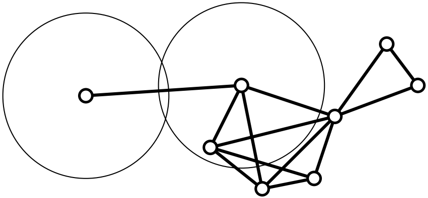
Computing Diameter+2 in Truly Subquadratic Time for Unit-Disk Graphs

Hsien-Chih Chang (Dartmouth)
Jie Gao (Rutgers)
Hung Le (U Mass Amherst)

SoCG 2024

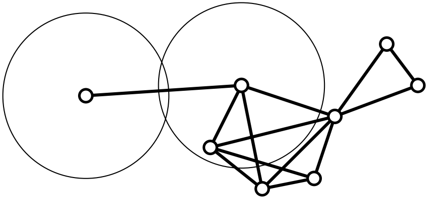
Diameter in Unit Disk Graphs

Unit disk graph (UDG): points in $\mathbb{R}^2$, connect an edge pq iff $|pq| \leq 1$.
A geometric intersection graph of disks of radius $1/2$.



Diameter in Unit Disk Graphs

Unit disk graph (UDG): points in $\mathbb{R}^2$, connect an edge pq iff $|pq| \leq 1$.
A geometric intersection graph of disks of radius $1/2$.



Q: Compute the diameter of an (unweighted) UDG in $O(n^{2-\varepsilon})$ time?

Fine-Grained Complexity for Diameter Problem

Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm for Diameter of a **general** graph.

- Even to distinguish between 2 and 3 in sparse graphs.

Fine-Grained Complexity for Diameter Problem

Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm for Diameter of a **general** graph.

- Even to distinguish between 2 and 3 in sparse graphs.

Planar graphs:

- $\tilde{O}(n^{11/6})$ [Cabello'17]
- $\tilde{O}(n^{5/3})$ [GKMSW'18].

Fine-Grained Complexity for Diameter Problem

Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm for Diameter of a **general** graph.

- Even to distinguish between 2 and 3 in sparse graphs.

Planar graphs:

- $\tilde{O}(n^{11/6})$ [Cabello'17]
- $\tilde{O}(n^{5/3})$ [GKMSW'18].

What about Diameter for geometric intersection graphs?

Fine-Grained Complexity for Diameter Problem

[BKKNP'22] Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm in a **geometric intersection** graph to decide if Diameter ≤ 3 for

Fine-Grained Complexity for Diameter Problem

[BKKNP'22] Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm in a **geometric intersection** graph to decide if Diameter ≤ 3 for

- unit segments in $\mathbb{R}^2$.
- congruent equilateral triangles (with rotation) in $\mathbb{R}^2$.

Fine-Grained Complexity for Diameter Problem

[BKKNP'22] Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm in a **geometric intersection** graph to decide if Diameter ≤ 3 for

- unit segments in $\mathbb{R}^2$.
- congruent equilateral triangles (with rotation) in $\mathbb{R}^2$.

or, decide if Diameter $\leq k$ for

Fine-Grained Complexity for Diameter Problem

[BKKNP'22] Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm in a **geometric intersection** graph to decide if Diameter ≤ 3 for

- unit segments in $\mathbb{R}^2$.
- congruent equilateral triangles (with rotation) in $\mathbb{R}^2$.

or, decide if Diameter $\leq k$ for

- axis-parallel line segments in $\mathbb{R}^2$.
- unit balls in $\mathbb{R}^3$.
- axis-parallel unit cubes in $\mathbb{R}^3$.

Fine-Grained Complexity for Diameter Problem

[BKKNP'22] Assuming SETH, there is **no** $O(n^{2-\varepsilon})$ time algorithm in a **geometric intersection** graph to decide if Diameter ≤ 3 for

- unit segments in $\mathbb{R}^2$.
- congruent equilateral triangles (with rotation) in $\mathbb{R}^2$.

or, decide if Diameter $\leq k$ for

- axis-parallel line segments in $\mathbb{R}^2$.
- unit balls in $\mathbb{R}^3$.
- axis-parallel unit cubes in $\mathbb{R}^3$.

Unit (axis-parallel) square graph in $\mathbb{R}^2$:

- $O(n \log n)$ time to decide if Diameter ≤ 2.

How fast can we find Diameter in UDG?

A unit disk graph can be a dense graph, i.e., $\Theta(n^2)$ edges.

How fast can we find Diameter in UDG?

A unit disk graph can be a dense graph, i.e., $\Theta(n^2)$ edges.

Exact diameter $\tilde{O}(n^2)$

- Run n single-source shortest path, each in $O(n \log n)$ time.
- $O(n^2 \sqrt{\log \log n / \log n})$ time. [CK'16]

How fast can we find Diameter in UDG?

A unit disk graph can be a dense graph, i.e., $\Theta(n^2)$ edges.

Exact diameter $\tilde{O}(n^2)$

- Run n single-source shortest path, each in $O(n \log n)$ time.
- $O(n^2 \sqrt{\log \log n / \log n})$ time. [CK'16]

Approximate diameter D in time $\tilde{O}(n)$ [CS'18]

- Weighted graph (edge weight $=$ Euclidean distance): $(1+\varepsilon)D$
- Unweighted graph: $(1+\varepsilon)D + (4+2\varepsilon)$.

How fast can we find Diameter in UDG?

A unit disk graph can be a dense graph, i.e., $\Theta(n^2)$ edges.

Exact diameter $\tilde{O}(n^2)$

- Run n single-source shortest path, each in $O(n \log n)$ time.
- $O(n^2\sqrt{\log\log n / \log n})$ time. [CK'16]

Approximate diameter D in time $\tilde{O}(n)$ [CS'18]

- Weighted graph (edge weight = Euclidean distance): $(1+\varepsilon)D$
- Unweighted graph: $(1+\varepsilon)D + (4+2\varepsilon)$.

Our results: $D+2$ approximation in time $\tilde{O}(n^{2-1/18})$ time.

Address Two Technical Elements for UDG

Inspired by [LP19, DHV22, LW23]:
Planar/Minor-free graphs $\Rightarrow$ Unit disk graphs

1. bounded VC-dimension for encoding distances.

Address Two Technical Elements for UDG

Inspired by [LP19, DHV22, LW23]:
Planar/Minor-free graphs $\Rightarrow$ Unit disk graphs

1. bounded VC-dimension for encoding distances.

Address Two Technical Elements for UDG

Inspired by [LP19, DHV22, LW23]:
Planar/Minor-free graphs $\Rightarrow$ Unit disk graphs

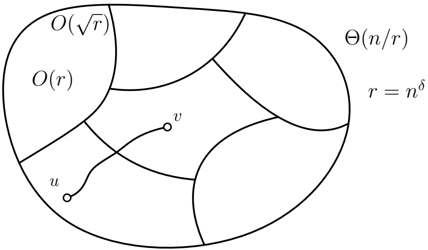
1. bounded VC-dimension for encoding distances.
2. "r-division" for UDG?

Computer Diameter in Planar/Minor-free Graphs

Compute **eccentricity** of $v \in V$: $ecc(v) = max_u d(u, v)$.

Computer Diameter in Planar/Minor-free Graphs

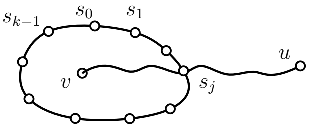
Compute **eccentricity** of $v \in V$: $ecc(v) = max_u d(u, v)$.
Use an r-division: decomposition into $\Theta(n/r)$ pieces each is connected with $O(r)$ vertices and $O(\sqrt{r})$ boundary vertices.



Compute distance $d(u, v)$ with u outside the current piece H.

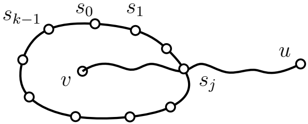
Computer Diameter in Planar/Minor-free Graphs

Compute distance $d(u, v)$ with u outside the current piece H with k boundary vertices $\{s_0, s_1, \cdots, s_{k-1}\}$, $k = O(\sqrt{r})$.



Computer Diameter in Planar/Minor-free Graphs

Compute distance $d(u, v)$ with u outside the current piece H with k boundary vertices $\{s_0, s_1, \cdots, s_{k-1}\}$, $k = O(\sqrt{r})$.



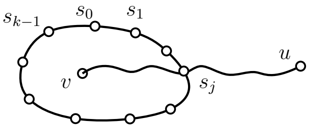
$$d(u, v) = d(u, s_j) + d(s_j, v) = \min_{0 \le i \le k-1} \{d(u, s_i) + d(s_i, v)\}$$

$$= \min_{0 \le i \le k-1} \{d(u, s_0) + \underline{d(u, s_i) - d(u, s_0)} + d(s_i, v)\}$$

$$= d(u, s_0) + \min_{0 \le i \le k-1} \{P_u[i] + d(s_i, v)\}$$

Computer Diameter in Planar/Minor-free Graphs

Compute distance $d(u, v)$ with u outside the current piece H with k boundary vertices $\{s_0, s_1, \cdots, s_{k-1}\}$, $k = O(\sqrt{r})$.

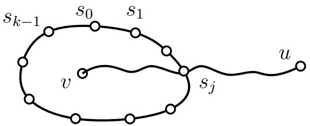


$$
\begin{aligned}
d(u, v) =& d(u, s_j) + d(s_j, v) = \min_{0 \le i \le k-1} \{d(u, s_i) + d(s_i, v)\} \\
=& \min_{0 \le i \le k-1} \{d(u, s_0) + \underline{d(u, s_i) - d(u, s_0)} + d(s_i, v)\} \\
=& d(u, s_0) + \min_{0 \le i \le k-1} \{P_u[i] + d(s_i, v)\}
\end{aligned}
$$

Pattern P_u, a k-dim vector, $P_u[i] \le d(s_i, s_0) = O(r)$.

Computer Diameter in Planar/Minor-free Graphs

Compute distance $d(u, v)$ with u outside the current piece H with k boundary vertices $\{s_0, s_1, \cdots, s_{k-1}\}$, $k = O(\sqrt{r})$.

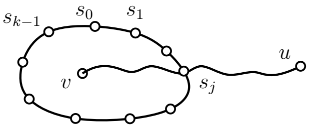


$$
\begin{aligned}
d(u,v) =& d(u,s_j) + d(s_j,v) = \min_{0\le i\le k-1}\{d(u,s_i) + d(s_i,v)\} \\
=& \min_{0\le i\le k-1}\{d(u,s_0) + \underline{d(u,s_i) - d(u,s_0)} + d(s_i,v)\} \\
=& d(u,s_0) + \min_{0\le i\le k-1}\{P_u[i] + d(s_i,v)\}
\end{aligned}
$$

Pattern P_u, a k-dim vector, $P_u[i] \le d(s_i, s_0) = O(r)$.
The set of all patterns is $O(kr)^d = O(r^{3d/2})$, d=VC-dim of patterns.

Computer Diameter in Planar/Minor-free Graphs

Compute distance $d(u, v)$ with u outside the current piece H with k boundary vertices $\{s_0, s_1, \cdots, s_{k-1}\}$.



$$\begin{aligned} d(u, v) =& d(u, s_0) + \min_{0 \le i \le k-1} \{P_u[i] + d(s_i, v)\} \\ =& d(u, s_0) + d(\mathbf{p}, v) \end{aligned}$$

Now, calculate the distance of v to every possible pattern **p** (sublinealy many).

Address Two Technical Elements for UDG

Inspired by [LP19, DHV22, LW23]:
Planar/Minor-free graphs $\Rightarrow$ Unit disk graphs

1. bounded VC-dimension for encoding distances.
2. "r-division" for UDG?

VC-dimension

Maximum # points shattered by a geometric shape.

VC-dimension of disks: 3

VC-dimension

Maximum # points shattered by a geometric shape.
VC-dimension of disks: 3

VC-dimension of two set systems in a UDG:

- Distance VC-dimension: the set of k-neighborhood in G, $\forall k \geq 1$.
- Distance encoding vector wrt k vertices, $\forall k \geq 1$.

VC-dimension

Maximum # points shattered by a geometric shape.
VC-dimension of disks: 3

VC-dimension of two set systems in a UDG:

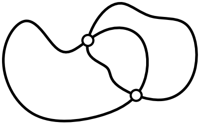
- Distance VC-dimension: the set of k-neighborhood in G, $\forall k \geq 1$.
- Distance encoding vector wrt k vertices, $\forall k \geq 1$.

In both cases the VC-dimension is 4.

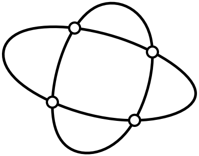
VC-dimension of a Pseudo Disk Graph

Our proof works for the intersection graph of pseudo disks as well.

- Two pseudo disks have at most 2 intersections at the boundary.



Pseudo disks

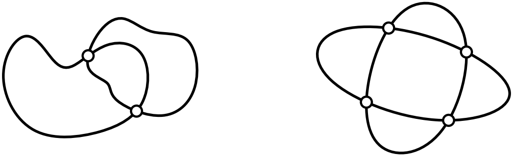


Not pseudo disks

VC-dimension of a Pseudo Disk Graph

Our proof works for the intersection graph of pseudo disks as well.

- Two pseudo disks have at most 2 intersections at the boundary.



Pseudo disks　　　Not pseudo disks

Literature: distance VC-dimension $= 4$

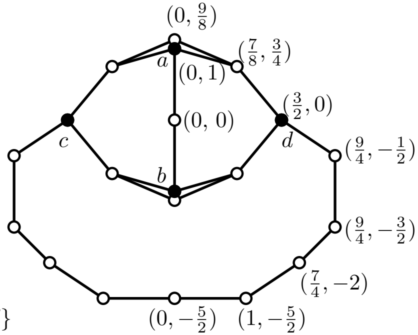
- Proof using geometry for unit disk graphs. [AACMMSS'21]
- For closed, bounded, convex, center-symmetric objects. [DKP'23]

VC-dimension in a UDG

4 vertices can be shattered.

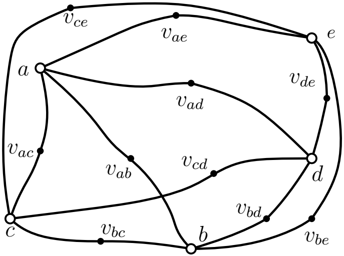
$$B((\tfrac{7}{8},\tfrac{3}{4}),1)=\{a,d\}$$
$$B((0,0),1)=\{a,b\}$$
$$B((0,-\tfrac{5}{2}),5)=\{a,d\}$$

$$B((\tfrac{3}{2},0),2)=\{a,b,d\}$$
$$B((0,\tfrac{9}{8}),2)=\{a,c,d\}$$
$$B((0,0),3)=\{a,b,c,d\}$$



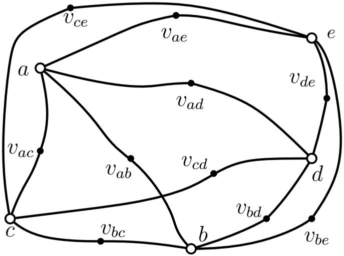
VC-dimension in a UDG

Assume 5 vertices a, b, c, d, e that can be shattered.
Build a K_5 graph – connect a, b through v_{ab} where a ball at v_{ab} 'scoops out' only vertices a, b.



VC-dimension in a UDG

Assume 5 vertices a, b, c, d, e that can be shattered.
Build a K_5 graph – connect a, b through v_{ab} where a ball at v_{ab} 'scoops out' only vertices a, b.



By planarity (Hanani–Tutte theorem), there is a crossing pair. $\Rightarrow$ argue a contradiction that the ball at v_{ab} includes one more vertex.

r-clustering for UDG

r-division for a planar graph:

- Recursively apply a $O(\sqrt{n})$ size balanced separator.

r-clustering for UDG

r-division for a planar graph:

- Recursively apply a $O(\sqrt{n})$ size balanced separator.

No classical separator for a unit disk graph: e.g., a dense clique.

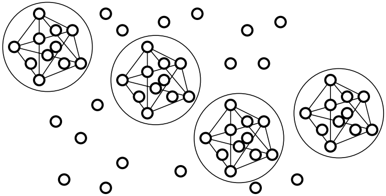
r-clustering for UDG

r-division for a planar graph:

- Recursively apply a $O(\sqrt{n})$ size balanced separator.

No classical separator for a unit disk graph: e.g., a dense clique.

- Clique-based balanced separator [BBKMZ'20, BKMT'23, Ber23]: $O(\sqrt{n})$ cliques



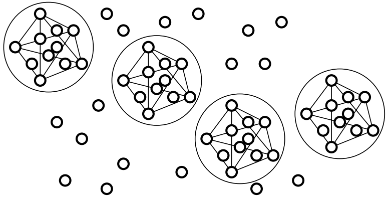
r-clustering for UDG

r-division for a planar graph:

- Recursively apply a $O(\sqrt{n})$ size balanced separator.

No classical separator for a unit disk graph: e.g., a dense clique.

- Clique-based balanced separator [BBKMZ'20, BKMT'23, Ber23]: $O(\sqrt{n})$ cliques



- 1st issue: # vertices in the boundary cliques can be a lot.
- 2nd issue: reduce # boundary cliques of a piece through recursion

Clique-based r-clustering

$(\mathcal{R}, \mathcal{C})$: $\mathcal{R}$ clusters, $\mathcal{C}$ cliques.

- $\#$ clusters: $O(n/\sqrt{r})$.
- Each cluster has $\leq r$ interior vertices; and $\leq r$ boundary cliques.
- Total boundary vertices $O(n/\sqrt{r})$.

Clique-based r-clustering

$(\mathcal{R}, \mathcal{C})$: $\mathcal{R}$ clusters, $\mathcal{C}$ cliques.

- # clusters: $O(n/\sqrt{r})$.
- Each cluster has $\leq r$ interior vertices; and $\leq r$ boundary cliques.
- Total boundary vertices $O(n/\sqrt{r})$.

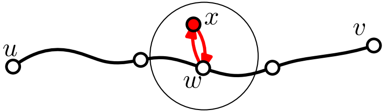
We cannot afford to enumerate over all vertices in the boundary – there can be $\Omega(n)$ many. $\Rightarrow$ We choose one representative vertex from each boundary clique.

Clique-based r-clustering

$(\mathcal{R}, \mathcal{C})$: $\mathcal{R}$ clusters, $\mathcal{C}$ cliques.

- # clusters: $O(n/\sqrt{r})$.
- Each cluster has $\leq r$ interior vertices; and $\leq r$ boundary cliques.
- Total boundary vertices $O(n/\sqrt{r})$.

We cannot afford to enumerate over all vertices in the boundary – there can be $\Omega(n)$ many. $\Rightarrow$ We choose one representative vertex from each boundary clique.



This gives length $\leq D + 2$.

Additional Results

Intersection graph of pseudo disks of ‘similar size’ with $O(1)$ complexity: $D+2$ in time $\tilde{O}(n^{2-1/18})$.

Additional Results

Intersection graph of pseudo disks of 'similar size' with $O(1)$ complexity: $D + 2$ in time $\tilde{O}(n^{2-1/18})$.

If in addition the pseudo disks have k-ply, then we can find **exact** diameter in time $\tilde{O}(k^{11/9} n^{2-1/18})$.

Additional Results

Intersection graph of pseudo disks of 'similar size' with $O(1)$ complexity: $D + 2$ in time $\tilde{O}(n^{2-1/18})$.

If in addition the pseudo disks have k-ply, then we can find **exact** diameter in time $\tilde{O}(k^{11/9} n^{2-1/18})$.

Distance oracle: $(+2)$-approximation with space $O(n^{2-1/18})$, $O(1)$ query time.

Open Questions

- Find Diameter of a UDG in truly subquadratic time? Or a conditional lower bound?
- Algorithms for pseudo disk graphs, e.g., disks of different radii?

Questions and Comments?