Network Applications of Discrete Curvature and Ricci Flow

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Wireless Networks

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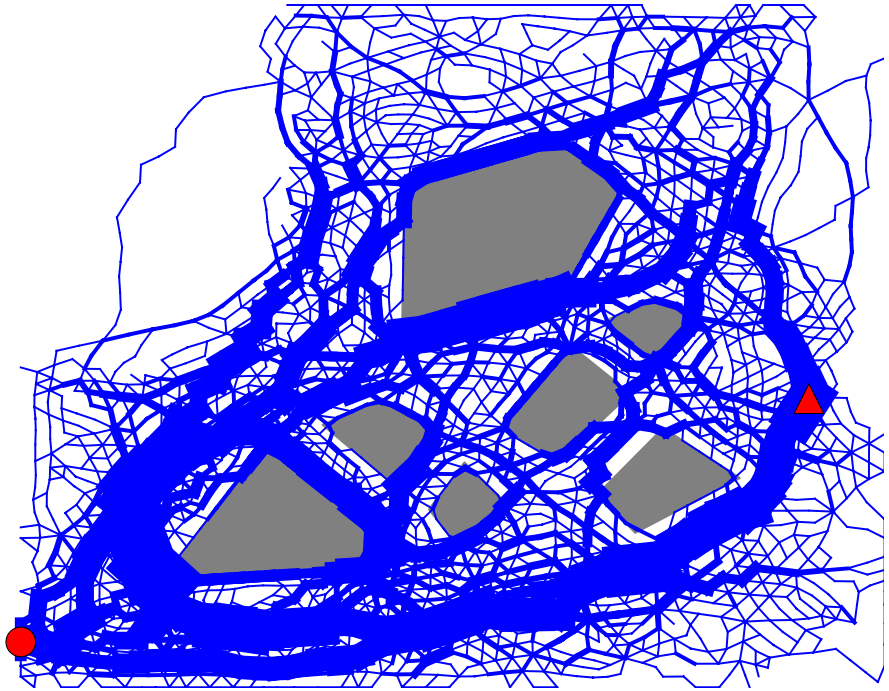

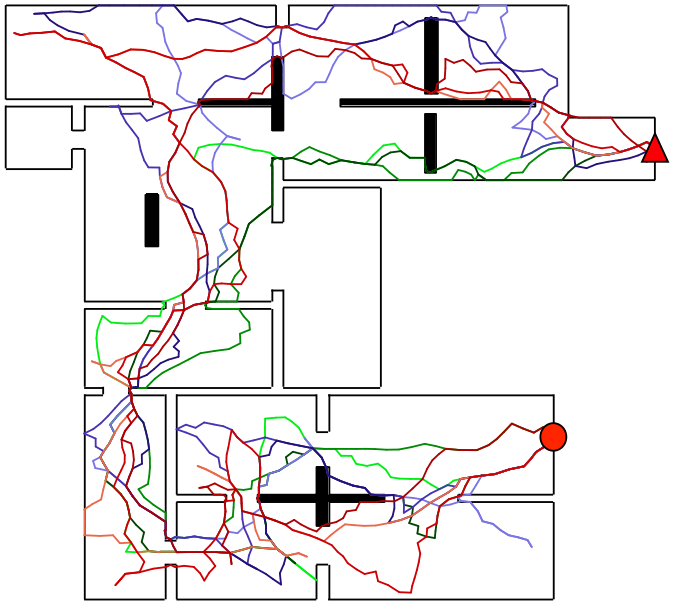

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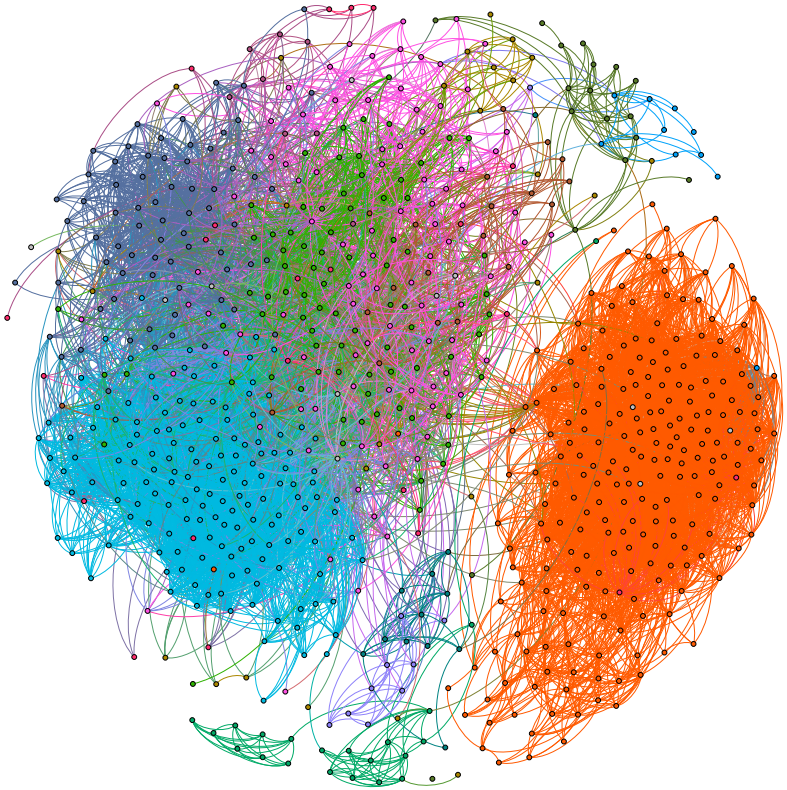

Outline

- Past work on wireless networks: Ricci flow on triangulations.

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- Recent work on Internet & Social networks: discrete graph curvatures.

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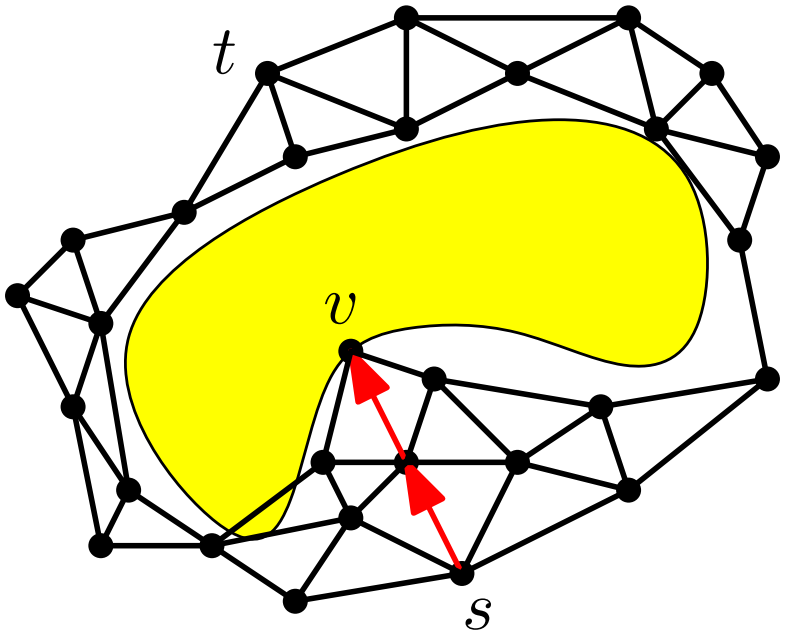
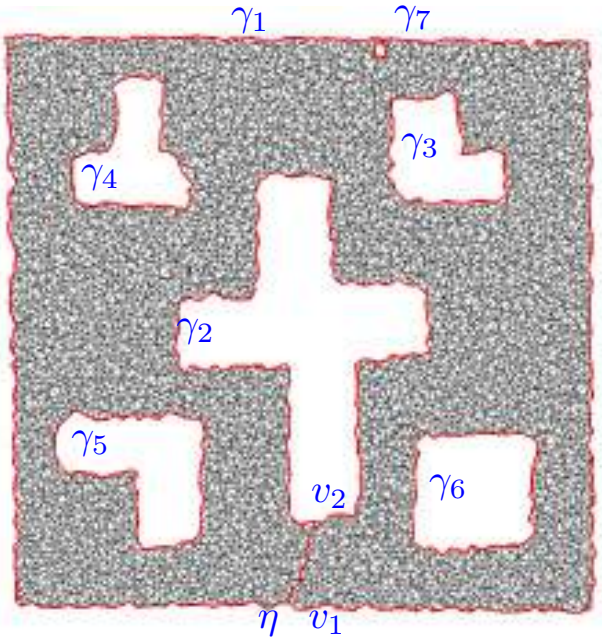
Geographical Greedy Routing

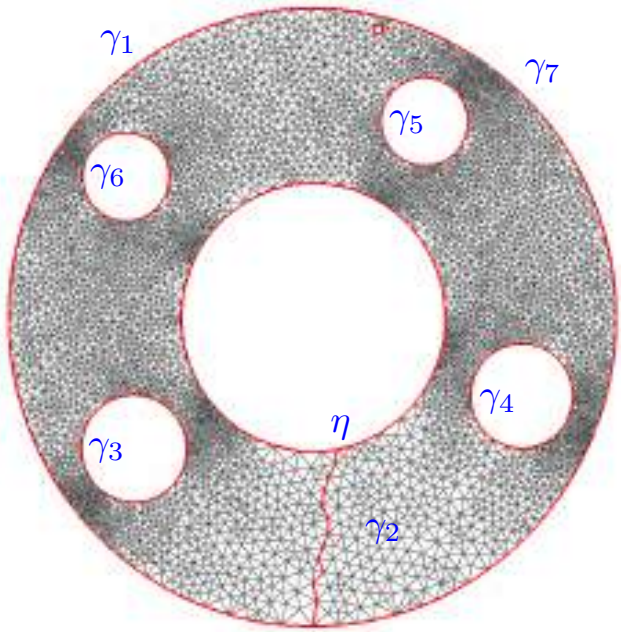
A message is sent to the neighbor closest to the destination.



Greedy Routing with Virtual Coordinates

Mapping the network into a 'circular domain' (all boundaries are circles). $\rightarrow$ Greedy routing has guaranteed delivery.



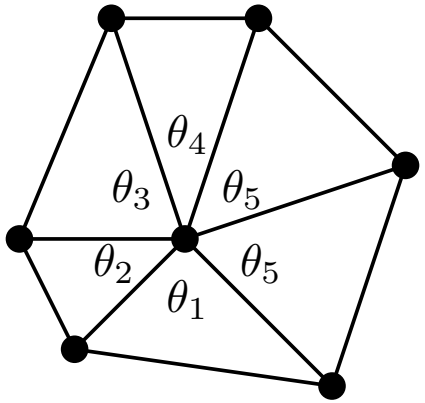


[Sarkar, Yin, G, Luo, Gu 2009]

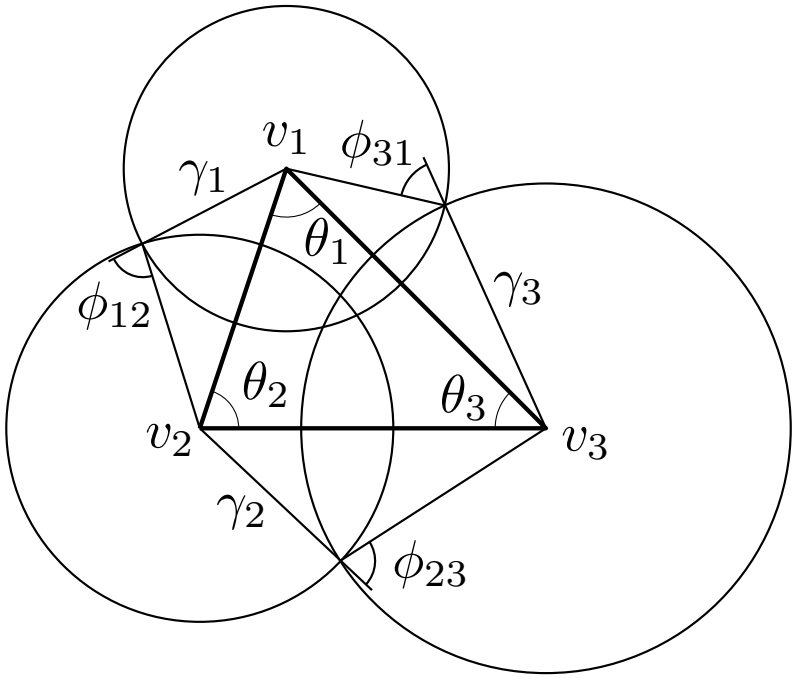
Discrete Gaussian Curvature on a Triangulation

Discrete Gaussian Curvature defined by

$$2\pi - \sum \theta_i$$



Circle Packing Metric



Ricci flow: change the circle radius to meet the target curvature (zero in the interior, and $2\pi/K$ on a hole with K nodes).

Circular Domain is Not Unique

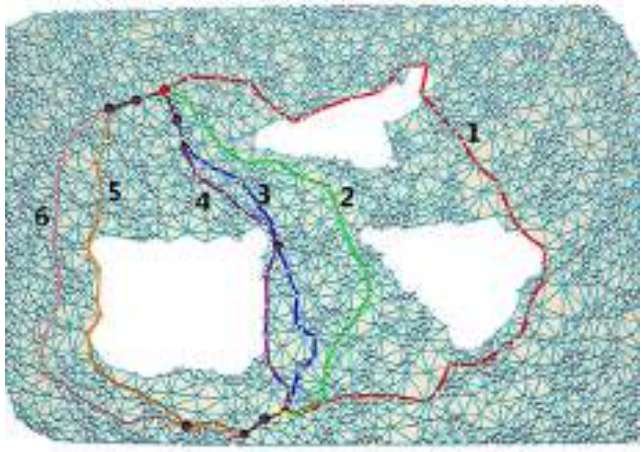
Apply a mobius transformation (which maps circles to circles).

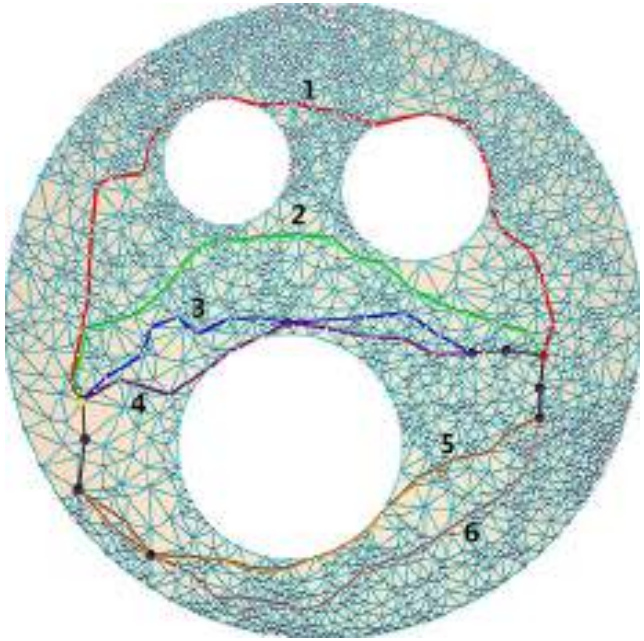
$$m(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Multi-Path Routing

Run greedy routing with different mobius transformation $\rightarrow$ a different path.

- Rapid response to failures.
- Multi-path routing with higher throughput.

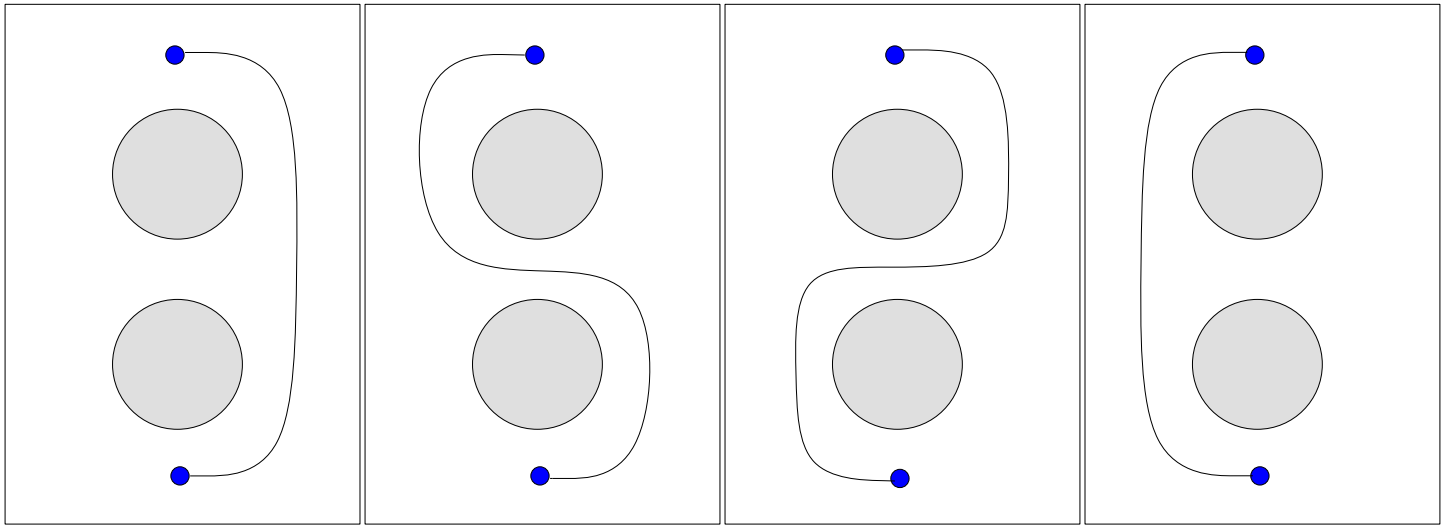




[Jiang, Ban, Goswami, Zeng, G, Gu 2011]

Homotopic Routing

Can we find a path with the specified homotopic type?



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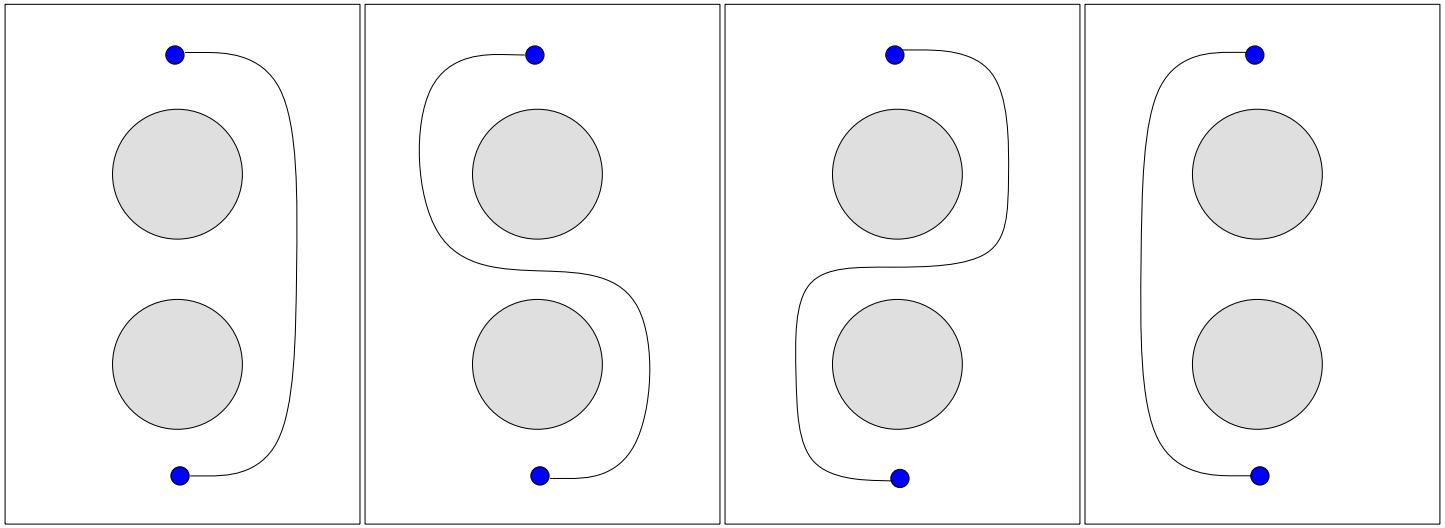
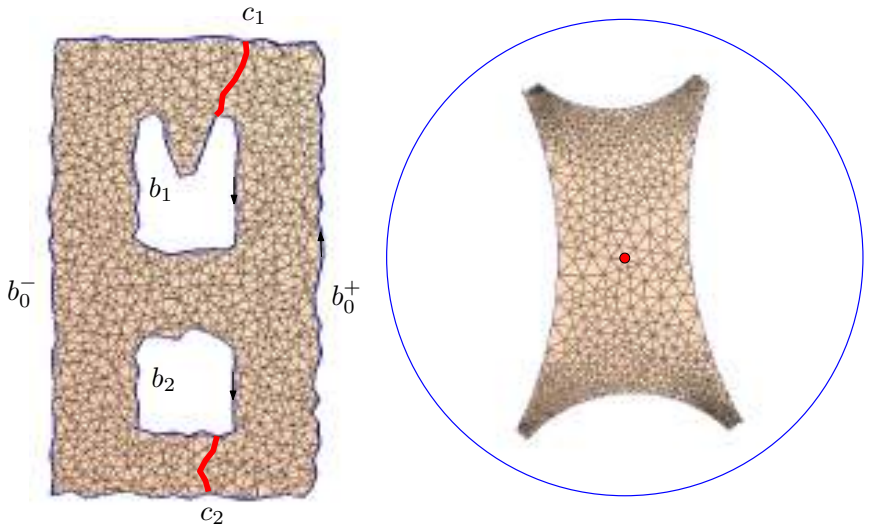


Our idea: use universal covering space and hyperbolic embedding.

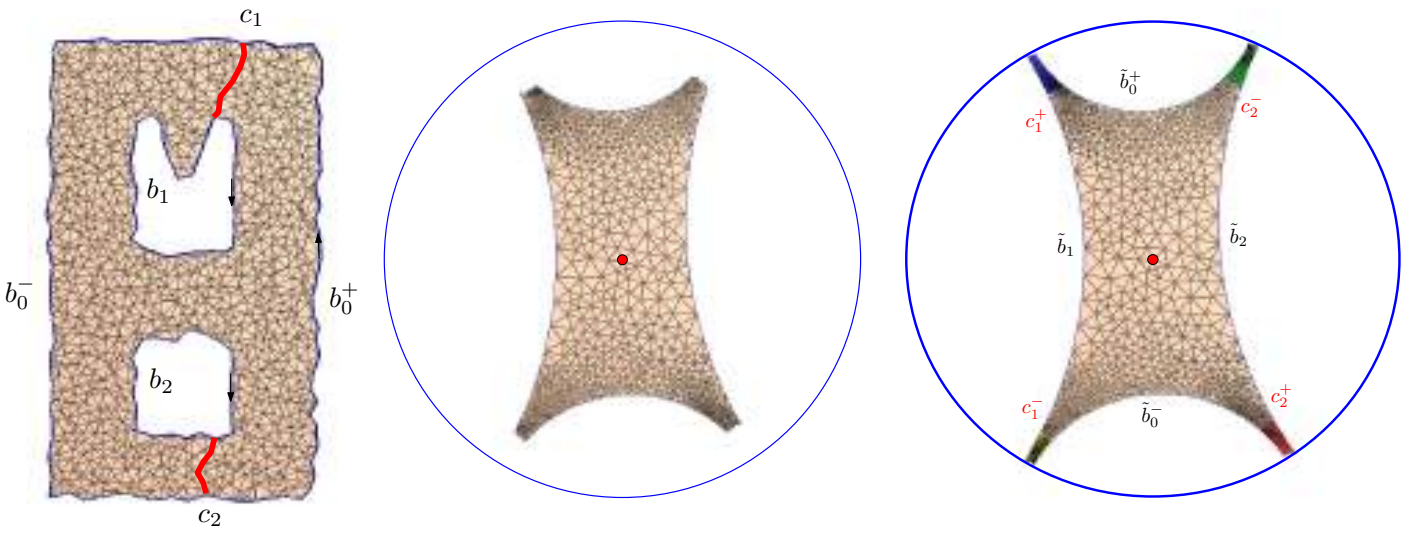
Homotopic Routing

- Cut a multi-hole domain open, embed it to a convex polygon in the hyperbolic plane.
- Greedy routing (using hyperbolic distance) guarantees delivery.



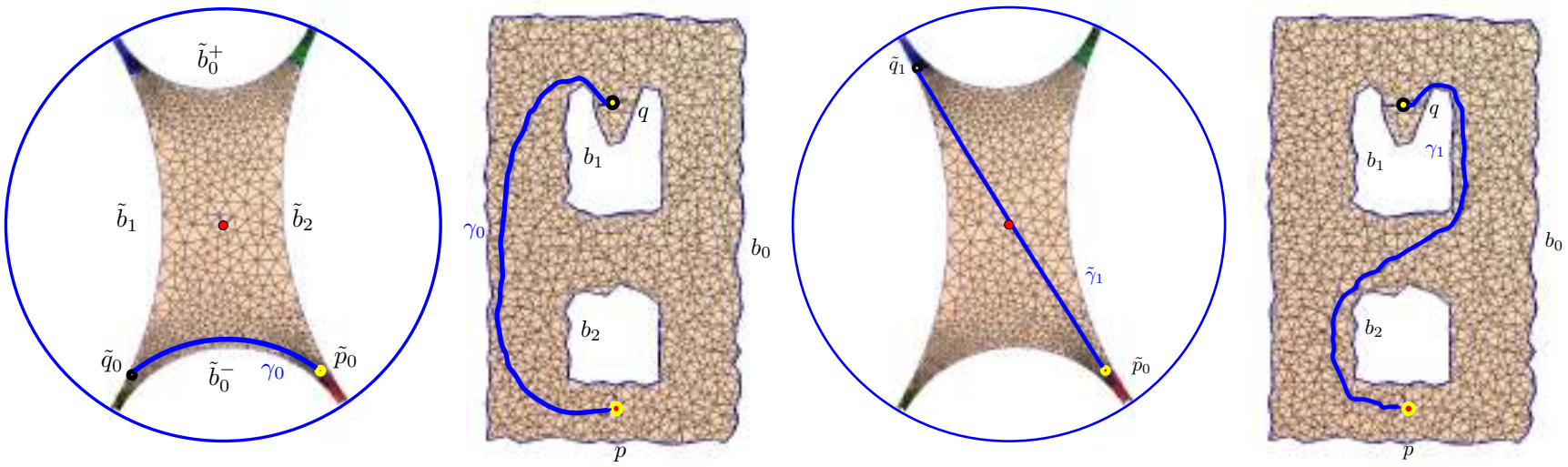
Homotopic Routing

- Use Mobius transformation to embed copies that could fill up the domain.



Homotopic Routing

- Greedy routing to the destination in different copies gives routes of different homotopy types.



[Zeng, Sarkar, Luo, Gu, G 2010]

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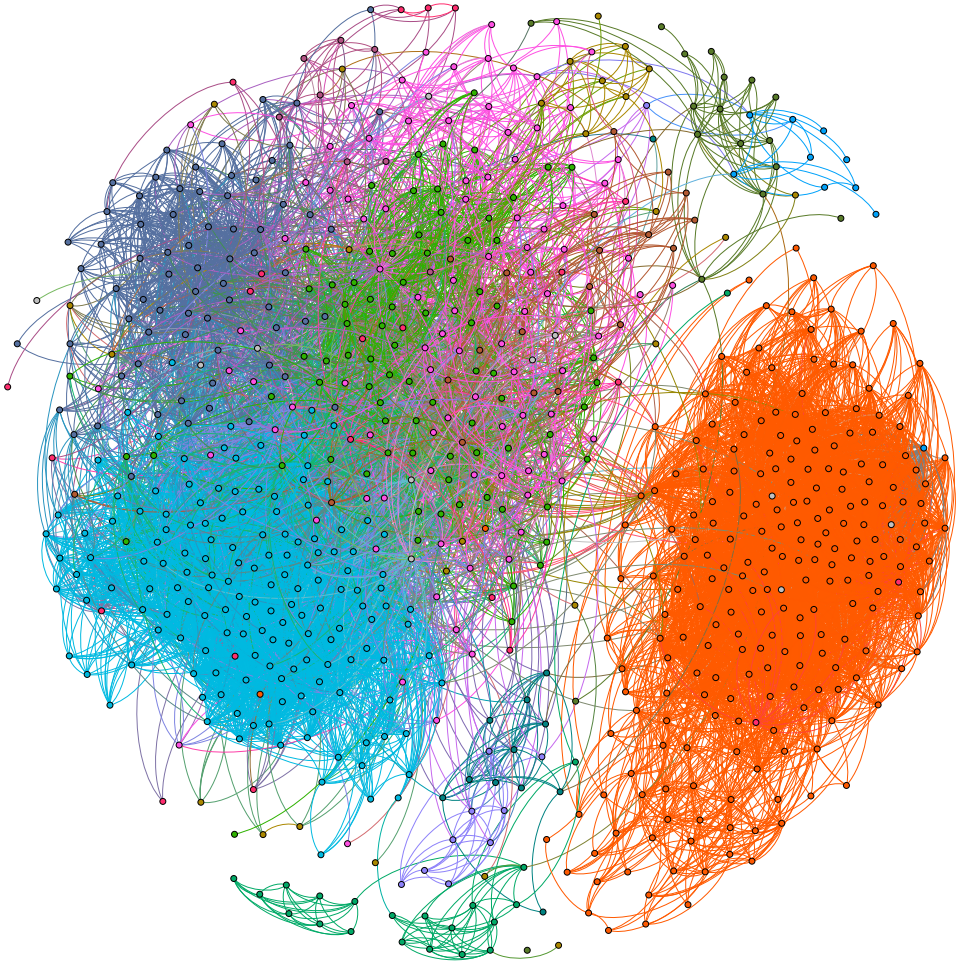
Until we are tired of this setting..

What about other graphs?



Coarse Geometry: Gromov's Curvature

It has been discovered (Baryshnikov, Narayan & Saniee) that the Internet topology has **negative curvature**, defined by Gromov's **thin triangle property**.

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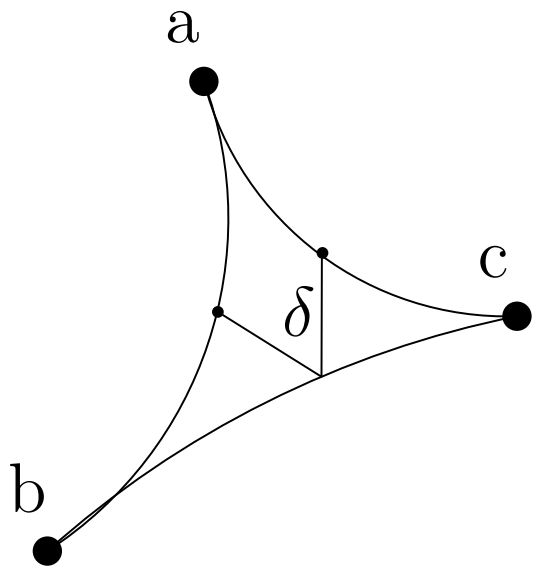

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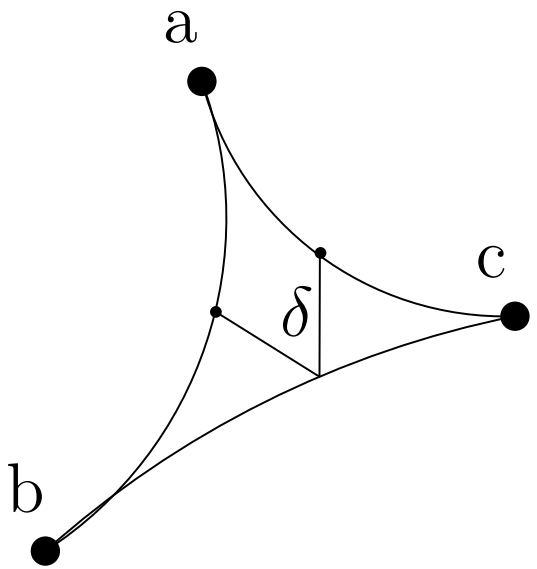

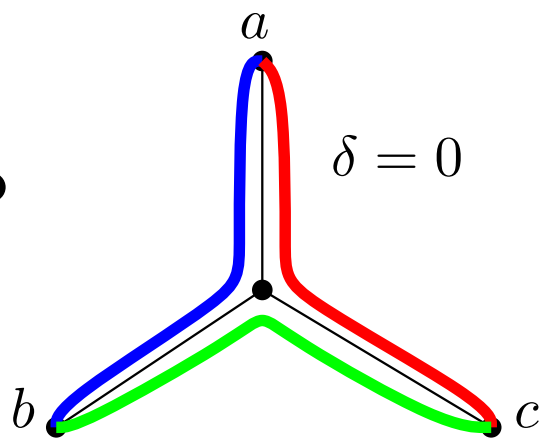

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A variety of data sets, both AS-level and router level topologies, show δ-hyperbolicity for small constant δ (2 or 3).

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- There is a 'core' in which all shortest path visit.
- Congestion inside the core is high;
- But one can use preprocessing to speed up shortest path queries.

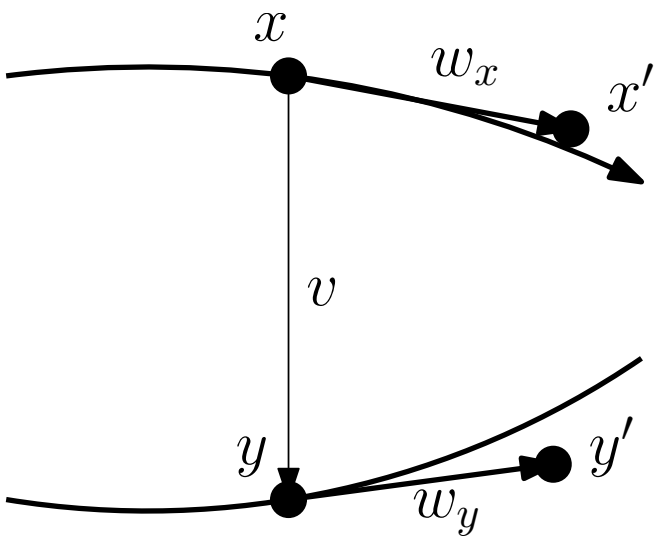
Global v.s. Local Curvature

Gromov hyperbolicity is a *global* measure. Here we examine *local* curvatures:

- Which edges are negatively curved?
- Local curvature and network congestion?
- Local curvature and node centrality?

Sectional Curvature in Geometry

Consider a tangent vector $v = xy$ and another tangent vector w_x at x. Transport w_x along v to be a tangent vector w_y at y.
If $|x'y'| < |xy|$, then sectional curvature is positive.



Ricci curvature: averaging over all directions w.

Discrete Ricci Curvature

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- Issue: how to match x's neighbors to y's neighbors?
- Assign uniform distribution μ_1, μ_2 on x' and y's neighbors.
- Use optimal transportation distance (earth-mover distance) from μ_1 to μ_2: the matching that minimizes the total transport distance.

Discrete Ricci Curvature

Definition (Ollivier, Lin & Yau)

Let (X, d) be a metric space and let m_1, m_2 be two probability measures on X. For any two distinct points $x, y \in X$, the (Ollivier-) Ricci curvature along xy is defined as

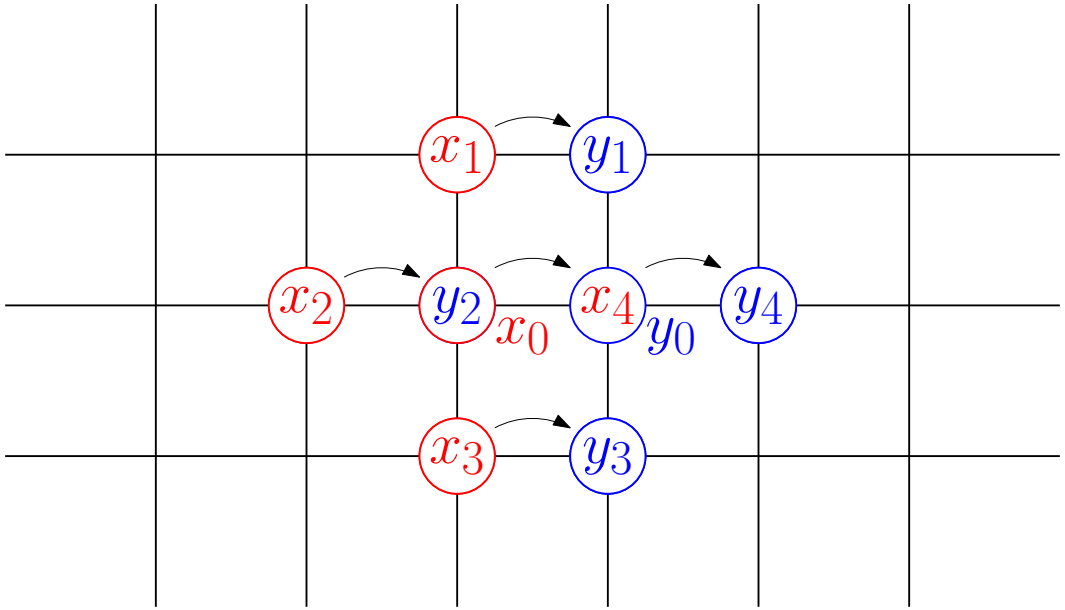
$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)},$$

where m_x (m_y) is a probability distribution defined on x (y) and its neighbors, $W_1(\mu_1, \mu_2)$ is the L_1 **optimal transportation distance** between two probability measure μ_1 and μ_2 on X:

$$W_1(\mu_1, \mu_2) := \inf_{\psi \in \Pi(\mu_1, \mu_2)} \int_{(u,v)} d(u, v) d\psi(u, v)$$

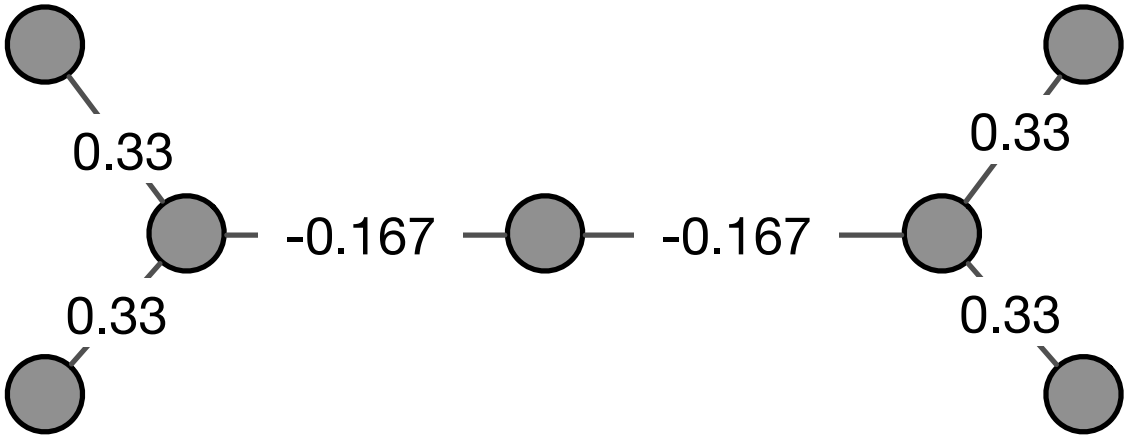
Examples

Zero curvature: 2D grid.



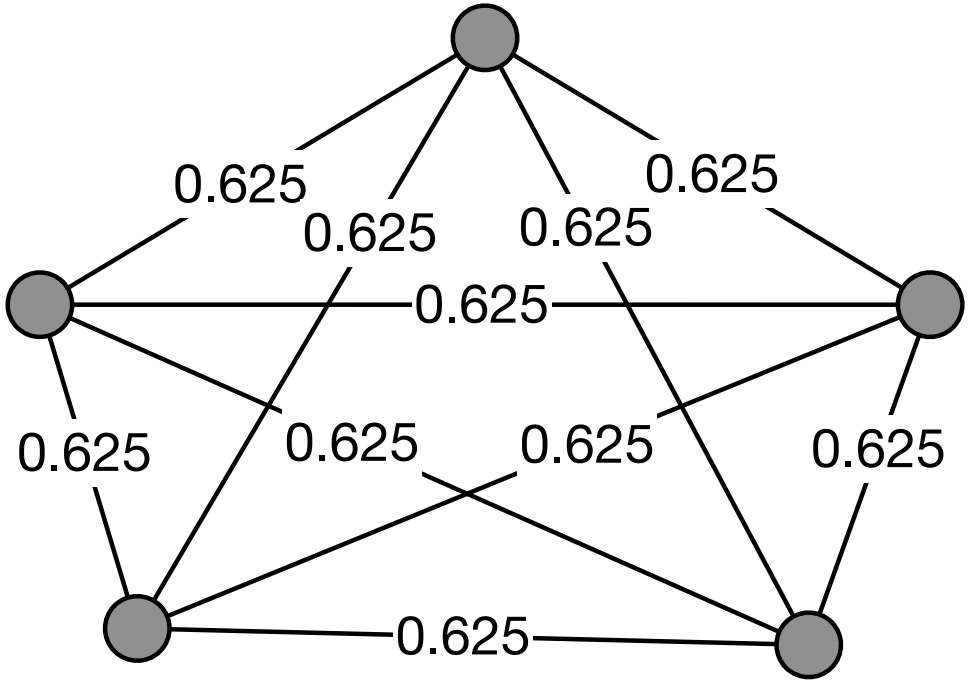
Examples

Negative curvature: tree: $\kappa(x, y) = 1/d_x + 1/d_y - 1$, d_x is degree of x.



Examples

Positive curvature: complete graph.



Data Sets

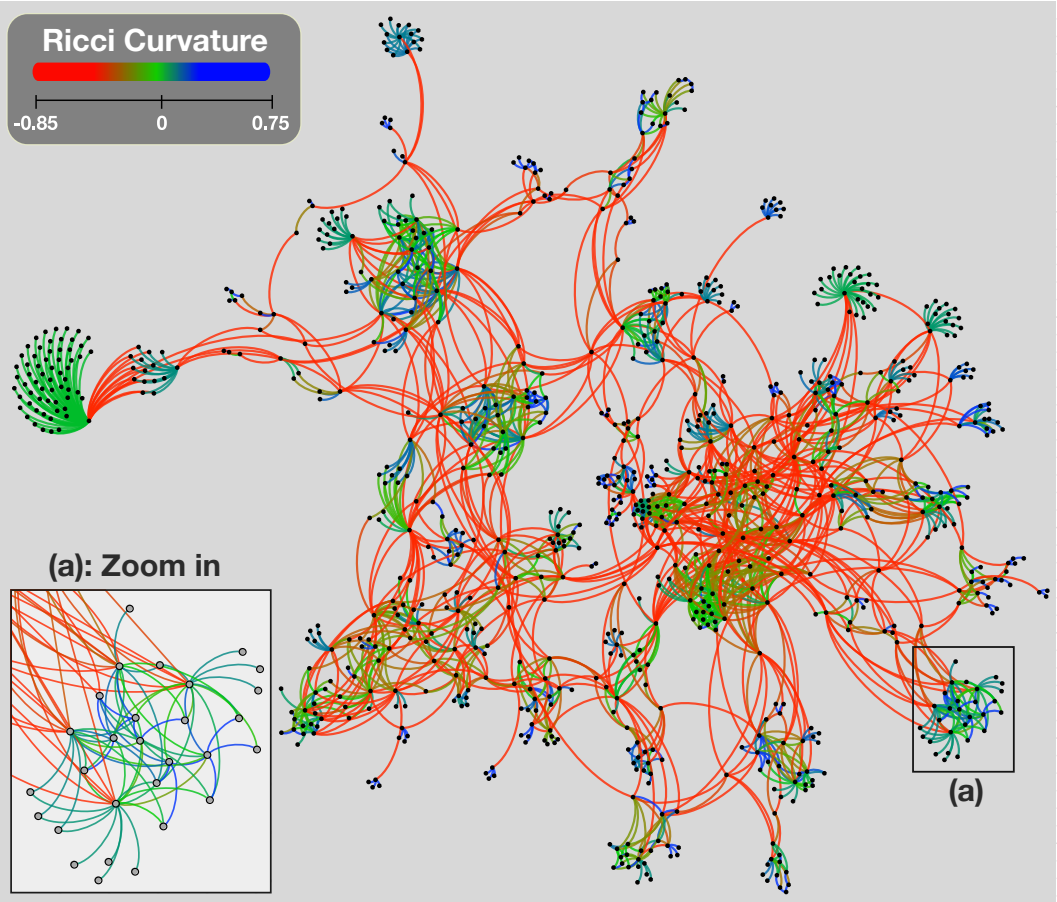
Data Set	Node	Edge	MaxDeg	AvgDeg	Diam	Mean SPL
RouteView	6474	12572	1458	3.88	9	3.71
Gnutella	6301	20777	97	6.59	9	4.64
PGP	10680	24340	205	4.56	24	7.48
Power grid	4941	6594	19	2.67	46	18.99

RocketFuel data set: router level network.

RochetFuel	Node	Edge	MaxDeg	AvgDeg	Diam	Mean SPL
AS: 7018	10145	14166	78	2.04	11	6.95
AS: 3257	843	1156	90	2.23	14	5.27
AS: 3967	895	2071	75	4.63	13	5.94
AS: 1221	2998	3789	106	2.53	12	5.53

Curvature Distribution

Negatively curved edges are like "backbones", maintaining the connectivity of clusters, in which edges are mostly positively curved.



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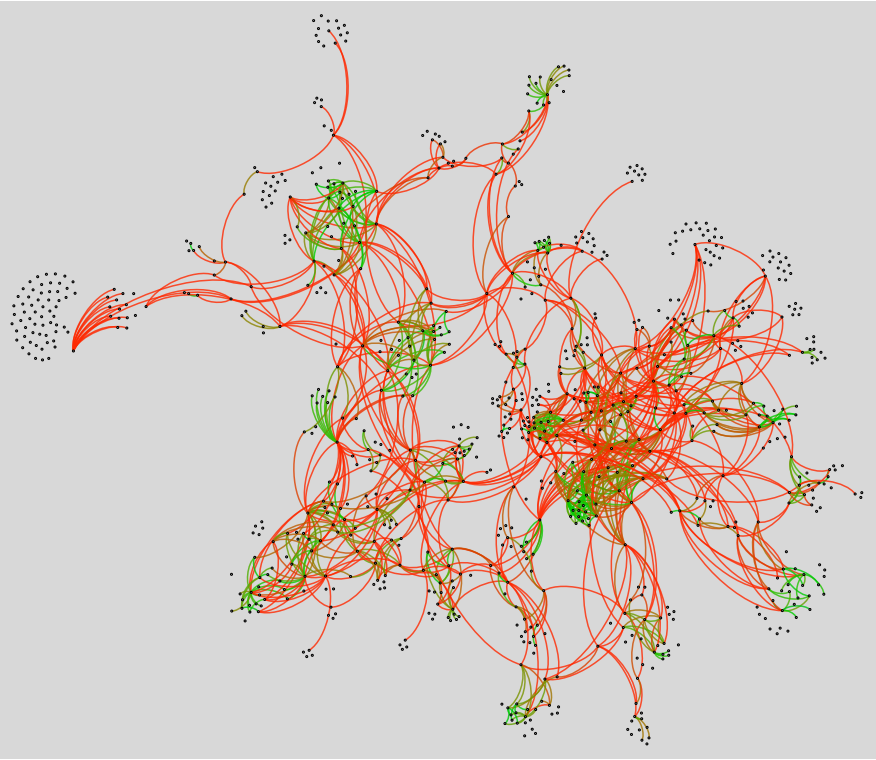

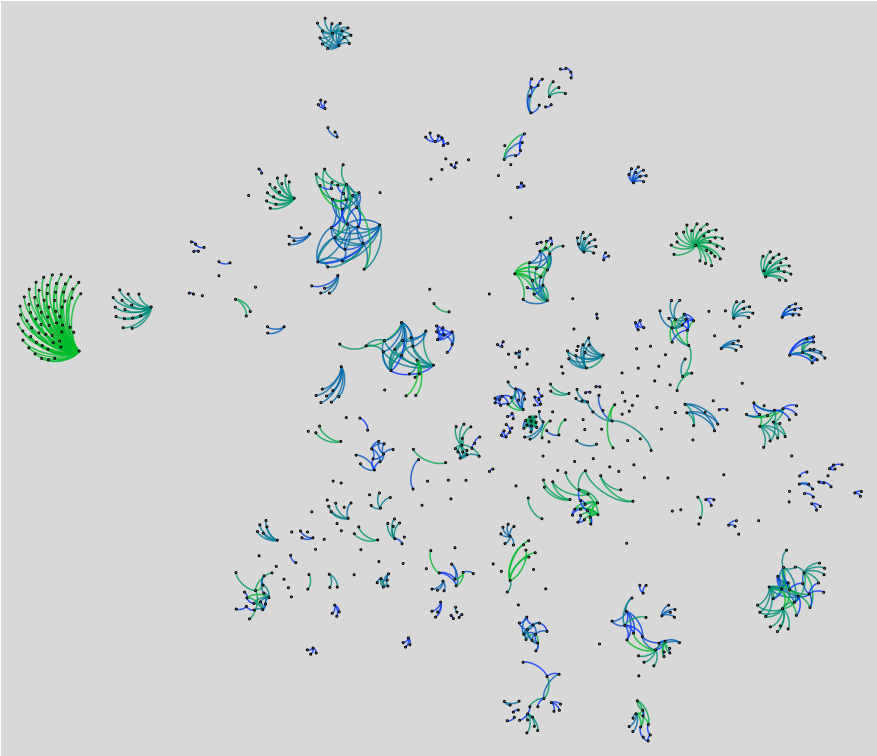

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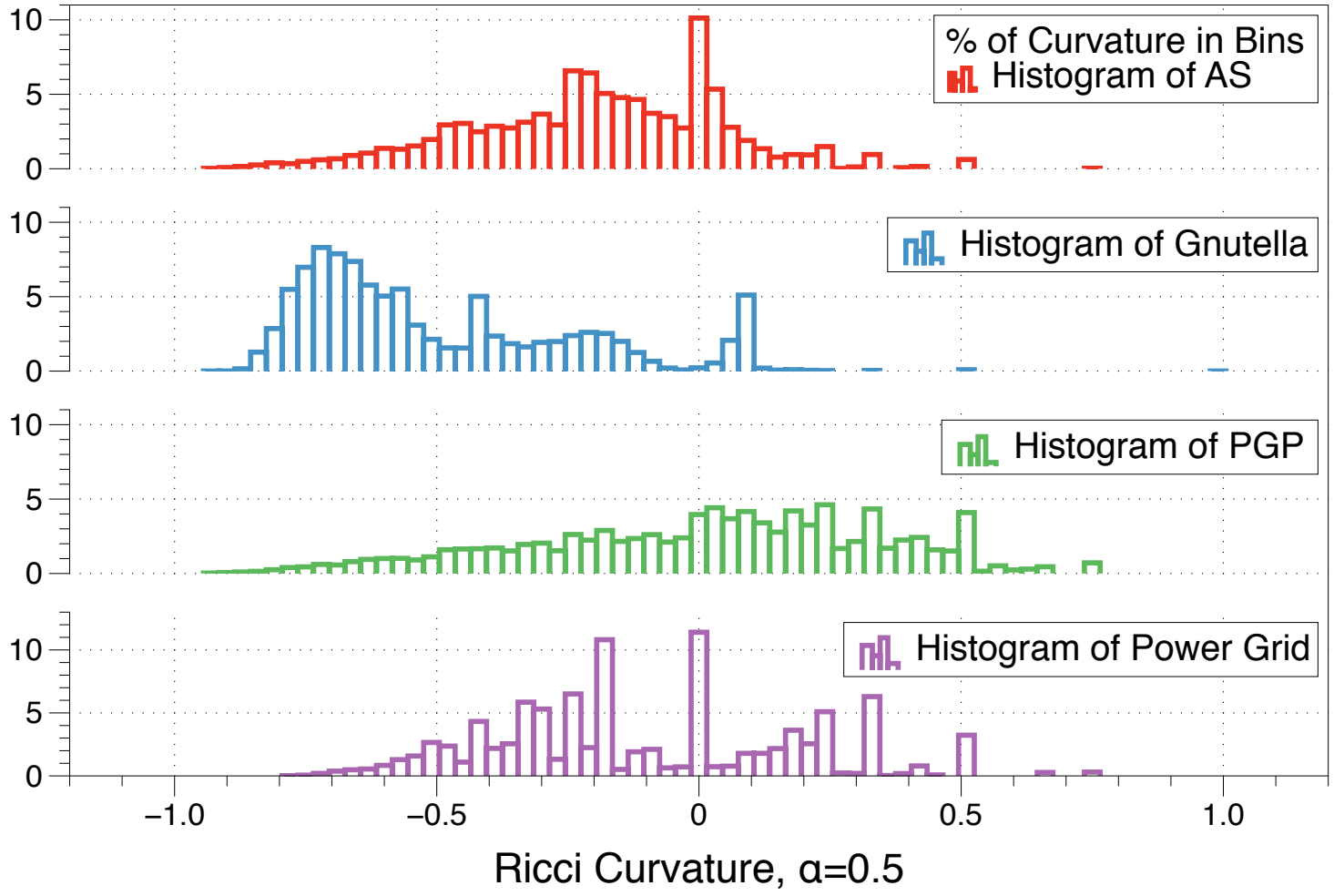
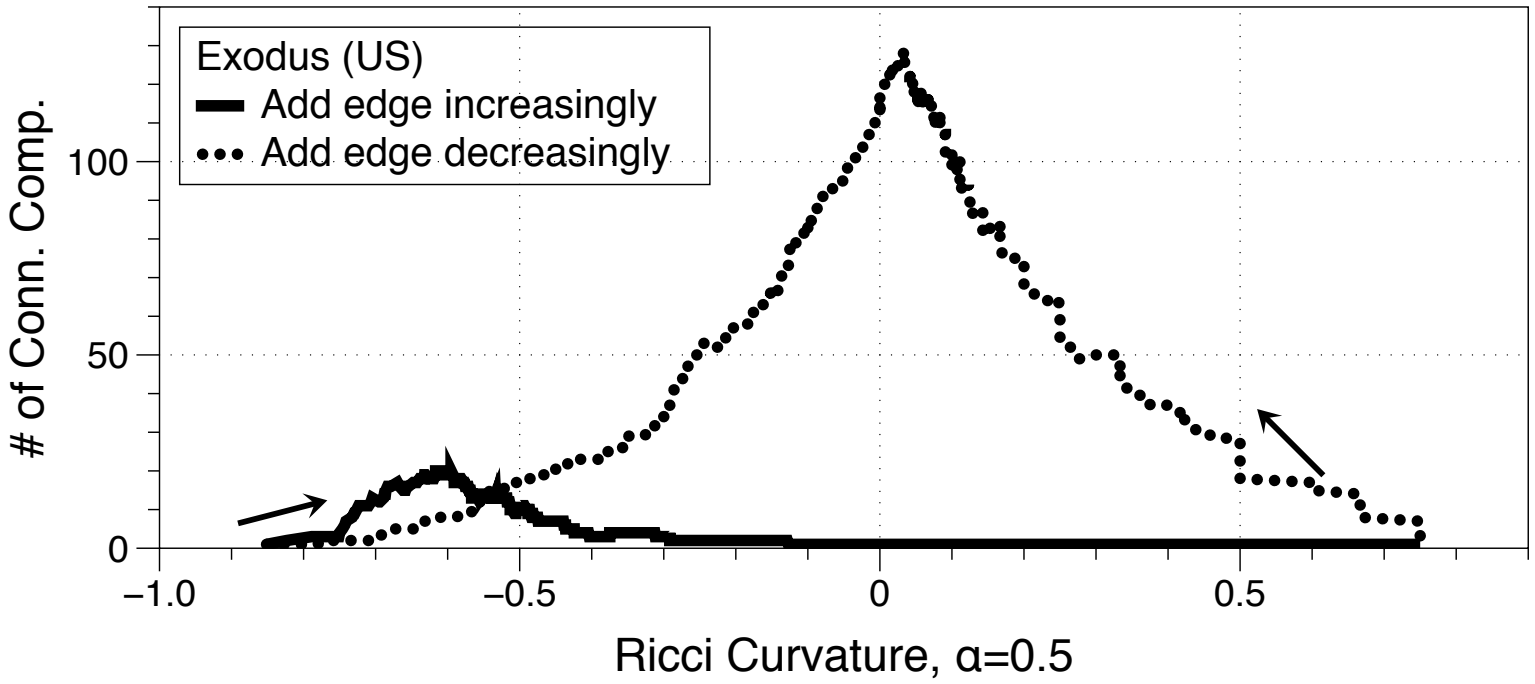


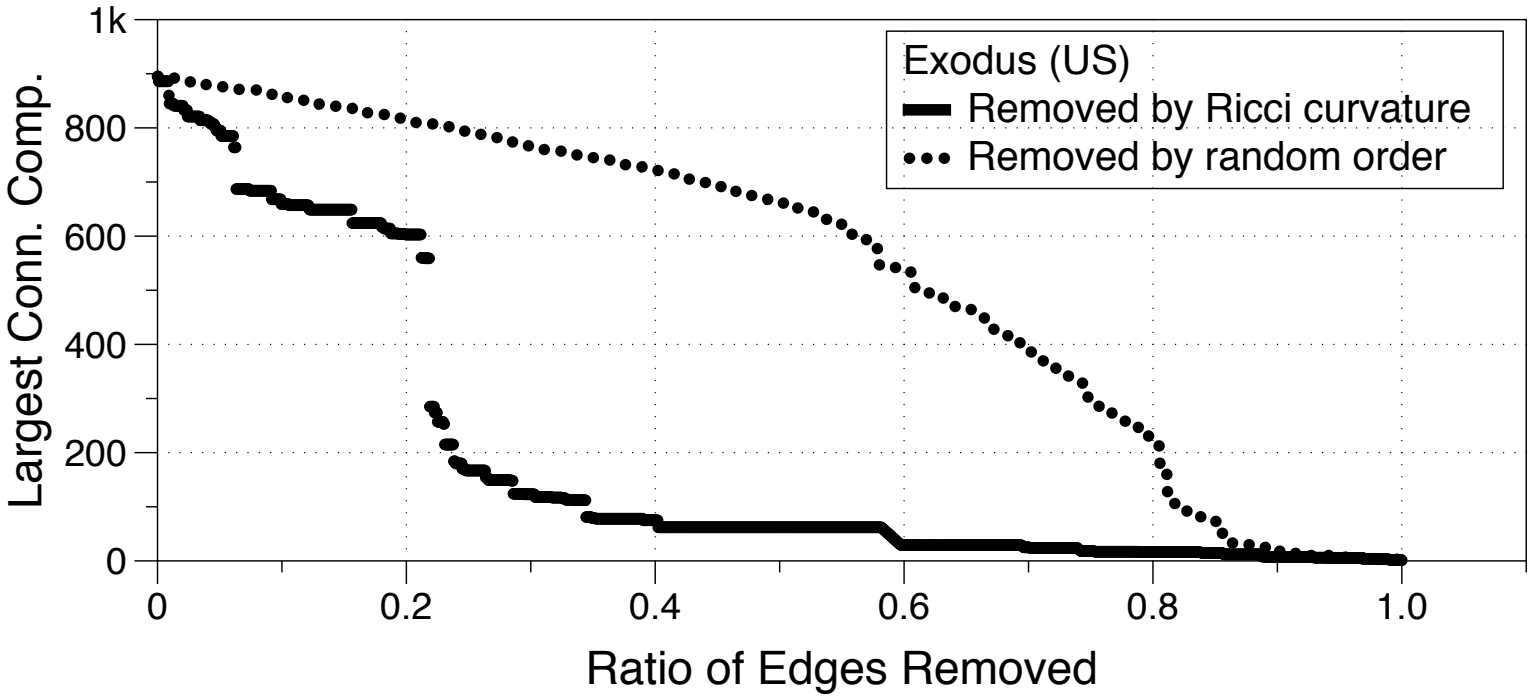
Network Connectivity

Negatively curved edges are well connected. Adding edges with increasing/decreasing curvature: few/many connected components.



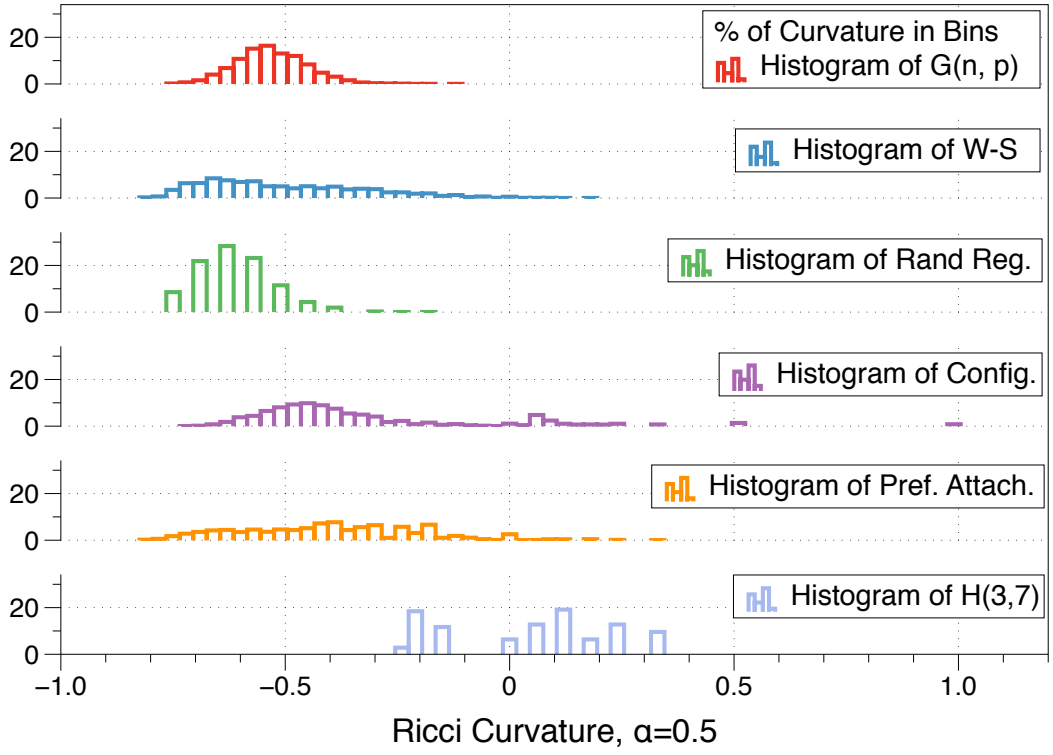
Robustness v.s. Vulnerabilty

Removing edges with increasing curvature: size of largest connected component drops quickly.



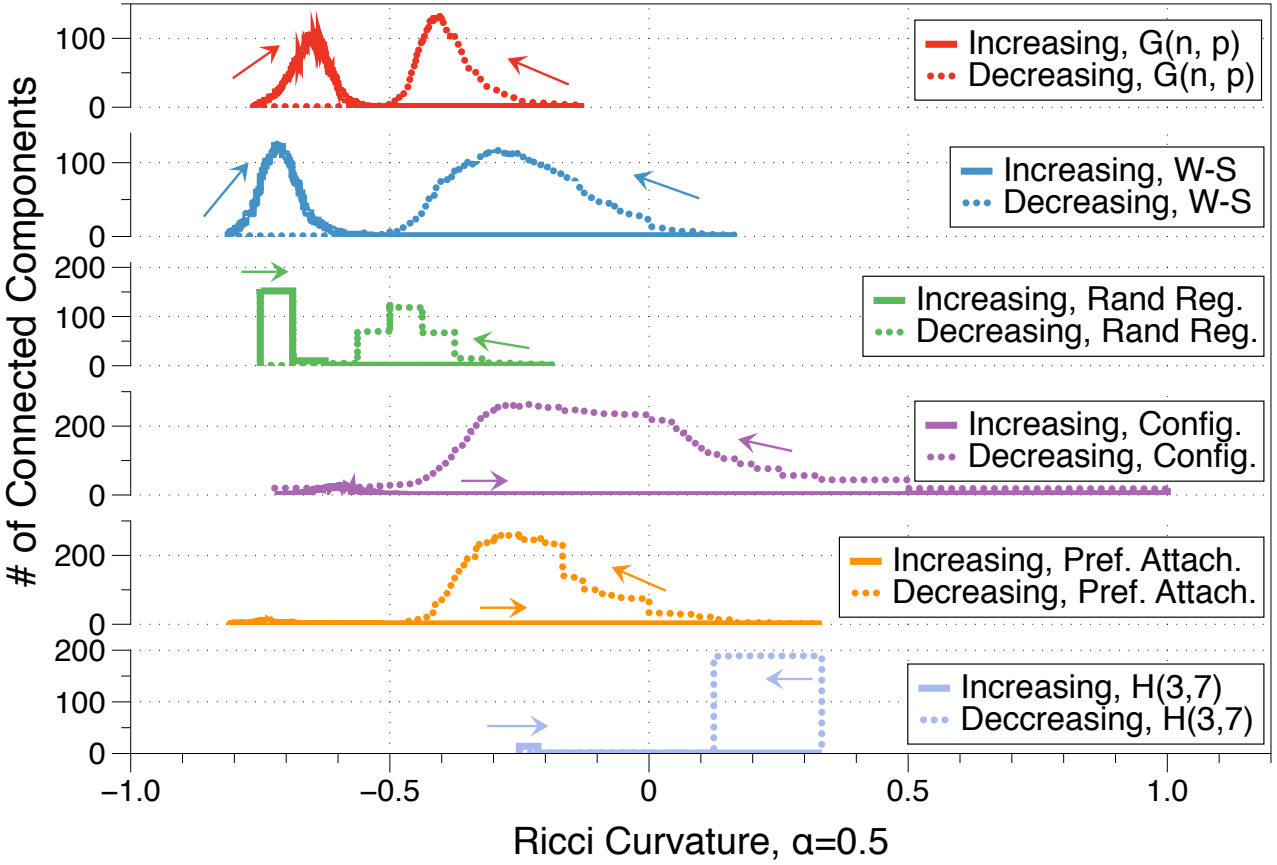
Model Networks

Erdos Renyi Graph, Watts Strogatz Graph, Random Regular Graph, Configuration Model, Preferential Attachment Model, Hyperbolic Grid $H(3,7)$



Network Connectivity Behavior

Small world property is not relevant; graph hyperbolicity and power law degree distribution appear to be more relevant.



Graph Isomorphism

Given a pair of graphs G_1, G_2, find a one-to-one correspondence of the vertices in G_1 to vertices in G_2 such that (u, v) is an edge in G_1 if and only if their corresponding nodes $f(u), f(v)$ are connected in G_2.

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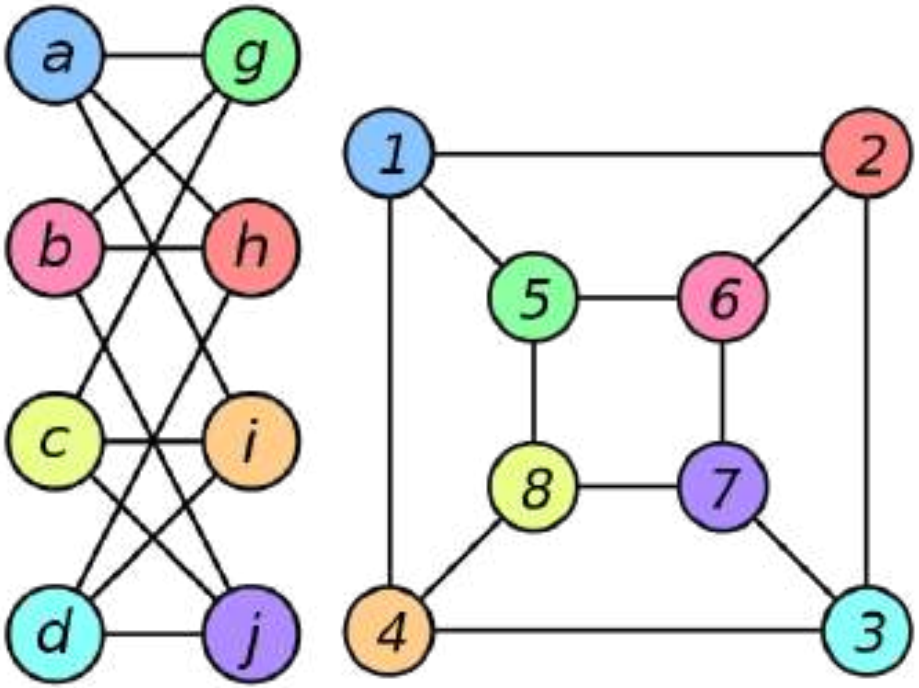

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- Subgraph isomorphism is NP-complete.
- Approximate graph isomorphism: find the best correspondence between vertices in G_1 and G_2.

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- Match two social networks on the same population – de-anonymization.

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Our Idea: Edge Weights Generated by Ricci flow

Given a graph G in which $d(x, y)$ is the weight of the edge xy and $\kappa(x, y)$ is the discrete Ricci curvature, we run
/* Ricci flow

$$d_{i+1}(x, y) = d_i(x, y) - \varepsilon \cdot \kappa_i(x, y) \cdot d_i(x, y)$$

/* Normalization

$$d_i(x, y) \leftarrow d_i(x, y) \cdot \frac{|E|}{\sum_{\overline{xy} \in E} d_i(x, y)}$$

Until convergence.

Our Idea: Edge Weights Generated by Ricci flow

Given a graph G in which $d(x, y)$ is the weight of the edge xy and $\kappa(x, y)$ is the discrete Ricci curvature, we run
/* Ricci flow

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Until convergence.
Ricci Flow Metric: Shortest path metric with edge weights computed above.

[Ni, Lin, G, Gu under submission]

Ricci Flow Metric

$$(d(x,y) - \kappa(x,y) \cdot d(x,y)) \cdot N \approx d(x,y)$$

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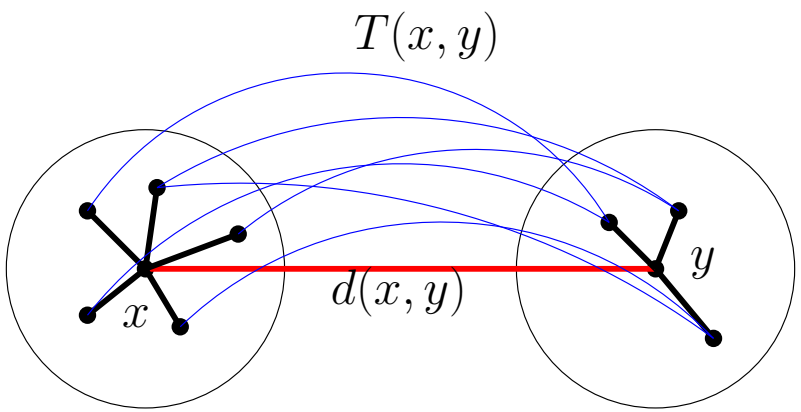
$$\frac{T(x,y)}{d(x,y)} \approx \frac{1}{N}.$$



If (x,y) is removed, there are likely alternative paths from x to y that give similar distance. $\rightarrow$ Much better robustness under edge insertion/deletion.

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- Identify k landmarks $\ell_1, \cdots, \ell_k$ on G_1 and their correspondences in G_2.

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Evaluations

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- Spring embedding;
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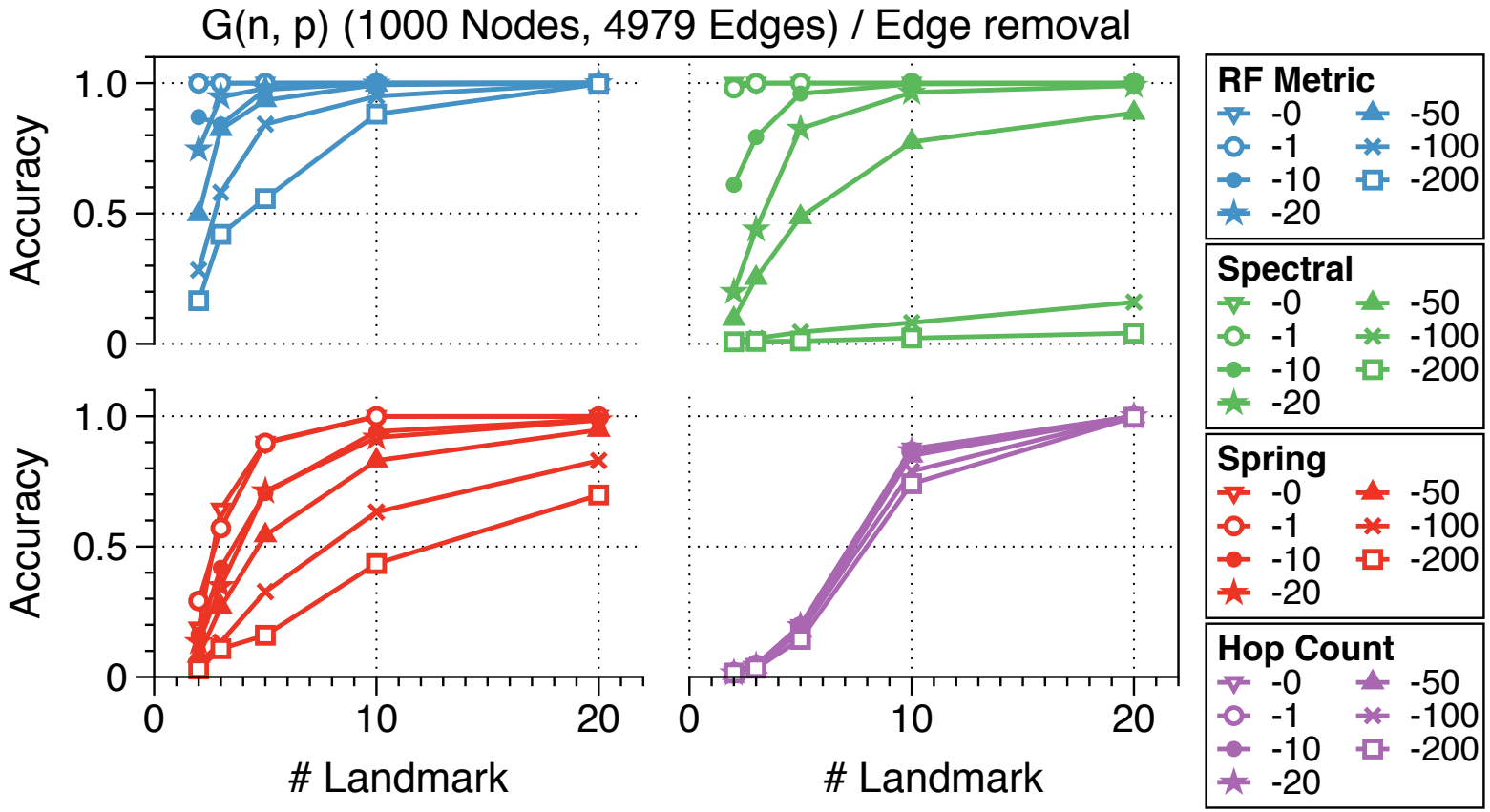
- Model graphs: random regular graphs, Kleinberg's small world graphs, Erdos-Renyi graphs, preferential attachent graphs.
- Internet AS graphs.
- Email graphs.
- Protein-protein interaction graph.

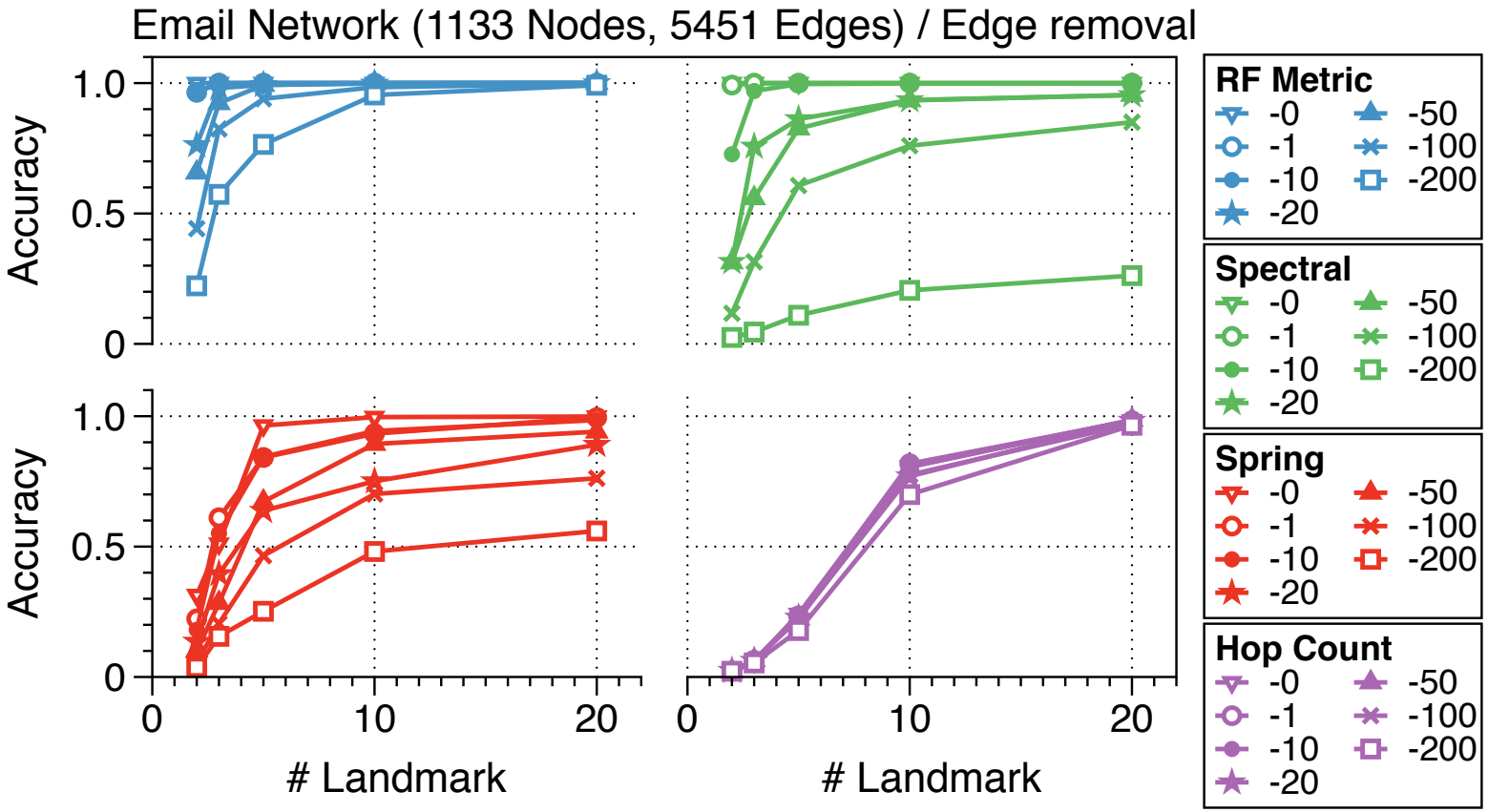
$G(n, p)$

Randomly remove some edges.



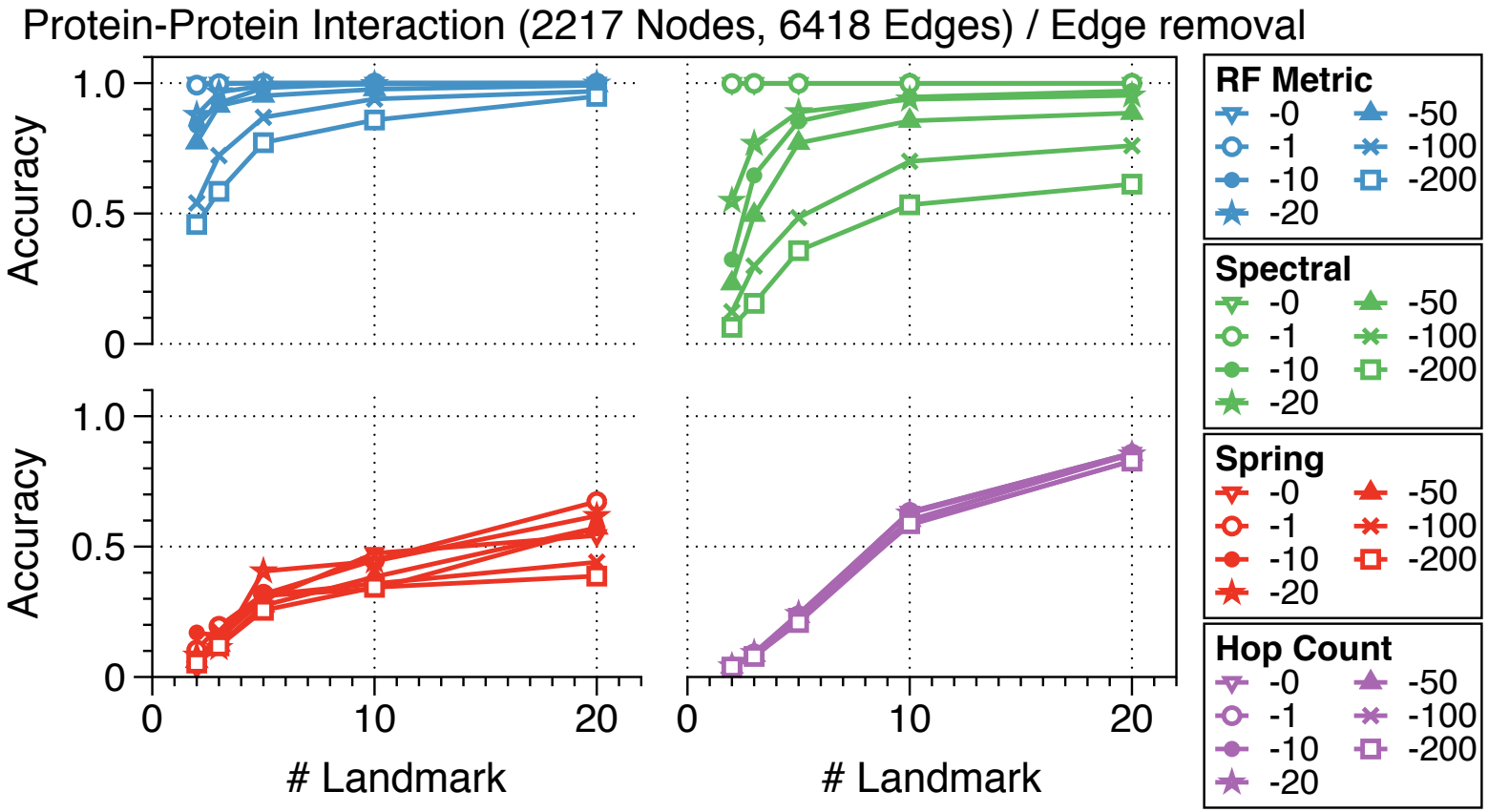
Email networks

Randomly remove some edges.



Protein-Protein networks

Randomly remove some edges.



Conclusion and On-going Work

Geometric approaches for network analysis has great potential.

Conclusion and On-going Work

Geometric approaches for network analysis has great potential.

- Speed up the computation.
- Find good approximation.
- More applications.

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- https://www3.cs.stonybrook.edu/∼jgao
- Questions and comments?