# The Weak Core, Partition-based Universal Stability, and their Risk Associations through a Partial Order

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# Coalitional Game and Core

- Consider a *coalitional game* with transferable utility (TU)

- Given a player set $N \equiv \{i_1, ..., i_n\}$,
     let $\mathcal{C}(N)$ be its set of nonempty subsets or *coalitions*

- Game $(N, \vec{v})$ is characterized by $\vec{v} \equiv (v(C))_{C \in \mathcal{C}(N)}$, with a *value* $v(C)$ attached to every coalition $C$

- Traditional *core* $\mathbb{X}^+(N, \vec{v})$ is made up of allocations $\vec{x} \equiv (x(i))_{i \in N}$ of *grand coalition* $N$ that make $\sum_{i \in N} x(i) = v(N)$ and

$$\sum_{i \in C} x(i) \geq v(C), \qquad \forall C \in \mathcal{C}(N)$$

# Universal Stability

- Unfortunately, only *balanced* games possess *nonempty* cores

- A game is *balanced* when $v(N)$ is above

$$
\begin{array}{ll}
\max & \sum_{C \in \mathcal{C}(N)} \delta(C) \cdot v(C) \\
\text{s.t.} & \sum_{C \in \mathcal{C}(N) \text{ and } C \ni i} \delta(C) = 1 \qquad \forall i \in N \\
& \delta(C) \geq 0 \qquad\qquad \forall C \in \mathcal{C}(N)
\end{array}
$$

- We aim at *stability* notions that are *universal*—*every* game $(N, \vec{v})$ *has* its own stable solutions

- Naturally, focus is on *partitions* $\mathcal{P} \equiv \{C_1, ..., C_p\}$ and *allocations* $\vec{x}$ *reasonably* associated with them

# Partition-allocation Pairs

- For any player set $N$, let set of *partitions* be $\mathscr{P}(N)$

- For $\mathcal{P} \in \mathscr{P}(N)$, let $\mathscr{X}(N, \vec{v}, \mathcal{P})$ be set of $(N, \vec{v})$'s *individually rational* allocations that are *efficient* in $\mathcal{P}$'s constituent coalitions:

$$\left\{ \vec{x} \in \prod_{i \in N} [v(\{i\}), +\infty) : \sum_{i \in C} x(i) = v(C), \quad \forall C \in \mathcal{P} \right\}$$

- For game $(N, \vec{v})$, its set of *partition-allocation* pairs is

$$\mathscr{Q}(N, \vec{v}) \equiv \bigcup_{\mathcal{P} \in \mathscr{P}(N)} \{\mathcal{P}\} \times \mathscr{X}(N, \vec{v}, \mathcal{P})$$

- A stability notion $\mathbb{S}$ supplies a subset $\mathbb{S}(N, \vec{v})$ of $\mathscr{Q}(N, \vec{v})$ to every game $(N, \vec{v})$; *universality* means $\mathbb{S}(N, \vec{v})$ is always *nonempty*

# An Illustrative Example

- Consideration of general $(\mathcal{P}, \vec{x})$ instead of special $(\{N\}, \vec{x})$ might still not be enough; *guiding principle* for core needs changing as well

- Suppose Alice wants to leave an unhappy marriage with Bob while also bringing along their daughter Carol; current core condoning *unilateral* change of status quo would allow her attempt to succeed as long as she and Carol could fare better afterwards

- In reality, Bob might try hard to prevent Alice and Carol from leaving him unless his fear of falls in living standards is assuaged

- Dissolution of union would be more realizable when intended change becomes *consensual*—Alice and Carol could still be better off after Bob is compensated enough to go on with his life style

# The Weak Core

- Stability in old $\mathbb{X}^+(N, \vec{v})$ stipulates blocking of *unilateral* changes:

  $$\{\vec{x} \in \mathscr{X}(N, \vec{v}, \{N\}) : \text{for any } \mathcal{P}' \in \mathscr{P}(N)\backslash\{\{N\}\},$$
  $$\forall C' \in \mathcal{P}' \text{ we have } \sum_{i \in C'} x(i) \geq v(C')\}$$

- *Traditional core* is more about blocking of "*entrance*" by one sub-coalition unsatisfied with potential *making* of grand coalition

- Blocking of *consensual* changes leads to *weak core* $\mathbb{X}^-(N, \vec{v})$:

  $$\{\vec{x} \in \mathscr{X}(N, \vec{v}, \{N\}) : \text{for any } \mathcal{P}' \in \mathscr{P}(N)\backslash\{\{N\}\},$$
  $$\exists C' \in \mathcal{P}' \text{ so that } \sum_{i \in C'} x(i) \geq v(C')\}$$

- *Weak core* is more about blocking of "*exit*" by one sub-coalition unsatisfied with potential *breaking* of grand coalition

# An Intermediate Concept

- Define a partition $\mathcal{P}$'s *worth*

$$\tilde{w}(N, \vec{v}, \mathcal{P}) \equiv \sum_{C \in \mathcal{P}} v(C)$$

and then *medium core* $\mathbb{X}^0(N, \vec{v})$ by

$$\begin{cases} \mathscr{X}(N, \vec{v}, \{N\}) & \text{when } v(N) \geq \max_{\mathcal{P}' \in \mathscr{P}(N) \setminus \{\{N\}\}} \tilde{w}(N, \vec{v}, \mathcal{P}') \\ \emptyset & \text{otherwise} \end{cases}$$

- It is easy to show resistance to *unilateral* blocking $\implies$ dominance in worth $\implies$ resistance to *consensual* blocking; thus

$$\mathbb{X}^+(N, \vec{v}) \subseteq \mathbb{X}^0(N, \vec{v}) \subseteq X^-(N, \vec{v})$$

Both inclusions could be *strict* for some games

# When $\mathbb{X}^0(N, \vec{v}) \setminus \mathbb{X}^+(N, \vec{v}) \neq \emptyset$

- Consider $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 5$, and $v(\{1, 2, 3\}) = 6$

- Here, $\{N\}$ is dominant in its worth of 6 over others which top at 5

- Yet, there is no *core* member as $x(1) + x(2) + x(3) = 6$ contradicts with $x(1) + x(2) \geq 5$, $x(1) + x(3) \geq 5$, and $x(2) + x(3) \geq 5$

# When $\mathbb{X}^-(N, \vec{v}) \setminus \mathbb{X}^0(N, \vec{v}) \neq \emptyset$

- Inspired by *Alice-Bob-Carol* story, consider $N = \{A, B, C\}$, $v(\{A\}) = 0$, $v(\{B\}) = 4$, $v(\{C\}) = 0$, $v(\{A, B\}) = 0$, $v(\{A, C\}) = 6$, $v(\{B, C\}) = 0$, and $v(\{A, B, C\}) = 8$

- Note $\vec{x} = (0, 6, 2)$ is in *weak core*—under this allocation plan, no attempt to split grand coalition $N$ can make all resulting sub-coalitions strictly better off

- The most competitive alternative comes from $\{\{A, C\}, \{B\}\}$; while sub-coalition $\{A, C\}$ has strong incentives to leave grand coalition, player $B$ would hold out

## More on Core Concepts

- Against every partition attempt, *traditional* core lets one *separatist* group to break up a union; meanwhile, *weak* core allows one *unionist* group to hold a union together; *medium* core is in between

- Solving a mixed integer programming (MIP) could help determine whether *weak* core is nonempty; even closer to Bondareva-Shapley theory for traditional core, solving a linear programming (LP) or its dual would help determine whether *medium* core is nonempty

- Peleg's (1986) *axiomatization* effort has reached $\{(\Gamma^+, \Sigma^+)\} = \mathfrak{S}_{\mathsf{ne}} \cap \mathfrak{S}_{\mathsf{ir}} \cap \mathfrak{S}_{\mathsf{rgp}} \cap \mathfrak{S}_{\mathsf{supa}}$; we can reach

$$\begin{cases} (\Gamma^-, \Sigma^-) & \in \mathfrak{S}_{\mathsf{ne}} \cap \mathfrak{S}_{\mathsf{ir}} \cap \mathfrak{S}_{\mathsf{po}} \cap \mathfrak{S}_{\mathsf{crgp}} \cap \mathfrak{S}_{\mathsf{supa}}^C \cap \mathfrak{S}_{\mathsf{wrgp}}^C \\ (\Gamma^0, \Sigma^0) & \in \mathfrak{S}_{\mathsf{ne}} \cap \mathfrak{S}_{\mathsf{ir}} \cap \mathfrak{S}_{\mathsf{po}} \cap \mathfrak{S}_{\mathsf{crgp}} \cap \mathfrak{S}_{\mathsf{supa}} \cap \mathfrak{S}_{\mathsf{wrgp}}^C \end{cases}$$

# An MIP linked to the Weak Core

- $\mathbb{X}^-(N, \vec{v})$ would be nonempty if and only if $v(N)$ is above optimal objective $z^-(N, \vec{v})$ of *mixed integer program* (MIP)

$$
\begin{array}{lll}
\min & \sum_{i \in N} x(i) \\
\text{s.t.} & \sum_{i \in C} x(i) + \left[ \sum_{i \in C} v(\{i\}) - v(C) \right] \cdot y(C) \\
& \qquad\qquad \geq \sum_{i \in C} v(\{i\}) & \forall C \in \mathcal{C}(N) \backslash \{N\} \\
& \sum_{C \in \mathcal{P}} y(C) \geq 1 & \forall \mathcal{P} \in \mathscr{P}(N) \backslash \{\{N\}\} \\
& x(i) \in [v(\{i\}), +\infty) & \forall i \in N \\
& y(C) \in \{0, 1\} & \forall C \in \mathcal{C}(N) \backslash \{N\}
\end{array}
$$

- With $|N|$ real variables $x(i)$ and $2^{|N|} - 2$ *binary* variables $y(C)$,

$$
\sum_{i \in C} x(i) \geq \left\{ \begin{array}{ll} v(C) & \text{when } y(C) = 1 \\ \sum_{i \in C} v(\{i\}) & \text{when } y(C) = 0 \end{array} \right.
$$

# Traditional Core and Credible Threats

- Following Ray (1989), let $\mathbb{X}^{+-}(\{i\}, v) = \{v\} \neq \emptyset$ for any single-player game $(\{i\}, v)$

- For any game $(N, \vec{v})$ with $|N| \geq 2$, recursively define

$$\mathbb{X}^{+-}(N, \vec{v}) \equiv \{\vec{x} \in \partial(N, \vec{v}) : \sum_{i \in C} x(i) \geq v(C), \\ \forall C \in \mathcal{C}(N) \backslash \{N\} \text{ with } \mathbb{X}^{+-}(C, \vec{v}|_{\mathcal{C}(C)}) \neq \emptyset\}$$

- A sub-coalition $C$ may be *threatening* when $v(C) > \sum_{i \in C} x(i)$; only one with $\mathbb{X}^{+-}(C, \vec{v}|_{\mathcal{C}(C)}) \neq \emptyset$ poses a *credible* threat

- Not only $\mathbb{X}^+(N, \vec{v}) \subseteq \mathbb{X}^{+-}(N, \vec{v})$, but *opposite* was shown by Ray (1989) to be true—*traditional-core* member may thus be understood as facing *no credible threat* in any partition attempt

# Weak Core and Credible Threats

- Let $\mathbb{X}^{--}(\{i\}, v) = \{v\} \neq \emptyset$ for any single-player game $(\{i\}, v)$

- For any game $(N, \vec{v})$ with $|N| \geq 2$, recursively define

$$\mathbb{X}^{--}(N, \vec{v}) \equiv \{\vec{x} \in \partial(N, \vec{v}) : \text{ for any } \mathcal{P}' \in \mathscr{P}(N) \backslash \{\{N\}\},$$
$$\exists C' \in \mathcal{P}' \text{ so that either } \mathbb{X}^{--}(C, \vec{v}|_{\mathcal{C}(C)}) = \emptyset$$
$$\text{or } \textstyle\sum_{i \in C'} x(i) \geq v(C')\}$$

- An imputation $\vec{x} \in \mathbb{X}^{--}(N, \vec{v})$ when in any partition attempt, *not all* constituent coalitions pose *credible threats*; whereas, a coalition $C$'s credibility is defined in *weak* sense of $\mathbb{X}^{--}(C, \vec{v}|_{\mathcal{C}(C)}) \neq \emptyset$

- It is easy to tell $\mathbb{X}^{-}(N, \vec{v}) \subseteq \mathbb{X}^{--}(N, \vec{v})$; we can prove *opposite*—when a *weak-core* member is pitted against a proper *partition*, there must exist *one constituent coalition* that *either* is sufficiently content with allocation associated with this member *or* is *not credible* for its *threat* to be taken seriously

# Fission and Fusion Resistances

- A partition $\mathcal{P}$'s *fission-down-to* neighborhood $\mathscr{I}(N, \mathcal{P})$ contains all partitions $\mathcal{P}'$ that constitute *splits* of $\mathcal{P}$'s constituent coalitions

- A partition-allocation pair $(\mathcal{P}, \vec{x})$ in feasible set $\mathscr{Q}(N, \vec{v})$ is *strong fission-resistant* when for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\forall C' \in \mathcal{P}' \backslash \mathcal{P} \quad \text{we have} \quad \sum_{i \in C'} x(i) \geq v(C')$$

- With symmetrically defined *fusion-up-to* neighborhood $\mathscr{U}(N, \mathcal{P})$, a pair $(\mathcal{P}, \vec{x})$ is *fusion-resistant* when for any $\mathcal{P}' \in \mathscr{U}(N, \mathcal{P})$,

$$\forall C' \in \mathcal{P}' \backslash \mathcal{P} \quad \text{we have} \quad \sum_{i \in C'} x(i) \geq v(C'),$$

which is merely about $\sum_{C \in \mathcal{P}, \, C \subsetneq C'} v(C) \geq v(C')$

## More Stability Notions

- Medium fission resistance is about for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\sum_{i \in N} x(i) = \sum_{C \in \mathcal{P}} \sum_{i \in C} x(i) = \sum_{C \in \mathcal{P}} v(C) = \tilde{w}(N, \vec{v}, \mathcal{P}) \geq \tilde{w}(N, \vec{v}, \mathcal{P}')$$

- Weak fission resistance is about for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\exists C' \in \mathcal{P}' \setminus \mathcal{P} \quad \text{so that} \quad \sum_{i \in C'} x(i) \geq v(C')$$

- Let $\mathbb{Q}^{\mathsf{i}*}(N, \vec{v})$ be set of all $(\mathcal{P}, \vec{x})$'s that are *fission*-resistant, with $* = +$ for *strong*, $0$ for *medium*, and $-$ for *weak*; let $\mathbb{Q}^{\mathsf{u}}(N, \vec{v})$ be set of all solution pairs that are *fusion*-resistant

- Define *stability* concepts $\mathbb{S}^*(N, \vec{v}) \equiv \mathbb{Q}^{\mathsf{i}*}(N, \vec{v}) \cap \mathbb{Q}^{\mathsf{u}}(N, \vec{v})$

# Other Stability-related Notions

- Each stability is corresponding-*core-compatible*:

$$\mathbb{S}^*(N, \vec{v}) \cap [\{\{N\}\} \times \mathscr{X}(N, \vec{v}, \{N\})]$$
$$= \mathbb{Q}^{\mathbf{i}*}(N, \vec{v}) \cap [\{\{N\}\} \times \mathscr{X}(N, \vec{v}, \{N\})] = \{\{N\}\} \times \mathbb{X}^*(N, \vec{v})$$

  Of course, these sets could be simultaneously *empty*

- For **every** $\vec{v} \in \Re^{\mathcal{C}(N)}$, no matter how "*poor*" it is,

$$\mathbb{S}^+(N, \vec{v}) \subseteq \mathbb{S}^0(N, \vec{v}) \subseteq \mathbb{S}^-(N, \vec{v}) \qquad \text{and} \qquad \mathbb{S}^{\mathbf{0}}(\mathbf{N}, \tilde{\mathbf{v}}) \neq \emptyset$$

- There is no *universality* guarantee for strong stability $\mathbb{S}^+$;
    still, it might allow stable solutions $(\mathcal{P}, \vec{x})$ other than
  $\mathcal{P} = \{N\}$ and $\vec{x} \in \mathbb{X}^+(N, \vec{v})$

## Fission-related Constructs

- Given a partition $\mathcal{P} \in \mathscr{P}(N)$, let *patched-up core* be

$$\mathbb{X}^{\mathsf{i}*}(N, \vec{v}, \mathcal{P}) \equiv \prod_{C \in \mathcal{P}} \mathbb{X}^*(C, \vec{v}|_{\mathcal{C}(C)})$$

- With $\mathbb{P}^{\mathsf{i}*}(N, \vec{v}) \equiv \Big\{ \mathcal{P} \in \mathscr{P}(N) : \mathbb{X}^{\mathsf{i}*}(N, \vec{v}, \mathcal{P}) \neq \emptyset \Big\}$,

$$\mathbb{Q}^{\mathsf{i}*}(N, \vec{v}) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathsf{i}*}(N, \vec{v})} \{\mathcal{P}\} \times \mathbb{X}^{\mathsf{i}*}(N, \vec{v}, \mathcal{P})$$

- Earlier inclusion relationships among *cores* would lead to

$$\mathbb{Q}^{\mathsf{i}+}(N, \vec{v}) \subseteq \mathbb{Q}^{\mathsf{i}0}(N, \vec{v}) \subseteq \mathbb{Q}^{\mathsf{i}-}(N, \vec{v})$$

# Some More Structures

- An alternative definition for $\mathbb{P}^{\mathbf{i0}}(N, \vec{v})$ turns out to be

$$\left\{\mathcal{P} \in \mathscr{P}(N) : \ \tilde{w}(N, \vec{v}, \mathcal{P}) \geq \tilde{w}(N, \vec{v}, \mathcal{P}'), \ \ \forall \mathcal{P}' \in \mathscr{I}(N, \mathcal{P})\right\}$$

- For $\mathbb{P}^{\mathbf{u}}(N, \vec{v})$ defined *similarly* except with $\mathscr{U}(N, \mathcal{P})$ replacing $\mathscr{I}(N, \mathcal{P})$, it would follow that

$$\mathbb{Q}^{\mathbf{u}}(N, \vec{v}) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathbf{u}}(N, \vec{v})} \{\mathcal{P}\} \times \mathscr{X}(N, \vec{v}, \mathcal{P})$$

- What lead to *universality* are $\mathbb{P}^{\mathbf{i0}}(N, \vec{v}) \cap \mathbb{P}^{\mathbf{u}}(N, \vec{v}) \neq \emptyset$ and

$$\mathbb{S}^*(N, \vec{v}) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathbf{i*}}(N, \vec{v}) \cap \mathbb{P}^{\mathbf{u}}(N, \vec{v})} \{\mathcal{P}\} \times \mathbb{X}^{\mathbf{i*}}(N, \vec{v}, \mathcal{P})$$

# Reasons behind Earlier Structures

- Very importantly, we can show

$$(\mathcal{P}, \vec{x}) \in \mathbb{Q}^{\mathsf{i}*}(N, \vec{v}) \iff (\{C\}, \vec{x}|_C) \in \mathbb{Q}^{\mathsf{i}*}(C, \vec{v}|_{\mathcal{C}(C)}), \ \forall C \in \mathcal{P}$$

- With $(\{N\}, \vec{x}) \in \mathbb{Q}^{\mathsf{i}*}(N, \vec{v}) \iff \vec{x} \in \mathbb{X}^*(N, \vec{v})$, this would result in

$$\begin{aligned}
(\mathcal{P}, \vec{x}) \in \mathbb{Q}^{\mathsf{i}*}(N, \vec{v}) &\iff (\{C\}, \vec{x}|_C) \in \mathbb{Q}^{\mathsf{i}*}(C, \vec{v}|_{\mathcal{C}(C)}), \ \forall C \in \mathcal{P} \\
&\iff \vec{x}|_C \in \mathbb{X}^*(C, \vec{v}|_{\mathcal{C}(C)}), \ \forall C \in \mathcal{P} \\
&\iff \vec{x} \in \mathbb{X}^{\mathsf{i}*}(N, \vec{v}, \mathcal{P})
\end{aligned}$$

- Concerning *fusion*, we can also establish

$$(\mathcal{P}, \vec{x}) \in \mathbb{Q}^{\mathsf{u}}(N, \vec{v}) \iff \mathcal{P} \in \mathbb{P}^{\mathsf{u}}(N, \vec{v})$$

# A Schematic Sketch of Various Entities
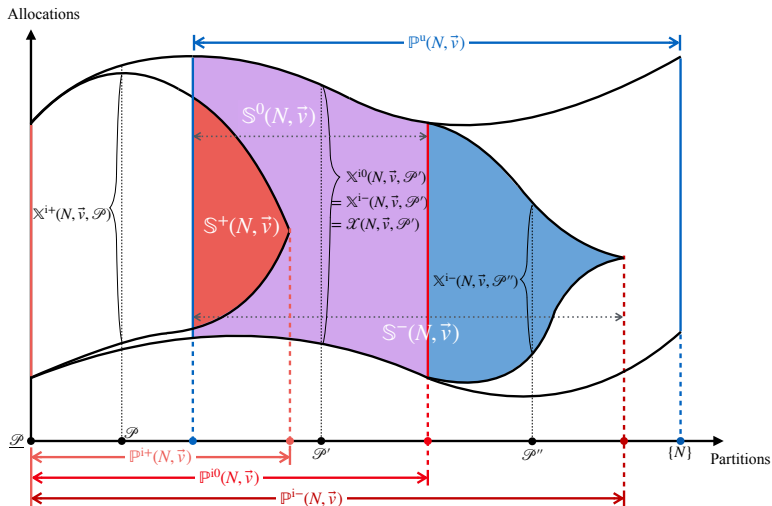
# Road to Medium Stability

- Set $\mathscr{P}(N)$ can be decomposed into $\mathscr{P}^n(N)$, $\mathscr{P}^{n-1}(N)$, $\cdots$, $\mathscr{P}^1(N) \equiv \{\{N\}\}$ depending on member partitions' *sizes*

- One-step fission and fusion *arcs* link two neighboring $\mathscr{P}^{p+1}(N)$ and $\mathscr{P}^p(N)$, with $\mathscr{I}(N, \mathcal{P})$ understandable as "*left*" branch stemming from a given $\mathcal{P}$ and $\mathscr{U}(N, \mathcal{P})$ "*right*" branch

- A *steepest ascending method* (SAM) can help reach a *mediumly stable* pair $(\mathcal{P}^0, \vec{x}^0)$ from *any* starting partition
    by incessantly moving from one $\mathcal{P}$ to a $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P}) \cup \mathscr{U}(N, \mathcal{P})$
  that *maximizes* $\tilde{w}(N, \vec{v}, \cdot)$ until *no improvement* is possible

- After $\mathcal{P}^0$ is identified, $\vec{x}^0$ can be any member of $\mathbb{X}^{\mathrm{i}0}(N, \vec{v}, \mathcal{P}^0)$

# A Graph Representation of Partitions
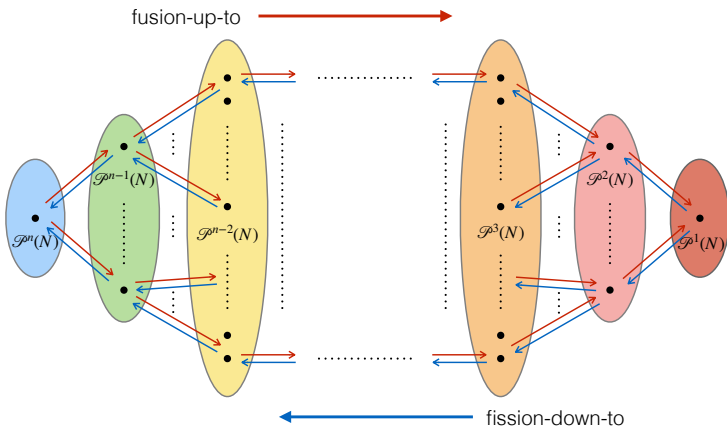
# Core Stability in Literature

- If *fission-down-to* neighborhood in *strong fission* resistance or *fusion-up-to* one in *fusion* resistance were replaced by *space of all other partitions*, we would obtain *all-temptation resistance*

- This super-strong resistance seems to have propped up so-called "*core stability*" in coalition formation literature since Gale and Shapley (1962); see, e.g., Pycia (2012)

- Since *universality* is clearly out of the question, focus has been on identifying *conditions* that induce existence of *stable partitions* (coalition structures); see, e.g., Greenberg and Weber (1993), Banerjee, Konishi, and Sonmez (2001), Bogomolnaia and Jackson (2002), Papai (2004), and Alcalde and Romero-Medina (2006)

- Core stability is still (strong-)*core-compatible* by our standard

## From Centripetality to Cooperation

- When $(N, \vec{v})$ is *strictly positive* with every $v(C) > 0$ except when $|C| = 1$ at which time only $v(C) \geq 0$ is required, we may take a *fractional* view on allocations with each $f(i) \equiv x(i)/v(C)$

- Earlier stability notions are *transplantable* here, after replacing $\mathscr{X}(N, \vec{v}, \mathcal{P})$ with $\mathscr{F}(N, \vec{v}, \mathcal{P})$, $\mathbb{X}^{\pm}(N, \vec{v})$ with $\mathbb{F}^{\pm}(N, \vec{v})$, and $\mathbb{X}^{i\pm}(N, \vec{v}, \mathcal{P})$ with $\mathbb{F}^{i\pm}(N, \vec{v}, \mathcal{P})$

- A *centripetality* partial order can be defined for games so that $(N, \vec{v}^1) \leq_{\mathsf{cp}} (N, \vec{v}^2)$ if and only if
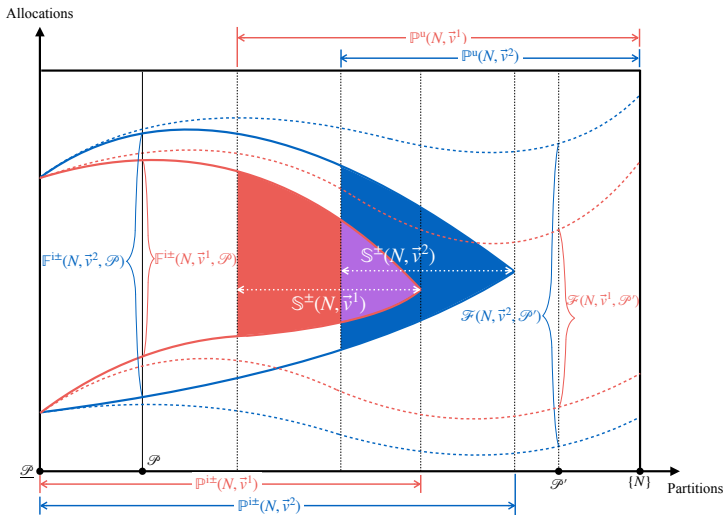
$$\frac{v^1(C^2)}{v^1(C^1)} \leq \frac{v^2(C^2)}{v^2(C^1)}, \qquad \text{when } C^1 \subseteq C^2$$

- Consequences turn out to be $\mathbb{F}^{i\pm}(N, \vec{v}^1, \mathcal{P}) \subseteq \mathbb{F}^{i\pm}(N, \vec{v}^2, \mathcal{P})$,

$$\mathbb{Q}^{i\pm}(N, \vec{v}^1) \subseteq \mathbb{Q}^{i\pm}(N, \vec{v}^2), \quad \text{and} \quad \mathbb{P}^{\mathsf{u}}(N, \vec{v}^1) \supseteq \mathbb{P}^{\mathsf{u}}(N, \vec{v}^2)$$

# An Illustration of Centripetality
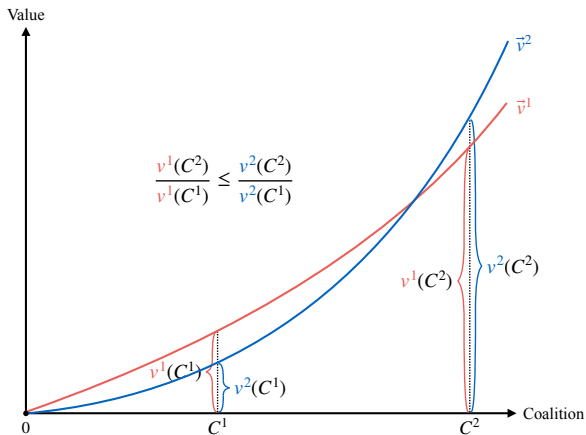
# A Schematic Sketch of Consequences

# From Risk Aversion to Centripetality

- With *centripetality* $\implies$ *cooperation* at hand, we can further demonstrate *risk aversion* $\implies$ *centripetality*

- Each coalition $C$ is associated with a *random outcome* $\Phi(C)$

- All players share a common strictly-positive-valued *reward function* $\tilde{R}$ that is *positively homogeneous* in sense that

$$\tilde{R}(f \cdot Y) = f \cdot \tilde{R}(Y), \qquad \text{if } f \geq 0$$

- When $Y = \phi(C) \cdot \sum_{i \in C} \Theta_i$ with *i.i.d.* $\Theta_i$'s and

$$\tilde{R}(Y) \equiv \mathbb{E}[Y] - \bar{r} \cdot \sqrt{\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2},$$

$\bar{r}^1 \leq \bar{r}^2$ would lead to $(N, \vec{v}^1) \leq_{\mathsf{cp}} (N, \vec{v}^2)$

# Another Law-invariant Occasion

- In another *law-invariant* case, we characterize $\Phi(C)$'s by corresponding $\tilde{k}(C)$'s, where each $\tilde{k}(C)$ is *inverse* of $\Phi(C)$'s *cumulative distribution function*

- Let reward function $\tilde{r}$ operating on above *quantiles* be parameterized by a *probability density function* $\bar{\mu}$ on $[0,1]$ so that

$$\tilde{r}(k) \equiv \int_0^1 \bar{a}(k, \alpha) \cdot \bar{\mu}(\alpha) \cdot d\alpha,$$

where $\bar{a}(\cdot, \alpha)$ is $\alpha$-level *conditional value at risk* defined as in

$$\bar{a}(k, \alpha) \equiv \frac{1}{1 - \alpha} \cdot \int_0^{1-\alpha} k(\beta) \cdot d\beta$$

# Risk Aversion Promotes Cooperation

- For case above, $\bar{\mu}^1 \leq_{\mathsf{lr}} \bar{\mu}^2$ would lead to $(N, \vec{v}^1) \leq_{\mathsf{cp}} (N, \vec{v}^2)$ under mild conditions on *quantile functions* $\tilde{k}(C)$

- Thus, for both cases, we can show that *risk aversion* promotes resulting coalitional game's *centripetality*

- This link, when combined with already-established link about *centripetality* promoting *cooperation*, would deliver on message

    risk aversion promotes cooperation

# Concluding Remarks

- Using *partition-allocation* pairs and *weak core* based on *consensual blocking*, as well as certain middle grounds, we have identified *stability notions* that are *universal*

- We wonder if there are *universal* stability concepts that are still compatible with plain *core*

- Concerning link from *risk aversion* to *cooperation*, more on causes of *centripetality* would be welcome

<div align="center">

Thank you! Comments and suggestions?

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