Ancient Solutions of the Affine Normal Flow

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1 Introduction

Consider a smooth, strictly convex hypersurface $\mathcal{L}$ locally parametrized by $F(x) \in \mathbb{R}^{n+1}$. The affine normal is a vector field $\xi = \xi_{\mathcal{L}}$ transverse to $\mathcal{L}$ and invariant under volume-preserving affine transformations of $\mathbb{R}^{n+1}$. The affine normal flow evolves such a hypersurface in time t by

$$\partial_t F(x,t) = \xi(x,t), \qquad F(x,0) = F(x).$$

In [7], Ben Chow proved that every smooth, strictly convex hypersurface in $\mathbb{R}^{n+1}$ converges in finite time under the affine normal flow to a point. In [1], Ben Andrews proved that the rescaled limit of the contracting hypersurface around the final point converges to an ellipsoid. Later, Andrews [2] also studied the case in which the initial hypersurface is compact and convex with no regularity assumed. In this case, the affine normal flow, unlike the Gauss curvature flow, is instantaneously smoothing. In other words, such an initial hypersurface under the affine normal flow will evolve to be smooth and strictly convex at any positive time before the extinction time.

In the present work, we develop the affine normal flow for any noncompact convex hypersurface $\mathcal{L}$ in $\mathbb{R}^{n+1}$ whose convex hull $\hat{\mathcal{L}}$ contains no lines (if $\hat{\mathcal{L}}$ contains a line, the affine normal flow does not move it at all). As in [2] we define the flow by treating the $\mathcal{L}$ as a limit of a nested sequence of smooth, compact, strictly convex hypersurfaces $\mathcal{L}^i$. Our main new result is to classify

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ancient solutions—solutions defined for time $(-\infty, T)$—for the affine normal flow.

Theorem 1.1. *Any ancient solution to the affine normal flow must be be either an elliptic paraboloid (which is a translating soliton) or an ellipsoid (which is a shrinking soliton).*

The proof of Theorem 1.1 relies on a decay estimate of Andrews for the cubic form C^i_{jk} of a compact hypersurface under the affine normal flow [1]. In particular, the norm squared $|C|^2$ of the cubic form with respect to the affine metric decays like $1/t$ from the initial time. For an ancient solution then, we may shift the initial time as far back as we like, and thus the cubic form C^i_{jk} is identically zero. Then a classical theorem of Berwald shows that the hypersurface must be a hyperquadric, and the paraboloid and ellipsoid are the only hyperquadrics which form ancient solutions to the affine normal flow (the hyperboloid, an expanding soliton, is not part of an ancient solution).

In order to apply this estimate in our case, we need local regularity estimates to ensure that for all positive time t, the evolving hypersurfaces $\mathcal{L}^i(t)$ converge locally in the C^∞ topology to $\mathcal{L}(t)$. Thus Andrews's pointwise bound on the cubic form survives in the limit. We work in terms of the support function. The C^2 estimates are provided by a speed bound of Andrews [2] and a Pogorelov-type Hessian bound similar similar to one in Gutiérrez-Huang [15]. These estimates provide uniform local parabolicity, and then Krylov's theory and standard bootstrapping provide local estimates to any order.

Another key ingredient is the use of barriers. Here the invariance of the affine normal flow under volume-preserving affine transformations is important. The main barriers we use are ellipsoids and a particular expanding soliton (a hyperbolic affine sphere) due to Calabi [4]. In particular, Gutiérrez-Huang's estimate can only be applied to solutions of PDEs which move in time by some definite amount. Calabi's example is a crucial element in constructing a barrier to guarantee the solution does not remain constant in time.

Solitons of the affine normal flow have been very well studied [4, 6]. They are precisely the affine spheres. The shrinking solitons of the affine normal flow are the elliptic affine spheres, and Cheng-Yau proved that any properly embedded elliptic affine sphere must be an ellipsoid [6]. Translating solitons are parabolic affine spheres, and again Cheng-Yau showed that any properly embedded parabolic affine sphere must be an elliptic paraboloid [6].

Expanding solitons are hyperbolic affine spheres, which behave quite differently. Cheng-Yau proved that every convex cone in $\mathbb{R}^{n+1}$ which contains no lines admits a unique (up to scaling) hyperbolic affine sphere which is asymptotic to the boundary of the cone [5, 6]. (For example, the hyperboloid is the hyperbolic affine sphere asymptotic to the standard round cone.) The converse is also true: every properly embedded hyperbolic affine sphere in $\mathbb{R}^{n+1}$ is asymptotic to the boundary of a convex cone containing no lines [6]. Our definition of the affine normal flow immediately provides an expanding soliton which is a weak (viscosity) solution, and our local regularity estimates show that this solution is smooth.

We should note that Cheng-Yau [6] proved results for hyperbolic affine sphere based on the affine metric. In particular, a hyperbolic affine sphere has complete affine metric if and only if it is properly embedded in $\mathbb{R}^{n+1}$ if and only if it is asymptotic to the boundary of a convex cone in $\mathbb{R}^{n+1}$ containing no lines. Our methods do not yet yield any insight into the affine metric of evolving hypersurfaces. If the initial hypersurface of the affine normal flow is the boundary of a convex cone containing no lines, then at any positive time, the solution is the homothetically expanding hyperbolic affine sphere asymptotic to the cone. Cheng-Yau's result implies the affine metric in this case is complete at any positive time t. It will be interesting to determine whether, under the affine normal flow, the affine metric is complete at any positive time for any noncompact properly embedded initial hypersurface. Presumably a parabolic version of the affine geometric gradient estimate of Cheng-Yau is needed, as suggested by Yau [25].

When restricted to an affine hyperplane, the support function of a hypersurface evolving under the affine normal flow satisfies

$$\partial_t s = -(\det \partial^2_{ij} s)^{-\frac{1}{n+2}}. \tag{1.1}$$

Gutiérrez and Huang [15] have studied a similar parabolic Monge-Ampère equation

$$\partial_t s = -(\det \partial^2_{ij} s)^{-1}.$$

They prove that any ancient entire solution to this equation which *a priori* satisfies bounds on the ellipticity must be an evolving quadratic polynomial. Our Theorem 1.1 reduces to an similar result for (1.1): The ellipsoid and paraboloid solitons provide ancient solutions to (1.1) which can be represented, up to possible affine coordinate changes, by

$$s = \left(-\frac{2n+2}{n+2}\,t\right)^{\frac{n+2}{2n+2}}\sqrt{1+|y|^2}, \qquad s = \frac{|y|^2}{2} - t$$

respectively. Our result doesn't require any *a priori* bounds on the ellipticity. We do not require our solutions to be entire, but they do solve a Dirichlet boundary condition. See Section 14 below.

We also mention a related theorem due to Jörgens [17] for $n = 2$, Calabi [3] for $n \leq 5$, and independently to Pogorelov [21] and Cheng-Yau [6] for all dimensions:

Theorem 1.2. *Any entire convex solution to*

$$\det \partial^2_{ij} u = c > 0$$

is an quadratic polynomial.

The graph of each such u is a parabolic affine sphere, and Cheng-Yau's classification provides the result. Our techniques do not yet yield an independent proof of this classical theorem: We do not yet know if the affine normal flow is unique for a given initial convex noncompact hypersurface. Even though any parabolic affine sphere may naturally be thought of as a translating soliton under the affine normal flow, the flow we define, with the parabolic affine sphere as initial condition, may not *a priori* be the same flow as the soliton solution, and thus may not come from an ancient solution in our sense.

It is also interesting to compare our noncompact affine normal flow with other geometric flows on noncompact hypersurfaces. In particular, Ecker-Huisken and Ecker have studied mean-curvature flow of entire graphs in Euclidean space [12] [13] and of spacelike hypersurfaces in Lorentzian manifolds [9] [10] [11]. In [13], Ecker-Huisken prove that under any entire graph of a locally Lipschitz function moves under the mean curvature flow in Euclidean space to be smooth at any positive time, and the solution exists for all time. Ecker proves long-time existence for any initial spacelike hypersurface in Minkowski space under the mean curvature flow [10] and proves instantaneous smoothing for some weakly spacelike hypersurfaces in [11].

In the present work, we prove instantaneous smoothing and long-time existence for the affine normal flow on noncompact hypersurfaces for any initial convex noncompact properly embedded hypersurface $\mathcal{L} \subset \mathbb{R}^{n+1}$ which contains no lines. In this case, the evolving hypersurface $\mathcal{L}(t)$ under the affine normal flow exists for all time $t > 0$ (Theorem 8.2) and is smooth for all $t > 0$ (Theorem 13.1). Moreover, the following maximum principle at infinity is satisfied: If $\mathcal{L}^1$ and $\mathcal{L}^2$ are convex properly embedded hypersurfaces whose convex hulls satisfy $\widehat{\mathcal{L}^1} \subset \widehat{\mathcal{L}^2}$, then for all $t > 0$, the convex hulls satisfy $\widehat{\mathcal{L}^1(t)} \subset \widehat{\mathcal{L}^2(t)}$. This sort of maximum principle at infinity does not hold for all evolution equations of noncompact hypersurfaces. In particular, there is an example due to Ecker [10], of two soliton solutions to the mean curvature flow in Minkowski space, for which this fails.

The affine normal flow is equivalent (up to a diffeomorphism) to the hypersurface flow by $K^{\frac{1}{n+2}}\nu$, where K is the Gauss curvature and ν is the inward unit normal. The techniques we use (the definitions and ellipticity estimates) should apply to flows of noncompact convex hypersurfaces by other power of the Gauss curvature. Andrews [2] addresses many aspects of the compact case of flow by powers of Gauss curvature. In particular, he verifies that for $\alpha \leq 1/n$, any convex compact hypersurface in $\mathbb{R}^{n+1}$ evolves under the flow by $K^\alpha \nu$ to be smooth and strictly convex at any positive time t. In essence, we verify this in the noncompact case for $\alpha = 1/(n+2)$ (see Theorem 13.1 below). We expect the same result to be true in the noncompact case for all $\alpha \leq 1/n$. We should note that for $\alpha > 1/n$, flat sides of any initial hypersurface remain non-strictly convex for some positive time. We note that in the case of the Gauss curvature flow in $\mathbb{R}^3$ ($\alpha = 1$), Daskalopoulos-Hamilton [8] study how the boundary of such a flat side evolves over time.

Our treatment of the affine normal flow is largely self-contained. In Sections 2 and 3, we recall the definition of the affine normal and the basic affine structure equations. We develop the computations necessary by using notation similar to that of e.g. Zhu [26]: let $F\colon U \to \mathbb{R}^{n+1}$ represent a local embedding of a hypersurface for $U \subset \mathbb{R}^n$ a domain. Then we derive the structure equations based on derivatives of F. Using this notation, we develop the affine normal flow of the basic quantities associated with the hypersurface in Sections 4, 5 and 6. The main estimate we need on the cubic form is found in Section 5. These evolution equations are all due to Andrews [1], and we include derivations of them for the reader's convenience. In Section 7, we introduce the support function and some basic results we will need. We define our affine normal flow on a noncompact convex hypersurface $\mathcal{L}$ in

Section 8, basically as a limit of compact convex hypersurfaces approaching $\mathcal{L}$ from the inside, and we verify that the soliton solutions behave properly under our definition in Section 9.

In Section 10, we turn to the estimates that are the technical heart of the paper. We prove an estimate of Andrews on the speed of the support function evolving under affine normal flow [2]. In particular, we verify that this estimate survives in the limit to our noncompact hypersurface. In Section 11, we prove a version of a Pogorelov-type estimate due to Gutiérrez-Huang [15], which bounds the Hessian of the evolving support function, and in Section 12, we construct barriers to ensure that Gutiérrez-Huang's estimate applies. Krylov's estimates then ensure the support function is smooth for all time $t > 0$. In Section 13, we verify that the evolving hypersurface is smooth as well, and relate the noncompact affine normal flow to a Dirichlet problem for the support function in Section 14. The main results are proved in Section 15.

Our treatment of noncompact hypersurfaces as limits of compact hypersurfaces is a bit different from the usual analysis on noncompact manifolds. Typically noncompact manifolds are exhausted by compact domains with boundary (e.g. geodesic balls on complete Riemannian manifolds or sublevel sets of a proper height function on a hypersurface considered as a Euclidean graph), and then a version of the maximum principle is shown to hold in the limit of the exhaustion. Our limiting process is extrinsic, on the other hand: We apply the maximum principle to $|C|^2$ to derive Andrews's pointwise bound on compact hypersurfaces without boundary, which in turn survives in the limiting noncompact hypersurface. It is still desirable to implement an approach by intrinsically exhausting the hypersurface, to be able to use the maximum principle more directly on the evolving noncompact hypersurface. Perhaps the description in Section 14 of the affine normal flow in terms of a Dirichlet problem for the support function will be of some use.

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Notation: Subscripts after a comma are used to denote covariant derivatives with respect to the affine metric. So the second covariant derivative of H is $H_{,ij}$, for example. Of course the first covariant derivative of a function is just ordinary differentiation, which commutes with the time derivative ∂_t.

∂_i will denote an ordinary space derivative. We use Einstein's summation convention that any paired indices, one up and one down, are to be summed from 1 to n. Unless otherwise noted, we raise and lower indices using the affine metric g_{ij}.

2 The affine normal

Here we define the affine normal to a hypersurface in a similar way to Nomizu-Sasaki [20], but using notation adapted to our purposes.

Let $F = F(x^1, \dots, x^n)$ be a local embedding of a smooth, strictly convex hypersurface in $\mathbb{R}^{n+1}$. Let $F : \Omega \to \mathbb{R}^{n+1}$, where Ω is a domain in $\mathbb{R}^n$. Let $\tilde{\xi}$ be a smooth transverse vector field to F. Now we may differentiate to determine

$$\partial^2_{ij} F = \tilde{g}_{ij}\tilde{\xi} + \tilde{\Gamma}^k_{ij}\partial_k F, \tag{2.1}$$

$$\partial_i \tilde{\xi} = \tilde{\tau}_i \tilde{\xi} - \tilde{A}^j_i \partial_j F. \tag{2.2}$$

It is straightforward to check that $\tilde{g}_{ij}$ is a symmetric tensor, $\tilde{\Gamma}^k_{ij}$ is a torsion free connection, $\tilde{\tau}_i$ is a one-form, and $\tilde{A}^j_i$ is an endomorphism of the tangent bundle. With respect to $\tilde{\xi}$, $\tilde{g}_{ij}$ is called the *second fundamental form* and $\tilde{A}^j_i$ is the *shape operator*.

Proposition 2.1. *There is a unique transverse vector field ξ, called the* affine normal*, which satisfies*

1. *ξ points inward. In other words, ξ and the hypersurface $F(\Omega)$ are on the same side of the tangent plane.*

2. *$\tau_i = 0$.*

3. *$\det g_{ij} = \det(\partial_1 F, \dots, \partial_n F, \xi)^2$. The determinant on the left is that of an $n \times n$ matrix, while the determinant on the right is that on R^{n+1}.*

Note we have dropped the tildes in quantities defined by the affine normal (the connection term is an exception: see the next section). Condition 1 implies that the second fundamental form g_{ij} is positive definite, and thus we say g_{ij} is the *affine metric*. Condition 2 is called that ξ is *equiaffine*. Condition 3 is that the volume form on the hypersurface induced by ξ and the volume form on $\mathbb{R}^{n+1}$ is the same as the volume form induced by the affine metric.

The following proof of Proposition 2.1 will be instructive in computing the affine normal later on.

Proof. Given an arbitrary inward-pointing transverse vector filed $\tilde{\xi}$, any other may be written as $\xi = \phi\tilde{\xi} + Z^i\partial_i F$, where ϕ is a positive scalar function and $Z^i\partial_i F$ is a tangent vector field.

Condition 3 determines ϕ in terms of $\tilde{\xi}$: Plug $\xi = \phi\tilde{\xi} + Z^i\partial_i F$ into (2.1), and the terms in the span of $\tilde{\xi}$ give

$$g_{ij} = \phi^{-1}\tilde{g}_{ij}. \tag{2.3}$$

Now Condition 3 shows that

$$\phi^{-n}\det\tilde{g}_{ij} = \det g_{ij} = \det(\partial_1 F, \ldots, \partial_n F, \xi)^2 = \phi^2\det(\partial_1 F, \ldots, \partial_n F, \tilde{\xi})^2,$$

and so

$$\phi = \left(\frac{\det\tilde{g}_{ij}}{\det(\partial_1 F, \ldots, \partial_n F, \tilde{\xi})^2}\right)^{\frac{1}{n+2}}. \tag{2.4}$$

Finally, we use the equiaffine condition to determine Z^i: Plug in for ξ, set $\tau_i = 0$, and consider the terms in the span of $\tilde{\xi}$ to find

$$\begin{aligned}
-A_i^j\partial_j F &= \partial_i(\phi\tilde{\xi} + Z^j\partial_j F)\\
&= \partial_i\phi\,\tilde{\xi} + \phi\,\partial_i\tilde{\xi} + \partial_i Z^j\,\partial_j F + Z^j\,\partial^2_{ij}F\\
&= \partial_i\phi\,\tilde{\xi} + \phi(\tilde{\tau}_i\tilde{\xi} - \tilde{A}_i^j\partial_j F) + \partial_i Z^j\,\partial_j F + Z^j(\tilde{g}_{ij}\xi + \tilde{\Gamma}^k_{ij}\partial_k F),\\
0 &= \partial_i\phi + \phi\,\tilde{\tau}_i + Z^j\tilde{g}_{ij},\\
Z^j &= -\tilde{g}^{ij}(\partial_i\phi + \phi\,\tilde{\tau}_i),
\end{aligned} \tag{2.5}$$

where $\tilde{g}^{ij}$ is the inverse matrix of $\tilde{g}_{ij}$. □

Corollary 2.1. *The affine normal is invariant under volume-preserving affine automorphisms of* $\mathbb{R}^{n+1}$. *In other words, if* Φ *is such an affine map, and* ξ *is the affine normal filed to a hypersurface* $F(\Omega)$*, then* $\Phi_*\xi$ *is the affine normal to* $(\Phi\circ F)(\Omega)$*.*

Proof. The defining conditions in the proposition are invariant under affine volume-preserving maps on $\mathbb{R}^{n+1}$. □

3 Affine structure equations

Consider a smooth, strictly convex hypersurface in $\mathbb{R}^{n+1}$ given by the image of an embedding $F = F(x^1, \ldots, x^n)$. The affine normal is an inward-pointing transverse vector field to the hypersurface, and we have the following structure equations:

$$
\begin{aligned}
\partial^2_{ij} F &= g_{ij}\xi + (\Gamma^k_{ij} + C^k_{ij})F_{,k} \qquad (3.1)\\
\xi_{,i} &= -A^k_i F_{,k} \qquad (3.2)
\end{aligned}
$$

Here g_{ij} is the *affine metric*, which is positive definite. Γ^k_{ij} are the Christoffel symbols of the metric. Since $\Gamma^k_{ij} + C^k_{ij}$ is a connection, then C^k_{ij} is a tensor called the *cubic form*. A^k_i is the *affine curvature*, or *affine shape operator*. Equation (3.1) shows immediately that

$$C^k_{ij} = C^k_{ji}.$$

Now consider the second covariant derivatives with respect to the affine metric

$$
\begin{aligned}
F_{,ij} &= \partial^2_{ij} F - \Gamma^k_{ij} F_{,k}\\
&= g_{ij}\xi + C^k_{ij} F_{,k} \qquad (3.3)\\
\xi_{,ij} &= -A^k_{i,j} F_{,k} - A^k_i F_{,kj}\\
&= -A^k_{i,j} F_{,k} - A_{ij}\xi - A^k_i C^\ell_{kj} F_{,\ell}
\end{aligned}
$$

Since $\xi_{,ij} = \xi_{,ji}$, we have

$$A_{ij} = A_{ji}$$

and the following Codazzi equation for the affine curvature:

$$A^k_{j,i} - A^k_{i,j} = A^\ell_i C^k_{\ell j} - A^\ell_j C^k_{\ell i}, \qquad (3.4)$$

$$A_{jk,i} = A_{ji,k} + A^l_i C_{ljk} - A^l_j C_{lik}.$$

Finally, consider the third covariant derivative of F

$$
\begin{aligned}
F_{,ijk} &= g_{ij}\xi_{,k} + C^\ell_{ij,k} F_\ell + C^\ell_{ij} F_{,\ell k}\\
&= -g_{ij} A^\ell_k F_{,\ell} + C^\ell_{ij,k} F_{,\ell} + C_{ijk}\xi + C^m_{ij} C^\ell_{mk} F_{,\ell}
\end{aligned}
$$

Recall the conventions for commuting covariant derivatives of tensors by using the Riemannian curvature R^{ℓ}_{ijk}:

$$v^h_{,ji} - v^h_{,ij} = R^h_{ijk} v^k, \quad \text{and} \quad w_{k,ji} - w_{k,ij} = -R^h_{ijk} w_h.$$

Therefore,

$$\begin{aligned}
-R^{\ell}_{jki} F_{,\ell} &= F_{,ikj} - F_{,ijk} \\
&= -g_{ik} A^{\ell}_j F_{,\ell} + C^{\ell}_{ik,j} F_{,\ell} + C_{ikj}\xi + C^m_{ik} C^{\ell}_{mj} F_{,\ell} \\
&\quad + g_{ij} A^{\ell}_k F_{,\ell} - C^{\ell}_{ij,k} F_{,\ell} - C_{ijk}\xi - C^m_{ij} C^{\ell}_{mk} F_{,\ell}
\end{aligned}$$

From the part of this equation in the span of ξ, we see

$$C_{ikj} = C_{ijk},$$

and so the cubic form is totally symmetric in all three indices. Lower the index $R_{jk\ell i} = R^m_{jki} g_{m\ell}$ and compute $2R_{jk\ell i} = R_{jk\ell i} - R_{jki\ell}$ to find

$$R_{jk\ell i} = \tfrac{1}{2} g_{ik} A_{j\ell} - \tfrac{1}{2} g_{ij} A_{k\ell} - \tfrac{1}{2} g_{\ell k} A_{ji} + \tfrac{1}{2} g_{\ell j} A_{ki} - C^m_{ik} C_{mj\ell} + C^m_{ij} C_{mk\ell}, \tag{3.5}$$

$$R^{\ell}_{jki} = \tfrac{1}{2}(g_{ik} A^{\ell}_j - g_{ij} A^{\ell}_k - \delta^{\ell}_k A_{ji} + \delta^{\ell}_j A_{ki}) - C^m_{ik} C^{\ell}_{mj} + C^m_{ij} C^{\ell}_{mk},$$

and the Ricci curvature of the affine metric

$$R_{ki} = g^{j\ell} R_{jk\ell i} = \tfrac{1}{2} g_{ik} H + \tfrac{n-2}{2} A_{ki} + C^{m\ell}_i C_{mk\ell}.$$

Note here that $H = A^i_i$ is the *affine mean curvature.*

On the other hand we may compute $0 = R_{jki\ell} + R_{jk\ell i}$ to find the following Codazzi equation for the cubic form:

$$C_{ij\ell,k} - C_{ik\ell,j} = \tfrac{1}{2} g_{ij} A_{k\ell} - \tfrac{1}{2} g_{ik} A_{j\ell} + \tfrac{1}{2} g_{\ell j} A_{ki} - \tfrac{1}{2} g_{\ell k} A_{ji}. \tag{3.6}$$

Thus far, we have only used equations (3.1) and (3.2) to derive the structure equations. The only constraint is that the transversal vector field ξ be equiaffine. The position vector and the Euclidean normal are also equiaffine. Another important property of the affine normal is the following *apolarity* condition

$$C^i_{ij} = 0, \tag{3.7}$$

which follows from taking the first covariant derivative of Condition 3 in Proposition 2.1:

$$\begin{aligned}
0 &= \partial_j \det(F_{,1},\ldots,F_{,n},\xi) \\
&= \det(F_{,1j},\ldots,F_{,n},\xi)+\cdots+\det(F_{,1},\ldots,F_{,nj},\xi)+\det(F_{,1},\ldots,F_{,n},\xi_{,j}) \\
&= C^1_{1j}\det(F_{,1},\ldots,F_{,n},\xi)+\cdots+C^n_{nj}\det(F_{,1},\ldots,F_{,n},\xi)+0 \\
&= \left(C^i_{ij}\right)\det(F_{,1},\ldots,F_{,n},\xi).
\end{aligned}$$

The apolarity condition and (3.3) imply the following formula for the affine normal in terms of the metric:

$$\xi = \frac{\Delta F}{n}.$$

4 Evolution of $\bar{g}_{ij}$, g_{ij} and K

Let M^n be an n-dimensional smooth manifold and let $F(\cdot,t) : M^n \mapsto R^{n+1}$ be a one-parameter family of smooth hypersurface immersions in R^{n+1}. We say that it is a solution of the affine normal flow if

$$\partial_t F = \frac{\partial F(x,t)}{\partial t} = \xi\ , \quad x \in M^n\ , \quad t > 0 \tag{4.1}$$

where ξ is affine normal flow on $F(\cdot,t)$.

In a local coordinate system $\{x_i\}$, $1 \le i \le n$. The Euclidean inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{n+1}$ induces the metric $\bar{g}_{ij}$ and the Euclidean second fundamental form h_{ij} on $F(\cdot,t)$. These can be computed as follows

$$\bar{g}_{ij} = \langle \partial_i F, \partial_j F\rangle$$

and

$$h_{ij} = \langle \partial^2_{ij} F, \nu\rangle,$$

where ν is the unit inward normal on $F(\cdot,t)$. The Gaussian curvature is

$$K = \frac{\det h_{ij}}{\det \bar{g}_{ij}}.$$

By (2.3) and (2.4), the affine metric is

$$g_{ij} = \frac{h_{ij}}{\phi}, \quad \text{where} \quad \phi = K^{\frac{1}{n+2}}.$$

(Note that $\det \bar{g}_{ij} = \det(\partial_1 F, \ldots, \partial_n F, \nu)^2$.) Proposition 2.1 shows that the affine normal is

$$\xi = -h^{ki}\,\partial_i\phi\,\partial_k F + \phi\nu = -g^{ki}\,\partial_i(\ln\phi)\,\partial_k F + \phi\nu. \tag{4.2}$$

(Note that ν is equiaffine.) Also recall the affine curvature $\{A_j^k\}$ is defined by

$$\partial_j \xi = -A_j^k\,\partial_k F. \tag{4.3}$$

As we'll see below in Section 7, the support function of a smooth convex hypersurface is defined by

$$s = -\langle F, \nu\rangle.$$

Proposition 4.1. *Under the affine normal flow,*

$$\begin{aligned}
\partial_t F_{,i} &= -A_i^k F_{,k}, \\
\partial_t \nu &= 0, \\
\partial_j \nu &= -h_{jl}\bar{g}^{lm} F_{,m}, \\
\partial_t \bar{g}_{ij} &= -(A_i^k \bar{g}_{kj} + A_j^k \bar{g}_{ki}), \\
\partial_t \bar{g}^{ij} &= A_k^i \bar{g}^{kj} + A_k^j \bar{g}^{ki}, \\
\partial_t \det \bar{g}_{ij} &= -2H \det \bar{g}_{ij}, \\
\partial_t h_{ij} &= -\phi A_{ij}, \\
\partial_t \det h_{ij} &= -H \det h_{ij}, \\
\partial_t K &= HK, \\
\partial_t \phi &= \frac{H}{n+2}\,\phi, \\
\partial_t g_{ij} &= -\frac{H}{n+2}\,g_{ij} - A_{ij}, \\
\partial_t s &= -\phi.
\end{aligned}$$

Proof. We interchange partial derivatives and use equation (4.1) to get

$$\partial_t F_{,i} = \partial^2_{ti} F = \partial_i \xi = -A_i^k F_{,k}.$$

Note we have also used the definition of affine curvature in equation (4.3).

Since $\partial_t \nu$ is a tangent vector,

$$\begin{aligned}
\partial_t \nu &= \langle \partial_t \nu, F_{,i}\rangle \bar{g}^{ij} F_{,j} \\
&= -\langle \nu, \partial^2_{ti} F\rangle \bar{g}^{ij} F_{,j} \\
&= -\langle \nu, -A_i^k F_{,k}\rangle \bar{g}^{ij} F_{,j} \\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
\partial_p \nu &= \langle \partial_p \nu, F_{,i} \rangle \bar{g}^{ij} F_{,j} \\
&= -\langle \nu, \partial^2_{pi} F \rangle \bar{g}^{ij} F_{,j} \\
&= -h_{pi} \bar{g}^{ij} F_{,j}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{g}_{ij} &= \partial_t \langle \partial_i F, \partial_j F \rangle \\
&= \langle \partial^2_{ti} F, \partial_j F \rangle + \langle \partial_i F, \partial^2_{tj} F \rangle \\
&= \langle -A^k_i \partial_k F, \partial_j F \rangle + \langle \partial_i F, -A^l_j \partial_l F \rangle \\
&= -A^k_i \bar{g}_{kj} - A^k_j \bar{g}_{ki}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \det \bar{g}_{ij} &= (\det \bar{g}_{lm}) \bar{g}^{ij} \partial_t \bar{g}_{ij} \\
&= (\det \bar{g}_{lm}) \bar{g}^{ij} (-A^k_i \bar{g}_{kj} - A^k_j \bar{g}_{ki}) \\
&= -2(\det \bar{g}_{lm}) H.
\end{aligned}
$$

$$
\begin{aligned}
\partial_t h_{ij} &= \partial_t \langle \partial^2_{ij} F, \nu \rangle \\
&= \langle \partial^3_{tij} F, \nu \rangle + \langle \partial^2_{ij} F, \partial_t \nu \rangle \\
&= \langle \partial^2_{ij} \xi, \nu \rangle \\
&= \langle \partial_i (-A^k_j \partial_k F), \nu \rangle \\
&= -A^k_j h_{ik}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \det h_{ij} &= (\det h_{lm}) h^{ij} \partial_t h_{ij} \\
&= (\det h_{lm}) h^{ij} (-A^k_j h_{ik}) \\
&= -(\det h_{lm}) H
\end{aligned}
$$

Recall the formulas for the Gaussian curvature K, the affine metric g_{ij} and ϕ:

$$
K = \frac{\det h_{ij}}{\det \bar{g}_{ij}}, \qquad g_{ij} = \frac{h_{ij}}{\phi}, \qquad \phi = K^{\frac{1}{n+2}}.
$$

Thus, lowering the index on A_i^k by the affine metric,

$$\partial_t h_{ij} = -A_i^k h_{kj} = -\phi h^{kl} A_{li} h_{kj} = -\phi A_{ij},$$

$$\begin{aligned}
\partial_t K &= \partial_t \left(\frac{\det h_{ij}}{\det \bar{g}_{ij}} \right) \\
&= \frac{(\partial_t \det h_{ij}) \det \bar{g}_{ij} - \det h_{ij} (\partial_t \det \bar{g}_{ij})}{(\det \bar{g}_{ij})^2} \\
&= HK.
\end{aligned}$$

and

$$\partial_t \phi = \frac{1}{n+2} H\phi.$$

Thus

$$\begin{aligned}
\partial_t g_{ij} &= \partial_t \left(\frac{h_{ij}}{\phi} \right) \\
&= (\partial_t h_{ij}) \left(\frac{1}{\phi} \right) - \frac{h_{ij}}{\phi^2} \partial_t \phi \\
&= (-\phi A_{ij}) \left(\frac{1}{\phi} \right) - \frac{h_{ij}}{\phi^2} \left(\frac{1}{n+2} H\phi \right) \\
&= -\frac{H}{n+2} g_{ij} - A_{ij}.
\end{aligned}$$

$$\partial_t s = -\langle \partial_t F, \nu \rangle - \langle F, \partial_t \nu \rangle = -\langle \phi\nu, \nu \rangle - 0 = -\phi.$$

□

5 Evolution of the cubic form

We use the structure equation (3.1) to compute the evolution of the cubic form. First, we need to find the evolution of the affine normal ξ and of the Christoffel symbols.

Proposition 5.1. *Under the affine normal flow,*

$$\begin{aligned}
\partial_t \xi &= -\frac{1}{n+2} g^{ij} H_{,i} F_{,j} + \frac{H}{n+2} \xi \\
&= \frac{1}{n+2} \Delta \xi + \frac{2}{n+2} H\xi + \frac{4}{n+2} A_i^m C_m^{ik} F_{,k}.
\end{aligned}$$

Proof. Recall $\xi = -g^{ki}(\ln\phi)_{,i}F_{,k} + \phi\nu$. First note

$$\partial_t g^{iq} = -g^{i\ell}(\partial_t g_{\ell m})g^{mq} = -g^{i\ell}\left(-\frac{H}{n+2}g_{\ell m} - A_{\ell m}\right)g^{mq} = \frac{H}{n+2}g^{iq} + A^{iq}. \tag{5.1}$$

Then compute using Proposition 4.1

$$\begin{aligned}
\partial_t\xi &= \partial_t\left(-g^{ki}(\ln\phi)_{,i}F_{,k} + \phi\nu\right)\\
&= -(\partial_t g^{ki})(\ln\phi)_{,i}F_{,k} - g^{ki}(\partial_t \ln\phi)_{,i})F_{,k} - g^{ki}(\ln\phi)_{,i}(\partial_t F_{,k}) + (\partial_t\phi)\nu + 0\\
&= -\left(\frac{H}{n+2}g^{ki} + A^{ki}\right)(\ln\phi)_{,i}F_{,k} - g^{ki}\left(\frac{H}{n+2}\right)_{,i}F_{,k}\\
&\quad + g^{ki}(\ln\phi)_{,i}A^{\ell}_{k}F_{,\ell} + \frac{H}{n+2}\phi\nu\\
&= -\frac{1}{n+2}g^{ij}H_{,i}F_{,j} + \frac{H}{n+2}\xi.
\end{aligned}$$

From equation (3.3), we have

$$\Delta\xi = g^{ij}\xi_{,ij} = g^{ij}(-A^k_{i,j}F_{,k} - A_{ij}\xi - A^k_i C^{\ell}_{kj}F_{,\ell}) = -H\xi + g^{ij}(-A^k_{i,j}F_{,k} - A^k_i C^{\ell}_{kj}F_{,\ell}).$$

Now

$$g^{ij}A^k_{i,j}F_{,k} = g^{ij}g^{kl}A_{il,j}F_{,k} = g^{ij}g^{kl}(A_{ij,l} + A^m_i C_{mlj} - A^m_l C_{mij})F_{,k} = g^{kl}H_{,l}F_{,k} + A^m_i C^{ik}_m F_{,k}.$$

Hence

$$\Delta\xi = -H\xi - g^{kl}H_{,l}F_{,k} - 2A^m_i C^{ik}_m F_{,k}$$

and

$$\left(\partial_t - \frac{1}{n+2}\Delta\right)\xi = \frac{2}{n+2}H\xi + \frac{4}{n+2}A^m_i C^{ik}_m F_{,k}.$$

$\square$

We also compute

$$\partial_t\Gamma^k_{ij} = \partial_t \tfrac{1}{2}g^{kl}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

Note $\partial_t\Gamma^k_{ij}$ is a tensor; therefore, we may choose normal coordinates so that

$\partial_k g_{ij} = \Gamma_{ij}^k = 0$ at time $t = 0$. In these coordinates,

$$
\begin{aligned}
\partial_t \Gamma_{ij}^k &= \tfrac{1}{2} g^{k\ell} \left[\partial_i \left(-\frac{H}{n+2} g_{j\ell} - A_{j\ell} \right) + \partial_j \left(-\frac{H}{n+2} g_{i\ell} - A_{i\ell} \right) \right. \\
&\quad \left. - \partial_\ell \left(-\frac{H}{n+2} g_{ij} - A_{ij} \right) \right] \\
&= -\frac{1}{2(n+2)} [(\partial_i H)\delta_j^k + (\partial_j H)\delta_i^k - g^{k\ell}(\partial_\ell H) g_{ij}] \\
&\quad - \tfrac{1}{2}(\partial_i A_j^k + \partial_j A_i^k - g^{k\ell}\partial_\ell A_{ij}) \\
&= -\frac{1}{2(n+2)} (H_{,i}\delta_j^k + H_{,j}\delta_i^k - g^{k\ell} H_{,\ell} g_{ij}) - \tfrac{1}{2}(A_{j,i}^k + A_{i,j}^k - g^{k\ell} A_{ij,\ell})
\end{aligned}
$$

Now compute the evolution of $F_{,ij}$

$$
\begin{aligned}
\partial_t F_{,ij} &= \partial_t \partial_{ij}^2 F - (\partial_t \Gamma_{ij}^k) F_{,k} - \Gamma_{ij}^k \partial_t F_{,k} \\
&= (\partial_t F)_{,ij} - (\partial_t \Gamma_{ij}^k) F_{,k} \\
&= \xi_{,ij} + \frac{1}{2(n+2)} (H_{,i}\delta_j^k + H_{,j}\delta_i^k - g^{k\ell} H_{,\ell} g_{ij}) F_{,k} \\
&\quad + \tfrac{1}{2}(A_{j,i}^k + A_{i,j}^k - g^{k\ell} A_{ij,\ell}) F_{,k} \\
&= -A_{i,j}^k F_{,k} - A_{ij}\xi - A_i^\ell C_{\ell j}^k F_{,k} + \tfrac{1}{2}(A_{j,i}^k + A_{i,j}^k - g^{k\ell} A_{ij,\ell}) F_{,k} \\
&\quad + \frac{1}{2(n+2)} (H_{,i}\delta_j^k + H_{,j}\delta_i^k - g^{k\ell} H_{,\ell} g_{ij}) F_{,k}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\partial_t F_{,ij} &= \partial_t (g_{ij}\xi + C_{ij}^k F_{,k}) \\
&= \left(-\frac{H}{n+2} g_{ij} - A_{ij} \right) \xi + g_{ij} \left(-\frac{1}{n+2} g^{k\ell} H_{,\ell} F_{,k} + \frac{H}{n+2}\xi \right) \\
&\quad + (\partial_t C_{ij}^k) F_{,k} - C_{ij}^\ell A_\ell^k F_{,k}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\partial_t C^k_{ij} &= -A^k_{i,j} - A^\ell_i C^k_{\ell j} + \tfrac{1}{2}(A^k_{j,i} + A^k_{i,j} - g^{k\ell}A_{ij,\ell}) \\
&\quad + \frac{1}{2(n+2)}(H_{,i}\delta^k_j + H_{,j}\delta^k_i - g^{k\ell}H_{,\ell}g_{ij}) \\
&\quad + C^\ell_{ij}A^k_\ell + \frac{1}{n+2}g_{ij}g^{k\ell}H_{,\ell} \\
&= -\tfrac{1}{2}A^\ell_i C^k_{\ell j} - \tfrac{1}{2}A^\ell_j C^k_{\ell i} - \tfrac{1}{2}g^{k\ell}A_{ij,\ell} \\
&\quad + \frac{1}{2(n+2)}(H_{,i}\delta^k_j + H_{,j}\delta^k_i - g^{k\ell}H_{,\ell}g_{ij}) \\
&\quad + C^\ell_{ij}A^k_\ell + \frac{1}{n+2}g_{ij}g^{k\ell}H_{,\ell}
\end{aligned}$$

The second line follows from the first by the Codazzi equation (3.4) for A^k_i. Furthermore,

$$\begin{aligned}
\partial_t C_{ijm} &= \partial_t(g_{km}C^k_{ij}) \\
&= \left(-\frac{H}{n+2}g_{km} - A_{km}\right)C^k_{ij} - \tfrac{1}{2}A^\ell_i C_{\ell jm} - \tfrac{1}{2}A^\ell_j C_{\ell im} - \tfrac{1}{2}A_{ij,m} \\
&\quad + \frac{1}{2(n+2)}(H_{,i}g_{jm} + H_{,j}g_{im} - H_{,m}g_{ij}) + C^\ell_{ij}A_{\ell m} + \frac{1}{n+2}g_{ij}H_{,m} \\
&= -\frac{H}{n+2}C_{ijm} + \frac{1}{2(n+2)}(H_{,i}g_{jm} + H_{,j}g_{im} + H_{,m}g_{ij}) \\
&\quad - \tfrac{1}{2}(A_{ij,m} - A^\ell_m C_{\ell ij}) - \tfrac{1}{2}A^\ell_i C_{\ell jm} - \tfrac{1}{2}A^\ell_j C_{\ell im} - \tfrac{1}{2}A^\ell_m C_{\ell ij}
\end{aligned}$$

Note the first term in the last line is totally symmetric by the Codazzi equation (3.4) for A^k_i.

Now we compute the Laplacian of the cubic form. We use apolarity (3.7), the Codazzi equations (3.4) and (3.6) for A and C respectively, and the curvature equation (3.5).

$$
\begin{aligned}
0 &= g^{jk}C_{ijk,\ell m} \\
&= g^{jk}(C_{i\ell k,jm} + \tfrac{1}{2}g_{ij}A_{k\ell,m} - \tfrac{1}{2}g_{i\ell}A_{jk,m} + \tfrac{1}{2}g_{kj}A_{\ell i,m} - \tfrac{1}{2}g_{k\ell}A_{ji,m}) \\
&= g^{jk}C_{i\ell j,km} + \tfrac{1}{2}A_{i\ell,m} - \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}nA_{\ell i,m} - \tfrac{1}{2}A_{\ell i,m} \\
&= g^{jk}(C_{i\ell j,mk} - R_{kmip}C^p_{\ell j} - R_{km\ell p}C^p_{ij} - R_{kmjp}C^p_{i\ell}) - \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}nA_{\ell i,m} \\
&= g^{jk}C_{i\ell j,mk} - \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}nA_{\ell i,m} \\
&\quad - C^{pk}_{\ell}[\tfrac{1}{2}(g_{ik}A_{mp} - g_{im}A_{kp} - g_{pk}A_{mi} + g_{pm}A_{ki}) - C^r_{ik}C_{mrp} + C^r_{im}C_{krp}] \\
&\quad - C^{pk}_{i}[\tfrac{1}{2}(g_{\ell k}A_{mp} - g_{\ell m}A_{kp} - g_{pk}A_{m\ell} + g_{pm}A_{k\ell}) - C^r_{\ell k}C_{mrp} + C^r_{\ell m}C_{krp}] \\
&\quad - g^{jk}C_{i\ell p}[\tfrac{1}{2}(g_{jk}A_{mp} - g_{jm}A_{kp} - g_{pk}A_{mj} + g_{pm}A_{kj}) - C^r_{jk}C_{mrp} + C^r_{jm}C_{jrp}] \\
&= g^{jk}C_{ij\ell,mk} - \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}nA_{\ell i,m} \\
&\quad + \tfrac{1}{2}g_{im}A^p_kC^k_{p\ell} - \tfrac{1}{2}A^j_iC_{m\ell j} + 2C^r_{ik}C^p_{mr}C^k_{p\ell} - C^r_{im}C^p_{kr}C^k_{p\ell} + \tfrac{1}{2}g_{\ell m}A^p_kC^k_{pi} \\
&\quad - \tfrac{1}{2}A_{k\ell}C^k_{mi} - C^r_{\ell m}C^p_{kr}C^k_{pi} - \tfrac{1}{2}nA^p_mC_{pi\ell} - \tfrac{1}{2}HC_{mi\ell} - C^r_{jm}C^j_{rp}C^p_{i\ell} \\
&= g^{jk}C_{i\ell m,jk} + \tfrac{1}{2}A_{m\ell,i} - \tfrac{1}{2}g_{im}A^k_{\ell,k} + \tfrac{1}{2}A_{mi,\ell} - \tfrac{1}{2}g_{\ell m}A^k_{i,k} - \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}nA_{\ell i,m} \\
&\quad + \tfrac{1}{2}g_{im}A^p_kC^k_{p\ell} - \tfrac{1}{2}A^j_iC_{m\ell j} + 2C^r_{ik}C^p_{mr}C^k_{p\ell} - C^r_{im}C^p_{kr}C^k_{p\ell} + \tfrac{1}{2}g_{\ell m}A^p_kC^k_{pi} \\
&\quad - \tfrac{1}{2}A_{k\ell}C^k_{mi} - C^r_{\ell m}C^p_{kr}C^k_{pi} - \tfrac{1}{2}nA^p_mC_{pi\ell} - \tfrac{1}{2}HC_{mi\ell} - C^r_{jm}C^j_{rp}C^p_{i\ell}
\end{aligned}
$$

Now the Codazzi equation (3.4) for A^k_i and the apolarity condition (3.7) imply

$$A^k_{i,k} = A^k_{k,i} + A^\ell_kC^k_{\ell i} - A^\ell_iC^k_{\ell k} = H_{,i} + A^p_kC^k_{pi}.$$

Apply this identity and the Codazzi equation (3.4) for A^k_i to the first two occurrences of the covariant derivatives of A to find

$$
\begin{aligned}
\Delta C_{i\ell m} &= g^{jk}C_{i\ell m,jk} = \tfrac{1}{2}g_{im}H_{,\ell} + \tfrac{1}{2}g_{\ell m}H_{,i} + \tfrac{1}{2}g_{i\ell}H_{,m} + \tfrac{1}{2}(n+2)(A^k_mC_{ki\ell} - A_{\ell i,m}) \\
&\quad - 2C^r_{ik}C^p_{mr}C^k_{p\ell} + C^r_{im}C^p_{kr}C^k_{p\ell} + C^r_{\ell m}C^p_{kr}C^k_{pi} + C^r_{jm}C^j_{rp}C^p_{i\ell} + \tfrac{1}{2}HC_{mi\ell}
\end{aligned}
$$

Together with the evolution equation of C, compute

$$
\begin{aligned}
\partial_tC_{ijk} &= \frac{1}{n+2}\Delta C_{ijk} - \frac{3H}{2(n+2)}C_{ijk} + \frac{2}{n+2}C^m_{i\ell}C^p_{km}C^\ell_{pj} \\
&\quad - \frac{1}{n+2}(C^m_{ik}C^p_{\ell m}C^\ell_{pj} + C^m_{jk}C^p_{\ell m}C^\ell_{pi} + C^m_{\ell k}C^\ell_{mp}C^p_{ij}) \\
&\quad - \frac{1}{2}(A^\ell_iC_{\ell jk} + A^\ell_jC_{\ell ik} + A^\ell_kC_{\ell ij})
\end{aligned}
$$

To compute $\partial_t |C|^2$, use (5.1) to show

$$\begin{aligned}
\partial_t |C|^2 &= \partial_t (C_{ijk} g^{iq} g^{jr} g^{ks} C_{qrs}) \\
&= 3C_{ijk} (\partial_t g^{iq}) C_q^{rs} + 2(\partial_t C_{ijk}) C^{ijk} \\
&= \frac{3H}{n+2} |C|^2 + 3C_{ik}^r A^{iq} C_{qr}^k + \frac{2}{n+2} \Delta C_{ijk} C^{ijk} - \frac{3H}{n+2} |C|^2 \\
&\quad + \frac{4}{n+2} C_{i\ell}^m C_{km}^p C_{pj}^\ell C^{ijk} - \frac{6}{n+2} C_{ik}^m C_{\ell m}^p C_{pj}^\ell C^{ijk} - 3A^{i\ell} C_{\ell k}^j C_{ij}^k \\
&= \frac{2}{n+2} \Delta C_{ijk} C^{ijk} + \frac{4}{n+2} C_{i\ell}^m C_{km}^p C_{pj}^\ell C^{ijk} - \frac{6}{n+2} |P|^2
\end{aligned}$$

for $P_{ij} = C_{i\ell}^k C_{jk}^\ell$. Finally compute $\Delta |C|^2 = 2\Delta C_{ijk} C^{ijk} + 2|\nabla C|^2$ to find

$$\partial_t |C|^2 = \frac{1}{n+2} \Delta |C|^2 - \frac{2}{n+2} |\nabla C|^2 + \frac{4}{n+2} C_{i\ell}^m C_{km}^p C_{pj}^\ell C^{ijk} - \frac{6}{n+2} |P|^2.$$

Now if $\mathcal{Y}_{ijkl} = C_{ij}^m C_{klm} - C_{ik}^m C_{jlm}$, we find

$$0 \le \frac{1}{2} |\mathcal{Y}|^2 = |P|^2 - C_{i\ell}^m C_{km}^p C_{pj}^\ell C^{ijk},$$

and so

$$\partial_t |C|^2 \le \frac{1}{n+2} \Delta |C|^2 - \frac{2}{n+2} |P|^2 \le \frac{1}{n+2} \Delta |C|^2 - \frac{2}{n(n+2)} |C|^4,$$

since $|C|^2 = P_i^i$ and thus Cauchy-Schwartz applied to the eigenvalues of P implies $|P|^2 \ge \frac{1}{n} |C|^4$. We note this estimate of Andrews [1] is a parabolic version of an estimate of Calabi [4] on the cubic form on affine spheres, and is related to Calabi's earlier interior C^3 estimates of solutions to the Monge-Ampére equation [3].

The maximum principle implies the following estimate for $|C|^2$ then: If $\mathcal{L}$ is any compact smooth strictly convex hypersurface evolving as $\mathcal{L}(t)$ under the affine normal flow, then

$$\sup_{\mathcal{L}(t)} |C|^2 \le \frac{1}{(\sup_{\mathcal{L}(0)} |C|^2)^{-1} + \frac{2}{n(n+2)} t}.$$

Thus we get the following bound independent of initial data:

Proposition 5.2 (Andrews [1]). *Let $\mathcal{L}$ be any compact smooth strictly convex hypersurface evolving under the affine normal flow. Then*

$$\sup_{\mathcal{L}(t)} |C|^2 \le \frac{n(n+2)}{2t}.$$

6 Evolution of the affine curvature

In this section, we treat the evolution of the affine curvature A^i_k, as computed by Andrews [1], and also the evolution of the affine conormal vector U. At each point, U is defined by

$$\langle U, \xi\rangle = 1, \qquad \langle U, F_{,i}\rangle = 0, \quad i = 1, \ldots, n. \tag{6.1}$$

(It should be clear that in this case, we are using the Euclidean inner product $\langle\cdot,\cdot\rangle$ only for notational convenience. As its name suggests, the conormal vector U is more naturally a vector in the dual space to $\mathbb{R}^{n+1}$, not a vector in $\mathbb{R}^{n+1}$ itself.)

Proposition 6.1. *Under the affine normal flow,*

$$\begin{aligned}
\partial_t U &= -\frac{H}{n+2}U = \frac{1}{n+2}\Delta U, \\
\partial_t A^k_i &= A^j_i A^k_j + \frac{1}{n+2}H_{,\ell i}\, g^{\ell k} + \frac{1}{n+2}H_{,\ell}\, g^{\ell j} C^k_{ji} + \frac{H}{n+2}A^k_i, \\
\partial_t A_{ij} &= \frac{1}{n+2}H_{,ij} + \frac{1}{n+2}H_{,\ell}\, g^{\ell k} C_{ijk}, \\
\partial_t H &= \frac{1}{n+2}\Delta H + |A|^2 + \frac{1}{n+2}H^2. \\
\partial_t A_{ij} &= \frac{1}{n+2}\Delta A_{ij} - \frac{1}{n+2}\Big(2A^{pk}C_{pmk}C^m_{ij} + 2A^{ml}C_{mij,l} + A^p_i C_{pml}C^{ml}_j \\
&\quad + C^{lp}_i C_{lpm}A^m_j - 2A^p_l C_{pmi}C^{ml}_j - g_{ij}A^k_m A^m_k + nA^m_i A_{mj}\Big)
\end{aligned}$$

Proof. Compute $\partial_t U$ by differentiating its defining equation (6.1):

$$\begin{aligned}
\langle \partial_t U, \xi\rangle &= -\langle U, \partial_t \xi\rangle = -\left\langle -\frac{1}{n+2}g^{ij}H_{,i}F_{,j} + \frac{H}{n+2}\xi\right\rangle = -\frac{H}{n+2}. \\
\langle \partial_t U, F_{,i}\rangle &= -\langle U, \partial_t F_{,i}\rangle = -\langle U, -A^k_i F_{,k}\rangle = 0, \\
\partial_t U &= -\frac{H}{n+2}U.
\end{aligned}$$

Similarly, covariantly differentiate in space to find

$$\begin{aligned}
\langle U_{,i}, \xi\rangle &= -\langle U, \xi_{,i}\rangle = -\langle U, -A^k_i F_{,k}\rangle = 0, \\
\langle U_{,i}, F_{,j}\rangle &= -\langle U, F_{,ij}\rangle = -\langle U, g_{ij}\xi + C^k_{ij}F_{,k}\rangle = -g_{ij}, \\
\langle U_{,ij}, \xi\rangle &= -\langle U_{,i}, \xi_{,j}\rangle = -\langle U_{,i}, -A^l_j F_{,l}\rangle = -g_{il}A^l_j = -A_{ij}, \\
\langle U_{,ij}, F_{,k}\rangle &= -\langle U_{,i}, F_{,kj}\rangle = -\langle U_{,i}, g_{kj}\xi + C^l_{kj}F_{,l}\rangle = g_{il}C^l_{kj} = C_{ijk}.
\end{aligned}$$

Now for $\Delta U = g^{ij}U_{,ij}$, we have $\langle g^{ij}U_{,ij}, \xi\rangle = -H$, $\langle g^{ij}U_{,ij}, F_{,k}\rangle = g^{ij}C_{ijk} = 0$ by the apolarity of the cubic form. So $\Delta U = -HU$ and

$$\partial_t U = \frac{1}{n+2}\,\Delta U.$$

To compute $\partial_t A^i_k$, we use the defining equation for A: $\xi_{,i} = -A^k_i F_{,k}$. Take ∂_t to find

$$\left(-\frac{1}{n+2}H_{,\ell}g^{\ell k}F_{,k} + \frac{H}{n+2}\xi\right)_{,i} = \partial_t\xi_{,i} = -\partial_t(A^k_i F_k) = -(\partial_t A^k_i)F_{,k} + A^k_i A^j_k F_{,j}.$$

So we have

$$\begin{aligned}
(\partial_t A^k_i)F_{,k} &= A^k_i A^j_k F_{,j} - \left(-\frac{1}{n+2}H_{,\ell}g^{\ell k}F_{,k} + \frac{H}{n+2}\xi\right)_{,i} \\
&= A^k_i A^j_k F_{,j} - \frac{1}{n+2}[-H_{,\ell i}g^{\ell k}F_{,k} - H_{,\ell}g^{\ell k}F_{,ki} \\
&\quad + H_{,i}\xi - HA^k_i F_{,k}] \\
&= \left(A^j_i A^k_j + \frac{1}{n+2}H_{,\ell i}g^{\ell k} + \frac{1}{n+2}H_{,\ell}g^{\ell j}C^k_{ji} + \frac{H}{n+2}A^k_i\right) F_{,k}.
\end{aligned}$$

Here we have used the structure equation (3.1).

$$\begin{aligned}
\partial_t A_{im} &= \partial_t(g_{km}A^k_i) \\
&= g_{km}\left(A^j_i A^k_j + \frac{1}{n+2}H_{,\ell i}g^{\ell k} + \frac{1}{n+2}H_{,\ell}g^{\ell j}C^k_{ji} + \frac{H}{n+2}A^k_i\right) \\
&\quad + A^k_i\left(-\frac{H}{n+2}g_{km} - A_{km}\right) \\
&= \frac{1}{n+2}H_{,mi} + \frac{1}{n+2}H_{,\ell}C^\ell_{im}.
\end{aligned} \tag{6.2}$$

Finally,

$$\begin{aligned}
\partial_t H &= \partial_t A^i_i \\
&= A^j_i A^i_j + \frac{1}{n+2}H_{,\ell i}\, g^{\ell i} + \frac{1}{n+2}H_{,\ell}\, g^{\ell j}C^i_{ji} + \frac{H}{n+2}A^i_i \\
&= \frac{1}{n+2}\Delta H + |A|^2 + \frac{H^2}{n+2}
\end{aligned}$$

by the apolarity condition $C^i_{ji} = 0$. □

$$
\begin{aligned}
\Delta A_{ij} &= g^{kl} A_{ij,kl} = g^{kl} A_{ji,kl} \\
&= g^{kl}(A_k^m C_{mij} + A_{jk,i} - A_i^m C_{mjk})_l \\
&= g^{kl}(A_{mk,l} C_{ij}^m + A_k^m C_{mij,l} + A_{jk,il} - A_{mi,l} C_{jk}^m - A_i^m C_{mjk,l})
\end{aligned}
$$

$$
\begin{aligned}
A_{jk,il} &= A_{jk,li} + R_{ilj}^m A_{mk} + R_{ilk}^m A_{jm} \\
&= A_{kj,li} - \left[\tfrac{1}{2}(-g_{lj} A_i^m + g_{ij} A_l^m + \delta_l^m A_{ij} - \delta_i^m A_{lj}) + C_{lj}^p C_{pi}^m - C_{ij}^p C_{pl}^m\right] A_{mk} \\
&- \left[\tfrac{1}{2}(-g_{lk} A_i^m + g_{ik} A_l^m + \delta_l^m A_{ik} - \delta_i^m A_{lk}) + C_{lk}^p C_{pi}^m - C_{ik}^p C_{pl}^m\right] A_{mj}
\end{aligned}
$$

$$
\begin{aligned}
A_{jk,li} &= A_{kl,ji} + (A_l^m C_{mjk} - A_j^m C_{mlk})_i \\
&= A_{kl,ji} + (A_{ml,i} C_{jk}^m + A_l^m C_{mjk,i} - A_{mj,i} C_{lk}^m - A_j^m C_{mlk,i})
\end{aligned}
$$

$$
\begin{aligned}
g^{kl} A_{jk,il} &= H_{,ij} + A_{ml,i} C_j^{ml} + A_l^m C_{mj,i}^l - \tfrac{1}{2} g_{ij} A_l^m A_m^l + \tfrac{n}{2} A_i^m A_{mj} \\
&- C_{lj}^p C_{pi}^m A_m^l + C_{ij}^p C_{pl}^m A_m^l + C_{ik}^p C_{pm}^k A_j^m
\end{aligned}
$$

$$
\begin{aligned}
A_l^m C_{mj,i}^l &= A^{mk} C_{mkj,i} \\
&= A^{mk} C_{mjk,i} \\
&= A^{mk}[C_{mji,k} + \tfrac{1}{2}(g_{mk} A_{ji} + g_{jk} A_{mi} - g_{mi} A_{jk} - g_{ji} A_{mk})] \\
&= A^{mk} C_{mij,k} + \tfrac{1}{2} H A_{ij} - \tfrac{1}{2} g_{ij} A_{mk} A^{mk}
\end{aligned}
$$

$$
\begin{aligned}
g^{kl} A_{jk,il} &= H_{,ij} + A_{ml,i} C_j^{ml} + A^{mk} C_{mij,k} + \tfrac{1}{2} H A_{ij} - \tfrac{1}{2} g_{ij} A_{mk} A^{mk} - \tfrac{1}{2} g_{ij} A_l^m A_m^l \\
&+ \tfrac{n}{2} A_i^m A_{mj} - C_{lj}^p C_{pi}^m A_m^l + C_{ij}^p C_{pl}^m A_m^l + C_{ik}^p C_{pm}^k A_j^m
\end{aligned}
$$

$$
\begin{aligned}
\Delta A_{ij} &= g^{kl}(A_{mk,l}C_{ij}^m + A_k^m C_{mij,l} + A_{jk,il} - A_{i,l}^m C_{mjk} - A_i^m C_{mjk,l}) \\
&= g^{kl}(A_{mk,l}C_{ij}^m + A_k^m C_{mij,l} + A_{jk,il} - A_{i,l}^m C_{mjk} - A_i^m C_{mjk,l})
\end{aligned}
$$

By the Codazzi equations (3.4) and (3.6),

$$
\begin{aligned}
g^{kl}A_{mk,l}C_{ij}^m &= H_{,m}C_{ij}^m + A^{pk}C_{pmk}C_{ij}^m, \\
g^{kl}A_{l,i}^m C_{mjk} &= g^{kl}A_{mi,l}C_{jk}^m + A_i^p C_{mpl}C_j^{ml} - A_l^p C_{mpi}C_j^{ml}, \\
g^{kl}A_i^m C_{mjk,l} &= \frac{1}{2}(HA_{ij} + A_i^m A_{mj} - A_i^m A_{mj} - nA_i^m A_{mj}) = \frac{1}{2}(HA_{ij} - nA_i^m A_{mj}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Delta A_{ij} &= g^{kl}(A_{mk,l}C_{ij}^m + A_k^m C_{mij,l} + A_{jk,il} - A_{i,l}^m C_{mjk} - A_i^m C_{mjk,l}) \\
&= H_{,m}C_{ij}^m + A^{pk}C_{pmk}C_{ij}^m + A^{ml}C_{mij,l} + H_{,ij} + A_{ml,i}C_j^{ml} + A^{mk}C_{mij,k} \\
&\quad + \tfrac{1}{2}HA_{ij} - \tfrac{1}{2}g_{ij}A_{mk}A^{mk} - \tfrac{1}{2}g_{ij}A_l^m A_m^l + \tfrac{n}{2}A_i^m A_{mj} - C_{lj}^p C_{pi}^m A_m^l \\
&\quad + C_{ij}^p C_{pl}^m A_m^l + C_{ik}^p C_{pm}^k A_j^m - A_{mi,l}C_j^{ml} - \tfrac{1}{2}(HA_{ij} - nA_i^m A_{mj}) \\
&= H_{,ij} + H_{,m}C_{ij}^m + 2A^{pk}C_{pmk}C_{ij}^m + 2A^{ml}C_{mij,l} + A_i^p C_{pml}C_j^{ml} + A_j^m C_{lpm}C_i^{lp} \\
&\quad - 2A^{pl}g^{mn}C_{pmi}C_{lnj} - g_{ij}A_m^k A_k^m + nA_i^m A_{mj}
\end{aligned}
$$

By the evolution equation (6.2) of A_{ij},

$$
\begin{aligned}
\partial_t A_{ij} &= \frac{1}{n+2}\Delta A_{ij} - \frac{1}{n+2}\Big(2A^{pk}C_{pmk}C_{ij}^m + 2A^{ml}C_{mij,l} + A_i^p C_{pml}C_j^{ml} \\
&\quad + C_i^{lp}C_{lpm}A_j^m - 2A_l^p C_{pmi}C_j^{ml} - g_{ij}A_m^k A_k^m + nA_i^m A_{mj}\Big).
\end{aligned}
$$

7 The support function

In this section, we recall some standard facts about the support function of a convex body in $\mathbb{R}^{n+1}$, derive the equation satisfied by the support function under the affine normal flow, and use convexity to prove local C^0 and C^1 estimates for support functions of a family of smooth bounded convex domains exhausting a general convex domain.

Below we will consider the following situation: Let $\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}^i$ be a convex domain in $\mathbb{R}^{n+1}$ exhausted by bounded convex domains $\mathcal{K}^i$. Our initial hypersurface $\mathcal{L} = \partial\mathcal{K}$ will then be considered as a limit of the more regular hypersurfaces $\mathcal{L}^i = \partial\mathcal{K}^i$. Let $\mathcal{L}^i(t)$ and $\mathcal{L}(t)$ denote the affine normal flow with initial hypersurface $\mathcal{L}^i$ and $\mathcal{L}$ respectively.

Then, for an initial convex hypersurface $\mathcal{L} = \partial\mathcal{K} = \partial\left(\bigcup_{i=1}^{\infty}\mathcal{K}^i\right)$, we want local uniform estimates of the affine normal flow $\mathcal{L}^i(t)$ as $\mathcal{L}^i(t) \to \mathcal{L}(t)$. In this section, we recall some standard facts about the support function and use convexity to prove C^0 and C^1 estimates locally in $\mathcal{D}^\circ(s_{\mathcal{K}})$.

Recall for $\mathcal{L} = \partial\mathcal{K}$, the support function is defined for $Y \in \mathbb{R}^{n+1}$ by

$$s(Y) = \sup_{x\in\mathcal{K}}\langle x, Y\rangle.$$

Here are some important properties of the support function (see Rockafellar [22]). First of all, recall equation (7.2) that in the case $\mathcal{L}$ is smooth and strictly convex, the total derivative of the support function $ds = F$ the embedding. In our case, $\mathcal{L}$ may not be smooth and strictly convex; but we may still recover the convex domain $\mathcal{K}$ from the support function. Take the *Legendre transform* of s : For $x \in \mathbb{R}^{n+1}$, let

$$\delta(x) = \sup_{Y\in\mathbb{R}^{n+1}}\langle x, Y\rangle - s(Y).$$

Then δ is the *indicator function* of the closed convex set $\bar{\mathcal{K}}$. In other words,

$$\delta(x) = \begin{cases} 0 & \text{for } x \in \bar{\mathcal{K}} \\ +\infty & \text{for } x \notin \bar{\mathcal{K}}. \end{cases}$$

Let $\mathcal{D}(s) = s^{-1}(-\infty, +\infty) \subset \mathbb{R}^{n+1}$ be the *domain* of the support function s, and let $\mathcal{D}^\circ(s)$ denote the interior of the domain. The support function of a convex domain $\mathcal{K}$ is always a convex, lower-semicontinuous function $s : \mathbb{R}^{n+1} \to (-\infty, +\infty]$ of homogeneity one. Moreover, any convex lower-semicontinuous function $s : \mathbb{R}^{n+1} \to (-\infty, +\infty]$ of homogeneity one is the support function of a closed convex set so long as s is not identically $+\infty$. The support function, since it is convex, is continuous on $\mathcal{D}^\circ(s)$ but may not be continuous on all of $\mathcal{D}(s)$.

The following lemma follows from the description above of the Legendre transform of the support function:

Lemma 7.1. *If Q_1 and Q_2 are closed convex subsets of $\mathbb{R}^{n+1}$, then $Q_1 \subset Q_2$ if and only if the support functions $s_{Q_1} \leq s_{Q_2}$ on all $\mathbb{R}^{n+1}$.*

All of our estimates will be uniform on compact subsets of $\mathcal{D}^\circ(s) \times (0, T]$ for some positive time T. So we need the following lemma to start

Lemma 7.2. *If $\mathcal{K}$ is a convex domain in $\mathbb{R}^{n+1}$ which contains no lines, then for the support function $s_\mathcal{K}$, $\mathcal{D}^\circ(s_\mathcal{K}) \neq \emptyset$.*

Proof. We prove the lemma by contradiction. If $\mathcal{D}^\circ(s_\mathcal{K}) = \emptyset$, then then since $\mathcal{D}(s_\mathcal{K})$ is a convex collection of rays, $\mathcal{D}(s_\mathcal{K})$ must be contained in a hyperplane $\mathcal{H} = \{Y : \langle Y, v\rangle = 0\}$. Since $s_\mathcal{K}|_\mathcal{H}$ is a convex function of homogeneity one on $\mathcal{H}$, there is a linear function $\langle Y, w\rangle$ which is $\leq s$ on $\mathcal{H}$. Now consider the line $L = \{w + \tau v : \tau \in \mathbb{R}\}$, whose support function is

$$s_L(Y) = \begin{cases} +\infty & \text{for } \langle Y, v\rangle \neq 0 \\ \langle Y, w\rangle & \text{for } \langle Y, v\rangle = 0. \end{cases}$$

By construction, $s_L \leq s_\mathcal{K}$ on $\mathbb{R}^{n+1}$, and so $L \subset \bar{\mathcal{K}}$ by Lemma 7.1. The convex hull of L and any open ball in $\mathcal{K}$ then contains another line contained in the open set $\mathcal{K}$, and this provides a contradiction. $\square$

Proposition 7.1. *Let*

$$\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}^i$$

be convex bodies so that $\mathcal{K}^i \subset \mathcal{K}^{i+1}$. Then the support functions $s = s_\mathcal{K}$, $s_i = s_{\mathcal{K}^i}$ satisfy $s_{i+1} \geq s_i$ and $s_i \to s$ everywhere, and the convergence is uniform on compact subsets of $\mathcal{D}^\circ(s)$. If, in addition, each $\mathcal{K}^i$ is bounded with smooth, strictly convex boundary, then the C^1 norm of s_i is uniformly bounded on each compact subset of $\mathcal{D}^\circ(s)$.

Proof. First of all, it is clear from the definition of s that $s_{i+1} \geq s_i$, and $s(Y) = \lim_{i\to\infty} s_i(Y)$ for all $Y \in \mathbb{R}_{n+1}$:

$$s(Y) = \sup_{x\in\mathcal{K}} \langle x, Y\rangle = \sup_{x\in\bigcup\mathcal{K}^i} \langle x, Y\rangle = \sup_i \sup_{x\in\mathcal{K}^i} \langle x, Y\rangle = \sup_i s_{\mathcal{K}^i}(Y) = \lim_{i\to\infty} s_i(Y)$$

since $\{s_i(Y)\}$ is an increasing sequence for all Y.

Let $C \subset \mathcal{D}^\circ$ be a compact subset. Choose a compact $C' \subset \mathcal{D}^\circ$ which contains a neighborhood of C. Note that on all of $\mathcal{D}^\circ$, for all i,

$$s_1 \leq s_i \leq s.$$

Thus for $Y \in \partial C$, we have

$$|ds_i(Y)| \leq \frac{\max_{\partial C'} |s| - \min_{\partial C} |s_1|}{\text{dist } (\partial C', \partial C)}.$$

(Proof: For every direction v, consider s_i restricted to the line L through Y with direction v. Then the directional derivative of s_i at Y is bounded above by the slope of the secant line of the graph of s_i through Y and a point in $L \cap \partial C'$.)

Since each s_i is convex, the same estimate is true on all of C. Therefore, the C^1 norm of all the s_i is bounded on C, and since we have pointwise convergence, Ascoli-Arzelá implies uniform convergence of $s_i \to s$ on C. $\square$

Now we recall the standard formulas for the support function of a domain with smooth and strictly convex boundary, in particular, relating it to the Gauss curvature. We also derive the parabolic Monge-Ampère equation the support function satisfies under the affine normal flow.

Recall above that $\partial_t s = -\phi = -K^{\frac{1}{n+2}}$. We now derive some standard formulas relating the Gauss curvature K to the support function s.

Recall $s(Y)$ is a convex function on $\mathbb{R}^{n+1}$ which is homogeneous of degree one. Let $F(x)$ denote a local embedding of a smooth, strictly convex hypersurface $\mathcal{L} = \partial \mathcal{K}$. Then at any $F(x) \in \mathcal{L}$ at which $s(Y) = \langle F(x), Y \rangle$, Y is perpendicular to the tangent space $T_{F(x)}\mathcal{L}$. By restricting to Y on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we have a natural parametrization of $\mathcal{L}$, which is given by the inverse of the Gauss map $-\nu$. For $F(x) \in \mathcal{L}$, let $Y = -\nu(x)$ be the outward normal. Then since $\mathcal{L}$ is strictly convex, $x \mapsto Y$ is a local diffeomorphism for $Y \in \mathbb{S}^n$, and we can consider $F = F(Y)$ for $Y \in \mathbb{S}^n$. We extend F to be homogeneous of order zero:

$$F\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}, \qquad F(Y) = F\left(\frac{Y}{|Y|}\right).$$

Then $s(Y) = -\langle F, \nu \rangle$ and thus

$$s(Y) = \langle F, Y \rangle \tag{7.1}$$

for all $Y \in \mathbb{R}^{n+1} \setminus \{0\}$.

It is useful to consider the support function restricted to an affine hyperplane of distance 1 to the origin in $\mathbb{R}^{n+1}$. We may choose coordinates so that

$$Y = (y, -1) = (y^1, \ldots, y^n, -1).$$

By projecting from this hyperplane to $\mathbb{S}^n$, we still have a local parametrization of our hypersurface $\mathcal{L}$, and (7.1) still holds. Now differentiate (7.1) to find for $i = 1, \ldots, n$

$$\frac{\partial s}{\partial y^i} = \left\langle \frac{\partial F}{\partial y^i}, Y \right\rangle + F^i = F^i$$

since Y is normal to $\mathcal{L}$. Moreover, we use Euler's formula

$$\sum_{i=1}^{n+1} y^i \frac{\partial F}{\partial y^i} = 0$$

to show

$$\begin{aligned}
\frac{\partial F}{\partial y^{n+1}} &= -\frac{1}{y^{n+1}} \sum_{i=1}^{n} y^i \frac{\partial F}{\partial y^i} = \sum_{i=1}^{n} y^i \frac{\partial F}{\partial y^i}, \\
\frac{\partial s}{\partial y^{n+1}} &= \left\langle \frac{\partial F}{\partial y^{n+1}}, Y \right\rangle + F^{n+1} \\
&= \left\langle \sum_{i=1}^{n} y^i \frac{\partial F}{\partial y^i}, Y \right\rangle + F^{n+1} \\
&= F^{n+1}
\end{aligned}$$

since Y is normal to the image of F. Thus at any $Y \in \mathbb{R}^{n+1} \setminus \{0\}$, the total derivative

$$ds = (F^1, \ldots, F^{n+1}) = F. \tag{7.2}$$

Now differentiate $\langle \frac{\partial F}{\partial y^i}, Y \rangle = 0$ to find for $i, j = 1, \ldots, n$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial y^j} \left\langle \frac{\partial F}{\partial y^i}, Y \right\rangle \\
&= \left\langle \frac{\partial^2 F}{\partial y^i \partial y^j}, Y \right\rangle + \frac{\partial F^j}{\partial y^i}, \\
\frac{\partial^2 s}{\partial y^i \partial y^j} = \frac{\partial F^j}{\partial y^i} &= -\left\langle \frac{\partial^2 F}{\partial y^i \partial y^j}, Y \right\rangle \\
&= \left\langle \frac{\partial^2 F}{\partial y^i \partial y^j}, \nu |Y| \right\rangle \\
&= \sqrt{1 + |y|^2} \left\langle \frac{\partial^2 F}{\partial y^i \partial y^j}, \nu \right\rangle \\
&= h_{ij} \sqrt{1 + |y|^2}.
\end{aligned}$$

Moreover, we compute for $i, j = 1, \dots, n$

$$\begin{aligned}
\bar{g}_{ij} &= \left\langle \frac{\partial F}{\partial y^i}, \frac{\partial F}{\partial y^j} \right\rangle \\
&= \sum_{k=1}^{n+1} \frac{\partial F^k}{\partial y^i} \frac{\partial F^k}{\partial y^j} \\
&= \frac{\partial F^{n+1}}{\partial y^i} \frac{\partial F^{n+1}}{\partial y^j} + \sum_{k=1}^{n} \frac{\partial^2 s}{\partial y^i \partial y^k} \frac{\partial^2 s}{\partial y^j \partial y^k} \\
&= \frac{\partial F^i}{\partial y^{n+1}} \frac{\partial F^j}{\partial y^{n+1}} + \sum_{k=1}^{n} \frac{\partial^2 s}{\partial y^i \partial y^k} \frac{\partial^2 s}{\partial y^j \partial y^k} \\
&= \left(\sum_{k=1}^{n} \frac{\partial F^i}{\partial y^k} y^k \right) \left(\sum_{l=1}^{n} \frac{\partial F^j}{\partial y^l} y^l \right) + \sum_{k=1}^{n} \frac{\partial^2 s}{\partial y^i \partial y^k} \frac{\partial^2 s}{\partial y^j \partial y^k} \\
&= \sum_{k,l=1}^{n} \frac{\partial^2 s}{\partial y^i \partial y^k} (y^k y^l + \delta^{kl}) \frac{\partial^2 s}{\partial y^j \partial y^l}, \\
\det \bar{g}_{ij} &= \det \left(\frac{\partial^2 s}{\partial y^i \partial y^k} \right) \det(y^k y^l + \delta^{kl}) \det \left(\frac{\partial^2 s}{\partial y^j \partial y^l} \right) \\
&= (1 + |y|^2) \det \left(\frac{\partial^2 s}{\partial y^i \partial y^j} \right)^2 .
\end{aligned}$$

So the Gaussian curvature

$$\begin{aligned}
K &= \frac{\det h_{ij}}{\det \bar{g}_{ij}} = (1 + |y|^2)^{-\frac{n+2}{2}} \left(\det \frac{\partial^2 s}{\partial y^i \partial y^j} \right)^{-1}, \\
\phi &= K^{\frac{1}{n+2}} = (1 + |y|^2)^{-\frac{1}{2}} \left(\det \frac{\partial^2 s}{\partial y^i \partial y^j} \right)^{-\frac{1}{n+2}} .
\end{aligned}$$

In order to address the evolution of s, we note *a priori* that there are two natural parametrizations F of our hypersurface. First, the affine normal flow defines a particular parametrization at time $t > 0$ given an initial parametrization at time $t = 0$. On the other hand, for any hypersurface $F(y, t)$, there is a natural parametrization in terms of the inverse of the Gauss map $-\nu$. These two parametrizations are compatible in the following sense:

Proposition 7.2. *Given a hypersurface $\mathcal{L} \subset \mathbb{R}^{n+1}$ parametrized by the inverse of its Gauss map $F\colon \mathbb{S}^n \to \mathcal{L}$, under the affine normal flow, $F(y,t)$ is still a parametrization by the inverse of the Gauss map.*

Proof. The two parametrizations are related by the Gauss map $-\nu$. Under the affine normal flow, ν satisfies $\partial_t \nu = 0$ by Proposition 4.1. $\square$

Thus if we assume the initial parametrization is via the inverse of the Gauss map, the formulas developed in this section are still valid under the affine normal flow (and in any case the two parametrization merely differ by a diffeomorphism).

Denote by $s(y)$

$$s(y) = s(y^1, \dots, y^n, -1) = \sqrt{1+|y|^2}\ s\left(\frac{Y}{|Y|}\right)$$

for $Y/|Y| \in \mathbb{S}^n$. Thus we find under the affine normal flow

$$\partial_t s(y) = \sqrt{1+|y|^2}\ \partial_t s\left(\frac{Y}{|Y|}\right) = -\phi\sqrt{1+|y|^2} = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}},$$

where we have used $\partial_t s = -\phi$ from Proposition 4.1.

We record this as

Proposition 7.3. *For any smooth solution to the affine normal flow, the support function $s(y)$ as defined above satisfies*

$$\partial_t s(y) = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}}. \tag{7.3}$$

8 The flow

There is no question about the definition of affine normal flow beginning at a smooth strictly convex compact hypersurface in $\mathbb{R}^{n+1}$ (this is true for any convex compact hypersurface by Andrews [2]). It is convenient to define the affine normal flow for an open convex domain in $\mathbb{R}^{n+1}$ by performing affine normal flow on the boundary of the domain. In this way we let $\Psi_t \mathcal{J} = \mathcal{J}(t)$ denote the affine normal flow of $\mathcal{J}$ a bounded domain with smooth strictly convex boundary in $\mathbb{R}^{n+1}$. For t larger than the extinction time, define $\Psi_t \mathcal{J} = \mathcal{J}(t) = \emptyset$.

Note to pass from a convex embedded hypersurface $\mathcal{L}$ to a domain $\mathcal{J}$ with $\mathcal{L} = \partial\mathcal{J}$, set $\mathcal{J}$ to be the interior of the convex hull $(\hat{\mathcal{L}})^\circ$.

Consider an open convex region $\mathcal{K} \subset \mathbb{R}^{n+1}$ which contains no lines. Then the boundary $\partial\mathcal{K}$ is a properly embedded convex hypersurface in $\mathbb{R}^{n+1}$. We define the affine normal flow on the hypersurface $\partial\mathcal{K}$ by its action on the interior of its convex hull $\mathcal{K}$. Now we define the affine normal flow on the hypersurface $\partial\mathcal{K}$ and on the region $\mathcal{K}$ by

$$\mathcal{K}(t) = \bigcup_{\mathcal{J}\subset\mathcal{K}} \mathcal{J}(t), \tag{8.1}$$

where each $\mathcal{J}$ in (8.1) is a bounded domain with smooth strictly convex boundary.

Lemma 8.1. *If $\mathcal{L}$ is a compact convex hypersurface in $\mathbb{R}^{n+1}$, then our definition of the affine normal flow $\mathcal{L}(t)$ corresponds with the usual one.*

Proof. If $\mathcal{L}$ is strictly convex and smooth, then this follows at once from the maximum principle. Otherwise, Andrews [2] shows that there is a viscosity solution $\tilde{\mathcal{L}}(t)$ to the affine normal flow which is unique provided that the Hausdorff distance from $\tilde{\mathcal{L}}(t)$ to $\mathcal{L}$ goes to zero as $t \to 0$. Moreover $\tilde{\mathcal{L}}(t)$ is smooth and strictly convex for positive t less than the extinction time.

Our definition $\mathcal{L}(t)$ is clearly a viscosity solution, and the Hausdorff convergence property is satisfied by Lemma 8.3 below. Therefore, Andrews's uniqueness result implies $\tilde{\mathcal{L}}(t) = \mathcal{L}(t)$, and so our definition coincides with the standard one in the compact case. □

Remark 8.1. *We recall (see e.g. [2]) that a* viscosity solution *to a hypersurface flow problem is a family of hypersurfaces $\mathcal{L}(t)$ with initial condition $\mathcal{L}(0) = \mathcal{L}$ so that: 1) If $\mathcal{J}$ is a smooth hypersurface contained in $\mathcal{L}$, then the evolving hypersurface $\mathcal{J}(t)$ is contained in $\mathcal{L}(t)$ for all $t \in [0, T]$, and 2) If $\mathcal{J}$ is a smooth hypersurface containing $\mathcal{L}$, then the evolving hypersurface $\mathcal{J}(t)$ contains $\mathcal{L}(t)$ for all $t \in [0, T]$. In short, a viscosity solution $\mathcal{L}(t)$ is one for which the maximum principle always works, even if $\mathcal{L}(t)$ does not have C^2 regularity.*

The following proposition depends on estimates proved by Ben Andrews in the case of compact hypersurfaces [2]. Below, we prove local versions of the estimates needed.

Proposition 8.1. *Let $\mathcal{K}^i$ and $\mathcal{K}$ be open convex bodies containing no lines so that*

$$\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}^i, \qquad \mathcal{K}^i \subset \mathcal{K}^{i+1}.$$

Then for all $t > 0$,

$$\mathcal{K}(t) = \bigcup_{i=1}^{\infty} \mathcal{K}^i(t).$$

Proof. Let $\mathcal{J} \subset \mathcal{K}$ be a bounded domain with smooth, strictly convex boundary. Then

$$\mathcal{J} = \bigcup_{i=1}^{\infty} \mathcal{J}^i, \qquad \mathcal{J}^i = \mathcal{J} \cap \mathcal{K}^i.$$

Then we claim that

$$\mathcal{J}(t) = \bigcup_{i=1}^{\infty} \mathcal{J}^i(t) \tag{8.2}$$

To prove the claim (8.2), we recall estimates of Andrews [2, Section 8] for compact convex hypersurfaces (we prove local versions of these estimates below).

By exhausting $\mathcal{J}^i = \bigcup_{j=1}^{\infty} \mathcal{I}^i_j$ by nested domains $\mathcal{I}^i_j$ with smooth, strictly convex boundary, the affine normal flow $\mathcal{J}^i(t)$ is defined as a limit as $j \to \infty$ of the affine normal flow $\mathcal{I}^i_j(t)$. The support functions $s_{\mathcal{I}^i_j} \to s_{\mathcal{J}^i}$ uniformly on compact subsets of $\mathbb{R}^{n+1}$ as $j \to \infty$. The resulting C^0 estimates automatically entail parabolic $C^{2,1}$ estimates for positive t (see below), and then Krylov's theory implies parabolic $C^{2+\alpha,1+\frac{\alpha}{2}}$ estimates. These estimates ensure that the limit $s_{\mathcal{J}^i}(t)$ of the $s_{\mathcal{I}^i_j}(t)$ exists and is smooth for $t > 0$. Andrews shows this solution is unique by applying barriers and the maximum principle.

The key point is that C^0 estimates on the support function of convex bounded regions imply local parabolic $C^{2+\alpha,1+\frac{\alpha}{2}}$ estimates of the affine normal flow for all times $t > 0$. Since $\mathcal{J} = \bigcup_{i=1}^{\infty} \mathcal{J}^i$, we have that $s_{\mathcal{J}^i} \to s_{\mathcal{J}}$ locally in C^0. Therefore, under the affine normal flow, $s_{\mathcal{J}^i}(t)$ converges to a limit $s(t)$ locally in parabolic $C^{2+\alpha,1+\frac{\alpha}{2}}$ for $t > 0$. Since $\mathcal{J} = \bigcup_{i=1}^{\infty} \mathcal{J}^i$, the limit $s(t)$ converges uniformly on convex sets to $s_{\mathcal{J}}(0)$ as $t \to 0$. Andrews's uniqueness argument then shows that $s(t) = s_{\mathcal{J}}(t)$ and the claim (8.2) is proved.

Now use (8.2) to compute

$$\mathcal{K}(t) = \bigcup_{\mathcal{J} \subset \mathcal{K}} \mathcal{J}(t) = \bigcup_{\mathcal{J} \subset \mathcal{K}} \left(\bigcup_{i=1}^{\infty} \mathcal{J}^i(t) \right) \subset \bigcup_{\mathcal{J} \subset \mathcal{K}} \left(\bigcup_{i=1}^{\infty} \mathcal{K}^i(t) \right) = \bigcup_{i=1}^{\infty} \mathcal{K}^i(t).$$

($\mathcal{J}$ of course represents bounded domains with smooth, strictly convex boundaries.) On the other hand,

$$\mathcal{K}(t) = \bigcup_{\mathcal{J}\subset\mathcal{K}} \mathcal{J}(t) = \bigcup_{\mathcal{J}\subset\bigcup_{i=1}^{\infty}\mathcal{K}^i} \mathcal{J}(t) \supset \bigcup_{i=1}^{\infty}\left(\bigcup_{\mathcal{J}\subset\mathcal{K}^i} \mathcal{J}(t)\right) = \bigcup_{i=1}^{\infty} \mathcal{K}^i(t).$$

This completes the proof of Proposition 8.1. $\square$

The following corollary ensures convexity

Corollary 8.1. *$\mathcal{K}(t)$ is convex for all $t > 0$ before the extinction time.*

Proof. Choose each $\mathcal{K}^i$ in the previous proposition to be a bounded domain with strictly convex smooth boundary. Then $\mathcal{K}(t)$ is an increasing union of convex sets. $\square$

We verify that our definition satisfies the semigroup property:

Lemma 8.2. *$\Psi_t\Psi_s\mathcal{K} = \Psi_{t+s}\mathcal{K}$.*

Proof. We work in terms of the support functions. Let $s_{\mathcal{K}}(Y, t)$ denote the support function of the domain $\Psi_t\mathcal{K}$. We claim

$$s_{\mathcal{K}}(Y, t+s) = s_{\Psi_s\mathcal{K}}(Y, t). \tag{8.3}$$

To prove the claim, write $\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}^i$, where each $\mathcal{K}^i$ is a bounded domain with smooth strictly convex boundary and $\mathcal{K}^i \subset \mathcal{K}^{i+1}$ for all i. Since the semigroup property holds for each $\mathcal{K}^i$, we have

$$\Psi_{t+s}\mathcal{K}^i = \Psi_t\Psi_s\mathcal{K}^i \quad\Longrightarrow\quad s_{\mathcal{K}^i}(Y, t+s) = s_{\Psi_s\mathcal{K}^i}(Y, t)$$

for all $t, s > 0$ and $Y \in \mathbb{R}^{n+1}$. Now let $i \to \infty$. Propositions 7.1 and 8.1 then prove the claim (8.3).

The lemma follows from (8.3) because any open convex domain can be recovered from its support function by taking the Legendre transform [22]. $\square$

We also have a lemma on the continuity of the flow:

Lemma 8.3. *For any $\tau \geq 0$, and $\mathcal{K}$ a convex body in $\mathbb{R}^{n+1}$ containing no lines,*

$$\mathcal{K}(\tau) = \bigcup_{t>\tau} \mathcal{K}(t).$$

Proof. By the semigroup property, we may assume $\tau = 0$. Consider any point $p \in \mathcal{K}$. Since $\mathcal{K}$ is open, there is a small ball around p contained in $\mathcal{K}$. This ball acts as a barrier under the affine normal flow, and $p \in \mathcal{K}(t)$ for $t > 0$ the extinction time of the affine normal flow of this ball. $\square$

By means of outer barriers, we show our definition actually corresponds to the usual definition of affine normal flow for a smooth, strictly convex hypersurface. Let $\mathrm{Aff}(n+1)$ denote the special affine group $\mathbf{SL}(n+1) \ltimes \mathbb{R}^{n+1}$.

Proposition 8.2. *Let $\mathcal{L} \subset \mathbb{R}^{n+1}$ be a properly embedded convex hypersurface which contains no lines. Assume in a neighborhood of a point $p \in \mathcal{L}$ that $\mathcal{L}$ is C^2 and strictly convex. Then*

$$\frac{\partial \mathcal{L}}{\partial t}(p) = \xi_p \textit{ mod } T_p(\mathcal{L}(t)).$$

Remark 8.2. *This proposition should also follow from the estimates proved below (what is still needed in addition is a local version of Andrews's speed bound).*

Proof. Note that of course the derivative $\frac{\partial \mathcal{L}}{\partial t}(p)$ is defined only when $\mathcal{L}(t)$ is locally parametrized. This parametrization defines the derivative, but different parametrizations may cause it to vary by an element of the the tangent space $T_p(\mathcal{L}(t))$.

Since Ψ_t is a semigroup, we may assume $t = 0$.

To proceed with the proof, we need a lemma on choosing nice coordinates.

Lemma 8.4. *Let $p \in \mathcal{L} \subset \mathbb{R}^{n+1}$, and let $\mathcal{L}$ be a C^2 strictly convex hypersurface near p. Then there is an element $\Phi \in \mathbf{SL}(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$ so that $p \mapsto 0$ and the image locally is*

$$\Phi(\mathcal{L}) = \left\{ x^{n+1} = \frac{\gamma}{2}|x|^2 + o(|x|^2) \right\} \tag{8.4}$$

for $x = (x^1, \cdots, x^n)$, $\gamma > 0$.

Proof. This amounts to using $\mathrm{Aff}(n+1)$ to choose coordinates. Use a rotation to set the inward-pointing normal to be e_{n+1}, and translate so that p is at the origin. We can still move the tangent plane $\{x^{n+1} = 0\}$ by an action of $\mathbf{SL}(n, \mathbb{R})$. Since $\mathcal{L}$ is strictly convex, we have

$$\mathcal{L} = \left\{ x^{n+1} = \sum_{i,j} a_{ij} x^i x^j + o(|x|^2) \right\}$$

for (a_{ij}) a positive definite symmetric matrix. Use the action of $\mathbf{SL}(n,\mathbb{R})$ to send the ellipsoid $a_{ij}x^ix^j \leq C$ to a sphere of the same volume. This amounts to setting $a_{ij} = \frac{\gamma}{2}\delta_{ij}$ for a positive constant γ. $\square$

Locally we write $\mathcal{L}(t) = \{x^{n+1} = f(t,x)\}$ for $f(0,x) = f(x)$ given in (8.4). Modulo a tangential piece, the affine normal to $\mathcal{L}$ at 0 is $\xi = (\det f_{ij})^{\frac{1}{n+2}} e_{n+1}$ (see e.g. Nomizu-Sasaki, page 48). Thus we want to show that

$$\frac{\partial f}{\partial t}(0) = (\det f_{ij})^{\frac{1}{n+2}}.$$

In other words, we want to show

$$\limsup_{t\to 0^+} \frac{f(t,0)-f(0,0)}{t} \leq (\det f_{ij})^{\frac{1}{n+2}} \leq \liminf_{t\to 0^+} \frac{f(t,0)-f(0,0)}{t}. \tag{8.5}$$

The left-hand inequality in (8.5) is equivalent to showing that for each smooth strictly convex hypersurface $\mathcal{H} \subset \hat{\mathcal{L}}$, $p \in \mathcal{H}$, the affine normal of $\mathcal{H}$ at p is $\leq (\det f_{ij})^{\frac{1}{n+2}}$. This is true by the definition. Consider $\mathcal{H}$ a compact, strictly convex, C^2 hypersurface so that $\mathcal{H} \subset \hat{\mathcal{L}}$ and $\mathcal{H}$ coincides with $\mathcal{L}$ in a neighborhood of the point p. Then the definition gives $\hat{\mathcal{L}}(t) \supset \mathcal{H}(t)$ for all small positive t. Therefore, the left-hand inequality in (8.5) is proved.

To show the right-hand inequality in (8.5), we find specific hyperboloid barriers whose e_{n+1} component of the affine normal at p approaches $(\det f_{ij})^{\frac{1}{n+2}}$.

So choose $\epsilon > 0$. Then consider hyperboloids of the form

$$\{x_{n+1} = G(x) = \sqrt{\alpha|x|^2+\beta} - \sqrt{\beta}\}$$

for $\alpha, \beta > 0$. Then compute the Hessian matrix $G_{ij}(0) = \beta^{-\frac{1}{2}}\alpha\delta_{ij}$. Fix α and β so that $\beta^{-\frac{1}{2}}\alpha = \gamma - \epsilon$ so that

$$G(x) = G_\beta(x) = \sqrt{\sqrt{\beta}(\gamma-\epsilon)|x|^2+\beta} - \sqrt{\beta}.$$

Then for x in a small ball B near 0, $G_1(x) \leq f(x)$ by (8.4). Now since $\mathcal{L}$ is convex, $\mathcal{L} \setminus B$ lies above the graph of a function $c|x|$ for c a positive constant. $G_\beta(x) \to 0$ as $\beta \to 0^+$, and moreover $\partial G_\beta/\partial\beta \geq 0$. So we may choose β close to zero so that $G_\beta(x) \leq f(x)$ on B and also $G_\beta(x) \leq c|x|$. Therefore, $\mathcal{L}$ lies above the graph of $G_\beta(x)$, and the graph of G_β is a barrier to all compact hypersurfaces contained in $\mathcal{L}$. The e_{n+1} component of the affine normal of the graph of G_β at 0 is given by

$$\left(\det \frac{\partial^2 G_\beta}{\partial x^i \partial x^j}(0)\right)^{\frac{1}{n+2}} = (\gamma - \epsilon)^{\frac{n}{n+2}}.$$

Therefore, by our definition of affine normal flow, we have

$$\liminf_{t \to 0^+} \frac{f(t,0) - f(0,0)}{t} \geq (\gamma - \epsilon)^{\frac{n}{n+2}}$$

for all $\epsilon > 0$. Now let $\epsilon \to 0$ and Proposition 8.2 is proved. □

An important and easy consequence of our definition is the following

Theorem 8.1 (Maximum principle at infinity). *Consider two convex domains $\mathcal{K}^1 \subset \mathcal{K}^2$. Then for all positive t, $\mathcal{K}^1(t) \subset \mathcal{K}^2(t)$.*

Remark 8.3. *There are other natural flows for which such a global maximum principal fails. For example, there is an example in Ecker [10] in which two spacelike soliton solutions to the mean curvature flow in Minkowski space cross at infinity in finite time.*

Theorem 8.2 (Long Time Existence). *Let $\mathcal{K}$ be an unbounded convex domain in $\mathbb{R}^{n+1}$ which contains no lines. Then for all $t > 0$, $\mathcal{K}(t) \neq \emptyset$.*

Proof. It is well known that such a $\mathcal{K}$ contains an infinite half-cylinder in $\mathbb{R}^{n+1}$. Therefore, $\mathcal{K}$ contains ellipsoids of unbounded volume, which in turn have unbounded extinction times (the ellipsoids are equivalent under the action of $\mathrm{Aff}(n+1)$ to spheres of unbounded volume; these have unbounded extinction times, as in Example 1). The maximum principle then completes the proof. □

Proposition 8.3. *If $\mathcal{K}$ is a convex domain in $\mathbb{R}^{n+1}$ which contains a line, then the affine normal flow leaves $\mathcal{K}$ unchanged.*

Proof. Let $p \in \mathcal{K}$ and recall $\mathcal{K}$ is open. Then $\mathcal{K}$ contains a round cylinder centered at p. This cylinder contains ellipsoids of arbitrarily large volume centered at p, which act as barriers to the affine normal flow. These barriers ensure that p is always in $\mathcal{K}(t)$. Thus $\mathcal{K}(t) = \mathcal{K}$ for all $t \geq 0$. □

9 Soliton solutions

It is well-known that solitons of the affine normal flow are the convex properly embedded affine spheres—see Proposition 9.1 below. These were classified by Cheng and Yau [6].

Proposition 9.1. *Under the affine normal flow, an expanding soliton is a hyperbolic affine sphere, a translating soliton is a parabolic affine sphere, and a shrinking soliton is an elliptic affine sphere.*

Proof. This is a simple local calculation. $F = F(x)$ is a local embedding of an expanding soliton which expands away from a central point P at a given point in time if and only if

$$\partial_t F = \xi = \lambda(F - P) + Z^i F_{,i}$$

for λ a positive constant and $Z^i F_{,i}$ a tangent vector field.

The equiaffine condition of the affine normal (as in Proposition 2.1), states that $\xi_{,j}$ is contained in the tangent space. Thus we compute

$$\xi_{,j} = \lambda F_{,j} + Z^i F_{,ij} + Z^i_{,j} F_{,i} = (\lambda \delta^i_j + Z^i_{,j}) F_{,i} + Z^i (g_{ij} \xi + C^k_{ij} F_{,k}).$$

By comparing both sides of the equation in the span of ξ, we find $Z^i g_{ij} = 0$, and so the tangential piece $Z^i = 0$. Therefore, $\xi = \lambda(F - P)$, which is the equation for a hyperbolic affine sphere centered at P. The cases of shrinking and translating solitons are essentially the same. $\square$

So shrinking solitons are elliptic affine spheres, and the only properly embedded examples are ellipsoids [6], which are images under $\mathrm{Aff}(n+1)$ of the Euclidean spheres discussed above. Since they are compact, Lemma 8.1 shows our definition corresponds with the classical one. We record the example of the round sphere.

Example 1. *For a sphere of radius r in $\mathbb{R}^{n+1}$, the affine normal ξ is $r^{-\frac{n}{n+2}}$ times the unit inward normal vector. So then, the affine normal flow $\partial_t F = \xi$ becomes the ODE $dr/dt = -r^{-\frac{n}{n+2}}$, and so the radius at time t satisfies*

$$r(t) = \left(r_0^{\frac{2n+2}{n+2}} - \frac{2n+2}{n+2} t \right)^{\frac{n+2}{2n+2}}.$$

The extinction time of a sphere with initial radius r_0 is

$$\frac{n+2}{2n+2} r_0^{\frac{2n+2}{n+2}}.$$

Note that if the initial radius (or the initial enclosed volume) of a family of spheres tends to ∞, then the extinction time goes to ∞.

Translating solitons are parabolic affine spheres, and the only properly embedded examples are elliptic paraboloids. Expanding solitons are hyperbolic affine spheres, and for every convex cone in $\mathbb{R}^{n+1}$ containing no lines, there is a homothetic family of hyperbolic affine spheres asymptotic to the cone (for the standard round cone, these are simply hyperboloids). In the next few examples, we verify that our definition of affine normal flow leads to the correct behavior for these solitons.

Example 2. *We consider the paraboloid $\mathcal{L} = \{x^{n+1} = |x|^2\}$. Our affine normal flow Ψ_t is invariant under the action of* $\mathrm{Aff}(n+1)$*. Consider a point $P = (\tilde{x}^1, \ldots, \tilde{x}^{n+1})$ on the paraboloid. Then the following map in* $\mathrm{Aff}(n+1)$ *preserves the paraboloid:*

$$(x^1, \ldots, x^n, x^{n+1}) \mapsto \left(x^1 + \tilde{x}^1, \ldots, x^n + \tilde{x}^n, x^{n+1} + \tilde{x}^{n+1} + 2\sum_{i=1}^{n} x^i \tilde{x}^i \right).$$

This map sends the origin to P, and also sends

$$(0, \ldots, 0, c) \mapsto P + (0, \ldots, 0, c).$$

Since Ψ_t is invariant under such transformations, each $\Psi_t \mathcal{L} = \mathcal{L}(t)$ must be a paraboloid $x^{n+1} = |x|^2 + c(t)$, and since Ψ_t is the usual affine normal flow pointwise, $\mathcal{L}(t)$ is the standard translating soliton.

Example 3. *Let $\mathcal{C}$ be a convex cone in $\mathbb{R}^{n+1}$ which contains no lines and has the origin as vertex. Then $\mathcal{C}$ is invariant under scaling by positive constants. Of course, such homothetic scalings are not in* $\mathrm{Aff}(n+1)$ *in general, but we can still use the scaling properties of the usual affine normal flow to determine how $\Psi_t \mathcal{C} = \mathcal{C}(t)$ scales in time.*

It is straightforward to check that for each compact convex region $\mathcal{K}$ and scale parameter $\lambda > 0$,

$$\Psi_t(\lambda \mathcal{K}) = \lambda \Psi_{t\lambda^{-(2n+2)/(n+2)}} \mathcal{K}.$$

Because

$$\mathcal{K} \subset \mathcal{C} \iff \lambda\mathcal{K} \subset \mathcal{C},$$

and by our definition of Ψ_t*, then*

$$\Psi_t(\mathcal{C}) = \Psi_t(\lambda\mathcal{C}) = \lambda\Psi_{t\lambda^{-(2n+2)/(n+2)}}\mathcal{C}$$

for all $\lambda > 0$*. Thus for all* $t > 0$*, we have*

$$\Psi_t\mathcal{C} = t^{\frac{n+2}{2n+2}}\Psi_1\mathcal{C}.$$

Thus the hypersurface evolving from any such cone $\mathcal{C}$ *scales as a hyperbolic affine sphere under the affine normal flow. Indeed, this expanding soliton is (homothetic scalings of) Cheng-Yau's hyperbolic affine sphere. The local regularity results below will prove that this viscosity solution is the same as Cheng-Yau's smooth solution.*

If the cone $\mathcal{C}$ is homogeneous, then we can use the full affine symmetry group to conclude much more. Below we analyze the affine normal flow of Calabi's example [4]. The case of the hyperboloid is similar (the symmetry group being the Lorentz group in this case).

Example 4. *Let* $\mathcal{C} = \mathcal{C}(0)$ *be the boundary of the first orthant. Then since* $\mathcal{C}$ *is invariant under multiplying all the coordinates by positive scalars, and the flow is invariant under Aff*$(n+1)$*,* $\mathcal{C}(t)$ *is invariant under the group*

$$G = \{(\lambda_1, \dots, \lambda_{n+1}) : \lambda_i > 0, \prod \lambda_i = 1\}$$

acting by

$$(x^1, \dots, x^{n+1}) \mapsto (\lambda_1 x^1, \dots, \lambda_{n+1}x^{n+1}).$$

At time ϵ*, Calabi's example does move (consider a hyperboloid containing the first orthant with vertex at the origin). By group invariance,* $\mathcal{C}(t)$ *must be of the form*

$$\left\{\prod_i x^i = \text{const.},\, x^i > 0\right\},$$

which is an orbit of G*. Proposition 8.2 shows that at time* $t > 0$*, our flow is the affine normal flow pointwise. The radial graph of* $\Psi_t\mathcal{C}$ *must solve an ODE in* t*, and so it must be the standard solution*

$$\mathcal{C}(t) = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i \geq 0, \prod_i x^i = c_n t^{\frac{n+2}{2}}\}$$

for $c_n = (n+1)^{\frac{1}{2}}(\frac{2}{n+2})^{\frac{n+2}{2}}$.

10 Andrews's Speed Estimate

The following proposition is essentially found in Andrews [2], following work of Tso [24], although the statement of the proposition in [2] is slightly incorrect.

Proposition 10.1. *Let s be the support function of a smooth strictly convex compact hypersurface evolving under affine normal flow. If $s(Y,t) \geq r > 0$ for all $Y \in \mathbb{S}^n$ and $t \in [0,T]$, then*

$$|\partial_t s| \leq \left(C + C' t^{-\frac{n}{2n+2}}\right) s$$

on $\mathbb{S}^n \times [0,T]$, where C and C' are constants only depending on r and n.

Proof. Consider the function

$$q = \frac{-\partial_t s}{s - r/2}.$$

We apply the maximum principle to $\log q = \log |\partial_t s| - \log(s - r/2)$. In particular, at a fixed time $t \in [0,T]$, consider a point $Y \in \mathbb{S}^n$ at which q attains its maximum. By changing coordinates, we may assume that this point $Y = (0, \ldots, 0, -1)$ is the south pole. Then, as in Tso, consider the coordinates $y = (y^1, \ldots, y^n)$ for s restricted to the hyperplane $\{(y^1, \ldots, y^n, -1)\}$. At $y = 0$, we have for $i = 1, \ldots, n$

$$(\log q)_i = 0 \quad \Longleftrightarrow \quad \frac{s_{ti}}{s_t} = \frac{s_i}{s - r/2} \tag{10.1}$$

The condition for $(\log q)|_{\mathbb{S}^n}$ to have a maximum at the south pole is

$$(\log q)_{ij} + (\log q)_{n+1} \delta_{ij} \leq 0 \tag{10.2}$$

as a symmetric matrix. Here we use subscripts to denote ordinary differentiation $f_i = \partial_{y^i} f$ and $f_t = \partial_t f$.

To compute the second term in (10.2), use Euler's identities

$$\sum_{i=1}^{n+1} y^i s_{ti} = s_t, \qquad \sum_{i=1}^{n+1} y^i s_i = s$$

at the point $Y = (0, \dots, 0, -1)$ to conclude $s_{tn+1} = -s_t$, $s_{n+1} = -s$, and

$$(\log q)_{n+1} = \frac{r/2}{s - r/2}.$$

For the first term in (10.2), compute

$$\begin{aligned}
(\log q)_{ij} &= \frac{s_{tij}}{s_t} - \frac{s_{ti}s_{tj}}{s_t^2} - \frac{s_{ij}}{s - r/2} + \frac{s_i s_j}{(s - r/2)^2} \\
&= \frac{s_{tij}}{s_t} - \frac{s_{ij}}{s - r/2}
\end{aligned}$$

at $y = 0$ by (10.1). Thus (10.2) becomes at $y = 0$

$$\frac{r/2}{s - r/2}\delta_{ij} + \frac{s_{tij}}{s_t} - \frac{s_{ij}}{s - r/2} \leq 0. \tag{10.3}$$

Now, we compute using the flow equation (7.3)

$$\begin{aligned}
(\log q)_t &= \partial_t \log |\partial_t s| - \partial_t \log(s - r/2) \\
&= -\frac{1}{n+2}\, \partial_t \log \det(s_{ij}) - \frac{s_t}{s - r/2} \\
&= -\frac{1}{n+2}\, s^{ij} s_{tij} - \frac{s_t}{s - r/2}
\end{aligned}$$

for s^{ij} the inverse matrix of s_{ij}. Then (10.3) implies that

$$\begin{aligned}
(\log q)_t &\leq \frac{r/2}{n+2} \cdot \frac{s_t}{s - r/2}\delta_{ij}s^{ij} - \frac{2n}{n+2} \cdot \frac{s_t}{s - r/2} \\
&= -\frac{r/2}{n+2}\, q\, \delta_{ij}s^{ij} + \frac{2n}{n+2}\, q, \\
q_t &\leq -\frac{r/2}{n+2}\, q^2\, \delta_{ij}s^{ij} + \frac{2n}{n+2}\, q^2.
\end{aligned}$$

Now if we let μ_i be the eigenvalues of s^{ij}, or equivalently the reciprocals of the eigenvalues of s_{ij}, then we see

$$|s_t| = (\det s_{ij})^{-\frac{1}{n+2}} = \left(\prod_{i=1}^{n} \mu_i\right)^{\frac{1}{n+2}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} \mu_i\right)^{\frac{n}{n+2}} = \left(\frac{1}{n}\, \delta_{ij}s^{ij}\right)^{\frac{n}{n+2}}$$

by the arithmetic-geometric mean inequality. Therefore,

$$\delta_{ij}s^{ij} \geq n|s_t|^{\frac{n+2}{n}} = nq^{\frac{n+2}{n}}(s-r/2)^{\frac{n+2}{n}} \geq nq^{\frac{n+2}{n}}(r/2)^{\frac{n+2}{n}}$$

since $s \geq r$. And so finally, at $y = 0$, and thus at any maximum point of $q|_{\mathbb{S}^n}$,

$$q_t \leq -\frac{n(r/2)^{\frac{2n+2}{n}}}{n+2}\, q^{\frac{3n+2}{n}} + \frac{2n}{n+2}\, q^2. \tag{10.4}$$

Now define $Q(t) = \max_{Y\in\mathbb{S}^n} q(Y,t)$. Then (10.4) implies that

$$Q_t \leq -Q^2\left(c_n r^{\frac{2n+2}{n}} Q^{\frac{n+2}{n}} - c_n'\right)$$

for constants c_n, c_n' depending only on n. Therefore,

$$Q \leq \max\left\{c_n r^{-\frac{2n+2}{n+2}}, c_n' r^{-1} t^{-\frac{n}{2n+2}}\right\} \tag{10.5}$$

for c_n, c_n' new constants depending only on n. The result easily follows.

Remark 10.1. *Q may not be differentiable as a function of t, but the above estimate (10.5) still holds—see e.g. Hamilton [16, Section 3].*

□

11 Gutiérrez-Huang's Second Derivative Estimate

In this section, we follow Gutiérrez-Huang [15] to find an upper bound of the Hessian of solutions to the equation satisfied by the support function under the affine normal flow

$$\partial_t s = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}}.$$

This is a parabolic version of an estimate of Pogorelov for elliptic Monge-Ampère equations. We will treat the slightly more general case

$$\partial_t u = -\rho(y)\left(\det \frac{\partial^2 u}{\partial y^i \partial y^j}\right)^{-\alpha}, \tag{11.1}$$

for $\rho(y)$ a smooth positive function on $\mathbb{R}^n$ and α a positive constant. Gutiérrez and Huang considered the case $\rho(y) = \alpha = 1$. The reason we introduce $\rho(y)$ is that the evolution of the support function of a hypersurface by a power of the Gauss curvature involves a term $\rho(y)$ which is a power of $1 + |y|^2$. The calculations are essentially the same as those in [15].

First we define a *bowl-shaped domain* in spacetime and its *parabolic boundary.* A set $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is bowl-shaped if there are constants $t_0 < T$ so that

$$\Omega = \bigcup_{t_0 \le t \le T} \Omega_t \times \{t\},$$

where each Ω_t is convex and $\Omega_{t_1} \subset \Omega_{t_2}$ whenever $t_1 < t_2$. The parabolic boundary of Ω is then $\partial\Omega \setminus (\Omega_T \times \{T\})$.

Proposition 11.1. *Let u be a smooth solution to (11.1) which is convex in y, and let Ω be a bowl-shaped domain in space-time $\mathbb{R}^n \times \mathbb{R}$ so that $u = 0$ on the parabolic boundary of Ω. Let β be any unit direction in space.*

Then at the maximum point P of the function

$$w = |u| \; \partial^2_{\beta\beta} u \; e^{\frac{1}{2}(\partial_\beta u)^2},$$

w is bounded by a constant depending on only α, ρ, $u(P)$, $\nabla u(P)$ and n.

Proof. Choose coordinates so that $\beta = (1, 0, \ldots, 0)$ and so that at a maximum point P of w, $u_{ij} = \frac{\partial^2 u}{\partial y^i \partial y^j}$ is diagonal (in order to bound all second derivatives $u_{\beta\beta}$, it suffices to focus only on the eigendirections of the Hessian of u).

Since w is positive in Ω and 0 on the parabolic boundary, there is a point P outside the parabolic boundary of Ω at which w assumes its maximum value. We work with $\log w$ instead of w. Then at P,

$$(\log w)_i = 0, \qquad (\log w)_t \ge 0, \qquad (\log w)_{ij} \le 0.$$

Here we use i, j, t subscripts for partial derivatives in y^i, y^j and t, and the last inequality is as a symmetric matrix. These equations become, at P,

$$\frac{u_i}{u} + \frac{u_{11i}}{u_{11}} + u_1 u_{1i} = 0, \tag{11.2}$$

$$\frac{u_t}{u} + \frac{u_{11t}}{u_{11}} + u_1 u_{1t} \ge 0, \tag{11.3}$$

$$\frac{u_{ij}}{u} - \frac{u_i u_j}{u^2} + \frac{u_{11ij}}{u_{11}} - \frac{u_{11i} u_{11j}}{u_{11}^2} + u_{1j} u_{1i} + u_1 u_{1ij} \le 0. \tag{11.4}$$

To use (11.3), we compute

$$
\begin{aligned}
u_{1t} &= \left[-\rho(\det u_{ij})^{-\alpha}\right]_1 \\
&= (\det u_{ij})^{-\alpha}(-\rho_1 + \alpha\rho\, u^{ij}u_{ij1}), \\
u_{11t} &= (\det u_{ij})^{-\alpha} \\
&\quad \times \left[2\alpha\rho_1\, u^{ij}u_{ij1} - \alpha^2\rho(u^{ij}u_{ij1})^2 - \rho_{11} - \alpha\rho\, u^{ik}u^{jl}u_{kl1}u_{ij1} + \alpha\rho\, u^{ij}u_{ij11}\right],
\end{aligned}
$$

where u^{ij} is the inverse matrix of the Hessian u_{ij}. Now plug into (11.3) and divide out by $(\det u_{ij})^{-\alpha}$ to find

$$
\begin{aligned}
\frac{1}{u_{11}}&\left[2\alpha\rho_1\, u^{ij}u_{ij1} - \alpha^2\rho(u^{ij}u_{ij1})^2 - \rho_{11} - \alpha\rho\, u^{ik}u^{jl}u_{kl1}u_{ij1} + \alpha\rho\, u^{ij}u_{ij11}\right] \\
&\qquad - \frac{\rho}{u} + u_1(-\rho_1 + \alpha\rho\, u^{ij}u_{ij1}) \geq 0 \qquad (11.5)
\end{aligned}
$$

The last term of the first line of (11.5) leads us to contract (11.4) with the positive-definite matrix u^{ij} so that at P:

$$
\begin{aligned}
0 &\geq u^{ij}\left(\frac{u_{ij}}{u} - \frac{u_iu_j}{u^2} + \frac{u_{11ij}}{u_{11}} - \frac{u_{11i}u_{11j}}{u_{11}^2} + u_{1j}u_{1i} + u_1u_{1ij}\right) \\
&= \frac{n}{u} - \frac{2u^{ij}u_iu_j}{u^2} + \frac{u^{ij}u_{11ij}}{u_{11}} - \frac{u^{ij}u_iu_1u_{1j}}{u} - \frac{u^{ij}u_ju_1u_{1i}}{u} \\
&\quad - u^{ij}u_1^2u_{1i}u_{1j} + u^{ij}u_{1j}u_{1i} + u^{ij}u_1u_{1ij} \qquad \text{(by (11.2))} \\
&= \frac{n}{u} - \frac{2u^{ij}u_iu_j}{u^2} + \frac{u^{ij}u_{11ij}}{u_{11}} - \frac{2u_1^2}{u} - u_1^2u_{11} + u_{11} + u^{ij}u_1u_{1ij} \\
&\qquad \text{(since } u_{ij} \text{ is diagonal at } P\text{)} \\
&\geq \frac{n}{u} - \frac{2u^{ij}u_iu_j}{u^2} - \frac{2u_1^2}{u} - u_1^2u_{11} + u_{11} + u^{ij}u_1u_{1ij} + \frac{1}{\alpha u} + \frac{\rho_1u_1}{\alpha\rho} \\
&\quad - u_1u^{ij}u_{ij1} - \frac{2\rho_1u^{ij}u_{ij1}}{\rho u_{11}} + \frac{\alpha(u^{ij}u_{ij1})^2}{u_{11}} + \frac{\rho_{11}}{\alpha\rho u_{11}} + \frac{u^{ik}u^{jl}u_{kl1}u_{ij1}}{u_{11}} \\
&\qquad \text{(by (11.5))} \\
&\geq \frac{n+\frac{1}{\alpha}}{u} - 2\sum_{i=1}^{n}\frac{u_i^2}{u^2u_{ii}} - \frac{2u_1^2}{u} - u_1^2u_{11} + u_{11} + \frac{\rho_1u_1}{\alpha\rho} \\
&\quad - \frac{\rho_1^2}{\alpha\rho^2u_{11}} + \frac{\rho_{11}}{\alpha\rho u_{11}} + \sum_{i,j=1}^{n}\frac{u_{ij1}^2}{u_{11}u_{ii}u_{jj}}
\end{aligned}
$$

by collecting terms, completing the square, and since u_{ij} is diagonal at P. Continue computing

$$\begin{aligned}
0 \quad \geq \quad & \frac{n+\frac{1}{\alpha}}{u} - 2\sum_{i=1}^{n}\frac{u_i^2}{u^2 u_{ii}} - \frac{2u_1^2}{u} - u_1^2 u_{11} + u_{11} + \frac{\rho_1 u_1}{\alpha\rho} \\
& - \frac{\rho_1^2}{\alpha\rho^2 u_{11}} + \frac{\rho_{11}}{\alpha\rho u_{11}} + \frac{u_{111}^2}{u_{11}^3} + 2\sum_{i=2}^{n}\frac{u_{11i}^2}{u_{11}^2 u_{ii}} \\
= \quad & \frac{n+\frac{1}{\alpha}}{u} - \frac{2u_1^2}{u^2 u_{11}} - \frac{2u_1^2}{u} - u_1^2 u_{11} + u_{11} + \frac{\rho_1 u_1}{\alpha\rho} \\
& - \frac{\rho_1^2}{\alpha\rho^2 u_{11}} + \frac{\rho_{11}}{\alpha\rho u_{11}} + \frac{u_1^2}{u_{11} u^2} + \frac{2u_1^2}{u} + u_1^2 u_{11}
\end{aligned}$$

by (11.2) and since u_{ij} is diagonal at P. Finally, collect terms so that

$$0 \geq u_{11} + \left(\frac{n+\frac{1}{\alpha}}{u} + \frac{\rho_1 u_1}{\alpha\rho}\right) + \frac{1}{u_{11}}\left(-\frac{u_1^2}{u^2} - \frac{\rho_1^2}{\alpha\rho^2} + \frac{\rho_{11}}{\alpha\rho}\right)$$

and multiply each side of the inequality by $u^2 u_{11} e^{u_1^2}$ to find a quadratic inequality

$$w^2 + aw + b \leq 0$$

for $w = |u| u_{11} e^{\frac{1}{2}u_1^2}$ at P the point in Ω at which the maximum of w is achieved. The coefficients a and b involve only n, α, ρ, $u(P)$ and $u_1(P)$, and so there is an upper bound of w on Ω depending on only these quantities. □

This bounds $\partial^2_{ij} s$ away from infinity, which, together with Andrews's speed estimate, shows that the ellipticity is locally uniformly controlled in the interior of appropriate bowl-shaped domains. In the next section, we use barriers constructed from Calabi's example to find appropriate bowl-shaped domains.

12 Applying Gutiérrez-Huang's Estimate

In this section, we find bowl-shaped domains which are uniformly large on any compact subset of $\mathcal{D}^\circ(s)$ for $s = s(\cdot, t)$ the support function of $\mathcal{K}(t)$. Upper barriers will be produced from Calabi's example to achieve this.

First, we give an outline of our approach: Given a sequence of smooth solutions s to $\partial_t s = -(\det s_{ij})^{-\frac{1}{n+2}}$, and a point y_0 in $\mathcal{D}^\circ(s_0)$, modify s by adding a linear function so that $s|_{t=0}$ has its minimum at y_0. Adding a linear function does not affect the flow. Then, if $s(y_0, 0) = p$, the sublevel set $\{(y,t) : s(y,t) \le p, 0 \le t \le T\}$ is a bowl-shaped domain with $(y_0, 0)$ at its vertex. In order to apply Gutiérrez-Huang's estimate, we must ensure that these bowl-shaped domains are uniformly large. This amounts to showing that s must decrease by a definite amount in a neighborhood of $(y_0, 0)$.

We achieve this by using barriers made out of Calabi's Example 4. For each of the $n+1$ faces in Calabi's initial orthant $\mathcal{C} = \mathcal{C}(0)$, consider the outward normal directions Y^i, $i = 1, \dots, n+1$. Under the affine normal flow of Calabi's example, the support function $s_{\mathcal{C}}(Y^i, t)$ remains constant in t for each $i = 1, \dots, n+1$. Thus Calabi's example, in and of itself, is inadequate as a barrier to move the support function in directions normal to these faces.

For our initial convex domain $\mathcal{K}$, we will obtain estimates only for those $Y \in \mathcal{D}^\circ(s_{\mathcal{K}})$ the interior of the domain of the support function. For such a Y, there is a supporting hyperplane to $\mathcal{K}$ with Y as its outer normal which intersects the hypersurface $\partial\mathcal{K}$ in a compact set W. Then an appropriate barrier can be constructed as the intersection

$$X = \bigcap_{i=1}^{n+1} \mathcal{C}^i$$

of $n+1$ affine images $\mathcal{C}^i$ of Calabi's example so that the boundary of X has one bounded face which contains W and is normal to Y, and $n+1$ unbounded faces (for example, a U-shaped well in $\mathbb{R}^2$ is the intersection of two affine images of the first quadrant). Under the affine normal flow, $X(t)$ must remain inside each $\mathcal{C}^i(t)$, and the explicit solution to Calabi's example then shows that the support function $s_X(Y)$ must move as t increases away from 0. The discussion below proves this geometric sketch by working with the support functions instead of the hypersurfaces involved.

Here are the details of the construction. Recall Calabi's Example 4 from above:

$$\mathcal{C}(t) = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i \ge 0, \prod_i x^i = c_n t^{\frac{n+2}{2}}\}$$

for $c_n = (n+1)^{\frac{1}{2}} (\frac{2}{n+2})^{\frac{n+2}{2}}$ and $t \ge 0$. Compute the support function for $t \ge 0$ and $Y = (y^1, \dots, y^{n+1})$:

$$s_{\mathcal{C}}(Y,t) = \begin{cases} +\infty & \text{if any } y^i > 0 \\ -(n+1)\left(c_n t^{\frac{n+2}{n}} \prod_{i=1}^{n+1} |y^i|\right)^{\frac{1}{n+1}} & \text{if all } y^i \le 0 \end{cases} \tag{12.1}$$

In our analysis, we restrict the homogeneity-one function $s_{\mathcal{C}}$ to an affine hyperplane $\sim \mathbb{R}^n$ so that $s_{\mathcal{C}}(0) = 0$ on a simplex in $\mathbb{R}^n$ and is $+\infty$ elsewhere.

Now consider the action of the affine group on the support function. If $\mathcal{K}$ is a convex body and $Y \in \mathbb{R}^{n+1}$, recall $s_{\mathcal{K}}(Y) = \sup_{x\in\mathcal{K}}\langle x, Y\rangle$. Then for any matrix A and vector b,

$$s_{A\mathcal{K}+b}(Y) = s_{\mathcal{K}}(Y)(A^{\top}Y) + \langle b, Y\rangle. \tag{12.2}$$

Therefore, for any simplex $\mathcal{S}$ in $\mathbb{R}^n$ and any linear function $\ell(y)$ on $\mathbb{R}^n$, there is an affine image of $\mathcal{C}(0)$ whose support function restricted to an affine $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is equal to $\ell(y)$ on its domain $\mathcal{S}$.

We will use $n+1$ of these copies of Calabi's example to construct our barrier. Consider a regular $(n+1)$-simplex in $\mathbb{R}^{n+1}$ with one vertex at the origin and so that the face opposite this vertex is in a hyperplane $y^{n+1} = c > 0$ and intersects the positive y^{n+1} axis. Then the $n+1$ remaining faces of this simplex form the graph of a piecewise-linear convex function P whose domain is a simplex $\mathcal{S}_n$ in $\mathbb{R}^n$ centered at the origin. Extend this function to be $+\infty$ outside $\mathcal{S}_n$. We refer to P as a *polyhedral pencil* function, after the shape of the region above its graph.

Now consider our convex body $\mathcal{K} = \cup_{m=1}^{\infty}\mathcal{K}^i$, and let $s_{\mathcal{K}}(y)$ denote the support function of $\mathcal{K}$ restricted to an affine slice of $\mathbb{R}^{n+1}$. Let $\mathcal{N}$ be a compact subset the interior of the domain of $s_{\mathcal{K}}(y)$ (i.e. $\mathcal{N}$ is the intersection of the affine hyperplane $\mathbb{R}^n$ with a compact subset of $\mathcal{D}^{\circ}(s_{\mathcal{K}})$). Then we know that $s_{\mathcal{K}^m} = s_m \to s = s_{\mathcal{K}}$ uniformly on $\mathcal{N}$ and that $|ds_m|$ is uniformly bounded on $\mathcal{N}$. This bound on the first derivatives of s_m means that there is a uniform $\lambda > 1$ so that by replacing $P(y)$ by $\lambda^n P(\lambda y)$, we have

$$P(y - \tilde{y}) + \sum_{j=1}^{n} \frac{\partial s_m}{\partial y^j}(\tilde{y})(y^j - \tilde{y}^j) + s_m(\tilde{y}) \geq s_m(y) \tag{12.3}$$

for all $\tilde{y} \in \mathcal{N}$, $y \in \mathbb{R}^n$.

So the polyhedral pencil function P provides an initial barrier at each point $\tilde{y} \in \mathcal{N}$. We do not have an explicit solution for the evolution of P, but we can conclude enough to apply Gutiérrez-Huang's estimates. Since P can be extended to be a convex, lower-semicontinuous function of homogeneity one on $\mathbb{R}^{n+1}$, there is a corresponding convex body $\mathcal{K}_P$ whose support function is P. The affine normal flow on $\mathcal{K}_P$ induces a natural flow on P: $P(Y, 0) = P(Y)$ for all $Y \in \mathbb{R}^{n+1}$, and $P(Y, t)$ is the support function of $\mathcal{K}_P(t)$. Assume that the affine hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is given by $\{y^{n+1} = -1\}$, and then we have

Lemma 12.1. $P(0,\dots,0,-1,t) < 0$ *for all* $t > 0$.

Note that in the notation $Y = (y,-1)$, $P(0,\dots,0,-1,t)$ is just $P(y,t)$ for $y = 0$. (So the Lemma may be restated as $P(0,t) < 0$ for all $t > 0$.) We use this notation for the proof.

Proof. Note $P(0,0) = 0$.

As a function of y, the domain of $P(y,0)$ consists of $n+1$ simplices in $\mathbb{R}^n$, and $P(y,0)$ is the restriction of a linear function on each one. $P(y,0)$ is then the minimum of $n+1$ copies $\mathcal{C}_1,\dots\mathcal{C}_{n+1}$ of Calabi's initial example, each properly modified by an affine transformation. Each of $\mathcal{C}_1(t),\dots\mathcal{C}_{n+1}(t)$ is an upper barrier for the evolution of P. $\mathcal{C}_k(0,t) = 0$ for all $t \ge 0$, however.

To show that $P(0,t) < 0$, we use the fact that $P(y,t)$ is always convex and less than each $\mathcal{C}_k(y,t)$. The explicit formula (12.1), together with (12.2), shows that each $\mathcal{C}_k(y,t) < 0$ for y near zero on a ray R_k leaving the origin—this is because, near the origin, the $-(\prod |y^i|)^{\frac{1}{n+1}}$ term in (12.1) will dominate any linear term coming from (12.2). Since $P(y,t)$ is convex in y and is less than each $\mathcal{C}_k(y,t)$, the graph of $P(y,t)$ must be below the convex hull of the graphs of $\{\mathcal{C}_k(y,t)\}_{k=1}^{n+1}$. Since 0 is in the convex hull of the rays $\{R_k\}_{k=1}^{n+1}$ (because P was constructed using a regular $(n+1)$-simplex in $\mathbb{R}^{n+1}$) and since $\mathcal{C}_k(y,t) < 0$ for $y \in R_k$ near 0, we conclude $P(0,t) < 0$ for each $t > 0$. $\square$

For each $\tilde{y} \in \mathcal{N}$, and for $m = 1,2,3,\dots$, consider

$$\tilde{s}_m(y) = s_m(y) - s_m(\tilde{y}) - \sum_{j=1}^{n} \frac{\partial s_m}{\partial y^j}(\tilde{y})(y^j - \tilde{y}^j).$$

Then at $t = 0$, $\tilde{s}_m(y,0)$ has its minimum value of 0 at $y = \tilde{y}$. As time goes forward, for each $T > 0$, the sublevel set $\{(y,t) : t \in (0,T],\ \tilde{s}_m(y,t) < 0\}$ is a bowl-shaped domain. This bowl-shaped domain must contain the sublevel set

$$\mathcal{B} = \{(y,t) : t \in (0,T],\ P(y - \tilde{y}) < 0\},$$

which contains $\{0\} \times (0,T]$ by Lemma 12.1. Note that $\mathcal{B}$ is (except for translation) independent of m and $\tilde{y} \in \mathcal{N}$. There is an increasing, positive function of $t > 0$ $\epsilon(t)$ so that for each $\tilde{y} \in \mathcal{N}$, Gutiérrez-Huang's estimates can be applied uniformly on the ball $B_{\epsilon(t)/2}(\tilde{y}) \times \{t\}$ to $\tilde{s}_m$—since $\tilde{s}_m$ satisfies the same flow equation (7.3) as s_m.

Since the second derivatives of $\tilde{s}_m$ are the same as those of s_m, Gutiérrez-Huang's estimate Proposition 11.1, Andrews's speed bound Proposition 10.1

and the convexity of s_m imply uniform C^2 estimates on s_m on each compact subset of $\mathcal{D}^\circ(s)\times(0,T]$, where T is chosen so that each $s_m \ge r$ on $[0,T]$ (this is possible for some T by choosing coordinates so that an evolving sphere centered at the origin as a uniform inner barrier.)

Proposition 12.1. *If T is chosen so that each $s_m \ge r$ on $\mathbb{S}^n\times[0,T]$, then on each compact subset of $\mathcal{D}^\circ(s)\times(0,T]$ there are uniform spatial C^2 estimates for s_m and the Hessian of s_m is uniformly bounded away from zero.*

Recall that $s_{\mathcal{K}^m}\to s_{\mathcal{K}}$ everywhere on $\mathbb{R}^{n+1}\times[0,T]$ by Propositions 8.1 and 7.1. The estimates will give greater regularity to this pointwise convergence.

Note that the locally uniform spatial C^2 estimates in Proposition 12.1 imply, by the evolution equation (7.3), locally uniform parabolic $C^{2,1}$ estimates (i.e. two derivatives in spatial coordinates and 1 in t). Then, since the logarithm of the Monge-Ampère operator is concave, Krylov's interior parabolic $C^{2+\alpha,1+\frac{\alpha}{2}}$ estimates [18] are available. Ascoli-Arzelá then shows that the convergence must be in $C^{2,1}$ on each compact subset of $\mathcal{D}^\circ(s)\times(0,T]$. Indeed, $s_{\mathcal{K}}$ is a $C^{2+\alpha,1+\frac{\alpha}{2}}$ solution on $\mathcal{D}^\circ(s)\times(0,T]$, and further bootstrapping shows $s_{\mathcal{K}}$ is smooth. See e.g. Gutiérrez-Huang [15] for details on defining the $C^{2+\alpha,1+\frac{\alpha}{2}}$ norm and on applying Krylov's estimates in the present case.

A remaining issue is long-time regularity. Since long-time existence is already guaranteed, we need only apply the estimates again starting at $t=T$. The only possible sticking point is that we still need to make sure that the same r satisfying $s_{\mathcal{K}^m}\ge r$ still works (in order to apply Andrews's speed estimate Proposition 10.1). This can be assured by an affine change of coordinates. As in the proof of Theorem 8.2, $\mathcal{K}$ contains ellipsoids of arbitrarily large volume. We can change the affine coordinates so that an appropriate ellipsoid becomes a sphere centered at the origin which is large enough to guarantee that $s_{\mathcal{K}}(Y,t)>2r$ for all $Y\in\mathbb{S}^n$ and $t\in[T,2T]$. This is certainly enough to ensure that we can choose new exhausting domains $\mathcal{K}^m$ satisfy $s_{\mathcal{K}^m}(Y,t)\ge r$ for all $Y\in\mathbb{S}^n$ and $t\in[T,2T]$.

Theorem 12.1. *If $\mathcal{K}$ is an unbounded convex domain in $\mathbb{R}^{n+1}$ which contains no lines, then, under the affine normal flow, the support function $s_{\mathcal{K}}=s_{\mathcal{K}}(Y,t)$ is smooth and spatially locally strictly convex on $\mathcal{D}^\circ(s_{\mathcal{K}})\times(0,\infty)$.*

13 Regularity of the hypersurface

We've seen in the previous sections that under the affine normal flow, if $\mathcal{K}$ is an unbounded convex domain in $\mathbb{R}^{n+1}$ containing no lines, the support function $s_{\mathcal{K}}$ evolves to be smooth and strictly convex for all positive time for all $Y \in \mathcal{D}^\circ(s_{\mathcal{K}})$. In this section, we verify that, for $t > 0$, every supporting hyperplane of the evolving hypersurface $\partial\mathcal{K}(t)$ has its normal vector in $\mathcal{D}^\circ(s_{\mathcal{K}})$. Therefore, since the smoothness and convexity of the hypersurface are determined by the regularity of the support function, the hypersurface $\partial\mathcal{K}(t)$ is smooth and strictly convex for all $t > 0$.

Theorem 13.1. *Let $\mathcal{K}$ be an unbounded convex domain in $\mathbb{R}^{n+1}$ which contains no lines. Then, under the affine normal flow, the hypersurface $\partial\mathcal{K}(t)$ is smooth and strictly convex for all $t > 0$.*

Moreover, if

$$\mathcal{K} = \bigcup_i \mathcal{K}^i, \qquad \mathcal{K}^i \subset \mathcal{K}^{i+1},$$

where each $\mathcal{K}^i$ is bounded and have smooth, strictly convex boundary, then for all $t > 0$, each $p \in \partial\mathcal{K}(t)$ has a neighborhood on which the sequence of hypersurfaces $\partial K^i(t)$ converges to $\partial\mathcal{K}(t)$ in the C^∞ topology.

The proof will depend on finding appropriate initial barriers. We begin with some easy results on the support function.

Lemma 13.1. *If $\mathcal{K}$ is an unbounded convex domain in $\mathbb{R}^{n+1}$, then for every nonzero $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$, there is a ray R perpendicular to Y_0 which is contained in the closure $\bar{\mathcal{K}}$.*

Proof. We work in terms of support functions. The support function $s_{\mathcal{K}}$ is a homogeneity-one, convex, lower-semicontinuous function on $\mathbb{R}^{n+1}$ with values in $(-\infty, +\infty]$. Since $\mathcal{K}$ is unbounded, $s_{\mathcal{K}}$ must assume the value $+\infty$, and the convexity of $s_{\mathcal{K}}$ implies $s_{\mathcal{K}}$ is infinite on an open half-space of $\mathbb{R}^{n+1}$.

$R \subset \bar{\mathcal{K}}$ if and only if $s_R \leq s_{\mathcal{K}}$ on all of $\mathbb{R}^{n+1}$. For the ray

$$R = \{w + \tau v : \tau \geq 0\}, \qquad s_R(Y) = \begin{cases} \langle Y, w\rangle \text{ for } \langle Y, v\rangle \leq 0 \\ +\infty \text{ for } \langle Y, v\rangle > 0. \end{cases}$$

Thus, given $s_{\mathcal{K}}$ and $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$, we seek an R so that $R \perp Y_0$ and $s_R \leq s_{\mathcal{K}}$.

Since $\mathcal{D}(s_{\mathcal{K}})$ is a convex cone in $\mathbb{R}^{n+1}$ with vertex at the origin, if $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$, then $\mathcal{D}(s_{\mathcal{K}})$ is contained in a closed half-space with Y_0 in its boundary. Thus there is a nonzero vector v so that

$$\mathcal{D}(s_{\mathcal{K}}) \subset \{Y : \langle Y, v\rangle \leq 0\}, \qquad \langle Y_0, v\rangle = 0.$$

In order to find R, we also need a vector w so that $\langle Y, w\rangle \leq s_{\mathcal{K}}(Y)$ for all $Y \in \mathcal{D}(s_{\mathcal{K}})$. This is easy: $\langle Y, w\rangle$ is the support function of the convex set $\{w\}$. So for any $w \in \bar{\mathcal{K}}$, $\langle Y, w\rangle \leq s_{\mathcal{K}}(Y)$ for all $Y \in \mathbb{R}^{n+1}$, and $R = \{v+\tau w : \tau \geq 0\}$ is the ray to be constructed. $\square$

Lemma 13.2. *If $\mathcal{K}$ is an unbounded convex domain in $\mathbb{R}^{n+1}$, $Y \in \partial\mathcal{D}(s_{\mathcal{K}})$, and R is any ray contained in $\bar{\mathcal{K}}$, there is a half-cylinder $\mathcal{Q}$ pointing in the direction of R which is contained in the open set $\mathcal{K}$.*

Proof. Let B be an open ball contained in $\mathcal{K}$. Then the convex hull of R and B contains such a half-cylinder. $\square$

We are now ready to prove Theorem 13.1.

Proof of Theorem 13.1. We show that for all $t > 0$, every supporting hyperplane of $\mathcal{K}(t)$ must have outward normal vector Y_0 lying in $\mathcal{D}^\circ(s_{\mathcal{K}})$. Then the smoothness and strict convexity of the support function $s_{\mathcal{K}(t)}$ imply that the hypersurface $\partial\mathcal{K}(t)$ is also smooth and strictly convex.

First we rule out the case $Y_0 \notin \overline{\mathcal{D}(s_{\mathcal{K}})}$. In this case, there is a closed half-space of $\mathbb{R}^{n+1}$ containing $\mathcal{D}(s_{\mathcal{K}})$ but excluding Y_0. In other words, there is a nonzero vector v so that

$$\mathcal{D}(s_{\mathcal{K}}) \subset \{Y : \langle Y, v\rangle \leq 0\}, \qquad \langle Y_0, v\rangle > 0.$$

Then, as in Lemmas 13.1 and 13.2 above, there is a half-cylinder $\mathcal{Q}$ in the direction of v contained in $\mathcal{K}$. Since there are ellipsoids of arbitrarily large volume inside $\mathcal{Q}$ to act as barriers, $\mathcal{Q}$ always intersects $\mathcal{K}(t)$. Since $\mathcal{Q}$ is in the direction of v and $\langle Y_0, v\rangle > 0$, this shows that $s_{\mathcal{K}(t)}(Y_0) = +\infty$ for all $t > 0$. Since $\mathcal{K}(t)$ is convex, this shows it has no supporting hyperplane with outward normal $Y_0 \notin \overline{\mathcal{D}(s_{\mathcal{K}})}$.

Finally, we show that if $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$ is a nonzero vector, then there is no supporting hyperplane to $\partial\mathcal{K}(t)$ with outward normal Y_0. By Proposition 13.1 below, $s_{\mathcal{K}(t)}(Y_0) = s_{\mathcal{K}}(Y_0)$ for all $t > 0$. Thus, we simply need to ensure that the hyperplane $\mathcal{P} = \{x : \langle Y_0, x\rangle = s_{\mathcal{K}}(Y_0)\}$ does not intersect $\overline{\mathcal{K}(t)}$ for

$t > 0$. To achieve this, we choose an affine image $\mathcal{I}$ of Calabi's example as an initial outer barrier. In particular, one of the faces of $\mathcal{I}$ can be chosen to be contained in the hyperplane $\mathcal{P}$. (Proof: The support function of $\mathcal{I}$ is a linear function on a cone over a simplex and is $+\infty$ elsewhere. To find such a function to be an upper barrier to $s_{\mathcal{K}}$ at Y_0, note that for any closed simplex contained in $\mathcal{D}(s_{\mathcal{K}})$, the support function $s_{\mathcal{K}}$ is continuous on this simplex by Theorem 10.2 in [22]. So for any cone $\mathcal{C}$ over a closed n-simplex containing Y_0 and contained in $\mathcal{D}(s_{\mathcal{K}})$, we may find a linear function as an upper barrier to $s_{\mathcal{K}}$ at Y_0. Then extend this function to be $+\infty$ outside $\mathcal{C}$.) The explicit solution to Calabi's example proves that $\mathcal{P}$ does not intersect $\overline{\mathcal{I}(t)} \supset \overline{\mathcal{K}(t)}$ for all $t > 0$.

Thus all the supporting hyperplanes of $\partial\mathcal{K}(t)$ have outward normal in $\mathcal{D}^\circ(s_{\mathcal{K}})$, and Theorem 13.1 is proved. $\square$

Proposition 13.1. *If $\mathcal{K}$ is a convex unbounded domain in $\mathbb{R}^{n+1}$ which does not contain any lines, then under the affine normal flow, the support function $s_{\mathcal{K}(t)}(Y_0) = s_{\mathcal{K}}(Y_0)$ for all $t > 0$ and $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$.*

Proof. It is obvious that $s_{\mathcal{K}(t)}(Y_0) \leq s_{\mathcal{K}}(Y_0)$ since the effect of the affine normal flow on support functions is to decrease them. We need only show $s_{\mathcal{K}(t)}(Y_0) \geq s_{\mathcal{K}}(Y_0)$ for $Y_0 \in \partial\mathcal{D}(s_{\mathcal{K}})$.

We achieve this by using ellipsoids as inner barriers. Assume that $s_{\mathcal{K}}(Y_0) < +\infty$ and let $\epsilon > 0$. Then there is an $x \in \mathcal{K}$ so that

$$\langle x, Y_0\rangle > s_{\mathcal{K}}(Y_0) - \epsilon.$$

Lemmas 13.1 and 13.2 ensure that there is a half-cylinder $\mathcal{Q} \subset \mathcal{K}$ which points in a direction v perpendicular to Y_0. Then, inside the convex hull of $\mathcal{Q}$ and $\{x\}$, there is another half-cylinder $\mathcal{Q}' \subset \mathcal{K}$ which points in the direction of v and whose central ray contains a point x' satisfying

$$\langle x', Y_0\rangle > s_{\mathcal{K}}(Y_0) - 2\epsilon.$$

Now there are ellipsoids of arbitrarily large volume contained in $\mathcal{Q}'$, and these inner barriers show that for all $t > 0$, there is a point x'' on the central ray of $\mathcal{Q}'$ which is contained in $\mathcal{K}(t)$. Now since $x'' - x'$ is perpendicular to Y_0,

$$s_{\mathcal{K}(t)}(Y_0) \geq \langle x'', Y_0\rangle = \langle x', Y_0\rangle > s_{\mathcal{K}}(Y_0) - 2\epsilon.$$

Thus $s_{\mathcal{K}(t)}(Y_0) \geq s_{\mathcal{K}}(Y_0)$ so long as $s_{\mathcal{K}}(Y_0) < +\infty$. The case $s_{\mathcal{K}}(Y_0) = +\infty$ is essentially the same. $\square$

14 A Dirichlet Problem

Proposition 13.1 above shows that the affine normal flow on noncompact domains can be recast as a Dirichlet boundary problem for the support function, although discontinuous and infinite boundary values are allowed. In the interior $\mathcal{D}^\circ(s_\mathcal{K})$, the support function evolves by the affine normal flow equation, while the value of the support function on the boundary $\partial\mathcal{D}(s_\mathcal{K})$ is fixed. At each positive time t, $s_{\mathcal{K}(t)}$ is lower-semicontinuous.

In terms of PDEs, we can take an affine slice of the domain of the support function. Consider first the case when $\mathcal{D}(s_\mathcal{K})$ contains no lines (this is true if and only if $\mathcal{K}$ contains a nonempty open convex cone). In this case, we can choose coordinates so that $\Omega = \{y \in \mathbb{R}^n : (y, -1) \in \mathcal{D}^\circ(s_\mathcal{K})\}$ is bounded. The support function, when restricted to this hyperplane, then satisfies the Dirichlet boundary problem for the flow equation

$$\frac{\partial s}{\partial t} = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}}$$

with initial condition given by $s_\mathcal{K}$. If $\mathcal{D}^\circ(s_\mathcal{K})$ does contain a line, then we must consider more than one affine hyperplane slice. Since $s_\mathcal{K}$ has homogeneity one, this amounts to considering $s_\mathcal{K}$ as a section of the tautological bundle over projective sphere $\mathbb{S}^n_P = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$, where $\mathbb{R}^+$ acts on $\mathbb{R}^{n+1}$ by homothetic scaling.

Alternately, we can consider $s = s_\mathcal{K}$ restricted to the Euclidean sphere $\mathbb{S}^n$. Define a subset of $\mathbb{S}^n$ to be convex if it is the intersection of $\mathbb{S}^n$ with a convex cone in $\mathbb{R}^{n+1}$ with vertex at the origin. Then $\Upsilon = \mathbb{S}^n \cap \mathcal{D}^\circ(s)$ is a convex domain in $\mathbb{S}^n$, and s evolves under the affine normal flow via a Dirichlet problem on Υ with equation, as in [2], for $s = s|_{\mathbb{S}^n}$

$$s_t = -\left[\det\left(s_{;ab} + s\delta_{ab}\right)\right]^{-\frac{1}{n+2}}.$$

Here $s_{;ab}$ denotes second covariant derivative of s with respect to the standard connection on $\mathbb{S}^n$ and the subscripts a, b indicate an orthonormal frame.

It is an interesting question to study under what condition this Dirichlet problem admits a unique solution. We plan to study this problem in detail later. We remark now that in the case that when s is continuous when restricted to the boundary $\partial\mathcal{D}(s)$, then s must be continuous on the closure $\overline{\mathcal{D}(s)}$ (see Lemma 14.1 below). Thus in the case s is continuous and finite when restricted to $\partial D(s)$, the Dirichlet problem has a unique solution by the maximum principle.

Lemma 14.1. *Let s be a convex, lower semicontinuous function from $\mathbb{R}^n$ to $(-\infty, \infty]$. If s is continuous when restricted to $\partial D(s)$, then it is continuous on the closure of its domain $\overline{\mathcal{D}(s)}$.*

Proof. Let $x_i \in \mathcal{D}^\circ(s)$, $x_i \to x \in \partial\mathcal{D}(s)$. Let $z \in \mathcal{D}^\circ(s)$ and let y_i be the intersection of $\partial D(s)$ and the ray from z to x_i. $y_i \to x$ and so $s(y_i) \to s(x)$. Moreover, s is convex restricted to each such ray, and so

$$s(y_i) - s(x_i) \geq \frac{|y_i - x_i|}{|x_i - z|}[s(x_i) - s(z)].$$

Thus, since $|y_i - x_i|/|x_i - z| \to 0$,

$$\limsup s(x_i) \leq \lim s(y_i) = s(x).$$

Lower semicontinuity then shows $\lim s(x_i) = s(x)$. □

15 Proofs of Theorems

Here we restate Theorem 1.1 a bit more precisely:

Theorem 15.1. *Let $\mathcal{L}(t)$ be a solution to the affine normal flow defined for all $t \in (-\infty, 0)$. Assume that at some $t_0 \in (-\infty, 0)$, the convex hull $\widehat{\mathcal{L}(t_0)}$ contains no lines. Then $\mathcal{L}(t)$ must be a paraboloid translating in time or an ellipsoid shrinking in time.*

Proof. Consider $\mathcal{L}'(t)$ defined for $t \in [\tau, 0)$ so that $\mathcal{L}'(\tau)$ is smooth, compact, and strictly convex. Then Proposition 5.2 and the semigroup property show that the cubic form

$$|C|^2_{\mathcal{L}'(t)} \leq \frac{c_n}{t - \tau}$$

for all $t \in [\tau, 0)$ for c_n a constant depending only on the dimension.

Theorem 13.1 then shows that $\mathcal{L}(t)$, for $t \in [\tau, 0)$, is locally a C^∞ limit of such $\mathcal{L}'(t)$. Thus $|C|^2_{\mathcal{L}(t)} \leq c_n/(t - \tau)$ also. Since $\mathcal{L}(t)$ is an ancient solution we may let $\tau \to -\infty$. So $C_{ijk} = 0$ identically on $\mathcal{L}(t)$ for all t.

A well-known classical theorem of Berwald (see e.g. Cheng-Yau [6] or Nomizu-Sasaki [20]) shows that $\mathcal{L}(t)$ must be a hyperquadric: an ellipsoid, a paraboloid, or a hyperboloid. Only the ellipsoid (a shrinking soliton) and the paraboloid (a translating soliton) are part of an ancient solution. □

We can also prove the following existence result on hyperbolic affine spheres which is essentially due to Cheng-Yau [5]. The essential step, due to Cheng-Yau, is to solve the Monge-Ampère equation

$$\det \partial_{ij}^2 \phi = \left(-\frac{1}{\phi}\right)^{\frac{1}{n+2}}, \qquad \phi|_{\partial\Omega} = 0, \qquad \partial_{ij}^2 \phi > 0$$

on a convex bounded domain $\Omega \subset \mathbb{R}^n$. The radial graph of $-\frac{1}{\phi}$ over Ω is then a hyperbolic affine sphere asymptotic to the boundary of the cone over Ω. We note that the proper embeddedness of the hyperbolic affine sphere is contained in Gigena [14] and Sasaki [23]. See [19] for a more complete discussion.

Theorem 15.2. *For every open convex cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$ which contains no lines, there is a properly embedded hyperbolic affine sphere in $\mathbb{R}^{n+1}$ asymptotic to the boundary of $\mathcal{C}$.*

Proof. Example 3 ensures that under the affine normal flow, the boundary $\partial\mathcal{C}$ evolves as an expanding soliton $\partial\mathcal{C}(t)$. The regularity result Theorem 13.1 ensures that for $t > 0$ the hypersurface $\partial\mathcal{C}(t)$ is smooth and strictly convex. Thus, for each $t > 0$, $\partial\mathcal{C}(t)$ is a hyperbolic affine sphere by Proposition 9.1. The discussion in Section 14 shows that $\partial\mathcal{C}(t)$ is asymptotic to the boundary of the cone $\mathcal{C}$. $\square$

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