $\chi_{t s_T s_I} \equiv (\chi_{t s_T s_I x})_{x=1,...,\bar{x}} \in \partial\Delta(\bar{x})$, we associate it with the x_t that satisfies

$$x_t(s_T, s_I, g) = x \text{ when } \sum_{x'=1}^{x-1} \chi_{t s_T s_I x'} \leq g < \sum_{x'=1}^{x} \chi_{t s_T s_I x'} \text{ for } x = 1, ..., \bar{x}, \tag{10}$$

and $x_t(s_T, s_I, 1) = \bar{x}$; also, we let $x_t(s_T, 0, g) = \infty$ which reflects the inventory-0-price-∞ association. We have effectively made each $\chi_{t s_T s_I x}$ the chance that choice x will be made by a type-s_T firm when in period t it is with inventory level $s_I \geq 1$. Now the search for equilibria $x^*_{[1\bar{t}]} \equiv (x^*_t)_{t=1,...,\bar{t}} \in (\mathcal{M}(S \times G), X))^{\bar{t}}$ can be reduced to that for the corresponding equilibria $\chi^*_{[1\bar{t}]} \equiv (\chi^*_t)_{t=1,...,\bar{t}} \in ((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$.

Given nonatomic game $\Gamma(\sigma_1)$, where $\sigma_1 \equiv (\sigma_{1 s_T s_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ is any initial state-variable profile that defines the game, analysis made in Appendix A tells us that we can construct a correspondence $\mathbf{Z}^D_D(\cdot|\sigma_1)$ on the space $((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$. A fixed point $\chi^*_{[1\bar{t}]}$ satisfying $\chi^*_{[1\bar{t}]} \in \mathbf{Z}^D_D(\chi^*_{[1\bar{t}]}|\sigma_1)$ will provide a Markov equilibrium $x^*_{[1\bar{t}]} \in (\mathcal{M}(S \times G, X))^{\bar{t}}$ to the game. Meanwhile, existence of fixed points can be verified.

Proposition 1 *For any $\sigma_1 \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$, the correspondence $\mathbf{Z}^D_D(\cdot|\sigma_1)$ has fixed points.*

With discrete type, inventory, and price spaces S_T, S_I, and X, the $\tilde{s}_t$ defined at (9) automatically satisfies (S1) of Yang [7]. From (9), we know $\tilde{s}_{I,t}(s_T, s_I, x, \mu, i) \neq \tilde{s}_{I,t}(s_T, s_I, x, \mu', i)$ can occur only when $\tilde{\lambda}_{s_T x}(\mu|_{S_T \times X}) \leq i \leq \tilde{\lambda}_{s_T x}(\mu'|_{S_T \times X})$ or $\tilde{\lambda}_{s_T x}(\mu'|_{S_T \times X}) \leq i \leq \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X})$. Due to the finiteness of the spaces, (P1), (7), and the fact that $\bar{\iota}$ is uniform on $I \equiv [0, 1]$, this chance will go to zero at a speed independent of the s_T, s_I, and x present, as μ' converges to μ. So $\tilde{s}_t$ satisfies (S2) there as well. With discrete type, inventory, and price spaces S_T, S_I, and X, the $\tilde{f}_t$ defined at (8) automatically satisfies (F1). By the finiteness of the spaces, (P1), and (7), it is clear that $\tilde{f}_t(s_T, s_I, x, \mu')$ will converge to $\tilde{f}_t(s_T, s_I, x, \mu)$ at a speed independent of the s_T, s_I, and x present, as μ' converges to μ. So $\tilde{f}_t$ satisfies (F2) as well.

The above indicate that the dynamic pricing game is compatible with the setting of Sections 3 to 5 of Yang [7]. For Theorem 2 there to be applicable, we only need that the $x_t^* \in \mathcal{M}(S \times G, X)$ corresponding to each obtained equilibrium component $\chi_t^* \in (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ through (10) is such that the prescribed price choice changes continuously in a $\bar{\gamma}$-measured probabilistic sense as the type or inventory level changes. But with discrete spaces S_T and S_I, this is automatically true.

Thus, when there are many of them, heterogeneous firms engaged in the aforementioned dynamic price competition do not have to fret about their inabilities to observe inventory levels of the other firms. They can all adopt the same pricing policy, basically a map from their types, inventory levels, and individualized pre-action shocks to price choices, as implied by an observation-blind equilibrium of the NG counterpart, and still expect to receive a total payoff that cannot on average be much improved by any one-time unilateral deviation. Note that the NG equilibrium may in turn be obtained from solving a fixed point problem, which in practice, can most likely be realized through an iterative procedure.

5 The Stationary Game

We now add production to the pricing game and study its stationary version. At the cost of $\bar{k}_{s_T}$, we suppose a type-s_T firm's inventory will be brought back to the full $\bar{s}_I$-level in one period's time when it has dropped down to 0. That is, the production lead time is one period and every firm exercises an (s^*, S^*) production policy with $s^* = 0$ and $S^* = \bar{s}_I$. The situation with $s^* \geq 1$ can be similarly handled. On top of the previous setup, we add a per-period discount factor $\bar{\phi} \in [0, 1)$.

For single-period payoff, we let $\tilde{f} : S_T \times S_I \times X \times \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x}) \longrightarrow \Re$ satisfy

$$\tilde{f}(s_T, s_I, x, \mu) = \mathbf{1}(s_I \geq 1) \cdot \bar{p}_x \cdot \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X}) - \bar{h}_{s_T} \cdot s_I - \bar{k}_{s_T} \cdot \mathbf{1}(s_I = 0). \tag{11}$$

The first two terms remain the same as those in (8). The extra term $\bar{k}_{s_T} \cdot \mathbf{1}(s_I = 0)$ reflects the total production cost needed to bring a type-s_T firm's inventory level back to $\bar{s}_I$ from 0. For state transition, we let $\tilde{s} \equiv (s_T, s_I) : S_T \times S_I \times X \times \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x}) \times I \longrightarrow S_T \times S_I$ satisfy

$$\begin{cases} \tilde{s}_T(s_T, s_I, x, \mu, i) = s_T, \\ \tilde{s}_I(s_T, s_I, x, \mu, i) = \mathbf{1}(s_I \geq 1) \cdot [s_I - \mathbf{1}(i \leq \tilde{\lambda}_{s_T x}(\mu|_{S_T \times X}))] + \mathbf{1}(s_I = 0) \cdot \bar{s}_I. \end{cases} \tag{12}$$

The T-portion of (12) is the same as that of (9). In the current I-portion, the first term has its counterpart in (9). The new term $\mathbf{1}(s_I = 0) \cdot \bar{s}_I$ reflects the locked-in production policy of bringing the inventory level back to full in one period's time.

For the stationary nonatomic version Γ of the dynamic pricing game, analysis made in Appendix B tells us that we can construct a correspondence $\mathbf{Z}_{DEF}^{DEF}$ on the space $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x}) \times \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$. A fixed point $(\chi^*, \xi^*, \sigma^*)$ satisfying $(\chi^*, \xi^*, \sigma^*) \in \mathbf{Z}_{DEF}^{DEF}(\chi^*, \xi^*, \sigma^*)$ will provide a Markov equilibrium $x^* \in \mathcal{M}(S \times G, X)$, all the while with σ^* serving as a consistent state-variable profile. Again, existence of fixed points can be verified.

Proposition 2 *The correspondence $\mathbf{Z}_{DEF}^{DEF}$ has fixed points.*

Using similar arguments as those employed after Proposition 1 of Section 4, we can verify that (S1-s), (S2-s), (F1-s), and (F2-s) as needed by Yang [7] are all true. Also, the $x^* \in \mathcal{M}(S \times G, X)$ corresponding to each obtained equilibrium component $\chi^* \in (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ through (10) is a member of $\mathcal{K}(S, G, \bar{\gamma}, X)$. With Theorem 3 of Yang [7] thus applicable, we can state the following. When production is allowed, we can learn from Proposition 2 about the existence of an NG equilibrium that prescribes for firms the dynamic pricing policy x^*. The latter would help the firms fare well in the long run under the environment σ^* which is consistent with all these firms adopting the policy x^*. When their number is finite but large enough, firms can still use the same pricing policy x^* to fare relatively well in the long run in an average sense that is defined through the joint type-inventory distribution σ^*.

6 Concluding Remarks

Yang [7] has established links between multi-period Markovian games and their NG counterparts. Here, the paper's results have been successfully brought to bear on dynamic pricing

situations with their own practical significances.

Appendices

A Technical Developments in Section 4

Use $\chi_{[1\bar{t}]} \equiv (\chi_t)_{t=1,...,\bar{t}} \in ((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ to denote a policy. For $t = 1, ..., \bar{t}$, suppose state-variable profile $\sigma_t \equiv (\sigma_{ts_Ts_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ has been determined. By (7) and (8) of Yang [7], the in-action environment $\mu_t \equiv (\mu_{ts_Ts_Ix})_{s_T=1,...,\bar{s}_T, s_I=1,...,s_I, x=1,...,\bar{x}} \in \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x})$, to be faced by all firms, will satisfy

$$\mu_{ts_Ts_Ix} = \sigma_{ts_Ts_I} \cdot \chi_{ts_Ts_Ix} \qquad \forall s_T = 1, ..., \bar{s}_T, s_I = 1, ..., \bar{s}_I, x = 1, ..., \bar{x}. \tag{A.1}$$

Combining this with (7), we see that the type-action marginal $\xi_t \equiv (\xi_{ts_Tx})_{s_T=1,...,\bar{s}_T, x=1,...,\bar{x}} = \mu_t|_{S_T \times X} \in \Delta(\bar{s}_T \cdot \bar{x})$ satisfies

$$\xi_{ts_Tx} = \sum_{s_I=1}^{\bar{s}_I} \mu_{ts_Ts_Ix} = \sum_{s_I=1}^{\bar{s}_I} \sigma_{ts_Ts_I} \cdot \chi_{ts_Ts_Ix} \qquad \forall s_T = 1, ..., \bar{s}_T, x = 1, ..., \bar{x}. \tag{A.2}$$

Let x_t be the element in $\mathcal{M}(S \times G, X)$ corresponding to the current $\chi_t \in (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ in the fashion of (10). From (10) and (11) of Yang [7] and (9), we see that $\sigma_{t+1} \equiv (\sigma_{t+1,s_Ts_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} = T_t(x_t) \circ \sigma_t \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ should satisfy

$$\begin{aligned} \sigma_{t+1,s_Ts_I} = \mathbf{1}(s_I = 0) \cdot \sigma_{ts_T0} + \mathbf{1}(s_I \geq 1) \cdot \sigma_{ts_Ts_I} \cdot \textstyle\sum_{x=1}^{\bar{x}} \chi_{ts_Ts_Ix} \cdot (1 - \tilde{\lambda}_{s_Tx}(\xi_t)) \\ + \mathbf{1}(s_I \leq \bar{s}_I - 1) \cdot \sigma_{ts_T,s_I+1} \cdot \textstyle\sum_{x=1}^{\bar{x}} \chi_{ts_T,s_I+1,x} \cdot \tilde{\lambda}_{s_Tx}(\xi_t), \end{aligned} \tag{A.3}$$

for $s_T = 1, ..., \bar{s}_T$ and $s_I = 0, 1, ..., \bar{s}_I$. The above (A.3) shows that a firm's type will not change. It also reflects that, a firm will start period $t+1$ with s_I items when, either it starts period t with s_I items and whatever price choice x it makes does not bring in any demand or $s_I \leq \bar{s}_I - 1$, the firm starts period t with $s_I + 1$ items, and whatever price choice x it makes does bring in one unit of demand.

At a particular $t = 1, ..., \bar{t}$, we can understand (A.2) as $\xi_t \equiv (\xi_{ts_Tx})_{s_T=1,...,\bar{s}_T, x=1,...,\bar{x}} = Z^E_{FD,t}(\sigma_t, \chi_t)$ where $Z^E_{FD,t}$ is a mapping from $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1)) \times (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ to $\Delta(\bar{s}_T \cdot \bar{x})$, and understand (A.3) as $\sigma_{t+1} = Z^F_{FDE,t}(\sigma_t, \chi_t, \xi_t)$ where $Z^F_{FDE,t}$ is a mapping from $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1)) \times (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x})$ to $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$. Under the fixed game-defining $\sigma_1 \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$, let us now use $\xi_{[1\bar{t}]} \equiv (\xi_t)_{t=1,...,\bar{t}} = Z^E_D(\chi_{[1\bar{t}]} \mid \sigma_1)$, where $Z^E_D(\cdot|\sigma_1)$ is a

mapping from $((\partial\Delta(\bar{x}))^{\bar{s}_T\cdot\bar{s}_I})^{\bar{t}}$ to $(\Delta(\bar{s}_T\cdot\bar{x}))^{\bar{t}}$, to denote the attainment of the ξ_t's through the iterations $\xi_t = Z^E_{FD,t}(\sigma_t,\chi_t)$ and $\sigma_{t+1} = Z^F_{FDE,t}(\sigma_t,\chi_t,\xi_t)$ over $t = 1,...,\bar{t}$.

Suppose a type-price-distribution sequence $\xi_{[1\bar{t}]} \equiv (\xi_t)_{t=1,...,\bar{t}} \in (\Delta(\bar{s}_T\cdot\bar{x}))^{\bar{t}}$ is given. Let $v^*_{ts_Ts_I}$ be the optimal total expected payoff a type-s_T firm can make from period t to $\bar{t}$ when it starts with inventory level $s_I = 0,1,...,\bar{s}_I$. As a terminal condition, we have

$$v^*_{\bar{t}+1,s_Ts_I} = 0, \qquad \forall s_T = 0,1,...,\bar{s}_T, s_I = 1,...,\bar{s}_I. \tag{A.4}$$

For $t = \bar{t},\bar{t}-1,...,1$ and $s_T = 1,...,\bar{s}_T$, we have, due to (8) and (9), $v^*_{ts_T0} = 0$; also, for $s_I = 1,...,\bar{s}_I$, we have $v^*_{ts_Ts_I}$ equal to

$$-\bar{h}_{s_T}\cdot s_I + \sup_{\chi\in\partial\Delta(\bar{x})}\left\{\sum_{x=1}^{\bar{x}}\chi_x\cdot[\tilde{\lambda}_{s_Tx}(\xi_t)\cdot(\bar{p}_x + v^*_{t+1,s_T,s_I-1}) + (1-\tilde{\lambda}_{s_Tx}(\xi_t))\cdot v^*_{t+1,s_Ts_I}]\right\}. \tag{A.5}$$

The above (A.5) allows any type-s_T firm to choose the best probabilities χ_x to use on price choices x; when a particular x has been settled on, there will be a $\tilde{\lambda}_{s_Tx}(\xi_t)$ chance for the firm to earn $\bar{p}_x$ in revenue and enter period $t+1$ with one less item in stock, and also a $1-\tilde{\lambda}_{s_Tx}(\xi_t)$ chance for it to earn nothing and enter period $t+1$ with the same stock level as it had to start period t with. Here, the impact of the current individual firm's decision on the overall environment's future evolution is ignored—ξ_t is given; yet, the future evolution of the current firm's own inventory level is a concern in its decision making.

For $t = \bar{t}+1,\bar{t},...,1$, define $v^*_t \equiv (v^*_{ts_Ts_I})_{s_T=1,...,\bar{s}_T,s_I=1,...,\bar{s}_I} \in \Re^{\bar{s}_T\cdot\bar{s}_I}$. At a particular $t = \bar{t},\bar{t}-1,...,1$, with the understanding that $v^*_{t+1,s_T0} = 0$ for any $s_T = 1,...,\bar{s}_T$, we can write (A.5) for any particular $s_T = 1,...,\bar{s}_T$ and $s_I = 1,...,\bar{s}_I$ as $v^*_{ts_Ts_I} = Z^V_{EV,ts_Ts_I}(\xi_t, v^*_{t+1})$, where Z^V_{EV,ts_Ts_I} is a mapping from $\Delta(\bar{s}_T\cdot\bar{x})\times\Re^{\bar{s}_T\cdot\bar{s}_I}$ to $\Re$. At the same time, we can use $\chi_{ts_Ts_I} \in \mathbf{Z}^D_{EV,ts_Ts_I}(\xi_t, v^*_{t+1})$ to denote $\chi_{ts_Ts_I}$'s optimality to (A.5), where $\mathbf{Z}^D_{EV,ts_Ts_I}$ is a correspondence from $\Delta(\bar{s}_T\cdot\bar{x})\times\Re^{\bar{s}_T\cdot\bar{s}_I}$ to $(\partial\Delta(\bar{x}))^{\bar{s}_T\cdot\bar{s}_I}$. Combining these over all the $s_T = 1,...,\bar{s}_T$ and $s_I = 1,...,\bar{s}_I$, we can write $v^*_t = Z^V_{EV,t}(\xi_t, v^*_{t+1})$ where $Z^V_{EV,t} = \prod_{s_T=1}^{\bar{s}_T}\prod_{s_I=1}^{\bar{s}_I} Z^V_{EV,ts_Ts_I}$ and $\chi_t \equiv (\chi_{ts_Ts_I})_{s_T=1,...,\bar{s}_T,s_I=1,...,\bar{s}_I} \in \mathbf{Z}^D_{EV,t}(\xi_t, v^*_{t+1})$ where $\mathbf{Z}^D_{EV,t} = \prod_{s_T=1}^{\bar{s}_T}\prod_{s_I=1}^{\bar{s}_I}\mathbf{Z}^D_{EV,ts_Ts_I}$. Suppose we start $v^*_{\bar{t}+1}$ with the all-zero vector which satisfies (A.4), and for $t = \bar{t},\bar{t}-1,...,1$ obtain v^*_t by applying $v^*_t = Z^V_{EV,t}(\xi_t, v^*_{t+1})$. Then, we can define a correspondence $\mathbf{Z}^D_E$ from $(\Delta(\bar{s}_T\cdot\bar{x}))^{\bar{t}}$ to $((\partial\Delta(\bar{x}))^{\bar{s}_T\cdot\bar{s}_I})^{\bar{t}}$ so that $\chi_{[1\bar{t}]} \equiv (\chi_t)_{t=1,...,\bar{t}} \in \mathbf{Z}^D_E(\xi_{[1\bar{t}]})$ if and only if $\chi_t \in \mathbf{Z}^D_{EV,t}(\xi_t, v^*_{t+1})$ for every $t = \bar{t},\bar{t}-1,...,1$.

Equate correspondence $\mathbf{Z}^D_D(\cdot|\sigma_1)$ on $((\partial\Delta(\bar{x}))^{\bar{s}_T\cdot\bar{s}_I})^{\bar{t}}$ with $\mathbf{Z}^D_E\circ Z^E_D(\cdot|\sigma_1)$, so that

$$\chi'_{[1\bar{t}]} \in \mathbf{Z}^D_D(\chi_{[1\bar{t}]}|\sigma_1) \qquad \text{if and only if} \qquad \chi'_{[1\bar{t}]} \in \mathbf{Z}^D_E\left(Z^E_D(\chi_{[1\bar{t}]}|\sigma_1)\right). \tag{A.6}$$

Due to definitions and arguments made from (20) to (23) of Yang [7] and the correspondence established at (10), we know that a fixed point $\chi^*_{[1\bar{t}]}$ satisfying $\chi^*_{[1\bar{t}]} \in \mathbf{Z}^D_D(\chi^*_{[1\bar{t}]}|\sigma_1)$ will provide a Markov equilibrium $x^*_{[1\bar{t}]} \in (\mathcal{M}(S \times G, X))^{\bar{t}}$ to the nonatomic game $\Gamma(\sigma_1)$.

Proof of Proposition 1: Let us use Kakutani's Theorem to prove existence of the fixed points. Since $((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ is a non-empty, compact, and convex subset of the Euclidean space $\Re^{\bar{s}_T \cdot \bar{s}_I \cdot \bar{x} \cdot \bar{t}}$, it will suffice if $\mathbf{Z}^D_D(\chi_{[1\bar{t}]}|\sigma_1)$ is non-empty and convex at every $\chi_{[1\bar{t}]} \in ((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ and as a correspondence, $\mathbf{Z}^D_D(\cdot|\sigma_1)$ is closed.

From (A.2), we see that each $Z^E_{FD,t}$ is a continuous mapping from $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1)) \times (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ to $\Delta(\bar{s}_T \cdot \bar{x})$. From (P1) and (A.3), we see that each $Z^F_{FDE,t}$ is a continuous mapping from $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1)) \times (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x})$ to $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$. When used iteratively, these will lead to the continuity of $Z^E_D(\cdot|\sigma_1)$ as a mapping from $((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ to $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$. Indeed, the dependence on $\sigma_1 \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ is also continuous, although this property is not used here.

For any $t = \bar{t} + 1, \bar{t}, ..., 1$, $s_T = 1, ..., \bar{s}_T$, and $s_I = 0, 1, ..., \bar{s}_I$, define Z^V_{E,ts_Ts_I} from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $\Re$ as follows. First, let $Z^V_{E,ts_Ts_I}(\xi_{[1\bar{t}]}) = 0$ whenever $t = \bar{t} + 1$ or $s_I = 0$. Then, for $t = \bar{t}, \bar{t} - 1, ..., 1$, let

$$Z^V_{E,ts_Ts_I}(\xi_{[1\bar{t}]}) = -\bar{h}_{s_T} \cdot s_I + \sup_{\chi \in \partial\Delta(\bar{x})} \left\{ \sum_{x=1}^{\bar{x}} \chi_x \cdot W_{ts_Ts_Ix}(\xi_{[1\bar{t}]}) \right\}, \tag{A.7}$$

for any $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$, where each

$$W_{ts_Ts_Ix}(\xi_{[1\bar{t}]}) = \tilde{\lambda}_{s_Tx}(\xi_t) \cdot (\bar{p}_x + Z^V_{E,t+1,s_T,s_I-1}(\xi_{[1\bar{t}]})) + (1 - \tilde{\lambda}_{s_Tx}(\xi_t)) \cdot Z^V_{E,t+1,s_Ts_I}(\xi_{[1\bar{t}]}). \tag{A.8}$$

By their definitions, we know that each $Z^V_{E,\bar{t}+1,s_Ts_I}$ is a continuous mapping from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $\Re$; so is each Z^V_{E,ts_T0}. Suppose for some $t = \bar{t}, \bar{t}-1, ..., 1$, each $Z^V_{E,t+1,s_Ts_I}$ for $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$ is a continuous mapping from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $\Re$. Then by (P1) and (A.8), we know that $W_{ts_Ts_Ix}$ for every $s_T = 1, ..., \bar{s}_T$, $s_I = 1, ..., \bar{s}_I$, and $x = 1, ..., \bar{x}$ is a continuous mapping from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $\Re$. Now in the optimization problem of (A.7), the objective is continuous in χ and $\xi_{[1\bar{t}]}$, while the feasible region is compact as a set of the χ's and even independent of $\xi_{[1\bar{t}]}$. So by Berge's Theorem, we can obtain the continuity of Z^V_{E,ts_Ts_I} as a mapping from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $\Re$ at every $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$. An induction process has been completed.

Comparing (A.5) with (A.7) and (A.8), we see that the $v^*_{ts_Ts_I}$'s defined for a given $\xi_{[1\bar{t}]}$ are exactly the same as the $Z^V_{E,ts_Ts_I}(\xi_{[1\bar{t}]})$'s. This way, $\chi_{[1\bar{t}]} \equiv ((\chi_{ts_Ts_I})_{s_T=1,...,\bar{s}_T, s_I=1,...,\bar{s}_I})_{t=1,...,\bar{t}} \in$

$\mathbf{Z}^D_E(\xi_{[1\bar{t}]})$ if and only if $\chi_{ts_Ts_I} \in \mathbf{Z}^D_{E,ts_Ts_I}(\xi_{[1\bar{t}]})$ for every $t = \bar{t}, \bar{t}-1, ..., 1$, $s_T = 1, ..., \bar{s}_T$, and $s_I = 1, ..., \bar{s}_I$, where

$$\mathbf{Z}^D_{E,ts_Ts_I}(\xi_{[1\bar{t}]}) = \text{argmax}_{\chi \in \partial\Delta(\bar{x})} \left\{ \sum_{x=1}^{\bar{x}} \chi_x \cdot W_{ts_Ts_Ix}(\xi_{[1\bar{t}]}) \right\}. \tag{A.9}$$

Since $W_{ts_Ts_Ix}$ and hence the objective of the optimization problem in (A.9) is continuous in $\xi_{[1\bar{t}]}$, the latter being also jointly continuous in χ, and the feasible region is compact as a set of the χ's and also independent of $\xi_{[1\bar{t}]}$, we know from Berge's Theorem that $\mathbf{Z}^D_{E,ts_Ts_I}(\xi_{[1\bar{t}]})$ is non-empty at every $\xi_{[1\bar{t}]}$ and that $\mathbf{Z}^D_{E,ts_Ts_I}$ as a correspondence is closed. Since the objective in the optimization problem of (A.9) is affine in $\chi \equiv (\chi_x)_{x=1,...,\bar{x}}$ and the feasible region is convex, we also know that every $\mathbf{Z}^D_{E,ts_Ts_I}(\xi_{[1\bar{t}]})$ is convex.

In combination, we have that $\mathbf{Z}^D_E(\xi_{[1\bar{t}]})$ is non-empty and convex at each $\xi_{[1\bar{t}]}$, and that $\mathbf{Z}^D_E$ as a correspondence from $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$ to $((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ is closed. Combine this further with the continuity of $Z^E_D(\cdot|\sigma_1)$ as a mapping from $((\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I})^{\bar{t}}$ to $(\Delta(\bar{s}_T \cdot \bar{x}))^{\bar{t}}$, and we can assert that $\mathbf{Z}^D_D(\cdot|\sigma_1) = \mathbf{Z}^D_E \circ Z^E_D(\cdot|\sigma_1)$ as defined in (A.6) has the desired properties. ∎

B Technical Developments in Section 5

Let $\chi \in (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ denote an action plan. When given a state-variable profile $\sigma \equiv (\sigma_{s_Ts_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ for the period, the in-action environment $\mu \in \Delta(\bar{s}_T \cdot \bar{s}_I \cdot \bar{x})$ would still be provided by (A.1), whose marginal $\xi = \mu|_{S_T \times X}$ is again defined by (A.2). Let x be the element in $\mathcal{M}(S \times G, X)$ that corresponds to the current χ through (10). Due to (37) and (38) of Yang [7] and (12), $\sigma' \equiv (\sigma'_{s_Ts_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} = T(x) \circ \sigma \in \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ should satisfy

$$\begin{aligned} \sigma'_{s_Ts_I} = \mathbf{1}(s_I = \bar{s}_I) \cdot \sigma_{s_T0} + \mathbf{1}(s_I \geq 1) \cdot \sigma_{s_Ts_I} \cdot \textstyle\sum_{x=1}^{\bar{x}} \chi_{s_Ts_Ix} \cdot (1 - \tilde{\lambda}_{s_Tx}(\xi)) \\ + \mathbf{1}(s_I \leq \bar{s}_I - 1) \cdot \sigma_{s_T,s_I+1} \cdot \textstyle\sum_{x=1}^{\bar{x}} \chi_{s_T,s_I+1,x} \cdot \tilde{\lambda}_{s_Tx}(\xi), \end{aligned} \tag{B.1}$$

for $s_T = 1, ..., \bar{s}_T$ and $s_I = 0, 1, ..., \bar{s}_I$. Besides the same last two terms in (A.3), the above (B.1) has a different first term reflecting the scheduled production that always restores the inventory level back to $\bar{s}_I$ from 0. Due to its stationarity, we can understand (A.2) as $\xi = Z^E_{FD}(\sigma, \chi)$. Also, denote (B.1) by $\sigma' = Z^F_{FDE}(\sigma, \chi, \xi)$ where Z^F_{FDE} is a mapping from $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1)) \times (\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x})$ to $\partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$.

Given joint type-price distribution $\xi \in \Delta(\bar{s}_T \cdot \bar{x})$ and vector $v \equiv (v_{s_T s_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \equiv \Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$, we are prompted by (11) and (12) to define

$$v'_{s_T 0} = -\bar{k}_{s_T} + \bar{\phi} \cdot v_{s_T \bar{s}_I}, \qquad \forall s_T = 1, ..., \bar{s}_T, \tag{B.2}$$

and for $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$, define $v'_{s_T s_I}$ as

$$-\bar{h}_{s_T} \cdot s_I + \sup_{\chi \in \partial\Delta(\bar{x})} \left\{ \sum_{x=1}^{\bar{x}} \chi_x \cdot [\tilde{\lambda}_{s_T x}(\xi) \cdot (\bar{p}_x + \bar{\phi} \cdot v_{s_T, s_I - 1}) + (1 - \tilde{\lambda}_{s_T x}(\xi)) \cdot \bar{\phi} \cdot v_{s_T s_I}] \right\}. \tag{B.3}$$

(B.2) reflects that, at the expense of $\bar{k}_{s_T}$, a type-s_T firm's inventory level will be automatically restored to $\bar{s}_I$ once it has reached 0. Meanwhile, (B.3) is almost the same as (A.5), except that the current version takes into account the discount factor $\bar{\phi}$. We can view (B.2) and (B.3) as $v' = Z^V_{EV}(\xi, v)$, where Z^V_{EV} is a mapping from $\Delta(\bar{s}_T \cdot \bar{x}) \times \Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$ to $\Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$.

Lemma 1 *At each fixed $\xi \in \Delta(\bar{s}_T \cdot \bar{x})$, the operator $Z^V_{EV}(\xi, \cdot)$ has a unique fixed point $Z^V_E(\xi)$. Also, the fixed-point operator Z^V_E is continuous on $\Delta(\bar{s}_T \cdot \bar{x})$.*

Proof: For $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$, define mapping $Z^V_{DEV, s_T s_I}$ from $\partial\Delta(\bar{x}) \times \Delta(\bar{s}_T \cdot \bar{x}) \times \Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$ to $\Re$, so that

$$Z^V_{DEV, s_T s_I}(\chi, \xi, v) = \sum_{x=1}^{\bar{x}} \chi_x \cdot \left[\tilde{\lambda}_{s_T x}(\xi) \cdot (\bar{p}_x + \bar{\phi} \cdot v_{s_T, s_I - 1}) + (1 - \tilde{\lambda}_{s_T x}(\xi)) \cdot \bar{\phi} \cdot v_{s_T s_I} \right]. \tag{B.4}$$

For any $u \equiv (u_{s_T s_I})_{s_T=1,...,\bar{s}_T, s_I=0,1,...,\bar{s}_I} \in \Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$, let $||u|| = \max_{s_T=1}^{\bar{s}_T} \max_{s_I=0}^{\bar{s}_I} |u_{s_T s_I}|$. We can check that

$$|Z^V_{DEV, s_T s_I}(\chi, \xi, v) - Z^V_{DEV, s_T s_I}(\chi, \xi, v')| \leq \bar{\phi} \cdot ||v - v'||. \tag{B.5}$$

For $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$, define mapping $Z^V_{EV, s_T s_I}$ from $\Delta(\bar{s}_T \cdot \bar{x}) \times \Re^{\bar{s}_T \cdot (\bar{s}_I+1)}$ to $\Re$, so that

$$Z^V_{EV, s_T s_I}(\xi, v) = \sup_{\chi \in \partial\Delta(\bar{x})} Z^V_{DEV, s_T s_I}(\chi, \xi, v). \tag{B.6}$$

Suppose, without loss of generality, that $Z^V_{EV, s_T s_I}(\xi, v) < Z^V_{EV, s_T s_I}(\xi, v')$. Let $\chi' \in \partial\Delta(\bar{x})$ be such that $Z^V_{EV, s_T s_I}(\xi, v') = Z^V_{DEV, s_T s_I}(\chi', \xi, v')$. Then by (B.5) and (B.6),

$$Z^V_{EV, s_T s_I}(\xi, v) \geq Z^V_{DEV, s_T s_I}(\chi', \xi, v) \geq Z^V_{DEV, s_T s_I}(\chi', \xi, v') - \bar{\phi} \cdot ||v - v'||, \tag{B.7}$$

which is equal to $Z^V_{EV, s_T s_I}(\xi, v') - \bar{\phi} \cdot ||v - v'||$. So by symmetry,

$$\left| Z^V_{EV, s_T s_I}(\xi, v) - Z^V_{EV, s_T s_I}(\xi, v') \right| \leq \bar{\phi} \cdot ||v - v'||. \tag{B.8}$$

Combining (B.2) to (B.8), we see that

$$||Z^V_{EV}(\xi, v) - Z^V_{EV}(\xi, v')|| \leq \bar{\phi} \cdot ||v - v'||. \tag{B.9}$$

In other words, Z^V_{EV} is a contraction mapping on the Banach space $\Re^{\bar{s}_T \cdot (\bar{s}_I + 1)}$ equipped with the $||\cdot||$-norm. Thus, it has a unique fixed point $Z^V_E(\xi)$ satisfying $Z^V_E(\xi) = Z^V_{EV}(\xi, Z^V_E(\xi))$. This point also happens to be $\lim_{n \to +\infty}(Z^V_{EV}(\xi, \cdot))^n(v_0)$ for any $v_0 \in \Re^{\bar{s}_T \cdot (\bar{s}_I + 1)}$.

At the same time, due to (P1) and (B.4), each $Z^V_{DEV, s_T s_I}$ is continuous in ξ. Let $||\cdot||$ also stand for the supremum norm of $\Re^{\bar{s}_T \cdot \bar{x}}$. Using arguments similar to those employed from (B.5) to (B.9), we can establish that, for any fixed $\xi \in \Delta(\bar{s}_T \cdot \bar{x})$ and any $\epsilon > 0$, there is $\delta > 0$, so that for any $\xi' \in \Delta(\bar{s}_T \cdot \bar{x})$ satisfying $||\xi - \xi'|| < \delta$,

$$||Z^V_{EV}(\xi, Z^V_E(\xi)) - Z^V_{EV}(\xi', Z^V_E(\xi))|| < \epsilon. \tag{B.10}$$

Consequently,

$$\begin{aligned} ||Z^V_E(\xi) - Z^V_E(\xi')|| &= ||Z^V_E(\xi) - \lim_{n \to +\infty}(Z^V_{EV}(\xi', \cdot))^n(Z^V_E(\xi))|| \\ &\leq \textstyle\sum_{n=1}^{+\infty} ||(Z^V_{;EV}(\xi', \cdot))^{n-1}(Z^V_E(\xi)) - (Z^V_{EV}(\xi', \cdot))^n(Z^V_E(\xi))|| \\ &= \textstyle\sum_{n=1}^{+\infty} ||(Z^V_{EV}(\xi', \cdot))^{n-1}(Z^V_{EV}(\xi, Z^V_E(\xi))) - (Z^V_{EV}(\xi', \cdot))^{n-1}(Z^V_{EV}(\xi', Z^V_E(\xi)))|| \\ &\leq \textstyle\sum_{n=1}^{+\infty} \bar{\phi}^{n-1} \cdot \epsilon = \epsilon/(1 - \bar{\phi}), \end{aligned} \tag{B.11}$$

where the first equality utilizes the fact that $Z^V_E(\xi') = \lim_{n \to +\infty}(Z^V_{EV}(\xi', \cdot))^n(Z^V_E(\xi))$, the first inequality is obtainable from a routine expansion, the second equality exploits the fact that $Z^V_E(\xi) = Z^V_{EV}(\xi, Z^V_E(\xi))$, the second inequality follows from combined uses of (B.9) and (B.10), and the last equality is an identity. ∎

Following the mapping Z^V_E, let us define correspondence $\mathbf{Z}^D_E$ from $\Delta(\bar{s}_T \cdot \bar{x})$ to $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$, so that $\chi \equiv (\chi_{s_T s_I})_{s_T = 1, \ldots, \bar{s}_T, s_I = 1, \ldots, \bar{s}_I} \in \mathbf{Z}^D_E(\xi)$ if and only if each $\chi_{s_T s_I}$ maximizes

$$\sum_{x=1}^{\bar{x}} \chi_x \cdot \left[\tilde{\lambda}_{s_T x}(\xi) \cdot (\bar{p}_x + \bar{\phi} \cdot Z^V_{E, s_T, s_I - 1}(\xi)) + (1 - \tilde{\lambda}_{s_T x}(\xi)) \cdot \bar{\phi} \cdot Z^V_{E, s_T s_I}(\xi)\right]. \tag{B.12}$$

Define correspondence $\mathbf{Z}^{DEF}_{DEF}$ on $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x}) \times \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$, so that

$$\begin{aligned} (\chi', \xi', \sigma') \in \mathbf{Z}^{DEF}_{DEF}(\chi, \xi, \sigma) \qquad & \text{if and only if} \\ \chi' \in \mathbf{Z}^D_E(\xi'), \qquad \xi' = Z^E_{FD}(\sigma', \chi), \qquad & \sigma' = Z^F_{FDE}(\sigma, \chi, \xi). \end{aligned} \tag{B.13}$$

Suppose $(\chi^*, \xi^*, \sigma^*)$ is a fixed point for $\mathbf{Z}^{DEF}_{DEF}$, so that $(\chi^*, \xi^*, \sigma^*) \in \mathbf{Z}^{DEF}_{DEF}(\chi^*, \xi^*, \sigma^*)$. The fact that $\sigma^* = Z^F_{FDE}(\sigma^*, \chi^*, \xi^*) = Z^F_{FDE}(\sigma^*, \chi^*, Z^E_{FD}(\sigma^*, \chi^*))$ indicates that σ^* is consistent

with x^*, an element of $\mathcal{M}(S \times G, X)$ that corresponds to χ^* through (10). Meanwhile, $\chi^* \in \mathbf{Z}_E^D(\xi^*)$ implies the validity of Yang's [7] (29). Therefore, $(\chi^*, \xi^*, \sigma^*)$ will provide a Markov equilibrium $x^* \in \mathcal{M}(S \times G, X)$ to the stationary nonatomic game Γ.

Proof of Proposition 2: We still use Kakutani's Theorem to prove existence of the fixed points. Since $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x}) \times \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ is a non-empty, compact, and convex subset of the Euclidean space $\Re^{\bar{s}_T + \bar{s}_T \cdot \bar{s}_I + \bar{s}_T \cdot \bar{x} + \bar{s}_T \cdot \bar{s}_I \cdot \bar{x}}$, it will suffice if $\mathbf{Z}_{DEF}^{DEF}(\chi, \xi, \sigma)$ is non-empty and convex at every (χ, ξ, σ) and as a correspondence, $\mathbf{Z}_{DEF}^{DEF}$ is closed.

From the proof of Proposition 1, we know that Z_{FD}^E and Z_{FDE}^F are continuous mappings. For $s_T = 1, ..., \bar{s}_T$, $s_I = 1, ..., \bar{s}_I$, and $x = 1, ..., \bar{x}$, let

$$W_{s_T s_I x}(\xi) = \tilde{\lambda}_{s_T x}(\xi) \cdot (\bar{p}_x + Z_{E,s_T,s_I-1}^V(\xi)) + (1 - \tilde{\lambda}_{s_T x}(\xi)) \cdot Z_{E,s_T s_I}^V(\xi). \tag{B.14}$$

The definition (B.12) dictates that, for any $s_T = 1, ..., \bar{s}_T$ and $s_I = 1, ..., \bar{s}_I$,

$$\mathbf{Z}_{E,s_T s_I}^D(\xi) = \text{argmax}_{\chi \equiv (\chi_x)_{x=1,..,\bar{x}} \in \partial\Delta(\bar{x})} \left\{ \sum_{x=1}^{\bar{x}} \chi_x \cdot W_{s_T s_I x}(\xi) \right\}. \tag{B.15}$$

By Lemma 1 and (B.14), each $W_{s_T s_I x}$ and hence the objective of the optimization problem in (B.15) is continuous in ξ, the latter being also jointly continuous in χ. Meanwhile, the feasible region of the same problem is compact as a set of the χ's and also independent of ξ. So by Berge's Theorem, $\mathbf{Z}_{E,s_T s_I}^D(\xi)$ is non-empty at every ξ and that $\mathbf{Z}_{E,s_T s_I}^D$ as a correspondence is closed. Since the objective in the optimization problem of (B.15) is affine in $\chi \equiv (\chi_x)_{x=1,...,\bar{x}}$ and the feasible region is convex, every $\mathbf{Z}_{E,s_T s_I}^D(\xi)$ is also convex.

In combination, we have that $\mathbf{Z}_E^D(\xi)$ is non-empty and convex at each ξ, and that $\mathbf{Z}_E^D$ as a correspondence from $\Delta(\bar{s}_T \cdot \bar{x})$ to $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I}$ is closed. Combine this further with the continuities of Z_{FD}^E and Z_{FDE}^F, and we can assert that $\mathbf{Z}_{DEF}^{DEF}$ as defined in (B.13) from $(\partial\Delta(\bar{x}))^{\bar{s}_T \cdot \bar{s}_I} \times \Delta(\bar{s}_T \cdot \bar{x}) \times \partial\Delta(\bar{s}_T \cdot (\bar{s}_I + 1))$ to itself has the desired properties. ∎

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