ERRATUM TO AFFINE MANIFOLDS, SYZ GEOMETRY AND THE "Y" VERTEX

JOHN LOFTIN, SHING-TUNG YAU, AND ERIC ZASLOW

1. Main result

The purpose of this erratum is to correct an error in the proof of the main result of [2]. A semi-flat Calabi-Yau structure on a smooth manifold M consists of an affine flat structure, together with an affine Kähler metric whose potential satisfies $\det \Phi_{ij} = 1$ in the local affine coordinates. The main result of [2] is

Main Result. *There exist many nontrivial semi-flat Calabi-Yau structures on the complement of a trivalent vertex of a graph inside a ball in* $\mathbb{R}^3$.

We present two separate constructions of the Main Result below: The first, presented in Section 3 below, uses hyperbolic affine sphere structures constructed in [3]. The second approach, proved in Section 4 below, is to use elliptic affine spheres constructed by

Theorem 2′. *Let U be a nonzero holomorphic cubic differential on $\mathbb{CP}^1$ with exactly 3 poles of order two. Let M be $\mathbb{CP}^1$ minus the pole set of U. At each pole of U, let w be a local coordinate so that $U = \frac{1}{w^2}\,dw^3$ in a neighborhood of the pole. Let there be a conformal background metric on M which is equal to*

$$\frac{|\log|w|^2|}{|w|}\,|dw|^2$$

on a neighborhood of each pole. Then there is a $\delta > 0$ so that for all $\epsilon \in (0,\delta)$, there is a smooth bounded function η on M satisfying

$$\Delta\eta + 4\|\epsilon U\|^2 e^{-2\eta} + 2e^{\eta} - 2\kappa = 0. \tag{1}$$

(Here Δ is the Laplace operator of the background metric, $\|\cdot\|$ is the induced norm on cubic differentials, and κ is the Gauss curvature.) The bound δ only depends on $\int_M |U|^{\frac{2}{3}}$.

This replaces Theorem 2 of [2], which has a gap in its proof. The only essential difference between Theorem 2 of [2] and Theorem 2′ is that in Theorem 2, we claimed an explicit value for the bound δ, while now in Theorem 2′, we must assume δ is a small positive number without explicit estimates. We stress that this small strengthening of the hypothesis does not affect the Main Result.

2. Erratum

The proof of Theorem 2 of [2] is incomplete. There is a sign error in the proof of Part 2 of Proposition 8 which we have not yet been able to overcome. We provide a separate proof of Theorem 2′, which is only slightly weaker, in Section 4 below. We also give another alternate proof of the Main Result in Section 3 below.

Theorem 2 was used to produce global solutions of the equation governing two-dimensional elliptic affine spheres,

$$\Delta u + 4\|U\|^2 e^{-2u} + 2e^u - 2\kappa = 0, \tag{2}$$

on $\mathbb{CP}^1$ for U a nonzero cubic differential with three poles of order 2, and Δ and κ the Laplacian and curvature of a background metric. Theorem 2 claimed to find a solution of (2) for each U which is small in the sense that

$$\int_{\mathbb{CP}^1} |U|^{\frac{2}{3}} < \frac{\pi\sqrt[3]{2}}{3}. \tag{3}$$

Below in the proof of Theorem 2′, we produce solutions to (2) for all such U satisfying

$$\int_{\mathbb{CP}^1} |U|^{\frac{2}{3}} < \delta$$

for a small $\delta > 0$. Then a result of Baues-Cortés [1] relates two-dimensional elliptic affine spheres to three-dimensional semi-flat Calabi-Yau metrics, and the three poles of U form the rays of the "Y" vertex. (The original bound $\frac{\sqrt[3]{2}}{3\pi}$ is sharp if it is true [2].)

The examples in Sections 2,4,5 of [2] and the construction of local radially symmetric solutions to equation (2) near a pole of U in Section 6.1 of [2] are unaffected by Theorem 2.

3. Hyperbolic affine spheres and metrics near the "Y" vertex

Our first alternative proof of the Main Result is based on the following analogue of the result of Baues-Cortés:

Proposition 1. *Assume n is even. Let H be a hyperbolic affine sphere in $\mathbb{R}^{n+1}$ centered at the origin with affine mean curvature -1. Replace H by an open subset of itself if necessary so that each ray through the origin hits H only once. Let*

$$\mathcal{C}' = \bigcup_{0<r<1} rH$$

be the union of all line segments from the origin to H. Then the function $\Phi\colon \mathcal{C}' \to \mathbb{R}$ given by

$$\Phi = \int (r^{n+1} - 1)^{\frac{1}{n+1}}\, dr$$

is convex and solves the real Monge-Ampère equation $\det \Phi_{ij} = 1$ *on* $\mathcal{C}' \subset \mathbb{R}^{n+1}$.

The proof of this proposition is the same as our proof of Baues-Cortés's result (which we label as Theorem 1 in [2]).

Proposition 1 then reduces the problem of finding a semi-flat Calabi-Yau structure on a neighborhood of the "Y" vertex of a graph in $\mathbb{R}^3$ to finding a hyperbolic affine sphere structure on S^2 minus 3 singular points. The next proposition, which follows from [3], provides many such examples.

Proposition 2. *Let U be a meromorphic cubic differential on $\mathbb{CP}^1$ with three poles, each of order at most 3. Then there is a background metric h and a solution u to*

$$\Delta u + 4\|U\|^2 e^{-2u} - 2e^u - 2\kappa = 0 \tag{4}$$

so that $e^u h$ is the metric of a hyperbolic affine sphere.

This proposition, together with a result of Wang [6], exhibits many hyperbolic affine sphere structures on S^2 minus the pole set of U. Thus we find, using Proposition 1, many nontrivial examples of semi-flat Calabi-Yau metrics on a neighborhood of the "Y" vertex of a graph in $\mathbb{R}^3$. Moreover, the holonomy type around each puncture on S^2 (but not the global holonomy representation) is determined in [3].

4. Construction of elliptic affine spheres

This section is devoted to the proof of Theorem $2'$, using the barrier methods of [4]. First of all, we note

Lemma 3. *A function η satisfies (1) if and only if $\nu = \eta - \log \epsilon$ satisfies*

$$\Delta \nu + 4\|U\|^2 e^{-2\nu} + 2\epsilon e^{\nu} - 2\kappa = 0. \tag{5}$$

It will be convenient to pass to a branched cover of $\mathbb{CP}^1$ under which the pullback of U has 6 poles of order one. For example, consider the map

$$\Pi\colon Z \mapsto Z^2 + Z^{-2},$$

for Z an inhomogeneous coordinate on $\mathbb{CP}^1$. Π is then a branched cover from $\mathbb{CP}^1 \to \mathbb{CP}^1$ with branch points over $Z = 2, -2, \infty$. If we allow the (order-two) poles of U to be at $2, -2, \infty$, then $V = \Pi^* U$ has six poles of order one at $0, \infty, 1, -1, i, -i$. Let G be the group of deck transformations of Π. Note $|G| = 4$.

This discussion shows that Theorem $2'$ is equivalent to the existence of a G-invariant bounded solution μ of

$$L_{h,\epsilon}(\mu) = \Delta_h \mu + 4\|V\|_h^2 e^{-2\mu} + 2\epsilon e^{\mu} - 2\kappa_h = 0 \tag{6}$$

for $\epsilon < \delta$ and for a G-invariant background metric h on $N = \Pi^{-1}M$ which is equal to $|\log |z|^2| \, |dz|^2$ near the poles of V. (Here the coordinate z is chosen so that $V = \frac{1}{z} dz^3$ near $z = 0$.)

Proposition 4. *There is a $\delta > 0$ so that there are smooth bounded G-invariant functions s, S on N satisfying*

$$s \leq S, \tag{7}$$

$$L_{h,\epsilon}(s) \geq 0, \tag{8}$$

$$L_{h,\epsilon}(S) \leq 0 \tag{9}$$

for all $\epsilon \in (0, \delta)$.

Proof. These barriers follow from the same arguments as in Section 3.1 of [4]. Recall z is a local coordinate near each pole of V so that $V = \frac{1}{z} \, dz^3$ near the pole $z = 0$, and that the background metric is equal to $|\log |z|^2| \, |dz|^2$ near $z = 0$. All the explicit formulas for barriers below are invariant under the action of G: a nontrivial element of G acts at a fixed point $z = 0$ by $z \mapsto -z$, and the formulas below only depend on $|z|$.

Let $\alpha \in (-1, 0)$, and let f be a G-invariant positive function which is equal to $|\log |z||^{\alpha}$ in a neighborhood of each pole of V and is equal to a constant outside a neighborhood of the poles. Then by the calculations in [4], for $\beta \ll 0$, $s = \beta f$ satisfies

$$L_{h,\epsilon}(s) \geq L_{h,0}(s) \geq 0.$$

(This computation uses the fact that V has no zeros, although the case in which V has zeros can be handled, too [4].)

On the other hand, to produce S, let k be a smooth G-invariant metric on $\mathbb{CP}^1$ which is equal to $|dz|^2$ near each pole of V. Let $\tilde{\kappa}$ be a G-invariant positive function on N which is equal to

$$\frac{1}{2|z|^2(\log |z|)^2}$$

in a neighborhood of each pole and so that

$$\int_N (\kappa_k - \tilde{\kappa}) dV_k = 0$$

for dV_k the volume form of k. Also, let ξ be a G-invariant function equal to $\log|\log|z|^2|$ near each pole $z=0$ and smooth away from the poles. Apply the Green's function of Δ_k to produce a G-invariant function $\tilde{g}$ so that

$$\Delta_k \tilde{g} = 2\kappa_k - 2\tilde{\kappa} - \Delta_k \xi.$$

Since $\Delta_k(\log|\log|z|^2|) = -\frac{1}{|z|^2(\log|z|)^2} = -2\tilde{\kappa}$ in the sense of distributions near $z=0$, $\tilde{g}$ is harmonic near $z=0$ and is thus smooth. By construction, $g = \tilde{g} + \xi$ satisfies

$$g = \log|\log|z|^2| + O(1)$$

near each pole of V. (We may also produce g by applying the Green's function to $2\kappa_k - 2\tilde{\kappa}$ as in [4].) Compute for a constant c,

$$L_{k,0}(g+c) = 4e^{-2g-2c}\|V\|_k^2 - 2\tilde{\kappa} < 0$$

on N for $c \gg 0$, since $e^{-2g}\|V\|_k^2$ and $\tilde{\kappa}$ have the same order of growth as $z \to 0$. If we relate the metrics h and k by $k = e^v h$, then compute for $S = g + c + v$

$$L_{h,0}(S) = e^v L_{k,0}(g+c) < 0.$$

The asymptotics of g and the definition of v ensure that S is bounded. Moreover,

$$L_{h,0}(S) \le -\frac{C}{|z|^2|\log|z||^3}$$

for a constant $C > 0$ near each pole of V. On the other hand, e^S is bounded, and so on all of N

$$L_{h,\epsilon}(S) = L_{h,0}(S) + 2\epsilon e^S < 0$$

for all positive ϵ which are small enough. □

Now we use the method of sub- and super-solutions to produce a G-invariant solution to $L_{h,\epsilon}(\mu) = 0$ on N. Exhaust N by compact submanifolds N_n with boundary

$$N = \bigcup_n N_n, \qquad N_n \subset \operatorname{int} N_{n+1}.$$

Assume each N_n is G-invariant. We proceed as in e.g. Schoen-Yau [5] Proposition V.1.1 (which is essentially the same result on compact manifolds without boundary)

Proposition 5. *For ϵ sufficiently small, there exists a G-invariant solution μ_n to (6) on N_n which satisfies $s \le \mu_n \le S$ on the interior of N_n.*

Proof. Write $L_{h,\epsilon}(\mu) = \Delta_h \mu + f(x,\mu)$ for $f(x,t) = 4\|V\|_h^2 e^{-2t} + 2\epsilon e^t - 2\kappa_h$ and $x \in N_n$. By compactness, there is a constant $\gamma > 0$ so that $F(x,t) = \gamma t + f(x,t)$ is increasing for $t \in [-A, A]$ for a constant $A = A_n$ satisfying $-A \leq s \leq S \leq A$ on N_n (for now, we suppress the dependence on n). Define an operator H by $H\mu = -\Delta_h \mu + \gamma\mu$, and define inductively s_k and S_k

$$\begin{array}{lll} s_0 = s, & H(s_k) = F(x, s_{k-1}), & s_k|_{\partial N_n} = S. \\ S_0 = S, & H(S_k) = F(x, S_{k-1}), & S_k|_{\partial N_n} = S. \end{array}$$

Note that the Dirichlet problems admit unique solutions to make s_k, S_k well-defined. The functions s_k, S_k are G-invariant by this uniqueness. As in [5], we use the maximum principle to show that

$$s = s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_k \leq \cdots \leq S_k \leq \cdots \leq S_2 \leq S_1 \leq S_0 = S.$$

Then the monotone pointwise limits $s_k \to s_\infty$ and $S_k \to S_\infty$ are each $C^{2,\alpha}$ solutions to (6) on the interior of N_n (use the standard elliptic theory as in [5]). Let $\mu_n = s_\infty$, which is G-invariant, since it is the limit of G-invariant functions. □

Proposition 6. *As $n \to \infty$, there is a subsequence n_i so that $\mu_{n_i} \to \mu$ in C^2_{loc} on N. μ is a G-invariant bounded solution to (6).*

Proof. Choose n and let $m \geq n$. Then μ_m are uniformly bounded solutions to (6) on N_n, as $|\mu_m| \leq A_n$ on N_n. So there are L^p estimates for μ_m and $\Delta_h \mu_m$ on N_n, which lead to local L^p_2 estimates on μ_m on each open subset of $\mathcal{O}_n \subset N_n$. Then we may use Sobolev embedding to get $C^{0,\alpha}$ estimates on μ_m and $\Delta_h \mu_m$:

$$\|\mu_m\|_{C^{0,\alpha}}, \|\Delta_h \mu_m\|_{C^{0,\alpha}} \leq C_n \text{ on } \mathcal{O}_n,$$

where we have shrunk $\mathcal{O}_n$ slightly. Then by the Schauder theory, there are uniform $C^{2,\alpha}$ estimates for μ_m on $\mathcal{O}_n$ (which is again shrunk a little). All these estimates are uniform in $m \geq n$ large on each $\mathcal{O}_n \subset N$. We may still assume $\mathcal{O}_n$ is still an exhaustion of N:

$$N = \bigcup_n \mathcal{O}_n, \qquad \mathcal{O}_n \Subset \mathcal{O}_{n+1}.$$

Then we may use the Ascoli-Arzelá Theorem and the usual diagonalization argument to produce a subsequence $\mu_{n_i} \to \mu$ converging in C^2 on any compact subset of N, and so μ is a solution to (6) bounded between s and S. μ is G-invariant, since it is a limit of functions which are all G-invariant. □

To complete the proof of Theorem $2'$, we simply note that all the estimates only depend on $\|U\|$, and since the space of admissible U on $\mathbb{CP}^1$ has one complex dimension, δ depends only on $\int_M |U|^{\frac{2}{3}}$.

Remark. It is presumably possible to solve (1) for a nontrivial cubic differential U on $\mathbb{CP}^1$ which admits ≤ 3 poles of order 2 and an arbitrary number of poles of order 1. It is not possible to have more than 3 poles of order 2, since the curvature κ of the background metric contributes a point mass at each pole of order 2 (while the corresponding background metric for poles of order 1 makes no such contribution). We have

$$\int_{\mathbb{CP}^1} 2\kappa \, dV = 8\pi - 2\pi\ell$$

for ℓ the number of poles of order 2. Integrating equation (1) then shows that there are no nontrivial solutions of (1) if $\ell > 3$. (In the present case, the essential fact is that the orbifold $\mathbb{CP}^1/G$ admits a positive-curvature orbifold metric.)

References

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