The Weak Core, Partition-based Universal Stability, and their Risk Associations through a Partial Order

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Coalitional Game and Core

- Consider a *coalitional game* with transferable utility (TU)
- Given a player set $N \equiv \{i_1, ..., i_n\}$,
  let $\mathcal{C}(N)$ be its set of nonempty subsets or *coalitions*
- Game (N, v) is characterized by $v \equiv (v(C))_{C \in \mathcal{C}(N)}$, with a *value* $v(C)$ attached to every coalition C
- Traditional *core* $\mathbb{X}^+(N, v)$ is made up of allocations $x \equiv (x(i))_{i \in N}$ of *grand coalition* N that make $\sum_{i \in N} x(i) = v(N)$ and

$$\sum_{i \in C} x(i) \geq v(C), \qquad \forall C \in \mathcal{C}(N)$$

Universal Stability

- Unfortunately, only *balanced* games possess *nonempty* cores
- A game is *balanced* when $v(N)$ is above

$$\begin{array}{lll} \max & \sum_{C \in \mathcal{C}(N)} \delta(C) \cdot v(C) & \\ \text{s.t.} & \sum_{C \in \mathcal{C}(N) \text{ and } C \ni i} \delta(C) = 1 & \forall i \in N \\ & \delta(C) \geq 0 & \forall C \in \mathcal{C}(N) \end{array}$$

- We aim at *stability* notions that are *universal*—*every* game (N, v) *has* its own stable solutions
- Naturally, focus is on *partitions* $\mathcal{P} \equiv \{C_1, ..., C_p\}$ and *allocations* x *reasonably* associated with them

Partition-allocation Pairs

- For any player set N, let set of *partitions* be $\mathscr{P}(N)$
- For $\mathcal{P} \in \mathscr{P}(N)$, let $\mathscr{X}(N, v, \mathcal{P})$ be set of (N, v)'s *individually rational* allocations that are *efficient* in $\mathcal{P}$'s constituent coalitions:

$$\left\{ x \in \prod_{i \in N} [v(\{i\}), +\infty) : \sum_{i \in C} x(i) = v(C), \quad \forall C \in \mathcal{P} \right\}$$

- For game (N, v), its set of *partition-allocation* pairs is

$$\mathscr{Q}(N, v) \equiv \bigcup_{\mathcal{P} \in \mathscr{P}(N)} \{\mathcal{P}\} \times \mathscr{X}(N, v, \mathcal{P})$$

- A stability notion $\mathbb{S}$ supplies a subset $\mathbb{S}(N, v)$ of $\mathscr{Q}(N, v)$ to every game (N, v); *universality* means $\mathbb{S}(N, v)$ is always *nonempty*

An Illustrative Example

- Consideration of general $(\mathcal{P}, x)$ instead of special $(\{N\}, x)$ might still not be enough; *guiding principle* for core needs changing as well
- Suppose Alice wants to leave an unhappy marriage with Bob while also bringing along their daughter Carol; current core condoning *unilateral* change of status quo would allow her attempt to succeed as long as she and Carol could fare better afterwards
- In reality, Bob might try hard to prevent Alice and Carol from leaving him unless his fear of falls in living standards is assuaged
- Dissolution of union would be more realizable when intended change becomes *consensual*—Alice and Carol could still be better off after Bob is compensated enough to go on with his life style

The Weak Core

- Stability in old $\mathbb{X}^+(N,v)$ stipulates blocking of *unilateral* changes:

$$\{x \in \mathscr{X}(N,v,\{N\}) : \text{ for any } \mathcal{P}' \in \mathscr{P}(N)\backslash\{\{N\}\}, \\ \forall C' \in \mathcal{P}' \text{ we have } \textstyle\sum_{i\in C'} x(i) \geq v(C')\}$$

- *Traditional core* is more about blocking of "*entrance*" by one sub-coalition unsatisfied with potential *making* of grand coalition

- Blocking of *consensual* changes leads to *weak core* $\mathbb{X}^-(N,v)$:

$$\{x \in \mathscr{X}(N,v,\{N\}) : \text{ for any } \mathcal{P}' \in \mathscr{P}(N)\backslash\{\{N\}\}, \\ \exists C' \in \mathcal{P}' \text{ so that } \textstyle\sum_{i\in C'} x(i) \geq v(C')\}$$

- *Weak core* is more about blocking of "*exit*" by one sub-coalition unsatisfied with potential *breaking* of grand coalition

An Intermediate Concept

- Define a partition $\mathcal{P}$'s *worth*

$$\tilde{w}(N, v, \mathcal{P}) \equiv \sum_{C \in \mathcal{P}} v(C)$$

and then *medium core* $\mathbb{X}^0(N, v)$ by

$$\begin{cases} \mathscr{X}(N, v, \{N\}) & \text{when } v(N) \geq \max_{\mathcal{P}' \in \mathscr{P}(N) \setminus \{\{N\}\}} \tilde{w}(N, v, \mathcal{P}') \\ \emptyset & \text{otherwise} \end{cases}$$

- It is easy to show resistance to *unilateral* blocking $\Longrightarrow$ dominance in worth $\Longrightarrow$ resistance to *consensual* blocking; thus

$$\mathbb{X}^+(N, v) \subseteq \mathbb{X}^0(N, v) \subseteq X^-(N, v)$$

Both inclusions could be *strict* for some games

When $\mathbb{X}^0(N,v) \setminus \mathbb{X}^+(N,v) \neq \emptyset$

- Consider $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 5$, and $v(\{1,2,3\}) = 6$
- Here, $\{N\}$ is dominant in its worth of 6 over others which top at 5
- Yet, there is no *core* member as $x(1) + x(2) + x(3) = 6$ contradicts with $x(1) + x(2) \geq 5$, $x(1) + x(3) \geq 5$, and $x(2) + x(3) \geq 5$

When $\mathbb{X}^{-}(N,v) \setminus \mathbb{X}^{0}(N,v) \neq \emptyset$

- Inspired by *Alice-Bob-Carol* story, consider $N = \{A, B, C\}$, $v(\{A\}) = 0$, $v(\{B\}) = 4$, $v(\{C\}) = 0$, $v(\{A, B\}) = 0$, $v(\{A, C\}) = 6$, $v(\{B, C\}) = 0$, and $v(\{A, B, C\}) = 8$
- Note $x = (0, 6, 2)$ is in *weak core*—under this allocation plan, no attempt to split grand coalition N can make all resulting sub-coalitions strictly better off
- The most competitive alternative comes from $\{\{A, C\}, \{B\}\}$; while sub-coalition $\{A, C\}$ has strong incentives to leave grand coalition, player B would hold out

More on Core Concepts

- Against every partition attempt, *traditional* core lets one *separatist* group to break up a union; meanwhile, *weak* core allows one *unionist* group to hold a union together; *medium* core is in between
- Solving a mixed integer programming (MIP) could help determine whether *weak* core is nonempty; even closer to Bondareva-Shapley theory for traditional core, solving a linear programming (LP) or its dual would help determine whether *medium* core is nonempty

An MIP linked to the Weak Core

- $\mathbb{X}^{-}(N, v)$ would be nonempty if and only if $v(N)$ is above optimal objective $z^{-}(N, v)$ of *mixed integer program* (MIP)

$$\begin{array}{lll} \min & \sum_{i \in N} x(i) & \\ \text{s.t.} & \sum_{i \in C} x(i) + \left[\sum_{i \in C} v(\{i\}) - v(C)\right] \cdot y(C) & \\ & \qquad \geq \sum_{i \in C} v(\{i\}) & \forall C \in \mathcal{C}(N) \setminus \{N\} \\ & \sum_{C \in \mathcal{P}} y(C) \geq 1 & \forall \mathcal{P} \in \mathscr{P}(N) \setminus \{\{N\}\} \\ & x(i) \in [v(\{i\}), +\infty) & \forall i \in N \\ & y(C) \in \{0, 1\} & \forall C \in \mathcal{C}(N) \setminus \{N\} \end{array}$$

- With $|N|$ real variables $x(i)$ and $2^{|N|} - 2$ *binary* variables $y(C)$,

$$\sum_{i \in C} x(i) \geq \begin{cases} v(C) & \text{when } y(C) = 1 \\ \sum_{i \in C} v(\{i\}) & \text{when } y(C) = 0 \end{cases}$$

Traditional Core and Credible Threats

- Following Ray (1989), let $\mathbb{X}^{+-}(\{i\}, v) = \{v\} \neq \emptyset$ for any single-player game $(\{i\}, v)$
- For any game (N, v) with $|N| \geq 2$, recursively define

$$\mathbb{X}^{+-}(N, v) \equiv \{x \in \partial(N, v) : \textstyle\sum_{i \in C} x(i) \geq v(C), \\ \forall C \in \mathcal{C}(N) \backslash \{N\} \text{ with } \mathbb{X}^{+-}(C, v|_C) \neq \emptyset\},$$

  where $v|_C$ stands for $(v(C'))_{C' \in \mathcal{C}(C)}$
- A sub-coalition C may be *threatening* when $v(C) > \sum_{i \in C} x(i)$; only one with $\mathbb{X}^{+-}(C, v|_C) \neq \emptyset$ poses a *credible* threat
- Not only $\mathbb{X}^{+}(N, v) \subseteq \mathbb{X}^{+-}(N, v)$, but *opposite* was shown by Ray (1989) to be true—*traditional-core* member may thus be understood as facing *no credible threat* in any partition attempt

Weak Core and Credible Threats

- Let $\mathbb{X}^{--}(\{i\}, v) = \{v\} \neq \emptyset$ for any single-player game $(\{i\}, v)$
- For any game (N, v) with $|N| \geq 2$, recursively define

$$\mathbb{X}^{--}(N, v) \equiv \{x \in \partial(N, v) : \text{ for any } \mathcal{P}' \in \mathscr{P}(N) \backslash \{\{N\}\}, \\ \exists C' \in \mathcal{P}' \text{ so that either } \mathbb{X}^{--}(C, v|_C) = \emptyset \\ \text{ or } \textstyle\sum_{i \in C'} x(i) \geq v(C')\}$$

- An imputation $x \in \mathbb{X}^{--}(N, v)$ when in any partition attempt, *not all* constituent coalitions pose *credible threats*; whereas, a coalition C's credibility is defined in *weak* sense of $\mathbb{X}^{--}(C, v|_C) \neq \emptyset$
- It is easy to tell $\mathbb{X}^{-}(N, v) \subseteq \mathbb{X}^{--}(N, v)$; we can prove *opposite*—when a *weak-core* member is pitted against a proper *partition*, there must exist *one constituent coalition* that *either* is sufficiently content with allocation associated with this member *or* is *not credible* for its *threat* to be taken seriously

Fission and Fusion Resistances

- A partition $\mathcal{P}$'s *fission-down-to* neighborhood $\mathscr{I}(N, \mathcal{P})$ contains all partitions $\mathcal{P}'$ that constitute *splits* of $\mathcal{P}$'s constituent coalitions
- A partition-allocation pair $(\mathcal{P}, x)$ in feasible set $\mathscr{Q}(N, v)$ is *strong fission-resistant* when for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\forall C' \in \mathcal{P}' \backslash \mathcal{P} \quad \text{we have} \quad \sum_{i \in C'} x(i) \geq v(C')$$

- With symmetrically defined *fusion-up-to* neighborhood $\mathscr{U}(N, \mathcal{P})$, a pair $(\mathcal{P}, x)$ is *fusion-resistant* when for any $\mathcal{P}' \in \mathscr{U}(N, \mathcal{P})$,

$$\forall C' \in \mathcal{P}' \backslash \mathcal{P} \quad \text{we have} \quad \sum_{i \in C'} x(i) \geq v(C'),$$

which is merely about $\sum_{C \in \mathcal{P},\, C \subsetneq C'} v(C) \geq v(C')$

More Stability Notions

- Medium fission resistance is about for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\sum_{i \in N} x(i) = \sum_{C \in \mathcal{P}} \sum_{i \in C} x(i) = \sum_{C \in \mathcal{P}} v(C) = \tilde{w}(N, v, \mathcal{P}) \geq \tilde{w}(N, v, \mathcal{P}')$$

- Weak fission resistance is about for any $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P})$,

$$\exists C' \in \mathcal{P}' \setminus \mathcal{P} \quad \text{so that} \quad \sum_{i \in C'} x(i) \geq v(C')$$

- Let $\mathbb{Q}^{\mathsf{i}*}(N, v)$ be set of all $(\mathcal{P}, x)$'s that are *fission*-resistant, with $* = +$ for *strong*, 0 for *medium*, and $-$ for *weak*; let $\mathbb{Q}^{\mathsf{u}}(N, v)$ be set of all solution pairs that are *fusion*-resistant

- Define *stability* concepts $\mathbb{S}^*(N, v) \equiv \mathbb{Q}^{\mathsf{i}*}(N, v) \cap \mathbb{Q}^{\mathsf{u}}(N, v)$

Other Stability-related Notions

- Each stability is corresponding-*core-compatible*:

$$\mathbb{S}^*(N,v) \cap [\{\{N\}\} \times \mathscr{X}(N,v,\{N\})] = \mathbb{Q}^{\mathsf{i}*}(N,v) \cap [\{\{N\}\} \times \mathscr{X}(N,v,\{N\})] = \{\{N\}\} \times \mathbb{X}^*(N,v)$$

  Of course, these sets could be simultaneously *empty*

- For **every** $v \in \Re^{\mathcal{C}(N)}$, no matter how "*poor*" it is,

$$\mathbb{S}^+(N,v) \subseteq \mathbb{S}^0(N,v) \subseteq \mathbb{S}^-(N,v) \qquad \text{and} \qquad \mathbb{S}^{\mathbf{0}}(\mathbf{N},\mathbf{v}) \neq \emptyset$$

- There is no *universality* guarantee for strong stability $\mathbb{S}^+$;
  still, it might allow stable solutions $(\mathcal{P},x)$ other than $\mathcal{P} = \{N\}$ and $x \in \mathbb{X}^+(N,v)$

Fission-related Constructs

- Given a partition $\mathcal{P} \in \mathscr{P}(N)$, let *patched-up core* be

$$\mathbb{X}^{\mathsf{i}*}(N, v, \mathcal{P}) \equiv \prod_{C \in \mathcal{P}} \mathbb{X}^{*}(C, v|_C)$$

- With $\mathbb{P}^{\mathsf{i}*}(N, v) \equiv \Big\{\mathcal{P} \in \mathscr{P}(N) : \ \mathbb{X}^{\mathsf{i}*}(N, v, \mathcal{P}) \neq \emptyset\Big\}$,

$$\mathbb{Q}^{\mathsf{i}*}(N, v) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathsf{i}*}(N,v)} \{\mathcal{P}\} \times \mathbb{X}^{\mathsf{i}*}(N, v, \mathcal{P})$$

- Earlier inclusion relationships among *cores* would lead to

$$\mathbb{Q}^{\mathsf{i}+}(N, v) \subseteq \mathbb{Q}^{\mathsf{i}0}(N, v) \subseteq \mathbb{Q}^{\mathsf{i}-}(N, v)$$

Some More Structures

- An alternative definition for $\mathbb{P}^{\mathsf{i}0}(N,v)$ turns out to be

$$\{\mathcal{P} \in \mathscr{P}(N) : \ \tilde{w}(N,v,\mathcal{P}) \geq \tilde{w}(N,v,\mathcal{P}'), \quad \forall \mathcal{P}' \in \mathscr{I}(N,\mathcal{P})\}$$

- For $\mathbb{P}^{\mathsf{u}}(N,v)$ defined *similarly* except with $\mathscr{U}(N,\mathcal{P})$ replacing $\mathscr{I}(N,\mathcal{P})$, it would follow that

$$\mathbb{Q}^{\mathsf{u}}(N,v) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathsf{u}}(N,v)} \{\mathcal{P}\} \times \mathscr{X}(N,v,\mathcal{P})$$

- What lead to *universality* are $\mathbb{P}^{\mathsf{i}0}(N,v) \cap \mathbb{P}^{\mathsf{u}}(N,v) \neq \emptyset$ and

$$\mathbb{S}^{*}(N,v) = \bigcup_{\mathcal{P} \in \mathbb{P}^{\mathsf{i}*}(N,v) \cap \mathbb{P}^{\mathsf{u}}(N,v)} \{\mathcal{P}\} \times \mathbb{X}^{\mathsf{i}*}(N,v,\mathcal{P})$$

Reasons behind Earlier Structures

- Very importantly, we can show

$$(\mathcal{P}, x) \in \mathbb{Q}^{\mathsf{i}*}(N, v) \iff (\{C\}, x|_C) \in \mathbb{Q}^{\mathsf{i}*}(C, v|_C),\ \forall C \in \mathcal{P}$$

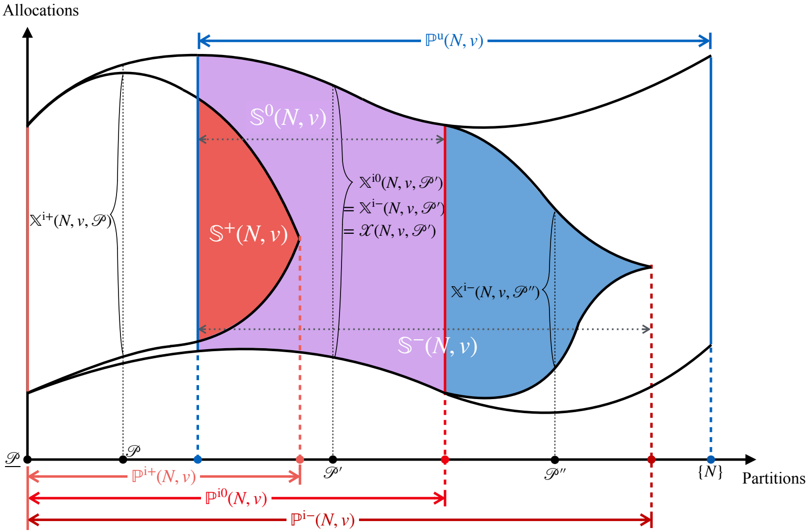
- With $(\{N\}, x) \in \mathbb{Q}^{\mathsf{i}*}(N, v) \iff x \in \mathbb{X}^*(N, v)$, this would result in

$$\begin{aligned}(\mathcal{P}, x) \in \mathbb{Q}^{\mathsf{i}*}(N, v) &\iff (\{C\}, x|_C) \in \mathbb{Q}^{\mathsf{i}*}(C, v|_C),\ \forall C \in \mathcal{P} \\ &\iff x|_C \in \mathbb{X}^*(C, v|_C),\ \forall C \in \mathcal{P} \\ &\iff x \in \mathbb{X}^{\mathsf{i}*}(N, v, \mathcal{P})\end{aligned}$$

- Concerning *fusion*, we can also establish

$$(\mathcal{P}, x) \in \mathbb{Q}^{\mathsf{u}}(N, v) \iff \mathcal{P} \in \mathbb{P}^{\mathsf{u}}(N, v)$$

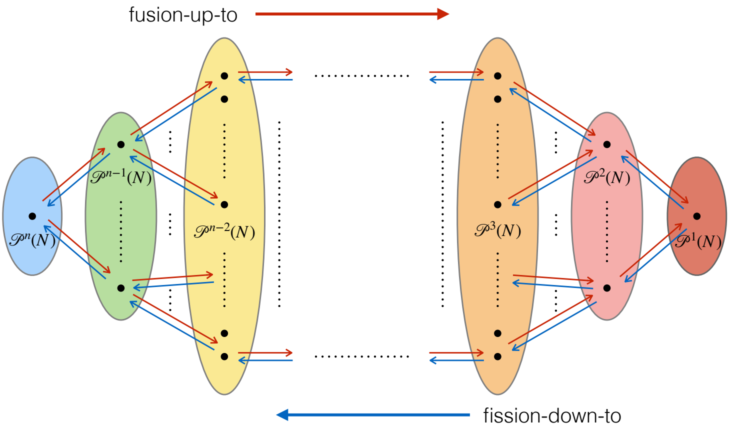
A Schematic Sketch of Various Entities



Road to Medium Stability

- Set $\mathscr{P}(N)$ can be decomposed into $\mathscr{P}^n(N)$, $\mathscr{P}^{n-1}(N)$, $\cdots$, $\mathscr{P}^1(N) \equiv \{\{N\}\}$ depending on member partitions' *sizes*
- One-step fission and fusion *arcs* link two neighboring $\mathscr{P}^{p+1}(N)$ and $\mathscr{P}^p(N)$, with $\mathscr{I}(N, \mathcal{P})$ understandable as "*left*" branch stemming from a given $\mathcal{P}$ and $\mathscr{U}(N, \mathcal{P})$ "*right*" branch
- A *steepest ascending method* (SAM) can help reach a *mediumly stable* pair $(\mathcal{P}^0, x^0)$ from *any* starting partition by incessantly moving from one $\mathcal{P}$ to a $\mathcal{P}' \in \mathscr{I}(N, \mathcal{P}) \cup \mathscr{U}(N, \mathcal{P})$ that *maximizes* $\tilde{w}(N, v, \cdot)$ until *no improvement* is possible
- After $\mathcal{P}^0$ is identified, x^0 can be any member of $\mathbb{X}^{\mathrm{i}0}(N, v, \mathcal{P}^0)$

A Graph Representation of Partitions



Core Stability in Literature

- If *fission-down-to* neighborhood in *strong fission* resistance or *fusion-up-to* one in *fusion* resistance were replaced by *space of all other partitions*, we would obtain *all-temptation resistance*
- This super-strong resistance seems to have propped up so-called "*core stability*" in coalition formation literature since Gale and Shapley (1962); see, e.g., Pycia (2012)
- Since *universality* is clearly out of the question, focus has been on identifying *conditions* that induce existence of *stable partitions* (coalition structures); see, e.g., Greenberg and Weber (1993), Banerjee, Konishi, and Sonmez (2001), Bogomolnaia and Jackson (2002), Papai (2004), and Alcalde and Romero-Medina (2006)
- Core stability is still (strong-)*core-compatible* by our standard

From Centripetality to Cooperation

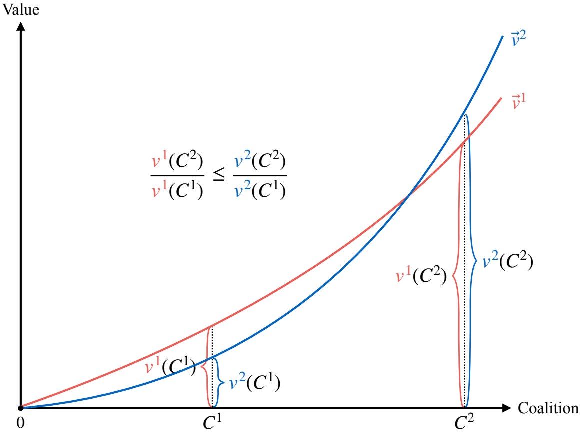
- When (N, v) is *strictly positive* with every $v(C) > 0$ except when $|C| = 1$ at which time only $v(C) \geq 0$ is required, we may take a *fractional* view on allocations with each $f(i) \equiv x(i)/v(C)$
- Earlier stability notions are *transplantable* here, after replacing $\mathscr{X}(N, v, \mathcal{P})$ with $\mathscr{F}(N, v, \mathcal{P})$, $\mathbb{X}^{\pm}(N, v)$ with $\mathbb{F}^{\pm}(N, v)$, and $\mathbb{X}^{\mathsf{i}\pm}(N, v, \mathcal{P})$ with $\mathbb{F}^{\mathsf{i}\pm}(N, v, \mathcal{P})$
- A *centripetality* partial order can be defined for games so that $(N, v^1) \leq_{\mathsf{cp}} (N, v^2)$ if and only if

$$\frac{v^1(C^2)}{v^1(C^1)} \leq \frac{v^2(C^2)}{v^2(C^1)}, \qquad \text{when } C^1 \subseteq C^2$$

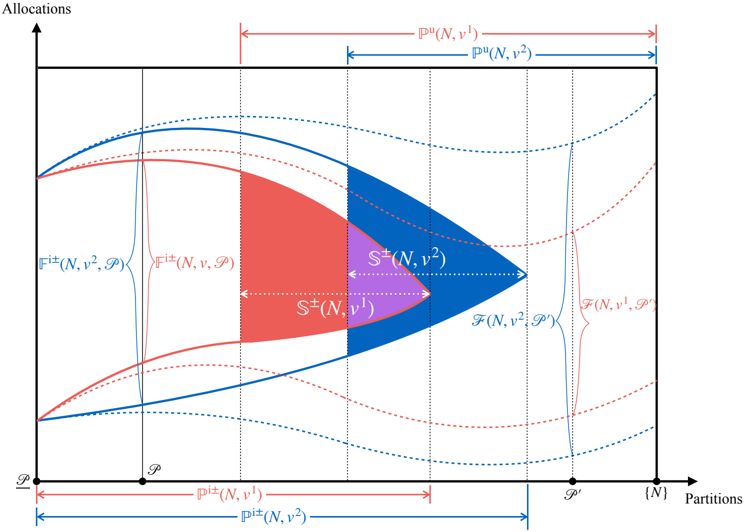
- Consequences turn out to be $\mathbb{F}^{\mathsf{i}\pm}(N, v^1, \mathcal{P}) \subseteq \mathbb{F}^{\mathsf{i}\pm}(N, v^2, \mathcal{P})$,

$$\mathbb{Q}^{\mathsf{i}\pm}(N, v^1) \subseteq \mathbb{Q}^{\mathsf{i}\pm}(N, v^2), \quad \text{and} \quad \mathbb{P}^{\mathsf{u}}(N, v^1) \supseteq \mathbb{P}^{\mathsf{u}}(N, v^2)$$

An Illustration of Centripetality



A Schematic Sketch of Consequences



From Risk Aversion to Centripetality

- With *centripetality* $\Longrightarrow$ *cooperation* at hand, we can further demonstrate *risk aversion* $\Longrightarrow$ *centripetality*
- Each coalition C is associated with a *random outcome* $\Phi(C)$
- All players share a common strictly-positive-valued *reward function* $\tilde{R}$ that is *positively homogeneous* in sense that

$$\tilde{R}(\rho \cdot Y) = \rho \cdot \tilde{R}(Y), \qquad \text{if } \rho \geq 0$$

- When $Y = \phi(C) \cdot \sum_{i \in C} \Theta_i$ with *i.i.d.* Θ_i's and

$$\tilde{R}(Y) \equiv \mathbb{E}[Y] - \bar{r} \cdot \sqrt{\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2},$$

$\bar{r}^1 \leq \bar{r}^2$ would lead to $(N, v^1) \leq_{\text{cp}} (N, v^2)$

Another Law-invariant Occasion

- In another *law-invariant* case, we characterize $\Phi(C)$'s by corresponding $\tilde{k}(C)$'s, where each $\tilde{k}(C)$ is *inverse* of $\Phi(C)$'s *cumulative distribution function*

- Let reward function $\tilde{r}$ operating on above *quantiles* be parameterized by a *probability density function* $\bar{\mu}$ on $[0,1]$ so that

$$\tilde{r}(k) \equiv \int_0^1 \bar{a}(k,\alpha) \cdot \bar{\mu}(\alpha) \cdot d\alpha,$$

where $\bar{a}(\cdot,\alpha)$ is α-level *conditional value at risk* defined as in

$$\bar{a}(k,\alpha) \equiv \frac{1}{1-\alpha} \cdot \int_0^{1-\alpha} k(\beta) \cdot d\beta$$

Risk Aversion Promotes Cooperation

- For case above, $\bar{\mu}^1 \leq_{\mathsf{lr}} \bar{\mu}^2$ would lead to $(N, v^1) \leq_{\mathsf{cp}} (N, v^2)$ under mild conditions on *quantile functions* $\tilde{k}(C)$
- Thus, for both cases, we can show that *risk aversion* promotes resulting coalitional game's *centripetality*
- This link, when combined with already-established link about *centripetality* promoting *cooperation*, would deliver on message

risk aversion promotes cooperation

Concluding Remarks

- Using *partition-allocation* pairs and *weak core* based on *consensual blocking*, as well as certain middle grounds, we have identified *stability notions* that are *universal*
- We wonder if there are *universal* stability concepts that are still compatible with plain *core*
- Concerning link from *risk aversion* to *cooperation*, more on causes of *centripetality* would be welcome

Thank you! Comments and suggestions?