Nonatomic Game with General Preferences over Returns

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Traditional and Nonatomic Games

- In traditional n-player game with *utilities* $\tilde{u}(i, a(i), a(-i))$, a *mixed strategy* profile $\delta^* \equiv (\delta^*(\cdot|i))_{i=1,2,\ldots,n}$ would be considered an *equilibrium* when for every player $i = 1, 2, \ldots, n$,

$$\delta^* \left(\text{argmax}\left\{u(i, a(i), \delta^*(-i)) : \ a(i) \in A\right\} |i\right) = 1,$$

where each $\delta^*(\cdot|i)$ is an *action distribution* in $\mathscr{P}(A)$ and

$$u(i, a(i), \delta^*(-i)) \equiv \int_{A^{n-1}} \tilde{u}(i, a(i), a(-i)) \cdot \prod_{j \neq i} \delta^*(da(j)|j)$$

- An NG involves a *continuum* players with *no* one having any *discernible* influence on others; and yet, all players *in aggregation* have real impacts on each and every one of them

Nonatomic Game with a Utility Function

- A conventional NG still relies on a real-valued *utility function* $u(t, a, \delta)$, where $t \in T$ is player *identity* or *type*, $a \in A$ is *action*, and δ represents *outside environment*

- In *semi-anonymous* version, $\delta \in \mathscr{P}(T \times A)$ is *joint distribution* over all players' *identities* and *actions*; in simpler *anonymous* version, $\delta \in \mathscr{P}(A)$ is distribution over all player' *actions* alone

- A game is also associated with a player distribution $\theta \in \mathscr{P}(T)$

- According to Schmeidler's (1973) *pioneering work*, a T-to-$\mathscr{P}(A)$ mapping σ^* would reach *equilibrium* when for almost every $t \in T$,

$$\sigma^* \left(\text{argmax} \left\{ u(t, a, \theta \odot \sigma^*) : \ a \in A \right\} |t\right) = 1,$$

where $[\theta \odot \sigma^*](T' \times A') = \int_{t \in T'} \sigma^*(A'|t) \cdot \theta(dt)$ for any T' and A'

Generalization with a Distributional View

- In Mas-Colell's (1984) *alternative view*, a $\delta^* \in \mathcal{D} \equiv \mathscr{P}(T \times A)$ would be considered an *equilibrium* when

$$\delta^* \left(\{ (t,a) \in T \times A : \ u(t,a',\delta^*) \not> u(t,a,\delta^*), \ \forall a' \in A \} \right) = 1$$

and $\delta^*|_T = \theta$, meaning that $\delta^*(T' \times A) = \theta(T')$ for any T'

- We *generalize* this to case where instead of real-valued utility, players have *general preferences* on *returns* they receive

- There is a *return function* $\rho(t,a,\delta)$ *not necessarily real-valued* and there is a *preference* $\psi(t)$ so that $(r, r') \in \psi(t)$ equates to $r \not\succ_t r'$

- A distribution $\delta^* \in \mathcal{D}$ with $\delta^*|_T = \theta$ would be *equilibrium* when

$$\delta^* \left(\{ (t,a) \in T \times A : \ \left(\rho(t,a',\delta^*), \rho(t,a,\delta^*) \right) \in \psi(t), \ \forall a' \in A \} \right) = 1$$

Our Return-based Preferences

- Our *general preferences* are not directly on *actions* and other players' *action profiles*, but rather,
  on *returns* that players *personally* and *locally* feel
- We do believe in influences that other players' *identities* and *actions* might have on a given player
- However, these influences ought to be limited to extent that they affect a particular player *personally* and *locally*—an individual player could *not* care less about which *external* player-action profiles would actually yield him current return
- Return might be a bundle of commodities, a stash of cash, a get-out-of-jail card, admittance to a free-trade pact, lifting of some economic sanctions, winning of a soccer match, etc.

Main Results

- Under *compactness*- and *continuity*-related assumptions, we can establish *existence* of equilibria in *distributional* nature
- Probably more interestingly, we can derive *upper hemi-continuity* of set $\mathbb{E}(\rho, \psi)$ of equilibria with respect to *return function* $\rho \equiv (\rho(t, a, \delta))_{t \in T, a \in A, \delta \in \mathcal{D}}$ and *preference profile* $\psi \equiv (\psi(t))_{t \in T}$
- Here, two profiles ψ and ψ' are considered close when this is true for every pair of $\psi(t)$ and $\psi'(t)$, distance of which as two subsets of product *return space* $R \times R$ is then measured in Hausdorff sense
- Game would be *anonymous* when other players influence a given player only through *action distribution* they form
- When game is *anonymous*, player distribution θ is *atomless*, and action space A is *finite*, we can show same for *pure equilibria*

Implications of Results

- We have *generalized* classical results such as Schmeidlier (1973) and Mas-Colell (1984) by considering *general preferences*
- Upper hemi-continuities of equilibrium sets with respect to *return function* and *preference profile* would *guard against* severity of player-characteristic *mis-specifications*
- In real life where game might be played repeatedly over time, a player t might have to be merely capable of understanding his own preference $\psi(t)$ over fixed return space R
- Trials and errors would likely help him reach an equilibrium with others *without* any one player t' fully *comprehending* preferences $\psi(t'')$ of others or all complexities embedded in return generation mechanism $\rho \equiv (\rho(t, a, \delta))_{t \in T, a \in A, \delta \in \mathcal{D}}$

Nonatomic Games and General Preferences

- Systematic research on nonatomic games (NGs) started with Schmeidler (1973), a distributional alternative of which was exploited by Mas-Colell (1984)
- General preferences were first considered by Schmeidler (1969) for a *competitive economy* involving traders and commodities
- Mas-Colell (1974) studied a finite economy where every trader would strictly prefer some other bundles to any given bundle and these form *preferred-to correspondences* (PTCs)
- Shafer and Sonnenschein's (1975) PTCs were *globalized* to extent of being dependent on *others' choices* as well

Preference-to-Correspondences with Externalities

- Many works concerning NGs with general preferences also introduced *externality* to PTCs—such $P(t, \alpha)$, i.e., sets of actions that individual players t would prefer to those prescribed by given player-action profiles $\alpha \equiv (\alpha(t'))_{t' \in T}$, depends on not only $\alpha(t)$ but also $(\alpha(t'))_{t' \in T \setminus \{t\}}$; see, e.g., Khan and Vohra (1984), Khan and Papageorgiou (1987), and Kim, Prikry, and Yannelis (1989)
- However, Balder (2000) challenged compatibility among (a) *irreflexivity* of preferences, (b) *atomeless-ness* of player set, and (c) certain *continuity* often assumed for PTCs
- Martins-da-Rocha and Topuzu (2008) proposed to circumvent this difficulty by working with *expanded* PTCs $P(t, a, \alpha)$ whose dependencies on current-player actions a are newly added

Some Formal Notions

- Given a space X with metric d_X, use $\mathscr{B}(X)$ for Borel σ-field and $\mathscr{P}(X)$ for space of *probabilities* defined on $(X, \mathscr{B}(X))$
- Space $\mathscr{P}(X)$ is endowed with Prokhorov metric $\pi_X \equiv d_{\mathscr{P}(X)}$, which also induces weak convergence; it will be *compact* when X is
- Let $\mathscr{C}(X, Y)$ be space of *continuous mappings* from X to Y; its members will be uniformly continuous when X is compact
- Players form a compact metric space T, *actions* form a compact metric space A, and *returns* form a compact metric space R; these form $\mathcal{D} \equiv \mathscr{P}(T \times A)$ as space of *joint player-action distributions*

Returns and Preferences

- Let $\mathcal{R} \equiv \mathscr{C}(T \times A \times \mathcal{D}, R)$ be space of *return functions*—in any circumstance, return is written as $\rho(t, a, \delta)$
- A preference ψ can be represented by a subset of $R \times R$ with connotations of *irreflexivity* and *transitivity*:
  (i) $(r, r) \in \psi$;
  (ii) $(r, r') \notin \psi$ and $(r', r'') \notin \psi$ would lead to $(r, r'') \notin \psi$
- This way, ψ would be equivalent to a *preference relationship* $\succ$ so that $(r, r') \in \psi$ if and only if $r \not\succ r'$
- We shall also add *closedness* of ψ which confers *continuity* on corresponding $\succ$; for convenience, define

$$\Psi \equiv \{\psi \subseteq R \times R : \ \psi \text{ is nonepty, closed, and enjoying (i) and (ii)}\}$$

Distance for Preferences

- Let Φ be space of nonempty closed subsets of $R \times R$ which contains set Ψ of all preferences

- A metric d_Φ for Φ can be defined using Hausdorff distance so that

$$d_\Phi(\phi_1, \phi_2) = \inf\left(\epsilon > 0 : \ \phi_1 \subseteq (\phi_2)^\epsilon \text{ and } \phi_2 \subseteq (\phi_1)^\epsilon\right)$$

- Φ is known to be *compact* under d_Φ; also Ψ can be shown as a *closed* subset of Φ and hence a *compact* set in its own right

- Let $\mathcal{P} \equiv \mathscr{C}(T, \Psi)$ be space of continuous mappings from T to Ψ—game has a given *preference profile* $\psi \equiv (\psi(t))_{t \in T} \in \mathcal{P}$

Game Definition

- Recall *player* space T, *action* space A, and *return* space R would help form space $\mathcal{D} \equiv \mathscr{P}(T \times A)$ of joint player-action *distributions*, space $\mathcal{R} \subset R^{T \times A \times \mathcal{D}}$ of *return functions*, space $\Psi \subset 2^{R \times R}$ of *preferences*, and space $\mathcal{P} \subset \Psi^T$ of *preference profiles*

- At a *player distribution* $\theta \in \mathscr{P}(T)$, *return function* $\rho \equiv (\rho(t, a, \delta))_{t \in T, a \in A, \delta \in \mathcal{D}} \in \mathcal{R}$, and *preference profile* $\psi \equiv (\psi(t))_{t \in T} \in \mathcal{P}$, tuple $(T, A, R, \theta, \rho, \psi)$ would define a nonatomic game (NG), say $\Gamma(\rho, \psi)$

- When $R = [0, 1]$ and each $\psi(t)$ is triangle $\{(r, r') \in [0, 1]^2 : \ r \leq r'$ or equivalently $r \not> r'\}$, we would have a *classical nonatomic game* with ρ serving as its *utility function*

A PTC Interpretation

- In general, R could be multi- or infinite-dimensional
- Our primitives could lead to a sort of expanded PTC through

$$P(t,a,\delta) \equiv \left\{a' \in A : \left(\rho(t,a',\delta), \rho(t,a,\delta)\right) \notin \psi(t)\right\}$$

  So ours is in some sense a special PTC-based model
- As argued earlier, PTCs in practice are likely generated in this fashion or something akin to it
- Besides, our modeling approach has dispensed with any *linear structure* for action space A; also, it would facilitate advances on *mixed equilibria* and equilibrium set's *upper hemi-continuity*

An Illustrative Example

- Let $T = [0,1]$, $A = \{-1,+1\}$, $R = [0,1]$, and each player t's preference $\psi(t)$ be *ordered* one of $\{(r,r') \in [0,1]^2 : \ r \leq r'\}$
- When a player t takes an action a in presence of a joint player-action distribution δ, suppose return $\rho(t,a,\delta)$ is

$$\sum_{n=1}^{+\infty} \{[\sin(nta)+1]/2\} \cdot \delta\left([(n-1)/n, n/(n+1)) \times \{-1\}\right) \\ + \sum_{n=1}^{+\infty} \{[\cos(nta)+1]/2\} \cdot \delta\left([(n-1)/n, n/(n+1)) \times \{+1\}\right),$$

  which would not bring too much difficulty to our approach
- In PTC setup, however, one has to undertake humongous task of figuring out whether $P(t,a,\delta)$ is $\emptyset$ or $\{-a\}$; besides, as action space is not *convex* and externality is not expressible through a *finite* number of statistics, this example would not yield to Martins-da-Rocha and Topuzu's (2008) analysis

Strategies and Equilibria

- Consider set of joint player-action distributions whose T-marginals are θ; namely, $\mathcal{D}|_T(\theta) \equiv \{\delta \in \mathcal{D} : \ \delta|_T = \theta\}$
- Each $\delta \in \mathcal{D}|_T(\theta)$ also spells out a *mixed strategy* as it gives sense of how likely a (t, a)-neighborhood would be visited
- Use $\rho(t, A', \delta)$ for *set* $\{\rho(t, a, \delta) : \ a \in A'\}$ *of returns* that player t, while facing δ, could expect to get by taking actions from set A'
- We shall consider $\delta^* \in \mathcal{D}|_T(\theta)$ an equilibrium for $\Gamma(\rho, \psi)$ when

$$\delta^* \left(\mathbb{B}(\delta^*|\rho, \psi)\right) = 1,$$

with $\mathbb{B}(\delta|\rho, \psi) \equiv \{(t, a) \in T \times A : \ \rho(t, A, \delta) \times \rho(t, \{a\}, \delta) \subseteq \psi(t)\}$

Equilibrium-related Definitions

- As in Mas-Collel (1984), we define correspondence $\mathbb{F}(\cdot|\rho,\psi) : \mathcal{D}|_T(\theta) \rightrightarrows \mathcal{D}|_T(\theta)$ so that

$$\mathbb{F}(\delta|\rho,\psi) \equiv \left\{\delta' \in \mathcal{D}|_T(\theta) :\ \delta'\left(\mathbb{B}(\delta|\rho,\psi)\right) = 1\right\}$$

  It is set of joint player-action distributions that could possibly arise when players *optimally respond to* a common environment δ

- $\delta^* \in \mathcal{D}|_T(\theta)$ would be a member to set $\mathbb{E}(\rho,\psi)$ of *equilibria* if and only if δ^* is a *fixed point* for $\mathbb{F}(\cdot|\rho,\psi)$ satisfying $\delta^* \in \mathbb{F}(\delta^*|\rho,\psi)$

- We can show that $\mathbb{E}(\cdot,\cdot)$ is both *nonempty* and *upper hemi-continuous* as a correspondence from $\mathcal{R} \times \mathcal{P}$ to $\mathcal{D}|_T(\theta)$

Pure Equilibrium

- Suppose action space A is a *finite* $\{a_1, ..., a_{|A|}\}$
- Suppose return $\rho(t, a, \delta) \equiv \tilde{\rho}(t, a, \delta|_A)$ for some $\tilde{\rho} : T \times A \times \tilde{\mathcal{D}} \to R$—game is *anonymous* so that other players influence a given player only through action distribution
- Suppose player distribution θ is *atomless* so that for any T' with $\theta(T') > 0$ there would be T'' with $0 < \theta(T'') < \theta(T')$
- Now at each $(\tilde{\rho}, \psi) \in \tilde{\mathcal{R}} \times \mathcal{P}$, tuple $(T, A, R, \theta, \tilde{\rho}, \psi)$ would help define an *anonymous* NG $\tilde{\Gamma}(\tilde{\rho}, \psi)$
- We have a similar result for $\tilde{\Gamma}(\tilde{\rho}, \psi)$'s *pure equilibria*

Rich Player Space and Cruder Traits

- Also considered is Khan et al.'s (2013) setup where players form a rich enough (*saturated*) space $(I, \mathscr{I}, \lambda)$ and T contains *traits* that help provide external environment

- Let game $\hat{\gamma} \equiv (\hat{\theta}, \hat{\rho}, \hat{\psi})$ be defined through

  (i) an I-to-T mapping $\hat{\theta}$ so that each $\hat{\theta}(i)$ is player i's trait;

  (ii) an I-to-$\mathscr{C}(A, \mathscr{P}(T \times A))$ mapping $\hat{\rho}$ so that each $\hat{\rho}(i)$ is player i's return function—under action $a \in A$ and joint trait-action distribution $\delta \in \mathscr{P}(T \times A)$, player i will receive a return of $\hat{\rho}(i, a, \delta) \equiv [\hat{\rho}(i)](a, \delta)$;

  (iii) an I-to-Ψ mapping $\hat{\psi}$ so that a $\hat{\psi}(i)$ is player i's preference

Pure Equilibria with Saturation

- We consider an I-to-A mapping $\hat{\alpha}^*$ equilibrium when

$$\hat{\rho}\left(i, A, \lambda \circ (\hat{\theta}, \hat{\alpha}^*)^{-1}\right) \times \hat{\rho}\left(i, \{\hat{\alpha}^*(i)\}, \lambda \circ (\hat{\theta}, \hat{\alpha}^*)^{-1}\right) \subseteq \hat{\psi}(i),$$

  for λ-almost very player i

- A saturated player space is capable of supplying player-to-action mapping $\hat{\alpha}^*$ that would, together with given player-to-trait mapping $\hat{\theta}$ already matching trait-marginal of a given joint trait-action distribution δ^*, weave out a player-to-trait/action-pair mapping whose law happens to be δ^*

- Equilibrium existence can be assured under *saturation*, though without upper hemi-continuity being guaranteed

Concluding Remarks

- Modeling general preferences *indirectly* through *returns* which are then influenced by player-action profiles *rather than directly* through latter has given us advantages in *realism* and *simplicity*

- Properties of *nonemptiness* and *upper hemi-continuity* of mixed-equilibrium sets in presence of *general preferences* have already been achieved for finite games; see, e.g., Yang (2018)

- While focusing on unleashing most potential benefits of indirect modeling of preferences through returns, we have not prioritized at attainment of *uttermost generality* for spaces and mappings—further generalizations could be just on horizon