Title:	Discrete Ricci Flow for Geometric Routing
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Years aud Authors of Summarized Original Work

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Problem Definition

The problem is concerned about computing virtual coordinates for greedy routing in a wireless ad hoc network. Consider a set of wireless nodes S densely deployed inside a geometric domain $\mathcal{R} \subseteq \mathbb{R}^2$. Nodes within comminication range can directly communication with each other. We ask whether one can compute a set of virtual coordinates for S such that greedy routing has guaranteed delivery? In particular, each node forwards the message to the neighbor whose distance to the destination, computed under the virtual coordinates and some metric function d, is the smallest. If such a neighbor can always be found, greedy routing successfully delivers the message to the destination. The problem can be phrased as finding a *greedy embedding* of S in some geometric space, such that greedy routing always succeeds.

In the setting of this entry we assume that the nodes are a dense sample of the domain $\mathcal{R}$ such that the communication graph on S contains a triangulated mesh Σ as a discrete approximation of $\mathcal{R}$.

Key Results

The key result is a family of distributed algorithms for computing the greedy embedding using discrete Ricci flow. Given a triangular mesh Σ with vertex set V, edge set E and face set F, we can define a piecewise linear metric by the edge lengths on Σ: $l : E \to \mathbb{R}^+$ that satisfies the triangle inequality for each triangle face. The piecewise linear metric determines the corner angles of the triangles on Σ, by the cosine law. The *discrete curvature* K_i at a vertex v_i is defined as the angle deficit on the mesh. If v_i is an interior vertex, $K_i = 2\pi - \sum_j \theta_j$, where θ_j's are the corner angles at v_i. If v_i is a vertex on the boundary, $K_i = \pi - \sum_j \theta_j$, where θ_j's are the corner angles at v_i. Thus, the curvature at an interior vertex v_i is 0 if the surface is flat at v_i. The curvature at a boundary vertex v_i is 0 if the boundary is locally a straight line at v_i. See Figure 1 (i). The famous Gauss-Bonnet theorem states that the total curvature is a topological invariant: $\sum_{v_i \in V} K_i = 2\pi\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic number[3] of Σ. Ricci flow is a process that deforms the surface metric to meet any target curvature that is admissible by the Gauss-Bonnet theorem.

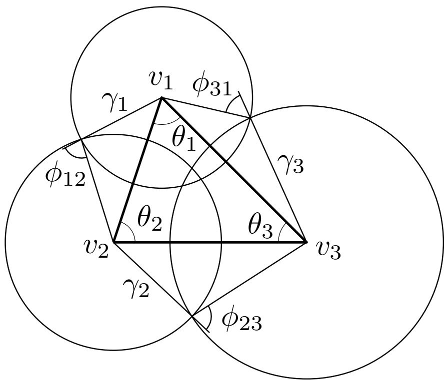


Fig. 1. The circle packing metric.

A conformal map in the continuous surface preserves the intersection angle of any two curves. In the discrete case, the "intersection angle" is defined using the circle packing metric [Thurston(1976), Stephenson(2005)]. We place a circle at each vertex v_i with radius γ_i such that for each edge e_{ij} the circles at v_i, v_j intersect or are tangent to each other. The intersection angle is denoted by $\phi(e_{ij})$. The pair of vertex radii and

[3] The Euler characterisitcs number of a surface is $2 - 2g - h$, where g is the genus or the number of handles and h is the number of holes.

the intersection angles on a mesh Σ, (Γ, Φ), is called a *circle packing metric* of Σ. See Figure 1 (ii). Two circle packing metrics (Γ_1, Φ_1) and (Γ_2, Φ_2) on the same mesh are *conformal equivalent*, if $\Phi_1 \equiv \Phi_2$. Therefore, a conformal deformation of a circle packing metric only modifies the vertex radii γ_i's and preserves the intersection angles. Note that the circle packing metric and the edge lengths (the piecewise linear metric) on one mesh can be converted to each other by using the cosine law.

Now we are ready to introduce the discrete Ricci flow algorithm. Let u_i to be $\log \gamma_i$ for each vertex. Then, the discrete Ricci flow, introduced in the work of [Chow and Luo(2003)], is defined as follows: $\frac{du_i(t)}{dt} = \bar{K}_i - K_i$, where $K_i, \bar{K}_i$ are the current and target curvature at vertex v_i respectively. Discrete Ricci flow can be formulated in the variational setting, namely, it is a negative gradient flow of some special energy form. $f(\mathbf{u}) = \int_{\mathbf{u_0}}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i$, where $\mathbf{u_0}$ is an arbitrary initial metric and $\bar{K}$ is the prescribed target curvature. The integration above is well defined, and called the *Ricci energy*. The discrete Ricci flow is the negative gradient flow of the discrete Ricci energy. The discrete metric which induces $\bar{K}$ is the minimizer of the energy. Computing the desired circle packing metric with prescribed curvature $\bar{K}$ is equivalent to minimizing the discrete Ricci energy. The discrete Ricci energy is strictly convex (namely, its Hessian is positive definite after a normalization). The global minimum uniquely exists, corresponding to the metric $\bar{\mathbf{u}}$, which induces $\bar{K}$. The discrete Ricci flow converges to this global minimum and the convergence is exponentially fast [Chow and Luo(2003)], i.e., $|\bar{K}_i - K_i(t)| < c_1 e^{-c_2 t}$, where c_1, c_2 are two positive constants. This represents a centralized algorithm for computing the discrete Ricci flow on Σ. In the following we describe the distributed algorithm for different type of greedy routing scenarios.

Discrete Ricci Flow Algorithm

To apply discrete Ricci flow for greedy routing, we take a triangular mesh Σ as a subgraph from the communication graph. All non-triangular faces are considered as network holes that will be mapped to circular holes in the embedding. All nodes not on hole boundaries have zero curvature under the mapping. Thus the embedding is denoted as a circular domain. With the virtual coordinates and Euclidean distance metric, greedy routing guarantees delivery[4].

In particular, we set all edge lengths to be initially 1, which determines the initial curvature at each node. In particular, we choose the circle packing metric by placing a circle of initial radius 1/2 on each node. The circles at adjacent nodes are tangent to each other. Thus the intersection angle is kept at 0. We now set the target curvature at interior nodes to be zero and at hole boundary nodes to be $2\pi/k$ with k as the number of nodes on the hole boundary. The algorithms runs in a gossip-style. In each round, each node exchanges it radius with neighbors and computes its own Gaussian curvature. The algorithm stops when the current curvature is within error ϵ from the specified target curvature.

At each gossip round, node v_i is associated with a disk with radius e^{u_i}, where u_i is a scalar value. The length of the edge connecting v_i and v_j equals to $e^{u_i} + e^{u_j}$. The corner angles of each triangle can be estimated using cosine law by each node locally. That is, the angle θ_i^{jk} in triangle $[v_i, v_j, v_k]$ is

$$\theta_i^{jk} = \cos^{-1} \frac{l_{ij}^2 + l_{ki}^2 - l_{jk}^2}{2 l_{ij} l_{ki}}$$

[4] For a node in the interior of the triangulation, if the corner angle is greater than $2\pi/3$ we will adopt greedy routing on an edge that has provably guaranteed delivery.

The curvature k_i at v_i is,

$$k_i = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk}, v_i \in \partial M \end{cases}$$

When the target curvature is not met, u_i is modified proportionally to the difference between the target curvature and the current curvature.

$$u_i \Leftarrow u_i + \delta(\bar{k}_i - k_i)$$

Once the curvatures are computed, the triangulation is then flattened out by a simple flooding from a triangle root. Given three edge lengths of the root triangle $[v_0, v_1, v_2]$, the node coordinates can be constructed directly. Then the neighboring triangle of the root, e.g. $[v_1, v_0, v_i]$, can be flattened, the virtual coordinates of v_i is the intersection of two circles, one is centered at v_0 with radius l_{0i}, the other is centered at v_1 with radius l_{1i}. In similar way, the neighbors of the newly flattened triangles can be further embedded. The virtual coordinates of the whole network are thus computed.

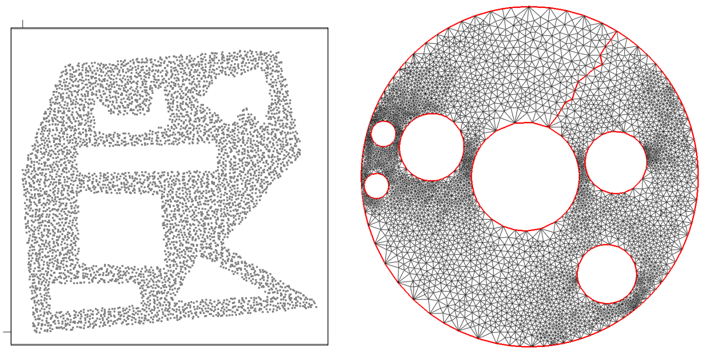


Fig. 2. (a) A network of 7000 nodes with many holes; (b) Virtual coordinates.

Discrete Hyperbolic Ricci Flow

The key result in conformal geometry says that any surface with a Riemannian metric admits a Riemannian metric of constant Gaussian curvature, which is conformal to the original metric. Such metric is called the uniformization metric. Thus, depending on the surface topology, the uniformization metric has either positive constant, zero, or negative constant curvature everywhere. Simply connected surfaces with constant curvature are only of three canonical types: the sphere (constant positive curvature everywhere), the Euclidean plane (zero curvature everywhere), and the hyperbolic plane (negative curvature everywhere). Discrete Ricci flow is a powerful tool to compute the uniformization metric.

In our setting, when the triangulation Σ has two or more holes, it has negative total curvature. Thus its uniformization metric is hyperbolic. To actually embed the surface and realize the uniformization metric, the holes in the network are cut open to get a simply connected triangulation T. Using discrete hyperbolic Ricci flow we embed T in a convex region S in hyperbolic space. Each node is given a hyperbolic coordinate. Each edge uv has a length $d(u, v)$ as the geodesic between u, v in the hyperbolic space. In this way, greedy routing with the hyperbolic metric, i.e., send the message to the neighbor closer to the destination measured by hyperbolic distance, has *guaranteed delivery*.

The hyperbolic Ricci flow is very similar to the Euclidean version with a few modifications. First all metrics are hyperbolic. The edge length $l_{ij}$of e_{ij} is determined by the hyperbolic cosine law:

$$\cosh l_{ij} = \cosh\gamma_i \cosh\gamma_j + \sinh\gamma_i \sinh\gamma_j \cos\phi_{ij}. \tag{1}$$

Let $u_i = \log\tanh\frac{\gamma_i}{2}$, *the discrete Ricci flow* is defined as

$$\frac{du_i(t)}{dt} = -K_i, \tag{2}$$

where K_i is the discrete Gaussian curvature at v_i. Once the hyperbolic metric is computed, we can embed the triangulation isometrically onto the Poincare disk.

Generalized Discrete Surface Ricci Flow

There are many schemes for discrete surface Ricci flow [Zhang et al(2014)Zhang, Guo, Zeng, Luo, Yau, a including tangential circle packing, Thurston's circle packing, inversive distance circle packing, Yamabe flow, virtual radius circle packing and mixed typed schemes. All of them can be unified as follows. The combinatorial structure of the triangulation is Σ, it is with one of three background geometries: Euclidean $\mathbb{E}^2$, hyperbolic $\mathbb{H}^2$, and spherical $\mathbb{S}^2$. Each vertex is associated with a circle, the vertex radii function is $\gamma : V \to \mathbb{R}^+$. Each vertex is also associated with a constant ϵ, which indicates the scheme. Each edge has a conformal structure coefficient $\eta : E \to \mathbb{R}$. So a circle packing metric is given by $(\Sigma, \gamma, \eta, \epsilon)$. The discrete conformal factor is given by

$$u_i = \begin{cases} \log\gamma_i & , \mathbb{E}^2 \\ \log\tanh\frac{\gamma_i}{2} & , \mathbb{H}^2 \\ \log\tan\frac{\gamma_i}{2} & , \mathbb{S}^2 \end{cases}$$

The length of $[v_i, v_j]$ is given by

$$\begin{cases} l_{ij}^2 & = 2\eta_{ij}e^{u_i+u_j} + \epsilon_i e^{2u_i} + \epsilon_j e^{2u_j} & , \mathbb{E}^2 \\ \cosh l_{ij} & = \frac{4\eta_{ij}e^{u_i+u_j} + (1+\epsilon_i e^{2u_i})(1+\epsilon_j e^{2u_j})}{(1-\epsilon_i e^{2u_i})(1-\epsilon_j e^{2u_j})} & , \mathbb{H}^2 \\ \cos l_{ij} & = \frac{4\eta_{ij}e^{u_i+u_j} + (1-\epsilon_i e^{2u_i})(1-\epsilon_j e^{2u_j})}{(1+\epsilon_i e^{2u_i})(1+\epsilon_j e^{2u_j})} & , \mathbb{S}^2 \end{cases}$$

The discrete Ricci flow is given by

$$\frac{du_i(t)}{dt} = \bar{K}_i - K_i(t),$$

where $\bar{K} : V \to \mathbb{R}$ is the prescribed target curvature, which is the negative gradient flow of the discrete Ricci energy

$$E(u) = \int^{\mathbf{u}} \sum_i (\bar{K}_i - K_i) du_i.$$

For the discrete surfaces with Euclidean back ground geometry, the Ricci energy is convex on the space $\sum_i u_i = 0$. For those with hyperbolic background geometry, the energy is convex. For spherical case, the energy is indefinite.

For Yamabe scheme (where $\epsilon \equiv 0$), the combinatorial structure Σ is Delaunay, if for each edge $[v_i, v_j]$ share by two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$, $\theta_{ij}^k + \theta_{ji}^l \leq \pi$. If during the Yamabe flow, the combinatorial structure can be updated to ensure the Delaunay condition, then for any $\bar{K} : V \to (-\infty, 2\pi)$ satisfying the Gauss-Bonnet constraint $\sum_{v \in V} \bar{K}(v) = 2\pi\chi(\Sigma)$, the Yamabe flow with surgery can lead to the discrete metric that realizes the target curvature, the convergence is exponentially fast. This theorem implies the discrete uniformization theorem: any closed polyhedral surface admits a polyhedral metric discretely conformal to the original one, which induces constant Gaussian curvature everywhere [Gu et al(2013)Gu, Luo, Sun, and Wu, Gu et al(2014)Gu, Guo, Luo, Sun, and Wu].

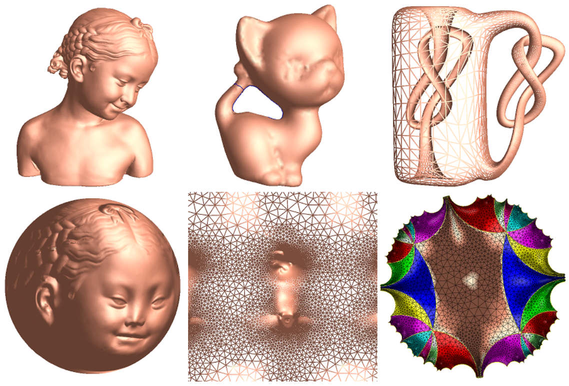
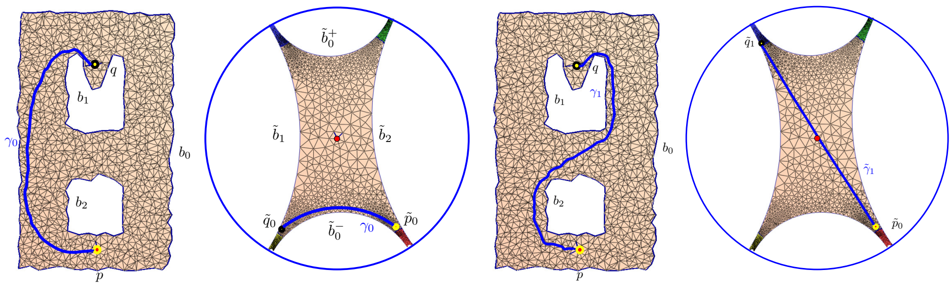


Fig. 3. Discrete surface uniformization.

Applications

The presented Ricci flow algorithms can be applied for a variety of routing primitives for large-scale wireless sensor networks with non-uniform node distribution. Beside guaranteed delivery [Sarkar et al(2009)Sarkar, Yin, Gao, Luo, and Gu], we can also achieve multiple additional desirable routing objectives, all derived from the unique property of a conformal mapping. For example, greedy routing on a circular domain may accumulate high traffic load on the interior hole boundaries. To alleviate that, we can reflect the network along a hole boundary using a Mobius transformation and map a copy of the network to cover the interior of the hole, recursively [Sarkar et al(2010)Sarkar, Zeng, Gao, and Gu]. See Figure 5. Routing on this covering space makes traffic load more balanced as hole boundaries essentially ‘disappear. In another case, when there are sudden link or node failures, we can apply a Mobius transformation to generate a different circular domain, with the sizes and positions of the holes rearranged, on which greedy routing generates a different path [Jiang et al(2011)Jiang, Ban, Goswami, adn Jie Gao, and Gu]. Thus quick recovery from a spontaneous failure is possible. The hyperbolic Ricci flow can be used to map the domain with the holes cut open to a convex polygon that can tile up the entire hyperbolic plane. This mapping supports greedy routing with specified ‘homotopy types, i.e., routes that go around holes in different ways [Zeng et al(2010)Zeng, Sarkar, Luo, Gu, and Gao]. See Figure 4. Hyperbolic embedding can be generalized to 3D sensor networks with complex topology as in the case of monitoring underground tunnels [Yu et al(2012)Yu, Yin, Han, Gao, and Gu]. Aditional applications include generation of ‘space filling curves for arbitrary do-



(a) shortest path 1 (b) geodesic of (a) (c) shortest path 2 (d) geodesic of (c)

Fig. 4. Computing the shortest paths using the hyperbolic embedding of a 3-connected domain with 1286 nodes. Two different paths are generated using greedy routing towards images of the destination in different patches.

mains [Ban et al(2013)Ban, Goswami, Zeng, Gu, and Gao], supporting greedy routing in mobile networks [Li et al(2013)Li, Zeng, Zhou, Gu, and Gao], etc.

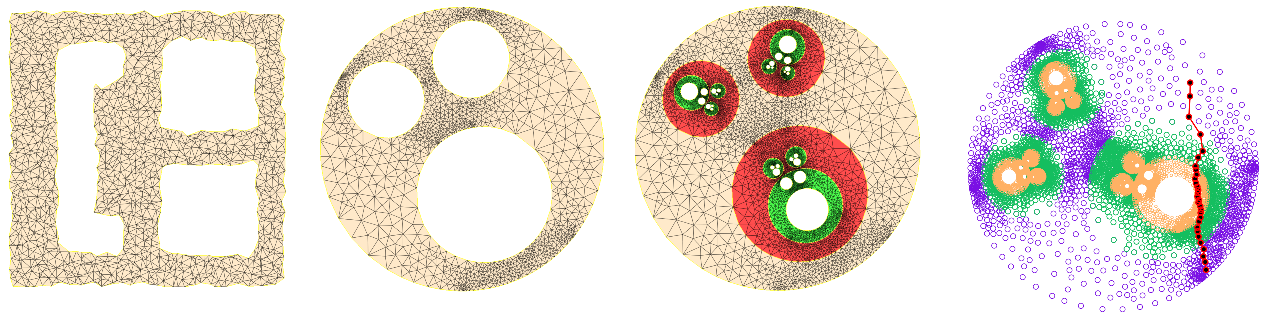


Fig. 5. 3-level circular reflections and a routing path.

Open Problems

Given a smooth surface S with a Riemannian metric $\mathbf{g}$, the smooth Ricci flow leads to the uniformization metric $e^{2\lambda}\mathbf{g}$, where λ is the smooth conformal factor. If the surface is tessellated to get a discrete surface M_0, and discrete Ricci flow is performed on M_0, one obtains discrete conformal factor function u_0. When M is subdivided by n times, the discrete conformal factor is u_n, whether $\lim_{n\to\infty} u_n = \lambda$.

Experimental Results

URLs to Code and Data Sets

http://www.cs.sunysb.edu/~gu/tutorial/RicciFlow.html

Cross-References

None is reported.

Recommended Reading

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