

THE MODULI SPACES OF EQUIVARIANT MINIMAL SURFACES IN \mathbb{RH}^3 AND \mathbb{RH}^4 VIA HIGGS BUNDLES

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ABSTRACT. In this article we introduce a definition for the moduli space of equivariant minimal immersions of the Poincaré disc into a non-compact symmetric space, where the equivariance is with respect to representations of the fundamental group of a compact Riemann surface of genus at least two. We then study this moduli space for the non-compact symmetric space \mathbb{RH}^n and show how $SO_0(n, 1)$ -Higgs bundles can be used to parametrise this space, making clear how the classical invariants (induced metric and second fundamental form) figure in this picture. We use this parametrisation to provide details of the moduli spaces for \mathbb{RH}^3 and \mathbb{RH}^4 , and relate their structure to the structure of the corresponding Higgs bundle moduli spaces.

1. INTRODUCTION

One rich area of study in surface theory lies in understanding more deeply the consequences of *non-abelian Hodge theory*. This theory provides a homeomorphism between two different moduli spaces. On one side is the *character variety* $\mathcal{R}(\pi_1\Sigma, G)$ of a non-compact semi-simple Lie group G , where Σ is a smooth closed oriented surface of genus $g \geq 2$. The character variety is the moduli space of G -conjugacy classes of reductive representations of $\pi_1\Sigma$ in G . On the other side is the moduli space $\mathcal{H}(\Sigma_c, G)$ of *polystable G -Higgs bundles* over a compact Riemann surface Σ_c . Here we use Σ_c to denote Σ equipped with a complex structure. The space $\mathcal{H}(\Sigma_c, G)$ parametrises solutions of an appropriate version of the self-dual Yang-Mills equations over Σ_c . These moduli spaces are homeomorphic (and diffeomorphic away from singularities), but $\mathcal{H}(\Sigma_c, G)$ also has a complex structure which depends upon Σ_c . This correspondence developed from the seminal work of Hitchin [19], Donaldson [13], Corlette [11] and Simpson [26] (with the case of stable Higgs bundles for arbitrary real reductive groups proven in [8]). The half of the non-abelian Hodge correspondence due to Hitchin and Simpson, in which the polystable Higgs bundle is shown to produce an equivariant harmonic map, is the Higgs bundle case of the Donaldson-Uhlenbeck-Yau correspondence. There are now many good surveys available of the principal results of non-abelian Hodge theory (see, for example, [15, 16, 28]).

Our main interest here lies in a consequence of this correspondence which has not had much attention until recently, namely, it can be used to describe moduli spaces of minimal surfaces. In the non-abelian Hodge correspondence a central role is played by equivariant (or “twisted”) harmonic maps: fix a Fuchsian representation c of $\pi_1\Sigma$ into the group $\text{Isom}^+(\mathcal{D})$ of oriented isometries of the Poincaré disc \mathcal{D} , so that $\Sigma_c \simeq \mathcal{D}/c(\pi_1\Sigma)$. Then for every irreducible representation $\rho : \pi_1\Sigma \rightarrow G$, there is a unique equivariant harmonic map $f : \mathcal{D} \rightarrow N$ into the non-compact symmetric space N associated to G [13, 11]. Equivariance means that

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$f \circ c(\delta) = \rho(\delta) \circ f$ for all $\delta \in \pi_1 \Sigma$. A map from a surface is minimal when it is both conformal and harmonic, and it is easy to show that f is conformal when the Higgs field Φ of the corresponding Higgs bundle satisfies $\text{tr}(\text{ad } \Phi^2) = 0$ and Φ does not vanish (more generally, f will have branch points at zeroes of Φ). Note that when ρ is discrete we obtain an incompressible minimal immersion of Σ into the locally symmetric space $N/\rho(\pi_1 \Sigma)$.

To obtain all equivariant minimal immersions one must allow c to range over all Fuchsian representations. The natural equivalence for equivariant immersions means we only care about the conjugacy class of c , and these are parametrised by the Teichmüller space \mathcal{T}_g of Σ . We will also insist that the representation ρ is irreducible, for otherwise the same map f can be equivariant with respect to more than one representation ρ . For the case where $N = \mathbb{R}\mathbb{H}^n$ (or $\mathbb{C}\mathbb{H}^n$) this is equivalent to the condition that f is *linearly full*, i.e., does not take values in a lower dimensional totally geodesic subspace. With this assumption we can embed the moduli space $\mathcal{M}(\Sigma, N)$ of such equivariant minimal immersions into the product space $\mathcal{T}_g \times \mathcal{R}(\pi_1 \Sigma, G)$ (this is explained more carefully in §2 below).

In this article we focus on the case where $N = \mathbb{R}\mathbb{H}^n$ and $G = SO_0(n, 1)$ is its group of orientation preserving isometries. We begin by showing how the induced metric and second fundamental form of an equivariant minimal surface in $\mathbb{R}\mathbb{H}^n$ determines the Higgs bundle for ρ . From the work of Aparacio & García-Prada [4], we see that each $SO_0(n, 1)$ -Higgs bundle (E, Φ) has $E = V \oplus 1$ where V is an $SO(n, \mathbb{C})$ -bundle (i.e., rank n with an orthogonal structure Q_V) and 1 denotes the trivial bundle. We show in Theorem 2.4 that this is the Higgs bundle for an equivariant minimal immersion if and only if it is polystable and V is constructed from an $SO(n-2, \mathbb{C})$ bundle (W, Q_W) and a cohomology class $\xi \in H^1(W, K^{-1})$, where K is the canonical bundle of Σ_c , as follows. As a smooth bundle V is just $K^{-1} \oplus W \oplus K$. This has a natural orthogonal structure Q_V given by Q_W on W together with the canonical pairing $K \times K^{-1} \rightarrow 1$ on $K^{-1} \oplus K$. The class ξ is the extension class of a rank $n-1$ holomorphic bundle V_{n-1}

$$0 \rightarrow K^{-1} \xrightarrow{\phi} V_{n-1} \rightarrow W \rightarrow 0.$$

We show there is a unique holomorphic structure V on $K^{-1} \oplus W \oplus K$ for which: (a) the projection $V \rightarrow K$ is holomorphic and has kernel isomorphic to V_{n-1} , (b) the orthogonal structure Q_V is holomorphic. The Higgs field Φ is then determined by ϕ and its dual with respect to Q_V . In terms of the geometry of the minimal surface, W is just the complexified normal bundle of the immersion, and ξ comes from the $(0, 2)$ part of the second fundamental form.

To use this data (W, Q_W, ξ) to parametrise $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^n)$ we must know when the Higgs bundle it produces is polystable (and indecomposable, to ensure that the representation is irreducible). This is very easy to do when $n = 3$, since in that case W is trivial. As a consequence we quickly arrive at a description of $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^3)$: it is diffeomorphic to the punctured tangent bundle of Teichmüller space, i.e, $T\mathcal{T}_g$ with its zero section removed. Moreover, the zero section corresponds to the totally geodesic minimal immersions, which necessarily lie in a copy of $\mathbb{R}\mathbb{H}^2$ and are not linearly full. This much was known already to Alessandrini & Li [3], but we include it here since this moduli space turns up at the boundary of $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^4)$. The moduli space of minimal surfaces in 3-dimensional hyperbolic space forms was also studied by Taubes [27], using the more traditional approach of classifying these surfaces by their metric and Hopf differential via the Gauss-Codazzi equations. We explain, in Remark 3.3, how our results fit in with Taubes' space of "minimal hyperbolic germs".

The structure of $\mathcal{M}(\Sigma, \mathbb{RH}^4)$ is more interesting. In this case the normal bundle $T\Sigma^\perp$ has an Euler number $\chi(T\Sigma^\perp)$ and we show that this integer invariant is bounded and indexes the connected components of the moduli space. To be precise, we prove:

Theorem 1.1. *The moduli space $\mathcal{M}(\Sigma, \mathbb{RH}^4)$ can be given the structure of a non-singular complex manifold of dimension $10g-9$. It has $4g-5$ connected components $\mathcal{M}_l(\Sigma, \mathbb{RH}^4)$, with integer index satisfying $|l| < 2(g-1)$. The component $\mathcal{M}_l(\Sigma, \mathbb{RH}^4)$ consists of all linearly full minimal immersions whose normal bundle has $\chi(T\Sigma^\perp) = l$.*

Our understanding of the structure goes beyond simply counting connected components. First, we show that each component is an open subvariety of a complex analytic family over Teichmüller space. In general, the fibre of $\mathcal{M}(\Sigma, \mathbb{RH}^4)$ over $c \in \mathcal{T}_g$ is isomorphic to the nilpotent cone in $\mathcal{H}(\Sigma_c, G)$, or to be precise, the open subset of this which corresponds to indecomposable Higgs bundles. By Hausel's Theorem [18] the nilpotent cone is a union of the unstable manifolds of the downwards gradient flow for the Higgs bundle energy (which is sometimes called the *Hitchin function*). The critical manifolds of this flow consist of Hodge bundles. The Hodge bundles correspond to *superminimal* surfaces and we describe the conditions on the parameters (W, Q_W, ξ) which determine these. This is similar to our findings for the moduli space of equivariant minimal surfaces in \mathbb{CH}^2 [25].

For $l \neq 0$ we show that each fibre of the connected component $\mathcal{M}_l(\Sigma, \mathbb{RH}^4)$, when thought of as a subvariety of $\mathcal{H}(\Sigma_c, G)$, contains precisely one connected critical manifold $S_{c,l}$. The fibre itself is a complex analytic vector bundle over $S_{c,l}$ and we conjecture that it agrees with the unstable manifold of $S_{c,l}$. Hence the topology of the component $\mathcal{M}_l(\Sigma, \mathbb{RH}^4)$ is given by the topology of $\cup_c S_{c,l}$. We show that each $S_{c,l}$ is an open subvariety of a vector bundle over the Picard variety $\text{Pic}_l(\Sigma_c)$, but are unable to determine its topology. Indeed, the topology of $\mathcal{H}(\Sigma, G)$ is itself not well understood and it boils down to the same problem: understand the topology of the critical manifolds.

The component $\mathcal{M}_0(\Sigma, \mathbb{RH}^4)$ is more complicated because the corresponding critical manifolds $S_{c,0}$ consist entirely of decomposable Hodge bundles, so these must be excluded from $\mathcal{M}_0(\Sigma, \mathbb{RH}^4)$. The excluded minimal immersions are precisely the minimal immersions into either a copy of \mathbb{RH}^3 or \mathbb{RH}^2 . Since either type has flat normal bundle, these can only lie on the boundary of $\mathcal{M}_0(\Sigma, \mathbb{RH}^4)$. There is only one copy of $\mathcal{M}(\Sigma, \mathbb{RH}^3)$ on this boundary, but multiple copies of $\mathcal{M}(\Sigma, \mathbb{RH}^2) \simeq \mathcal{T}_g$ since one can alter how ρ acts in the normal bundle by any element of $\text{Hom}(\pi_1\Sigma, S^1)$.

Finally, we observe that there appears to be no way of using boundary points to get between connected components (i.e., between maps with distinct values of $\chi(T\Sigma^\perp)$) without collapsing the map to a constant along the way. Such a collapse corresponds to the asymptotic limit of downwards Morse flow as the Higgs field tends to zero. Limits of this type lie on the boundary of all connected components.

By combining the results here with those of our earlier paper [25] we can make a reasonable conjecture for the structure of $\mathcal{M}(\Sigma, N)$ for an general rank one non-compact symmetric space. Namely, we expect it admits the complex structure of an open subvariety of a complex analytic family over Teichmüller space. We expect the fibre over $[c] \in \mathcal{T}_g$ is a disjoint union of vector bundles or punctured vector bundles, each over a base which consists of the Hodge bundles in one critical manifold of the Hitchin function. It is not hard to show that for $N = \mathbb{CH}^n$ the Hodge bundles always correspond to superminimal immersions and we expect the same to hold for the other rank one symmetric spaces. Given this, we conjecture that the connected components of $\mathcal{M}(\Sigma, N)$ are indexed by the topological invariants of linearly full

superminimal immersions. It is reasonable to expect to classify these for rank one symmetric spaces.

Outside the case of rank one symmetric spaces there is a great deal yet to be done. Labourie conjectured in [22] that every *Hitchin representation* into a split real form should admit a unique minimal surface, which would imply that the moduli space of Hitchin representations (whose parametrisation is due to Hitchin [20]) provides components of $\mathcal{M}(\Sigma, N)$ when the isometry group G of N is a split real form. Labourie recently proved his conjecture for split real forms of rank-two complex simple Lie groups [23]. Rank-two phenomena also appear a bit indirectly below as in harmonic sequence theory in Subsection 4.1 and the geometry of rank-two holomorphic vector bundles in Appendix B. It should be quite interesting to develop the theory further in higher rank.

When $G = Sp(4, \mathbb{R})$ the Hitchin representations are all *maximal* (i.e., have maximal Toledo invariant) and Collier [9] extended Labourie's result to cover all maximal representations in $Sp(4, \mathbb{R})$. Very recently Collier and collaborators have further extended this uniqueness result to maximal representations in $PSp(4, \mathbb{R})$ [1] and $SO(2, n)$ [10]. Since $PSp(4, \mathbb{R}) \simeq SO_0(2, 3)$ an adaptation of the techniques of this current paper may shed some light on the minimal surfaces for non-maximal representations. Finally, let us mention that Higgs bundles have also been used to study minimal surfaces by Baraglia [5, 6] (his concept of cyclic surfaces is central to Labourie's proof), while Alessandrini & Li characterised AdS 3-manifolds using minimal surfaces into the Klein quadric and Higgs bundles for $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ [2].

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2. EQUIVARIANT MINIMAL SURFACES IN $\mathbb{R}\mathbb{H}^n$ AND HIGGS BUNDLES.

2.1. Equivariant minimal surfaces. Let Σ be a compact oriented surface of genus $g \geq 2$ and let $c : \pi_1 \Sigma \rightarrow \text{Isom}^+(\mathcal{D})$ be a Fuchsian representation into the group of orientation preserving isometries of the Poincaré disc \mathcal{D} . Let $\Sigma_c = \mathcal{D}/c(\pi_1 \Sigma)$ be the corresponding compact Riemann surface. For any non-compact (and for simplicity, irreducible) globally symmetric space N , with isometry group G , we say a minimal immersion $f : \mathcal{D} \rightarrow N$ is equivariant with respect to a representation $\rho : \pi_1 \Sigma \rightarrow G$ when $f \circ c(\delta) = \rho(\delta) \circ f$ for all $\delta \in \pi_1 \Sigma$. In general one wants to allow branched minimal immersions, which for this part of the discussion we will do. Naturally, we want to consider such triples (f, c, ρ) to be equivalent when they are related by isometries of the domain or codomain. Accordingly, we will write

$$(f, c, \rho) \sim (f', c', \rho'),$$

whenever there is $\gamma \in \text{Isom}^+(\mathcal{D})$ and $g \in G$ for which

$$c' = \gamma c \gamma^{-1}, \quad \rho' = g \rho g^{-1}, \quad f' = g f \gamma,$$

where $f_\gamma(z) = f(\gamma^{-1}z)$. The equivalence class will be written $[f, c, \rho]$. The set of these equivalence classes for which ρ is irreducible and f is oriented will be our *moduli space of equivariant minimal surfaces in N* , which we will denote by $\mathcal{M}(\Sigma, N)$. We choose ρ to be irreducible to avoid having a multiplicity of maps which only differ by changing a reductive factor in the decomposition of ρ . For example, for every totally geodesic embedding of $\mathbb{R}\mathbb{H}^2 \simeq \mathcal{D}$ in $\mathbb{R}\mathbb{H}^n$ one can take ρ to be c post-composed with an embedding of $\text{Isom}^+(\mathcal{D}) \simeq SO_0(2, 1)$ into $SO_0(n, 1)$, but one can also alter this by any reductive representation of $\pi_1\Sigma$ into $SO(n-2)$ and acting in the normal bundle of the immersion. Such behaviour turns up at the boundary of the moduli space and creates singularities and lower dimensional strata there.

The moduli space of Fuchsian representations up to conjugacy is Teichmüller space \mathcal{T}_g , while the moduli space of reductive representations ρ up to conjugacy is the character variety $\mathcal{R}(\pi_1\Sigma, G)$. The subset of irreducible representations is an open submanifold. For a fixed conformal structure, standard uniqueness theorems for the harmonic metric (e.g. [11, 13]) apply. It follows that we have an injective map

$$F : \mathcal{M}(\Sigma, N) \rightarrow \mathcal{T}_g \times \mathcal{R}(\pi_1\Sigma, G), \quad [f, c, \rho] \mapsto ([c], [\rho]), \quad (2.1)$$

where the square brackets denote conjugacy classes. The topology we will use for $\mathcal{M}(\Sigma, N)$ is the one induced by this injection. Indeed, we can use this to put a real analytic structure on $\mathcal{M}(\Sigma, N)$. A conjugacy class $[c] \in \mathcal{T}_g$ is sometimes called a *marked conformal structure*: we will denote the subset of equivariant minimal surfaces with fixed marked conformal structure $[c]$ by $\mathcal{M}(\Sigma_c, N)$.

Now recall the central result of non-abelian Hodge theory, which describes the relationship with Higgs bundles (see, for example, [16]). This says that for each Fuchsian representation c there is a bijective correspondence between *polystable G -Higgs bundles* over Σ_c and reductive representations $\rho : \pi_1\Sigma \rightarrow G$, up to their respective equivalence classes. This correspondence gives a homeomorphism from the Higgs bundle moduli space $\mathcal{H}(\Sigma_c, G)$ to $\mathcal{R}(\pi_1\Sigma, G)$ which is real analytic away from singularities.

This central result works by assigning to each polystable Higgs bundle an equivariant harmonic map (to be precise, a triple $[f, c, \rho]$ where f is harmonic). It identifies the Higgs field Φ with the restriction of df to $T^{1,0}\mathcal{D}$, which we will denote by ∂f . It is a well-known fact that, using $g^{\mathbb{C}}$ to denote the complex bilinear extension of the metric g of N , f is weakly conformal precisely when $g^{\mathbb{C}}(\partial f, \partial f) = 0$. Since the metric on N comes from the Killing form on G , it follows that this harmonic map f is weakly conformal, and therefore a branched minimal immersion, precisely when the Higgs field Φ satisfies $\text{tr}(\text{ad } \Phi^2) = 0$. Therefore the map from $\mathcal{M}(\Sigma_c, N)$ to the moduli space $\mathcal{H}(\Sigma_c, G)$ of polystable G -Higgs bundles, which assigns to each equivariant minimal surface its Higgs bundle data, is injective and its image is the complex analytic subvariety given by function $\text{tr}(\text{ad } \Phi^2) = 0$. When G has real rank one (i.e., N is a rank one symmetric space) this level set $\text{tr}(\text{ad } \Phi^2) = 0$ is the *nilpotent cone*.

Now it is reasonable to expect that as $[c]$ varies over Teichmüller space the Higgs bundle moduli spaces form a *complex analytic family* in the sense of Kodaira & Spencer [21], and moreover that the function $\text{tr}(\text{ad } \Phi^2)$ is holomorphic on this family. Given this, $\mathcal{M}(\Sigma, N)$ would acquire the structure of a complex analytic family with fibres $\mathcal{M}(\Sigma_c, N)$. This has been shown for the case $N = \mathbb{C}\mathbb{H}^2$ in [25] by focussing more directly on the properties which characterise a $PU(2, 1)$ -Higgs bundle for which $\text{tr}(\text{ad } \Phi^2) = 0$.

Remark 2.1. It is very interesting to note that the mapping class group of Σ acts naturally on $\mathcal{M}(\Sigma, N)$. Recall that, by the Dehn-Nielsen theorem, the mapping class group is isomorphic to

the group $\text{Out}(\pi_1\Sigma) = \text{Aut}(\pi_1\Sigma)/\text{Inn}(\pi_1\Sigma)$ of outer automorphisms of $\pi_1\Sigma$, where $\text{Inn}(\pi_1\Sigma)$ is the subgroup of automorphisms given by conjugation. For any $\tau \in \text{Aut}(\pi_1\Sigma)$ and equivariant minimal surface (f, c, ρ) it is easy to check that $(f, \tau^*c, \tau^*\rho)$ is again an equivariant minimal surface and that the equivalence class $[f, c, \rho]$ is unchanged when τ is an inner automorphism. Note that $\text{Out}(\pi_1\Sigma)$ acts similarly on $\mathcal{T}_g \times \mathcal{R}(\pi_1\Sigma, G)$ and that the embedding F in (2.1) is equivariant with respect to this action.

2.2. Equivariant minimal surfaces in $\mathbb{R}\mathbb{H}^n$. We will now restrict our attention to the case where $N = \mathbb{R}\mathbb{H}^n$ and $G = SO_0(n, 1)$, the connected component of the identity in $SO(n, 1)$, for $n \geq 3$. Our aim here is to characterise the Higgs bundles which correspond to linearly full minimal immersions and show how the Higgs bundle data relates to the metric and second fundamental form of the immersion. Recall (from e.g., [4]) that an $SO_0(n, 1)$ -Higgs bundle over Σ_c is uniquely determined by an equivalence class of data (V, Q_V, φ) where V is a rank n holomorphic vector bundle, Q_V is an orthogonal structure on V and $\phi \in H^0(K \otimes \text{Hom}(1, V))$. The Higgs bundle itself is the rank $n + 1$ bundle $E = V \oplus 1$ with orthogonal structure

$$Q_E = \begin{pmatrix} Q_V & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.2)$$

and Higgs field

$$\Phi = \begin{pmatrix} 0 & \phi \\ \phi^t & 0 \end{pmatrix}$$

where $\phi^t \in H^0(K \otimes \text{Hom}(V, 1))$ is the dual of ϕ with respect to Q_E . Notice that we use a different convention from [4] for the sign of the orthogonal structure on the trivial summand. This fits better with the the interpretation of Q_E as the complex bilinear extension of a Lorentz metric on $\mathbb{R}^{n,1}$ given below. From now on we will simply refer to (V, Q_V, ϕ) as the $SO_0(n, 1)$ -Higgs bundle.

Now let us recall how the equivariant harmonic map (f, c, ρ) is related to such a Higgs bundle. Let $\mathbb{C}^{n,1}$ denote \mathbb{C}^{n+1} as a pseudo-Hermitian vector space with inner product

$$\langle v, w \rangle = v_1\bar{w}_1 + \dots + v_n\bar{w}_n - v_{n+1}\bar{w}_{n+1}.$$

A pair $(c, \rho) \in \mathcal{T}_g \times \mathcal{R}(\pi_1\Sigma, G)$ provides an action of $\pi_1\Sigma$ on the trivial bundle $\mathcal{D} \times \mathbb{C}^{n,1}$ and the quotient is a flat $SO_0(n, 1)$ -bundle $E = \mathcal{D} \times_{\pi_1\Sigma} \mathbb{C}^{n,1}$ over Σ_c , with pseudo-Hermitian metric $\langle \cdot, \cdot \rangle$, a real involution $\bar{\cdot} : E \rightarrow E$ and a flat pseudo-Hermitian connexion ∇^E . The first two equip E with an orthogonal structure Q_E (given by $Q_E(\sigma, \sigma) = \langle \sigma, \bar{\sigma} \rangle$). The connexion equips E with a holomorphic structure $\bar{\partial}_E = (\nabla^E)^{0,1}$ with respect to which Q_E is holomorphic.

From this the $SO_0(n, 1)$ -Higgs bundle (V, Q_V, ϕ) is obtained as follows. We first identify $\mathbb{R}\mathbb{H}^n$ with the pseudo-sphere

$$S^{n,1} = \{v \in \mathbb{R}^{n,1} : \langle v, v \rangle = -1, v_{n+1} > 0\},$$

in Minkowski space $\mathbb{R}^{n,1}$ so that we can consider $T\mathbb{R}\mathbb{H}^n \subset \mathbb{R}\mathbb{H}^n \times \mathbb{R}^{n,1}$, with metric g obtained by restriction of $\langle \cdot, \cdot \rangle$. We can consider the quotient $V = (f^{-1}T^{\mathbb{C}}\mathbb{R}\mathbb{H}^n)/\pi_1\Sigma$ as a subbundle of E . A choice of orientation of $S^{n,1}$ fixes a trivial line subbundle 1 such that $E = V \oplus 1$ and the decomposition is orthogonal. It follows that the orthogonal projection of ∇^E onto V , which we will denote simply by ∇ , coincides with the pullback of the Levi-Civita connexion on $\mathbb{R}\mathbb{H}^n$. Therefore the holomorphic structure $\bar{\partial}_V$ of V agrees with that induced by $\nabla^{0,1}$.

The Higgs field carries the information of the differential of f . To be precise, the differential of f extends complex linearly to $df : T^{\mathbb{C}}\mathcal{D} \rightarrow T^{\mathbb{C}}\mathbb{R}\mathbb{H}^n$ and thus has a type decomposition

$$df = \partial f + \bar{\partial} f, \quad \partial f : T^{1,0}\mathcal{D} \rightarrow T^{\mathbb{C}}\mathbb{R}\mathbb{H}^n, \quad \bar{\partial} f : T^{0,1}\mathcal{D} \rightarrow T^{\mathbb{C}}\mathbb{R}\mathbb{H}^n.$$

By equivariance we can think of ∂f as a section of $\mathcal{E}^{1,0}(V)$. The harmonicity of f ensures that $\phi = \partial f$ is a holomorphic section of $K \otimes \text{Hom}(1, V) \simeq K \otimes V$.

In the reverse direction, when (V, Q_V, ϕ) is polystable and $\text{tr}(\Phi^2) = 0$ we obtain a weakly conformal equivariant map. This map will be an immersion if and only if ϕ has no zeroes. To understand more precisely the structure of (V, Q_V, ϕ) for a minimal immersion, we begin by characterising those which have $\text{tr}(\Phi^2) = 0$ and ϕ nowhere vanishing.

Lemma 2.1. *An $SO_0(n, 1)$ -Higgs bundle (V, Q_V, ϕ) satisfies $\text{tr}(\Phi^2) = 0$ precisely when $\phi^t \circ \phi = 0$. In that case, when ϕ has no zeroes (V, Q_V, ϕ) uniquely determines, and is determined by, a triple (W, Q_W, ξ) where (W, Q_W) is a holomorphic rank $n - 2$ bundle with orthogonal structure Q_W and $\xi \in H^1(\text{Hom}(W, K^{-1}))$.*

Proof. It is a simple exercise to show that $\text{tr}(\Phi^2) = 0$ if and only if $\phi^t \circ \phi = 0$. Now let $V_{n-1} = \ker(\phi^t)$ and define $W = V_{n-1}/\text{im}(\phi)$. Since ϕ has no zeroes this describes V_{n-1} as an extension bundle

$$0 \rightarrow K^{-1} \xrightarrow{\phi} V_{n-1} \rightarrow W \rightarrow 0. \quad (2.3)$$

This determines, and is determined by, an extension class $\xi \in H^1(\text{Hom}(W, K^{-1}))$. Further, $\phi^t \circ \phi = 0$ implies that $\text{im}(\phi)$ is Q_V -isotropic since $Q_V(\phi(Z), \phi(Z)) = Q_V(Z, \phi^t \circ \phi(Z)) = 0$. Hence Q_V descends to an orthogonal structure Q_W on W .

Conversely, given (W, Q_W, ξ) we model V smoothly on $K^{-1} \oplus W \oplus K$ and give it the orthogonal structure

$$Q_V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & Q_W & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.4)$$

i.e., using Q_W and the canonical pairing $K^{-1} \times K \rightarrow 1$. Then V has a unique holomorphic structure for which: (i) $V_{n-1} = K^{-1} \oplus W$ is given by the extension class ξ (and this also fixes ϕ), (ii) Q_V is holomorphic. To see this, we observe that any holomorphic structure for which the flag $K^{-1} \subset V_{n-1} \subset V$ is holomorphic must correspond to a $\bar{\partial}$ -operator of the form

$$\bar{\partial}_V = \begin{pmatrix} \bar{\partial} & \alpha_1 & \alpha_2 \\ 0 & \bar{\partial}_W & \alpha_3 \\ 0 & 0 & \bar{\partial} \end{pmatrix}$$

where the matrix indicates how the operator acts on V as the direct sum $K^{-1} \oplus W \oplus K$. Here $\bar{\partial}_W$ denotes the holomorphic structure on W , $\bar{\partial}$ denotes the holomorphic structure on both K and K^{-1} , and α_j are $(0, 1)$ -forms taking values in the appropriate bundle homomorphisms (so $\alpha_1 \in \mathcal{E}^{0,1}(\text{Hom}(W, K^{-1}))$ and so forth). A straightforward calculation shows that for Q_V to be holomorphic we must have $\alpha_1 = -\alpha_3^t$ and $\alpha_2 = 0$. With respect to this structure the class $[\alpha_1] \in H^{0,1}(\text{Hom}(W, K^{-1}))$ corresponds to ξ under the Dolbeault isomorphism. Now it is easy to check that with these conditions two holomorphic structures $\bar{\partial}_V, \bar{\partial}'_V$ of this form are equivalent if and only if $[\alpha_1] = [\alpha'_1]$. Hence (V, Q_V, ϕ) is determined by the data (W, Q_W, ξ) . \square

We will also need to understand when (V, Q_V, ϕ) is polystable. Fortunately there is a simple characterisation due to Aparicio & García-Prada.

Theorem 2.2 ([4], Prop 2.3 & Thm 3.1). *When $n > 2$ an $SO_0(n, 1)$ Higgs bundle is stable if for any isotropic subbundle $W \subset V$ with $\phi^t(W) = 0$ we have $\deg(W) < 0$. It is polystable if it is a direct sum of stable G -Higgs bundles where G is either $SO_0(k, 1)$, $SO(k)$ or $U(k) \subset SO(2k)$.*

Remark 2.2. We will say that the Higgs bundle is *decomposable* when it is a direct sum of more than one G -Higgs bundle. Unlike the case of $G = GL(n, \mathbb{C})$, stability does not imply indecomposability. For example, it is shown in [4, Prop. 3.2] that if the decomposition above involves only Higgs subbundles for $SO_0(k, 1)$ or $SO(l)$ with $l \neq 2$, then (V, Q_V, ϕ) is stable. Clearly irreducible representations correspond precisely to indecomposable Higgs bundles. These also provide smooth points in the moduli space of $SO_0(n, 1)$ -Higgs bundles (and hence in the character variety) by Thm 5.5 and Cor 4.4 of [4].

The geometric meaning for f of irreducibility of ρ is the following. We say f is *linearly full* if its image does not lie in a totally geodesic copy of \mathbb{RH}^k in \mathbb{RH}^n for some $k < n$.

Lemma 2.3. *An equivariant minimal immersion $f : \mathcal{D} \rightarrow \mathbb{RH}^n$ is linearly full if and only if the representation ρ is irreducible.*

Proof. Clearly if f is not linearly full its image lies in some copy of $S^{k,1} \subset \mathbb{R}^{k,1} \subset \mathbb{R}^{n,1}$, and this must be preserved by ρ , hence ρ is reducible. Conversely, suppose ρ is reducible, then its Higgs bundle is decomposable. Since the Higgs field is non-trivial there must be at least one (and therefore precisely one) indecomposable subbundle with group $SO_0(k, 1)$ for some $k < n$. The other Higgs bundle summands have compact group structures, and therefore their Higgs fields are trivial. Since the Higgs field represents df , the map f takes values in the totally geodesic copy of \mathbb{RH}^k which corresponds to the $SO_0(k, 1)$ -Higgs summand. \square

Now we describe more explicitly how the Higgs bundle data is related to the classical minimal surface data, namely its induced metric $\gamma = f^*g$ and its second fundamental form \mathbb{I} . Given an immersion f we have a smooth orthogonal decomposition

$$V = T^{\mathbb{C}}\Sigma_c \oplus W \quad (2.5)$$

where $W = (T^{\perp}\Sigma)^{\mathbb{C}}$. We will treat \mathbb{I} as a W -valued complex bilinear form on $T^{\mathbb{C}}\Sigma$. Then f is minimal precisely when $\mathbb{I}^{1,1} = 0$ in which case \mathbb{I} is completely determined by $\mathbb{I}^{2,0}$. By the Codazzi equations this is a holomorphic W -valued quadratic form on Σ_c (see the appendix A). In the case $n = 3$ it is essentially the Hopf differential (see §3 below).

Now we write $T^{\mathbb{C}}\Sigma_c = T^{1,0}\Sigma_c \oplus T^{0,1}\Sigma_c$ and let $\gamma^{\mathbb{C}}$ denote the complex bilinear extension of the metric. This gives the orthogonal structure on $T^{\mathbb{C}}\Sigma_c$, for which both $T^{1,0}\Sigma_c$ and $T^{0,1}\Sigma_c$ are isotropic. Let $\hat{\gamma} : T^{0,1}\Sigma \rightarrow K$ denote the isomorphism $\bar{Z} \rightarrow \gamma^{\mathbb{C}}(\cdot, \bar{Z})$. It has inverse

$$\hat{\gamma}^{-1} : K \rightarrow T^{0,1}\Sigma_c; \quad dz \mapsto \bar{Z} / \|\bar{Z}\|_{\gamma}^2, \quad (2.6)$$

whenever $dz(Z) = 1$. In particular, we obtain an isomorphism $T^{\mathbb{C}}\Sigma \simeq K^{-1} \oplus K$ for which the orthogonal structure $\gamma^{\mathbb{C}}$ makes both K^{-1}, K isotropic and pairs them canonically. Thus we have a smooth isomorphism

$$V = (f^{-1}T\mathbb{RH}^n)^{\mathbb{C}} \simeq K^{-1} \oplus W \oplus K. \quad (2.7)$$

The complex bilinear extension of the metric on $T\mathbb{RH}^n$ provides the orthogonal structure Q_W on W . Let $\bar{\partial}$ denote the holomorphic structure on both K^{-1} and K , and let $\bar{\partial}_W$ denote the holomorphic structure on W coming from the connexion in the normal bundle. Using the isomorphism $\hat{\gamma}$ above we associate \mathbb{I} to a $\text{Hom}(K, W)$ -valued $(0, 1)$ form β , defined locally by

$$\beta(\bar{Z}) : dz \mapsto \mathbb{I}(\bar{Z}, \bar{Z}) / \|\bar{Z}\|_{\gamma}^2, \quad (2.8)$$

for $Z = \partial/\partial z$.

Theorem 2.4. *Let $[f, c, \rho]$ be an equivariant minimal surface in $\mathbb{R}\mathbb{H}^n$, $n \geq 3$, with Higgs bundle (V, Q_V, ϕ) . Then (V, Q_V, ϕ) is given by the data $(W, Q_W, [-\beta^t])$ as in Lemma 2.1, where W is the complexified normal bundle, Q_W is the complex bilinear extension of the normal bundle metric. The cohomology class $[-\beta^t]$ comes from the adjoint β^t of $\beta \in \mathcal{E}_{\Sigma_c}^{0,1}(\text{Hom}(K, W))$ with respect to Q_V via the Dolbeault isomorphism $H^1(\text{Hom}(W, K^{-1})) \simeq H^{0,1}(\text{Hom}(W, K^{-1}))$. Conversely, if the Higgs bundle determined by (W, Q_W, ξ) is stable and indecomposable then it determines a unique linearly full equivariant minimal immersion $[f, c, \rho]$.*

Proof. Recall that with respect to the orthogonal decomposition (2.5) the Levi-Civita connexion can be block-decomposed as

$$\nabla = \begin{pmatrix} \nabla^\Sigma & -B^t \\ B & \nabla^\perp \end{pmatrix}$$

where $\nabla^\Sigma, \nabla^\perp$ denote respectively the induced connexions on the tangent bundle and the normal bundle, and $B \in \mathcal{E}_\Sigma^1(\text{Hom}(T^{\mathbb{C}}\Sigma, W))$ represents the second fundamental form, i.e., $B(X) : Y \rightarrow \mathbb{I}(X, Y)$. Its adjoint B^t is with respect to the metric on $f^{-1}T\mathbb{R}\mathbb{H}^n$.

It follows that with respect to the decomposition (2.7) we can write

$$\bar{\partial}_V = \begin{pmatrix} \bar{\partial} & -\beta^t & 0 \\ \alpha & \bar{\partial}_W & \beta \\ 0 & -\alpha^t & \bar{\partial} \end{pmatrix}, \quad Q_V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & Q_W & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\alpha \in \mathcal{E}_{\Sigma_c}^{0,1}(\text{Hom}(K^{-1}, W))$ and $\beta \in \mathcal{E}_{\Sigma_c}^{0,1}(\text{Hom}(K, W))$ are given locally by

$$\alpha(\bar{Z}) : Z \rightarrow \mathbb{I}(Z, \bar{Z}), \quad \beta(\bar{Z}) : dz \rightarrow \mathbb{I}(\bar{Z}, \bar{Z})/\|\bar{Z}\|_\gamma^2.$$

However, $\mathbb{I}(Z, \bar{Z}) = 0$ since f is minimal, hence $\bar{\partial}_V$ is given by

$$\bar{\partial}_V = \begin{pmatrix} \bar{\partial} & -\beta^t & 0 \\ 0 & \bar{\partial}_W & \beta \\ 0 & 0 & \bar{\partial} \end{pmatrix}, \quad (2.9)$$

From the proof of Lemma 2.1 it follows that $\xi = [-\beta^t]$.

The converse is just non-abelian Hodge theory together with Lemmas 2.1 and 2.3. \square

Note that the isomorphism $W \simeq W^*$ induced by Q_W allows us to identify $\text{Hom}(W, K^{-1})$ with $\text{Hom}(K, W)$, and this identifies β^t with β . From now on we will make this identification.

Regarding the case where $[\beta] = 0$ we make the following observation (cf. Theorem 3.1 of [25]).

Proposition 2.5. *The extension class $[\beta]$ above is trivial if and only if \mathbb{I} is identically zero, i.e., if and only if the map f is a totally geodesic embedding.*

In particular, the case where $[\beta] = 0$ corresponds precisely to those triples $[f, c, \rho]$ for which the representation ρ factors through an embedding of $SO_0(2, 1)$ into $SO_0(n, 1)$. In this case ρ is reducible: it is just the Fuchsian representation c post-composed with this embedding.

Proof. It suffices to show that there is a Hermitian metric h on $\text{Hom}(K, W)$ with respect to which β is harmonic. This metric is none other than the metric induced by the immersion, for with respect to that metric the Hodge star

$$\mathcal{E}^{0,1}(K^{-1} \otimes W) \xrightarrow{\bar{*}} \mathcal{E}^{1,0}(W^* \otimes K)$$

maps $\beta(\bar{Z})d\bar{z}$ to $-ih(\cdot, \beta(\bar{Z}))dz$. Now using (2.8) and the definition of Q_W we see that

$$h(\cdot, \beta(\bar{Z})) = Q_W(\cdot, \mathbb{I}(Z, Z)).$$

But $\mathbb{I}(Z, Z)$ is holomorphic, and therefore $\bar{\partial}\bar{\kappa}\beta = 0$. Hence β is harmonic. \square

Lemma 2.1 and Theorem 2.4 show we should be able to parametrise equivariant minimal immersions by their data $([c], W, Q_W, [\beta])$. The major difficulty in general is to identify in a satisfying way the conditions which ensure stability of the Higgs bundle. In the cases where $n = 3$ or $n = 4$ the bundle (W, Q_W) is simple enough that we can do this, and this is the purpose of the remainder of the article.

3. EQUIVARIANT MINIMAL SURFACES IN $\mathbb{R}\mathbb{H}^3$.

Theorem 2.4 leads very quickly to a description of the moduli space $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^3)$. For in this case $f^{-1}T\mathbb{R}\mathbb{H}^3/\pi_1\Sigma$ is $SO(3)$ -bundle with decomposition into $T\Sigma \oplus T\Sigma^\perp$. Since f is an oriented immersion $T\Sigma^\perp$ is trivial. Therefore, as a smooth bundle,

$$V \simeq K^{-1} \oplus 1 \oplus K,$$

with orthogonal structure

$$Q_V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is well-known that ρ has associated with it a \mathbb{Z}_2 invariant, which we will denote by $w_2(\rho)$, equal to the second Steifel-Whitney class of the $SO(3)$ -bundle associated to (V, Q_V) (see, for example, [4]). In our case this $SO(3)$ -bundle is the bundle of oriented frames of $f^{-1}T\mathbb{R}\mathbb{H}^3/\pi_1\Sigma$, and the question is whether or not this lifts to a $Spin(3)$ -bundle. The decomposition $T\Sigma \oplus T\Sigma^\perp$ gives a reduction of structure group to $SO(2)$, and therefore it does lift because the first Chern class of $T\Sigma$ is even. Hence $w_2(\rho) = 0$.

Remark 3.1. If we had allowed the possibility that f has branch points then these would occur on a divisor $D \subset \Sigma_c$, and the splitting of V would replace K^{-1} by $K^{-1}(D)$. In that case $w_2(\rho) = \deg(K^{-1}(D)) \bmod 2 = \deg(D) \bmod 2$.

Let ν be a unit normal field for f and recall that the *Hopf differential* of a minimal surface in a 3-manifold is the quadratic holomorphic differential $q = \langle \mathbb{I}^{2,0}, \nu \rangle$. In that case the quantity β in Theorem 2.4 can be written as $\bar{q} \otimes \hat{\gamma}^{-1}$. Here we interpret $\hat{\gamma}^{-1} \in \Gamma(K^{-1}\bar{K}^{-1})$ (cf. (2.6)). Thus we can write the holomorphic structure for V in the form

$$\bar{\partial}_V = \begin{pmatrix} \bar{\partial} & -\bar{q} \otimes \hat{\gamma}^{-1} & 0 \\ 0 & \bar{\partial} & \bar{q} \otimes \hat{\gamma}^{-1} \\ 0 & 0 & \bar{\partial} \end{pmatrix}. \quad (3.1)$$

and the holomorphic structure depends only upon

$$[\bar{q} \otimes \hat{\gamma}^{-1}] \in H^{0,1}(\Sigma_c, K^{-1}) \simeq H^1(\Sigma_c, K^{-1}).$$

By Proposition 2.5 this cohomology class is trivial if and only if $q = 0$.

With respect to the smooth isomorphism

$$E \simeq (K^{-1} \oplus 1 \oplus K) \oplus 1$$

the Higgs field Φ and orthogonal structure Q_E have the form

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q_E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is not hard to check that this Higgs bundle $(V \oplus 1, \Phi)$ is stable if and only if $q \neq 0$, and that when $q = 0$ it is polystable with decomposition

$$(V_2 \oplus 1, \Phi') \oplus (1, 0),$$

where $V_2 = K^{-1} \oplus K$ and Φ' is obtained from Φ by striking out the second row and second column.

Now non-abelian Hodge theory provides the converse: to every equivalence class of data (c, ξ) , where c is a marked conformal structure and $\xi \in H^1(\Sigma_c, K^{-1})$, we obtain an equivariant minimal surface $[f, c, \rho]$ in \mathbb{RH}^3 . This is determined only up to the equivalence of such triples above. As a consequence we can equip the moduli space $\mathcal{M}(\Sigma, \mathbb{RH}^3)$ with the structure of a complex manifold. Since $H^1(\Sigma_c, K^{-1})$ is the tangent space to Teichmüller space \mathcal{T}_g at c , we deduce the following.

Theorem 3.1. *The moduli space of equivariant linearly full minimal immersions into \mathbb{RH}^3 can be identified with the bundle of punctured tangent spaces over Teichmüller space,*

$$\mathcal{M}(\Sigma, \mathbb{RH}^3) \simeq \{X \in T\mathcal{T}_g : X \neq 0\}.$$

In particular, this gives it the structure of a non-singular connected complex manifold of complex dimension $6(g-1)$.

Note that when $\mathcal{M}(\Sigma, \mathbb{RH}^3)$ is completed by totally geodesic immersions ($q = 0$), which comprise all the non-full minimal immersions in this case, we obtain the full tangent space to Teichmüller space. We will denote this completed space by $\bar{\mathcal{M}}(\Sigma, \mathbb{RH}^3)$.

Remark 3.2. A generalization of Theorem 3.1 in which branch points are allowed was already known to Alessandrini & Li [3] using a slightly different approach. With that generality the problem is equivalent to finding a parametrisation of the components of the nilpotent cone. This was essentially done many years earlier by Donagi, Ein & Lazarsfeld in [12], albeit with some translation required to adapt their results for $GL(2, \mathbb{C})$ -Higgs bundles to $PSL(2, \mathbb{C})$ -Higgs bundles. Recall that the non-abelian Hodge theory for this case is due entirely to Hitchin [19] and Donaldson [13].

Remark 3.3. Taubes studied a similar moduli space to $\bar{\mathcal{M}}(\Sigma, \mathbb{RH}^3)$ in [27], which he called the moduli space \mathcal{H} of *minimal hyperbolic germs*. Each of these is a pair (γ, \mathbb{I}) consisting of a metric on Σ and a trace-free symmetric bilinear form \mathbb{I} which together satisfy the Gauss-Codazzi equations for a minimal immersion into a 3-space of constant negative curvature -1 (it is easy to show that these are the only two pieces of information needed for these equations). He showed that his moduli space is smooth and has real dimension $12(g-1)$, the same as the real dimension of $T\mathcal{T}_g$. Now, the pair (γ, \mathbb{I}) is all that is needed to construct the Higgs bundles above, and the Gauss-Codazzi equations are exactly the zero curvature equations for the related connexion ∇^E (see appendix A below). It follows that there is a bijection between $\bar{\mathcal{M}}(\Sigma, \mathbb{RH}^3)$ and \mathcal{H} which assigns to each equivariant minimal immersion its metric and second fundamental form. Indeed, Taubes shows that there is a smooth map $\mathcal{H} \rightarrow \mathcal{R}(\pi_1 \Sigma, SO(3, \mathbb{C}))$, given by mapping (γ, \mathbb{I}) to the (conjugacy class of the) holonomy of a

flat connexion which essentially plays the role of ∇^E after the isomorphism between $SO_0(3,1)$ and $SO(3, \mathbb{C})$. Hence there is a smooth map

$$\mathcal{H} \rightarrow \mathcal{T}_g \times \mathcal{R}(\pi_1 \Sigma, SO(3, \mathbb{C})),$$

by taking the conformal class of γ for the first factor. It should be possible to use Taubes' calculations to show that the image can be smoothly identified with $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^3)$.

4. EQUIVARIANT MINIMAL SURFACES IN $\mathbb{R}\mathbb{H}^4$.

For $n = 4$ the normal bundle $T\Sigma^\perp$ is an $SO(2)$ -bundle, and therefore comes with a canonical complex structure J compatible with the orientation. Locally this is given by

$$J\nu_1 = \nu_2, \quad J\nu_2 = -\nu_1,$$

with respect to an oriented orthonormal local frame ν_1, ν_2 for the normal bundle. It follows that the complexified normal bundle splits into a direct sum of line subbundles given by the eigenspaces of J . Let $L \subset W$ be the line subbundle for eigenvalue i , then the eigenbundle for $-i$ is \bar{L} and is therefore smoothly isomorphic to the dual L^{-1} of L . Therefore $W \simeq L \oplus L^{-1}$, and this is an orthogonal decomposition since J is an isometry on the normal bundle. Since $(T\Sigma^\perp, J) \simeq L$ as a complex line bundle, the Euler number of $T\Sigma^\perp$ satisfies

$$\chi(T\Sigma^\perp) = \deg(L). \quad (4.1)$$

Note also that Q_W is just the canonical pairing $L \times L^{-1} \rightarrow 1$. Moreover, since J is parallel for the normal bundle connexion ∇^\perp these line subbundles are ∇^\perp -invariant. It follows that the holomorphic structure $\bar{\partial}_W$ on W diagonalises with respect to this splitting so that L, L^{-1} are holomorphic line subbundles. The projections of W onto L and L^{-1} given, respectively, by

$$\sigma \mapsto \frac{1}{2}(\sigma - iJ\sigma), \quad \sigma \mapsto \frac{1}{2}(\sigma + iJ\sigma),$$

are consequently holomorphic. In particular, define

$$\theta_2 = \frac{1}{2}(\Pi^{2,0} - iJ\Pi^{2,0}), \quad \theta_1 = \frac{1}{2}(\Pi^{2,0} + iJ\Pi^{2,0}). \quad (4.2)$$

These are holomorphic sections of K^2L and K^2L^{-1} respectively.

From Theorem 2.4 the holomorphic structure on V is determined by the cohomology class of $\beta \in \mathcal{E}^{0,1}(\text{Hom}(K, W))$ given by (2.8). Since $W \simeq L \oplus L^{-1}$ holomorphically, we can represent the holomorphic structure $\bar{\partial}_V$ with respect to the smooth isomorphism

$$V \simeq K^{-1} \oplus L \oplus L^{-1} \oplus K, \quad (4.3)$$

by

$$\bar{\partial}_V = \begin{pmatrix} \bar{\partial} & -\beta_2 & -\beta_1 & 0 \\ 0 & \bar{\partial}_1 & 0 & \beta_1 \\ 0 & 0 & \bar{\partial}_2 & \beta_2 \\ 0 & 0 & 0 & \bar{\partial} \end{pmatrix}, \quad (4.4)$$

where $\bar{\partial}_1, \bar{\partial}_2$ are the holomorphic structures on L, L^{-1} and β_1, β_2 are the components of β with respect to the splitting $W \simeq L \oplus L^{-1}$. In particular,

$$\beta_j(\bar{Z}) : dz \mapsto \overline{\theta_j(Z, \bar{Z})} / \|Z\|_\gamma^2,$$

with $\beta_1(\bar{Z})$ taking values in $K^{-1}L$ and $\beta_2(\bar{Z})$ taking values in $K^{-1}L^{-1}$. The forms $-\beta_1, -\beta_2$ determine extension bundles

$$0 \rightarrow K^{-1} \rightarrow V_1 \rightarrow L^{-1}, \quad 0 \rightarrow K^{-1} \rightarrow V_2 \rightarrow L,$$

respectively, which are holomorphic subbundles of V . It is easy to see that the only Q_V -isotropic subbundles of V are V_1, V_2 or subbundles of these.

Let us now consider the conditions under which a minimal immersion can be constructed from an $SO_0(4, 1)$ -Higgs bundle of the type just described. That is, we fix a line bundle L , cohomology classes

$$[\beta_1] \in H^1(\Sigma_c, K^{-1}L), \quad [\beta_2] \in H^1(\Sigma_c, K^{-1}L^{-1}),$$

and construct $(V, \bar{\partial}_V, Q_V, \phi)$ as above, and let (E, Φ) denote the $SO_0(4, 1)$ -Higgs bundle obtained. We need to ascertain when this bundle is stable and indecomposable. For the remainder of this section we assume that $\deg(L) \geq 0$.

Lemma 4.1. *Suppose that $\deg(L) \geq 1$. Then $(V, \bar{\partial}_V, Q_V, \phi)$ gives a stable $SO_0(4, 1)$ -Higgs bundle if and only if $\deg(L) < 2(g - 1)$ and every line subbundle of V_2 has negative degree. All such Higgs bundles are indecomposable.*

Proof. According to the stability condition in Theorem 2.2, we must ensure that every isotropic subbundle of $\ker(\phi^t)$ has negative degree. As a smooth bundle $\ker(\phi^t) = K^{-1} \oplus L \oplus L^{-1}$ and the isotropic subbundles are therefore V_1, V_2 and their subbundles. Now recall that for any extension bundle

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$$

the degree is $\deg(\mathcal{F}_1) + \deg(\mathcal{F}_2)$ and when $\deg(\mathcal{F}_2) > \deg(\mathcal{F}_1)$ every holomorphic line subbundle has degree no greater than $\deg(\mathcal{F}_2)$. Moreover there is a line subbundle of degree equal to $\deg(\mathcal{F}_2)$ if and only if the extension is trivial. Since we are assuming $\deg(L) \geq 1$, V_1, V_2 both have negative degree if and only if $\deg(L) < \deg(K)$. Further, line subbundles of V_1 necessarily have negative degree when $\deg(L) > 0$. \square

Notice that, in particular, we must have $[\beta_2] \neq 0$.

Lemma 4.2. *Suppose that $\deg(L) = 0$. Then $(V, \bar{\partial}_V, Q_V, \phi)$ gives a stable $SO_0(4, 1)$ -Higgs bundle if and only if both $[\beta_1], [\beta_2]$ are non-zero. It is also indecomposable except when $L \simeq 1$ and $[\beta_1] = [\beta_2]$.*

Proof. That $[\beta_1], [\beta_2]$ are both non-zero follows from similar reasoning to the proof of the previous lemma. Now suppose (E, Φ) is decomposable. From Lemma 2.3 f maps into either a copy of $\mathbb{R}\mathbb{H}^2$ or $\mathbb{R}\mathbb{H}^3$. Since the extensions V_1, V_2 are non-trivial it must be the latter. Thus

$$(E, \Phi) = (V' \oplus 1, \Phi') \oplus (V'', 0),$$

where the first summand is a stable $SO_0(3, 1)$ -Higgs bundle and $V'' \simeq 1$. Hence W is trivial and therefore so is L . In particular, the normal bundle $T\Sigma^\perp$ has a global orthonormal frame ν_1, ν_2 for which ν_1 is a unit normal to f inside the tangent space to this $\mathbb{R}\mathbb{H}^3$, and ν_2 is a unit normal to the $\mathbb{R}\mathbb{H}^3$, so that ν_2 is parallel. It follows that

$$\langle \mathbb{I}(X, Y), \nu_2 \rangle = \langle \nabla_X Y, \nu_2 \rangle = -\langle Y, \nabla_X \nu_2 \rangle = 0.$$

From (4.2) we have in general, locally,

$$\theta_1(Z, Z) = \frac{1}{2}(A_1 - iA_2)(\nu_1 + i\nu_2)dz^2, \quad \theta_2(Z, Z) = \frac{1}{2}(A_1 + iA_2)(\nu_1 - i\nu_2)dz^2, \quad (4.5)$$

where $A_j = \langle \mathbb{I}(Z, Z), \nu_j \rangle$. So when $A_2 = 0$ we have

$$\theta_1(Z, Z) = \frac{1}{2}A_1(\nu_1 + i\nu_2)dz^2, \quad \theta_2(Z, Z) = \frac{1}{2}A_1(\nu_1 - i\nu_2)dz^2.$$

Now $L \simeq 1 \simeq L^{-1}$ and the isomorphism identifies $\nu_1 + i\nu_2$ with $\nu_1 - i\nu_2$, therefore θ_1 is identified with θ_2 , hence β_1 is identified with β_2 . \square

To parametrise the space of pairs $([\beta_1], [\beta_2])$ which correspond to these stability conditions we need to introduce some new spaces. For a fixed complex structure Σ_c and integer l satisfying $2(1-g) < l < 2(g-1)$, let $\mathcal{V}_{c,l}$ denote the holomorphic vector bundle over $\text{Pic}_l(\Sigma_c)$ (the moduli space of degree l line bundles over Σ_c) whose fibre over $L \in \text{Pic}_l(\Sigma_c)$ is $H^1(\Sigma_c, K^{-1}L^{-1})$. It is easy to check that, since $\deg(K^{-1}L^{-1}) < 0$, each fibre has positive dimension

$$h^1(\Sigma_c, K^{-1}L^{-1}) = 3(g-1) + l. \quad (4.6)$$

This dimension is independent of the choice of L and therefore Grauert's result [17, section 10.5] ensures that we do obtain a holomorphic vector bundle. To each $\xi \in \mathcal{V}_{c,l}$ we assign the extension bundle $0 \rightarrow K^{-1} \rightarrow \xi \rightarrow L \rightarrow 0$ characterised by it. Every line subbundle of ξ has degree bounded above by $\deg(L)$, so there is a well-defined integer function

$$\mu : \mathcal{V}_{c,l} \rightarrow \mathbb{Z}, \quad \mu(\xi) = \max\{\deg(\lambda) : \lambda \subset \xi \text{ a holomorphic line subbundle}\}.$$

Lemma 4.3. *For $1 \leq l < 2(g-1)$ the set $\mathcal{V}_{c,l}^0 = \{\xi \in \mathcal{V}_{c,l} : \mu(\xi) < 0\}$ is a non-empty Zariski open subvariety.*

The proof follows from Prop. 1.1 in [24], but we defer this to an appendix to avoid digression.

Now for $1 \leq l < 2(g-1)$ let $\mathcal{M}(\Sigma_c, \mathbb{R}\mathbb{H}^4)_l$ denote the moduli space of equivariant minimal immersions for fixed Fuchsian representation c and whose normal bundle has Euler number l . By the discussion above elements of this set are parametrised by the data $(L, [\beta_1], [\beta_2])$ satisfying the stability conditions in Lemma 4.1, i.e., by the variety

$$\mathcal{W}_{c,l} = \{(L, \xi_1, \xi_2) \in \iota^* \mathcal{V}_{c,-l} \oplus \mathcal{V}_{c,l} : \xi_2 \in \mathcal{V}_{c,l}^0\},$$

where $\iota : \text{Pic}_l(\Sigma_c) \rightarrow \text{Pic}_{-l}(\Sigma_c)$ maps L to L^{-1} . For $2(1-g) < l < 0$ define

$$\mathcal{W}_{c,l} = \{(L, \xi_1, \xi_2) \in \iota^* \mathcal{V}_{c,-l} \oplus \mathcal{V}_{c,l} : \xi_1 \in \iota^* \mathcal{V}_{c,l}^0\},$$

Finally, let $\mathcal{V}_{c,0}^+$ denote the bundle $\mathcal{V}_{c,0}$ without its zero section and define

$$\mathcal{W}_{c,0} = \{(L, \xi_1, \xi_2) \in \iota^* \mathcal{V}_{c,0}^+ \oplus \mathcal{V}_{c,0}^+\} \setminus \{(1, \xi, \xi)\}.$$

By the same reasoning as in [25, §6], it can be shown that each family

$$\mathcal{V}_l = \cup_{c \in \mathcal{T}_g} \mathcal{V}_{c,l},$$

is a complex analytic family over Teichmüller space. Let \mathcal{W}_l denote the open subset whose fibres are $\mathcal{W}_{c,l}$. Each \mathcal{V}_l is clearly a connected complex manifold, and by (4.6) each has dimension $10g-9$. Thus \mathcal{W}_l must also be connected, since each fibre is Zariski open in a vector space.

As a consequence of the results of this section we have, for each integer l satisfying $2(1-g) < l < 2(g-1)$, a map $F_l : \mathcal{W}_l \rightarrow \mathcal{M}(\Sigma, N)$ which assigns to each point (c, L, ξ_1, ξ_2) an indecomposable $SO_0(4, 1)$ -Higgs bundle, and therefore a point $[f, c, \rho]$ in $\mathcal{M}(\Sigma, N)$.

Theorem 4.4. *For each $2(1-g) < l < 2(g-1)$ the map $F_l : \mathcal{W}_l \rightarrow \mathcal{M}(\Sigma, N)$ is an injective local diffeomorphism. Hence we can smoothly identify $\mathcal{M}(\Sigma, N)$ with the disjoint union of open sets $\cup_l \mathcal{W}_l$. Consequently, $\mathcal{M}(\Sigma, N)$ can be given the structure of a non-singular complex manifold of dimension $10g-9$ with $4g-5$ connected components indexed by the Euler number of the normal bundle $\chi(T\Sigma^\perp)$, with $|\chi(T\Sigma^\perp)| < 2(g-1)$.*

We can relate this structure to the Morse theoretic study of the moduli space of $SO_0(n, 1)$ -Higgs bundles in [4]. This is done in §4.2 below, after we have identified which equivariant minimal immersions correspond to Hodge bundles.

Proof. By construction we have taken the smooth structure of $\mathcal{M}(\Sigma, N)$ from its inclusion in $\mathcal{T}_g \times \mathcal{R}(\pi_1\Sigma, G)$. We know each F_l is injective, so it suffices to show that $\ker(dF_l)$ is injective at each point. By post-composing F_l with the inclusion we can write $F_l = (\pi_l, \psi_l)$ where

$$\pi_l : \mathcal{W}_l \rightarrow \mathcal{T}_g,$$

is simply the fibration of \mathcal{W}_l over \mathcal{T}_g , and

$$\psi_l : \mathcal{W}_l \rightarrow \mathcal{R}(\pi_1\Sigma, G) \simeq \mathcal{H}(\Sigma_c, G),$$

assigns the Higgs bundle for fixed choice of c . It follows that dF_l is injective precisely when $d\psi_l$ is injective on the tangent spaces to the fibres $\mathcal{W}_{c,l}$ of π_l . Therefore it suffices to show that for fixed c, l the map

$$\psi : \mathcal{W}_{c,l} \rightarrow \mathcal{H}(\Sigma_c, G),$$

is an immersion at each point. So fix a point (L, ξ_1, ξ_2) and let $\bar{\partial}_L$ be a $\bar{\partial}$ -operator on L compatible with its holomorphic structure. Let β_j be representatives for ξ_j . Let

$$\hat{\mathcal{W}}_{c,l} = \mathcal{E}^{0,1}(\mathbb{C}) \times \mathcal{E}^{0,1}(LK^{-1}) \times \mathcal{E}^{0,1}(L^{-1}K^{-1}),$$

and define

$$\hat{\psi} : \hat{\mathcal{W}}_{c,l} \rightarrow \mathcal{H}(\Sigma_c, G), \quad (\alpha, \eta_1, \eta_2) \mapsto \psi(L_\alpha, [\eta_1]_\alpha, [\eta_2]_\alpha),$$

where L_α is L with the holomorphic structure $\bar{\partial}_L + \alpha$ and $[\eta_j]_\alpha$ is the Dolbeault cohomology class of η_j with respect to L_α . Clearly ψ will be an immersion at (L, ξ_1, ξ_2) if

$$\frac{d}{dt} \hat{\psi}(t\alpha, \beta_1 + t\eta_1, \beta_2 + t\eta_2) = 0 \tag{4.7}$$

implies (α, η_1, η_2) is tangent to the fibre of the projection $\hat{\mathcal{W}}_{c,l} \rightarrow \mathcal{W}_{c,l}$ at the point $(0, \beta_1, \beta_2)$. Now let $\bar{\partial}_V$ denote the operator in (4.4), then $\hat{\psi}(t\alpha, \beta_1 + t\eta_1, \beta_2 + t\eta_2)$ is the Higgs bundle determined by $\bar{\partial}_V + tA$ where

$$A = \begin{pmatrix} 0 & -\eta_2 & -\eta_1 & 0 \\ 0 & \alpha & 0 & \eta_1 \\ 0 & 0 & -\alpha & \eta_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and neither the Higgs field nor the orthogonal structure Q_V depend upon t (the inclusion $K^{-1} \rightarrow V$ is holomorphic for every t). Therefore the left hand side of (4.7) is represented by the equivalence class of the $\text{End}(V)$ -valued $(0,1)$ -form A with respect to infinitesimal gauge transformations. This class is trivial if there is a curve of gauge transformations $g(t) \in \Gamma(\text{End}(V))$ which is Q_V -orthogonal, transforms operators of the shape (4.4) into operators of the same shape, and satisfies

$$A = -g^{-1} \dot{g} g^{-1} \bar{\partial}_V g + g^{-1} \bar{\partial}_V \dot{g}, \tag{4.8}$$

where $\dot{g} = (dg/dt)(0)$. A straightforward calculation shows that the conditions on g imply that, with respect to the smooth decomposition (4.3), it has the form

$$g = \begin{pmatrix} 1 & -av & -a^{-1}u & 0 \\ 0 & a & 0 & u \\ 0 & 0 & a^{-1} & v \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where a is a smooth \mathbb{C}^* -valued function on Σ_c and $u \in \Gamma(K^{-1}L)$, $v \in \Gamma(K^{-1}L^{-1})$ satisfy $\bar{\partial}(uv) = 0$. Since K^{-2} has no globally holomorphic sections, either $u = 0$ or $v = 0$. We will

treat the case $u = 0$: the other case follows *mutatis mutandis*. In this case g has a 2×2 block decomposition and we deduce that (4.8) holds if and only if $\eta_1 = 0$ and

$$\begin{pmatrix} 0 & -\eta_2 \\ 0 & \alpha \end{pmatrix} = -g_1^{-1} \dot{g}_1 g_1^{-1} \bar{\partial}_{V_2} g_1 + g_1^{-1} \bar{\partial}_{V_2} \dot{g}_1,$$

where

$$\bar{\partial}_{V_2} = \begin{pmatrix} \bar{\partial} & -\beta_2 \\ 0 & \bar{\partial}_L + \alpha \end{pmatrix} \quad g_1 = \begin{pmatrix} 1 & -av \\ 0 & a \end{pmatrix}.$$

However, this is precisely the condition that the deformation of the holomorphic structure of V_2 on $K^{-1} \oplus L$ along the curve $[\bar{\partial}_L + t\alpha, -\beta_2 - t\eta_2]$ is constant. Hence $(\alpha, 0, \eta_2)$ is tangent to the fibre of $\hat{\mathcal{W}}_{c,l} \rightarrow \mathcal{W}_{c,l}$ at the point $(0, \beta_1, \beta_2)$. We conclude that $d\psi$ has trivial kernel at (L, ξ_1, ξ_2) . \square

4.1. Superminimal maps and Hodge bundles. An important geometric invariant of any equivariant minimal immersion f is the holomorphic quartic differential

$$U_4 = \langle \mathbb{I}^{2,0}, \mathbb{I}^{0,2} \rangle = Q_W(\mathbb{I}^{2,0}, \mathbb{I}^{2,0}). \quad (4.9)$$

This vanishes at points where either $\mathbb{I}^{2,0}$ is zero or it is Q_W -isotropic.

Remark 4.1. It is easy to check that points where $\mathbb{I}^{2,0}$ is isotropic are points where f has *circular ellipse of curvature* (i.e., the image of the unit circle in $T\mathcal{D}$ under the map $T\mathcal{D} \rightarrow T\mathcal{D}^\perp$; $X \mapsto \mathbb{I}(X, X)$ is a circle).

By a simple adaptation of the harmonic sequence arguments used in [7] for minimal immersions into S^n , it can be shown that an equivariant minimal immersion $f : \mathcal{D} \rightarrow \mathbb{R}\mathbb{H}^4$ is determined up to congruence by the induced metric γ and the holomorphic quartic differential U_4 . The following definition comes from the harmonic sequence theory.

Definition 4.5. *We will say f is superminimal when U_4 vanishes identically.*

Let v_γ denote the area form for the induced metric $\gamma = f^*g$. This is $\pi_1\Sigma$ -invariant and therefore lives on Σ . We will call its integral over Σ the area of the equivariant minimal immersion. From the Gauss equation (A.8) we deduce that it satisfies

$$\int_\Sigma v_\gamma = 4\pi(g-1) - \int_\Sigma \|\mathbb{I}^{2,0}\|_\gamma^2 v_\gamma. \quad (4.10)$$

For superminimal immersions we can relate the last term to the Euler number of the normal bundle $\chi(T\Sigma^\perp)$.

Theorem 4.6. *Suppose $f : \mathcal{D} \rightarrow \mathbb{R}\mathbb{H}^4$ is equivariant superminimal. Then one of the following holds: (i) $\chi(T\Sigma^\perp) > 0$ and*

$$\kappa^\perp = \|\mathbb{I}^{2,0}\|_\gamma^2 = -(1 + \kappa_\gamma), \quad (4.11)$$

or, (ii) $\chi(T\Sigma^\perp) < 0$ and

$$\kappa^\perp = -\|\mathbb{I}^{2,0}\|_\gamma^2 = 1 + \kappa_\gamma, \quad (4.12)$$

or, (iii) $\chi(T\Sigma^\perp) = 0$ and f is totally geodesic. In all cases the area of f is

$$\int_\Sigma v_\gamma = 2\pi[2(g-1) - |\chi(T\Sigma^\perp)|]. \quad (4.13)$$

In particular, there are superminimal immersions for every value of $\chi(T\Sigma^\perp)$ with $|\chi(T\Sigma^\perp)| < 2(g-1)$, but these are only linearly full when $\chi(T\Sigma^\perp) \neq 0$.

Proof. Since $\mathbb{I}^{2,0} = \theta_1 + \theta_2$ and L, L^{-1} are isotropic and paired by Q_W , we have $U_4 = 2\theta_1\theta_2$. So $U_4 = 0$ if and only if either $\theta_1 = 0$ or $\theta_2 = 0$. Now $\theta_j = 0$ if and only if $\beta_j = 0$. By the stability conditions in Lemma 4.2, if $\chi(T\Sigma^\perp) = \deg(L) > 0$ then $[\beta_2] \neq 0$ and therefore it must be $\theta_1 = 0$. From Lemma A.2 in appendix A this implies that the normal curvature satisfies (4.11). Similarly, for $\deg(L) < 0$ we have $\theta_2 = 0$, which yields (4.12). From Theorem 4.4 we know that when $l = \chi(T\Sigma^\perp) \neq 0$ we have families of linearly full immersions with either $[\beta_1] = 0$ (provided $l > 0$) or $[\beta_2] = 0$ (provided $l < 0$).

Now if $\deg(L) = 0$ then one of (4.11) or (4.12) must still hold, but $\int_\Sigma \kappa^\perp v_\gamma = 0$, hence $\kappa^\perp = 0$ and therefore $\mathbb{I}^{2,0} = 0$, i.e., f is totally geodesic. \square

By combining (4.10) with Lemma A.1 we obtain an area bound for every equivariant minimal immersion, based on the connected component of the moduli space in which it lies.

Corollary 4.7. *The area of an equivariant minimal immersion $f : \mathcal{D} \rightarrow \mathbb{RH}^4$ is bounded above by the area of any superminimal immersion whose normal bundle has the same Euler number:*

$$\int_\Sigma v_\gamma \leq 2\pi[2(g-1) - |\chi(T\Sigma^\perp)|]. \quad (4.14)$$

The next result allows us to relate the structure of $\mathcal{M}(\Sigma, \mathbb{RH}^4)$ to the topology of the moduli spaces of Higgs bundles $\mathcal{H}(\Sigma_c, SO_0(4,1))$.

Proposition 4.8. *An equivariant minimal immersion $f : \mathcal{D} \rightarrow \mathbb{RH}^4$ is superminimal if and only if its Higgs bundle is a Hodge bundle.*

Proof. We observed above that $U_4 = 0$ if and only if at least one of β_1, β_2 is identically zero. According to Proposition 7.5 in [4] the Higgs bundle (V, Q_V, ϕ) is a Hodge bundle when V decomposes into a direct sum $V = \oplus_r (W_r \oplus W_{-r})$ of holomorphic subbundles for which these subbundles W_r are eigenbundles of an infinitesimal gauge transformation $\psi \in \Gamma(\text{End}(V))$ with the properties that $\psi^t = -\psi$, $\nabla\psi = 0$ and

$$\left[\begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \phi \\ \phi^t & 0 \end{pmatrix} \right] = i \begin{pmatrix} 0 & \phi \\ \phi^t & 0 \end{pmatrix}.$$

This last condition is equivalent to $\psi\phi = i\phi$. The index r indicates that W_r corresponds to the eigenvalue ir for $r \in \mathbb{R}$. In particular: (i) $\text{im } \phi \subset W_1$, (ii) $Q_V(W_a, W_b) = 0$ unless $b = -a$, in which case Q_V pairs them dually. When V has rank 4 there are only two possibilities: W_1 has rank either one or two. In the former case V must be decomposable with $W_1 = \text{im } \phi = K^{-1}$ and Higgs bundle decomposition

$$(V' \oplus 1, \Phi') \oplus (L \oplus L^{-1}, 0),$$

where $V' = K^{-1} \oplus K$ and Φ' is just Φ restricted to V' . In this case polystability requires $\deg(L) = 0$ and the corresponding minimal immersion is totally geodesic into a copy of \mathbb{RH}^2 . When W_1 has rank two it is either V_1 or V_2 , since it is Q_V -isotropic and contains $\text{im } \phi$. The holomorphic splitting $V = W_1 \oplus W_{-1}$ then implies that either $\beta_1 = 0$ or $\beta_2 = 0$.

Conversely, suppose $\beta_1 = 0$, then we have a holomorphic splitting $V = V_2 \oplus V_{-2}$ where $V_{-2} = L^{-1} \oplus K$. Define ψ to have i -eigenspace V_2 and $-i$ -eigenspace V_{-2} . This ensures that ψ is skew-symmetric for Q_V and that $\psi\phi = i\phi$. It is also ∇ -parallel since it acts as the complex structure on K^{-1} (which is Kähler) and as J on L (which is parallel). Hence (V, Q_V, ϕ) is a Hodge bundle. The case of $\beta_2 = 0$ is argued similarly. \square

4.2. Some remarks on the structure of $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^4)$. Now that we have identified the Hodge bundles we can gain more insight into the structure of $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^4)$ and, in particular, how its topology is related to that of each Higgs bundle moduli space $\mathcal{H}(\Sigma_c, SO_0(4, 1))$. This is very similar to the structure observed for $N = \mathbb{C}\mathbb{H}^2$ in [25, §6.3]. First let us note that for $G = SO_0(4, 1)$ the topology of $\mathcal{H}(\Sigma_c, G)$ itself is nowhere near as well understood as the case $G = PU(2, 1)$. It is not even clear how many connected components it has (see [4]), although one does know that it is disconnected by the invariant $w_2(\rho) \in \mathbb{Z}_2$, which for us equals $\chi(T\Sigma^\perp) \bmod 2$.

We consider $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^4)$ as a fibre bundle over \mathcal{T}_g with fibres $\mathcal{M}(\Sigma_c, \mathbb{R}\mathbb{H}^4)$. From the discussion leading to Theorem 4.4 each of these fibres is a disjoint union of the open sets $\mathcal{W}_{c,l}$. We want to describe the boundary of $\mathcal{W}_{c,l}$ via its embedding into $\mathcal{H}(\Sigma_c, G)$. We have already observed that the image of $\mathcal{M}(\Sigma_c, \mathbb{R}\mathbb{H}^4)$ is the nilpotent cone. Now we recall Hausel's theorem [18], which asserts that the nilpotent cone agrees with the downwards gradient flow of the Hitchin function $\|\Phi\|_{L^2}^2$. This is a perfect Morse function on the Higgs bundle moduli space and its critical manifolds consist of precisely the Hodge bundles. The previous proposition identifies these with the superminimal immersions. In particular, for $l > 0$ in each $\mathcal{W}_{c,l}$ the superminimal immersions correspond to the locus $[\beta_1] = 0$ (for $l < 0$ this locus is $[\beta_2] = 0$). Let $S_{c,l} \subset \mathcal{W}_{c,l}$ denote this locus, and notice that this is analytically isomorphic to $\mathcal{V}_{c,l}^0$ defined in Lemma 4.3. Then we can think of $\mathcal{W}_{c,l}$ as a vector bundle over $S_{c,l}$ with fibre isomorphic to $H^1(K^{-1}L)$, i.e., one moves along each fibre by only changing $[\beta_1]$.

We conjecture that for $l \neq 0$ $\mathcal{W}_{c,l}$ is the unstable manifold of the downwards gradient flow from the critical submanifold $S_{c,l}$. This is consistent with the bound on area in Corollary 4.7 since, with an appropriate normalisation, the Hitchin function equals the area when the Higgs bundle corresponds to a minimal immersion. Aparicio & García-Prada have shown in [4, Thm 8.4] that the smooth minima of the Hitchin functional are absolute minima, i.e., their Higgs fields are zero. Polystability obliges the corresponding representations take values in a maximal compact subgroup of G . Of course these do not correspond to minimal immersions, but rather to constant (harmonic) maps of \mathcal{D} into $\mathbb{R}\mathbb{H}^4$. We expect that these lie on the boundary “at infinity” of the vector bundle $\mathcal{W}_{c,l} \rightarrow S_{c,l}$. Indeed, suppose $l > 0$ and $(L, \xi_1, \xi_2) \in \mathcal{W}_{c,l}$, with $\xi_1 \neq 0$ and corresponding Higgs bundle (V, Q_V, ϕ) . Then the scaling $(L, t\xi_1, \xi_2)$, $t > 0$, keeps us inside $\mathcal{W}_{c,l}$ and does not change (V, Q_V) (since the holomorphic structure of an extension bundle depends only on the projective line generated by its extension class). Therefore only the Higgs field can vary, and we predict that it tends to zero in the limit as $t \rightarrow \infty$. In particular, one cannot use the downward gradient flow to pass from one critical manifold $S_{c,l}$ to another: the flow is always down to absolute minima. This makes sense: one does not expect to be able to change $\chi(T\Sigma^\perp)$ without breaking the condition that the map is an immersion.

The situation with the boundary of $\mathcal{W}_{c,0}$ is slightly different. Recall that this is an open subset of the vector bundle $\mathcal{V}_{c,0} \rightarrow \text{Pic}_0(\Sigma_c)$ with parameters (L, ξ_1, ξ_2) , determined by the conditions $\xi_1 \neq 0$, $\xi_2 \neq 0$ with $\xi_1 \neq \xi_2$ when $L \simeq 1$. Notice that if only one of ξ_1, ξ_2 vanishes then we do not even get polystable Higgs bundles, i.e., this takes us out of $\mathcal{H}(\Sigma_c, G)$ altogether. We know from Theorem 4.6 that $\mathcal{W}_{c,0}$ contains no superminimal immersions, so there are no Hodge bundles. Indeed, the proof of Proposition 4.8 shows that the Hodge bundles are given by the locus $\{(L, 0, 0)\}$, i.e., the zero section of the bundle $\mathcal{V}_{c,l}$, and this parametrises all totally geodesic immersions into a copy of $\mathbb{R}\mathbb{H}^2$. By the proof of Lemma 4.2 the locus $(1, \xi, \xi)$, $\xi \neq 0$, describes all minimal immersions into a totally geodesic copy of $\mathbb{R}\mathbb{H}^3$. Notice that this

is wholly consistent with the description of $\mathcal{M}(\Sigma_c, \mathbb{R}\mathbb{H}^3)$ given in Theorem 3.1. Therefore one part of the boundary of \mathcal{W}_0 consists of all minimal immersions which are not linearly full, as one expects. The other boundary is the asymptotic limit along the fibres which takes us to the zero Higgs field limit, as above.

In summary, the boundary of the connected component $\mathcal{W}_0 \subset \mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^4)$ (i.e., the minimal immersions with flat normal bundle) in $\mathcal{T}_g \times \mathcal{R}(\pi_1 \Sigma, G)$ contains multiple copies of the moduli space $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^2)$, one for each pair (c, L) of marked conformal structure c and degree zero holomorphic line bundle L over Σ_c , but it contains only one copy of $\mathcal{M}(\Sigma, \mathbb{R}\mathbb{H}^3)$.

APPENDIX A. THE GAUSS-CODAZZI-RICCI EQUATIONS.

The zero curvature equations for the connexion ∇^E on E yield the Gauss-Codazzi-Ricci equations for the minimal immersion $f : \mathcal{D} \rightarrow \mathbb{R}\mathbb{H}^n$. It is convenient to calculate these in a local orthonormal frame, adapted to the splitting (2.7). In a conformal coordinate chart (U, z) on (Σ, γ) , let $Z = \partial/\partial z$ and for any smooth section σ of E write $Z\sigma$ to mean $\nabla_Z^E \sigma$ and so forth. Let f_0 denote the length -1 section of E which corresponds to the map f , so that f_0 generates the trivial line bundle in the summand $E = V \oplus 1$. Let $s = \|Z\|_\gamma$ in the induced metric γ , so that locally $\gamma = 2s^2|dz|^2$. Then $\|\bar{Z}\|_\gamma = s$ so that

$$f_1 = s^{-1}Zf_0, \quad f_{-1} = s^{-1}\bar{Z}f_0,$$

locally smoothly frame K^{-1} and K respectively. Now choose an oriented orthonormal frame ν_1, \dots, ν_{n-2} for $T\Sigma^\perp$: this provides a complex frame for W . Finally, let η_{jk} be the connexion 1-forms for the connexion in the normal bundle W , i.e.,

$$\eta_{jk}(\bar{Z}) = \langle \bar{Z}\nu_j, \nu_k \rangle$$

Altogether $f_1, \nu_1, \dots, \nu_{n-2}, f_{-1}, f_0$ provides a $U(n, 1)$ -frame for $E = V \oplus 1$. In this local frame the holomorphic structure on E is determined by the equations

$$\bar{Z}f_1 = -(\bar{Z} \log s)f_1 + sf_0, \tag{A.1}$$

$$\bar{Z}\nu_j = \sum_k \eta_{jk}(\bar{Z})\nu_k - s^{-1}\langle \mathbb{I}(\bar{Z}, \bar{Z}), \nu_j \rangle f_1 \tag{A.2}$$

$$\bar{Z}f_{-1} = (\bar{Z} \log s)f_{-1} + s^{-1}\mathbb{I}(\bar{Z}, \bar{Z}) \tag{A.3}$$

$$\bar{Z}f_0 = sf_{-1}. \tag{A.4}$$

Note that if we consider dz as a local section of K then $dz = s^{-1}f_{-1}$ and therefore by comparison with (2.8)

$$\beta(\bar{Z}) : f_{-1} \mapsto s^{-1}\mathbb{I}(\bar{Z}, \bar{Z}).$$

In this local frame the zero curvature equations for ∇ are

$$-s^{-2}Z\bar{Z} \log s^2 + s^{-4}\|\mathbb{I}(Z, Z)\|^2 + 1 = 0, \tag{A.5}$$

$$\bar{Z}\langle \mathbb{I}(Z, Z), \nu_k \rangle + \sum_j \eta_{jk}(\bar{Z})\langle \mathbb{I}(Z, Z), \nu_j \rangle = 0, \tag{A.6}$$

$$R^\perp(Z, \bar{Z}) + s^{-2}[\mathbb{I}(Z, Z) \otimes \mathbb{I}(Z, Z)^* - \mathbb{I}(\bar{Z}, \bar{Z}) \otimes \mathbb{I}(\bar{Z}, \bar{Z})^*] = 0, \tag{A.7}$$

where $R^\perp(X, Y) = [\nabla_X^\perp, \nabla_Y^\perp] - \nabla_{[X, Y]}^\perp$ is the curvature in the normal bundle, η is the $\text{End}(W)$ valued connexion 1-form for the normal bundle connexion,

$$\eta : \nu_j \mapsto \langle d\nu_j, \nu_k \rangle \nu_k,$$

and

$$\mathbb{I}(Z, Z) \otimes \mathbb{I}(Z, Z)^* : W \rightarrow W; \quad \sigma \mapsto \mathbb{I}(Z, Z) \langle \sigma, \mathbb{I}(Z, Z) \rangle.$$

Note that $R^\perp = d\eta + \eta \wedge \eta$. The local expression for the Gaussian curvature for the induced metric $\gamma = 2s^2|dz|^2$ is

$$\kappa_\gamma = -s^{-2}Z\bar{Z} \log s^2,$$

so that the Gauss equation in global form is

$$\kappa_\gamma = -1 - \|\mathbb{I}^{2,0}\|_\gamma^2. \quad (\text{A.8})$$

The Codazzi equation simply says $\nabla_{\frac{1}{Z}}\mathbb{I}(Z, Z) = 0$, i.e., $\mathbb{I}^{2,0}$ is a holomorphic quadratic differential with values in the complexified normal bundle W . It follows that $\kappa_\gamma = -1$ either everywhere (for totally geodesic embeddings) or only at isolated points.

In general the Ricci equation does not simplify further. But for maps into $\mathbb{R}\mathbb{H}^4$ the normal bundle curvature can be represented by a scalar κ^\perp . This is defined by

$$\langle R^\perp \nu_1, \nu_2 \rangle = \kappa^\perp v_\gamma, \quad (\text{A.9})$$

where v_γ is the area form with respect to the induced metric γ . The left hand side is well-defined globally because the normal bundle connexion is $SO(2)$ -invariant. This definition ensures that the normal bundle curvature is related to the Euler number of the normal bundle by

$$\chi(T\Sigma^\perp) = \frac{1}{2\pi} \int_\Sigma \kappa^\perp v_\gamma. \quad (\text{A.10})$$

Lemma A.1. *For a minimal immersion f into $\mathbb{R}\mathbb{H}^4$*

$$(\kappa^\perp)^2 = \|\mathbb{I}^{2,0}\|_\gamma^4 - \|U_4\|_\gamma^2 = (1 + \kappa_\gamma)^2 - \|U_4\|_\gamma^2, \quad (\text{A.11})$$

where $U_4 = \langle \mathbb{I}^{2,0}, \mathbb{I}^{0,2} \rangle$.

Proof. Set $A_j = \langle \mathbb{I}(Z, Z), \nu_j \rangle$, and write $U_4 = u_4 dz^4$, so that

$$\|\mathbb{I}(Z, Z)\|^2 = |A_1|^2 + |A_2|^2, \quad u_4 = A_1^2 + A_2^2,$$

Then the local form of the Ricci equation is

$$i\kappa^\perp s^2 - s^{-2}(A_1\bar{A}_2 - \bar{A}_1A_2) = 0. \quad (\text{A.12})$$

Define $s_2 = \sqrt{|A_1|^2 + |A_2|^2}$ so that $s^{-2}s_2$ is the local expression for $\|\mathbb{I}^{2,0}\|_\gamma$. Then

$$\begin{aligned} s_2^4 &= |A_1|^4 + 2|A_1|^2|A_2|^2 + |A_2|^4, \\ |u_4|^2 &= |A_1|^4 + A_1^2\bar{A}_2^2 + \bar{A}_1^2A_2^2 + |A_2|^4. \end{aligned}$$

Therefore, by (A.12),

$$(\kappa^\perp)^2 = s^{-8}(s_2^4 - |u_4|^2). \quad (\text{A.13})$$

This gives the global equation (A.11). \square

Lemma A.2. *Suppose f is superminimal, i.e., $U_4 = 0$. Then either $\kappa^\perp = \|\mathbb{I}^{2,0}\|_\gamma^2$ or $\kappa^\perp = -\|\mathbb{I}^{2,0}\|_\gamma^2$.*

Proof. Since $u_4 = A_1^2 + A_2^2 = (A_1 + iA_2)(A_1 - iA_2)$ this vanishes if and only if either $A_1 = iA_2$ or $A_1 = -iA_2$. By (4.5), if $\theta_1 = 0$ then $A_1 = iA_2$ and therefore (A.12) becomes

$$i\kappa^\perp = is^{-4}(|A_1|^2 + |A_2|^2),$$

and therefore $\kappa^\perp = \|\mathbb{I}^{2,0}\|_\gamma^2$. Similarly, when $\theta_2 = 0$ the opposite equality is obtained. \square

APPENDIX B. PROOF OF LEMMA 4.3.

We follow the terminology of [24]. For any holomorphic vector bundle ξ of rank two over Σ_c define

$$\begin{aligned} s(\xi) &= c_1(\xi) - 2 \max\{\deg(\lambda) : \lambda \subset \xi \text{ a holomorphic line subbundle}\} \\ &= c_1(\xi) - 2\mu(\xi). \end{aligned}$$

It is known that s is lower semi-continuous on algebraic families of vector bundles. We are interested in non-trivial extension bundles of the form $0 \rightarrow K^{-1} \rightarrow \xi \rightarrow L \rightarrow 0$ where $1 \leq l < 2(g-1)$ for $l = \deg(L)$. We want to show that the set \mathcal{V}_l^0 of all such extensions with $\mu(\xi) < l$, equivalently $s(\xi) > l - 2(g-1)$, is non-empty and open. By lower semi-continuity, it suffices to show that this is non-empty.

Since $s(K \otimes \xi) = s(\xi)$ it is equivalent to consider extension bundles of the form

$$0 \rightarrow 1 \rightarrow \xi \rightarrow \lambda \rightarrow 0,$$

for which $d = \deg(\lambda) = l + 2(g-1)$ and $s(\xi) > l - 2(g-1)$. By Serre duality $H^1(\lambda^{-1}) \simeq H^0(K\lambda)^*$ and by Riemann-Roch this space has dimension

$$h^0(K\lambda) = l + 3(g-1).$$

Consider the embedding

$$\varphi_\lambda : \Sigma_c \rightarrow \mathbb{P}H^0(K\lambda)^*,$$

which assigns to each $p \in \Sigma_c$ the hyperplane $H^0(K\lambda(-p))$ of sections of $K\lambda$ which vanish at p (this is well-defined since the degree of $K\lambda$ is sufficiently high for it to be very ample). Relative to this the l -th secant variety $\text{Sec}_l(\Sigma_c)$ is the subvariety of $\mathbb{P}H^0(K\lambda)^*$ whose elements correspond to linear forms which vanish on $H^0(K\lambda(-D))$ for some effective divisor D of degree l . The following result is Prop. 1.1 of [24] for the case where $d = l + 2(g-1)$ and using the value $s = l + 2 - 2(g-1)$. In particular, the conditions $s \equiv d \pmod{2}$ and $4 - d \leq s \leq d$ of that proposition hold when $l \geq 1$.

Lemma B.1. *The bundle ξ has $s(\xi) \geq l + 2 - 2(g-1)$ if and only if $\xi \notin \text{Sec}_l(\Sigma_c)$.*

Now

$$\dim(\text{Sec}_l(\Sigma_c)) = 2l - 1 < l + 3g - 4 = \dim(\mathbb{P}H^0(K\lambda)^*),$$

and therefore $\text{Sec}_l(\Sigma_c)$ is a proper subvariety. It follows that there exist non-zero $\xi \in H^1(\lambda^{-1})$ with $s(\xi) \geq l + 2 - 2(g-1) > l - 2(g-1)$. We deduce that $\mathcal{V}_l^0 \neq \emptyset$.

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