Solutions to
Exercise Manual for Math 135
Spring 2024 Edition

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Contents

1 Chapter 1: Algebra and Precalculus Review

§1.1, 1.2, 1.3, 1.4, 7.2, Appendix B

Ex. A-1 Algebra/Precalculus Fa17 Exam

Find all solutions to the following equation.

$$2\ln(x) = \ln\left(\frac{x^5}{5-x}\right) - \ln\left(\frac{x^3}{2+x}\right)$$

exf17

Solution **A-1**

The term on the left is equivalent to $2\ln(x) = \ln(x^2)$ for $x > 0$. Then we move all terms to the left and combine using logarithm rules.

$$\ln(x^2) - \ln\left(\frac{x^5}{5-x}\right) + \ln\left(\frac{x^3}{2+x}\right) = 0$$
$$\ln\left(x^2 \cdot \frac{5-x}{x^5} \cdot \frac{x^3}{2+x}\right) = 0$$
$$\ln\left(\frac{5-x}{2+x}\right) = 0$$

Exponentiating each side gives $\frac{5-x}{2+x} = 1$, whence $5 - x = 2 + x$, and so $x = \frac{3}{2}$.

Ex. A-2 Algebra/Precalculus Sp19 Exam

The number N of bacteria at time t grows exponentially, so that $N(t) = N_0 e^{kt}$ for some constants N_0 and k. Suppose an initial population of 100 bacteria grows to 500 after 2 hours. How many hours does it take for an initial population of 150 bacteria to grow to 300?

exs19

Solution **A-2**

We are given that $N_0 = 100$ and $N(2) = 500$. Hence $500 = 100e^{2k}$, and solving for k gives $k = \frac{1}{2}\ln(5)$. Now solving the equation $300 = 150e^{kt}$ for t gives $t = \frac{2\ln(2)}{\ln(5)}$.

Ex. A-3 Algebra/Precalculus Fa19 Exam

Solve the inequality $\dfrac{3x-6}{x+4} > 0$. Write your answer using interval notation.

exf19

Solution **A-3**

We solve the inequality using the method of sign charts. The cut points for our number line are $x = 2$ (obtained by solving $3x - 6 = 0$) and $x = -4$ (obtained by solving $x + 4 = 0$).

interval	test point	sign of $\frac{3x-6}{x+4}$	truth of inequality
$(-\infty, -4)$	$x = -5$	$\frac{\ominus}{\ominus} = \oplus$	true
$(-4, 2)$	$x = 0$	$\frac{\ominus}{\oplus} = \ominus$	false
$(2, \infty)$	$x = 3$	$\frac{\oplus}{\oplus} = \oplus$	true

The inequality is not satisfied at either cut point $x = -4$ or $x = 2$. Hence the solution to our inequality is the set $(-\infty, -4) \cup (2, \infty)$.

Ex. A-4 Algebra/Precalculus Sp20 Exam

Solve the inequality $\dfrac{3x+6}{x-4} < 0$. *Write your answer using interval notation.*

exs20

Solution **A-4**

We solve the inequality using the method of sign charts. The cut points for our number line are $x = -2$ (obtained by solving $3x + 6 = 0$) and $x = 4$ (obtained by solving $x - 4 = 0$).

A-4

interval	test point	sign of $\frac{3x+6}{x-4}$	truth of inequality
$(-\infty, -2)$	$x = -3$	$\frac{\ominus}{\ominus} = \oplus$	false
$(-2, 4)$	$x = 0$	$\frac{\oplus}{\ominus} = \ominus$	true
$(4, \infty)$	$x = 5$	$\frac{\oplus}{\oplus} = \oplus$	false

The inequality is not satisfied at either cut point $x = -2$ or $x = 4$. Hence the solution to our inequality is the set $(-2, 4)$.

Ex. A-5 **Algebra/Precalculus** Sp20 Exam

Find the domain of the function $f(x) = \dfrac{\ln(80-x)}{\sqrt{x}-5}$. *Write your answer using interval notation.*

exs20

Solution A-5

The expression $\ln(80-x)$ is defined only for $80 - x > 0$, or on the interval $(-\infty, 80)$. The expression $\sqrt{x}$ is defined only for $x \geq 0$, or on the interval $[0, \infty)$. Both expressions are thus defined on the intersection of these two intervals: $[0, 80)$. Finally, we must exclude any values of x for which $\sqrt{x} - 5 = 0$, so just $x = 25$. Hence the domain of f is $[0, 25) \cup (25, 80)$.

Ex. A-6 **Algebra/Precalculus** Su20 Exam

Let $f(x) = 8 - \dfrac{1}{5x}$. Fully simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ with $h \neq 0$. In your work, make clear where you use the assumption $h \neq 0$.

exsu20

Solution A-6

We have the following.

$$\frac{f(x+h)-f(x)}{h} = \frac{\left(8 - \frac{1}{5(x+h)}\right) - \left(8 - \frac{1}{5x}\right)}{h} = \frac{-\frac{1}{5(x+h)} + \frac{1}{5x}}{h} \cdot \frac{5x(x+h)}{5x(x+h)} = \frac{-x + (x+h)}{5hx(x+h)} = \frac{h}{5hx(x+h)}$$

At this point, since $h \neq 0$, we may cancel the common factor of h and obtain our final answer.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{5x(x+h)}$$

Ex. A-7 **Algebra/Precalculus** Su20 Exam

For both parts of this problem, consider the following inequality.

$$\frac{(x-3)(x-6)}{x-5} < 0$$

Your goal is to identify an error in a false solution of this inequality, and then to solve the inequality yourself.

(a) A student submits the following work for solving this equality.

> "First we multiply both sides by $(x-5)$. On the left side, this factor cancels, and on the right side we get 0. So we have $(x-3)(x-6) < 0$. The graph of $y = (x-3)(x-6)$ is a parabola that opens upward and crosses the x-axis at $x = 3$ and $x = 6$. This means that the graph is below the x-axis between these two x-values. So the solution to $(x-3)(x-6) < 0$ is the interval $(3, 6)$. But since the original inequality was undefined at $x = 5$, we also have to exclude 5. So the final answer is $(3, 5) \cup (5, 6)$."

The student's teacher does not give full credit for this solution, simply noting that $x = 4$ is included in the student's answer, but $x = 4$ does not satisfy the original inequality. So the final answer must be wrong.

What is the student's error? Be as specific as possible and explain why this is an error. ***To explain why the given solution is wrong, it is not enough to simply write the correct solution and observe that the two solutions are different.***

exsu20

Algebra/Precalculus Su20 Exam

(b) Solve the original inequality. Write your answer using interval notation.

exsu20

Solution A-7

(a) The quantity $(x-5)$ may take negative values (i.e., if $x<5$), and in that case, multiplying both sides of the inequality by $(x-5)$ would reverse the direction of the inequality. The student's work implicitly assumes that $(x-5)$ is positive throughout, evident by the student's not reversing the direction of the inequality. The student's primary error is then never properly considering the case in which $(x-5)<0$.

(b) We will use the method of sign charts. The cut points for our number line are $x=3$, $x=5$, and $x=6$.

interval	test point	sign of $\frac{(x-3)(x-6)}{(x-5)}$	truth of inequality
$(-\infty,3)$	$x=0$	$\frac{\ominus\ominus}{\ominus}=\ominus$	true
$(3,5)$	$x=4$	$\frac{\oplus\ominus}{\ominus}=\oplus$	false
$(5,6)$	$x=5.5$	$\frac{\oplus\ominus}{\oplus}=\ominus$	true
$(6,\infty)$	$x=7$	$\frac{\oplus\oplus}{\oplus}=\oplus$	false

Hence the solution to the inequality is $(-\infty,3)\cup(5,6)$.

Ex. A-8 Algebra/Precalculus Fa20 Exam

Fully simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{x+2}$ and $h\neq 0$. Write your answer without square roots or fractional exponents in the numerator.

exf20

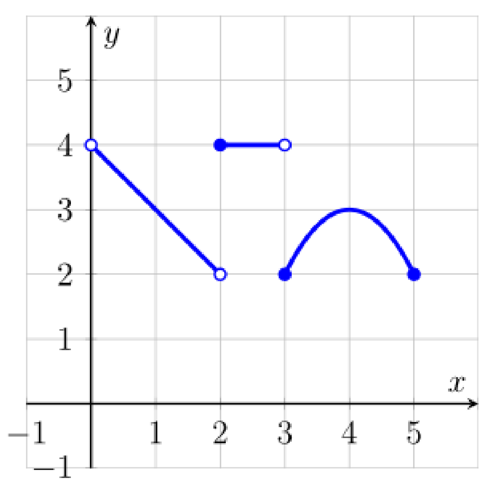
Solution A-8

Calculate the composition, then rationalize the numerator.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} = \frac{(x+h+2)-(x+2)}{h\left(\sqrt{x+h+2}+\sqrt{x+2}\right)} \\ &= \frac{h}{h\left(\sqrt{x+h+2}+\sqrt{x+2}\right)} = \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}\end{aligned}$$

Ex. A-9 Algebra/Precalculus Fa20 Exam

For each part, use the graph of $y=g(x)$ below.



(a) Find the domain of $g(x)$. Write your answer in interval notation.

(b) Calculate $g(g(4))$.

(c) As $x\to 2$, which of the left-sided and right-sided limits of $g(x)$ exist?

exf20

Solution **A-9**

(a) The domain is the set of allowed x-values for $g(x)$. Hence the domain is $(0, 5]$.

(b) $g(g(4)) = g(3) = 2$.

(c) Both the left-sided and right-sided limits of $g(x)$ exist as $x \to 2$.

Ex. A-10 **Algebra/Precalculus** Fa20 **Exam**

While solving the logarithmic equation

$$\log_2(3x + 1) = 3$$

a student wrote the following steps (this work contains two distinct errors):

$$\log_2(3x) + \log_2(1) = 3 \tag{1}$$
$$\log_2(3x) + 0 = 3 \tag{2}$$
$$3x = 3^2 \tag{3}$$
$$x = 3 \tag{4}$$

(a) Identify the lines in which the two errors occur and describe each error.

(b) What is the correct solution to the original equation?

exf20

Solution **A-10**

(a) Line (1) has an error: logarithms do not distribute over sums. That is, $\log_a(x+y) \neq \log_a(x)+\log_a(y)$ in general.

Line (3) has an error: the right side should be 2^3 instead of 3^2 since $\log_b(y) = x$ is equivalent to $y = b^x$.

(b) Exponentiating the equation $\log_2(3x+1) = 3$ immediately gives $3x+1 = 2^3 = 8$. Then solving for x gives $x = \frac{7}{3}$.

Ex. A-11 **Algebra/Precalculus** Fa20 **Exam**

Fully simplify the difference quotient $\dfrac{f(3 + h) - f(3)}{h}$ for $f(x) = \dfrac{6}{9 - 2x}$ and $h \neq 0$. Your answer cannot contain a complex fraction (fraction within a fraction).

exf20

Solution **A-11**

We have the following.

$$\frac{f(3 + h) - f(3)}{h} = \frac{\frac{6}{9-2(3+h)} - 2}{h} = \frac{\frac{6}{3-2h} - 2}{h} = \frac{6 - 2(3 - 2h)}{h(3 - 2h)} = \frac{4h}{h(3 - 2h)} = \frac{4}{3 - 2h}$$

Ex. A-12 **Algebra/Precalculus** Fa20 **Exam**

Suppose we have all of the following:

$$\log_3(x) = A \quad , \quad \log_3(y) = B \quad , \quad \log_{b^5}(z) = C$$

Write each of the following in terms of A, B, and C. Your final answer cannot contain any "log" symbol.

(a) $\log_3\left(\dfrac{\sqrt{x}}{9y^4}\right)$

(b) $\log_b(z)$

exf20

Solution **A-12**

(a) We use basic properties of logarithms.

$$\log_3\left(\frac{\sqrt{x}}{9y^4}\right) = \log_3(\sqrt{x}) - \log_3(9) - \log_3(y^4) = \frac{1}{2}\log_3(x) - 2 - 4\log_3(y) = \frac{1}{2}A - 2 - 4B$$

(b) By definition of the logarithm, we have:

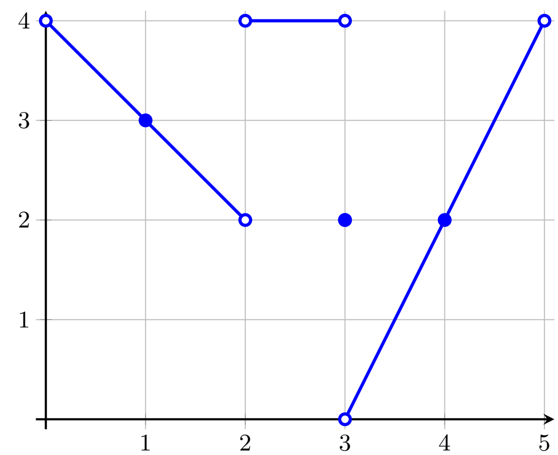
$$\log_{b^5}(z) = C \iff z = (b^5)^C$$

A-12

Hence $z = b^{5C}$. Now taking the log (base b) of both sides gives $\log_b(z) = 5C$.

Ex. A-13 **Algebra/Precalculus** Fa20 **Exam**

For each part, use the graph of $y = g(x)$ below.



(a) Calculate $g(g(1.5))$.

(b) Find the range of $g(x)$. Write your answer in interval notation.

exf20

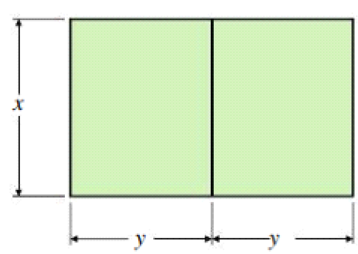
Solution **A-13**

(a) $g(g(1.5)) = g(2.5) = 4$.

(b) The range is the set of values of $g(x)$ (i.e., the y-values). Hence the range is $(0, 4]$.

Ex. A-14 **Algebra/Precalculus** Sp21 **Exam**

Suppose you have exactly 840 ft of fencing that will be used to build an enclosure that consists of two identical rectangular pens that share a common fence. Let x be the (vertical) length of each pen and let y be the (horizontal) width of each pen. See the figure below.



(a) Find an expression for $F(x)$, the area of one individual pen, as a function of x.

(b) Now suppose that, for each of the two pens, the sum of the length and width must not exceed 250 ft. In the context of this problem, what is the domain of F? Write your answer in interval notation.

exs21

Solution **A-14**

(a) Since we have 840 total fencing, we have that $3x + 4y = 840$, or $y = \frac{1}{4}(840 - 3x)$. The area of one individual pen is xy. Hence $F(x) = \frac{1}{4}x(840 - 3x)$.

(b) For one pen we are given that $x + y \leq 250$, or $x + \frac{1}{4}(840 - 3x) \leq 250$. Rearranging this inequality gives $x \leq 160$. Of course, the length x must be non-negative (so $x \geq 0$) and the width y must also be non-negative (so $\frac{1}{4}(840 - 3x) \geq 0$, or $x \leq 280$). Putting these restrictions altogether gives the domain of F as $[0, 160]$ (or $0 \leq x \leq 160$).

Ex. A-15 Algebra/Precalculus Sp21 Exam

Suppose $\log_3(x) = A$ and $\log_3(y) = B$. Rewrite the expression below in terms of A and B. Your final answer may not contain any logarithm symbol.

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right)$$

exs21

Solution A-15

We have the following:

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right) = \log_3(27) + \log_3(\sqrt{x}) - \log_3(y^4) = 3 + \frac{1}{2}\log_3(x) - 4\log_3(y) = 3 + \frac{1}{2}A - 4B$$

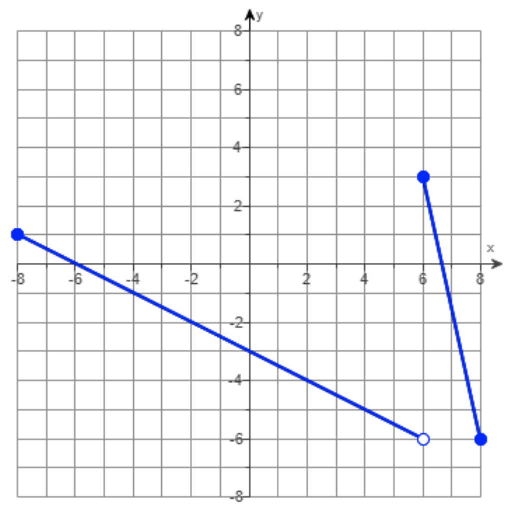
Ex. A-16 Algebra/Precalculus Fa21 Exam

The graph of $y = f(x)$ is given below.

Note that f is piecewise linear. An explicit formula for $f(x)$ can be written in the following form, where A and B are constants.

$$f(x) = \begin{cases} y_1(x) & \text{if } -8 \le x < A \\ y_2(x) & \text{if } B \le x \le 8 \end{cases}$$

Calculate each of A, B, $y_1(x)$, and $y_2(x)$.



exf21

Solution A-16

We see that the graph of f consists of two line segments, one valid for $-8 \le x < 6$ (hence $A = 6$) and the other valid for $6 \le x \le 8$ (hence $B = 6$).

We find $y_1(x)$ by finding the equation of the line through $(-8, 1)$ and $(6, -6)$. We find $y_2(x)$ by finding the equation of the line through $(6, 3)$ and $(8, -6)$. So using point-slope form, we have the following:

$$y_1(x) = 1 + \frac{-6-1}{6-(-8)}(x - (-8)) = 1 - \frac{1}{2}(x+8)$$
$$y_2(x) = -6 + \frac{-6-3}{8-6}(x-8) = -6 - \frac{9}{2}(x-8)$$

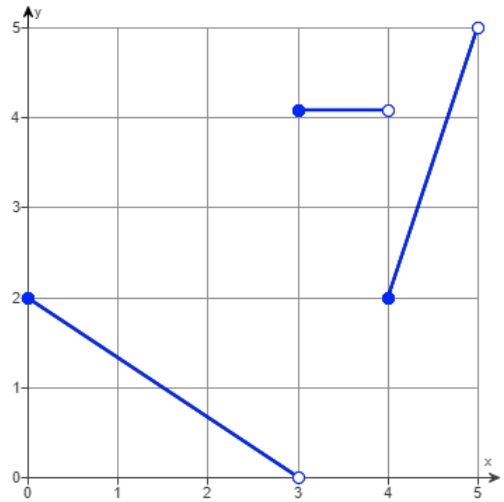
Ex. A-17 Algebra/Precalculus Fa21 Exam

For each part, use the graph of $y = f(x)$.

exf21

Algebra/Precalculus Fa21 **Exam**

(a) Calculate $f(f(2))$.
(b) State the domain of f in interval notation.
(c) State the range of f in interval notation.



exf21

Solution **A-17**

(a) Since f is piecewise linear, we can use point-slope form to find an equation for f valid for $0 \leq x < 3$.

$$f(x) = 2 + \frac{0-2}{3-0}(x-0) = 2 - \frac{2}{3}x$$

Hence we find $f(2) = 2 - \frac{2}{3} \cdot 2 = \frac{2}{3}$, whence $f(f(2)) = f(\frac{2}{3}) = 2 - \frac{2}{3} \cdot \frac{2}{3} = \frac{14}{9}$.
(b) The domain of f is $[0, 5)$.
(c) The range of f is $(0, 5)$.

Ex. A-18 **Algebra/Precalculus** Fa21 **Exam**

Suppose $\log_3(x) = A$ and $\log_3(y) = B$. Rewrite the expression below in terms of A and B. Your final answer may not contain any logarithm symbol.

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right)$$

exf21

Solution **A-18**

We have the following:

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right) = \log_3(27) + \log_3(\sqrt{x}) - \log_3(y^4) = 3 + \frac{1}{2}\log_3(x) - 4\log_3(y) = 3 + \frac{1}{2}A - 4B$$

Ex. A-19 **Algebra/Precalculus** Fa21 **Exam**

Rewrite the expression below as a single logarithm. Assume x and y are positive.

$$\frac{1}{2}\left(\log_5(x) - 7\log_5(y)\right) + 3\log_5(x-1)$$

exf21

Solution **A-19**

We have the following:

$$\frac{1}{2}\left(\log_5(x) - 7\log_5(y)\right) + 3\log_5(x-1) = \frac{1}{2}\log_5\left(\frac{x}{y^7}\right) + \log_5\left((x-1)^3\right)$$
$$= \log_5\left(\frac{x^{1/2}}{y^{7/2}}\right) + \log_5\left((x-1)^3\right) = \log_5\left(\frac{x^{1/2}(x-1)^3}{y^{7/2}}\right)$$

Ex. A-20 **Algebra/Precalculus** Fa21 **Exam**

Suppose $\cos(\theta) = \frac{A}{7}$ with $0 < A < 7$ and $\sin(\theta) < 0$. Find $\sec(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ in terms of A.

exf21

Solution **A-20**

By definition of secant,

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{7}{A}$$

Using the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$ and recalling that $\sin(\theta) < 0$, we have

$$\sin(\theta) = -\sqrt{1 - \cos(\theta)^2} = -\sqrt{1 - \frac{A^2}{49}} = -\frac{\sqrt{49 - A^2}}{7}$$

By definition of tangent,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\sqrt{1 - \frac{A^2}{49}}}{\frac{A}{7}} = -\frac{\sqrt{49 - A^2}}{A}$$

Ex. A-21 **Algebra/Precalculus** Fa21 **Exam**

A bacteria colony has an initial population of 3500. The population grows exponentially and triples every 7 hours. Recall that this means the population P at time t satisfies $P(t) = P_0 e^{kt}$ for some constants P_0 and k.

(a) Find the exact value of the growth constant k.

(b) Find the population after 25 hours.

(c) Find the time (in hours) when the population will be 12,600.

exf21

Solution **A-21**

(a) We are given that $P(7) = 3P(0)$, or $e^{7k} = 3$. Hence $k = \frac{1}{7}\ln(3)$.

(b) $P(25) = 3500e^{25k} = 3500 \cdot 3^{25/7} \approx 177040$.

(c) We have to solve the equation $12600 = 3500e^{kt}$ for t. Dividing by 3500 and taking logarithms gives $t = 7 \cdot \dfrac{\ln(18/5)}{\ln(3)} \approx 8.16$.

Ex. A-22 **Algebra/Precalculus** Fa21 **Exam**

A rectangular box is constructed according to the following rules.

- the length of the box is twice its width
- the height of the box is 5 feet more than three times the length

Let ℓ, w, and h denote the length, width, and height of the box, respectively, measured in feet.

(a) Write the height of the box in terms of w.

(b) Write an expression for $V(w)$, the volume of the box measured in cubic feet, as a function of its width.

(c) Suppose the rules also require that the sum of the box's width and height to be less than 26 feet. Under this condition, what is the domain of the function $V(w)$?

exf21

Solution **A-22**

(a) The first condition gives $\ell = 2w$, and the second condition gives $h = 3\ell + 5$. Hence $h = 3(2w) + 5 = 6w + 5$.

(b) The volume of the box is $V(w) = \ell \cdot w \cdot h = 2w \cdot w \cdot (6w + 5)$.

(c) We are given that $w + h < 26$, or $w + 6w + 5 < 26$. Solving for w gives $w < 3$. Since width must also be non-negative, we find that the domain of $V(w)$ is $0 \le w < 3$, or $w \in [0, 3)$ in interval notation.

Ex. A-23 **Algebra/Precalculus** Fa21 **Exam**

Let $f(x) = \dfrac{2}{3x}$ and assume $h \neq 0$. Fully simplify each of the following expressions:

(a) $f(x + h)$ (b) $f(x + h) - f(x)$ (c) $\dfrac{f(x + h) - f(x)}{h}$

exf21

Solution **A-23**

(a) $f(x+h) = \dfrac{2}{3(x+h)}$

(b) $f(x+h) - f(x) = \dfrac{2}{3(x+h)} - \dfrac{2}{3x}$

(c) We have the following.

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{2}{3(x+h)} - \frac{2}{3x}}{h} = \frac{2x-2(x+h)}{3hx(x+h)} = \frac{-2h}{3hx(x+h)} = \frac{-2}{3x(x+h)}$$

Ex. A-24 **Algebra/Precalculus** Fa21 **Exam**

Find the domain of the function $f(x) = \sqrt{x^2+x-6} + \ln(10-x)$. Write your answer using interval notation.

exf21

Solution **A-24**

We examine the square root and the logarithm separately.

The argument of the square root cannot be negative, hence we must have $x^2+x-6 \geq 0$. This is equivalent to $(x+3)(x-2) \geq 0$. To solve this inequality, we construct a sign chart and test each of the intervals $(-\infty,-3)$, $(-3,2)$, and $(2,\infty)$. We find that the solution to the inequality is $(-\infty,-3] \cup [2,\infty)$.

The argument of the logarithm cannot be negative or zero, hence we must have $10-x>0$, or $x<10$ (or $(-\infty,10)$ in interval notation).

The domain of f is the intersection of the solutions to these two inequalities.

$$(-\infty,-3] \cup [2,10)$$

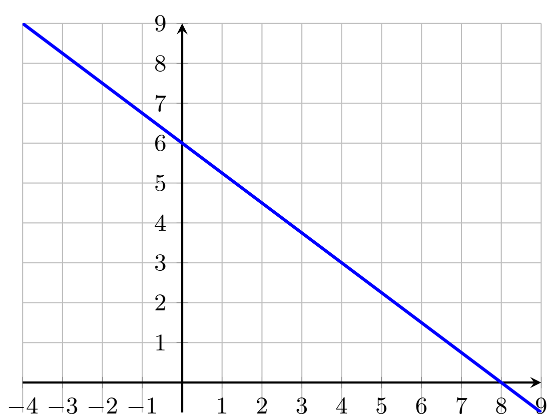
Ex. A-25 **Algebra/Precalculus** Sp22 **Exam**

For each part, use the graph of $y=g(x)$ given below and let $f(x) = 8x^2-4x+15$.

(a) Find an expression for $g(x)$.

(b) Calculate the y-intercept of the graph of $y = f(g(x))$.

(c) Calculate $g(f(x))$.



exs22

Solution **A-25**

(a) We observe from the figure that the graph of $y=g(x)$ is a line that passes through the points $(0,6)$ and $(8,0)$. Hence an equation for this line in point-slope form is

$$g(x) = 0 + \frac{0-6}{8-0}(x-8) = -\frac{3}{4}(x-8)$$

(b) The desired y-intercept is the point $(0, f(g(0)))$. Note that since the y-intercept of g is $(0,6)$, we have $g(0)=6$. Hence $f(g(0)) = f(6) = 8\cdot 6^2 - 4\cdot 6 + 15 = 264$.

(c) We have

$$g(f(x)) = -\frac{3}{4}(f(x)-8) = -\frac{3}{4}\left(8x^2-4x+7\right)$$

Ex. A-26 **Algebra/Precalculus** Sp22 **Exam**

A 100-gram sample of a radioactive substance decays to 65% of its initial mass in 15 hours. Recall that the mass of the sample M at time t satisfies $M(t) = M_0 e^{kt}$ for some constants M_0 and k.

(a) Find the growth constant k.

(b) Find the mass of the sample after 22 hours.

(c) Find the time in hours when the sample will have a mass of 41 grams.

exs22

Solution **A-26**

(a) We are given that $M(15) = 0.65M(0)$, which is equivalent to $M_0 e^{15k} = 0.65M_0$. Canceling the constant M_0, taking logarithms, and solving for k gives

$$k = \frac{\ln(0.65)}{15}$$

(b) We are given $M_0 = 100$, and so the mass at $t = 22$ is

$$M(22) = M_0 e^{22k} = 100e^{\ln(0.65)/15} = 100 \cdot (0.65)^{15}$$

(c) We must solve the equation $M(t) = 41$, or $100e^{kt} = 41$. Dividing by 100, taking logarithms, and solving for t gives

$$t = \frac{\ln(0.41)}{k} = 15 \cdot \frac{\ln(0.41)}{\ln(0.65)}$$

Ex. A-27 **Algebra/Precalculus** Sp22 **Exam**

A rectangular box is constructed according to the following rules.

- The length of the box is 5 times its width.
- The volume of the box is 110 cubic feet.

Let L, W, and H be the length, width, and height of the box (measured in feet), respectively.

(a) Write an equation in terms of L, W, and H that expresses the first constraint.

(b) Write an equation in terms of L, W, and H that expresses the second constraint.

(c) Write an expression for $S(W)$, the total surface area of the box as a function of W.

(d) Suppose the rules also require that the sum of the box's length and width be less than 78 feet. What is the domain of $S(W)$ in this context?

exs22

Solution **A-27**

(a) $L = 5W$

(b) $LWH = 110$

(c) The total surface area in terms of L, and W, and H is

$$S = 2(LW + LH + WH)$$

Putting the first constraint into the second gives $5W^2H = 110$, which then gives $H = \frac{22}{W^2}$. Now substituting our expressions for L and H in terms of W into our expression for S gives

$$S(W) = 2\left(5W \cdot W + 5W \cdot \frac{22}{W^2} + W \cdot \frac{22}{W^2}\right) = 10W^2 + \frac{264}{W}$$

(d) The new rule implies the constraint $L + W < 78$, or $6W < 78$ (given $L = 5W$). Hence $W < 13$. Of course, since W represents a distance, we must also have $W \geq 0$. Hence the domain of $S(W)$ in this context is $0 \leq W < 13$, or the interval $[0, 13)$.

Ex. A-28 **Algebra/Precalculus** Sp22 **Exam**

Suppose $\log_{16}(x) = A$ and $\log_{16}(y) = B$. Rewrite the expression below in terms of A and B. Your final answer may not contain any logarithm symbol.

$$\log_{16}\left(\frac{4x^7}{\sqrt[9]{y}}\right)$$

exs22

Solution **A-28**

Using various logarithm rules and the identity $4 = 16^{1/2}$ gives the following.

$$\begin{aligned}\log_{16}\left(\frac{4x^7}{\sqrt[9]{y}}\right) &= \log_{16}(4x^7) - \log_{16}(\sqrt[9]{y}) \\ &= \log_{16}(4) + \log_{16}(x^7) - \log_{16}(y^{1/9}) \\ &= \log_{16}(16^{1/2}) + 7\log_{16}(x) - \frac{1}{9}\log_{16}(y) \\ &= \frac{1}{2} + 7A - \frac{1}{9}B\end{aligned}$$

Ex. A-29 **Algebra/Precalculus** Sp22 **Exam**

Let $f(x) = \sqrt{3x}$ and assume $h \neq 0$. Fully simplify each of the following expressions:

(a) $f(x+h)$ (b) $f(x+h) - f(x)$ (c) $\dfrac{f(x+h) - f(x)}{h}$

exs22

Solution **A-29**

(a) $f(x+h) = \sqrt{3(x+h)}$

(b) $f(x+h) - f(x) = \sqrt{3(x+h)} - \sqrt{3x}$

(c) Rationalize the numerator, then simplify.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} = \frac{3(x+h) - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)} = \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}}$$

Ex. A-30 **Algebra/Precalculus** Sp22 **Exam**

Consider the function $f(x) = \dfrac{x-6}{x^2 - 9x + 20}$.

(a) Solve the equation $f(x) = 0$.

(b) List all numbers that are not in the domain of $f(x)$.

(c) Solve the inequality $f(x) > 0$ and write your answer using interval notation.

exs22

Solution **A-30**

(a) The equation $f(x) = 0$ is equivalent to $x - 6 = 0$, and so the only solution is $x = 6$.

(b) Since $f(x)$ is rational, its domain is the set of all real numbers except where the denominator vanishes. The equation $x^2 - 9x + 20 = 0$ is equivalent to $(x-4)(x-5) = 0$, whence the only numbers not in the domain of $f(x)$ are $x = 4$ and $x = 5$.

(c) We construct a sign chart whose cut points are those x-values where $f(x) = 0$ or where $f(x)$ is undefined. Hence the cut points are $x = 4$, $x = 5$, and $x = 6$. We then examine the sign of $f(x) = \frac{x-6}{(x-4)(x-5)}$ on each of the corresponding sub-intervals.

A-30

interval	test point	sign of $f(x)$	truth of inequality
$(-\infty, 4)$	$x = 0$	$\frac{\ominus}{\ominus\ominus} = \ominus$	false
$(4, 5)$	$x = 4.5$	$\frac{\ominus}{\oplus\ominus} = \oplus$	true
$(5, 6)$	$x = 5.5$	$\frac{\ominus}{\oplus\oplus} = \ominus$	false
$(6, \infty)$	$x = 7$	$\frac{\oplus}{\oplus\oplus} = \oplus$	true

None of the cut points satisfy the inequality. Hence the solution to the inequality $f(x) > 0$ is $(4, 5) \cup (6, \infty)$.

Ex. A-31 **Algebra/Precalculus** Sp22 **Exam**

Find all solutions to the following equation in the interval $[0, 2\pi)$.

$$2\sin(\theta)\cos(\theta) - \cos(\theta) = 0$$

exs22

Solution **A-31**

Factoring gives $\cos(\theta)\,(2\sin(\theta) - 1) = 0$, whence solutions to the equation are solutions to $\cos(\theta) = 0$ or $\sin(\theta) = \frac{1}{2}$.

Recall that on the unit circle, a point (x, y) corresponds to the point $(\cos(\theta), \sin(\theta))$. Hence solving the equation $\cos(\theta) = 0$ is equivalent to solving $x = 0$ on the unit circle; we get the two solutions $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Solving the equation $\sin(\theta) = \frac{1}{2}$ is equivalent to solving $y = \frac{1}{2}$ on the unit circle; we get the two solutions $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hence the original equation has 4 solutions in the given interval: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Ex. A-32 **Algebra/Precalculus** Sp22 **Exam**

Complete each of the following algebra exercises.

(a) Fully factor the polynomial $5x^4 + 25x^3 - 180x^2$.

(b) Solve the rational equation below.

$$\frac{4}{x+5} + \frac{9x}{x^2 - 25} = \frac{6}{x-5}$$

(c) Simplify the complex fraction below by writing it as a simple fraction.

$$\frac{\dfrac{4}{x} - \dfrac{2}{xy}}{8 + \dfrac{7}{y}}$$

exs22

Solution **A-32**

(a) $5x^4 + 25x^3 - 180x^2 = 5x^2(x^2 + 5x - 36) = 5x^2(x + 9)(x - 4)$

(b) Observe that $x^2 - 25 = (x - 5)(x + 5)$, hence $x^2 - 25$ serves as a common denominator for all terms. Multiplying each side of the equation by $x^2 - 25$ and canceling common factors gives

$$4(x - 5) + 9x = 6(x + 5)$$

Expanding each side and collecting like terms gives $7x - 50 = 0$, whence the only solution is $x = \frac{50}{7}$.

(c) Observe that the common denominator of the terms $\frac{4}{x}$, $\frac{2}{xy}$, 8, and $\frac{7}{y}$ is xy. We multiply the complex fraction by $\frac{xy}{xy}$ and distribute.

$$\frac{\dfrac{4}{x} - \dfrac{2}{xy}}{8 + \dfrac{7}{y}} \cdot \frac{xy}{xy} = \frac{4y - 2}{8xy + 7x}$$

Ex. A-33 **Algebra/Precalculus** Su22 **Exam**

Complete each of the following algebra exercises.

(a) Simplify $\left(\dfrac{27x^{3/5}}{x^{-3}z^{15}}\right)^{-1/3}$, leaving positive exponents and integer coefficients.

(b) Simplify $\dfrac{x^2-9}{3-\sqrt{6-x}}$ for $x \neq -3$. (All common factors must be canceled.)

(c) Factor the expression completely: $5x^9 - 14x^8 - 3x^7$.

(d) Fully simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for $f(x) = \dfrac{2}{x} - 3$ and $h \neq 0$.

exsu22

Solution **A-33**

(a) We have the following:

$$\left(\frac{27x^{3/5}}{x^{-3}z^{15}}\right)^{-1/3} = \frac{27^{-1/3}x^{-1/5}}{xz^{-5}} = \frac{z^5}{3x^{6/5}}$$

(b) Rationalize the denominator. Then cancel common factors.

$$\begin{aligned}\frac{x^2-9}{3-\sqrt{6-x}} &= \frac{x^2-9}{3-\sqrt{6-x}} \cdot \frac{3+\sqrt{6-x}}{3+\sqrt{6-x}} = \frac{(x-3)(x+3)\left(3+\sqrt{6-x}\right)}{9-(6-x)} \\ &= \frac{(x-3)(x+3)\left(3+\sqrt{6-x}\right)}{3+x} = (x-3)\left(3+\sqrt{6-x}\right)\end{aligned}$$

(c) We have the following:

$$5x^9 - 14x^8 - 3x^7 = x^7(5x^2 - 14x - 3) = x^7(5x+1)(x-3)$$

(d) Multiply all terms by the LCD $x(x+h)$, and cancel common factors.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{2}{x+h} - 3 - \left(\frac{2}{x} - 3\right)}{h} = \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} \\ &= \frac{2x - 2(x+h)}{hx(x+h)} = \frac{2x - 2x - 2h}{hx(x+h)} = \frac{-2h}{hx(x+h)} = \frac{-2}{x(x+h)}\end{aligned}$$

Ex. A-34 **Algebra/Precalculus** Su22 **Exam**

For each part, find all solutions to the given equation.

(a) $\sqrt{2x+1} + 1 = x$

(b) $(10-x^2)^{1/2} - x^2(10-x^2)^{-1/2} = 0$

(c) $2 + \sin(\theta) = 2\cos(\theta)^2$ (find solutions in $[0, 2\pi)$ only)

exsu22

Solution **A-34**

(a) Subtract 1 from either side, square both sides, and solve for x.

$$\begin{aligned}\sqrt{2x+1} + 1 &= x \\ \sqrt{2x+1} &= x - 1 \\ 2x + 1 = (x-1)^2 &= x^2 - 2x + 1 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0\end{aligned}$$

Hence we obtain candidate solutions of $x = 0$ and $x = 4$. However, checking these candidates in the original equation, we see that only $x = 4$ is a solution.

A-34

(b) Multiply all terms by $(10-x^2)^{1/2}$, and then solve for x.

$$\begin{gathered}(10-x^2)^{1/2}-x^2(10-x^2)^{-1/2}=0\\(10-x^2)^1-x^2\cdot 1=0\\10=2x^2\\x=\sqrt{5}\quad\text{or}\quad x=-\sqrt{5}\end{gathered}$$

(c) Use the Pythagorean identity on the right side, then rearrange and factor.

$$\begin{gathered}2+\sin(\theta)=2\cos(\theta)^2\\2+\sin(\theta)=2(1-\sin(\theta)^2)\\2+\sin(\theta)=2-2\sin(\theta)^2\\2\sin(\theta)^2+\sin(\theta)=0\\\sin(\theta)\,(2\sin(\theta)+1)=0\end{gathered}$$

Hence we have two possible equations to solve: $\sin(\theta)=0$ (which has solutions $\theta=0$ and $\theta=\pi$ in the given interval) and $\sin(\theta)=-\frac{1}{2}$ (which has solutions $\theta=\frac{7\pi}{6}$ and $\theta=\frac{11\pi}{6}$ in the given interval). So there are four solutions in total.

Ex. A-35 Algebra/Precalculus Su22 Exam

Find the domain of the function $f(x)=\ln(x^2-20)$. Write your answer using interval notation.

exsu22

Solution A-35

The domain of $f(x)$ consists of those x-values such that $x^2-20>0$. To solve this non-linear inequality, we find the cut points: solutions to $x^2-20=0$, or $x=-\sqrt{20}$ and $x=\sqrt{20}$. We then make a sign chart, testing each of the following intervals: $(-\infty,-\sqrt{20})$, $(-\sqrt{20},\sqrt{20})$, and $(\sqrt{20},\infty)$.

For these three intervals, we use the test points -5, 0, and 5, respectively. Hence we find that x^2-20 is positive on the first and third of these intervals only. Hence the domain of $f(x)$ is $(-\infty,-\sqrt{20})\cup(\sqrt{20},\infty)$.

Ex. A-36 Algebra/Precalculus Su22 Exam

The length of a rectangular box is three times its width, and the total surface area of the box is 200 in^2. Let W be the width of the box in inches. Find the volume of the box in terms of W.

exsu22

Solution A-36

Let L, W, and H be the length, width, and height of the box, respectively. Then we immediately have $L=3W$. For the surface area we have:

$$2(LW+LH+WH)=200$$

Substituting $L=3W$ into this equation and collecting like terms gives:

$$3W^2+4WH=100$$

Solving for H then gives:

$$H=\frac{100-3W^2}{4W}$$

Hence the volume of the box is

$$V=LWH=3W\cdot W\cdot\frac{100-3W^2}{4W}=\frac{3}{4}\left(100W-3W^3\right)$$

Ex. A-37 Algebra/Precalculus Su22 Exam

For each part, write an equation for the line in the xy-plane that satisfies the given description.

(a) The line through the point $(-2,10)$ with slope -3.

exsu22

Algebra/Precalculus Su22 **Exam**

(b) The line through the points $(3, 5)$ and $(-1, 4)$.

(c) The line through the point $(5, 1)$ and perpendicular to the line $x + 3y = 10$.

(d) The horizontal line through the point $(-2, 15)$.

exsu22

Solution **A-37**

Use point-slope form for all answers.

(a) $y - 10 = -3(x + 2)$

(b) The slope of the line is $m = \frac{4-5}{-1-3} = \frac{1}{4}$, hence an equation of the line is $y - 5 = \frac{1}{4}(x - 3)$.

(c) The given line can be written as $y = -\frac{1}{3}x + \frac{10}{3}$, whence the slope of the given line is $-\frac{1}{3}$, and so the slope of the desired line is 3. Hence an equation of the desired line is $y - 1 = 3(x - 5)$.

(d) $y = 15$

Ex. A-38 **Algebra/Precalculus** Su22 **Exam**

The number of bacteria in a certain colony grows exponentially. Recall that this means the number of bacteria N at time t is $N(t) = N_0 e^{kt}$, where N_0 and k are constants. Suppose there are initially 500 bacteria, and the number of bacteria triples every 2 hours. How much time must pass before the number of bacteria increases from 500 to 5000?

exsu22

Solution **A-38**

Let $N(t) = N_0 e^{kt}$ be the number of bacteria at time t (measured in hours). Then we have that $N(2) = 3N_0$, or $N_0 e^{2k} = 3N_0$. Canceling N_0 and solving for k gives:

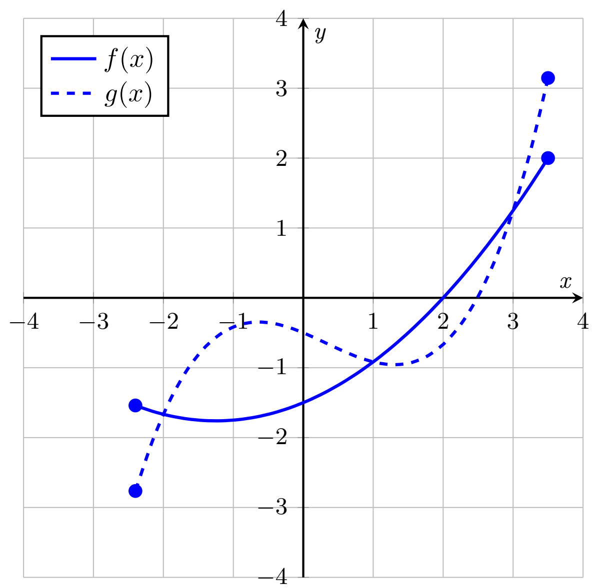
$$k = \frac{1}{2}\ln(3)$$

Now we want to find the value of T such that $N(T) = 5000$, with $N_0 = 500$. Hence we must solve the equation $5000 = 500e^{kT}$, where k is the value we found previously. We obtain:

$$T = 2 \cdot \frac{\ln(10)}{\ln(3)}$$

Ex. A-39 **Algebra/Precalculus** Su22 **Exam**

For each part, use the graphs of $y = f(x)$ and $y = g(x)$ below.



(a) Calculate $f(2)$.

(b) Estimate the value of $g(0) - f(0)$.

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(c) Find all solutions to the equation $f(x) = g(x)$.

(d) Solve the inequality $g(x) > f(x)$. Write your answer using interval notation.

exsu22

Solution **A-39**

(a) $f(2) = 0$

(b) $g(0) - f(0) = 1$

(c) The solutions are the x-values of the points where the graphs intersect: $x = -2, 1, 3$.

(d) The solution is the set of x-values where the graph of g lies above that of f: $(-2, 1) \cup (3, 3.5]$.

Ex. A-40 Algebra/Precalculus

Sp18 Quiz

Complete the following algebra exercises.

(a) Simplify the expression $\dfrac{x^3 - 4x}{x^3 - x^2 - 6x}$ as much as possible.

(b) Write the expression $\dfrac{\sqrt{xy^3}}{(x^{2/3}y^{-5/2})^6}$ in the form $x^a y^b$.

(c) Let $f(x) = 3x^2$. Simplify the expression $\dfrac{f(x+h) - f(x)}{h}$ as much as possible.

(d) Evaluate the expression $\log_6(9) + \log_6(4)$.

(e) Write the solution to the inequality $x^2 - 3x + 2 < 0$ using interval notation.

(f) Find all values of θ in the interval $[0, 2\pi)$ such that $2\sin(2\theta) = 1$.

(g) Let $f(x) = \dfrac{1}{x}$. Simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ as much as possible.

qs18

Solution **A-40**

(a) We have

$$\frac{x^3 - 4x}{x^3 - x^2 - 6x} = \frac{x(x^2 - 4)}{x(x^2 - x - 6)} = \frac{x(x+2)(x-2)}{x(x-3)(x+2)} = \frac{x-2}{x-3}$$

(b) We have

$$\frac{\sqrt{xy^3}}{(x^{2/3}y^{-5/2})^6} = \frac{x^{1/2}y^{3/2}}{x^4 y^{-15}} = x^{1/2-4}y^{3/2-(-15)} = x^{-7/2}y^{33/2}$$

(c) We have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 - 3x^2}{h} = \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \frac{6xh + 3h^2}{h} = 6x + 3h\end{aligned}$$

(d) We have

$$\log_6(9) + \log_6(4) = \log_6(36) = 2$$

(e) We have that $x^2 - 3x + 2 = (x-1)(x-2)$, thus the graph of $y = x^2 - 3x + 2$ is a parabola that opens upward and crosses the x-axis at $x = 1$ and $x = 2$. Hence $y < 0$ precisely when x lies between the two roots. Thus the solution to the inequality $x^2 - 3x + 2 < 0$ is $x \in (1, 2)$.

Alternatively, the inequality is equivalent to $(x-1)(x-2) < 0$. There are three possibilities to consider: $x < 1$, $1 < x < 2$, and $2 < x$. We find that $(x-1)(x-2) < 0$ only in the second case $1 < x < 2$, i.e., $x \in (1, 2)$ in interval notation.

(f) We have that $\sin(2\theta) = \frac{1}{2}$, whose reference angle solution is $2\theta = \frac{\pi}{6}$. The sine is positive also in the second quadrant, whence there is an additional solution given by $2\theta = \frac{5\pi}{6}$. Periodicity then gives all possible solutions.

$$\theta = \frac{\pi}{12} + \pi n \quad , \theta = \frac{5\pi}{12} + \pi n$$

where n is an integer. The only solutions that lie in the interval $[0, 2\pi)$ are

$$\theta = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}$$

A-40

(g) We have

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \frac{x-(x+h)}{hx(x+h)} = \frac{x-x-h}{hx(x+h)} = \frac{-h}{hx(x+h)} = -\frac{1}{x(x+h)}$$

Ex. A-41 **Algebra/Precalculus** Sp18 **Quiz**

For each part, find an equation of the described line.

(a) The line whose slope is -3 and which passes through the point $(1, 4)$.

(b) The line that passes through the point $(-\pi, 1)$ with slope $\sqrt{2}$.

qs18

Solution A-41

(a) Point-slope form gives the equation as $y - 4 = -3(x - 1)$, equivalent to $y = -3x + 7$.

(b) $y - 1 = \sqrt{2}(x + \pi)$

Ex. A-42 **Algebra/Precalculus** Sp18 **Quiz**

Find all solutions to the following equation.

$$\log_2(x) + \log_2(x-3) = 2$$

qs18

Solution A-42

Combine the logarithms using the identity $\log_a(x)+\log_a(y) = \log_a(xy)$. Then undo the logarithms by exponentiation, and solve the resulting equation.

$$\begin{aligned} \log_2(x) + \log_2(x-3) &= 2 \\ \log_2(x(x-3)) &= 2 \\ x(x-3) &= 2^2 \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \end{aligned}$$

Hence the two candidate solutions are $x = 4$ and $x = -1$. However, $x = -1$ does not satisfy the original equation, and so $x = 4$ is the only solution.

Ex. A-43 **Algebra/Precalculus** Su22 **Quiz**

Complete the following algebra exercises.

(a) Find all solutions to the given equation.

$$2x^{5/2} + x^{3/2} + x^{1/2} = 0$$

(b) Simplify the expression; assume $x \neq -10$.

$$\frac{x^3 + 10x^2}{\sqrt{15-x}-5}$$

(c) Simplify the expression; assume any common factors are non-zero.

$$\frac{\frac{x-1}{x+1} + \frac{6}{x}}{\frac{2}{x^2+x} + \frac{1}{x+1}}$$

qsu22

Solution A-43

(a) We first factor out $x^{1/2}$ from the left-hand side.

$$x^{1/2}\left(2x^2 + x + 1\right) = 0$$

A-43

Thus either $x^{1/2} = 0$ (whence $x = 0$) or $2x^2 + x + 1 = 0$. However, the discriminant of this quadratic is

$$\Delta = 1^2 - 4 \cdot 2 \cdot 1 = -7 < 0$$

Since the discriminant is negative, the equation $2x^2 + x + 1 = 0$ has no (real) solution. Thus the only solution of the original equation is $x = 0$.

(b) We rationalize the denominator, and then cancel common factors.

$$\begin{aligned} \frac{x^3 + 10x^2}{\sqrt{15 - x} - 5} \cdot \frac{\sqrt{15 - x} + 5}{\sqrt{15 - x} + 5} &= \frac{(x^3 + 10x^2)\left(\sqrt{15 - x} + 5\right)}{15 - x - 25} \\ &= \frac{x^2(x + 10)\left(\sqrt{15 - x} + 5\right)}{-(x + 10)} \\ &= -x^2\left(\sqrt{15 - x} + 5\right) \end{aligned}$$

(c) Since $x^2 + x = x(x+1)$, we see that the LCD of all terms is $x^2 + x$. So we multiply all terms by that LCD. Then we expand, factor, and cancel common factors.

$$\frac{\frac{x-1}{x+1} + \frac{6}{x}}{\frac{2}{x^2+x} + \frac{1}{x+1}} \cdot \frac{x(x+1)}{x(x+1)} = \frac{(x-1)x + 6(x+1)}{2 + 1 \cdot x} = \frac{x^2 + 5x + 6}{x + 2} = \frac{(x+2)(x+3)}{x+2} = x + 3$$

Ex. A-44 **Algebra/Precalculus** Fa22 **Quiz**

Find all solutions to the given equation.

$$\log_2(x - 3) + 2 = \log_2(x + 9)$$

qf22

Solution **A-44**

Combine the logarithms. Then exponentiate.

$$\begin{aligned} \log_2(x - 3) + 2 &= \log_2(x + 9) \\ 2 &= \log_2(x + 9) - \log_2(x - 3) \\ 2 &= \log_2\left(\frac{x + 9}{x - 3}\right) \\ 4 &= \frac{x + 9}{x - 3} \\ 4x - 12 &= x + 9 \\ x &= 7 \end{aligned}$$

Ex. A-45 **Algebra/Precalculus** Fa22 **Quiz**

Fully simplify the given expression. Assume any common factors are non-zero.

$$\frac{100}{x^2 - 25} - \frac{2x}{x + 5}$$

qf22

Solution **A-45**

Find a common denominator.

$$\begin{aligned} \frac{100}{x^2 - 25} - \frac{2x}{x + 5} &= \frac{100}{(x-5)(x+5)} - \frac{2x}{x+5} \cdot \frac{x-5}{x-5} = \frac{100 - 2x(x-5)}{(x-5)(x+5)} \\ &= \frac{100 - 2x^2 + 10x}{(x-5)(x+5)} = \frac{-2(x-10)(x+5)}{(x-5)(x+5)} = \frac{-2(x-10)}{x-5} \end{aligned}$$

Ex. A-46 **Algebra/Precalculus** Fa22 **Quiz**

Use rationalization to simplify the expression below. All common factors must be canceled.

$$\frac{3-\sqrt{2-x}}{x+7}$$

qf22

Solution **A-46**

Rationalize the numerator and cancel common factors.

$$\frac{3-\sqrt{2-x}}{x+7}\cdot\frac{3+\sqrt{2-x}}{3+\sqrt{2-x}}=\frac{9-(2-x)}{(x+7)(3+\sqrt{2-x})}=\frac{x+7}{(x+7)(3+\sqrt{2-x})}=\frac{1}{3+\sqrt{2-x}}$$

Ex. A-47 **Algebra/Precalculus**

For each of the following problems, zero or more of the choices are exact answers. Identify all of the exact answers, and ***explain why the other choices are wrong***. If the exact value of the correct answer does not appear as one of the choices, find the exact value of the correct answer.

(a) Find all real numbers x such that $x^2 = 2$.

A. 1.41 **B.** $\sqrt{2}$ **C.** ± 1.41 **D.** 1.41 and -1.41 **E.** $\pm\sqrt{2}$

(b) Find all real numbers t such that $t^3 + 4 = 0$.

A. -1.59 **B.** ± 1.59 **C.** $\pm\sqrt[3]{-4}$ **D.** $-2^{2/3}$ **E.** no real solution

(c) Find the circumference of a circle whose radius is 1.

A. 6.28 **B.** ± 6.283185 **C.** $\frac{44}{7}$ **D.** none of the above

(d) Find all real solutions to the equation $2^x = 3$.

A. 1.585 **B.** ± 1.585 **C.** 3^{-2} **D.** $\log_2(3)$ **E.** $\log_3(2)$ **F.** $\frac{\ln(3)}{\ln(2)}$ **G.** $\frac{1}{2}\log_2(9)$

unit1

Solution **A-47**

(a) ***Choice E.***

Choices A, C, and D do not give exact solutions. Choice B is missing the solution $-\sqrt{2}$.

(b) ***Choice D.***

Every real number has a unique cube root, and so the only solution is $x = (-4)^{1/3}$, which may be written as $(-4)^{1/3} = (-1)^{1/3}\cdot(2^2)^{1/3} = -2^{2/3}$. Choice A is not an exact solution. Choice B is neither exact nor correct since each real number has only one cube root, not two. Choice C is incorrect because $-\sqrt[3]{-4}$ is not a solution. Choice E is incorrect since there is a solution.

(c) ***Choice D.***

All of the other choices are incorrect since they are all not exact. The exact answer is 2π.

(d) ***Choices D, F, and G.***

Choice A is not exact. Choice B is incorrect because there is only one solution, not two. Choices C and E are incorrect because neither 3^{-2} nor $\log_3(2)$ is a solution.

Ex. A-48 **Algebra/Precalculus**

Simplify each of the following expressions according to the instructions.

(a) Positive exponents and integer coefficients only (assume $x, y > 0$): $\left(\frac{x^8y^{-4}}{16y^{4/3}}\right)^{-1/4}$

(b) Positive exponents only (assume $a, b > 0$): $\frac{(9ab)^{3/2}}{(27a^3b^{-4})^{2/3}}\cdot\left(\frac{3a^{-2}}{4b^{1/3}}\right)^{-1}$

(c) Common factors canceled (assume $h \neq 5$): $\frac{2h-10}{\sqrt{5}-\sqrt{h}}$

(d) Expand and fully simplify: $\left(\sqrt{9s^2+4}+2\right)\left(\sqrt{9s^2+4}-2\right)$

unit1

Algebra/Precalculus

(e) Factor completely: $5y^2(y-3)^5 + 10y(y-3)^4$

(f) Factor completely: $3x^3 + x^2 - 12x - 4$

(g) Factor completely: $3x^{-1/2} + 4x^{1/2} + x^{3/2}$

(h) Common factors canceled, positive exponents only ($x \neq y$ and $x, y \neq 0$): $\dfrac{y^{-1} - x^{-1}}{x^{-2} - y^{-2}}$

(i) Common factors canceled ($u \neq 1$ and $u \neq -2$): $\dfrac{\dfrac{4}{u-1} - \dfrac{4}{u+2}}{\dfrac{3}{u^2+u-2} + \dfrac{3}{u+2}}$

unit1

Solution **A-48**

(a) $\dfrac{2y^{4/3}}{x^2}$

(b) $4a^{3/2}b^{9/2}$

(c) $-2(\sqrt{5} + \sqrt{h})$

(d) $9s^2$

(e) $5y(y-1)(y-2)(y-3)^4$

(f) $(3x+1)(x-2)(x+2)$

(g) $x^{-1/2}(x+1)(x+3)$

(h) $\dfrac{-xy}{x+y}$

(i) $\dfrac{4}{u}$

Ex. A-49 Algebra/Precalculus

For each given function $f(x)$, fully simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$. Assume $h \neq 0$.

(a) $f(x) = 2x^2 - 2x$ (b) $f(x) = 9 - 5x$ (c) $f(x) = -4$ (d) $f(x) = \dfrac{1}{x}$

unit1

Solution **A-49**

(a) $\dfrac{2(x+h)^2 - 2(x+h) - (2x^2 - 2x)}{h} = 4x - 2 + 2h$

(b) $\dfrac{9 - 5(x+h) - (9 - 5x)}{h} = -5$

(c) $\dfrac{-4 - (-4)}{h} = 0$

(d) $\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \dfrac{-1}{x(x+h)}$

Ex. A-50 Algebra/Precalculus

Solve each equation or inequality. (Parts (b) – (d) are related!)

(a) $p^2 = p + 1$

(b) $2u^2 - 3u + 1 = 0$

(c) $2x^{5/2} - 3x^{3/2} + x^{1/2} = 0$

(d) $2\sin(\theta)^2 - 3\sin(\theta) + 1 = 0$

(e) $2x = x^2$

(f) $\dfrac{1-x}{1+x} + \dfrac{1+x}{1-x} = 6$

(g) $3\cos(x) + 2\sin(x)^2 = 3$

(h) $|2x+1| = 1$

(i) $|3x-5| = 4x$

(j) $\dfrac{x+5}{x-2} = \dfrac{5}{x+2} + \dfrac{28}{x^2-4}$

(k) $t^2 - 4t - 5 > 0$

(l) $\dfrac{x-4}{2x+1} < 0$

(m) $\dfrac{x-4}{2x+1} < 5$

unit1

Solution **A-50**

(a) Equivalently, $p^2 - p - 1 = 0$. The quadratic formula then gives the solutions: $p = \frac{1+\sqrt{5}}{2}$ or $p = \frac{1-\sqrt{5}}{2}$.

(b) Factoring gives $(2u-1)(u-1) = 0$, and so the solutions are $u = \frac{1}{2}$ or $u = 1$.

(c) Factoring gives $x^{1/2}(2x-1)(x-1) = 0$, and so the solutions are $x = 0$, $x = \frac{1}{2}$, or $x = 1$.

(d) Letting $u = \sin(\theta)$, we see the equation is equivalent to $2u^2 - 3u + 1 = 0$, which, from part (b), has solutions $u = \frac{1}{2}$ or $u = 1$. So we seek all solutions to the equations $\sin(\theta) = \frac{1}{2}$ and $\sin(\theta) = 1$.

The equation $\sin(\theta) = \frac{1}{2}$ has solutions $\theta = \frac{\pi}{6} + 2\pi n$ or $\theta = \frac{5\pi}{6} + 2\pi n$, where n is any integer. The equation

A-50

$\sin(\theta) = 1$ has solutions $\theta = \frac{\pi}{2} + 2\pi n$, where n is any integer.

(e) Equivalently, we have $0 = x^2 - 2x = x(x-2)$, and so the solutions are $x = 0$ or $x = 2$.

(f) Clearing denominators gives $(1-x)^2 + (1+x)^2 = 6(1-x)(1+x)$. Expanding each side and collecting like terms gives $2x^2 + 2 = 6 - 6x^2$. Equivalently, $8x^2 = 4$, and so the solutions are $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$.

(g) Using the identity $\sin(x)^2 = 1 - \cos(x)^2$ gives the equivalent equation $2\cos(x)^2 - 3\cos(x) + 1 = 0$. Factoring gives $(2\cos(x) - 1)(\cos(x) - 1) = 0$. Hence we must solve the equations $2\cos(x) - 1 = 0$ and $\cos(x) - 1 = 0$.

The equation $2\cos(x) - 1 = 0$ has solutions $x = \frac{\pi}{3} + 2\pi n$ or $x = -\frac{\pi}{3} + 2\pi n$ where n is any integer. The equation $\cos(x) - 1 = 0$ has solutions $x = 2\pi n$ where n is any integer.

(h) The given equation is equivalent to one of the equations $2x + 1 = 1$ or $2x + 1 = -1$. The solution to the former is $x = 0$ and the solution to the latter $x = -1$. Both candidate solutions are solutions to the original equation.

(i) The given equation is equivalent to one of the equations $3x - 5 = 4x$ or $3x - 5 = -4x$. The solution to the former is $x = -5$ and the solution to the latter is $x = \frac{5}{7}$. Only $x = \frac{5}{7}$ is a solution to the original equation.

(j) Clearing denominators gives $(x+5)(x+2) = 5(x-2)+28$, which is equivalent to $x^2+2x-8 = 0$, or $(x+4)(x-2) = 0$. Hence the candidate solutions are $x = -4$ or $x = 2$. However the expressions on each side of the original equation are undefined at $x = 2$, and so only $x = -4$ is a solution.

(k) Factoring gives $(t+1)(t-5) > 0$, and so we consider a sign chart with cut points $t = -1$ and $t = 5$. That is, we test each of the intervals $(-\infty, -1)$, $(-1, 5)$, and $(5, \infty)$ with a single test point each to check whether the inequality is satisfied on that interval. Testing the points -2, 0, and 6, we find that the inequality is *not* satisfied only on the interval $(-1, 5)$. Hence the solution is $(-\infty, -1) \cup (5, \infty)$.

(l) The numerator vanishes when $x = 4$ and the denominator vanishes when $x = -\frac{1}{2}$. So we consider a sign chart with cut points $x = -\frac{1}{2}$ and $x = 4$. That is, we test each of the intervals $(-\infty, -\frac{1}{2})$, $(-\frac{1}{2}, 4)$, and $(4, \infty)$ with a single test point each to check whether the inequality is satisfied on that interval. Testing the points -1, 0, and 5, we find that the inequality is satisfied only on the interval $(-\frac{1}{2}, 4)$. Hence the solution is $(-\frac{1}{2}, 4)$.

(m) First we subtract 5 from each side of the inequality to get it in the form $F(x) > 0$ or $F(x) < 0$. We find that the inequality is equivalent to $\frac{-9x-9}{2x+1} < 0$. The numerator vanishes when $x = -1$ and the denominator vanishes when $x = -\frac{1}{2}$. So we consider a sign chart with cut points $x = -1$ and $x = -\frac{1}{2}$. That is, we test each of the intervals $(-\infty, -1)$, $(-1, -\frac{1}{2})$, and $(-\frac{1}{2}, \infty)$ with a single test point each to check whether the inequality is satisfied on that interval. Testing the points -2, $-\frac{3}{4}$, and 0, we find that the inequality is *not* satisfied only on the interval $(-1, -\frac{1}{2})$. Hence the solution is $(-\infty, -1) \cup (-\frac{1}{2}, \infty)$.

Ex. A-51 **Algebra/Precalculus**

Find an equation of each described line.

(a) line through the point $(4, -6)$ with slope 3

(b) line through the points $(1, 2)$ and $(-3, 4)$

(c) line through the point $(5, 5)$ and perpendicular to the line described by $2x - 4y = 3$

(d) line through the point $(-1, -2)$ and parallel to the line described by $3x + 8y = 1$

(e) horizontal line through the point $(3, -1)$

(f) vertical line through the point $(2, -4)$

unit1

Solution **A-51**

(a) $y - (-6) = 3(x - 4)$

(b) The slope is $m = \frac{4-2}{-3-1} = -\frac{1}{2}$, whence an equation of the line is $y - 2 = -\frac{1}{2}(x - 1)$.

(c) The slope of the given line is $\frac{1}{2}$, whence the equation of the desired line is $m = -2$, whence an equation of the desired line is $y - 5 = -2(x - 5)$.

(d) The slope of the given line is $-\frac{3}{8}$, whence the equation of the desired line is $m = -\frac{3}{8}$, whence an equation of the desired line is $y - (-2) = -\frac{3}{8}(x - (-1))$.

(e) $y = -1$

(f) $x = 2$

Ex. A-52 **Algebra/Precalculus**

If $f(x)$ and $g(x)$ are functions, then $f(g(x))$ is also a function, called the composition of f and g. We also write $f \circ g$ to mean $f(g(x))$. Similarly, $g \circ f$ means $g(f(x))$.

(a) Let $f(x) = \sin(3x) + 7$ and $g(x) = e^{2x} + 1$. Write expressions for both $f(g(x))$ and $g(f(x))$.

(b) Let $h(x) = \log_{10}(\sin(\sqrt{x}) + 1)$. Find four functions f_1, f_2, f_3, and f_4 such that $h(x) = f_4(f_3(f_2(f_1(x))))$. You may not use the function $f(x) = x$ for any of your choices.

unit1

Solution **A-52**

(a) $f(g(x)) = \sin(3e^{2x} + 3) + 7$ and $g(f(x)) = e^{2\sin(3x)+14} + 1$.

(b) One possible choice is the following: $f_1(x) = \sqrt{x}$, $f_2(x) = \sin(x)$, $f_3(x) = x + 1$, and $f_4(x) = \log_{10}(x)$.

Ex. A-53 **Algebra/Precalculus**

For each of the following function pairs, find a simplified formula for $f \circ g$ and $g \circ f$. Then find the domain of f, g, $f \circ g$, and $g \circ f$.

(a) $f(x) = \sin(x)$ and $g(x) = 2x + 3$

(b) $f(x) = \dfrac{2+x}{1-2x}$ and $g(x) = \dfrac{x-2}{2x+1}$

unit1

Solution **A-53**

(a) The domain of both f and g is all real numbers. Hence the domain of $(f \circ g)(x) = \sin(2x + 3)$ and $(g \circ f) = 2\sin(x) + 3$ is also all real numbers.

(b) With some algebra we find the following:

$$(f \circ g)(x) = \frac{2 + \frac{x-2}{2x+1}}{1 - 2 \cdot \frac{x-2}{2x+1}} = x \quad , \quad (g \circ f)(x) = \frac{\frac{2+x}{1-2x} - 2}{2 \cdot \frac{2+x}{1-2x} + 1} = x$$

The domain of f is $x \neq \frac{1}{2}$ and the domain of g is $x \neq -\frac{1}{2}$. If x is in the domain of $f(g(x))$, then x must be in the domain of g (so $x \neq -\frac{1}{2}$) and $g(x)$ is in the domain of f (so $g(x) \neq \frac{1}{2}$). The equation $g(x) = \frac{1}{2}$ has no solution, so it is always true that $g(x) \neq \frac{1}{2}$. Hence the domain of $f \circ g$ is $x \neq -\frac{1}{2}$. Similarly, the domain of $g \circ f$ is $x \neq \frac{1}{2}$.

Ex. A-54 **Algebra/Precalculus**

Find the exact value of each expression. Your final answer cannot contain "log" or "ln".

(a) $\log_2(48) - \log_2(6)$ (b) $\log_2(48) - \log_4(144)$ (c) $\ln(\log_{10}(10^e))$ (d) $3^{\log_3(4e) - \log_3(e)}$

unit1

Solution **A-54**

(a) $\log_2(48) - \log_2(6) = \log_2(48/6) = \log_2(8) = \log_2(2^3) = 3$

(b) First use the change-of-base formula to write

$$\log_4(144) = \frac{\log_2(144)}{\log_2(4)} = \frac{1}{2}\log_2(144) = \log_2(144^{1/2}) = \log_2(12)$$

Now we have

$$\log_2(48) - \log_4(144) = \log_2(48) - \log_2(12) = \log_2(48/12) = \log_2(4) = \log_2(2^2) = 2$$

(c) $\ln(\log_{10}(10^e)) = \ln(e) = 1$

(d) $3^{\log_3(4e) - \log_3(e)} = 3^{\log_3(4e/e)} = 3^{\log_3(4)} = 4$

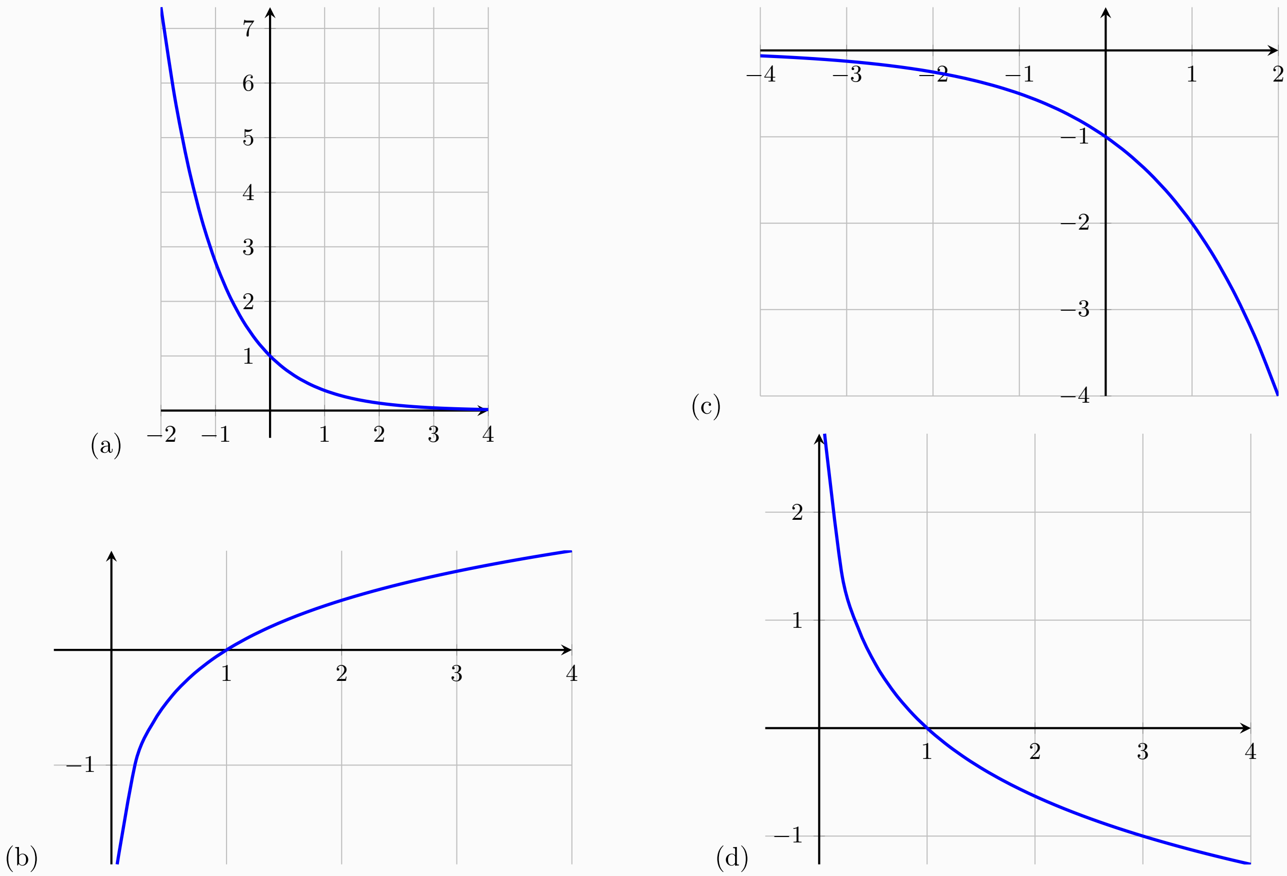
Ex. A-55 **Algebra/Precalculus**

Sketch the graph of each of the following functions.

(a) $f(x) = e^{-x}$ (b) $f(x) = \log_5(x)$ (c) $f(x) = -2^x$ (d) $f(x) = \log_{1/3}(x)$

unit1

Solution **A-55**



Ex. A-56 **Algebra/Precalculus**

Find all solutions to the following equations.

(a) $3^{x^2-x} = 9$ (b) $e^{2x+3} = 1$ (c) $\log_3(x) + \log_3(2x+1) = 1$

unit1

Solution **A-56**

(a) We have $3^{x^2-x} = 9 = 3^2$, and so $x^2 - x = 2$, or $(x+1)(x-2) = 0$. So the solutions are $x = -1$ or $x = 2$.

(b) We have $e^{2x+3} = 1 = e^0$, and so $2x + 3 = 0$. So the solution is $x = -\frac{3}{2}$.

(c) We have the following work.

$$\begin{aligned} \log_3(x) + \log_3(2x+1) &= 1 \\ \log_3\left(x(2x+1)\right) &= 1 \\ x(2x+1) &= 3 \\ 2x^2 + x - 3 &= 0 \\ (2x+3)(x-1) &= 0 \end{aligned}$$

Hence the candidate solutions are $x = 1$ and $x = -\frac{3}{2}$. Substitution of $x = -\frac{3}{2}$ in the original equation gives nonsense since the domain of all logarithms is strictly positive numbers. Hence the only solution is $x = 1$.

Ex. A-57 **Algebra/Precalculus**

Suppose $\log_{b^3}(5) = \frac{1}{6}$. Find the exact value of $\sqrt{b-16}$.

unit1

Solution **A-57**

By definition of logarithm, the equation $\log_{b^3}(5) = \frac{1}{6}$ is equivalent to $5 = (b^3)^{1/6}$. Hence $5 = b^{1/2}$, or $b = 25$. It follows that $\sqrt{b-16} = \sqrt{25-16} = 3$.

Ex. A-58 **Algebra/Precalculus**

Write the exact values of the sine, cosine, and tangent of each of the following angles: $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, π, $-\frac{\pi}{6}$, and $-\frac{3\pi}{4}$. *(You should do this without any reference or calculator.)*

unit1

Solution **A-58**

Refer to the table below.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
π	0	-1	0
$-\frac{\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
$-\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1

Ex. A-59 **Algebra/Precalculus**

Graph each of the following curves.

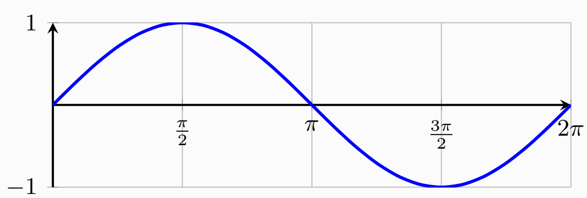
(a) $y = \sin(\theta)$

(b) $y = 3\cos(\pi\theta)$

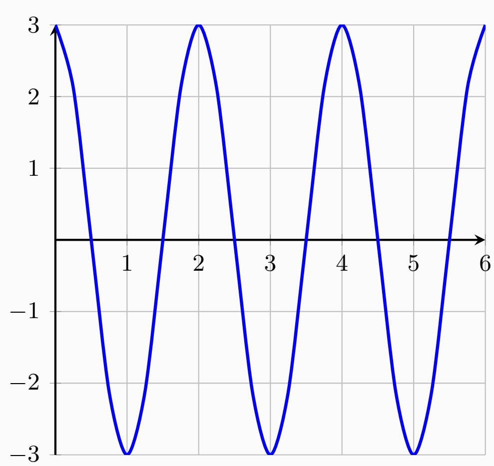
unit1

Solution **A-59**

(a) The graph below shows one period of $y = \sin(\theta)$, on the interval $\theta \in [0, 2\pi]$.



(b) The graph below shows three periods of $y = 3\cos(\pi\theta)$, on the interval $\theta \in [0, 6]$ (the period is 2). This graph can be calculated by starting with the graph of $y = \cos(\theta)$, and then stretching the graph vertically by a factor of 3 and shrinking the graph horizontally by a factor of π.



Ex. A-60 **Algebra/Precalculus**

A bank pays 6% annual interest compounded continuously. How long will it take for \$835 to triple?

unit1

Solution **A-60**

The value of the investment is $P(t) = P_0 e^{rt}$ with $r = 0.06$ and $P_0 = 835$. We desire the value of t such that $P(t) = 3P_0$, or $P_0 e^{rt} = 3P_0$, or $e^{rt} = 3$. Solving for t gives

$$t = \frac{\ln(3)}{r} = \frac{\ln(3)}{0.06}$$

Ex. A-61 **Algebra/Precalculus**

The number of bacteria in a certian petri dish obeys a law of exponential growth. Suppose there are initially 1000 bacteria and the number of bacteria doubles every 20 minutes. When will the number of bacteria reach 5000?

unit1

Solution **A-61**

The bacteria population is generally $P(t) = P_0 e^{kt}$ for unknown constants P_0 and k, with t measured in minutes. We are told that $P(0) = 1000$ and $P(20) = 2000$. Hence we have the equations $P_0 = 1000$ and $P_0 e^{20k} = 2000$. Substituting $P_0 = 1000$ into the second equation gives $1000e^{20k} = 2000$, or $e^{20k} = 2$. Solving for k gives

$$k = \frac{\ln(2)}{20}$$

We now desire the value of t such that $P(t) = 5000$, and so we must solve the equation $1000e^{kt} = 5000$, or $e^{kt} = 5$. Solving for t gives

$$t = \frac{\ln(5)}{k} = 20 \cdot \frac{\ln(5)}{\ln(2)}$$

Ex. A-62 **Algebra/Precalculus**

A rectangular box is constructed according to the following rules.

- the length of the box is twice its width
- the height of the box is 5 feet more than three times the length

(a) If x is the width of the box in feet, write an expression for $V(x)$, the volume of the box in cubic feet as a function of its width.

(b) Suppose the rules also require that the sum of the box's width and height to be no more than 26 feet. Under this condition, what is the domain of the function $V(x)$?

unit1

Solution **A-62**

(a) If $w = x$ is the width of the box, then the length is $\ell = 2x$ and the height is $h = 3\ell + 5 = 6x + 5$. Hence the volume of the box is $V(x) = \ell wh = (2x) \cdot x \cdot (6x + 5) = 2x^2(6x + 5)$.

(b) We must have that $w + h \leq 26$, or $x + (6x + 5) \leq 26$. This is equivalent to $x \leq 3$. Of course, the width must also be non-negative, and so we must have $x \geq 0$. Hence the domain of $V(x)$ is $0 \leq x \leq 3$, or the interval $[0, 3]$.

Ex. A-63 **Algebra/Precalculus**

The total cost (in \$) of producing q units of some product is $C(q) = 30q^2 + 400q + 500$.

(a) Compute the cost of making 20 units.

(b) Compute the cost of making the 20th unit.

(c) What is the initial setup cost?

unit1

Solution **A-63**

(a) $C(20) = 30(20)^2 + 400(20) + 500 = 20500$

(b) $C(20) - C(19) = (30(20)^2 + 400(20) + 500) - (30(19)^2 + 400(19) + 500) = 1570$

(c) The initial setup cost (or sunk cost) is $C(0) = 500$.

Ex. A-64 **Algebra/Precalculus**

The speed of blood that is a distance r from the central axis in an artery of radius R is $v(r) = C(R^2 - r^2)$, where C is some constant.

(a) What is the speed of the blood on the central axis?

(b) What is the speed halfway between the central axis and the artery wall?

unit1

Solution **A-64**

(a) $v(0) = CR^2$

(b) $v\left(\frac{R}{2}\right) = C\left(R^2 - \left(\frac{R}{2}\right)^2\right) = C\left(R^2 - \frac{1}{4}R^2\right) = \frac{3}{4}CR^2$

Ex. A-65 **Algebra/Precalculus**

An account in a certain bank pays 5% annual interest, compounded continuously. An initial deposit of \$200 is made into the account. How many years does it take for the \$200 to double?

unit1

Solution **A-65**

The value of the account t years after the initial deposit is $P(t) = 200e^{0.05t}$. The time taken to double in value is the time T such that $P(T) = 400$. Solving the equation $200e^{0.05T} = 400$ gives $T = \ln(2)/0.05 = 20\ln(2)$ years.

Ex. A-66 **Algebra/Precalculus**

A radioactive frog hops out of a pond full of nuclear waste. If its level of radioactivity declines to $\frac{1}{3}$ of its original value in 30 days, when will its level of radioactivity reach $\frac{1}{100}$ of its original value?
Hint: Use the exponential growth formula $P(t) = P_0e^{rt}$.

unit1

Solution **A-66**

Let $P(t)$ denote the radioactivity of the frog t days after jumping out of the pond and let P_0 denote the initial radioactivity. We are given that $P(30) = \dfrac{1}{3}P_0$, or $e^{30r} = \dfrac{1}{3}$, whence $r = -\dfrac{\ln(3)}{30}$. Given this value of r, the frog reaches $\dfrac{1}{100}$ of its original radioactivity at time T, where $P(T) = \dfrac{1}{100}P_0$, or $e^{rT} = \dfrac{1}{100}$. We thus find that

$$T = -\frac{\ln(100)}{r} = 30 \cdot \frac{\ln(100)}{\ln(3)}$$

Ex. A-67 **Algebra/Precalculus**

Complete each of the following exercises from various topics in algebra and precalculus.

(a) Simplify the expression $\dfrac{|2-x|}{x-2}$ for $x > 2$.

(b) Find all solutions to the equation $2^{x^2-2x} = 8$.

(c) Simplify the expression $2^{\log_2(3)-\log_2(5)}$.

(d) Find an equation of the line through the point $(-1, 4)$ with slope 2.

(e) Find the domain of $f(x) = \dfrac{\ln(x)}{x-2}$. Write your answer in interval notation.

(f) Solve the inequality $\dfrac{3x+6}{x(x-4)} \le 0$. Write your answer in interval notation.

unit1

Solution **A-67**

(a) If $x > 2$, then $2 - x < 0$, and so $|2-x| = -(2-x) = x - 2$. Hence $\dfrac{|2-x|}{x-2} = \dfrac{x-2}{x-2} = 1$.

(b) The equation is equivalent to $2^{x^2-2x} = 2^3$, or $x^2 - 2x = 3$. After some algebra we have $(x-3)(x+1) = 0$, and so the solutions are $x = -1$ and $x = 3$.

(c) We have $2^{\log_2(3)-\log_2(5)} = 2^{\log_2(3/5)} = 3/5$.

(d) Use point-slope form of a line: $y - 4 = 2(x+1)$.

A-67

(e) Note that the domain of $\ln(x)$ is $(0,\infty)$. Hence the domain of f is $(0,2)\cup(2,\infty)$ (the value $x=2$ must be excluded since $f(x)$ is undefined for $x=2$ due to division by 0).

(f) We solve the inequality using the method of sign charts. The cut points for our number line are $x=-2$ (obtained by solving $3x+6=0$); $x=0$ and $x=4$ (each obtained by solving $x(x-4)=0$).

interval	test point	sign of $\frac{3x+6}{x(x-4)}$	truth of inequality
$(-\infty,-2)$	$x=-3$	$\frac{\ominus}{\ominus\ominus}=\ominus$	true
$(-2,0)$	$x=-1$	$\frac{\oplus}{\ominus\ominus}=\oplus$	false
$(0,4)$	$x=1$	$\frac{\oplus}{\oplus\ominus}=\ominus$	true
$(4,\infty)$	$x=5$	$\frac{\oplus}{\oplus\oplus}=\oplus$	false

The inequality is satisfied at $x=-2$ also but no other cut point. Hence the solution to our inequality is the set $(-\infty,-2]\cup(0,4)$.

Ex. A-68 **Algebra/Precalculus** ⋆Challenge

Let $f(x)=\dfrac{2}{3-\sqrt{x}}$. Fully simplify the difference quotient $\dfrac{f(4+h)-f(4)}{h}$ for $h\neq 0$ (i.e., simplify the expression all common factors of h have been canceled.)

challenge

Solution A-68

We calculate the composition, find a common denominator, rationalize the numerator, and then expand the denominator. Our goal is to cancel the factor of h.

$$\frac{f(4+h)-f(4)}{h}=\frac{\frac{2}{3-\sqrt{4+h}}-2}{h}=\frac{2-2(3-\sqrt{4+h})}{h\left(3-\sqrt{4+h}\right)}=\frac{-4+2\sqrt{4+h}}{h\left(3-\sqrt{4+h}\right)}\cdot\frac{-4-2\sqrt{4+h}}{-4-2\sqrt{4+h}}$$

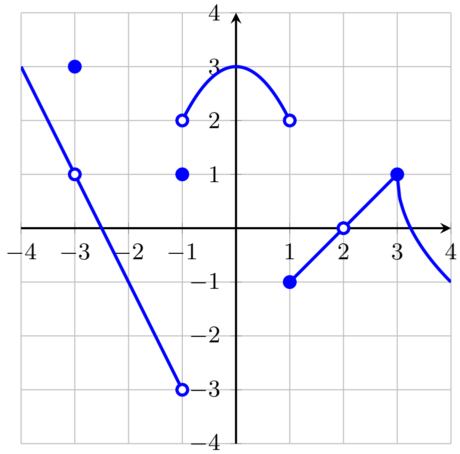
$$=\frac{(16-4(4+h))}{h\left(3-\sqrt{4+h}\right)\left(-4-2\sqrt{4+h}\right)}=\frac{-4h}{-2h\left(2-h+\sqrt{4+h}\right)}=\frac{2}{2-h+\sqrt{4+h}}$$

2 Chapter 2: Limits

§2.1, 2.2: Introduction to Limits

Ex. B-1 **2.1/2.2** Sp19 **Exam**

The graph of $y = f(x)$ is given below. Find all values of a in the interval $(-4, 4)$ for which $\lim_{x\to a} f(x)$ does not exist, or determine that no such values of a exist.



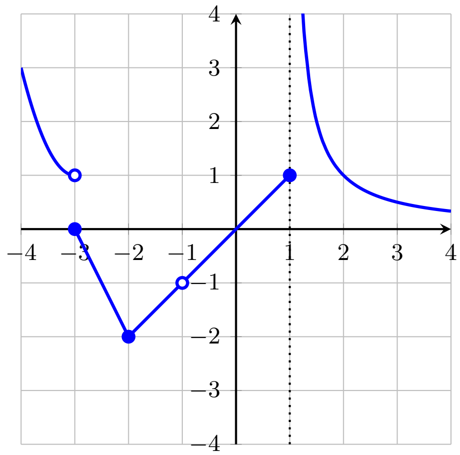
exs19

Solution **B-1**

The values of a for which $\lim_{x\to a} f(x)$ does not exist are $a = -1$ and $a = 1$ only. (At both of these values of a, the left-limit and right-limit are not equal.)

Ex. B-2 **2.1/2.2** Sp20 **Exam**

The graph of $y = f(x)$ is given below. Find all values of a in $(-4, 4)$ such that $\lim_{x\to a} f(x)$ does not exist.



exs20

Solution **B-2**

$a = -3$ and $a = 1$ only.

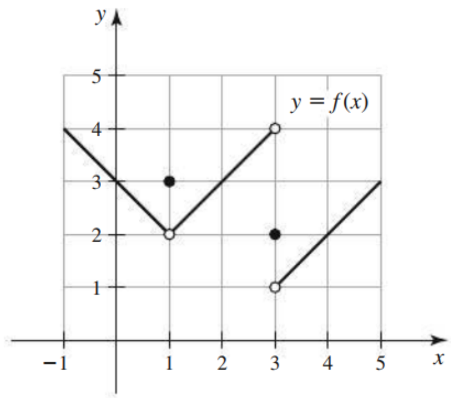
Ex. B-3 **2.1/2.2** Su20 **Exam**

For each part, use the graph of $f(x)$ below.

exsu20

2.1/2.2

Su20 Exam



(a) Calculate $\lim_{x\to 3} f(x)$ or determine that the limit does not exist.

(b) Find all values of a such that both $\lim_{x\to a} f(x)$ exists and this limit is not equal to $f(a)$.

exsu20

Solution B-3

(a) $\lim_{x\to 3} f(x)$ does not exist.

(b) $a = 1$ only.

Ex. B-4 2.1/2.2

Su20 Exam

Consider the function below.

$$f(x) = \begin{cases} x^2 + 4x - 1 & x < 2 \\ 11 & x = 2 \\ 19 - x^3 & x > 2 \end{cases}$$

A student correctly calculates that $\lim_{x\to 2} f(x) = 11$ and enters this as their final answer on an online exam, initially getting full credit. However, after inspecting the student's work, the teacher overrides this score and gives no credit. The teacher writes the comment "you have not correctly justified your answer." The student wrote the following:

"Since $f(x)$ is defined for all x and $f(2) = 11$, the answer is $\lim_{x\to 2} f(x) = 11$."

(a) Why is the student's justification incorrect?

(b) Write a complete and correct justification for the statement $\lim_{x\to 2} f(x) = 11$.

exsu20

Solution B-4

(a) Even though the student's final answer is correct, the value of a function at $x = a$ is irrelevant to the calculation of $\lim_{x\to a} f(x)$. (For instance, it's possible for $f(a)$ and $\lim_{x\to a} f(x)$ to be different.) So the justification is incorrect.

(b) Note that $x = 2$ is the transition point of the piecewise-defined function $f(x)$. So we will justify the statement $\lim_{x\to 2} f(x) = 11$ using one-sided limits.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(x^2 + 4x - 1\right) = 4 + 8 - 1 = 11$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(19 - x^3\right) = 19 - 8 = 11$$

Since the left-limit and right-limit are both equal to 11, we conclude that $\lim_{x\to 2} f(x) = 11$.

Ex. B-5 2.1/2.2

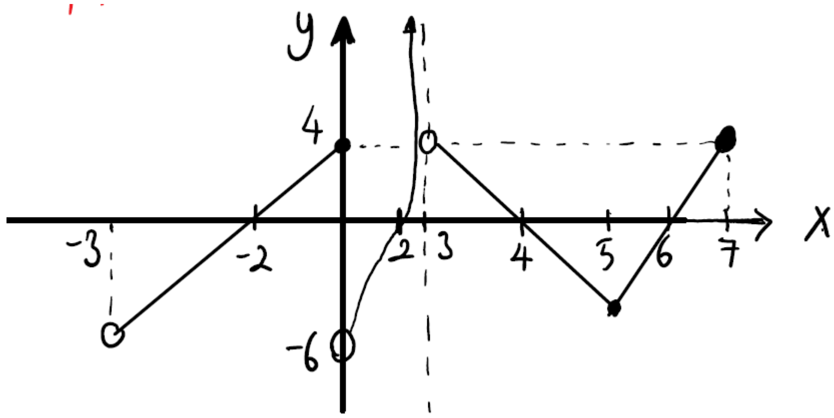
Sp21 Exam

For each part, use the graph of $y = f(x)$.

exs21

2.1/2.2

Sp21 Exam



(a) Calculate $f(f(2))$.

(b) Find where $f(x) = 0$.

(c) State the domain of f in interval notation.

(d) State the range of f in interval notation.

(e) For each part below, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(i) $\lim_{x\to 0^-} f(x)$ (ii) $\lim_{x\to 0^+} f(x)$ (iii) $\lim_{x\to 0} f(x)$ (iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$

exs21

Solution B-5

(a) $f(f(2)) = f(0) = 4$

(b) $x = -2$, $x = 2$, $x = 4$, $x = 6$

(c) $(-3, 3) \cup (3, 7]$

(d) $(-6, \infty)$

(e) (i) 4

(ii) -6

(iii) does not exist

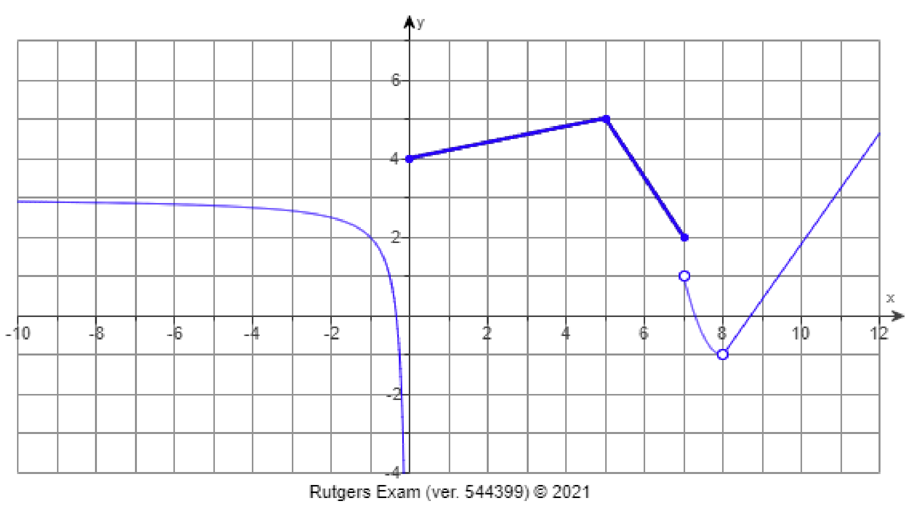
(iv) $+\infty$

(v) 4

Ex. B-6 2.1/2.2, 2.3, 2.4, 2.5

Fa21 Exam

For each part, use the graph of $y = f(x)$.



exf21

2.1/2.2, 2.3, 2.4, 2.5 Fa21 Exam

(a) List the x-values where f is not continuous or determine that f is continuous for all x.

(b) List all vertical asymptotes of f.

(c) List all horizontal asymptotes of f.

(d) Calculate $\lim_{x\to 8} f(x)$ or determine that the limit does not exist.

(e) At $x = 7$, which of the one-sided limits of f exist?

exf21

Solution B-6

(a) $x = 0, 7, 8$ only

(b) $x = 0$ only

(c) $y = 3$ only

(d) $\lim_{x\to 8} f(x) = -1$

(e) Both the left- and right-limits of $f(x)$ at $x = 7$ exist.

Ex. B-7 2.1/2.2 Fa21 Exam

The position of a particle (measured in feet) after t seconds is modeled by the following function.

$$h(t) = -16t^2 + 96t + 100$$

(a) Calculate the average velocity of the particle (in feet per second) between $t = 4$ and $t = 5$.

(b) Find an equation of the secant line between $(4, h(4))$ and $(5, h(5))$.

exf21

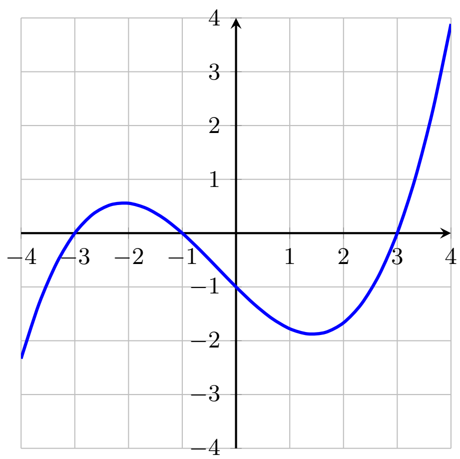
Solution B-7

(a) $\overline{v} = \frac{\Delta h}{\Delta t} = \frac{h(5)-h(4)}{5-4} = \frac{-16(25-16)+96(5-4)}{1} = -48$

(b) The slope of the secant line is -48 and the secant line passes through $(4, h(4)) = (4, 228)$. Hence an equation of the secant line is $y = 228 - 48(t - 4)$.

Ex. B-8 2.1/2.2, 3.7, 4.3/4.4 Fa21 Exam

For each part, use the graph of $y = g(x)$.



(a) How many solutions does the equation $g'(x) = 0$ have?

(b) Order the following quantities from least to greatest: $g'(-2.5)$, $g'(-2)$, $g'(0)$, and $g'(4)$. In your answer, write these quantities symbolically; do not give a numerical estimate.

(c) What is the sign of $g''(-3)$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(d) Let $h(x) = g(x)^2$. What is the sign of $h'(-4)$(negative, positive, or zero)? If there is not enough information to determine the value, explain why.

exf21

Solution **B-8**

(a) The function g is differentiable for all x and has two local extrema (one local min and one local max). So $g'(x) = 0$ has two solutions.

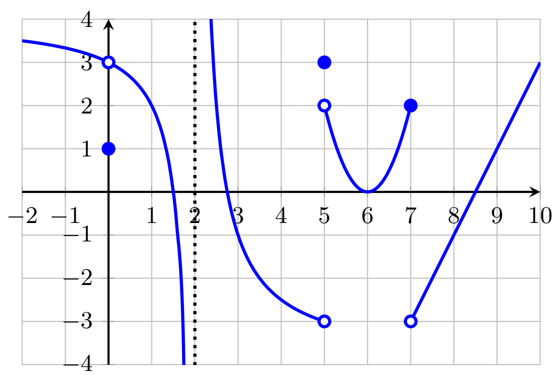
(b) We note the following: $g'(-2.5)$ is small and positive, $g'(-2) = 0$, $g'(0)$ is small and negative, and $g'(4)$ is large and positive. Thus the correct order is: $g'(0)$, $g'(-2)$, $g'(-2.5)$, $g'(4)$.

(c) The function g is concave down in an interval containing $x = -3$. Thus $g''(-3)$ is positive.

(d) We have $h'(x) = 2g(x)g'(x)$, whence $h'(-4) = 2g(-4)g'(-4)$. Observe that $g(-4) < 0$ and $g'(-4) > 0$. Thus $h'(-4) < 0$.

Ex. B-9 **2.1/2.2** Sp22 **Exam**

For each part, use the graph of $y = g(x)$ below to calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*



(a) $\lim_{x\to 0} g(x)$ (b) $\lim_{x\to 2^-} g(x)$ (c) $\lim_{x\to 5^-} g(x)$ (d) $\lim_{x\to 5^+} g(x)$ (e) $\lim_{x\to 7} g(x)$

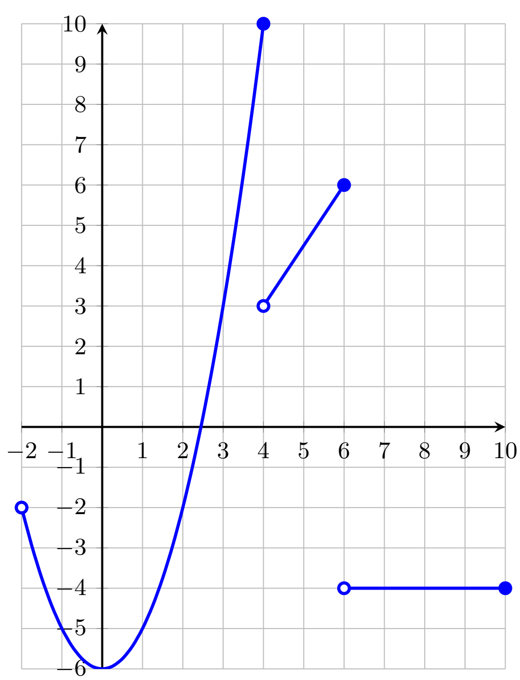
exs22

Solution **B-9**

(a) 3 (b) $-\infty$ (c) -3 (d) 2 (e) does not exist

Ex. B-10 **2.1/2.2** Sp18 **Quiz**

Evaluate the limits using the given graph.



qs18

2.1/2.2 Sp18 Quiz

(a) $\lim_{x\to -2^+} f(x)$ (b) $\lim_{x\to 4^-} f(x)$ (c) $\lim_{x\to 4^+} f(x)$ (d) $\lim_{x\to 6} f(x)$

qs18

Solution **B-10**

(a) -2 (b) $= 10$ (c) $= 3$ (d) does not exist

Ex. B-11 2.1/2.2 Fa22 Quiz

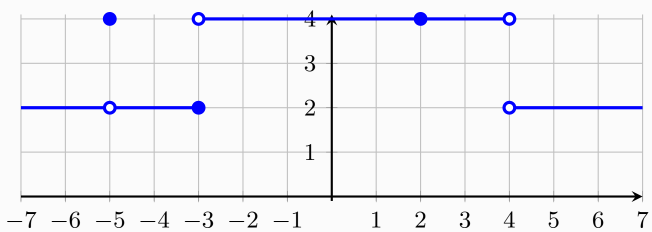
On the axes below, sketch the graph of a function $f(x)$ that satisfies the following properties:

- the domain of $f(x)$ is $[-7, 4) \cup (4, 7]$
- $\lim_{x\to -5} f(x) \neq f(-5)$
- $\lim_{x\to -3^-} f(x) = f(-3)$ but $\lim_{x\to -3} f(x)$ does not exist
- $\lim_{x\to 2} f(x) = f(2) = 4$
- $\lim_{x\to 4^+} f(x) = 2$ but $\lim_{x\to 4} f(x)$ does not exist

qf22

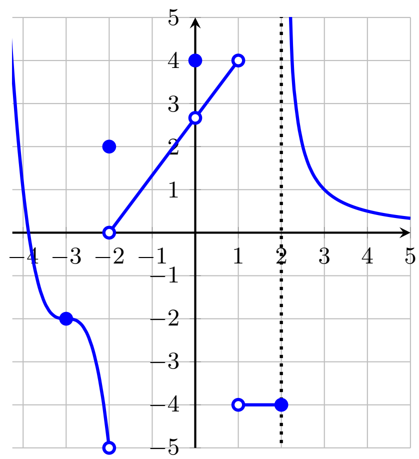
Solution **B-11**

There are many such solutions. Here is one.



Ex. B-12 2.1/2.2

For each part, use the graph of $y = f(x)$ below to calculate the limit or determine the limit does not exist.



(a) $\lim_{x\to -3^-} f(x)$ (e) $\lim_{x\to -2^-} f(x)$ (i) $\lim_{x\to 0^-} f(x)$ (m) $\lim_{x\to 1^-} f(x)$ (q) $\lim_{x\to 2^-} f(x)$

(b) $\lim_{x\to -3^+} f(x)$ (f) $\lim_{x\to -2^+} f(x)$ (j) $\lim_{x\to 0^+} f(x)$ (n) $\lim_{x\to 1^+} f(x)$ (r) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to -3} f(x)$ (g) $\lim_{x\to -2} f(x)$ (k) $\lim_{x\to 0} f(x)$ (o) $\lim_{x\to 1} f(x)$ (s) $\lim_{x\to 2} f(x)$

(d) $f(-3)$ (h) $f(-2)$ (l) $f(0)$ (p) $f(1)$ (t) $f(2)$

c2s1

Solution **B-12**

(a) -2 (b) -2 (c) -2 (d) -2

(e) -5 (f) 0 (g) DNE (h) 2

(i) $\frac{8}{3}$ (j) $\frac{8}{3}$ (k) $\frac{8}{3}$ (l) 4

(m) 4 (n) -4 (o) DNE (p) DNE

(q) -4 (r) ∞ (s) DNE (t) -4

Ex. B-13 **2.1/2.2**

Suppose $\lim_{x\to 2}\left(\frac{f(x)-3}{x-2}\right) = 5$ and $\lim_{x\to 2} f(x)$ exists (and is equal to $f(2)$). What is the value of $f(2)$? Explain your answer.

c2s1

Solution **B-13**

Direct substitution of $x = 2$ in the limit gives the undefined expression $\left(\frac{f(2)-3}{0}\right)$. If $f(2)$ were anything other 3, this would give us an expression of the form $\frac{c}{0}$ where $c \neq 0$. This would mean that the limit could not exist. (We will study infinite limits in more detail in §2.4.)

The only possible way that the given limit could exist (we know it exists because it is equal to 5) while still having the division-by-0 is if direction substitution of $x = 2$ were to give "$\frac{0}{0}$". Hence we must have $f(2) = 3$.

Ex. B-14 **2.1/2.2** **⋆Challenge**

Suppose $\lim_{x\to 0}(f(x) + g(x))$ exists. Is it true that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ also exist? Explain your answer.

challenge

Solution **B-14**

No. Let $f(x)$ be *any* function such that $\lim_{x\to 0} f(x)$ does not exist. (For example, $f(x) = \frac{|x|}{x}$.) Let $g(x) = -f(x)$. Then

$$\lim_{x\to 0}(f(x) + g(x)) = \lim_{x\to 0}(f(x) - f(x)) = \lim_{x\to 0}(0) = 0$$

Hence $\lim_{x\to 0}(f(x) + g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

§2.3: Techniques for Computing Limits

Ex. C-1 2.3 Fa17 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to7}\left(\dfrac{\frac{1}{7}-\frac{1}{x}}{x-7}\right)$ (b) $\lim_{x\to0}\left(\dfrac{\sin(7x)}{\tan(2x)}\right)$ (c) $\lim_{x\to-1}\left(\dfrac{|x+1|}{x+1}\right)$

exf17

Solution C-1

(a) We have the following work.

$$\lim_{x\to7}\left(\frac{\frac{1}{7}-\frac{1}{x}}{x-7}\right)=\lim_{x\to7}\left(\frac{\frac{1}{7}-\frac{1}{x}}{x-7}\cdot\frac{7x}{7x}\right)=\lim_{x\to7}\left(\frac{x-7}{7x(x-7)}\right)=\lim_{x\to7}\left(\frac{1}{7x}\right)=\frac{1}{49}$$

(b) We have the following work.

$$\lim_{x\to0}\left(\frac{\sin(7x)}{\tan(2x)}\right)=\lim_{x\to0}\left(\frac{\sin(7x)}{7x}\cdot\frac{2x}{\sin(2x)}\cdot\cos(2x)\cdot\frac{7}{2}\right)=1\cdot1\cdot1\cdot\frac{7}{2}=\frac{7}{2}$$

(c) We first examine the corresponding one-sided limits.

$$\lim_{x\to-1^-}\left(\frac{|x+1|}{x+1}\right)=\lim_{x\to-1^-}\left(\frac{-(x+1)}{x+1}\right)=-1$$
$$\lim_{x\to-1^+}\left(\frac{|x+1|}{x+1}\right)=\lim_{x\to-1^+}\left(\frac{+(x+1)}{x+1}\right)=+1$$

The one-sided limits are not equal, thus the desired limit does not exist.

Ex. C-2 2.3 Sp18 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to0}\left(\dfrac{(2x+9)^2-81}{x}\right)$ (b) $\lim_{x\to3^-}\left(\dfrac{|x-3|}{x-3}\right)$ (c) $\lim_{x\to1}\left(\dfrac{5-\sqrt{32-7x}}{x-1}\right)$

exs18

Solution C-2

(a) Expand the numerator and cancel common factors.

$$\lim_{x\to0}\left(\frac{(2x+9)^2-81}{x}\right)=\lim_{x\to0}\left(\frac{4x^2+36x}{x}\right)=\lim_{x\to0}(4x+46)=36$$

(b) If $x\to3^-$, then we may assume that $x<3$, or $x-3<0$. For such values of x, we have that $|x-3|=-(x-3)$. So now we have

$$\lim_{x\to3^-}\left(\frac{|x-3|}{x-3}\right)=\lim_{x\to3^-}\left(\frac{-(x-3)}{x-3}\right)=-1$$

(c) Rationalize the numerator and cancel common factors.

$$\lim_{x\to1}\left(\frac{5-\sqrt{32-7x}}{x-1}\right)=\lim_{x\to1}\left(\frac{5-\sqrt{32-7x}}{x-1}\cdot\frac{5+\sqrt{32-7x}}{5+\sqrt{32-7x}}\right)=\lim_{x\to1}\left(\frac{25-(32-7x)}{(x-1)(5+\sqrt{32-7x})}\right)$$
$$=\lim_{x\to1}\left(\frac{7(x-1)}{(x-1)(5+\sqrt{32-7x})}\right)=\lim_{x\to1}\left(\frac{7}{5+\sqrt{32-7x}}\right)=\frac{7}{10}$$

Ex. C-3 2.3, 3.1 Fa18 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{u\to 4}\left(\frac{(u+6)^2-25u}{u-4}\right)$

(b) $\lim_{s\to 1} g(s)$ where $g(s)=\begin{cases}\sqrt{1-s} & s\le 1\\ \frac{s^2-s}{s-1} & s>1\end{cases}$

(c) $\lim_{h\to 0}\left(\frac{\sin(7+h)-\sin(7)}{h}\right)$

Hint: Use the definition of the derivative.

(d) $\lim_{x\to 6}\left(\frac{\frac{1}{36}-x^{-2}}{x^2-36}\right)$

exf18

Solution C-3

(a) Expand the numerator and cancel common factors.

$$\lim_{u\to 4}\left(\frac{(u+6)^2-25u}{u-4}\right)=\lim_{u\to 4}\left(\frac{u^2-13u+36}{u-4}\right)=\lim_{u\to 4}\left(\frac{(u-9)(u-4)}{u-4}\right)=\lim_{u\to 4}(u-9)=-5$$

(b) We examine the one-sided limits.

$$\lim_{s\to 1^-} g(s)=\lim_{s\to 1^-}(\sqrt{1-s})=\sqrt{1-1}=0$$
$$\lim_{s\to 1^+} g(s)=\lim_{s\to 1^+}\left(\frac{s^2-s}{s-1}\right)=\lim_{s\to 1^+}\left(\frac{s(s-1)}{s-1}\right)=\lim_{s\to 1^+}(s)=1$$

Since the left-limit and right-limit are not equal, $\lim_{s\to 1} g(s)$ does not exist.

(c) Let $f(x)=\sin(x)$. Then by definition of the derivative,

$$f'(7)=\lim_{h\to 0}\left(\frac{\sin(7+h)-\sin(7)}{h}\right)$$

Since $f'(x)=\cos(x)$, the limit is $\cos(7)$.

(d) Find a common denominator and cancel common factors.

$$\lim_{x\to 6}\left(\frac{\frac{1}{36}-x^{-2}}{x^2-36}\cdot\frac{36x^2}{36x^2}\right)=\lim_{x\to 6}\left(\frac{x^2-36}{36x^2(x^2-36)}\right)=\lim_{x\to 6}\left(\frac{1}{36x^2}\right)=\frac{1}{1296}$$

Ex. C-4 2.3 Sp19 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 5}\left(\frac{x-5}{x^2-2x-15}\right)$

(b) $\lim_{x\to 0}\left(\frac{\sin(9x)}{\sin(16x)}\right)$

exs19

Solution C-4

(a) Cancel common factors.

$$\lim_{x\to 5}\left(\frac{x-5}{x^2-2x-15}\right)=\lim_{x\to 5}\left(\frac{x-5}{(x-5)(x+3)}\right)=\lim_{x\to 5}\left(\frac{1}{x+3}\right)=\frac{1}{8}$$

(b) Use the special limit $\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right)=1$ and some algebra.

$$\lim_{x\to 0}\left(\frac{\sin(9x)}{\sin(16x)}\right)=\lim_{x\to 0}\left(\frac{\sin(9x)}{9x}\cdot\frac{16x}{\sin(16x)}\cdot\frac{9}{16}\right)=1\cdot 1\cdot\frac{9}{16}=\frac{9}{16}$$

Ex. C-5 2.3 Fa19 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim\limits_{x\to 5}\left(\dfrac{x^2-3x-10}{x^2-x-20}\right)$ (b) $\lim\limits_{x\to 0}\left(\dfrac{\sin^2(4x)}{x^2}\right)$ (c) $\lim\limits_{x\to 4}\left(\dfrac{3-\sqrt{2x+1}}{x-4}\right)$

exf19

Solution C-5

(a) Cancel common factors.

$$\lim_{x\to 5}\left(\frac{x^2-3x-10}{x^2-x-20}\right)=\lim_{x\to 5}\left(\frac{(x-5)(x+2)}{(x-5)(x+4)}\right)=\lim_{x\to 5}\left(\frac{x+2}{x+4}\right)=\frac{5+2}{5+4}=\frac{7}{9}$$

(b) Use the special limit $\lim\limits_{\theta\to 0}\left(\dfrac{\sin(a\theta)}{a\theta}\right)=1$.

$$\lim_{x\to 0}\left(\frac{\sin^2(4x)}{x^2}\right)=\left(\lim_{x\to 0}\frac{\sin(4x)}{x}\right)^2=\left(\lim_{x\to 0}\left(\frac{\sin(4x)}{4x}\cdot 4\right)\right)^2=(1\cdot 4)^2=16$$

(c) First rationalize the numerator.

$$\lim_{x\to 4}\left(\frac{3-\sqrt{2x+1}}{x-4}\right)=\lim_{x\to 4}\left(\frac{9-(2x+1)}{(x-4)\left(3+\sqrt{2x+1}\right)}\right)=\lim_{x\to 4}\left(\frac{-2(x-4)}{(x-4)\left(3+\sqrt{2x+1}\right)}\right)$$
$$=\lim_{x\to 4}\left(\frac{-2}{3+\sqrt{2x+1}}\right)=\frac{-2}{3+\sqrt{9}}=-\frac{1}{3}$$

Ex. C-6 2.3 Sp20 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim\limits_{x\to 0}\left(\dfrac{(4x+1)^2-1}{x}\right)$

(b) $\lim\limits_{x\to 0}\left(\dfrac{9x\cos(2x)}{\sin(4x)}\right)$

(c) $\lim\limits_{x\to -1}\left(\dfrac{4-\sqrt{16x+32}}{x+1}\right)$

(d) $\lim\limits_{x\to 4^-}\left(\dfrac{|x^2-16|}{4-x}\right)$

(e) $\lim\limits_{x\to 3} g(x)$, where $g(x)=\begin{cases}\dfrac{x-3}{x^3-9x} & x<3\\ 18 & x=3\\ \dfrac{x-2}{x^2+9} & x>3\end{cases}$

exs20

Solution C-6

(a) Expand and cancel common factors.

$$\lim_{x\to 0}\left(\frac{(4x+1)^2-1}{x}\right)=\lim_{x\to 0}\left(\frac{16x^2+8x+1-1}{x}\right)=\lim_{x\to 0}(16x+8)=8$$

(b) Rearrange the terms and use the special trigonometric limit and direct substitution.

$$\lim_{x\to 0}\left(\frac{9x\cos(2x)}{\sin(4x)}\right)=\lim_{x\to 0}\left(\frac{4x}{\sin(4x)}\cdot\frac{9}{4}\cdot\cos(2x)\right)=1\cdot\frac{9}{4}\cdot 1=\frac{9}{4}$$

(c) Rationalize the numerator and cancel common factors.

$$\lim_{x\to -1}\left(\frac{4-\sqrt{16x+32}}{x+1}\right)=\lim_{x\to -1}\left(\frac{16-(16x+32)}{(x+1)\left(4+\sqrt{16x+32}\right)}\right)=\lim_{x\to -1}\left(\frac{-16}{4+\sqrt{16x+32}}\right)=\frac{-16}{4+4}=-2$$

(d) Note that the limit symbol "$x\to 4^-$" means that we may assume that both x is arbitrarily close to 4 and $x<4$. For values of x just slightly less than 4, the values of x^2-16 are negative. Hence under the assumptions of this

C-6

limit, we have $|x^2-16| = -(x^2-16) = 16-x^2 = (4-x)(4+x)$. So now we have

$$\lim_{x\to 4^-}\left(\frac{|x^2-16|}{4-x}\right) = \lim_{x\to 4^-}\left(\frac{(4-x)(4+x)}{4-x}\right) = \lim_{x\to 4^-}(4+x) = 8$$

(e) Compute the left- and right-limits and verify whether they are equal. For the left-limit cancel common factors. For the right-limit, use direct substitution. The function value $g(3)$ is irrelevant to this problem.

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}\left(\frac{x-3}{x^3-9x}\right) = \lim_{x\to 3^-}\left(\frac{1}{x(x+3)}\right) = \frac{1}{18}$$
$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}\left(\frac{x-2}{x^2+9}\right) = \frac{1}{18}$$

The left- and right-limits are both equal to $\frac{1}{18}$, hence $\lim\limits_{x\to 3} g(x) = \frac{1}{18}$ also.

Ex. C-7 2.3 Su20 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim\limits_{x\to 3}\left(\dfrac{x^3+2x^2-15x}{x^2-9}\right)$ (b) $\lim\limits_{x\to 0}\left(\dfrac{\sin(6x)^2}{x^2\cos(2x)}\right)$

exsu20

Solution C-7

(a) Factor and cancel.

$$\lim_{x\to 3}\left(\frac{x^3+2x^2-15x}{x^2-9}\right) = \lim_{x\to 3}\left(\frac{x(x+5)(x-3)}{(x+3)(x-3)}\right) = \lim_{x\to 3}\left(\frac{x(x+5)}{x+3}\right) = \frac{3\cdot 8}{6} = 4$$

(b) First we regroup terms and add factors of 6 to use the special trigonometric limt.

$$\lim_{x\to 0}\left(\frac{\sin(6x)^2}{x^2\cos(2x)}\right) = \lim_{x\to 0}\left(\frac{\sin(6x)}{6x}\cdot\frac{\sin(6x)}{6x}\cdot\frac{36}{\cos(2x)}\right)$$

Now we compute the limit of each factor and use the special limit $\lim\limits_{x\to 0}\left(\dfrac{\sin(6x)}{6x}\right) = 1$.

$$\lim_{x\to 0}\left(\frac{\sin(6x)}{6x}\cdot\frac{\sin(6x)}{6x}\cdot\frac{36}{\cos(2x)}\right) = 1\cdot 1\cdot\frac{36}{1} = 36$$

Ex. C-8 2.3 Su20 Exam

The parts of this problem are related.

(a) Suppose $x<3$. Write an algebraic expression that is equivalent to $|x-3|$ but without absolute value symbol.

(b) Calculate $\lim\limits_{x\to 2}\left(\dfrac{|x-3|-1}{x-2}\right)$. Explain why your work to part (a) is relevant here and precisely where you use it.

exsu20

Solution C-8

(a) If $x<3$, then $x-3<0$, whence $|x-3| = -(x-3) = 3-x$.

(b) Part (a) is relevant here since we want to calculate a limit as $x\to 2$ and $x=2$ satisfies the inequality $x<3$. Hence, for all x sufficiently close to 2 (on both sides), we have $|x-3| = 3-x$. Now we may compute the limit.

$$\lim_{x\to 2}\left(\frac{|x-3|-1}{x-2}\right) = \lim_{x\to 2}\left(\frac{(3-x)-1}{x-2}\right) = \lim_{x\to 2}\left(\frac{2-x}{x-2}\right) = \lim_{x\to 2}(-1) = -1$$

Ex. C-9 2.3 Su20 Exam

The parts of this problem are related.

(a) Consider the function below.

$$f(x) = \begin{cases} \dfrac{x-1}{3-\sqrt{10-x}} & x \neq 1 \\ -6 & x = 1 \end{cases}$$

Show that $\lim_{x\to 1} f(x) \neq f(1)$.

(b) Now consider the similar function below.

$$g(x) = \begin{cases} \dfrac{x-1}{3-\sqrt{10-x}} & x \neq 1 \\ b & x = 1 \end{cases}$$

where b is an unspecified constant. Explain how to determine whether the following statement is true: $\lim_{x\to 1} g(x) \neq g(1)$. How does your work for part (a) change, if at all, to determine the truth of the statement? Explain your answer.

exsu20

Solution C-9

(a) Rationalize the denominator.

$$\begin{aligned} \lim_{x\to 1} f(x) &= \lim_{x\to 1}\left(\frac{x-1}{3-\sqrt{10-x}}\cdot\frac{3+\sqrt{10-x}}{3+\sqrt{10-x}}\right) = \lim_{x\to 1}\left(\frac{(x-1)(3+\sqrt{10-x})}{9-(10-x)}\right) \\ &= \lim_{x\to 1}\left(\frac{(x-1)(3+\sqrt{10-x}}{x-1}\right) = \lim_{x\to 1}\left(3+\sqrt{10-x}\right) = 3+\sqrt{10-1} = 6 \end{aligned}$$

Observe that $6 \neq -6 = f(1)$.

(b) The function value $g(1)$ has no effect on our calculation of $\lim_{x\to 1} g(x)$, which is equal to $\lim_{x\to 1} f(x) = 6$. Hence our work from part (a) does not change – we need only check whether $b = 6$.

Ex. C-10 2.3 Fa20 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to 5}\left(\frac{25-x^2}{x-5}\right)$

(b) $\displaystyle\lim_{x\to 4}\left(\frac{\frac{1}{x}-\frac{1}{4}}{4-x}\right)$

exf20

Solution C-10

(a) Factor and cancel.

$$\lim_{x\to 5}\left(\frac{25-x^2}{x-5}\right) = \lim_{x\to 5}\left(\frac{(5-x)(5+x)}{x-5}\right) = \lim_{x\to 5}\left(-(5+x)\right) = -10$$

(b) Simplify, factor, and cancel.

$$\lim_{x\to 4}\left(\frac{\frac{1}{x}-\frac{1}{4}}{4-x}\right) = \lim_{x\to 4}\left(\frac{4-x}{4x(4-x)}\right) = \lim_{x\to 4}\left(\frac{1}{4x}\right) = \frac{1}{16}$$

Ex. C-11 2.3 Fa20 Exam

A student is asked to solve a certain limit and determines the limit does not exist. (This may or may not be the correct answer.) They write the following for their justification:

> "I used the direct substitution property to evaluate the limit. I noticed the denominator gives me a zero, therefore the limit does not exist."

Explain why the student's justification is incorrect.

exf20

2.3 Fa20 Exam

Note: To solve this problem, it is not necessary to be given the actual limit the student was asked to compute.

exf20

Solution C-11

If direct substitution property gives "$\frac{0}{0}$" this only means that the limit cannot be computed by direct substitution (since $\frac{0}{0}$ is undefined); this does not necessarily mean that the limit does not exist.

Additionally, we also know that there are many limits which arise in this manner that actually do exist. For example, the limit $\lim_{x\to 0}\left(\frac{x}{x}\right)$ gives "$\frac{0}{0}$" upon direct substitution of $x = 0$, but this limit exists and is equal to 1.

Ex. C-12 2.3 Fa20 Exam

Determine whether $\lim_{x\to 0} f(x)$ exists, where $f(x) = \begin{cases} 3e^x - 7 & x < 0 \\ 4 + \sin(x) & x \geq 0 \end{cases}$.

exf20

Solution C-12

We examine the one-sided limits at $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3e^x - 7) = 3 - 7 = -4$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (4 + \sin(x)) = 4 + 0 = 4$$

Since the left- and right-limits are not equal, $\lim_{x\to 0} f(x)$ does not exist.

Ex. C-13 2.3 Fa20 Exam

A student is asked to calculate the following limit:

$$L = \lim_{x\to 0}\left(\frac{x\cos x}{\sin(3x)}\right)$$

Analyze their work below, which contains two distinct errors. ***Note:*** The correct answer is $\frac{1}{3}$, not 0.

$$L = \lim_{x\to 0}\left(\frac{x\cos(x)}{3\sin(x)}\right) \tag{1}$$
$$= \left[\lim_{x\to 0}\left(\frac{1}{3}\right)\right]\left[\lim_{x\to 0}\left(\frac{x}{\sin(x)}\right)\right]\left[\lim_{x\to 0}(\cos(x))\right] \tag{2}$$
$$= \left(\frac{1}{3}\right)(1)(0) \tag{3}$$
$$= 0 \tag{4}$$

Identify the lines in which the two errors occur and describe each error.

exf20

Solution C-13

Line (1) has an error: in general, $\sin(3x) \neq 3\sin(x)$. (These quantities are equal for some but not all values of x. In particular, $\sin(3x) \neq 3\sin(x)$ for x close to 0 but not equal to 0.)

Line (3) has an error: $\lim_{x\to 0}\cos(x) \neq 0$. (By direct substitution property, this limit is 1.)

Ex. C-14 2.3 Fa20 Exam

Consider the function $f(x)$ below, where $g(x)$ is an unspecified function with domain $[4, \infty)$.

$$f(x) = \begin{cases} 4 & x \leq 0 \\ \dfrac{x-4}{\frac{1}{4} - \frac{1}{x}} & 0 < x < 4 \\ 16 & x = 4 \\ g(x) & x > 4 \end{cases}$$

exf20

2.3 Fa20 Exam

(a) Show that $\lim_{x\to 4^-} f(x) = f(4)$.

(b) Suppose $g(4) = 16$. Is it necessarily true that $\lim_{x\to 4} f(x)$ exists? Justify your response.

exf20

Solution C-14

(a) Use the "second piece" of f to compute the limit.

$$\lim_{x\to 4^-}\left(\frac{x-4}{\frac{1}{4}-\frac{1}{x}}\right) = \lim_{x\to 4^-}\left(\frac{4x(x-4)}{x-4}\right) = \lim_{x\to 4^-}(4x) = 4\cdot 4 = 16$$

Since $f(4) = 16$, we have shown the desired statement.

(b) No. For instance, let g be the following function:

$$g(x) = \begin{cases} 16 & x = 4 \\ 0 & x > 4 \end{cases}$$

Then $g(4) = 16$, but $\lim_{x\to 4} f(x)$ does not exist because $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} g(x) = 0$, which is not equal to $\lim_{x\to 4^-} f(x) = 16$.

The main issue here is that we really need the right limit, not the function value, of g at $x = 4$ to be equal to 16.

Ex. C-15 2.3 Fa20 Exam

A student is asked to solve a certain limit and determines the limit does not exist. (This may or may not be the correct answer.) They write the following for their justification:

> "I used the direct substitution property to evaluate the limit. I obtained the expression "$\frac{0}{0}$", which is undefined. Therefore the limit does not exist."

Is the student's justification correct? Explain.

Note: To solve this problem, it is not necessary to be given the actual limit the student was asked to compute.

exf20

Solution C-15

If direct substitution property gives "$\frac{0}{0}$" this only means that the limit cannot be computed by direct substitution (since $\frac{0}{0}$ is undefined); this does not necessarily mean that the limit does not exist.

Additionally, we also know that there are many limits which arise in this manner that actually do exist. For example, the limit $\lim_{x\to 0}\left(\frac{x}{x}\right)$ gives "$\frac{0}{0}$" upon direct substitution of $x = 0$, but this limit exists and is equal to 1.

Ex. C-16 2.3 Fa20 Exam

Consider the limit $\lim_{x\to 3}\left(\frac{(5x-c)(x+4)}{x-3}\right)$, where c is an unspecified constant.

(a) For what value(s) of c does this limit exist? Explain.

(b) Suppose the limit exists. What is its value? Show all work.

exf20

Solution C-16

(a) Since direct substitution of $x = 3$ gives 0 in the denominator, the only hope we have of this limit existing is if we get cancellation. That is, there must be a common factor in numerator and denominator to cancel. (Alternatively, we must have a "$\frac{0}{0}$" form upon substitution of $x = 3$.) This means that the numerator must be 0 if $x = 3$.

$$0 = (5\cdot 3 - c)(3+4) = (15-c)\cdot 7 \Longrightarrow c = 15$$

(b) If the limit exists, then we must have $c = 15$, in which case we have:

$$\lim_{x\to 3}\left(\frac{(5x-15)(x+4)}{x-3}\right) = \lim_{x\to 3}\left(\frac{5(x-3)(x+4)}{x-3}\right) = \lim_{x\to 3}(5(x+4)) = 35$$

Ex. C-17 2.3 Sp21 Exam

Suppose $\lim_{x\to 0} f(x) = 4$. Calculate $\lim_{x\to 0}\left(\frac{xf(x)}{\sin(5x)}\right)$ or show that the limit does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

exs21

Solution C-17

We have

$$\lim_{x\to 0}\left(\frac{xf(x)}{\sin(5x)}\right) = \lim_{x\to 0}\left(\frac{1}{5}\cdot\frac{5x}{\sin(5x)}\cdot f(x)\right) = \frac{1}{5}\cdot 1\cdot 4 = \frac{4}{5}$$

Ex. C-18 2.3 Sp21 Exam

Consider the following limit, where a is an unspecified constant.

$$\lim_{x\to -3}\left(\frac{x^2-a}{x^3+x^2-6x}\right)$$

(a) Find the value of a for which this limit exists.

(b) For this value of a, calculate the value of the limit.

exs21

Solution C-18

(a) Direct substitution of $x=-3$ gives the undefined expression "$\frac{9-a}{0}$". If the given limit exists, then the only possibility is that this undefined expression is, in fact "$\frac{0}{0}$". (If the expression were "$\frac{\text{nonzero}}{0}$", we would have a vertical asymptote at $x=-3$ instead.) Hence $9-a=0$, and so $a=9$.

(b) With $a=9$, we have the following.

$$\lim_{x\to -3}\left(\frac{x^2-9}{x^3+x^2-6x}\right) = \lim_{x\to -3}\left(\frac{(x-3)(x+3)}{x(x-2)(x+3)}\right) = \lim_{x\to -3}\left(\frac{x-3}{x(x-2)}\right) = -\frac{2}{5}$$

Ex. C-19 2.3 Sp21 Exam

Consider the following function, where k is an unspecified constant.

$$g(x) = \begin{cases} xe^{x+4} - 7\ln(x+5) & x < -4 \\ -4\cos(\pi x) & -4 < x < 5 \\ 10 & x = 5 \\ \sqrt{2x-5}+k & 5 < x \end{cases}$$

Note that $g(-4)$ is undefined.

(a) Does $\lim_{x\to -4} g(x)$ exist? Choose the best answer below.

(i) Yes, $\lim_{x\to -4} g(x)$ exists and is equal to ______.

(ii) Yes, $\lim_{x\to -4} g(x)$ exists but we cannot determine its value with the given information.

(iii) No, $\lim_{x\to -4} g(x)$ does not exist because the corresponding one-sided limits are not equal.

(iv) No, $\lim_{x\to -4} g(x)$ does not exist because $g(-4)$ does not exist.

(v) No, $\lim_{x\to -4} g(x)$ does not exist because the limit is infinite.

(b) Calculate the following limits. Your answer may contain k.

(i) $\lim_{x\to 5^-} g(x)$ (ii) $\lim_{x\to 5^+} g(x)$

(c) Is it possible to choose a value of k so that $\lim_{x\to 5} g(x)$ exists? If so, what is that value of k?

exs21

Solution **C-19**

(a) Choice (i). Note the following:

$$\lim_{x\to -4^-} g(x) = \lim_{x\to -4^-} (xe^{x+4} - 7\ln(x+5)) = -4\cdot 1 - 7\cdot 0 = -4$$
$$\lim_{x\to -4^+} g(x) = \lim_{x\to -4^+} (-4\cos(\pi x)) = -4\cdot\cos(-4\pi) = -4$$

The left- and right-limits at $x = -4$ are both equal to -4, hence $\lim_{x\to -4} g(x) = -4$. (Note that the function value $g(-4)$, which is undefined, is irrelevant.)

(b) We have the following:

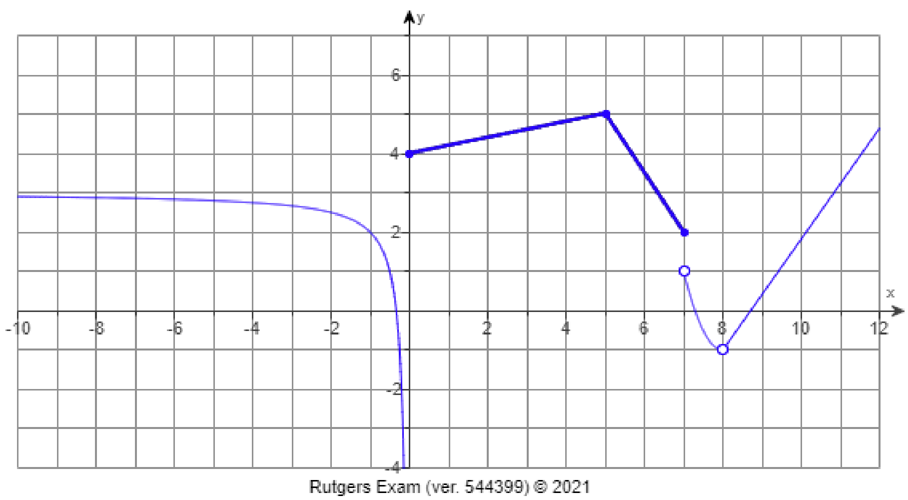
(i) $\lim_{x\to 5^-} g(x) = -4\cos(5\pi) = 4$

(ii) $\lim_{x\to 5^+} g(x) = \lim_{x\to 5^+} (\sqrt{2x-5}+k) = \sqrt{5}+k$

(c) Yes. From part (b), we need $4 = \sqrt{5}+k$, or $k = 4-\sqrt{5}$. (Again, the function value $g(5)$, which is 10, is irrelevant.)

Ex. B-6 **2.1/2.2, 2.3, 2.4, 2.5** Fa21 **Exam**

For each part, use the graph of $y = f(x)$.



(a) List the x-values where f is not continuous or determine that f is continuous for all x.

(b) List all vertical asymptotes of f.

(c) List all horizontal asymptotes of f.

(d) Calculate $\lim_{x\to 8} f(x)$ or determine that the limit does not exist.

(e) At $x = 7$, which of the one-sided limits of f exist?

exf21

Solution **B-6**

(a) $x = 0, 7, 8$ only

(b) $x = 0$ only

(c) $y = 3$ only

(d) $\lim_{x\to 8} f(x) = -1$

(e) Both the left- and right-limits of $f(x)$ at $x = 7$ exist.

Ex. C-20 2.3 Fa21 Exam

Suppose $\lim_{x\to 6} |f(x)| = 2$. Which of the following statements must be true about $\lim_{x\to 6} f(x)$?

(i) $\lim_{x\to 6} f(x)$ does not exist.

(ii) $\lim_{x\to 6} f(x) = 2$.

(iii) $\lim_{x\to 6} f(x)$ exists and is equal to either 2 or -2, but there is not enough information to determine which of these possibilities must be true.

(iv) There is not enough information about $f(x)$ to determine whether $\lim_{x\to 6} f(x)$ exists.

(v) $\lim_{x\to 6} f(x) = -2$.

exf21

Solution C-20

Choice (iv). Consider these two examples, both of which satisfy the hypothesis $\lim_{x\to 6} |f(x)| = 2$.

- $f(x) = 2$. Then $\lim_{x\to 6} f(x)$ exists and is equal to 2.
- $f(x) = 2$ for $x < 6$ and $f(x) = -2$ for $x \geq 2$. Then $\lim_{x\to 6} f(x)$ does not exist (the left- and right-limits at $x = 6$ are not equal).

Thus it is not possible to determine whether $\lim_{x\to 6} f(x)$ exists.

Ex. C-21 2.3 Fa21 Exam

Consider the following function, where k is an unspecified constant.

$$f(x) = \frac{4x^2 - kx}{x^2 + 12x + 32}$$

(a) Find the value of k for which $\lim_{x\to -4} f(x)$ exists.

(b) For the value of k described in part (a), evaluate $\lim_{x\to -4} f(x)$.

exf21

Solution C-21

(a) Direct substitution of $x = -4$ into $f(x)$ gives the undefined expression "$\frac{64+4k}{0}$". If the number $64 + 4k$ were non-zero, then we would conclude there is a vertical asymptote for f at $x = -4$. However, since $\lim_{x\to -4} f(x)$ exists, we must have $64 + 4k = 0$, whence $k = -16$.

(b) With $k = -16$, we have the following.

$$\lim_{x\to -4} \left(\frac{4x^2 + 16x}{x^2 + 12x + 32} \right) = \lim_{x\to -4} \left(\frac{4x(x+4)}{(x+8)(x+4)} \right) = \lim_{x\to -4} \left(\frac{4x}{x+8} \right) = -4$$

Ex. C-22 2.3 Fa21 Exam

Suppose $\lim_{x\to 0} \left(\frac{f(x)}{x} \right) = 8$. Calculate $\lim_{x\to 0} \left(\frac{f(x)}{\sin(6x)} \right)$ or show that the limit does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

exf21

Solution C-22

We have the following:

$$\lim_{x\to 0} \left(\frac{f(x)}{\sin(6x)} \right) = \lim_{x\to 0} \left(\frac{1}{6} \cdot \frac{f(x)}{x} \cdot \frac{6x}{\sin(6x)} \right) = \frac{1}{6} \cdot 8 \cdot 1 = \frac{4}{3}$$

Ex. C-23 2.3, 2.4 Sp22 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

exs22

2.3, 2.4 Sp22 Exam

(a) $\lim_{x\to3}\left(\dfrac{x-3}{10-\sqrt{x+97}}\right)$

(b) $\lim_{x\to6}\left(\dfrac{36-x^2}{\frac{1}{x}-\frac{1}{6}}\right)$

(c) $\lim_{x\to0}\left(\dfrac{x^2\csc(3x)}{\cos(7x)\sin(4x)}\right)$

(d) $\lim_{x\to2^-}\left(\dfrac{6x^2-7x}{x^2-4}\right)$

exs22

Solution C-23

(a) Rationalize the denominator, cancel common factors, and use direct substitution.

$$\begin{aligned}\lim_{x\to3}\left(\frac{x-3}{10-\sqrt{x+97}}\right)&=\lim_{x\to3}\left(\frac{x-3}{10-\sqrt{x+97}}\cdot\frac{10+\sqrt{x+97}}{10+\sqrt{x+97}}\right)=\lim_{x\to3}\left(\frac{(x-3)(10+\sqrt{x+97})}{100-(x+97)}\right)\\&=\lim_{x\to3}\left(\frac{(x-3)(10+\sqrt{x+97})}{-(x-3)}\right)=\lim_{x\to3}\left(10+\sqrt{x+97}\right)=10+\sqrt{100}=20\end{aligned}$$

(b) Cancel common factors and use direct substitution.

$$\lim_{x\to6}\left(\frac{36-x^2}{\frac{1}{x}-\frac{1}{6}}\right)=\lim_{x\to6}\left(\frac{6x(36-x^2)}{6-x}\right)=\lim_{x\to6}\left(\frac{6x(6-x)(6+x)}{6-x}\right)=\lim_{x\to6}\left(6x(6+x)\right)=432$$

(c) Write in terms of sine and cosine, regroup terms, and use the special trigonometric limits.

$$\lim_{x\to0}\left(\frac{x^2\csc(3x)}{\cos(7x)\sin(4x)}\right)=\lim_{x\to0}\left(\frac{3x}{\sin(3x)}\cdot\frac{4x}{\sin(4x)}\cdot\frac{1}{12\cos(7x)}\right)=1\cdot1\cdot\frac{1}{12\cdot1}=\frac{1}{12}$$

(d) Direct substitution of $x=2$ gives the undefined expression "$\frac{10}{0}$". Since this is a nonzero number divided by zero, we know the one-sided limit is infinite, and so all we must do is sign analysis to determine the sign of the infinity. As $x\to2$, the numerator approaches 10, so the numerator is positive. The denominator factors as $(x-2)(x+2)$. The second factor $(x+2)$ goes to 4 (and is thus positive) as $x\to2$. The first factor $(x-2)$ goes to 0 but remains negative as $x\to2^-$.

Putting this altogether, the expression inside the limit has a negative value $\left(\frac{\oplus}{\ominus\oplus}=\ominus\right)$ as $x\to2^-$. So the desired limit is $-\infty$.

Ex. C-24 2.3 Su22 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to5}\left(\dfrac{6x+10}{x^2-25}-\dfrac{4}{x-5}\right)$

(b) $\lim_{x\to6}\left(\dfrac{x-\sqrt{5x+6}}{6-x}\right)$

(c) $\lim_{x\to\infty}\left(\dfrac{5e^{2x}-3e^x}{9e^{3x}-12}\right)$

exsu22

Solution C-24

(a) Find a common denominator, factor, and then cancel common factors.

$$\lim_{x\to5}\left(\frac{6x+10}{x^2-25}-\frac{4}{x-5}\right)=\lim_{x\to5}\left(\frac{6x+10-4(x+5)}{x^2-25}\right)=\lim_{x\to5}\left(\frac{2(x-5)}{(x-5)(x+5)}\right)=\lim_{x\to5}\left(\frac{2}{x+5}\right)=\frac{2}{10}=\frac{1}{5}$$

(b) Rationalize the numerator, then cancel common factors.

$$\begin{aligned}&\lim_{x\to6}\left(\frac{x-\sqrt{5x+6}}{6-x}\cdot\frac{x+\sqrt{5x+6}}{x+\sqrt{5x+6}}\right)=\lim_{x\to6}\left(\frac{x^2-(5x+6)}{(6-x)(x+\sqrt{5x+6})}\right)=\\&=\lim_{x\to6}\left(\frac{(x-6)(x+1)}{(6-x)(x+\sqrt{5x+6})}\right)=\lim_{x\to6}\left(\frac{-(x+1)}{x+\sqrt{5x+6}}\right)=-\frac{7}{12}\end{aligned}$$

(c) The dominant term of the denominator is e^{3x}. So divide all terms by e^{3x} and take limits.

$$\lim_{x\to\infty}\left(\frac{5e^{2x}-3e^x}{9e^{3x}-12}\right)=\lim_{x\to\infty}\left(\frac{5e^{-x}-e^{-2x}}{9-12e^{-3x}}\right)=\frac{0-0}{9-0}=0$$

Ex. C-25 2.3 Fa22 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 8}\left(\frac{(x-2)^2-36}{x-8}\right)$ (b) $\lim_{x\to 5}\left(\frac{40-8x}{\sqrt{19-3x}-2}\right)$ (c) $\lim_{x\to 2^-}\left(\frac{4+x}{x^2+x-6}\right)$

exf22

Solution C-25

(a) Expand the numerator. Then cancel common factors.

$$\lim_{x\to 8}\left(\frac{(x-2)^2-36}{x-8}\right)=\lim_{x\to 8}\left(\frac{x^2-4x-32}{x-8}\right)=\lim_{x\to 8}\left(\frac{(x-8)(x+4)}{x-8}\right)=\lim_{x\to 8}(x+4)=12$$

(b) Rationalize the denominator. Then cancel common factors.

$$\begin{aligned}\lim_{x\to 5}\left(\frac{40-8x}{\sqrt{19-3x}-2}\right)&=\lim_{x\to 5}\left(\frac{40-8x}{\sqrt{19-3x}-2}\cdot\frac{\sqrt{19-3x}+2}{\sqrt{19-3x}+2}\right)\\&=\lim_{x\to 5}\left(\frac{8(5-x)\left(\sqrt{19-3x}+2\right)}{19-3x-4}\right)=\lim_{x\to 5}\left(\frac{8(5-x)\left(\sqrt{19-3x}+2\right)}{3(5-x)}\right)\\&=\lim_{x\to 5}\left(\frac{8}{3}\left(\sqrt{19-3x}+2\right)\right)=\frac{8}{3}(\sqrt{4}+2)=\frac{32}{3}\end{aligned}$$

(c) Direct substitution of $x=2$ gives the undefined expression "$\frac{6}{0}$" (i.e., a nonzero number divided by 0). Hence the one-sided limit is infinite. Observe that the denominator is $x^2+x-6=(x+3)(x-2)$. As $x\to 2^-$, the factor $(x+3)$ is positive and the factor $(x-2)$ is negative. Thus the entire fraction has the following sign as $x\to 2^-$: $\frac{6}{\oplus\ominus}=\ominus$. Thus the limit is equal to $-\infty$.

Ex. C-26 2.3 Sp18 Quiz

Evaluate each of the following limits or show why it does not exist.

(a) $\lim_{x\to 2}\left(\frac{2x^2-3x-2}{x^2+2x-8}\right)$ (b) $\lim_{x\to 4}\left(\frac{3-\sqrt{x+5}}{x-4}\right)$

qs18

Solution C-26

(a) We have the following work.

$$\lim_{x\to 2}\left(\frac{2x^2-3x-2}{x^2+2x-8}\right)=\lim_{x\to 2}\left(\frac{(x-2)(2x+1)}{(x-2)(x+4)}\right)=\lim_{x\to 2}\left(\frac{2x+1}{x+4}\right)=\frac{5}{6}$$

(b) We have the following work.

$$\lim_{x\to 4}\left(\frac{3-\sqrt{x+5}}{x-4}\right)=\lim_{x\to 4}\left(\frac{3-\sqrt{x+5}}{x-4}\cdot\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}\right)=\lim_{x\to 4}\left(\frac{4-x}{(x-4)(3+\sqrt{x+5})}\right)=\lim_{x\to 4}\left(\frac{-1}{3+\sqrt{x+5}}\right)=\frac{-1}{6}$$

Ex. C-27 2.3 Sp20 Quiz

Evaluate each of the following limits or show why it does not exist.

(a) $\lim_{x\to 1}\left(\frac{\sqrt{7x+9}-4}{x-1}\right)$ (b) $\lim_{x\to 5}\left(\frac{\frac{1}{5}-\frac{1}{x}}{\frac{x}{5}-\frac{5}{x}}\right)$ (c) $\lim_{x\to 3}\left(\frac{3-\sqrt{12-x}}{x-3}\right)$

qs20

Solution **C-27**

(a) Rationalize the numerator and cancel common factors.

$$\lim_{x\to 1}\left(\frac{\sqrt{7x+9}-4}{x-1}\right)=\lim_{x\to 1}\left(\frac{7x+9-16}{(x-1)(\sqrt{7x+9}+4)}\right)=\lim_{x\to 1}\left(\frac{7}{\sqrt{7x+9}+4}\right)=\frac{7}{8}$$

(b) Multiply all terms by $5x$ (common denominator) and then cancel common factors.

$$\lim_{x\to 5}\left(\frac{\frac{1}{5}-\frac{1}{x}}{\frac{x}{5}-\frac{5}{x}}\right)=\lim_{x\to 5}\left(\frac{x-5}{x^2-25}\right)=\lim_{x\to 5}\left(\frac{1}{x+5}\right)=\frac{1}{10}$$

(c) Rationalize the numerator and cancel common factors.

$$\lim_{x\to 3}\left(\frac{3-\sqrt{12-x}}{x-3}\right)=\lim_{x\to 3}\left(\frac{9-(12-x)}{(x-3)\left(3+\sqrt{12-x}\right)}\right)=\lim_{x\to 3}\left(\frac{1}{3+\sqrt{12-x}}\right)=\frac{1}{6}$$

Ex. C-28 **2.3** Su22 **Quiz**

For each part, calculate the limit or show that it does not exist.

(a) $\lim\limits_{x\to 9}\left(\dfrac{x^3-81x}{(x-4)^2-25}\right)$ (b) $\lim\limits_{x\to 1}\left(\dfrac{\sqrt{x+3}-2}{x-1}\right)$

qsu22

Solution **C-28**

(a) Expand, factor, and cancel common factors.

$$\lim_{x\to 9}\left(\frac{x^3-81x}{(x-4)^2-25}\right)=\lim_{x\to 9}\left(\frac{x(x^2-81)}{x^2-8x-9}\right)=\lim_{x\to 9}\left(\frac{x(x-9)(x+9)}{(x-9)(x+1)}\right)$$
$$=\lim_{x\to 9}\left(\frac{x(x+9)}{x+1}\right)=\frac{9\cdot 18}{10}=16.2$$

(b) Rationalize the numerator. Then cancel common factors.

$$\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right)=\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)=\lim_{x\to 1}\left(\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}\right)$$
$$=\lim_{x\to 1}\left(\frac{x-1}{(x-1)(\sqrt{x+3}+2)}\right)=\lim_{x\to 1}\left(\frac{1}{\sqrt{x+3}+2}\right)=\frac{1}{\sqrt{4}+2}=\frac{1}{4}$$

Ex. C-29 **2.3** Su22 **Quiz**

Calculate $\lim\limits_{x\to 0} f(x)$ or show the limit does not exist, where $f(x)$ is the function given below. Your work must be coherent and clearly explain your answer.

$$f(x)=\begin{cases}10e^x & x<0\\ 7 & x=0\\ \dfrac{\sin(10x)}{x} & x>0\end{cases}$$

qsu22

Solution **C-29**

Since $x=0$ is a transition point of the piecewise-defined function f, we examine the one-sided limits.

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(10e^x)=10e^0=10$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(\frac{\sin(10x)}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sin(10x)}{10x}\cdot 10\right)=1\cdot 10=10$$

C-29

Since the left- and right-limits at $x = 0$ are both equal to 10, we find that $\lim_{x\to 0} f(x) = 10$.

Ex. C-30 2.3 Fa22 Quiz

For each part, calculate the limit or determine it does not exist. You must show all work, and your work will be graded on its correctness and coherence.

(a) $\lim_{x\to 6}\left(\frac{x^2-36}{2x^2-11x-6}\right)$

(b) $\lim_{x\to 2}\left(\frac{\frac{3x+1}{x-1}-7}{x-2}\right)$

qf22

Solution C-30

(a) Cancel common factors.

$$\lim_{x\to 6}\left(\frac{x^2-36}{2x^2-11x-6}\right) = \lim_{x\to 6}\left(\frac{(x-6)(x+6)}{(2x+1)(x-6)}\right) = \lim_{x\to 6}\left(\frac{x+6}{2x+1}\right) = \frac{12}{13}$$

(b) Clear the denominators and cancel common factors.

$$\lim_{x\to 2}\left(\frac{\frac{3x+1}{x-1}-7}{x-2}\right) = \lim_{x\to 2}\left(\frac{3x+1-7(x-1)}{(x-2)(x-1)}\right)$$
$$= \lim_{x\to 2}\left(\frac{-4x+8}{(x-2)(x-1)}\right) = \lim_{x\to 2}\left(\frac{-4(x-2)}{(x-2)(x-1)}\right) = \lim_{x\to 2}\left(\frac{-4}{x-1}\right) = -4$$

Ex. C-31 2.3 Fa22 Quiz

Calculate $\lim_{x\to -3}\left(\frac{\sqrt{2x+15}-3}{x^2+8x+15}\right)$ or determine that it does not exist. If the limit is infinite, write "$+\infty$" or "$-\infty$" as your answer, as appropriate, instead of "DNE".

qf22

Solution C-31

Rationalize the numerator and cancel common factors.

$$\lim_{x\to -3}\left(\frac{\sqrt{2x+15}-3}{x^2+8x+15}\cdot\frac{\sqrt{2x+15}+3}{\sqrt{2x+15}+3}\right) = \lim_{x\to -3}\left(\frac{2x+15-9}{(x+5)(x+3)(\sqrt{2x+15}+3)}\right)$$
$$= \lim_{x\to -3}\left(\frac{2(x+3)}{(x+5)(x+3)(\sqrt{2x+15}+3)}\right) = \lim_{x\to -3}\left(\frac{2}{(x+5)(\sqrt{2x+15}+3)}\right) = \frac{2}{2\cdot(3+3)} = \frac{1}{6}$$

Ex. C-32 2.3

For each part, calculate the limit or show that it does not exist.

c2s3

2.3

(a) $\lim_{x\to 2}\left(\frac{x^2+3x-1}{x+\sin(\pi x)}\right)$

(b) $\lim_{x\to 1}(x^4-9x)^{1/3}$

(c) $\lim_{x\to -3}\left(\frac{x^2-9}{x^3+x^2-6x}\right)$

(d) $\lim_{x\to 1}\left(\frac{\sqrt{23-7x}-4}{x-1}\right)$

(e) $\lim_{h\to 0}\left(\frac{(x+h)^{-2}-x^{-2}}{h}\right)$

(f) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$

(g) $\lim_{x\to 1}\left(\frac{\frac{1}{x}-1}{\sqrt{x}-1}\right)$

(h) $\lim_{x\to 0}|x|$

(i) $\lim_{x\to 8}\left(\frac{|x-8|}{x-8}\right)$

(j) $\lim_{x\to 8^-}\left(\frac{|x^2-64|}{x-8}\right)$

(k) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

(l) $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

(m) $\lim_{x\to 0}\left(\frac{\sin(\pi x)}{x}\right)$

(n) $\lim_{x\to 0}\left(\frac{\sec(x)-1}{x\sec(x)}\right)$

(o) $\lim_{x\to 0}\left(\frac{1-\cos(x)}{\sin(x)}\right)$

(p) $\lim_{x\to 2}\left(\frac{\sin(6-3x)}{5x-10}\right)$

(q) $\lim_{x\to \pi}\left(\frac{\tan(x-\pi)}{x-\pi}\right)$

(r) $\lim_{x\to 0}\left(\frac{\sin(2x)^2\cos(3x)}{\tan(5x)\sin(7x)}\right)$

(s) $\lim_{x\to -1}g(x)$ where

$$g(x)=\begin{cases}4x-5 & \text{if } x<-1\\ x^3+x & \text{if } x\geq -1\end{cases}$$

(t) $\lim_{x\to 2}f(x)$ where

$$f(x)=\begin{cases}\dfrac{x^2-2x}{x-2} & \text{if } x<2\\ \sqrt{x+2} & \text{if } x>2\end{cases}$$

c2s3

Solution **C-32**

(a) Direct substitution.

$$\lim_{x\to 2}\left(\frac{x^2+3x-1}{x+\sin(\pi x)}\right)=\frac{2^2+3(2)-1}{2+\sin(2\pi)}=\frac{9}{2}$$

(b) Direct substitution.

$$\lim_{x\to 1}(x^4-9x)^{1/3}=(1^4-9(1))^{1/3}=(-8)^{1/3}=-2$$

(c) Factor and cancel.

$$\lim_{x\to -3}\left(\frac{x^2-9}{x^3+x^2-6x}\right)=\lim_{x\to 3}\left(\frac{(x-3)(x+3)}{x(x-2)(x+3)}\right)=\lim_{x\to 3}\left(\frac{x-3}{x(x-2)}\right)\frac{-3-3}{(-3)(-3-2)}=-\frac{2}{5}$$

(d) Rationalize the numerator, then factor and cancel.

$$\begin{aligned}\lim_{x\to 1}\left(\frac{\sqrt{23-7x}-4}{x-1}\right)&=\lim_{x\to 1}\left(\frac{\sqrt{23-7x}-4}{x-1}\cdot\frac{\sqrt{23-7x}+4}{\sqrt{23-7x}+4}\right)=\lim_{x\to 1}\left(\frac{(23-7x)-16}{(x-1)(\sqrt{23-7x}+4)}\right)\\&=\lim_{x\to 1}\left(\frac{-7(x-1)}{(x-1)(\sqrt{23-7x}+4)}\right)=\lim_{x\to 1}\left(\frac{-7}{\sqrt{23-7x}+4}\right)=-\frac{7}{\sqrt{23-7}+4}=-\frac{7}{8}\end{aligned}$$

(e) Find a common denominator, expand the numerator, then factor and cancel.

$$\begin{aligned}\lim_{h\to 0}\left(\frac{(x+h)^{-2}-x^{-2}}{h}\right)&=\lim_{h\to 0}\left(\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\right)=\lim_{h\to 0}\left(\frac{x^2-(x+h)^2}{hx^2(x+h)^2}\right)=\lim_{h\to 0}\left(\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\right)\\&=\lim_{h\to 0}\left(\frac{-h(2x+h)}{hx^2(x+h)^2}\right)=\lim_{h\to 0}\left(\frac{-(2x+h)}{x^2(x+h)^2}\right)=-\frac{2x}{x^2\cdot x^2}=-\frac{2}{x^3}\end{aligned}$$

(f) Find a common denominator, then factor and cancel.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)=\lim_{x\to 0}\left(\frac{(x+1)-1}{x(x+1)}\right)=\lim_{x\to 0}\left(\frac{x}{x(x+1)}\right)=\lim_{x\to 0}\left(\frac{1}{x+1}\right)=\frac{1}{0+1}=1$$

(g) Multiply all terms by the common denominator and rationalize the denominator. Then factor and cancel.

$$\lim_{x\to 1}\left(\frac{\frac{1}{x}-1}{\sqrt{x}-1}\right)=\lim_{x\to 1}\left(\frac{\frac{1}{x}-1}{\sqrt{x}-1}\cdot\frac{x(\sqrt{x}+1)}{x(\sqrt{x}+1)}\right)=\lim_{x\to 1}\left(\frac{(1-x)(\sqrt{x}+1)}{x(x-1)}\right)=\lim_{x\to 1}\left(\frac{-(\sqrt{x}+1)}{x}\right)=-2$$

C-32

(h) Write as a piecewise function and compute the one-sided limits.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Hence we have

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = -0 = 0$$
$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} (x) = 0$$

The one-sided limits exist and are equal, so $\lim_{x\to 0} |x| = 0$.

(i) Write as a piecewise function and compute the one-sided limits.

$$\frac{|x-8|}{x-8} = \begin{cases} \frac{-(x-8)}{x-8} & \text{if } x-8 < 0 \\ \frac{x-8}{x-8} & \text{if } x-8 > 0 \end{cases} = \begin{cases} -1 & \text{if } x < 8 \\ 1 & \text{if } x > 8 \end{cases}$$

Hence we have

$$\lim_{x\to 8^-} \left(\frac{|x-8|}{x-8}\right) = \lim_{x\to 8^-} (-1) = -1$$
$$\lim_{x\to 8^+} \left(\frac{|x-8|}{x-8}\right) = \lim_{x\to 8^+} (1) = 1$$

The one-sided limits exist but are not equal. Hence $\lim_{x\to 8} \left(\frac{|x-8|}{x-8}\right)$ does not exist.

(j) Note that $|x^2-64| = |x-8| \cdot |x+8|$. So if $x \to 8^-$, we consider x to be less than (but near) 8. So $(x-8)$ is a small negative number, whence $|x-8| = -(x-8)$. So now we have:

$$\lim_{x\to 8^-} \left(\frac{|x^2-64|}{x-8}\right) = \lim_{x\to 8^-} \left(\frac{-(x-8)(x+8)}{x-8}\right) = \lim_{x\to 8^-} (-(x+8)) = -16$$

(k) Write as a piecewise function and compute the one-sided limits.

$$\frac{1}{x} - \frac{1}{|x|} = \begin{cases} \frac{1}{x} - \frac{1}{-x} & \text{if } x < 0 \\ \frac{1}{x} - \frac{1}{x} & \text{if } x > 0 \end{cases} = \begin{cases} \frac{2}{x} & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Hence we have

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^+} (0) = 0$$

(l) From the work done in the previous part, we have

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-} \left(\frac{2}{x}\right) = -\infty$$

We will study infinite limits in more detail later. For now, we argue as follows.

If x is very small, then $\frac{2}{x}$ is very large. (Think of taking the reciprocal of a very small number: what happens?) But since $x \to 0^-$, we have that x is also negative. Hence $\frac{2}{x}$ is a negative, arbitrarily large number as $x \to 0^-$. Hence the limit is $-\infty$.

(m) Multiply by $\frac{\pi}{\pi}$ and use the special limit $\lim_{\theta\to 0} \left(\frac{\sin(a\theta)}{a\theta}\right) = 1$.

$$\lim_{x\to 0} \left(\frac{\sin(\pi x)}{x}\right) = \lim_{x\to 0} \left(\pi \cdot \frac{\sin(\pi x)}{\pi x}\right) = \pi \cdot \lim_{x\to 0} \left(\frac{\sin(\pi x)}{\pi x}\right) = \pi \cdot \lim_{\theta\to 0} \left(\frac{\sin(\theta)}{\theta}\right) = \pi \cdot 1 = \pi$$

C-32

(n) Write in terms of sine and cosine only first. Then use the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$. Finally use the limit $\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1$.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sec(x)-1}{x\sec(x)}\right) &= \lim_{x\to 0}\left(\frac{\frac{1}{\cos(x)}-1}{x\cdot\frac{1}{\cos(x)}}\right) = \lim_{x\to 0}\left(\frac{1-\cos(x)}{x}\right) = \lim_{x\to 0}\left(\frac{1-\cos(x)}{x}\cdot\frac{1+\cos(x)}{1+\cos(x)}\right)\\ &= \lim_{x\to 0}\left(\frac{1-\cos(x)^2}{x(1+\cos(x))}\right) = \lim_{x\to 0}\left(\frac{\sin(x)^2}{x(1+\cos(x))}\right) = \lim_{x\to 0}\left(\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{1+\cos(x)}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin(x)}{1+\cos(x)}\right) = 1\cdot\frac{0}{1+1} = 0\end{aligned}$$

(o) Use the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1-\cos(x)}{\sin(x)}\right) &= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\sin(x)}\cdot\frac{1+\cos(x)}{1+\cos(x)}\right) = \lim_{x\to 0}\left(\frac{1-\cos(x)^2}{\sin(x)(1+\cos(x))}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin(x)^2}{\sin(x)(1+\cos(x))}\right) = \lim_{x\to 0}\left(\frac{\sin(x)}{1+\cos(x)}\right) = \frac{0}{1+1} = 0\end{aligned}$$

(p) Make the change of variable $\theta = 6-3x$ (note that if $x\to 2$, then $\theta\to 0$). Then use the limit $\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1$.

$$\lim_{x\to 2}\left(\frac{\sin(6-3x)}{5x-10}\right) = \lim_{x\to 2}\left(\frac{\sin(6-3x)}{-\frac{5}{3}(6-3x)}\right)\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{-\frac{5}{3}\theta}\right) = -\frac{3}{5}\cdot\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = -\frac{3}{5}\cdot 1 = -\frac{3}{5}$$

(q) Write in terms of sine and cosine only first. Make the change of variable $\theta = x-\pi$ (note that if $x\to pi$, then $\theta\to 0$). Finally use the limit $\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1$.

$$\lim_{x\to\pi}\left(\frac{\tan(x-\pi)}{x-\pi}\right) = \lim_{x\to\pi}\left(\frac{\sin(x-\pi)}{(x-\pi)\cos(x-\pi)}\right) = \lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta\cos(\theta)}\right) = \lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right)\cdot\lim_{\theta\to 0}\left(\frac{1}{\cos(\theta)}\right) = 1\cdot\frac{1}{1} = 1$$

(r) Write in terms of sine and cosine only first. Then repeatedly use the limit $\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1$ with appropriate changes of variable and added factors.

$$\begin{aligned}&\lim_{x\to 0}\left(\frac{\sin(2x)^2\cos(3x)}{\tan(5x)\sin(7x)}\right) =\\ &= \lim_{x\to 0}\left(\frac{\sin(2x)\sin(2x)\cos(5x)\cos(3x)}{\sin(5x)\sin(7x)}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin(2x)}{2x}\cdot\frac{\sin(2x)}{2x}\cdot\frac{5x}{\sin(5x)}\cdot\frac{7x}{\sin(7x)}\cdot\frac{\cos(5x)\cos(3x)}{1}\cdot\frac{(2x)(2x)}{(5x)(7x)}\right)\end{aligned}$$

Each limit of the form $\lim_{x\to 0}\left(\frac{\sin(ax)}{ax}\right)$ or $\lim_{x\to 0}\left(\frac{ax}{\sin(ax)}\right)$ is equal to 1. So continuing our work, we have

$$= 1\cdot 1\cdot 1\cdot 1\cdot\lim_{x\to 0}\left(\frac{\cos(5x)\cos(3x)}{1}\cdot\frac{4x^2}{35x^2}\right) = \left(\frac{\cos(5x)\cos(3x)}{1}\cdot\frac{4}{35}\right) = 1\cdot 1\cdot\frac{4}{35} = \frac{4}{35}$$

(s) Compute the one-sided limits.

$$\begin{aligned}\lim_{x\to -1^-} g(x) &= \lim_{x\to -1^-}(4x-5) = 4(-1)-5 = -9\\ \lim_{x\to -1^+} g(x) &= \lim_{x\to -1^+}(x^3+x) = (-1)^3+(-1) = -2\end{aligned}$$

The one-sided limits exist but are not equal. Hence $\lim_{x\to -1} g(x)$ does not exist.

C-32

(t) Compute the one-sided limits. For the left limit, factor and cancel.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{x^2-2x}{x-2}\right) = \lim_{x\to 2^-}\left(\frac{x(x-2)}{x-2}\right) = \lim_{x\to 2^-}(x) = 2$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(\sqrt{x+2}) = \sqrt{2+2} = 2$$

The one-sided limits exist and are equal. Hence $\lim\limits_{x\to -1} f(x) = 2$.

Ex. C-33 2.3

For each part, calculate the limit or show that it does not exist.

(a) $\lim\limits_{x\to 0}\left(\frac{\sin(5x)}{3x}\cos(4x)\right)$ (b) $\lim\limits_{x\to -2}\left(\frac{x^2+3x+2}{x^2+x-2}\right)$ (c) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$

unit2

Solution C-33

(a) Use the special limit $\lim\limits_{x\to 0}\left(\frac{\sin(ax)}{ax}\right) = 1$ for any $a \neq 0$.

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{3x}\cos(4x)\right) = \lim_{x\to 0}\left(\frac{5}{3}\cdot\frac{\sin(5x)}{5x}\cdot\cos(4x)\right) = \frac{5}{3}\cdot 1\cdot 1 = \frac{5}{3}$$

(b) Cancel common factors, and then use direct substitution.

$$\lim_{x\to -2}\left(\frac{x^2+3x+2}{x^2+x-2}\right) = \lim_{x\to -2}\left(\frac{(x+2)(x+1)}{(x+2)(x-1)}\right) = \lim_{x\to -2}\left(\frac{x+1}{x-1}\right) = \frac{-2+1}{-2-1} = \frac{1}{3}$$

(c) Find a common denominator, cancel common factors, and then use direct substitution.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right) = \lim_{x\to 0}\left(\frac{x+1-1}{x^2+x}\right) = \lim_{x\to 0}\left(\frac{1}{x+1}\right) = \frac{1}{0+1} = 1$$

Ex. C-34 2.3

Let $f(x) = \begin{cases} \dfrac{3x^2-15x}{x-5} & \text{if } x < 5 \\ \sqrt{a+3x} & \text{if } x > 5 \end{cases}$, where a is an unspecified constant.

(a) For what value of a does $\lim\limits_{x\to 5} f(x)$ exist? Explain.

(b) Given that $\lim\limits_{x\to 5} f(x)$ exists, what is its value?

other

Solution C-34

(a) If $\lim\limits_{x\to 5} f(x)$ exists, then the left- and right-limits at $x = 5$ must be equal.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}\left(\frac{3x^2-15x}{x-5}\right) = \lim_{x\to 5^-}\left(\frac{3x(x-5)}{x-5}\right) = \lim_{x\to 5^-}(3x) = 15$$
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\left(\sqrt{a+3x}\right) = \sqrt{a+15}$$

So we must have $15 = \sqrt{a+15}$, whence $a = 210$.

(b) If $\lim\limits_{x\to 5} f(x)$ exists, then the limit must be equal to $\lim\limits_{x\to 5^-} f(x) = 15$.

Ex. C-35 2.3

Consider the limit $\lim_{x\to -2}\left(\frac{x^4-a}{x^2-2x-8}\right)$, where a is an unspecified constant.

(a) For what value of a does this limit exist? Explain.

(b) Given that the limit does exist, what is its value?

other

Solution C-35

(a) Direct substitution of $x=-2$ gives the undefined expression "$\frac{16-a}{0}$". If $a\neq 16$, then this expression is of the form "$\frac{\text{non-zero }\#}{0}$", which indicates that both corresponding one-sided limits are infinite, and so $\lim_{x\to -2}\left(\frac{x^4-a}{x^2-2x-8}\right)$ does not exist. Therefore, $a=16$ is the only possible value of a for which the limit may exist.

Indeed, for $a=16$, we have the following:

$$\lim_{x\to -2}\left(\frac{x^4-16}{x^2-2x-8}\right)=\lim_{x\to -2}\left(\frac{(x+2)(x-2)(x^2+4)}{(x+2)(x-4)}\right)=\lim_{x\to -2}\left(\frac{(x-2)(x^2+4)}{x-4}\right)=\frac{-4\cdot 8}{-6}=\frac{16}{3}$$

Thus the limit exists if and only if $a=16$.

(b) If the limit exists, it must be equal to $\frac{16}{3}$, as shown in part (a).

Ex. C-36 2.3

For each of the following, evaluate the limit or explain why it does not exist. Show all work.

(a) $\lim_{h\to 0}\left(\frac{(x+h)^{-2}-x^{-2}}{h}\right)$ (b) $\lim_{x\to 3}\left(\frac{4}{x-3}-\frac{8}{x^2-4x+3}\right)$ (c) $\lim_{x\to 0}\left(\frac{\sin(7x)^2\cos(9x)}{\tan(3x)\sin(4x)}\right)$

other

Solution C-36

(a) Find a common denominator, factor, and cancel common factors.

$$\lim_{h\to 0}\left(\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\right)=\lim_{h\to 0}\left(\frac{x^2-(x+h)^2}{hx^2(x+h)^2}\right)=\lim_{h\to 0}\left(\frac{-h(2x+h)}{hx^2(x+h)^2}\right)=\lim_{h\to 0}\left(\frac{-(2x+h)}{x^2(x+h)^2}\right)=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

(b) Combine into a single fraction, factor, and cancel common factors.

$$\lim_{x\to 3}\left(\frac{4}{x-3}-\frac{8}{x^2-4x+3}\right)=\lim_{x\to 3}\left(\frac{4(x-1)-8}{(x-3)(x-1)}\right)=\lim_{x\to 3}\left(\frac{4(x-3)}{(x-3)(x-1)}\right)=\lim_{x\to 3}\left(\frac{4}{x-1}\right)=\frac{4}{3-1}=2$$

(c) Rearrange the terms, and use the special limit $\lim_{\theta\to 0}\left(\frac{\sin(a\theta)}{a\theta}\right)=1$ several times.

$$\lim_{x\to 0}\left(\frac{\sin(7x)^2\cos(9x)}{\tan(3x)\sin(4x)}\right)=\lim_{x\to 0}\left(\frac{\sin(7x)}{7x}\cdot\frac{\sin(7x)}{7x}\cdot\frac{3x}{\sin(3x)}\cdot\frac{4x}{\sin(4x)}\cdot\frac{7x\cdot 7x}{3x\cdot 4x}\cdot\cos(9x)\cos(3x)\right)=\frac{49}{12}$$

Ex. C-37 2.3 ⋆Challenge

Calculate the following limit or determine that it does not exist.

$$\lim_{x\to a}\left(\frac{\cos\left(\frac{\pi a}{2x}\right)}{x-a}\right)$$

challenge

Solution C-37

This exercise is marked as "Challenge" because the limit is challenging to compute with only the techniques from §2.3 and §3.5. Later in the course we will learn l'Hospital's Rule, which makes this limit much easier.

Our first goal is to write the expression inside the limit symbol as an equivalent expression for which we can use the

C-37

special limit

$$\lim_{\theta \to 0} \left(\frac{\sin(\theta)}{\theta} \right) = 1$$

First we use the identity $\cos(\alpha) = \sin\left(\frac{\pi}{2} - \alpha\right)$.

$$\frac{\cos\left(\frac{\pi a}{2x}\right)}{x-a} = \frac{\sin\left(\frac{\pi}{2} - \frac{\pi a}{2x}\right)}{x-a}$$

Next we change our variable from x to θ, defined by:

$$\theta = \frac{\pi}{2} - \frac{\pi a}{2x}$$

To change variable, we have to change both the limit symbol and the expression for which we are computing the limit. Given our definition of θ, some algebra shows that

$$x = \frac{2a\theta}{\pi - 2\theta}$$

Second, observe that if $x \to a$, then $\theta \to 0$. So altogether we have the following.

$$\lim_{x \to a} \left(\frac{\cos\left(\frac{\pi a}{2x}\right)}{x-a} \right) = \lim_{x \to a} \left(\frac{\sin\left(\frac{\pi}{2} - \frac{\pi a}{2x}\right)}{x-a} \right) = \lim_{\theta \to 0} \left(\frac{\sin(\theta)}{\frac{2a\theta}{\pi - 2\theta}} \right) = \lim_{\theta \to 0} \left(\frac{\pi - 2\theta}{2a} \cdot \frac{\sin(\theta)}{\theta} \right)$$

Now use the special limit ($\lim_{\theta \to 0} \left(\frac{\sin(\theta)}{\theta} \right) = 1$).

$$\lim_{x \to a} \frac{\cos\left(\frac{\pi a}{2x}\right)}{x-a} = \lim_{\theta \to 0} \left(\frac{\pi - 2\theta}{2a} \cdot \frac{\sin(\theta)}{\theta} \right) = \frac{\pi - 0}{2a} \cdot 1 = \frac{\pi}{2a}$$

§2.4: Infinite Limits

Ex. D-1 2.4 Sp20 Exam

Consider the following function, where a and b are unspecified constants.

$$f(x) = \frac{x^2 + ax + b}{x - 2}$$

Is the line $x = 2$ necessarily a vertical asymptote of $f(x)$? Explain your answer. *Your answer may contain either English, mathematical symbols, or both.*

For example, let $a = 0$ and $b = -4$. Then $f(x) = x + 2$ for $x \neq 2$, and so f has no vertical asymptote at $x = 2$.

exs20

Solution D-1

No. If $x - 2$ is also a factor of the numerator $x^2 + ax + b$ (i.e., if substitution of $x = 2$ into the numerator gives 0), then the limit $\lim_{x\to 2} f(x)$ would not be infinite, and so $x = 2$ would not be a vertical asymptote.

Ex. D-2 2.4, 4.7 Sp20 Exam

Which of the following limits are equal to $+\infty$? Select all that apply.

(a) $\displaystyle\lim_{x\to 5^-}\left(\frac{x^2+25}{5-x}\right)$

(b) $\displaystyle\lim_{x\to 5^+}\left(\frac{x^2+25}{5-x}\right)$

(c) $\displaystyle\lim_{x\to -3^-}\left(\frac{x^3}{|x+3|}\right)$

(d) $\displaystyle\lim_{x\to 0^-}\left(\frac{x^4-2x-5}{\sin(x)}\right)$

(e) $\displaystyle\lim_{x\to 1^+}\left(\frac{x^6-x^2}{x-1}\right)$

exs20

Solution D-2

Direct substitution of each x-value gives $\frac{\text{non-zero \#}}{0}$ only for (a)–(d). A sign analysis of numerator and denominator then shows that only (a) and (d) are equal to $+\infty$. As for (e), we apply L'Hospital's Rule and find

$$\lim_{x\to 1^+}\left(\frac{x^6-x^2}{x-1}\right) \overset{H}{=} \lim_{x\to 1^+}\left(\frac{6x^5-2x}{1}\right) = 4$$

Hence only (a) and (d) are correct choices.

Ex. D-3 2.4 Sp20 Exam

Consider the function below.

$$f(x) = \frac{x^3 + 2x^2 - 13x + 10}{x^2 - 1}$$

Show that $x = -1$ is a vertical asymptote of f, but $x = 1$ is *not* a vertical asymptote of f.

exs20

Solution D-3

For $x = -1$, direct substitution gives the form "$\frac{24}{0}$", i.e., a nonzero divided by 0. Hence both one-sided limits of f at $x = -1$ are infinite, and so $x = -1$ is a vertical asymptote.

For $x = 1$, direct substitution gives the indeterminate form $\frac{0}{0}$, which *may* indicate a vertical asymptote but not necessarily. So we use L'Hospital's Rule.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{x^3+2x^2-13x+10}{x^2-1}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{3x^2+4x-13}{2x}\right) = \frac{-6}{2} = -3$$

Since this limit is not infinite, $x = 1$ is not a vertical asymptote.

Ex. D-4 2.4 Sp20 Exam

Determine which of the following limits are equal to $-\infty$. Select all that apply.

exs20

2.4 Sp20 Exam

(a) $\lim\limits_{x\to 6^-}\left(\dfrac{x^2-5x-6}{x-6}\right)$

(b) $\lim\limits_{x\to 6^-}\left(\dfrac{x^2-5x-6}{x^2-12x+36}\right)$

(c) $\lim\limits_{x\to \infty}\left(\dfrac{x^2-5x-6}{x-6}\right)$

(d) $\lim\limits_{x\to \infty}\left(\dfrac{x^2-5x-6}{x^2-12x+36}\right)$

exs20

Solution D-4

Choice (b) only.

(a) Direct substitution gives $\frac{0}{0}$, so use L'Hospital's Rule.

$$\lim_{x\to 6^-}\left(\frac{x^2-5x-6}{x-6}\right) \overset{H}{=} \lim_{x\to 6^-}\left(\frac{2x-5}{1}\right) = 7$$

(b) Direct substitution gives $\frac{0}{0}$, so use L'Hospital's Rule.

$$\lim_{x\to 6^-}\left(\frac{x^2-5x-6}{x^2-12x+36}\right) \overset{H}{=} \lim_{x\to 6^-}\left(\frac{2x-5}{2x-12}\right) = \frac{7}{0^-} = -\infty$$

(c) Factor out the highest power of numerator and denominator.

$$\lim_{x\to\infty}\left(\frac{x^2-5x-6}{x-6}\right) = \lim_{x\to\infty}\left(\frac{x^2}{x}\cdot\frac{1-\frac{5}{x}-\frac{6}{x^2}}{1-\frac{6}{x}}\right) = \lim_{x\to\infty}\left(x\cdot\frac{1-\frac{5}{x}-\frac{6}{x^2}}{1-\frac{6}{x}}\right) = \infty$$

(d) Factor out the highest power of numerator and denominator.

$$\lim_{x\to\infty}\left(\frac{x^2-5x-6}{x^2-12x+36}\right) = \lim_{x\to\infty}\left(\frac{1-\frac{5}{x}-\frac{6}{x^2}}{1-\frac{12}{x}+\frac{36}{x^2}}\right) = \frac{1-0-0}{1-0+0} = 1$$

Ex. D-5 2.4 Su20 Exam

Let $h(x) = \dfrac{f(x)}{g(x)}$, where f and g are continuous and $\lim\limits_{x\to a} g(x) = 0$. Is the following true or false?

"The line $x = a$ is necessarily a vertical asymptote of $h(x)$."

You must justify your answer. This means that if your answer is "true", you should explain why the above statement is always true. If your answer is "false", you should give an example to show that the above statement is sometimes false.

exsu20

Solution D-5

False. The issue here is that if $\lim\limits_{x\to a} f(x) = 0$ also, then h may or may not have a vertical asymptote at $x = a$.

For an explicit example, let $f(x) = g(x) = x$. Then f and g are continuous for all x and $\lim\limits_{x\to 0} g(x) = 0$, but $\dfrac{f(x)}{g(x)}$ does not have a vertical asymptote at $x = 0$ since $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = 1$.

Ex. D-6 2.4 Su20 Exam

Suppose that as x increases to 1, the values of $f(x)$ get larger and larger, and the values stay positive. Is the following true or false?

"Therefore, $\lim\limits_{x\to 1^-} f(x) = +\infty$."

You must justify your answer. This means that if your answer is "true", you should explain why the above statement is always true. If your answer is "false", you should give an example to show that the above statement is sometimes false.

exsu20

Solution **D-6**

False. The issue here is that the phrase "larger and larger" does not imply "arbitarily large", which is the more accurate description of what it means for a limit to be infinite.

For an explicit example, let $f(x) = x$. Then the values of $f(x)$ get larger and larger (i.e., increase) as x increases to 1. But $\lim_{x\to 1^-} f(x) = 1$.

Ex. D-7 **2.4, 2.6** Su20 **Exam**

Let $f(x) = \dfrac{9x - x^3}{x^2 + x - 6}$.

(a) Calculate all vertical asymptotes of f. Justify your answer.

(b) Where is f discontinuous?

(c) For each point at which f is discontinuous, determine what value should be reassigned to f, if possible, to guarantee that f will be continuous there.

exsu20

Solution **D-7**

(a) Putting the denominator to 0 gives $x^2 + x - 6 = 0$, with solutions $x = -3$ or $x = 2$. Direct substitution of $x = 2$ into f gives the (undefined) expression "$\frac{10}{0}$" (i.e., a non-zero number divided by zero). Hence $x = 2$ is a vertical asymptote. However, for $x = -3$, we observe the following.

$$\lim_{x\to -3}\left(\frac{9x - x^3}{x^2 + x - 6}\right) = \lim_{x\to -3}\left(\frac{x(3-x)(3+x)}{(x-2)(x+3)}\right) = \lim_{x\to -3}\left(\frac{x(3-x)}{x-2}\right) = \frac{18}{5}$$

Since this limit is not infinite, the line $x = -3$ is not a vertical asymptote. The only vertical asymptote is $x = 2$.

(b) Since f is a ratio two continuous functions, f is discontinuous only where its denominator is 0. Hence f is discontinuous only at $x = 2$ and $x = -3$.

(c) From our work in part (a), we know that $x = 2$ is a vertical asymptote. Thus it is impossible to redefine $f(2)$ to make f continuous at $x = 2$. (Why? The limit $\lim_{x\to 2} f(x)$ does not exist.)

However, for $x = -3$, we have $\lim_{x\to -3} f(x) = \dfrac{18}{5}$. Hence if we redefine $f(-3)$ to be $\dfrac{18}{5}$, then f becomes continuous at $x = -3$.

Ex. D-8 **2.4, 2.5** Su20 **Exam**

Let $f(x) = \dfrac{3 + 7e^{2x}}{1 - e^x}$. Calculate each of the following limits.

(a) $\lim_{x\to -\infty} f(x)$ (b) $\lim_{x\to +\infty} f(x)$ (c) $\lim_{x\to 0^-} f(x)$

exsu20

Solution **D-8**

(a) We recall that $\lim_{x\to -\infty}(e^x) = 0$, whence $\lim_{x\to -\infty}(e^{2x}) = 0$ also since $e^{2x} = (e^x)^2$. So we immediately have:

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\left(\frac{3 + 7e^{2x}}{1 - e^x}\right) = \frac{3 + 7\cdot 0}{1 - 0} = 3$$

(b) We recall that $\lim_{x\to +\infty}(e^x) = +\infty$, whence $\lim_{x\to +\infty}(e^{2x}) = +\infty$ also since $e^{2x} = (e^x)^2$. This would give the indeterminate form "$\dfrac{\infty}{-\infty}$" in our limit, so we instead factor out the "highest power" (or dominant term) as $x \to +\infty$ of the numerator and denominator separately. For the numerator, the dominant term is e^{2x}. For the denominator, the dominant term is e^x. So now we have:

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty}\left(\frac{e^{2x}}{e^x}\cdot\frac{3e^{-2x} + 7}{e^{-x} - 1}\right) = \lim_{x\to +\infty}\left(e^x\cdot\frac{3e^{-2x} + 7}{e^{-x} - 1}\right)$$

D-8

Now we recall that $\lim_{x\to+\infty}(e^{-x})=0$, whence $\lim_{x\to+\infty}(e^{-2x})=0$ also since $e^{2x}=(e^x)^2$. So our limit is:

$$\lim_{x\to+\infty}\left(e^x\cdot\frac{3e^{-2x}+7}{e^{-x}-1}\right)=\lim_{x\to+\infty}(e^x)\cdot\lim_{x\to+\infty}\left(\frac{3e^{-2x}+7}{e^{-x}-1}\right)=(+\infty)\cdot\frac{0+7}{0-1}=-\infty$$

(c) Direct substitution of $x=0$ into $f(x)$ gives the (undefined) expression "$\frac{10}{0}$", which means that both one-sided limits at $x=0$ are infinite. So we perform a sign analysis to determine whether the limit is positive or negative infinity.

As $x\to0^-$ the numerator $(3+7e^{2x})\to10$, which is positive. For the denominator, however, we note that e^x is an increasing function for all x. Hence $1=e^0>e^x$ (or $1-e^x>0$) for all $x<0$. (We can deduce this from a simple graph of $y=e^x$. Alternatively, a test point shows that $1-e^x>0$ for all x sufficiently close to and less than 0.) Hence the denominator is positive as $x\to0^-$. Putting this altogether gives the following:

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\left(\frac{3+7e^{2x}}{1-e^x}\right)=\frac{10}{0^+}=+\infty$$

Ex. D-9 2.4 Fa20 Exam

Consider the function $f(x)=\dfrac{(ax-6)(x+1)}{x-2}$, where a is an unspecified constant.

(a) For which value(s) of a does f have a vertical asymptote? What is the equation of this vertical asymptote?

(b) For which value(s) of a does f have a horizontal asymptote? What is the equation of this horizontal asymptote?

exf20

Solution D-9

(a) *The function f has a vertical asymptote if and only if $a\neq3$. The vertical asymptote is $x=2$. Proof below.*

The function f has a vertical asymptote at $x=2$ (where denominator is 0), as long as the denominator is not also a factor of the numerator. (Recall that if this happens, then the common factors would cancel and we would have a removable discontinuity, not a vertical asymptote.) Hence the numerator of f must be nonzero if we substitute $x=2$.

$$(2a-6)(2+1)\neq0\Longrightarrow a\neq3$$

So f has a vertical asymptote at $x=2$ if and only if $a\neq3$.

(b) *The function f has a horizontal asymptote if and only if $a=0$. The horizontal asymptote is $y=-6$. Proof below.*

If $a\neq0$, we have the following:

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\left(\frac{x^2}{x}\cdot\frac{\left(a-\frac{6}{x}\right)\left(1+\frac{1}{x}\right)}{1-\frac{2}{x}}\right)=\lim_{x\to\pm\infty}(x)\cdot\frac{(a-0)(1+0)}{1-0}=\pm\infty\cdot a=\pm\infty$$

(or the signs are reversed if $a<0$). So there is no horizontal asymptote if $a\neq0$. Also note that if $a\neq0$, the numerator of f has degree 2 and the denominator of f has degree 1. From precalculus, you may have learned that this implies f has no horizontal asymptote.

However, if $a=0$, then we have

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\left(\frac{-6(x+1)}{x-2}\right)=-6$$

So there is a horizontal asymptote at $y=-6$. (Note that in the case $a=0$, the numerator and denominator both have degree 1, whence there must be a horizontal asymptote.)

Ex. D-10 2.4 Fa20 Exam

For which value(s) of n, if any, is the following statement true: $\lim_{x\to2^-}(2-x)^n=+\infty$? Explain your answer.

exf20

Solution **D-10**

The statement is true if and only if $n > 0$.

If $n > 0$, then $\lim_{x\to 2^-}(2-x)^n = 0$ by direct substitution property. If $n = 0$, then $\lim_{x\to 2^-}(2-x)^n = 1$ since $(2-x)^0 = 1$ for any $x \neq 2$. If $n < 0$, then $n = -m$ for some positive m. So we can equivalently examine the limit:

$$\lim_{x\to 2^-}\left(\frac{1}{(2-x)^m}\right)$$

If $x \to 2^-$, then this means x is close to 2 and $x < 2$, whence $(2-x)^m$ has limit 0 as $x \to 2^-$ but remains positive. Hence the limit above is $+\infty$.

Ex. D-11 **2.4** Sp21 **Exam**

Determine whether the following statement is true or false. Explain your answer in 1 or 2 sentences. Your answer should contain English with few mathematical symbols.

> "Suppose f and g are functions with $g(3) = 1$. Put $H(x) = \dfrac{f(x)}{g(x)-1}$. Then H must have a vertical asymptote at $x = 3$."

exs21

Solution **D-11**

False. Let $f(x) = x-3$ and $g(x) = x-2$. Then $g(3) = 1$ but $H(x) = \frac{f(x)}{g(x)-1} = \frac{x-3}{x-3}$ does not have a vertical asymptote at $x = 3$ since $\lim_{x\to 3} H(x) = 1$ (i.e., the limit exists and is finite).

Other acceptable explanations:

- "Since the limit of f and g (and hence the limit of H) as $x \to 3$ does not depend on the function values $f(3)$ and $g(3)$, we cannot say for sure whether H has a vertical asymptote at $x = 3$. There is not enough information."
- "If $f(3) = 0$, then direct substitution of $x = 3$ into H gives the indeterminate form $\frac{0}{0}$, which does not necessarily indicate a vertical asymptote. There may be some algebraic cancellation that allows the limit $\lim_{x\to 3} H(x)$ to exist."

Ex. D-12 **2.4** Sp21 **Exam**

Let $f(x) = \dfrac{(x+a)(x-3)}{(x-2)(x+1)}$, where a is an unspecified, **positive** constant. For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 2^-} f(x)$ (c) $\lim_{x\to 2^+} f(x)$ (d) $\lim_{x\to 2} f(x)$

exs21

Solution **D-12**

(a) Use direct substitution.

$$\lim_{x\to 0} f(x) = \frac{(0+a)(0-3)}{(0-2)(0+1)} = \frac{3a}{2}$$

(b) Substitution of $x = 2$ gives "$\frac{-(2+a)}{0}$". Since $a > 0$, this expression is "$\frac{\text{nonzero}}{0}$", which means $x = 2$ is a vertical asymptote of f. So we must perform a sign analysis.

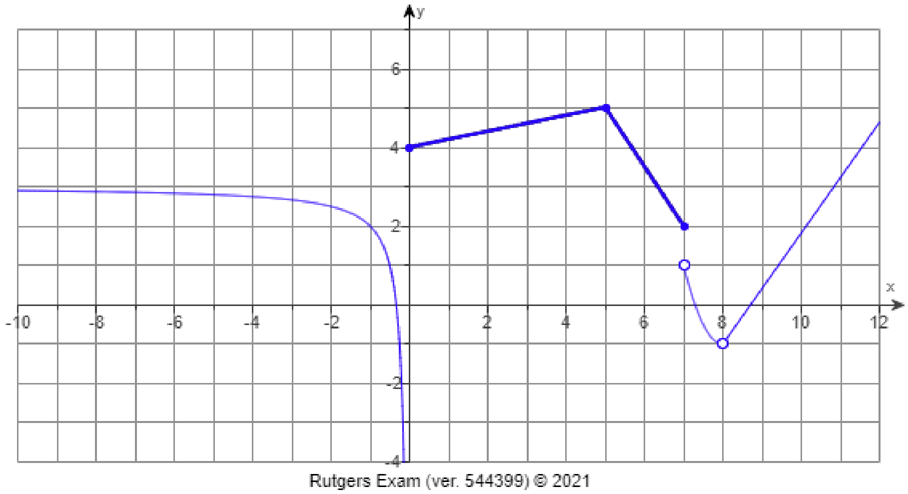
We have $-(2+a) < 0$, and so the numerator is negative as $x \to 2$. For the denominator, we note that since $x \to 2^-$ (i.e., $x < 2$), we have $x+1 > 0$ and $x-2 < 0$. Hence the entire expression for $f(x)$ is positive as $x \to 2^-$. Hence $\lim_{x\to 2^-} f(x) = \infty$.

(c) As in part (c), we perform a sign analysis. However, since $x \to 2^+$, we have $x - 2 > 0$ now. Hence $\lim_{x\to 2^-} f(x) = -\infty$.

(d) The limits in parts (b) and (c) are not equal, so $\lim_{x\to 2} f(x)$ does not exist.

Ex. B-6 2.1/2.2, 2.3, 2.4, 2.5 Fa21 Exam

For each part, use the graph of $y = f(x)$.



(a) List the x-values where f is not continuous or determine that f is continuous for all x.

(b) List all vertical asymptotes of f.

(c) List all horizontal asymptotes of f.

(d) Calculate $\lim_{x\to 8} f(x)$ or determine that the limit does not exist.

(e) At $x = 7$, which of the one-sided limits of f exist?

exf21

Solution B-6

(a) $x = 0, 7, 8$ only

(b) $x = 0$ only

(c) $y = 3$ only

(d) $\lim_{x\to 8} f(x) = -1$

(e) Both the left- and right-limits of $f(x)$ at $x = 7$ exist.

Ex. D-13 2.4, 2.5, 4.7 Fa21 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 1} \left(\dfrac{x^4 - x}{\ln(77x - 76)} \right)$

(b) $\lim_{x\to -\infty} \left(\dfrac{\sqrt{36x^2 + 63}}{31x} \right)$

(c) $\lim_{x\to 2^+} f(x)$, with $f(x) = \begin{cases} 1 + 4x & x \le 2 \\ \dfrac{x^2 - 4}{x - 2} & x > 2 \end{cases}$

(d) $\lim_{x\to 5^-} \left(\dfrac{\cos(\pi x)}{x^2 - 25} \right)$

exf21

Solution D-13

(a) Direct substitution gives "$\frac{0}{0}$", and so we use L'Hospital's Rule.

$$\lim_{x\to 1} \left(\frac{x^4 - x}{\ln(77x - 76)} \right) \overset{H}{=} \lim_{x\to 1} \left(\frac{4x^3 - 1}{\frac{1}{77x-76} \cdot 77} \right) = \frac{3}{77}$$

(b) We factor out x^2 from inside the square root in the numerator. Observe that since x goes to *negative* infinity, we have $\sqrt{x^2} = |x| = -x$.

$$\lim_{x\to -\infty} \left(\frac{\sqrt{36x^2 + 63}}{31x} \right) = \lim_{x\to -\infty} \left(\frac{-x\sqrt{36 + \frac{63}{x^2}}}{31x} \right) = \lim_{x\to -\infty} \left(\frac{-\sqrt{36 + \frac{63}{x^2}}}{31} \right) = \frac{-6}{31}$$

D-13

(c) We factor and cancel.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^+}\left(\frac{(x-2)(x+2)}{x-2}\right) = \lim_{x\to 2^+}(x+2) = 4$$

(d) Direct substitution gives "$\frac{-1}{0}$", whence the one-sided limit must be infinite. Observe that the numerator is negative (goes to -1) as $x \to 5^-$, and the denominator goes to 0 but remains negative as $x \to 5^-$. (For instance, use test points such as $x = 4.99$.) Hence the desired limit is $\frac{-1}{0^-} = +\infty$.

Ex. D-14 2.4, 4.7 Fa21 Exam

For each part, find all vertical asymptotes of the given function.

(a) $f(x) = \dfrac{x^2-8x+15}{x^2-9}$

(b) $g(x) = \dfrac{e^{x+3}-1}{x^2-9}$

exf21

Solution D-14

(a) First factor and cancel.

$$f(x) = \frac{x^2-8x+15}{x^2-9} = \frac{(x-3)(x-5)}{(x-3)(x+3)} = \frac{x-5}{x+3}$$

Hence $f(x)$ has a vertical asymptote at $x = -3$ only.

(b) We note that the denominator of $g(x)$ equals 0 only when $x = -3$ or $x = 3$. Direct substitution of $x = 3$ gives the expression "$\frac{e^6-1}{0}$" (nonzero number divided by 0), and so $x = 3$ is a vertical asymptote of $g(x)$. However, we have the following for $x = -3$ after using L'Hospital's Rule:

$$\lim_{x\to -3} g(x) = \lim_{x\to -3}\left(\frac{e^{x+3}-1}{x^2-9}\right) \overset{H}{=} \lim_{x\to -3}\left(\frac{e^{x+3}}{2x}\right) = -\frac{1}{6}$$

Since this limit is not infinite, there is no vertical asymptote at $x = -3$.

Ex. C-23 2.3, 2.4 Sp22 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to 3}\left(\frac{x-3}{10-\sqrt{x+97}}\right)$

(b) $\displaystyle\lim_{x\to 6}\left(\frac{36-x^2}{\frac{1}{x}-\frac{1}{6}}\right)$

(c) $\displaystyle\lim_{x\to 0}\left(\frac{x^2\csc(3x)}{\cos(7x)\sin(4x)}\right)$

(d) $\displaystyle\lim_{x\to 2^-}\left(\frac{6x^2-7x}{x^2-4}\right)$

exs22

Solution C-23

(a) Rationalize the denominator, cancel common factors, and use direct substitution.

$$\lim_{x\to 3}\left(\frac{x-3}{10-\sqrt{x+97}}\right) = \lim_{x\to 3}\left(\frac{x-3}{10-\sqrt{x+97}}\cdot\frac{10+\sqrt{x+97}}{10+\sqrt{x+97}}\right) = \lim_{x\to 3}\left(\frac{(x-3)(10+\sqrt{x+97})}{100-(x+97)}\right)$$
$$= \lim_{x\to 3}\left(\frac{(x-3)(10+\sqrt{x+97})}{-(x-3)}\right) = \lim_{x\to 3}\left(10+\sqrt{x+97}\right) = 10+\sqrt{100} = 20$$

(b) Cancel common factors and use direct substitution.

$$\lim_{x\to 6}\left(\frac{36-x^2}{\frac{1}{x}-\frac{1}{6}}\right) = \lim_{x\to 6}\left(\frac{6x(36-x^2)}{6-x}\right) = \lim_{x\to 6}\left(\frac{6x(6-x)(6+x)}{6-x}\right) = \lim_{x\to 6}(6x(6+x)) = 432$$

(c) Write in terms of sine and cosine, regroup terms, and use the special trigonometric limits.

$$\lim_{x\to 0}\left(\frac{x^2\csc(3x)}{\cos(7x)\sin(4x)}\right) = \lim_{x\to 0}\left(\frac{3x}{\sin(3x)}\cdot\frac{4x}{\sin(4x)}\cdot\frac{1}{12\cos(7x)}\right) = 1\cdot 1\cdot\frac{1}{12\cdot 1} = \frac{1}{12}$$

C-23

(d) Direct substitution of $x = 2$ gives the undefined expression "$\frac{10}{0}$". Since this is a nonzero number divided by zero, we know the one-sided limit is infinite, and so all we must do is sign analysis to determine the sign of the infinity. As $x \to 2$, the numerator approaches 10, so the numerator is positive. The denominator factors as $(x-2)(x+2)$. The second factor $(x+2)$ goes to 4 (and is thus positive) as $x \to 2$. The first factor $(x-2)$ goes to 0 but remains negative as $x \to 2^-$.

Putting this altogether, the expression inside the limit has a negative value ($\frac{\oplus}{\ominus\oplus} = \ominus$) as $x \to 2^-$. So the desired limit is $-\infty$.

Ex. D-15 **2.4, 2.5** Sp22 **Exam**

For the function f below, find its domain and all vertical and horizontal asymptotes.

$$f(x) = \frac{x^2 - 8x + 12}{3x^2 - 8x + 4}$$

exs22

Solution **D-15**

Since f is a rational function, its domain is all real numbers except where the denominator vanishes. Observe that $(3x^2 - 8x + 4) = (3x-2)(x-2)$, hence the denominator vanishes at $x = \frac{2}{3}$ and $x = 2$. The domain of f is $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 2) \cup (2, \infty)$.

Since f is continuous on its domain, vertical asymptotes can occur only at either $x = \frac{2}{3}$ or $x = 2$. Observe that direct substitution of $x = \frac{2}{3}$ into $f(x)$ gives an expression of "$\frac{\text{nonzero }\#}{0}$". Hence the one-sided limits of f at $x = \frac{2}{3}$ must each be infinite, and so $x = \frac{2}{3}$ is a vertical asymptote of f.

For $x = 2$, however, we have the following:

$$\lim_{x\to 2}\left(\frac{x^2 - 8x + 12}{3x^2 - 8x + 4}\right) = \lim_{x\to 2}\left(\frac{(x-2)(x-6)}{(3x-2)(x-2)}\right) = \lim_{x\to 2}\left(\frac{x-6}{3x-2}\right) = \frac{2-6}{6-2} = -1$$

Since this limit is finite, we conclude $x = 2$ is not a vertical asymptote of f.

For the horizontal asymptotes, we must compute the limits at infinity.

$$\lim_{x\to\pm\infty}\left(\frac{x^2 - 8x + 12}{3x^2 - 8x + 4}\right) = \lim_{x\to\pm\infty}\left(\frac{1 - \frac{8}{x} + \frac{12}{x^2}}{3 - \frac{8}{x} + \frac{4}{x^2}}\right) = \frac{1-0+0}{3-0+0} = \frac{1}{3}$$

So the only horizontal asymptote of f is the line $y = \frac{1}{3}$.

Ex. D-16 **2.4, 2.5** Su22 **Exam**

Consider the function $f(x) = \dfrac{x^3 - 3x + 1}{x^2 - 2x + 1}$.

(a) Find all horizontal asymptotes of f, if any.

(b) Find all vertical asymptotes of f. Then calculate $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$, where $x = a$ is the rightmost vertical asymptote of f.

exsu22

Solution **D-16**

(a) We compute the limits of f at infinity. To this end, we factor the highest powers of numerator and denominator separately.

$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2}\cdot\frac{1 - \frac{3}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x} + \frac{1}{x^2}}\right) = \lim_{x\to\pm\infty}\left(x\cdot\frac{1 - \frac{3}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x} + \frac{1}{x^2}}\right) = (\pm\infty)\cdot\frac{1-0+0}{1-0+0} = \pm\infty$$

These limits are not finite. Thus f has no horizontal asymptote.

(b) Since f is a rational function, vertical asymptotes can occur only where the denominator is 0. The only solution to $x^2 - 2x + 1 = (x-1)^2 = 0$ is $x = 1$. Substitution of $x = 1$ into f gives the undefined expression "$\frac{-1}{0} = \frac{\text{nonzero }\#}{0}$", whence $x = 1$ is, indeed, a vertical asymptote for f.

D-16

Now we compute the left- and right-limits using sign analysis.

$$\lim_{x\to 1^-}\left(\frac{x^3-3x+1}{(x-1)^2}\right)=\frac{-1}{0^+}=-\infty$$
$$\lim_{x\to 1^+}\left(\frac{x^3-3x+1}{(x-1)^2}\right)=\frac{-1}{0^+}=-\infty$$

(For this function, the analysis was simplified since the denominator is the perfect square $(x-1)^2$ and thus never negative.)

Ex. D-17 **2.4** Fa22 **Exam**

Find all vertical asymptotes of the function $f(x)=\dfrac{x^3-36x}{x^3-12x^2+36x}$.

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf22

Solution D-17

Since $f(x)$ is a rational function, VA's can occur only where the denominator of $f(x)$ vanishes.

$$x^3-12x^2+36x=0\Longrightarrow x(x^2-12x+36)=x(x-6)^2=0$$

Thus $f(x)$ can have a VA at $x=0$ or $x=6$ only.

For $x=0$, we note the following:

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\left(\frac{x^3-36x}{x^3-12x^2+36x}\right)=\lim_{x\to 0}\left(\frac{x(x-6)(x+6)}{x(x-6)^2}\right)=\lim_{xto0}\left(\frac{x+6}{x-6}\right)=\frac{0+6}{0-6}=-1$$

Since this limit is finite, we find that the line $x=0$ is not a VA for $f(x)$.

For $x=6$, we note the following:

$$\lim_{x\to 6}f(x)=\lim_{x\to 6}\left(\frac{x+6}{x-6}\right)$$

At this point, direct substitution of $x=6$ gives the expression "$\frac{12}{0}$" (i.e., a nonzero number divided by 0). This immediately implies that each corresponding one-sided limit is infinite. Thus the line $x=6$ is a VA for $f(x)$.

Ex. D-18 **2.4, 2.5** Su22 **Quiz**

Calculate all of the vertical and horizontal asymptotes of $f(x)=\dfrac{x^2-100}{10x-x^2}$.

Then find the two one-sided at $x=a$, where $x=a$ is the leftmost vertical asymptote of f.

qsu22

Solution D-18

First we calculate the horizontal asymptotes.

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\left(\frac{x^2\left(1-\frac{100}{x^2}\right)}{x^2\left(\frac{10}{x}-1\right)}\right)=\lim_{x\to\pm\infty}\left(\frac{1-\frac{100}{x^2}}{\frac{10}{x}-1}\right)=\frac{1-0}{0-1}=-1$$

Hence the only horizontal asymptote of f is the line $y=-1$.

For vertical asymptotes, we see that $10x-x^2=0$ has solutions $x=0$ and $x=10$. However, we have:

$$\lim_{x\to 10}f(x)=\lim_{x\to 10}\left(\frac{(x-10)(x+10)}{-x(x-10)}\right)=\lim_{x\to 10}\left(\frac{x+10}{-x}\right)=-2$$

Since this limit is finite, we see that $x=10$ is not a vertical asymptote. However, we also see that $x=0$ is, indeed, a vertical asymptote since direct substitution of $x=0$ into $\frac{x+10}{-x}$ gives "$\frac{\text{nonzero }\#}{0}$". This also means each of the one-sided limits at $x=0$ is infinite.

We now compute the corresponding one-sided limits. Note that as $x\to 0$, the expression $x+10$ is positive (tends to

D-18

10). However, the expression $-x$ stays positive as $x \to 0^-$ and negative as $x \to 0^+$. Hence we have:

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\left(\frac{x+10}{-x}\right) = \frac{10}{0^+} = +\infty$$
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\left(\frac{x+10}{-x}\right) = \frac{10}{0^-} = -\infty$$

Ex. D-19 2.4 Fa22 Quiz

Find all vertical asymptotes of $f(x)$. You must justify your answers precisely.

$$f(x) = \frac{\sin(2x)}{x^2 - 10x}$$

qf22

Solution D-19

We put the denominator to 0.

$$x^2 - 10x = x(x-10) = 0 \implies x = 0 \text{ or } x = 10$$

Thus $x = 0$ and $x = 10$ are our candidate VA's. For $x = 10$, we note that direct substitution into f gives "$\frac{\sin(20)}{0}$" (i.e., a nonzero number divided by 0). Thus $x = 10$ is a VA of f. However, for $x = 0$ we have:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin(2x)}{2x}\cdot\frac{2}{x-10}\right) = 1\cdot\frac{2}{0-10} = -\frac{1}{5}$$

We have used the special limit $\lim_{\theta\to 0}\left(\frac{\sin(a\theta)}{a\theta}\right) = 1$. Thus $x = 0$ is not a VA of f.

Ex. D-20 2.4

For each part, calculate the limit or show that it does not exist.

(a) $\lim_{x\to 0^+}\left(\frac{x^2-x+4}{2x+\sin(x)}\right)$ (b) $\lim_{x\to 3^-}\left(\frac{2x^2+8}{x^2-9}\right)$ (c) $\lim_{x\to 4^+}\left(\frac{|16-x^2|}{x-4}\right)$

c2s4

Solution D-20

(a) Substitution of $x = 0$ gives $\frac{4}{0}$, which indicates that the one-sided limit is infinite. Now we do sign analysis to determine the sign of infinity. If $x \to 0^+$, we may assume x is a small positive number. In that case, both terms in the denominator (i.e., both $2x$ and $\sin(x)$) are also small positive numbers. Hence $2x + \sin(x)$ approaches 0 but remains positive. So we have

$$\lim_{x\to 0^+}\left(\frac{x^2-x+4}{2x+\sin(x)}\right) = \frac{4}{0^+} = +\infty$$

(b) Substitution of $x = 3$ gives $\frac{26}{0}$, which indicates that the one-sided limit is infinite. Now we do sign analysis to determine the sign of infinity. If $x \to 3^-$, we may assume x is slightly less than 3. In that case, the denominator (i.e., $x^2 - 9$) is a small negative number. Hence $x^2 - 9$ approaches 0 but remains negative. So we have

$$\lim_{x\to 3^-}\left(\frac{2x^2+8}{x^2-9}\right) = \frac{26}{0^-} = -\infty$$

(c) Substitution of $x = 4$ gives $\frac{0}{0}$, which does not necessarily indicate an infinite limit, but rather that there may be algebraic cancellation. Note that if $x \to 4^+$, then we may assume x is slightly greater than 4. This means x^2 is slightly greater than 16, so that $x^2 - 16 > 0$. Hence $|16 - x^2| = x^2 - 16$. So now we have

$$\lim_{x\to 4^+}\left(\frac{|16-x^2|}{x-4}\right) = \lim_{x\to 4+}\left(\frac{x^2-16}{x-4}\right) = \lim_{x\to 4^+}(x+4) = 4+4 = 8$$

Ex. D-21 **2.4**

For each part, find the vertical asymptotes of $f(x)$. Then find both corresponding one-sided limits at each vertical asymptote.

(a) $f(x) = \dfrac{(x-1)(2x+5)}{(x+1)(3x-6)}$

(b) $f(x) = \dfrac{x^2-18x+81}{x^2-81}$

(c) $f(x) = \dfrac{(x-4)\sin(x)}{x^3-8x^2+16x}$

(d) $f(x) = \ln(x)$

(e) $f(x) = \dfrac{2e^x+3}{1-e^x}$

(f) $f(x) = e^{-1/x}$

c2s4

Solution **D-21**

(a) Candidate vertical asymptotes occur at x-values where $(x+1)(3x-6) = 0$ Hence the candidate vertical asymptotes are the lines $x = -1$ and $x = 2$. Direct substitution of either $x = -1$ or $x = 2$ into $f(x)$ gives "nonzero number divided by 0", hence both $x = -1$ and $x = 2$ are vertical asymptotes. Now for the one-sided limits.

First we calculate the one-sided limits at $x = -1$. Substitution of $x = -1$ gives $\frac{(-2)(3)}{0}$, which indicates that both one-sided limits are infinite. So we perform a sign analysis on each factor in $f(x)$. Remember that factors that approach a non-zero number have a definite sign. But factors that approach 0 have a sign that is determined by whether the one-sided limit is from the left or the right. (So this means that $x+1$ can be negative or positive depending on whether the limit is from the left or the right.)

$$\lim_{x\to -1^-}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \frac{\ominus\oplus}{\ominus\ominus}\infty = -\infty$$
$$\lim_{x\to -1^+}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \frac{\ominus\oplus}{\oplus\ominus}\infty = \infty$$

Now we do the same with $x = 2$. Substitution of $x = 2$ gives $\frac{(1)(9)}{0}$, which again indicates the one-sided limits are infinite. So we perform a sign analysis on each factor.

$$\lim_{x\to 2^-}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \frac{\oplus\oplus}{\oplus\ominus}\infty = -\infty$$
$$\lim_{x\to 2^+}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \frac{\oplus\oplus}{\oplus\oplus}\infty = \infty$$

(b) Setting the denominator to 0, we see that the only candidate asymptotes are $x = -9$ and $x = 9$. Direct substitution of $x = -9$ gives "$\frac{18^2}{0}$" (nonzero number divided by 0), whence $x = -9$ is a vertical asymptote. Direct substitution of $x = 9$, however, gives "$\frac{0}{0}$", and so we need more analysis.

For $x = 9$, we have the following:

$$\lim_{x\to 9}\left(\frac{x^2-18x+81}{x^2-81}\right) = \lim_{x\to 9}\left(\frac{(x-9)^2}{(x-9)(x+9)}\right) = \lim_{x\to 9}\left(\frac{x-9}{x+9}\right) = \frac{0}{18} = 0$$

Since this limit is not infinite, we conclude that $x = 9$ is not a vertical asymptote.

For $x = -9$, we use the simplified form of f: $f(x) = \dfrac{x-9}{x+9}$. We already know that the one-sided limits are infinite. We now perform a sign analysis. Testing $x = -9.01$ (for the left limit) and $x = -8.99$ (for the right limit), we find the following:

$$\lim_{x\to -9^-} f(x) = \frac{-18}{0^-} = +\infty \quad , \quad \lim_{x\to -9^+} f(x) = \frac{-18}{0^+} = -\infty$$

(c) Setting the denominator to 0, we have $x^3 - 8x + 16x = x(x-4)^2 = 0$, and so the only candidate vertical asymptotes are $x = 0$ and $x = 4$. Direct substitution of either $x = 0$ or $x = 4$ gives "$\frac{0}{0}$", which means we need more analysis.

For $x = 0$ we have

$$\lim_{x\to 0}\left(\frac{(x-4)\sin(x)}{x^3-8x^2+16x}\right) = \lim_{x\to 0}\left(\frac{\sin(x)}{x(x-4)}\right) = \lim_{x\to 0}\left(\frac{\sin(x)}{x}\cdot\frac{1}{x-4}\right) = 1\cdot\frac{1}{0-4} = -\frac{1}{4}$$

D-21

Since this limit is not infinite, $x = 0$ is not a vertical asymptote.

For $x = 4$, we use the simplified form of f: $f(x) = \dfrac{\sin(x)}{x(x-4)}$. Direct substitution of $x = 4$ in the simplified form gives "nonzero number divided by 0", whence $x = 4$ is a vertical asymptote. Observe that $\pi < 4 < 2\pi$, which means that $\sin(4) < 0$. So now testing $x = 3.99$ and $x = 4.01$ for the left- and right-limits, respectively, we have the following.

$$\lim_{x\to 4^-} f(x) = \frac{\ominus}{(4)(0^-)} = +\infty \quad , \quad \lim_{x\to 4^+} f(x) = \frac{\ominus}{(4)(0^+)} = -\infty$$

(d) Since f has the DSP on its domain, candidate vertical asymptotes occur at x-values not in the domain of f or at the boundary of the domain of f. Since the domain of f is $(0, \infty)$, the only candidate vertical asymptote is $x = 0$. Now recall the basic property of $\ln(x)$ that $\lim\limits_{x\to 0^+} \ln(x) = -\infty$. (The left-sided limit makes no sense to consider since $\ln(x)$ is not defined for $x < 0$.) Hence $x = 0$ is a vertical asymptote.

(e) Candidate vertical asymptotes occur at x-values where $1 - e^x = 0$. Hence the only candidate vertical asymptote is the line $x = 0$. Substitution of $x = 0$ gives "$\frac{5}{0}$" (nonzero number divided by 0), whence $x = 0$ is a vertical asymptote. Now for the one-sided limits.

Note that if x is a small negative number (i.e., $x \to 0^-$), then e^x is slightly less than 1, and so $1 - e^x$ is slightly positive. Hence we have

$$\lim_{x\to 0^-} \left(\frac{2e^x + 3}{1 - e^x}\right) = \frac{5}{0^+} = +\infty$$

Simlarly, if x is a small positive number (i.e., $x \to 0^+$), then e^x is slightly greater than 1, and so $1 - e^x$ is slightly negative. Hence we have

$$\lim_{x\to 0^+} \left(\frac{2e^x + 3}{1 - e^x}\right) = \frac{5}{0^-} = -\infty$$

(f) Since f has the DSP on its domain, candidate vertical asymptotes occur at x-values not in the domain of f or at the boundary of the domain of f. Since the domain of f is $(-\infty, 0) \cup (0, \infty)$, the only candidate vertical asymptote is $x = 0$. Note that f is not a fraction, so substitution of $x = 0$ alone does not yet determine whether $x = 0$ is a vertical asymptote. So we look at the one-sided limits.

First we recall two basic one-sided limits.

$$\lim_{x\to 0^-} \left(\frac{-1}{x}\right) = \frac{-1}{0^-} = +\infty \quad , \quad \lim_{x\to 0^+} \left(\frac{-1}{x}\right) = \frac{-1}{0^+} = -\infty$$

Letting $u = -\frac{1}{x}$, we have the following.

$$\lim_{x\to 0^-} e^{-1/x} = \lim_{u\to\infty} e^u = \infty$$
$$\lim_{x\to 0^+} e^{-1/x} = \lim_{u\to -\infty} e^u = 0$$

Hence the line $x = 0$ is a vertical asymptote. (Note that the limit is infinite only one one side, but this is okay!)

Ex. D-22 **2.4**

Find all vertical asymptotes of $f(x) = \dfrac{x^2 + x - 2}{x^2 - 4x + 3}$. Then at each vertical asymptote, calculate the corresponding one-sided limits of $f(x)$.

unit2

Solution **D-22**

Since $f(x)$ is a rational function, it is continuous on its domain, and so a vertical asymptote can occur only where the denominator vanishes. Solving $x^2 - 4x + 3 = (x-1)(x-3) = 0$, we find that a vertical asymptote can occur only at $x = 1$ or $x = 3$. Now we check each of these x-values individually.

D-22

For $x = 1$, we observe:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{(x-1)(x+2)}{(x-1)(x-3)}\right) = \lim_{x\to 1}\left(\frac{x+2}{x-3}\right) = -\frac{3}{2}$$

Since this limit is not infinite, there is no vertical asymptote at $x = 1$.

For $x \neq 1$, we may write $f(x) = \frac{x+2}{x-3}$. For $x = 3$ in particular, we see that direct substitution gives the (undefined) expression "$\frac{5}{0}$" (or "$\frac{\text{nonzero}}{0}$"). Hence both the left- and right-limit at $x = 3$ are infinite, and so $x = 3$ is a true vertical asymptote.

As for those one-sided limits, we observe that if $x \to 3$, then $(x+2) \to 5$, and so the numerator of $f(x)$ remains positive. However, the denominator $(x-3)$ remains negative if $x \to 3^-$ and remains positive if $x \to 3^+$. In summary,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(\frac{x+2}{x-3}\right) = \frac{\oplus}{\ominus}\infty = -\infty$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}\left(\frac{x+2}{x-3}\right) = \frac{\oplus}{\oplus}\infty = +\infty$$

Ex. D-23 **2.4, 2.5** ⋆Challenge

For each function, find all horizontal asymptotes and vertical asymptotes. Then, at each vertical asymptote, calculate both one-sided limits.

(a) $f(x) = \dfrac{4x^3+4x^2-8x}{x^3+3x^2-4}$ (b) $f(x) = \dfrac{4x^3-\sqrt{x^6+17}}{5x^3-40}$

challenge

Solution D-23

(a) First we factor the denominator. Let $p(x) = x^3+3x^2-4$ and observe that $p(1) = 0$, whence $x-1$ is a factor of $p(x)$. Performing long division of polynomials then gives $p(x) = (x-1)(x^2+4x+4) = (x-1)(x+2)^2$. So for $x \neq 1$ and $x \neq -2$, we have:

$$f(x) = \frac{4x^3+4x^2-8x}{x^3+3x^2-4} = \frac{4x(x+2)(x-1)}{(x-1)(x+2)^2} = \frac{4x}{x+2}$$

Hence the only vertical asymptote of $f(x)$ is the line $x = -2$.

Precisely, we have that $\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{4x}{x+2}\right) = \frac{4}{3}$. Since this limit is finite, $x = 1$ is not a vertical asymptote of $f(x)$.

For the one-sided limits we have:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^-} = +\infty$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^+} = -\infty$$

As for the horizontal asymptotes, we have the following:

$$\lim_{x\to\pm\infty}\left(\frac{4x}{x+2}\right) = \lim_{x\to\pm\infty}\left(\frac{4}{1+\frac{2}{x}}\right) = \frac{4}{1+0} = 4$$

Hence the only horizontal asymptote of $f(x)$ is the line $y = 4$.

(b) Observe that the only solution to $5x^3-40 = 0$ is $x = 2$, whence the only candidate vertical asymptote of $f(x)$ is $x = 2$. Direct substitution of $x = 2$ into $f(x)$ gives "$\frac{23}{0}$", which indicates the one-sided limits at $x = 2$ are both infinite, and so $x = 2$ is, indeed, a true vertical asymptote.

For the one-sided limits we have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^-} = -\infty$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^+} = +\infty$$

D-23

As for the horizontal asymptotes, we first perform some algebra to rewrite $f(x)$.

$$\frac{4x^3 - \sqrt{x^6+17}}{5x^3-40} = \frac{4x^3 - \sqrt{x^6}\sqrt{1+\frac{17}{x^6}}}{5x^3-40} = \frac{4 - \frac{|x|^3}{x^3}\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}$$

For $x > 0$, we note that $\frac{|x|^3}{x^3} = \frac{x^3}{x^3} = 1$. For $x < 0$, we note that $\frac{|x|^3}{x^3} = \frac{-x^3}{x^3} = -1$. So now we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4+\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4+\sqrt{1+0}}{5-0} = 1$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4-\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4-\sqrt{1+0}}{5-0} = \frac{3}{5}$$

Thus the two horizontal asymptotes of $f(x)$ are $y = 1$ and $y = \frac{3}{5}$.

§2.5: Limits at Infinity

Ex. E-1 2.5 Sp19 Exam

Find the equation of each horizontal asymptote, if any, of $f(x) = \dfrac{4x^3 - 3x^2}{2x^3 + 9x + 1}$.

exs19

Solution E-1

We compute the limits at infinity.

$$\lim_{x\to\pm\infty}\left(\frac{4x^3 - 3x^2}{2x^3 + 9x + 1}\right) = \lim_{x\to\pm\infty}\left(\frac{4 - \frac{3}{x}}{2 + \frac{9}{x^2} + \frac{1}{x^3}}\right) = \frac{4 - 0}{2 + 0 + 0} = 2$$

So the only horizontal asymptote of f is the line $y = 2$.

Ex. E-2 2.5, 4.7 Sp19 Exam

The parts of this problem *are* related!

(a) Show that $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-3}\right) = 1$.

(b) Calculate the following limit or show it does not exist.

$$\lim_{x\to\infty}\left(\frac{x}{x-3}\right)^x$$

Hint: First use part (a) to identify the appropriate indeterminate form.

exs19

Solution E-2

(a) We have the following.

$$\lim_{x\to\infty}\left(\frac{x}{x-3}\right) = \lim_{x\to\infty}\left(\frac{1}{1 - \frac{3}{x}}\right) = \frac{1}{1-0} = 1$$

(b) The result of part (a) implies that as $x \to \infty$, our limit has the indeterminate form 1^∞. Let L be the desired limit. Then we have the following.

$$\ln(L) = \lim_{x\to\infty} \ln\left[\left(\frac{x}{x-3}\right)^x\right] = \lim_{x\to\infty}\left[x \ln\left(\frac{x}{x-3}\right)\right] = \lim_{x\to\infty}\left[\frac{\ln\left(\frac{x}{x-3}\right)}{\frac{1}{x}}\right]$$

As $x \to \infty$, we now have the indeterminate form $\frac{0}{0}$, so we may use L'Hospital's Rule.

$$\ln(L) \overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{x-3}{x}\cdot\frac{(x-3)\cdot 1 - x\cdot 1}{(x-3)^2}}{\frac{-1}{x^2}}\right) = \lim_{x\to\infty}\left(\frac{3x}{x-3}\right) = \lim_{x\to\infty}\left(\frac{3}{1-\frac{3}{x}}\right) = 3$$

We have found that $\ln(L) = 3$, whence $L = e^3$.

Ex. E-3 2.5 Sp20 Exam

Find all horizontal asymptotes of

$$f(x) = \frac{12x + 5}{\sqrt{16x^2 + x + 1}}$$

or determine that there are no horizontal asymptotes.

exs20

Solution E-3

First we do some algebra before computing the relevant limits.

$$\frac{12x+5}{\sqrt{16x^2+x+1}} = \frac{x}{\sqrt{x^2}}\cdot\frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}} = \frac{x}{|x|}\cdot\frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}}$$

E-3

For the limit $x \to \infty$, we have $|x| = x$, whence $\frac{|x|}{x} = 1$. For the limit $x \to -\infty$, we have $|x| = -x$, whence $\frac{|x|}{x} = -1$. So now we have the following.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{12+\frac{5}{x}}{\sqrt{16+\frac{1}{x}+\frac{1}{x^2}}}\right) = \frac{12+0}{\sqrt{16+0+0}} = 3$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty}\left(-\frac{12+\frac{5}{x}}{\sqrt{16+\frac{1}{x}+\frac{1}{x^2}}}\right) = -\frac{12+0}{\sqrt{16+0+0}} = -3$$

Hence the horizontal asymptotes of f are $y = 3$ and $y = -3$.

Ex. E-4 2.5 Sp20 Exam

Suppose the function f has domain $(-\infty, \infty)$. Give a brief explanation of how you would find all horizontal asymptotes of f. Note that for this problem, f is unspecified; you should not assume it has any particular form. *Your answer may contain either English, mathematical symbols, or both.*

exs20

Solution E-4

Compute the limits $A = \lim\limits_{x\to-\infty} f(x)$ and $B = \lim\limits_{x\to+\infty} f(x)$. If A exists and is finite, then $y = A$ is a horizontal asymptote of f. Similarly for B. (Note that f can have zero, one, or two horizontal asymptotes.)

Ex. E-5 2.5 Su20 Exam

Let $f(x) = \dfrac{(x-3)(2x+1)}{(5x+2)(3x-10)}$. Calculate all horizontal asymptotes of f.

exsu20

Solution E-5

We must calculate the limits of f at infinity. First we assume $x \neq 0$ and factor out the highest power of numerator and denominator separately to prepare the calculation of those limits. In particular, we factor out x from each term.

$$\frac{(x-3)(2x+1)}{(5x+2)(3x-10)} = \frac{x^2}{x^2}\cdot\frac{\left(1-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{\left(5+\frac{2}{x}\right)\left(3-\frac{10}{x}\right)} = \frac{\left(1-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{\left(5+\frac{2}{x}\right)\left(3-\frac{10}{x}\right)}$$

Now we note that $\lim_{x\to-\infty}\left(\frac{1}{x}\right) = \lim_{x\to+\infty}\left(\frac{1}{x}\right) = 0$. Hence we have

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{\left(1-\frac{3}{x}\right)\left(2+\frac{1}{x}\right)}{\left(5+\frac{2}{x}\right)\left(3-\frac{10}{x}\right)}\right) = \frac{(1-0)(2+0)}{(5+0)(3-0)} = \frac{2}{15}$$

Hence f has a single horizontal asymptote: $y = \dfrac{2}{15}$.

Ex. D-8 2.4, 2.5 Su20 Exam

Let $f(x) = \dfrac{3+7e^{2x}}{1-e^x}$. Calculate each of the following limits.

(a) $\lim\limits_{x\to-\infty} f(x)$ (b) $\lim\limits_{x\to+\infty} f(x)$ (c) $\lim\limits_{x\to0^-} f(x)$

exsu20

Solution D-8

(a) We recall that $\lim\limits_{x\to-\infty}(e^x) = 0$, whence $\lim\limits_{x\to-\infty}(e^{2x}) = 0$ also since $e^{2x} = (e^x)^2$. So we immediately have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{3+7e^{2x}}{1-e^x}\right) = \frac{3+7\cdot 0}{1-0} = 3$$

(b) We recall that $\lim\limits_{x\to+\infty}(e^x) = +\infty$, whence $\lim\limits_{x\to+\infty}(e^{2x}) = +\infty$ also since $e^{2x} = (e^x)^2$. This would give the indeterminate form "$\dfrac{\infty}{-\infty}$" in our limit, so we instead factor out the "highest power" (or dominant term) as

D-8

$x \to +\infty$ of the numerator and denominator separately. For the numerator, the dominant term is e^{2x}. For the denominator, the dominant term is e^x. So now we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{e^{2x}}{e^x}\cdot\frac{3e^{-2x}+7}{e^{-x}-1}\right) = \lim_{x\to+\infty}\left(e^x\cdot\frac{3e^{-2x}+7}{e^{-x}-1}\right)$$

Now we recall that $\lim\limits_{x\to+\infty}(e^{-x}) = 0$, whence $\lim\limits_{x\to+\infty}(e^{-2x}) = 0$ also since $e^{2x} = (e^x)^2$. So our limit is:

$$\lim_{x\to+\infty}\left(e^x\cdot\frac{3e^{-2x}+7}{e^{-x}-1}\right) = \lim_{x\to+\infty}(e^x)\cdot\lim_{x\to+\infty}\left(\frac{3e^{-2x}+7}{e^{-x}-1}\right) = (+\infty)\cdot\frac{0+7}{0-1} = -\infty$$

(c) Direct substitution of $x = 0$ into $f(x)$ gives the (undefined) expression "$\frac{10}{0}$", which means that both one-sided limits at $x = 0$ are infinite. So we perform a sign analysis to determine whether the limit is positive or negative infinity.

As $x \to 0^-$ the numerator $(3 + 7e^{2x}) \to 10$, which is positive. For the denominator, however, we note that e^x is an increasing function for all x. Hence $1 = e^0 > e^x$ (or $1 - e^x > 0$) for all $x < 0$. (We can deduce this from a simple graph of $y = e^x$. Alternatively, a test point shows that $1 - e^x > 0$ for all x sufficiently close to and less than 0.) Hence the denominator is positive as $x \to 0^-$. Putting this altogether gives the following:

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left(\frac{3+7e^{2x}}{1-e^x}\right) = \frac{10}{0^+} = +\infty$$

Ex. E-6 2.5 Fa20 Exam

Calculate all horizontal asymptotes of the function $h(x) = \dfrac{\sqrt{3x^2+x+10}}{2-5x}$.

exf20

Solution E-6

For $x \neq 0$, we have the following algebra:

$$\frac{\sqrt{3x^2+x+10}}{2-5x} = \frac{\sqrt{x^2}\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{x\left(\frac{2}{x}-5\right)} = \frac{|x|}{x}\cdot\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}$$

We have used the identity $\sqrt{x^2} = |x|$. To compute the horizontal asymptotes, we compute the limits of h at infinity. For $x \to \infty$, we may assume that $x > 0$, and so $|x| = x$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left(\frac{|x|}{x}\cdot\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = \lim_{x\to\infty}\left(\frac{x}{x}\cdot\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = \lim_{x\to\infty}\left(\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = -\frac{\sqrt{3}}{5}$$

For $x \to -\infty$, we may assume that $x < 0$, and so $|x| = -x$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left(\frac{|x|}{x}\cdot\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = \lim_{x\to\infty}\left(\frac{-x}{x}\cdot\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = \lim_{x\to\infty}\left(-\frac{\sqrt{3+\frac{1}{x}+\frac{10}{x^2}}}{\frac{2}{x}-5}\right) = \frac{\sqrt{3}}{5}$$

Hence the two horizontal asymptotes are $y = -\frac{\sqrt{3}}{5}$ (as $x \to \infty$) and $y = \frac{\sqrt{3}}{5}$ (as $x \to -\infty$).

Ex. E-7 2.5 Fa20 Exam

Suppose the line $y = 3$ is a horizontal asymptote for f. Which of the following statements MUST be true? Select all that apply.

(a) $f(x) \neq 3$ for all x in the domain of f

(b) $f(3)$ is undefined

(c) $\lim\limits_{x\to3} f(x) = \infty$

(d) $\lim\limits_{x\to\infty} f(x) = 3$

(e) none of the above

exf20

Solution **E-7**

Choice (e) only.

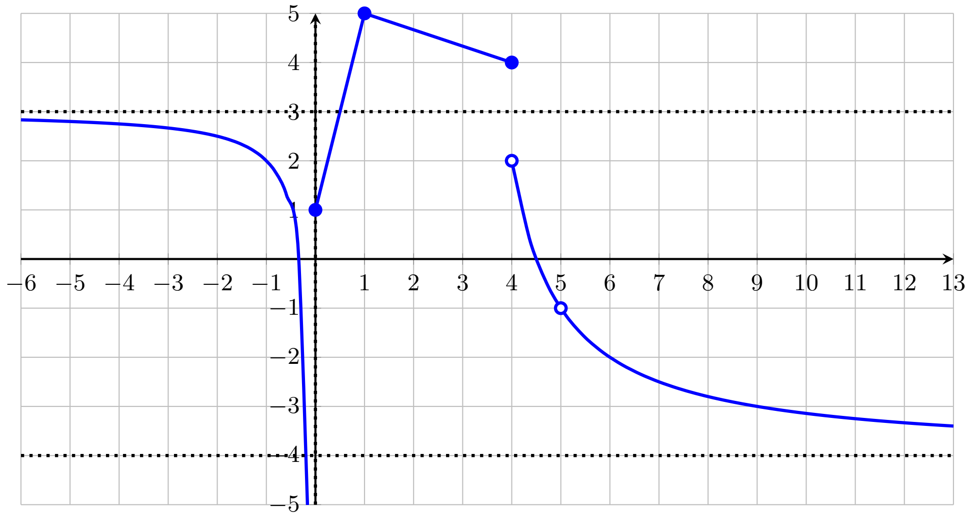
For choices (a), (b), and (c), consider $f(x) = 3$ (constant function). Then f has a horizontal asymptote at $y = 3$, but none of (a), (b), and (c) is true.

For choice (d), consider $f(x) = e^x + 3$. Then f has a horizontal asymptote at $y = 3$ because $\lim_{x\to-\infty} f(x) = 3$, but choice (d) is false since $\lim_{x\to\infty} f(x) = \infty$.

Hence choice (e) must be correct.

Ex. E-8 **2.5, 2.6, 3.1/3.2** Sp21 **Exam**

Use the graph of f below to answer the following questions. Dashed lines indicate the location of asymptotes.



(a) Calculate $\lim_{x\to\infty} f(x)$.

(b) Calculate $\lim_{x\to-\infty} f(x)$.

(c) List the values of x where f is not continuous.

(d) List the values of x where f is not differentiable.

(e) What is the sign of $f'(-1)$? (choices: positive, negative, zero, does not exist)

(f) What is the sign of $f'(0.5)$? (choices: positive, negative, zero, does not exist)

exs21

Solution **E-8**

(a) $\lim_{x\to\infty} f(x) = -4$

(b) $\lim_{x\to-\infty} f(x) = 3$

(c) $x = 0$, $x = 4$, $x = 5$

(d) $x = 0$, $x = 1$, $x = 4$, $x = 5$

(e) negative

(f) positive

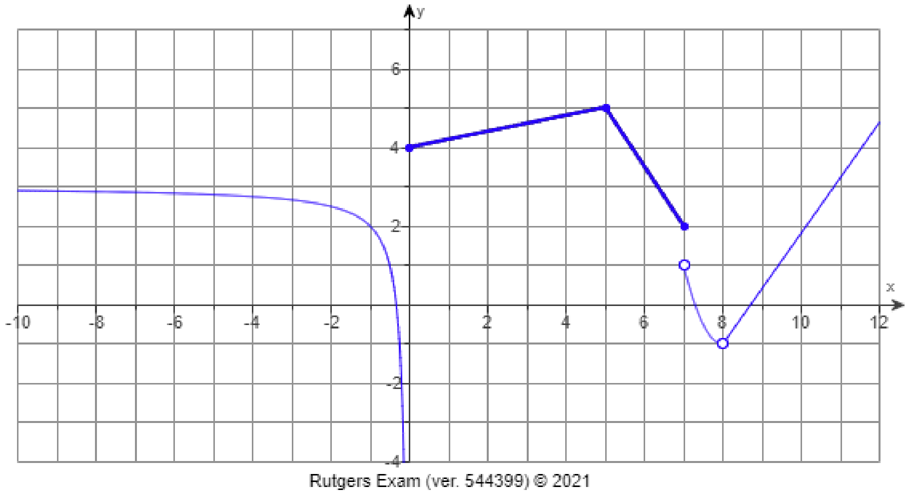
Ex. B-6 **2.1/2.2, 2.3, 2.4, 2.5** Fa21 **Exam**

For each part, use the graph of $y = f(x)$.

exf21

2.1/2.2, 2.3, 2.4, 2.5

Fa21 Exam



(a) List the x-values where f is not continuous or determine that f is continuous for all x.

(b) List all vertical asymptotes of f.

(c) List all horizontal asymptotes of f.

(d) Calculate $\lim_{x\to 8} f(x)$ or determine that the limit does not exist.

(e) At $x = 7$, which of the one-sided limits of f exist?

exf21

Solution B-6

(a) $x = 0, 7, 8$ only

(b) $x = 0$ only

(c) $y = 3$ only

(d) $\lim_{x\to 8} f(x) = -1$

(e) Both the left- and right-limits of $f(x)$ at $x = 7$ exist.

Ex. E-9 2.5 Fa21 Exam

Let $f(x) = \dfrac{8 + 6e^x}{9e^x - \pi^6}$.

(a) Evaluate $\lim_{x\to\infty} f(x)$.

(b) Evaluate $\lim_{x\to-\infty} f(x)$.

(c) List all vertical asymptotes of f.

exf21

Solution E-9

(a) Divide each term by e^x and recall that $\lim_{x\to\infty} e^{-x} = 0$.

$$\lim_{x\to\infty}\left(\frac{8+6e^x}{9e^x-\pi^6}\right) = \lim_{x\to\infty}\left(\frac{8e^{-x}+6}{9-\pi^6e^{-x}}\right) = \frac{0+6}{9-0} = \frac{2}{3}$$

(b) Recall that $\lim_{x\to-\infty} e^x = 0$.

$$\lim_{x\to-\infty}\left(\frac{8+6e^x}{9e^x-\pi^6}\right) = \frac{8+0}{0-\pi^6} = -\frac{8}{\pi^6}$$

(c) The denominator vanishes if $x = \ln(\frac{\pi^6}{9})$, and the numerator does not vanish at this x-value. Hence the only vertical asymptote of f is the line $x = \ln(\frac{\pi^6}{9})$.

Ex. D-13 2.4, 2.5, 4.7 Fa21 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

exf21

2.4, 2.5, 4.7

Fa21 Exam

(a) $\lim_{x\to 1}\left(\dfrac{x^4-x}{\ln(77x-76)}\right)$

(b) $\lim_{x\to -\infty}\left(\dfrac{\sqrt{36x^2+63}}{31x}\right)$

(c) $\lim_{x\to 2^+} f(x)$, with $f(x)=\begin{cases}1+4x & x\le 2\\ \dfrac{x^2-4}{x-2} & x>2\end{cases}$

(d) $\lim_{x\to 5^-}\left(\dfrac{\cos(\pi x)}{x^2-25}\right)$

exf21

Solution D-13

(a) Direct substitution gives "$\frac{0}{0}$", and so we use L'Hospital's Rule.

$$\lim_{x\to 1}\left(\frac{x^4-x}{\ln(77x-76)}\right)\overset{H}{=}\lim_{x\to 1}\left(\frac{4x^3-1}{\frac{1}{77x-76}\cdot 77}\right)=\frac{3}{77}$$

(b) We factor out x^2 from inside the square root in the numerator. Observe that since x goes to *negative* infinity, we have $\sqrt{x^2}=|x|=-x$.

$$\lim_{x\to-\infty}\left(\frac{\sqrt{36x^2+63}}{31x}\right)=\lim_{x\to-\infty}\left(\frac{-x\sqrt{36+\frac{63}{x^2}}}{31x}\right)=\lim_{x\to-\infty}\left(\frac{-\sqrt{36+\frac{63}{x^2}}}{31}\right)=\frac{-6}{31}$$

(c) We factor and cancel.

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2^+}\left(\frac{(x-2)(x+2)}{x-2}\right)=\lim_{x\to 2^+}(x+2)=4$$

(d) Direct substitution gives "$\frac{-1}{0}$", whence the one-sided limit must be infinite. Observe that the numerator is negative (goes to -1) as $x\to 5^-$, and the denominator goes to 0 but remains negative as $x\to 5^-$. (For instance, use test points such as $x=4.99$.) Hence the desired limit is $\frac{-1}{0^-}=+\infty$.

Ex. E-10 2.5

Sp22 Exam

Find all horizontal asymptotes of the function $g(x)=\dfrac{2e^x-15}{5e^{3x}+8}$.

exs22

Solution E-10

To find the horizontal asymptotes, we must compute the limits at infinity. For the limit at $-\infty$, recall that $\lim_{x\to-\infty} e^x=0$. Thus we have:

$$\lim_{x\to-\infty}\left(\frac{2e^x-15}{5e^{3x}+8}\right)=\frac{0-15}{0+8}=-\frac{15}{8}$$

For the limit at $+\infty$, recall that $\lim_{x\to+\infty} e^{-x}=0$. Divide each term by e^{3x} and use this special limit to obtain the following:

$$\lim_{x\to+\infty}\left(\frac{2e^x-15}{5e^{3x}+8}\right)=\lim_{x\to+\infty}\left(\frac{2e^{-2x}-15e^{-3x}}{5+8e^{-3x}}\right)=\frac{0-0}{5+0}=0$$

Hence the horizontal asymptotes of g are the lines $y=0$ and $y=-\frac{15}{8}$.

Ex. D-15 2.4, 2.5

Sp22 Exam

For the function f below, find its domain and all vertical and horizontal asymptotes.

$$f(x)=\frac{x^2-8x+12}{3x^2-8x+4}$$

exs22

Solution D-15

Since f is a rational function, its domain is all real numbers except where the denominator vanishes. Observe that $(3x^2-8x+4)=(3x-2)(x-2)$, hence the denominator vanishes at $x=\frac{2}{3}$ and $x=2$. The domain of f is $(-\infty,\frac{2}{3})\cup(\frac{2}{3},2)\cup(2,\infty)$.

D-15

Since f is continuous on its domain, vertical asymptotes can occur only at either $x = \frac{2}{3}$ or $x = 2$. Observe that direct substitution of $x = \frac{2}{3}$ into $f(x)$ gives an expression of "$\frac{\text{nonzero }\#}{0}$". Hence the one-sided limits of f at $x = \frac{2}{3}$ must each be infinite, and so $x = \frac{2}{3}$ is a vertical asymptote of f.

For $x = 2$, however, we have the following:

$$\lim_{x\to 2}\left(\frac{x^2-8x+12}{3x^2-8x+4}\right) = \lim_{x\to 2}\left(\frac{(x-2)(x-6)}{(3x-2)(x-2)}\right) = \lim_{x\to 2}\left(\frac{x-6}{3x-2}\right) = \frac{2-6}{6-2} = -1$$

Since this limit is finite, we conclude $x = 2$ is not a vertical asymptote of f.

For the horizontal asymptotes, we must compute the limits at infinity.

$$\lim_{x\to\pm\infty}\left(\frac{x^2-8x+12}{3x^2-8x+4}\right) = \lim_{x\to\pm\infty}\left(\frac{1-\frac{8}{x}+\frac{12}{x^2}}{3-\frac{8}{x}+\frac{4}{x^2}}\right) = \frac{1-0+0}{3-0+0} = \frac{1}{3}$$

So the only horizontal asymptote of f is the line $y = \frac{1}{3}$.

Ex. D-16 **2.4, 2.5** Su22 **Exam**

Consider the function $f(x) = \dfrac{x^3-3x+1}{x^2-2x+1}$.

(a) Find all horizontal asymptotes of f, if any.

(b) Find all vertical asymptotes of f. Then calculate $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$, where $x = a$ is the rightmost vertical asymptote of f.

exsu22

Solution D-16

(a) We compute the limits of f at infinity. To this end, we factor the highest powers of numerator and denominator separately.

$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2}\cdot\frac{1-\frac{3}{x^2}+\frac{1}{x^3}}{1-\frac{2}{x}+\frac{1}{x^2}}\right) = \lim_{x\to\pm\infty}\left(x\cdot\frac{1-\frac{3}{x^2}+\frac{1}{x^3}}{1-\frac{2}{x}+\frac{1}{x^2}}\right) = (\pm\infty)\cdot\frac{1-0+0}{1-0+0} = \pm\infty$$

These limits are not finite. Thus f has no horizontal asymptote.

(b) Since f is a rational function, vertical asymptotes can occur only where the denominator is 0. The only solution to $x^2-2x+1 = (x-1)^2 = 0$ is $x = 1$. Substitution of $x = 1$ into f gives the undefined expression "$\frac{-1}{0} = \frac{\text{nonzero }\#}{0}$", whence $x = 1$ is, indeed, a vertical asymptote for f.

Now we compute the left- and right-limits using sign analysis.

$$\lim_{x\to 1^-}\left(\frac{x^3-3x+1}{(x-1)^2}\right) = \frac{-1}{0^+} = -\infty$$
$$\lim_{x\to 1^+}\left(\frac{x^3-3x+1}{(x-1)^2}\right) = \frac{-1}{0^+} = -\infty$$

(For this function, the analysis was simplified since the denominator is the perfect square $(x-1)^2$ and thus never negative.)

Ex. E-11 **2.5** Fa22 **Exam**

Find all horizontal asymptotes of the function $h(x) = \dfrac{6x+5}{\sqrt{4x^2-9}}$.

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf22

Solution E-11

We must compute the limits at infinity. First we complete some algebraic manipulations by first factoring out the

E-11

largest powers of x in numerator and denominator of $h(x)$, separately. Note that $\sqrt{x^2} = |x|$.

$$\frac{6x+5}{\sqrt{4x^2+9}} = \frac{x\left(6+\frac{5}{x}\right)}{\sqrt{x^2\left(4+\frac{9}{x^2}\right)}} = \frac{x}{\sqrt{x^2}} \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}} = \frac{x}{|x|} \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}$$

Now we compute the necessary limits. Note that as $x \to \infty$, we have $|x| = x$, and so $x/|x| = x/x = 1$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(\frac{x}{|x|} \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = \lim_{x\to+\infty} \left(1 \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = 1 \cdot \frac{6+0}{\sqrt{4+0}} = \frac{6}{2} = 3$$

Now note that as $x \to -\infty$, we have $|x| = -x$, and so $x/|x| = x/(-x) = -1$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{x}{|x|} \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = \lim_{x\to-\infty} \left(-1 \cdot \frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = -1 \cdot \frac{6+0}{\sqrt{4+0}} = \frac{6}{2} = -3$$

Thus the HA's of $h(x)$ are the lines $y = 3$ and $y = -3$.

Ex. D-18 **2.4, 2.5** Su22 **Quiz**

Calculate all of the vertical and horizontal asymptotes of $f(x) = \dfrac{x^2-100}{10x-x^2}$.

Then find the two one-sided at $x = a$, where $x = a$ is the leftmost vertical asymptote of f.

qsu22

Solution **D-18**

First we calculate the horizontal asymptotes.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left(\frac{x^2\left(1-\frac{100}{x^2}\right)}{x^2\left(\frac{10}{x}-1\right)}\right) = \lim_{x\to\pm\infty} \left(\frac{1-\frac{100}{x^2}}{\frac{10}{x}-1}\right) = \frac{1-0}{0-1} = -1$$

Hence the only horizontal asymptote of f is the line $y = -1$.

For vertical asymptotes, we see that $10x - x^2 = 0$ has solutions $x = 0$ and $x = 10$. However, we have:

$$\lim_{x\to10} f(x) = \lim_{x\to10} \left(\frac{(x-10)(x+10)}{-x(x-10)}\right) = \lim_{x\to10} \left(\frac{x+10}{-x}\right) = -2$$

Since this limit is finite, we see that $x = 10$ is not a vertical asymptote. However, we also see that $x = 0$ is, indeed, a vertical asymptote since direct substitution of $x = 0$ into $\frac{x+10}{-x}$ gives "$\frac{\text{nonzero \#}}{0}$". This also means each of the one-sided limits at $x = 0$ is infinite.

We now compute the corresponding one-sided limits. Note that as $x \to 0$, the expression $x + 10$ is positive (tends to 10). However, the expression $-x$ stays positive as $x \to 0^-$ and negative as $x \to 0^+$. Hence we have:

$$\lim_{x\to0-} f(x) = \lim_{x\to0-} \left(\frac{x+10}{-x}\right) = \frac{10}{0^+} = +\infty$$

$$\lim_{x\to0+} f(x) = \lim_{x\to0+} \left(\frac{x+10}{-x}\right) = \frac{10}{0^-} = -\infty$$

Ex. E-12 **2.5** Su22 **Quiz**

Calculate the limit below.

$$\lim_{x\to-\infty} \left(\frac{2-3e^x+4e^{-x}}{5+7e^x-15e^{-x}}\right)$$

qsu22

Solution **E-12**

Recall that $\lim\limits_{x\to-\infty} e^{-x} = \infty$ and $\lim\limits_{x\to-\infty} e^x = 0$. Hence the "leading terms" of numerator and denominator are each

E-12

"e^{-x}". We factor out these leading terms, and then compute the limit.

$$\lim_{x\to-\infty}\left(\frac{e^{-x}\left(2e^x-3e^{2x}+4\right)}{e^{-x}\left(5e^x+7e^{2x}-15\right)}\right)=\lim_{x\to-\infty}\left(\frac{2e^x-3e^{2x}+4}{5e^x+7e^{2x}-15}\right)=\frac{0-0+4}{0+0-15}=-\frac{4}{15}$$

Ex. E-13 2.5 Fa22 Quiz

Find all horizontal asymptotes of $g(x)$. You must justify your answers precisely.

$$g(x)=\frac{3e^{-2x}+4e^{5x}-10}{6e^{-9x}-7e^{8x}+1}$$

qf22

Solution E-13

We must compute the limits at infinity. For $x\to-\infty$, the dominant term in the denominator is e^{-9x}. So we divide all terms by e^{-9x} (equivalently, multiply all terms by e^{9x}) to obtain the following:

$$\lim_{x\to-\infty}g(x)=\lim_{x\to-\infty}\left(\frac{e^{9x}}{e^{9x}}\cdot\frac{3e^{-2x}+4e^{5x}-10}{6e^{-9x}-7e^{8x}+1}\right)=\lim_{x\to-\infty}\left(\frac{3e^{7x}+4e^{14x}-10e^{9x}}{6-7e^{17x}+e^{9x}}\right)=\frac{0+0-0}{6-0+0}=0$$

For $x\to\infty$, the dominant term in the denominator is e^{8x}. So we divide all terms by e^{8x} (equivalently, multiply all terms by e^{-8x}) to obtain the following:

$$\lim_{x\to\infty}g(x)=\lim_{x\to\infty}\left(\frac{e^{-8x}}{e^{-8x}}\cdot\frac{3e^{-2x}+4e^{5x}-10}{6e^{-9x}-7e^{8x}+1}\right)=\lim_{x\to\infty}\left(\frac{3e^{-10x}+4e^{-3x}-10e^{-8x}}{6e^{-17x}-7+e^{-8x}}\right)=\frac{0+0-0}{0-7+0}=0$$

Thus the only HA of g is $y=0$.

Ex. E-14 2.5

For each part, calculate the limit or show that it does not exist.

(a) $\displaystyle\lim_{x\to\infty}\left(\frac{3x-5}{x+1}\right)$

(b) $\displaystyle\lim_{x\to-\infty}\left(\frac{3x}{\sqrt{4x^2+9}}\right)$

(c) $\displaystyle\lim_{x\to\infty}\left(\frac{(x-3)(2x+4)(x-5)}{(3x+1)(4x-7)(x+2)}\right)$

(d) $\displaystyle\lim_{x\to-\infty}\left(\frac{(x-3)(2x+4)(x-5)}{(3x+1)(4x-7)(x+2)}\right)$

(e) $\displaystyle\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)$

(f) $\displaystyle\lim_{x\to\infty}e^{-x^3}$

c2s5

Solution E-14

(a) Factor out dominant terms.

$$\lim_{x\to\infty}\left(\frac{3x-5}{x+1}\right)=\lim_{x\to\infty}\left(\frac{x}{x}\cdot\frac{3-\frac{5}{x}}{1+\frac{1}{x}}\right)=\lim_{x\to\infty}\left(\frac{x}{x}\right)\cdot\lim_{x\to\infty}\left(\frac{3-\frac{5}{x}}{1+\frac{1}{x}}\right)=1\cdot\frac{3-0}{1+0}=3$$

(b) Factor out dominant terms. Recall that $\sqrt{x^2}=|x|$. If $x\to-\infty$, we may assume $x<0$, so that $|x|=-x$.

$$\lim_{x\to-\infty}\left(\frac{3x}{\sqrt{4x^2+9}}\right)=\lim_{x\to-\infty}\left(\frac{3x}{\sqrt{x^2\left(4+\frac{9}{x^2}\right)}}\right)=\lim_{x\to-\infty}\left(\frac{x}{-x}\cdot\frac{3}{\sqrt{4+\frac{9}{x^2}}}\right)=\lim_{x\to-\infty}\left(\frac{-3}{\sqrt{4+\frac{9}{x^2}}}\right)=-\frac{3}{2}$$

(c) Factor out dominant terms.

$$\lim_{x\to\infty}\left(\frac{(x-3)(2x+4)(x-5)}{(3x+1)(4x-7)(x+2)}\right)=\lim_{x\to\infty}\left(\frac{x^3}{x^3}\cdot\frac{(1-\frac{3}{x})(2+\frac{4}{x})(1-\frac{5}{x})}{(3+\frac{1}{x})(4-\frac{7}{x})(1+\frac{2}{x})}\right)=1\cdot\frac{(1-0)(2+0)(1-0)}{(3+0)(4-0)(1+0)}=\frac{1}{6}$$

(d) Same work as part (c). Note that the sign of the infinity symbol is irrelevant in the solution since all of the reciprocals (terms like $\frac{1}{x}$) go to 0 whether $x\to\infty$ or $x\to-\infty$. So the limit is equal to $\frac{1}{6}$.

(e) As $x\to\infty$, we have that $\frac{1}{x}\to0$. Since the cosine function is continuous, we have

$$\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)=\cos\left(\lim_{x\to\infty}\frac{1}{x}\right)=\cos(0)=1$$

E-14

(f) Note that $-x^3 \to -\infty$ as $x \to \infty$. So we have

$$\lim_{x\to\infty} e^{-x^3} = \lim_{u\to-\infty} e^u = 0$$

Ex. E-15 2.5

For each function, find all horizontal asymptotes.

(a) $f(x) = \dfrac{(x-1)(2x+5)}{(x+1)(3x-6)}$

(b) $f(x) = \ln(x)$

(c) $f(x) = \dfrac{2e^x+3}{1-e^x}$

(d) $f(x) = e^{-1/x}$

c2s5

Solution E-15

(a) To calculate the limits as $x \to \pm\infty$, we factor out dominant terms.

$$\lim_{x\to-\infty}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \lim_{x\to-\infty}\left(\frac{x^2}{x^2}\cdot\frac{(1-\frac{1}{x})(2+\frac{5}{x})}{(1+\frac{1}{x})(3-\frac{6}{x})}\right) = 1\cdot\frac{(1-0)(2+0)}{(1+0)(3-0)} = \frac{2}{3}$$

Note that the work would be identical if we had $x\to\infty$ (all the reciprocals still approach 0). Hence we have

$$\lim_{x\to\infty}\left(\frac{(x-1)(2x+5)}{(x+1)(3x-6)}\right) = \frac{2}{3}$$

The only horizontal asymptote is the line $y = \frac{2}{3}$.

(b) The domain of $\ln(x)$ is $(0,\infty)$, so it only makes sense to consider a horizontal asymptote of f as $x\to\infty$. Since $\ln(x)\to\infty$ as $x\to\infty$, we see that there are no horizontal asymptotes.

(c) Recall that $e^x \to 0$ as $x\to-\infty$. So we have the following.

$$\lim_{x\to-\infty}\left(\frac{2e^x+3}{1-e^x}\right) = \frac{0+3}{1-0} = 3$$

Recall that $e^x\to\infty$ as $x\to\infty$. So we have the following.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{e^x}{e^x}\cdot\frac{2+3e^{-x}}{e^{-x}-1}\right) = \lim_{x\to\infty}\left(\frac{2+3e^{-x}}{e^{-x}-1}\right) = \frac{2+0}{0-1} = -2$$

Hence the horizontal asymptotes are the lines $y=-2$ and $y=3$.

(d) Note that $-\frac{1}{x}\to 0$ as $x\to\pm\infty$. Since e^x is continuous, we have:

$$\lim_{x\to\pm\infty} e^{-1/x} = e^0 = 1$$

Hence the line $y=1$ is the only horizontal asymptote.

Ex. E-16 2.5

Find all horizontal asymptotes of $f(x) = \dfrac{\sqrt[4]{16x^4+7x+5}}{3x-8}$.

other

Solution E-16

We calculate the limits of f at infinity. First we do some algebra by factoring out the highest power in numerator and denominator.

$$\frac{\sqrt[4]{16x^4+7x+5}}{3x-8} = \frac{\sqrt[4]{x^4\left(16+\frac{7}{x^3}+\frac{5}{x^4}\right)}}{x\left(3-\frac{8}{x}\right)} = \frac{|x|}{x}\cdot\frac{\sqrt[4]{16+\frac{7}{x^3}+\frac{5}{x^4}}}{3-\frac{8}{x}}$$

We have used the identity $\sqrt[4]{x^4} = |x|$. For $x\to-\infty$, we can assume $x<0$, whence $|x| = -x$ in that case. Similarly,

E-16

we have $|x| = x$ for the limit $x \to \infty$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{-x}{x}\cdot\frac{\sqrt[4]{16+\frac{7}{x^3}+\frac{5}{x^4}}}{3-\frac{8}{x}}\right) = \lim_{x\to-\infty}\left(-1\cdot\frac{\sqrt[4]{16+\frac{7}{x^3}+\frac{5}{x^4}}}{3-\frac{8}{x}}\right) = -\frac{\sqrt[4]{16+0+0}}{3+0} = -\frac{2}{3}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-+\infty}\left(\frac{-x}{x}\cdot\frac{\sqrt[4]{16+\frac{7}{x^3}+\frac{5}{x^4}}}{3-\frac{8}{x}}\right) = \lim_{x\to+\infty}\left(1\cdot\frac{\sqrt[4]{16+\frac{7}{x^3}+\frac{5}{x^4}}}{3-\frac{8}{x}}\right) = \frac{\sqrt[4]{16+0+0}}{3+0} = \frac{2}{3}$$

Hence the horizontal asymptotes are the lines $y = -\frac{2}{3}$ and $y = \frac{2}{3}$.

Ex. D-23 **2.4, 2.5** ⋆Challenge

For each function, find all horizontal asymptotes and vertical asymptotes. Then, at each vertical asymptote, calculate both one-sided limits.

(a) $f(x) = \dfrac{4x^3+4x^2-8x}{x^3+3x^2-4}$ (b) $f(x) = \dfrac{4x^3-\sqrt{x^6+17}}{5x^3-40}$

challenge

Solution D-23

(a) First we factor the denominator. Let $p(x) = x^3 + 3x^2 - 4$ and observe that $p(1) = 0$, whence $x - 1$ is a factor of $p(x)$. Performing long division of polynomials then gives $p(x) = (x-1)(x^2+4x+4) = (x-1)(x+2)^2$. So for $x \neq 1$ and $x \neq -2$, we have:

$$f(x) = \frac{4x^3+4x^2-8x}{x^3+3x^2-4} = \frac{4x(x+2)(x-1)}{(x-1)(x+2)^2} = \frac{4x}{x+2}$$

Hence the only vertical asymptote of $f(x)$ is the line $x = -2$.

> Precisely, we have that $\lim_{x\to1} f(x) = \lim_{x\to1}\left(\frac{4x}{x+2}\right) = \frac{4}{3}$. Since this limit is finite, $x = 1$ is not a vertical asymptote of $f(x)$.

For the one-sided limits we have:

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^-} = +\infty$$
$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^+} = -\infty$$

As for the horizontal asymptotes, we have the following:

$$\lim_{x\to\pm\infty}\left(\frac{4x}{x+2}\right) = \lim_{x\to\pm\infty}\left(\frac{4}{1+\frac{2}{x}}\right) = \frac{4}{1+0} = 4$$

Hence the only horizontal asymptote of $f(x)$ is the line $y = 4$.

(b) Observe that the only solution to $5x^3 - 40 = 0$ is $x = 2$, whence the only candidate vertical asymptote of $f(x)$ is $x = 2$. Direct substitution of $x = 2$ into $f(x)$ gives "$\frac{23}{0}$", which indicates the one-sided limits at $x = 2$ are both infinite, and so $x = 2$ is, indeed, a true vertical asymptote.

For the one-sided limits we have:

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^-} = -\infty$$
$$\lim_{x\to2^-} f(x) = \lim_{x\to2^+}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^+} = +\infty$$

D-23

As for the horizontal asymptotes, we first perform some algebra to rewrite $f(x)$.

$$\frac{4x^3 - \sqrt{x^6+17}}{5x^3-40} = \frac{4x^3 - \sqrt{x^6}\sqrt{1+\frac{17}{x^6}}}{5x^3-40} = \frac{4 - \frac{|x|^3}{x^3}\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}$$

For $x > 0$, we note that $\frac{|x|^3}{x^3} = \frac{x^3}{x^3} = 1$. For $x < 0$, we note that $\frac{|x|^3}{x^3} = \frac{-x^3}{x^3} = -1$. So now we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4+\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4+\sqrt{1+0}}{5-0} = 1$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4-\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4-\sqrt{1+0}}{5-0} = \frac{3}{5}$$

Thus the two horizontal asymptotes of $f(x)$ are $y = 1$ and $y = \frac{3}{5}$.

Ex. E-17 **2.5** ⋆Challenge

Find all horizontal asymptotes of $f(x) = \dfrac{2x}{x-\sqrt{x^2+10}}$.

challenge

Solution E-17

As $x \to \pm\infty$, we see that $f(x)$ has an "$\frac{\infty}{\infty}$"-form (or equivalent variant). So first we factor out dominant terms from numerator and denominator to write f in an algebraically equivalent way for $x \neq 0$. Recall that $\sqrt{x^2} = |x|$.

$$f(x) = \frac{2x}{x-\sqrt{x^2+10}} = \frac{2x}{x-\sqrt{x^2\left(1+\frac{10}{x^2}\right)}} = \frac{2x}{x-|x|\sqrt{1+\frac{10}{x^2}}} = \frac{2}{1-\frac{|x|}{x}\sqrt{1+\frac{10}{x^2}}}$$

Now we calculate the horizontal asymptotes. For the limit $x \to -\infty$, we may assume x is negative, whence $|x| = -x$ and $\frac{|x|}{x} = -1$. So we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{2}{1+\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{1+\sqrt{1+0}} = \frac{2}{1+1} = 1$$

So the line $y = 1$ is a horizontal asymptote.

Now for the limit $x \to +\infty$, we may assume x is positive, whence $|x| = x$, and $\frac{|x|}{x} = 1$. So we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{2}{1-\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{1-\sqrt{1+0}} = \frac{2}{1-1} = \frac{2}{0}$$

This is an undefined expression, but recall that a limit of the form "$\frac{c}{0}$" (with $c \neq 0$) indicates that the limit is infinite. So there is no other horizontal asymptote.

Bonus: What is the value of this last limit? The above limit must be either $+\infty$ or $-\infty$. Observe that $1 + \frac{10}{x^2} > 1$ for all $x \neq 0$, which implies that $\sqrt{1+\frac{10}{x^2}} > 1$ for all such x, and so

$$1 - \sqrt{1+\frac{10}{x^2}} < 0$$

Thus as $x \to +\infty$, we see that $1 - \sqrt{1+\frac{10}{x^2}}$ approaches 0 but remains negative. Hence we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{2}{1-\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{0^-} = -\infty$$

§2.6: Continuity

Ex. F-1 **2.6** Fa17 **Exam**

Find the values of the constants a and b so that the following function is continuous for all x. If this is not possible, explain why.

$$f(x) = \begin{cases} ax+b & \text{if } x<1 \\ -2 & \text{if } x=1 \\ 3\sqrt{x}+b & \text{if } x>1 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf17

Solution **F-1**

The first two "pieces" of $f(x)$ are continuous for all x regardless of the values of a and b since polynomials are continuous for all x. The "piece" $3\sqrt{x}+b$ is continuous regardless of the value of b as long as $x \geq 0$. Hence each piece is continuous on each of its "pieces" separately on the respective intervals. We need only force continuity at $x=1$ to guarantee f is continuous for all x.

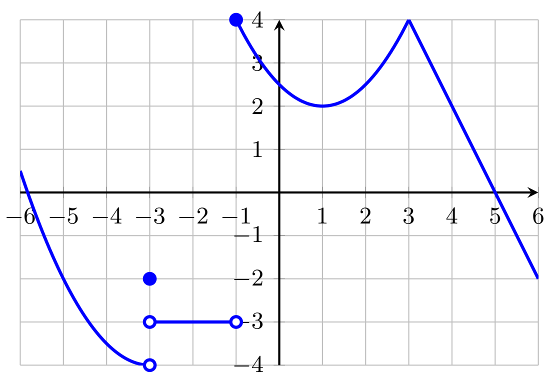
We calculate the left-limit, right-limit, and function value of f at $x=1$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(ax+b) = a+b \\ \lim_{x\to 1^-+} f(x) &= \lim_{x\to 1^+}(3\sqrt{x}+b) = 3+b \\ f(1) &= -2 \end{aligned}$$

These numbers must be equal, so $a+b = 3+b = -2$. Hence $a=3$ and $b=-5$.

Ex. F-2 **2.6, 3.1/3.2** Sp18 **Exam**

For each part, use the graph of $y=f(x)$ below.



(a) Find where $f(x)$ is *not* continuous in the interval $(-5,5)$.
(b) Find where $f(x)$ is *not* differentiable in the interval $(-5,5)$.
(c) Find where $f'(x)=0$ in the interval $(-5,5)$.
(d) Find where $f'(x)<0$ in the interval $(-5,5)$.

exs18

Solution **F-2**

(a) $x=-3$, $x=-1$
(b) $x=-3$, $x=-1$, $x=3$

Recall that continuity is necessary for differentiability. So any points of discontinuity are also points of non-differentiability. At $x=3$, the graph has a sharp corner, which means the function is not differentiable there.

(c) At each x-values in the interval $(-3,-1)$ and at $x=1$.

Recall that if $f'(a)=0$, then the graph of $y=f(x)$ has a horizontal tangent line at $x=a$. That is, the slope of the graph of $f(x)$ is 0.

F-2

(d) On each of the intervals $(-5,-3)$, $(-1,1)$, and $(3,5)$.

Ex. F-3 2.6 Sp18 Exam

Each part of this question refers to the function $f(x)$ below, where a and b are unspecified constants.

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} & \text{if } x < 0 \\ 2x+3 & \text{if } 0 \le x < 1 \\ b & \text{if } x = 1 \\ \dfrac{x^2-1}{x-1} & \text{if } 1 < x \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

(a) Find the value of a so that f is continuous at $x=0$. If this is not possible, explain why.

(b) Find the value of b so that f is continuous at $x=1$. If this is not possible, explain why.

exs18

Solution F-3

(a) We require that the left-limit, right-limit, and function value all be equal at $x=0$. We have the following.

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left(\frac{\sin(ax)}{x}\right) = \lim_{x\to 0^-}\left(a\cdot\frac{\sin(ax)}{ax}\right) = a\cdot 1 = a \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(2x+3) = 3 \\ f(0) &= (2x+3)\big|_{x=0} = 3 \end{aligned}$$

So we must have that $a=3$.

(b) We require that the left-limit, right-limit, and function value all be equal at $x=1$. We have the following.

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(2x+3) = 5 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{(x-1)(x+1)}{x-1}\right) = \lim_{x\to 1^+}(x+1) = 2 \\ f(0) &= b \end{aligned}$$

So we must have that $5=2=b$, which is impossible.

(It is impossible to find such a value of b because $\lim_{x\to 1} f(x)$ does not exist.)

Ex. F-4 2.6 Fa18 Exam

Find the values of the constants a and b so that the following function is continuous at $x=0$. If this is not possible, explain why.

$$f(x) = \begin{cases} \dfrac{4-\sqrt{16+49x^2}}{ax^2} & \text{if } x < 0 \\ -23 & x = 0 \\ \dfrac{\tan(2bx)}{x} & \text{if } x > 0 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf18

Solution **F-4**

We require that the left-limit, right-limit, and function value all be equal to $x = 0$. We have the following.

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left(\frac{4-\sqrt{16+49x^2}}{ax^2}\right) = \lim_{x\to 0^-}\left(\frac{16-(16+49x^2)}{ax^2(4+\sqrt{16+49x^2})}\right)\\
&= \lim_{x\to 0^-}\left(\frac{-49}{a(4+\sqrt{16+49x^2})}\right) = \frac{-49}{a(4+\sqrt{16+0})} = -\frac{49}{8a}\\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left(\frac{\tan(2bx)}{x}\right) = \lim_{x\to 0^+}\left(\frac{\sin(2bx)}{2bx}\cdot\frac{2b}{\cos(2bx)}\right)\\
&= \left(\lim_{x\to 0^+}\frac{\sin(2bx)}{2bx}\right)\left(\lim_{x\to 0^+}\frac{2b}{\cos(2bx)}\right) = 1\cdot\frac{2b}{1} = 2b\\
f(0) &= -23
\end{aligned}$$

Hence we must have that

$$-\frac{49}{8a} = -23 = 2b$$

and so the constants a and b are:

$$a = \frac{49}{184} \quad , \quad b = -\frac{23}{2}$$

Ex. F-5 **2.6** Sp19 **Exam**

Find the value of k that makes $f(x)$ continuous at $x = 1$. If no such value of k exists, write "does not exist".

$$f(x) = \begin{cases} k\cos(\pi x) - 3x^2 & \text{if } x \le 1 \\ 8e^x - k\ln(x) & \text{if } x > 1 \end{cases}$$

exs19

Solution **F-5**

We require that the left-limit, right-limit, and function value at $x = 1$ be equal to ensure continuity at $x = 1$.

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}\left(k\cos(\pi x) - 3x^2\right) = k\cos(\pi) - 3 = -k-3\\
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}\left(8e^x - k\ln(x)\right) = 8e^1 - k\ln(1) = 8e\\
f(1) &= \left(k\cos(\pi x) - 3x^2\right)\Big|_{x=1} = k\cos(\pi) - 3 = -k-3
\end{aligned}$$

Hence we must have $-k-3 = 8e$, or $k = -8e-3$.

Ex. F-6 **2.6** Sp19 **Exam**

Consider the function $f(x)$ below.

$$f(x) = \begin{cases} \dfrac{4-\sqrt{2x+10}}{x-3} & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$$

Is $f(x)$ continuous at $x = 3$? Explain your answer. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

exs19

Solution **F-6**

First we calculate the limit of $f(x)$ as $x \to 3$.

$$\begin{aligned}
\lim_{x\to 3} f(x) &= \lim_{x\to 3}\left(\frac{4-\sqrt{2x+10}}{x-3}\right) = \lim_{x\to 3}\left(\frac{4-\sqrt{2x+10}}{x-3}\cdot\frac{4+\sqrt{2x+10}}{4+\sqrt{2x+10}}\right) = \lim_{x\to 3}\left(\frac{16-(2x+10)}{(x-3)\left(4+\sqrt{2x+10}\right)}\right)\\
&= \lim_{x\to 3}\left(\frac{-2(x-3)}{(x-3)\left(4+\sqrt{2x+10}\right)}\right) = \lim_{x\to 3}\left(\frac{-2}{4+\sqrt{2x+10}}\right) = \frac{-2}{4+\sqrt{2\cdot 3+10}} = -\frac{1}{4}
\end{aligned}$$

Since $\lim_{x\to 3} f(x) \neq f(3) = 1$, f is not continuous at $x = 3$.

Ex. F-7 2.6 Fa19 Exam

Find the values of a and b that make f continuous at $x = 1$ or determine that no such values exist.

$$f(x) = \begin{cases} -3x + ax^2 & x < 1 \\ b & x = 1 \\ 4ax - 1 & x > 1 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf19

Solution F-7

We require that the left-limit, right-limit, and function value at $x = 1$ be equal to ensure continuity at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(-3x + ax^2\right) = -3 + a$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4ax - 1) = 4a - 1$$
$$f(1) = b$$

Hence we must have that

$$-3 + a = 4a - 1 = b$$

We solve the equation $-3 + a = 4a - 1$ first, then find b using the equation $b = 4a - 1$. Hence, the constants a and b must be $a = -\frac{2}{3}$ and $b = -\frac{11}{3}$.

Ex. F-8 2.6 Sp20 Exam

Determine where f is continuous. *Write your answer using interval notation.*

$$f(x) = \begin{cases} 9 - 16x & x < 0 \\ 3x^2 - x^3 & 0 \le x \le 3 \\ 1 - e^{x-3} & x > 3 \end{cases}$$

exs20

Solution F-8

Observe that f is clearly continuous for all x except possibly $x = 0$ or $x = 3$. For these transition points, we check whether the corresponding left-limit, right-limit, and function value are equal. For $x = 0$ we have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (9 - 16x) = 9 - 0 = 9$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3x^2 - x^3) = 0 - 0 = 0$$
$$f(0) = (3x^2 - x^3)\big|_{x=0} = 0$$

Since these three values are not all equal, f is discontinuous at $x = 0$. For $x = 3$ we have:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (3x^2 - x^3) = 27 - 27 = 0$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (1 - e^{x-3}) = 1 - 1 = 0$$
$$f(3) = (3x^2 - x^3)\big|_{x=3} = 27 - 27 = 0$$

Since these three values are all equal, f is continuous at $x = 3$. Hence f is continuous on $(-\infty, 0) \cup (0, \infty)$.

Ex. F-9 2.6 Sp20 Exam

Find the value of k that makes f continuous at $x = -2$ or determine that no such value of k exists.

$$f(x) = \begin{cases} 3x^2 + k & x < -2 \\ -10 & x = -2 \\ kx^3 - 6 & x > -2 \end{cases}$$

exs20

2.6 Sp20 Exam

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exs20

Solution F-9

For f to be continuous at $x = -2$, the corresponding left-limit, right-limit, and function value must all be equal. Those three values in terms of k are given by the following:

$$\begin{aligned} \lim_{x\to-2^-} f(x) &= \lim_{x\to-2^-} (3x^2+k) = 12+k \\ \lim_{x\to-2^+} f(x) &= \lim_{x\to-2^+} (kx^3-6) = -8k-6 \\ f(-2) &= -10 \end{aligned}$$

If f is to be continuous at $x = -2$, we must have $12 + k = -8k - 6 = -10$. This is equivalent to the following set of two equations in the single unknown k.

$$\begin{aligned} 12 + k &= -10 \\ -8k - 6 &= -10 \end{aligned}$$

This set of equations has no solution. (Indeed, the first equation gives $k = -22$, which does not satisfy the second equation.) Hence there is no value of k that makes f continuous at $x = -2$.

Ex. F-10 2.6 Sp20 Exam

Consider the function $f(x)$, where k is an unspecified constant. Find the value of k for which f continuous for all x, or show that no such value of k exists.

$$f(x) = \begin{cases} 38 + kx & x < 3 \\ kx^2 + x - k & x \geq 3 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exs20

Solution F-10

First we calculate the left-limit, right-limit, and function value at $x = 3$.

$$\begin{aligned} \lim_{x\to3^-} f(x) &= \lim_{x\to3^-} (38+kx) = 38+3k \\ \lim_{x\to3^+} f(x) &= \lim_{x\to3^+} \left(kx^2+x-k\right) = 8k+3 \\ f(3) &= 8k+3 \end{aligned}$$

To make f continuous at $x = 3$, the left-limit, right-limit, and function value at $x = 3$ must all be equal. Hence we must have

$$38 + 3k = 8k + 3$$

Hence $k = 7$.

Ex. F-11 2.6 Su20 Exam

In a certain parking garage, the cost of parking is \$20 per hour or any fraction thereof. For example, if you are in the garage for two hours and fifteen minutes, you pay \$60 (\$20 for the first hour, \$20 for the second hour, and \$20 for the fifteen-minute portion of the third hour). Let $P(t)$ be the cost of parking for t hours, where t is any non-negative real number. For example, $P(2.25) = 60$. Is the following true or false?

"$P(t)$ is a continuous function of t."

You must justify your answer.

exsu20

Solution F-11

False. The function $P(t)$ has a jump discontinuity at each non-negative integer (i.e., at $t = 0$, $t = 1$, $t = 2$, etc.).

For instance, the cost of parking for 1 hour or less is \$20. However, as soon as you are in the garage one moment past

F-11

1 hour, the price jumps to \$40. Mathematically, this means all of the following: $\lim_{t\to 1^-} P(t) = 20$, $\lim_{t\to 1^+} P(t) = 40$, and $P(1) = 20$. Hence $P(t)$ is not continuous at $t = 1$. (A similar argument holds for any other non-negative integer value of t.)

Ex. F-12 **2.6** Su20 **Exam**

Consider the following function, where a and b are unspecified constants.

$$f(x) = \begin{cases} 3 & x \le -1 \\ ax^2 + 2x + b & -1 < x \le 2 \\ 14 - ax & x > 2 \end{cases}$$

Find the values of a and b for which f is continuous for all x, or determine that no such values exist. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

exsu20

Solution **F-12**

Each piece of f is continuous for all x, so we need only force continuity at the transition points, $x = -1$ and $x = 2$. At each of these x-values, to have continuity, the left-limit, right-limit, and function value must all be equal. For $x = -1$, we must have:

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} (3) = 3 \\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \left(ax^2 + 2x + b\right) = a - 2 + b \\ f(-1) &= (3)\big|_{x=-1} = 3 \end{aligned}$$

Hence we obtain $a - 2 + b = 3$, or $a + b = 5$. Now for $x = 2$, we must have:

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \left(ax^2 + 2x + b\right) = 4a + 4 + b \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (14 - ax) = 14 - 2a \\ f(2) &= \left(ax^2 + 2x + b\right)\big|_{x=2} = 4a + 4 + b \end{aligned}$$

Hence we obtain $4a + 4 + b = 14 - 2a$, or $6a + b = 10$.

To find a and b we solve the simultaneous system of equations:

$$\begin{aligned} a + b &= 5 \\ 6a + b &= 10 \end{aligned}$$

Subtracting the first equation from the second gives $5a = 5$, whence $a = 1$. Back-substitution then gives $b = 4$.

Ex. D-7 **2.4, 2.6** Su20 **Exam**

Let $f(x) = \dfrac{9x - x^3}{x^2 + x - 6}$.

(a) Calculate all vertical asymptotes of f. Justify your answer.

(b) Where is f discontinuous?

(c) For each point at which f is discontinuous, determine what value should be reassigned to f, if possible, to guarantee that f will be continuous there.

exsu20

Solution **D-7**

(a) Putting the denominator to 0 gives $x^2 + x - 6 = 0$, with solutions $x = -3$ or $x = 2$. Direct substitution of $x = 2$ into f gives the (undefined) expression "$\frac{10}{0}$" (i.e., a non-zero number divided by zero). Hence $x = 2$ is a vertical

D-7

asymptote. However, for $x = -3$, we observe the following.

$$\lim_{x\to -3}\left(\frac{9x-x^3}{x^2+x-6}\right) = \lim_{x\to -3}\left(\frac{x(3-x)(3+x)}{(x-2)(x+3)}\right) = \lim_{x\to -3}\left(\frac{x(3-x)}{x-2}\right) = \frac{18}{5}$$

Since this limit is not infinite, the line $x = -3$ is not a vertical asymptote. The only vertical asymptote is $x = 2$.

(b) Since f is a ratio two continuous functions, f is discontinuous only where its denominator is 0. Hence f is discontinuous only at $x = 2$ and $x = -3$.

(c) From our work in part (a), we know that $x = 2$ is a vertical asymptote. Thus it is impossible to redefine $f(2)$ to make f continuous at $x = 2$. (Why? The limit $\lim_{x\to 2} f(x)$ does not exist.)

However, for $x = -3$, we have $\lim_{x\to -3} f(x) = \frac{18}{5}$. Hence if we redefine $f(-3)$ to be $\frac{18}{5}$, then f becomes continuous at $x = -3$.

Ex. F-13 **2.6** Fa20 **Exam**

Determine where the following function is continuous. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & x < 3 \\ 0 & x = 3 \\ 5x-9 & 3 < x < 4 \\ 11 & x = 4 \\ 27-x^2 & x > 4 \end{cases}$$

exf20

Solution **F-13**

Each piece of f is a rational function (actually, a polynomial) on their respective domains. So each piece is continuous. Hence we need only check continuity at $x = 3$ and $x = 4$. For $x = 3$, we have the following:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5x-9) = 6 \quad , \quad f(3) = 0$$

Since the right-limit and function value are not equal at $x = 3$, f is not continuous at $x = 3$. (Note: we don't even have to consider the left-limit here. However, the left-limit is 6.) For $x = 4$, we have the following:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5x-9) = 11 \quad , \quad \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(27-x^2) = 11 \quad , \quad f(4) = 11$$

Since the left-limit, right-limit, and function value at $x = 4$ are all equal, f is continuous at $x = 4$. Hence f is continuous on $(-\infty, 3) \cup (3, \infty)$.

Ex. F-14 **2.6** Fa20 **Exam**

Consider the function f below, where A, B, and C are unspecified constants.

$$f(x) = \begin{cases} 2x^3 + Ax & x < -1 \\ C & x = -1 \\ Bx^2 + 4 & x > -1 \end{cases}$$

(a) Calculate $\lim_{x\to -1^-} f(x)$.

(b) Calculate $\lim_{x\to -1^+} f(x)$.

(c) How must A and B be related if $\lim_{x\to -1} f(x)$ exists?

(d) Suppose $C = 10$ and f is continuous for all x. Find the values of A and B.

exf20

Solution **F-14**

(a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2x^3 + Ax) = -2 - A$

(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (Bx^2 + 4) = B + 4$

(c) The left- and right-limits must be equal, so we must have that $-2 - A = B + 4$.

(d) To have continuity at $x = -1$, we must have the left-limit, right-limit, and function value all equal. That is, we must have

$$-2 - A = B + 4 = 10$$

Solving for A and B then gives $A = -12$ and $B = 6$.

Ex. F-15 **2.6** Fa20 **Exam**

Which of the following equations expresses the fact that $f(x)$ is continuous at $x = 6$. (There is only one correct choice.)

(a) $\lim_{x\to 6} f(6) = f(6)$

(b) $\lim_{x\to 6} f(6) = 6$

(c) $\lim_{x\to 6} f(x) = f(6)$

(d) $\lim_{x\to 6} f(x) = 6$

(e) $\lim_{x\to 6} f(x) = 0$

(f) $\lim_{x\to 6} f(x) = \infty$

(g) $\lim_{x\to\infty} f(x) = f(6)$

(h) $\lim_{x\to\infty} f(x) = \infty$

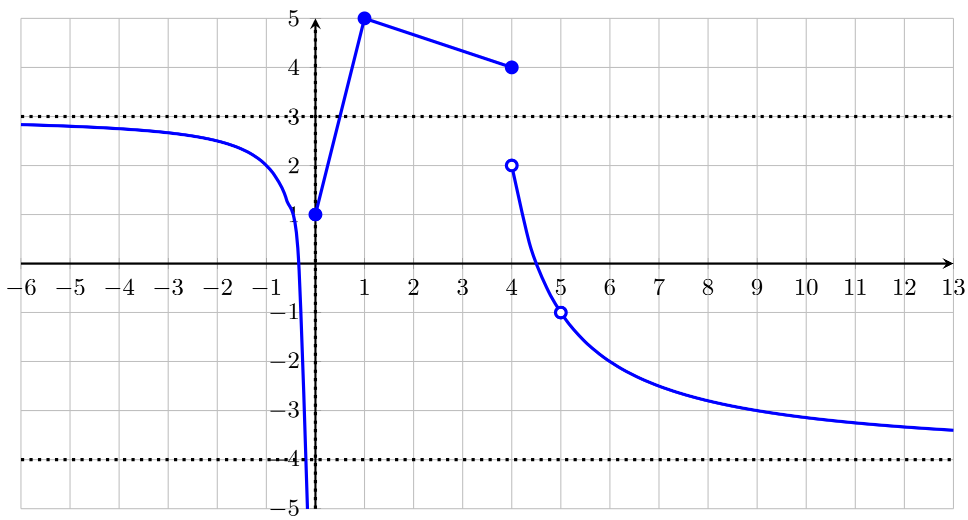
exf20

Solution **F-15**

Choice (c). (Definition of continuity.)

Ex. E-8 **2.5, 2.6, 3.1/3.2** Sp21 **Exam**

Use the graph of f below to answer the following questions. Dashed lines indicate the location of asymptotes.



(a) Calculate $\lim_{x\to\infty} f(x)$.

(b) Calculate $\lim_{x\to -\infty} f(x)$.

(c) List the values of x where f is not continuous.

(d) List the values of x where f is not differentiable.

(e) What is the sign of $f'(-1)$? (choices: positive, negative, zero, does not exist)

(f) What is the sign of $f'(0.5)$? (choices: positive, negative, zero, does not exist)

exs21

Solution **E-8**

(a) $\lim_{x\to\infty} f(x) = -4$

(b) $\lim_{x\to -\infty} f(x) = 3$

E-8

(c) $x = 0$, $x = 4$, $x = 5$
(d) $x = 0$, $x = 1$, $x = 4$, $x = 5$
(e) negative
(f) positive

Ex. F-16 2.6 Sp21 Exam

Consider the function g below, where a and b are unspecified constants. Assume that g is continuous for all x.

$$g(x) = \begin{cases} be^x + a + 1 & x \leq 0 \\ ax^2 + b(x+3) & 0 < x \leq 1 \\ a\cos(\pi x) + 7bx & 1 < x \end{cases}$$

(a) What relation must hold between a and b for g to be continuous at $x = 0$? Your answer should be an equation involving a and b.
(b) What relation must hold between a and b for g to be continuous at $x = 1$? Your answer should be an equation involving a and b.
(c) Calculate the values of a and b.

exs21

Solution F-16

(a) The left- and right-limits of $g(x)$ at $x = 0$ must be equal.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (be^x + a + 1) = b + a + 1$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \big(ax^2 + b(x+3)\big) = 3b$$

Hence we must have $b + a + 1 = 3b$, or $a = 2b - 1$.

(b) The left- and right-limits of $g(x)$ at $x = 1$ must be equal.

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \big(ax^2 + b(x+3)\big) = a + 4b$$
$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (a\cos(\pi x) + 7bx) = -a + 7b$$

Hence we must have $a + 4b = -a + 7b$, or $2a - 3b = 0$.

(c) The equations from parts (a) and (b) must be true simultaneously. Putting the equation from part (a) into the equation from part (b) gives $2(2b - 1) - 3b = 0$, whence $b = 2$. Part (a) then implies $a = 3$.

Ex. F-17 2.6 Fa21 Exam

Consider the piecewise-defined function $f(x)$ below; A and B are unspecified constants and $g(x)$ is an unspecified function with domain $[94, \infty)$.

$$f(x) = \begin{cases} Ax^2 + 8 & x < 75 \\ \ln(B) + 6 & x = 75 \\ \dfrac{x - 75}{\sqrt{x+6} - 9} & 75 < x < 94 \\ 19 & x = 94 \\ g(x) & x > 94 \end{cases}$$

(a) Find $\lim_{x\to 75^-} f(x)$ in terms of A and B.
(b) Find $\lim_{x\to 75^+} f(x)$ in terms of A and B.
(c) Find the exact values of A and B for which f is continuous at $x = 75$.
(d) Suppose $g(94) = 19$. What does this imply about $\lim_{x\to 94} f(x)$? Select the best answer.

(i) $\lim_{x\to 94} f(x)$ exists.

exf21

2.6 Fa21 Exam

(ii) $\lim_{x\to 94} f(x)$ does not exist.

(iii) It gives no information about $\lim_{x\to 94} f(x)$.

exf21

Solution **F-17**

(a) $\lim_{x\to 75^-} f(x) = \lim_{x\to 75^-} (Ax^2+8) = A\cdot 75^2 + 8 = 5625A + 8$

(b) We have the following:

$$\begin{aligned}\lim_{x\to 75^+} f(x) &= \lim_{x\to 75^+}\left(\frac{x-75}{\sqrt{x+6}-9}\right) = \lim_{x\to 75^+}\left(\frac{x-75}{\sqrt{x+6}-9}\cdot\frac{\sqrt{x+6}+9}{\sqrt{x+6}+9}\right)\\ &= \lim_{x\to 75^+}\left(\frac{(x-75)(\sqrt{x+6}+9)}{x+6-81}\right) = \lim_{x\to 75^+}\left(\sqrt{x+6}+9\right)\\ &= \sqrt{81}+9 = 18\end{aligned}$$

(c) We need the left-limit, right-limit, and function value of $f(x)$ at $x = 75$ all to be equal. Thus we must have:

$$5625A + 8 = 18 = \ln(B) + 6$$

Thus $A = \frac{10}{5625}$ and $B = e^{12}$.

(d) ***Choice (iii)***. Note that $\lim_{x\to 94^-} f(x) = \lim_{x\to 94^-}\left(\frac{x-75}{\sqrt{x+6}-9}\right) = 19$ (use direct substitution). So for $\lim_{x\to 94} f(x)$ to exist, we require only that $19 = \lim_{x\to 94^+} f(x) = \lim_{x\to 94^+} g(x)$. However, we are given no information at all about this right-limit of g since the function value $g(94)$ is irrelevant to its value.

Ex. F-18 2.6 Fa21 Exam

Consider the following function.

$$f(x) = \frac{x^2-x-6}{x^3-2x^2-3x}$$

(a) Where is f discontinuous?

(b) At the leftmost x-value where f is discontinuous, what type of discontinuity does f have (removable, jump, infinite (vertical asymptote), or other)?

(c) At the rightmost x-value where f is discontinuous, what type of discontinuity does f have (removable, jump, infinite (vertical asymptote), or other)?

exf21

Solution **F-18**

First we note the following:

$$f(x) = \frac{x^2-x-6}{x^3-2x^2-3x} = \frac{(x+2)(x-3)}{x(x+1)(x-3)}$$

(a) The function f is continuous on its domain, hence discontinuous at $x = -1, 0, 3$ only.

(b) ***Choice (iii).*** Direct substitution of $x = -1$ into $f(x)$ gives the undefined expression "$\frac{-6}{0}$", indicating a vertical asymptote at $x = -1$.

(c) ***Choice (i).*** We see that $\lim_{x\to 3} f(x) = \lim_{x\to 3}\left(\frac{x+2}{x(x+1)}\right) = \frac{5}{12}$. Since this limit exists, f has a removable discontinuity at $x = 3$.

Ex. F-19 2.6 Fa21 Exam

Let $f(x)$ be the following function, where k is an unspecified constant. Find the value of k that makes f continuous at $x = 2$ or determine that no such value of k exists.

$$f(x) = \begin{cases} 27x - kx^2 & x < 2 \\ -6 & x = 2 \\ 3x^3 + k & x > 2 \end{cases}$$

exf21

2.6 Fa21 Exam

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exf21

Solution **F-19**

We first compute the left-limit, right-limit, and function value at $x = 2$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (27x - kx^2) = 54 - 4k$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x^3 + k) = 24 + kf(2) = -6$$

If f is to be continuous at $x = 2$, these quantities must all be equal. Hence we must have $54-4k = -6$ and $24+k = -6$. However, this is impossible since the first equation gives $k = 15$ and the second equation gives $k = -30$. There is no value of k that satisfies both equations simultaneously. Hence there is no value of k for which f is continuous at $x = 2$.

Ex. F-20 2.6 Sp22 Exam

Determine where $f(x)$ is continuous. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

$$f(x) = \begin{cases} \dfrac{(x+1)^2 - 16}{2x-6} & \text{if } x < 3 \\ 3 - \ln(x-2) & \text{if } x \geq 3 \end{cases}$$

exs22

Solution **F-20**

Each "piece" of f is obviously continuous on each of their respective open intervals. The only issue is whether f is continuous at $x = 3$. So we analyze the one-sided limits at $x = 3$. For the left-limit we expand the numerator and cancel common factors.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \left(\frac{(x+1)^2 - 16}{2x-6}\right) = \lim_{x\to 3^-} \left(\frac{x^2 + 2x - 15}{2(x-3)}\right)$$
$$= \lim_{x\to 3^-} \left(\frac{(x-3)(x+5)}{2(x-3)}\right) = \lim_{x\to 3^-} \left(\frac{x+5}{2}\right) = \frac{3+5}{2} = 4$$

For the right-limit we use direct substitution.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3 - \ln(x-2)) = 3 - \ln(1) = 3$$

Since the left- and right-limits at $x = 3$ are not equal, f is discontinuous at $x = 3$. Hence f is continuous on $(-\infty, 3) \cup (3, \infty)$.

Ex. F-21 2.6 Sp22 Exam

Consider the function $f(x)$ defined below, where A and B are unspecified constants. Find the values of A and B for which f is continuous at $x = 2$, or determine that no such values exist.

$$f(x) = \begin{cases} Ax + B - 4 & \text{if } x < 2 \\ 9 & \text{if } x = 2 \\ Ax^2 - 5 & \text{if } x > 2 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exs22

Solution **F-21**

For f to be continuous at $x = 2$, we must have that the left-limit, right-limit, and function value at $x = 2$ are all equal.

F-21

Each of these quantities is given below.

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (Ax + B - 4) = 2A + B - 4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (Ax^2 - 5) = 4A - 5 \\ f(2) &= 9\end{aligned}$$

Since these three quantities must be equal, we have the following equations.

$$\begin{aligned}2A + B - 4 &= 9 \\ 4A - 5 &= 9\end{aligned}$$

The second equation gives $A = 3.5$, and back-substitution in the first equation gives $B = 6$.

Ex. F-22 2.6 Su22 Exam

Consider the function $f(x) = \dfrac{\sin(7x)}{x^2 - 5x}$.

(a) Find the domain of f. Write your answer using interval notation.

(b) Find the x-values where f is discontinuous.

(c) For each value of x where f is discontinuous, classify the type of discontinuity as "removable", "jump", "infinite", or "essential". Clearly label your work and justify your answers.

exsu22

Solution F-22

(a) The domain of f is all real numbers except where $x^2 - 5x = 0$ (i.e., $x = 0$ or $x = 5$). Hence the domain of f is $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$.

(b) Since f is a quotient of continuous functions, f is continuous for all x except where the denominator is 0. Hence f is discontinuous at both $x = 0$ and $x = 5$.

(c) Substitution of $x = 5$ into f gives the undefined expression "$\frac{\sin(35)}{0} = \frac{\text{nonzero }\#}{0}$". Hence $x = 5$ is a vertical asymptote for f, and so f has an infinite discontinuity at $x = 5$.

For $x = 0$, we have the following:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin(7x)}{x}\cdot\frac{1}{x-5}\right) = \lim_{x\to 0}\left(\frac{\sin(7x)}{7x}\cdot\frac{7}{x-5}\right) = 1\cdot\frac{7}{0-5} = -\frac{7}{5}$$

Since this limit is finite, we see that f has a removable discontinuity at $x = 0$.

Ex. F-23 2.6 Su22 Exam

Consider the limit $\displaystyle\lim_{x\to 3}\left(\frac{x^3 - 4x^2 + ax}{x^2 - 9}\right)$, where a is an unspecified constant.

(a) For what values of a does this limit exist? Explain your answer.

(b) Given that the limit does exist, what is its value?

exsu22

Solution F-23

(a) Direct substitution of $x = 3$ gives the undefined expression "$\frac{-9+3a}{0}$". If $-9 + 3a \neq 0$, then $x = 3$ is a vertical asymptote, whence the limit could not exist. Since the limit does exist, we must have $-9 + 3a = 0$, or $a = 3$.

(b) Put $a = 3$, factor, and cancel common factors.

$$\lim_{x\to 3}\left(\frac{x^3 - 4x^2 + 3x}{x^2 - 9}\right) = \lim_{x\to 3}\left(\frac{x(x-3)(x-1))}{(x-3)(x+3)}\right) = \lim_{x\to 3}\left(\frac{x(x-1)}{x+3}\right) = \frac{3\cdot 2}{6} = 1$$

Ex. F-24 2.6 Su22 Exam

Consider the function below, where a and b are unspecified constants. Find the values of a and b for which f is continuous for all x, or determine that no such values exist.

$$f(x) = \begin{cases} ax^2 + 3x + b & x < -1 \\ 2 + ax + \sin\left(\frac{\pi x}{2}\right) & -1 \le x < 4 \\ b(x-3)^2 + 1 & x \ge 4 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

exsu22

Solution F-24

Each piece of f is continuous on their respective intervals. So if f is to be continuous for all x, f must be continuous at the transition points $x = -1$ and $x = 4$.

For $x = -1$, the left-limit, right-limit, and function value must be equal.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (ax^2 + 3x + b) = a - 3 + b$$
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \left(2 + ax + \sin\left(\frac{\pi x}{2}\right)\right) = 1 - a$$
$$f(-1) = \left(2 + ax + \sin\left(\frac{\pi x}{2}\right)\right)\Big|_{x=-1} = 1 - a$$

So we must have $a - 3 + b = 1 - a$, or $2a + b = 4$. For $x = 4$, the left-limit, right-limit, and function value must be equal.

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \left(2 + ax + \sin\left(\frac{\pi x}{2}\right)\right) = 2 + 4a \quad \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \left(b(x-3)^2 + 1\right) = b + 1$$
$$f(4) = \left(b(x-3)^2 + 1\right)\Big|_{x=4} = b + 1$$

So we must have $2 + 4a = b + 1$, or $4a - b = -1$. Thus we must solve the simultaneous set of equations:

$$\begin{aligned} 2a + b &= 4 \\ 4a - b &= -1 \end{aligned}$$

Adding the equations gives $6a = 3$, whence $a = \frac{1}{2}$. Then the first equation gives $b = 3$.

Ex. F-25 2.6 Fa22 Exam

On the axes provided, sketch the graph of a function $f(x)$ that satisfies all of the following properties. ***Note:*** Make sure to read these properties carefully!

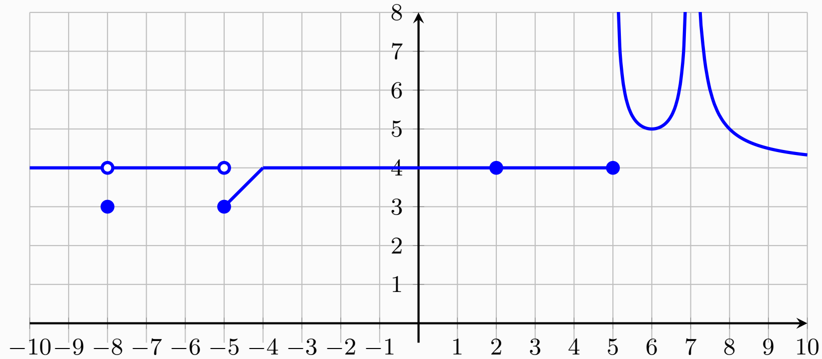
- the domain of $f(x)$ is $[-10, 7) \cup (7, 10]$
- $\lim_{x\to -8} f(x)$ exists but f is discontinuous at $x = -8$
- $\lim_{x\to -5^+} f(x) = f(-5)$ but $\lim_{x\to -5} f(x)$ does not exist
- $\lim_{x\to 2^-} f(x) = 4$ and f is continuous at $x = 2$
- the line $x = 5$ is a vertical asymptote for f (***Note:*** $x = 5$ is in the domain of f.)
- $\lim_{x\to 7} f(x) = +\infty$ (***Note:*** $x = 7$ is not in the domain of f.)

exf22

Solution F-25

There are many such solutions. Here is one.

F-25



Ex. F-26 **2.6** Fa22 **Exam**

Consider the function below, where a and b are unspecified constants.

$$f(x) = \begin{cases} \dfrac{\sin(4x)\sin(6x)}{x^2} & x < 0 \\ ax + b & 0 \le x \le 1 \\ \dfrac{5x+2}{x-1} - \dfrac{2x+5}{x^2-x} & x > 1 \end{cases}$$

(a) Calculate $\lim\limits_{x\to 0^-} f(x)$.

(b) Calculate $\lim\limits_{x\to 1^+} f(x)$.

(c) Find the values of a and b for which f is continuous for all x, or determine that no such values exist. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

exf22

Solution **F-26**

(a) Rearrange the terms and use the special trigonometric limit $\lim\limits_{\theta\to 0}\left(\dfrac{\sin(a\theta)}{a\theta}\right) = 1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin(4x)\sin(6x)}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{\sin(4x)}{4x}\cdot\frac{\sin(6x)}{6x}\cdot 4\cdot 6\right) = 1\cdot 1\cdot 4\cdot 6 = 24$$

(b) Find a common denominator. Then cancel common factors.

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}\left(\frac{5x+2}{x-1} - \frac{2x+5}{x^2-x}\right) = \lim_{x\to 1^+}\left(\frac{5x^2+2x}{x^2-x} - \frac{2x+5}{x^2-x}\right) \\ &= \lim_{x\to 1^+}\left(\frac{5x^2-5}{x^2-x}\right) = \lim_{x\to 1^+}\left(\frac{5(x-1)(x+1)}{x(x-1)}\right) = \lim_{x\to 1^+}\left(\frac{5(x+1)}{x}\right) = \frac{5(1+1)}{1} = 10\end{aligned}$$

(c) If f is to be continuous at $x = 0$, the left-limit, right-limit, and function value of f at $x = 0$ must be equal.

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= 24 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(ax+b) = b \\ f(0) &= (ax+b)\big|_{x=0} = b\end{aligned}$$

Thus we must have $b = 24$. If f is to be continuous at $x = 1$, the left-limit, right-limit, and function value of f at $x = 1$ must be equal.

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(ax+b) = a+b \\ \lim_{x\to 1^+} f(x) &= 10 \\ f(0) &= (ax+b)\big|_{x=1} = a+b\end{aligned}$$

Thus we must have $a + b = 10$. Given $b = 24$, we find that $a = -14$.

Ex. F-27 **2.6** Sp18 **Quiz**

Consider the following function.

$$f(x) = \begin{cases} x^3 + 27 & \text{if } x \leq -3 \\ \dfrac{x+3}{2-\sqrt{1-x}} & \text{if } -3 < x < 1 \\ 4 & \text{if } x = 1 \\ x^2 + 2x - 1 & \text{if } 1 < x \end{cases}$$

(a) Find all points where f is discontinuous. *Be sure to give a full justification here.*

(b) For each x-value you found in part (a), determine what value should be assigned to f, if any, to guarantee that f will be continuous there. Justify your answer.

(For example, if you claim f is discontinuous at $x = a$, then you should determine the value that should be assigned to $f(a)$, if any, to guarantee that f will be continuous at $x = a$.)

qs18

Solution **F-27**

(a) For $x \neq -3$ and $x \neq 1$, note that each piece individually is continuous on the given intervals. For $x = -3$, we have the following:

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (x^3 + 27) = (-3)^3 + 27 = 0$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \left(\frac{x+3}{2-\sqrt{1-x}} \cdot \frac{2+\sqrt{1-x}}{2+\sqrt{1-x}} \right) = \lim_{x \to -3^+} \left(\frac{(x+3)(2+\sqrt{1-x})}{x+3} \right)$$

$$= \lim_{x \to -3^+} \left(2 + \sqrt{1-x}\right) = 4$$

$$f(-3) = (x^3 + 27)\big|_{x=-3} = (-3)^3 + 27 = 0$$

Since these three numbers are not all equal, f is discontinuous at $x = -3$.

For $x = 1$, we have the following:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\frac{x+3}{2-\sqrt{1-x}} \right) = \frac{1+3}{2-\sqrt{1-1}} = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2x - 1) = 1^2 + 2(1) - 1 = 2$$

$$f(1) = 4$$

Since these three numbers are not all equal, f is discontinuous at $x = 1$. In summary, we have found that f is continuous for all real numbers except $x = -3$ and $x = 1$.

(b) Since the one-sided limits at $x = -3$ are not equal, the two-sided limit $\lim\limits_{x \to -3} f(x)$ does not exist. Hence it is not possible to assign a value to $f(-3)$ to make f continuous at $x = -3$.

The one-sided limits at $x = 1$ are equal, and so $\lim\limits_{x \to 1} f(x) = 2$. Hence if we re-assign $f(1)$ the value of 2, then f would be continuous at $x = 1$.

Ex. F-28 **2.6** Sp20 **Quiz**

Find the values of a and b for which f is continuous for all x, or show that no such values of a and b exist. You must use proper calculus methods and clearly explain your work using limits.

$$f(x) = \begin{cases} ax^2 - bx - 6 & \text{if } x < 3 \\ b & \text{if } x = 3 \\ 10x - x^3 & \text{if } x > 3 \end{cases}$$

qs20

Solution **F-28**

Since each piece of f is continuous, we need only force continuity at $x = 3$. So we calculate the left-limit, right-limit,

F-28

and function value at $x = 3$ and set these three quantities equal to each other.

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} (ax^2 - bx - 6) = 9a - 3b - 6\\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (10x - x^3) = 3\\ f(3) &= b\end{aligned}$$

From the right-limit and function value, we immediately find that $b = 3$. We must also have $9a - 3b - 6 = b$, whence $a = 2$.

Ex. F-29 2.6 Sp20 Quiz

Determine where $f(x)$ is continuous. Write your answer using interval notation.

$$f(x) = \begin{cases} 4x^2 - 10 & \text{if } x < -1\\ 6\sin\left(\frac{\pi x}{2}\right) & \text{if } -1 \le x \le 4\\ x - 4^{x-3} & \text{if } x > 4\end{cases}$$

qs20

Solution F-29

Observe that f is clearly continuous for all x except possibly $x = -1$ or $x = 4$. For these transition points, we check whether the corresponding left-limit, right-limit, and function value are equal. For $x = -1$ we have:

$$\begin{aligned}\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} (4x^2 - 10) = 4 - 10 = -6\\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} (6\sin(\tfrac{\pi x}{2})) = 6\cdot(-1) = -6\\ f(-1) &= 6\sin(\pi x/2)\big|_{x=-1} = 6\cdot(-1) = -6\end{aligned}$$

Since these three values are all equal, f is continuous at $x = -1$. For $x = 4$ we have:

$$\begin{aligned}\lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-} (6\sin(\tfrac{\pi x}{2})) = 6\cdot 0 = 0\\ \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} (x - 4^{x-3}) = 4 - 4^1 = 0\\ f(4) &= 6\sin(\tfrac{\pi x}{2})\big|_{x=4} = 6\cdot 0 = 0\end{aligned}$$

Since these three values are all equal, f is continuous at $x = 4$. Hence the final answer is that f is continuous on $(-\infty, \infty)$.

Ex. F-30 2.6 Su22 Quiz

Consider the function $f(x)$ below, where a and b are unspecified constants.

$$f(x) = \begin{cases} ax^2 - 7x + b & x < 2\\ 10 & x = 2\\ ae^{x-2} + b\ln(x-1) & x > 2\end{cases}$$

Find the values of a and b for which f is continuous for all x, or determine that no such values exist. Write "NONE" in the answer boxes if no such values exist.

In your work, you must use proper notation and limit-based methods to solve this problem. Solutions that have work that does not have proper notation or which is not based on limits will not receive full credit.

qsu22

Solution F-30

Each piece of f is continuous on the corresponding interval. So we need only impose continuity on f at $x = 2$ to ensure f is continuous for all x. Thus we must have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

F-30

We now calculate these quantities.

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \left(ax^2 - 7x + b\right) = 4a - 14 + b \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \left(ae^{x-2} + b\ln(x-1)\right) = ae^0 + b\ln(1) = a \\ f(2) &= 10\end{aligned}$$

So we must have $4a - 14 + b = a = 10$. We immediately find that $a = 10$. Then the equation $4a - 14 + b = 10$ (with $a = 10$) gives us $b = -16$.

Ex. F-31 2.6 Su22 Quiz

Let $f(x) = \dfrac{x^3 - 7x^2 + 10x}{x^2 - 6x}$.

(a) Find the domain of f. Write your answer using interval notation.

(b) Find all values of x where f is discontinuous.

(c) For each value of x where f is discontinuous, classify the type of discontinuity as "removable", "jump", "infinite", or "essential". Clearly label your work and justify your answers.

qsu22

Solution F-31

(a) Since f is a rational function, its domain is all real numbers except where $x^2 - 6x = 0$, i.e., the set $(-\infty, 0) \cup (0, 6) \cup (6, \infty)$.

(b) Since f is a rational function, it is a continuous precisely on its domain. Hence f is discontinuous at both $x = 0$ and $x = 6$.

(c) We examine the limits of f at $x = 0$ and $x = 6$. For $x = 0$, we have:

$$\lim_{x\to 0}\left(\frac{x^3 - 7x^2 + 10x}{x^2 - 6x}\right) = \lim_{x\to 0}\left(\frac{x(x-2)(x-5)}{x(x-6)}\right) = \lim_{x\to 0}\left(\frac{(x-2)(x-5)}{x-6}\right) = -\frac{3}{5}$$

Since this limit is finite, we conclude that f has a removable discontinuity at $x = 0$.

For $x = 6$, we simply observe that direct substitution of $x = 6$ into f gives the expression "$\frac{\text{nonzero }\#}{0}$", which implies $x = 6$ is a vertical asymptote. Hence f has an infinite discontinuity at $x = 6$.

Ex. F-32 2.6 Fa22 Quiz

Find the value of A that makes $f(x)$ continuous for all x, or determine that no such value exists. Write "DNE" if no such value of A exists. Your solution must be based on limits to receive full credit.

$$f(x) = \begin{cases} \dfrac{\sin(Ax)}{x} - 2 & \text{if } x < 0 \\ 9 & \text{if } x = 0 \\ 3x^3 - A\cos(x) + 10 & \text{if } x > 0 \end{cases}$$

qf22

Solution F-32

We must have that the left-limit, right-limit, and function value at $x = 0$ are all equal. First we compute each of these.

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left(\frac{\sin(Ax)}{x} - 2\right) = \lim_{x\to 0^-}\left(\frac{\sin(Ax)}{Ax}\cdot A - 2\right) = 1\cdot A - 2 = A - 2 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left(3x^3 - A\cos(x) + 10\right) = 0 - A + 10 = 10 - A \\ f(0) &= 9\end{aligned}$$

Hence we must have $A - 2 = 10 - A = 9$. However, this is impossible as the equation $A - 2 = 9$ implies $A = 11$ and the equation $10 - A = 9$ implies $A = 1$. Hence there is no value of A for which f is continuous.

Ex. F-33 **2.6**

Determine where $f(x)$ is continuous.

$$f(x) = \begin{cases} 3x^2 - x + 1 & \text{if } x < -2 \\ 15 + \sin(2\pi x) & \text{if } -2 \le x < 3 \\ 2x - 4 & \text{if } 3 \le x \end{cases}$$

c2s6

Solution **F-33**

Each "piece" of $f(x)$ is continuous on its respective open interval, whence f is continuous for all x except possibly where these "pieces" transition (i.e., $x = -2$ or $x = 3$). Recall that for $f(x)$ to be continuous at $x = a$, we must have that the left-limit, right-limit, and function value of f (all at $x = a$) are equal.
For $x = -2$, we have the following:

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (3x^2 - x + 1) = 3(-2)^2 - (-2) = 15 \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} (15 + \sin(2\pi x)) = 15 + \sin(-4\pi) = 15 \\ f(-2) &= (15 + \sin(2\pi x))\big|_{x=-2} = 15 + \sin(-4\pi) = 15 \end{aligned}$$

Hence f is continuous at $x = -2$. For $x = 3$, we have the following:

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} (15 + \sin(2\pi x)) = 15 + \sin(-6\pi) = 15 \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} (2x - 4) = 2(3) - 4 = 2 \\ f(3) &= (2x - 4)\big|_{x=3} = 2(3) - 4 = 2 \end{aligned}$$

Hence f is not continuous at $x = 3$. The function f is continuous on $(-\infty, 3) \cup (3, \infty)$.

Ex. F-34 **2.6**

Let $f(x) = \dfrac{x^3 - 9x}{x + 3}$.

(a) What is the domain of f?

(b) Find all points where f is discontinuous.

(c) For each point where f is discontinuous, classify the type of discontinuity as removable, jump, infinite, or other.

c2s6

Solution **F-34**

(a) Since $f(x)$ is a rational function, its domain is all real numbers except where the denominator is 0. Hence the domain of f is $(-\infty, -3) \cup (-3, \infty)$.

(b) Since f is a rational function, f is discontinuous only at points not in its domain. Hence f is discontinuous only at $x = -3$.

(c) We have the following:

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \left(\frac{x^3 - 9x}{x + 3} \right) = \lim_{x \to -3} \left(\frac{x(x-3)(x+3)}{x + 3} \right) = \lim_{x \to -3} (x(x-3)) = 18$$

Since this limit exists, f has a removable discontinuity at $x = -3$.

Ex. F-35 **2.6**

Let $f(x) = \dfrac{\sqrt{2x^2 + 1} - 1}{x^2(x - 3)}$.

(a) What is the domain of f?

(b) Find all points where f is discontinuous.

(c) For each point where f is discontinuous, classify the type of discontinuity as removable, jump, infinite, or other.

c2s6

Solution **F-35**

(a) Note that $2x^2 + 1 \geq 0$ for all x, so the only points not in the domain of f are those for which $x^2(x-3) = 0$. Hence the domain is $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$.

(b) Since f is an algebraic function, f is discontinuous only at points not in its domain. Hence f is discontinuous only at $x = 0$ and $x = 3$.

(c) For $x = 0$ we have:

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0}\left(\frac{\sqrt{2x^2+1}-1}{x^2(x-3)}\right) = \lim_{x\to 0}\left(\frac{\sqrt{2x^2+1}-1}{x^2(x-3)} \cdot \frac{\sqrt{2x^2+1}+1}{\sqrt{2x^2+1}+1}\right) \\ &= \lim_{x\to 0}\left(\frac{2x^2}{x^2(x-3)(\sqrt{2x^2+1}+1)}\right) = \lim_{x\to 0}\left(\frac{2}{(x-3)(\sqrt{2x^2+1}+1)}\right) = -\frac{1}{3}\end{aligned}$$

Since this limit exists, f has a removable discontinuity at $x = 0$. For $x = 3$, direct substitution gives the undefined form $\frac{\sqrt{19}-1}{0}$, or $\frac{c}{0}$ (with $c \neq 0$). This indicates that the left- and right-limits of $f(x)$ at $x = 3$ are both infinite. Hence f has an infinite discontinuity (vertical asymptote) at $x = 3$.

Ex. F-36 **2.6**

Find the values of the constants a and b that make f continuous for all real numbers.

$$f(x) = \begin{cases} ax^2 - x & \text{if } x < 4 \\ 6 & \text{if } x = 4 \\ x^3 + bx & \text{if } x > 4 \end{cases}$$

c2s6

Solution **F-36**

Any values of a and b make each individual piece continuous for all real numbers. Hence we need only force continuity at $x = 4$.

$$\begin{aligned}\lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-}(ax^2 - x) = 16a - 4 \\ \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+}(x^3 + bx) = 64 + 4b \\ f(4) &= 6\end{aligned}$$

If f is to be continuous at $x = 4$, these three values must be equal. Hence we obtain the two equations $16a - 4 = 6$ (whence $a = \frac{10}{16}$) and $64 + 4b = 6$ (whence $b = -\frac{29}{2}$).

Ex. F-37 **2.6**

Find the values of the constants a and b that make f continuous for all real numbers.

$$f(x) = \begin{cases} ax + 2b & \text{if } x \leq 0 \\ x^2 + 3a - b & \text{if } 0 < x \leq 2 \\ 3x - 5 & \text{if } x > 2 \end{cases}$$

c2s6

Solution **F-37**

Any values of a and b make each individual piece continuous for all real numbers. Hence we need only force continuity at $x = 0$ and $x = 2$. For $x = 0$, we have:

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}(ax + 2b) = 2b \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(x^2 + 3a - b) = 3a - b \\ f(0) &= 2b\end{aligned}$$

F-37

Hence we must have $2b = 3a - b$. For $x = 2$, we have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 + 3a - b) = 4 + 3a - b$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x - 5) = 1$$
$$f(2) = 4 + 3a - b$$

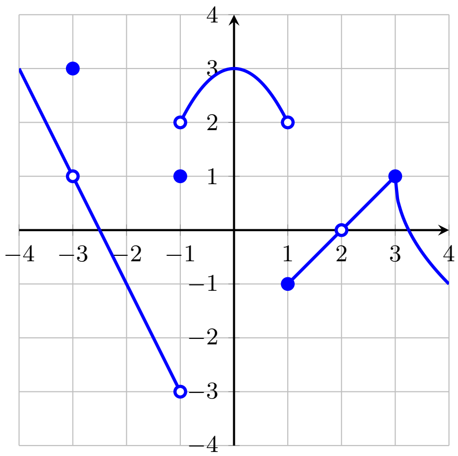
Hence we must have $4 + 3a - b = 1$ to have continuity at $x = 2$. Putting these two conditions together gives us a system of two simultaneous equations:

$$2b = 3a - b$$
$$4 + 3a - b = 1$$

We can solve this system by substitution or elimination. By way of substitution, we find that the first equation is equivalent to $a = b$, whence the second equation is $4+2a = 1$ (with solution $a = -\frac{3}{2}$). Hence we must have $a = b = -\frac{3}{2}$.

Ex. F-38 2.6

The figure below shows the graph of $y = f(x)$. Find all values of x in the interval $(-4, 4)$ at which f is not continuous.



unit2

Solution F-38

The function $f(x)$ is discontinuous at $x = -3$, $x = -1$, $x = 1$, and $x = 2$.

Ex. F-39 2.6

Find the values of the constants a and b that make f continuous at $x = 9$.

$$f(x) = \begin{cases} \sin(2\pi x) - 2ax & \text{if } x < 9 \\ b & \text{if } x = 9 \\ \dfrac{x-9}{\sqrt{x}-3} & \text{if } x > 9 \end{cases}$$

unit2

Solution F-39

First we calculate the left-limit, right-limit, and function value of $f(x)$ at $x = 9$.

$$\lim_{x\to 9^-} f(x) = \lim_{x\to 9^-} (\sin(2\pi x) - 2ax) = \sin(18\pi) - 18a = -18a$$
$$\lim_{x\to 9^+} f(x) = \lim_{x\to 9^+} \left(\frac{x-9}{\sqrt{x}-3}\right) = \lim_{x\to 9^+} \left(\sqrt{x}+3\right) = 6$$
$$f(9) = b$$

These three values must be equal for f to be continuous at $x = 9$. Hence $-18a = 6 = b$, and so $a = -1/3$ and $b = 6$.

Ex. F-40 2.6

Consider the function $f(x)$, where a and b are unspecified constants.

$$f(x) = \begin{cases} \dfrac{2x}{\sin(ax)} & \text{if } x < 0 \\ x - 4 & \text{if } 0 \le x < 5 \\ b & \text{if } x = 5 \\ \dfrac{4 - \sqrt{3x+1}}{x-5} & \text{if } x > 5 \end{cases}$$

(a) Find the value of a so that f is continuous at $x = 0$, or show that no such value exists.

(b) Find the value of b so that f is continuous at $x = 5$, or show that no such value exists.

other

Solution F-40

(a) We require that the left-limit, right-limit, and function value all be equal at $x = 0$. We have the following.

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \left(\frac{2x}{\sin(ax)} \right) = \lim_{x \to 0^-} \left(\frac{2}{a} \cdot \frac{ax}{\sin(ax)} \right) = \frac{2}{a} \cdot 1 = \frac{2}{a} \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x - 4) = -4 \\ f(0) &= (x-4)\big|_{x=0} = -4 \end{aligned}$$

So we must have that $\frac{2}{a} = -4$, or $a = -\frac{1}{2}$.

(b) We require that the left-limit, right-limit, and function value all be equal at $x = 5$. We have the following.

$$\begin{aligned} \lim_{x \to 5^-} f(x) &= \lim_{x \to 5^-} (x - 4) = 1 \\ \lim_{x \to 5^+} f(x) &= \lim_{x \to 5^+} \left(\frac{4 - \sqrt{3x+1}}{x-5} \right) = \lim_{x \to 5^+} \left(\frac{4 - \sqrt{3x+1}}{x-5} \cdot \frac{4 + \sqrt{3x+1}}{4 + \sqrt{3x+1}} \right) \\ &= \lim_{x \to 5^+} \left(\frac{15 - 3x}{(x-5)(4 + \sqrt{3x+1})} \right) = \lim_{x \to 5^+} \left(\frac{-3}{4 + \sqrt{3x+1}} \right) = \frac{-3}{8} \\ f(0) &= b \end{aligned}$$

So we must have that $1 = -\frac{3}{8} = b$, which is impossible.

(It is impossible to find such a value of b because $\lim_{x \to 5} f(x)$ does not exist.)

Ex. F-41 2.6

Consider the function

$$f(x) = \begin{cases} ax^2 - 3b & \text{if } x \le -1 \\ \cos(\pi x) + ax & \text{if } -1 < x < 2 \\ 2b - x^3 & \text{if } x \ge 2 \end{cases}$$

where a and b are unspecified constants. For what values of a and b, if any, is f continuous for all x?

other

Solution F-41

We must have continuity both at $x = -1$ and $x = 2$. For $x = -1$, we have:

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} \left(ax^2 - 3b \right) = a - 3b \\ \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (\cos(\pi x) + ax) = -1 - a \\ f(-1) &= \left(ax^2 - 3b \right)\big|_{x=-1} = a - 3b \end{aligned}$$

F-41

So we must have $a - 3b = -1 - a$, or $2a - 3b = -1$. For $x = 2$, we have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (\cos(\pi x) + ax) = 1 + 2a$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(2b - x^3\right) = 2b - 8$$
$$f(2) = \left(2b - x^3\right)\big|_{x=2} = 2b - 8$$

So we must have $1 + 2a = 2b - 8$, or $2a - 2b = -9$. Thus a and b must satisfy the simultaneous set of equations:

$$2a - 3b = -1$$
$$2a - 2b = -9$$

Subtracting the equations gives $b = -8$, and substituting into the first equation gives $a = -12.5$.

Ex. F-42 **2.6** ⋆Challenge

Consider $f(x) = \dfrac{\tan(2x)}{|5x|}$.

(a) Where is f not continuous?

(b) Is it possible to redefine f at $x = 0$ to make f continuous there? Explain your answer.

Hint: For the limit of f as $x \to 0$, examine the one-sided limits first.

challenge

Solution F-42

(a) The numerator $\tan(2x)$ is continuous precisely on its domain, hence not continuous wherever $\cos(2x) = 0$, that is, wherever $2x$ is an odd multiple of $\frac{\pi}{2}$. The denominator $|5x|$ vanishes when $x = 0$, and so $f(x)$ is also not continuous when $x = 0$. Hence $f(x)$ is not continuous at the following x-values:

$$x = \ldots, \frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

(b) We must compute the limit $\lim\limits_{x\to 0} f(x)$. Observe that $|x| = -x$ for $x < 0$ and $|x| = x$ for $x > 0$. So we have:

$$\lim_{x\to 0^-}\left(\frac{\tan(2x)}{|5x|}\right) = \lim_{x\to 0^-}\left(\frac{\sin(2x)}{2x}\cdot\frac{2x}{-5x}\right) = 1\cdot\frac{-2}{5} = -\frac{2}{5}$$
$$\lim_{x\to 0^+}\left(\frac{\tan(2x)}{|5x|}\right) = \lim_{x\to 0^+}\left(\frac{\sin(2x)}{2x}\cdot\frac{2x}{5x}\right) = 1\cdot\frac{2}{5} = \frac{2}{5}$$

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist, and so there is no value which we can give to $f(0)$ to ensure continuity of $f(x)$ at $x = 0$.

Ex. F-43 **2.6** ⋆Challenge

Find the values of the constants a and b that make f continuous at $x = 0$. *You may assume* $a > 0$.

$$f(x) = \begin{cases} \dfrac{1-\cos(ax)}{x^2} & , \quad x < 0 \\ 2a + b & , \quad x = 0 \\ \dfrac{x^2 - bx}{\sin(x)} & , \quad x > 0 \end{cases}$$

challenge

Solution **F-43**

We need only force continuity at $x = 0$.

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left(\frac{1-\cos(ax)}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{1-\cos(ax)}{x^2}\cdot\frac{1+\cos(ax)}{1+\cos(ax)}\right)\\
&= \lim_{x\to 0^-}\left(\frac{1-\cos(ax)^2}{x^2(1+\cos(ax))}\right) = \lim_{x\to 0^-}\left(\frac{\sin(ax)^2}{x^2(1+\cos(ax))}\right)\\
&= \lim_{x\to 0^-}\left(\left(\frac{\sin(ax)}{x}\right)^2\cdot\frac{1}{1+\cos(ax)}\right)\\
&= \lim_{x\to 0^-}\left(\left(a\cdot\frac{\sin(ax)}{ax}\right)^2\cdot\frac{1}{1+\cos(ax)}\right) = (a\cdot 1)^2\cdot\frac{1}{1+1} = \frac{a^2}{2}\\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left(\frac{x^2-bx}{\sin(x)}\right) = \lim_{x\to 0^+}\left(\frac{x}{\sin(x)}\cdot(x-b)\right) = 1\cdot(0-b) = -b\\
f(0) &= 2a+b
\end{aligned}$$

If f is to be continuous at $x = 0$, these three values must be equal. Hence we obtain the following system of equations:

$$\begin{aligned}
\frac{a^2}{2} &= 2a+b\\
-b &= 2a+b
\end{aligned}$$

The second equation is equivalent to $a = -b$, and substituting into the first equation gives $\frac{a^2}{2} = a$. Dividing by a (which we are told is positive!) gives $\frac{a}{2} = 1$, or $a = 2$. Hence we must have $a = 2$ and $b = -2$.

3 Chapter 3: Derivatives

§3.1, 3.2: Introduction to the Derivative

Ex. G-1 **3.1/3.2** Sp18 **Exam**

The parts of this question are independent of each other.

(a) Given the function $g(x)$, state the definition of $g'(4)$.

(b) Let $F(x) = \dfrac{1}{3x-5}$. Calculate $F'(2)$ directly from the definition. Show all work. *If you simply quote a rule, you will receive no credit. You must use the definition of derivative.*

exs18

Solution **G-1**

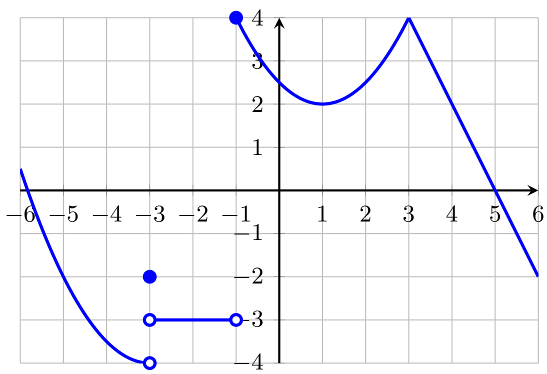
(a) $g'(4) = \lim_{h\to 0}\left(\dfrac{g(4+h)-g(4)}{h}\right)$

(b) Start with the definition of derivative, then simplify and cancel.

$$F'(2) = \lim_{h\to 0}\left(\frac{F(2+h)-F(2)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{3(2+h)-5}-1}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{3h+1}-1}{h}\right) = \lim_{h\to 0}\left(\frac{-3}{3h+1}\right) = -3$$

Ex. F-2 **2.6, 3.1/3.2** Sp18 **Exam**

For each part, use the graph of $y = f(x)$ below.



(a) Find where $f(x)$ is *not* continuous in the interval $(-5, 5)$.

(b) Find where $f(x)$ is *not* differentiable in the interval $(-5, 5)$.

(c) Find where $f'(x) = 0$ in the interval $(-5, 5)$.

(d) Find where $f'(x) < 0$ in the interval $(-5, 5)$.

exs18

Solution **F-2**

(a) $x = -3$, $x = -1$

(b) $x = -3$, $x = -1$, $x = 3$

Recall that continuity is necessary for differentiability. So any points of discontinuity are also points of non-differentiability. At $x = 3$, the graph has a sharp corner, which means the function is not differentiable there.

(c) At each x-values in the interval $(-3, -1)$ and at $x = 1$.

Recall that if $f'(a) = 0$, then the graph of $y = f(x)$ has a horizontal tangent line at $x = a$. That is, the slope of the graph of $f(x)$ is 0.

(d) On each of the intervals $(-5, -3)$, $(-1, 1)$, and $(3, 5)$.

Ex. G-2 **3.1/3.2** Sp18 **Exam**

Find an equation of each line that is both tangent to the graph of $f(x) = 4x^2 - 3x - 1$ and parallel to the line $y = 13x - 5$.

exs18

Solution **G-2**

The slope of the line $y = 13x - 5$ is 13, hence the slope of the desired tangent line is also 13 since parallel lines have equal slope. Hence we must solve the equation $f'(x) = 13$.

$$f'(x) = 8x - 3 = 13 \implies x = 2$$

Observe that $f(2) = 9$. Hence the desired tangent line is $y = 9 + 13(x - 2)$.

Ex. G-3 **3.1/3.2** Sp19 **Exam**

Let $g(x) = 6 - \dfrac{9}{x}$. Calculate $g'(3)$ directly from the limit definition of the derivative. *If you simply quote a rule, you will receive no credit. You must use the definition of derivative.*

exs19

Solution **G-3**

Start with the definition of derivative and compute the limit using algebra.

$$g'(3) = \lim_{h\to 0}\left(\frac{g(3+h) - g(3)}{h}\right) = \lim_{h\to 0}\left(\frac{\left(6 - \frac{9}{3+h}\right) - \left(6 - \frac{9}{3}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{3 - \frac{9}{3+h}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(3+h) - 9}{h(3+h)}\right) = \lim_{h\to 0}\left(\frac{3h}{h(3+h)}\right) = \lim_{h\to 0}\left(\frac{3}{3+h}\right) = \frac{3}{3+0} = 1$$

Ex. G-4 **3.1/3.2** Sp20 **Exam**

Let $f(x) = \dfrac{x+8}{x-3}$. Use the limit definition of derivative to calculate $f'(2)$. *If you simply quote a rule, you will receive no credit. You must use the definition of derivative.*

exs20

Solution **G-4**

Start with the definition of derivative and compute the limit using algebra.

$$f'(2) = \lim_{h\to 0}\left(\frac{f(2+h) - f(2)}{h}\right) = \lim_{h\to 0}\left(\frac{\dfrac{h+10}{h-1} - (-10)}{h}\right) = \lim_{h\to 0}\left(\frac{h+10+10(h-1)}{h(h-1)}\right)$$

$$= \lim_{h\to 0}\left(\frac{11h}{h(h-1)}\right) = \lim_{h\to 0}\left(\frac{11}{h-1}\right) = \frac{11}{0-1} = -11$$

Ex. G-5 **3.1/3.2** Sp20 **Exam**

Which statement is true about the graph of $f(x) = |x| + 91$ at the point $(0, 91)$?

(a) The graph has a tangent line at $y = 91$.
(b) The graph has infinitely many tangent lines.
(c) The graph has no tangent line.
(d) The graph has two tangent lines: $y = x + 91$ and $y = -x + 91$.
(e) None of the above statements is true.

exs20

Solution **G-5**

Choice C. Since $f(x)$ is not differentiable at $x = 0$, $f'(0)$ doesn't exist. So there is no tangent line at $x = 0$.

Ex. G-6 **3.1/3.2, 4.1, 4.9** Sp20 **Exam**

Suppose the derivative of f is $f'(x) = 3x^2 - 6x - 9$ and that $f(1) = 10$.

(a) Find an equation of the line tangent to the graph of $y = f(x)$ at $x = 1$.
(b) Find the critical points of f.
(c) Where does f have a local minimum value? local maximum value?

exs20

3.1/3.2, 4.1, 4.9 Sp20 **Exam**

(d) Calculate $f(0)$.

(e) Calculate the absolute maximum value of f on the interval $[0,6]$. At what x-value does it occur?

exs20

Solution **G-6**

(a) We have $f'(1) = 3 - 6 - 9 = -12$, whence an equation of the tangent line is $y = 10 - 12(x-1)$.

(b) Solving $f'(x) = 0$, we find that the critical points of f are $x = -1$ and $x = 3$.

(c) A sign chart for $f'(x)$ reveals that $f'(x)$ is positive on the intervals $(-\infty, -1)$ and $(3, \infty)$; and $f'(x)$ is negative on the interval $(-1, 3)$. Since f' changes from positive to negative at $x = -1$, a local maximum occurs at $x = -1$. Since f' changes from negative to positive to $x = 3$, a local minimum occurs at $x = 3$.

(d) We find $f(x)$ by finding the most general antiderivative of $f'(x)$.

$$f(x) = \int f'(x)\,dx = x^3 - 3x^2 - 9x + C$$

The initial condition $f(1) = 10$ implies $1 - 3 - 9 + C = 10$, or $C = 21$. Hence

$$f(x) = x^3 - 3x^2 - 9x + 21$$

So $f(0) = 21$.

(e) The absolute maximum of f on $[0,6]$ can occur only at an endpoint (0 or 6) or a critical number (-1 or 3). Calculating the values of f at these x-values gives: $f(0) = 21$, $f(-1) = 26$, $f(3) = -6$, and $f(6) = 75$. Hence the absolute maximum of f on $[0,6]$ is 75, occurring at $x = 6$.

Ex. G-7 **3.1/3.2** Su20 **Exam**

Explain the relationship between $f'(3)$ and the line tangent to the graph of $y = f(x)$ at $x = 3$.

exsu20

Solution **G-7**

The slope of the tangent line at $x = 3$ is $f'(3)$.

Ex. G-8 **3.1/3.2** Su20 **Exam**

Suppose $f'(7)$ exists. What can be said about the limit $\lim_{x\to 7} f(x)$?

exsu20

Solution **G-8**

Since f is differentiable at $x = 7$, f must also be continuous at $x = 7$. Hence $\lim_{x\to 7} f(x)$ exists and is equal to $f(7)$.

Ex. G-9 **3.1/3.2** Su20 **Exam**

Consider the following limit.

$$\lim_{h\to 0}\left(\frac{(4+h)^{3/2} - 8}{h}\right)$$

Use the limit definition of derivative to identify this limit as the derivative of some function $f(x)$ at the point $x = a$. Then calculate the value of the limit.

exsu20

Solution **G-9**

Let $f(x) = x^{3/2}$ and $a = 4$. Then, by the definition of derivative, the given limit is equal to $f'(4)$. To compute the limit, we use the power rule to note that $f'(x) = \frac{3}{2}x^{1/2}$. So $f'(4)$, and hence the given limit, is equal to $\frac{3}{2} \cdot 4^{1/2} = 3$.

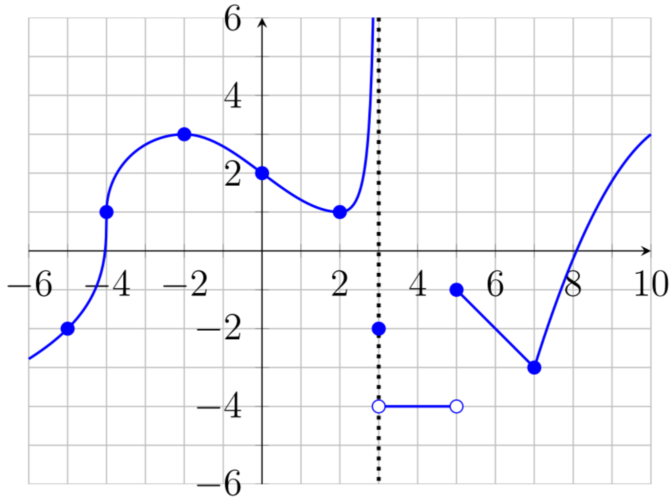
Ex. G-10 **3.1/3.2** Su20 **Exam**

Use the graph of $y = f(x)$ below to answer the following questions.

exsu20

3.1/3.2

Su20 Exam



(a) In the interval $(-6, 10)$, where is f not differentiable?

(b) Calculate a reasonable estimate of $f'(0)$. Explain your reasoning.

(c) In the interval $(-6, 10)$, where is $f'(x) = 0$?

(d) In the interval $(-6, 10)$, where is $f'(x) < 0$?

(e) In the interval $(-6, 10)$, where is $f'(x) > 0$?

exsu20

Solution **G-10**

(a) $x = -4$, $x = 3$, $x = 5$, and $x = 7$

(b) We use the secant line through the points $(-2, 3)$ and $(2, 1)$ to estimate $f'(0)$. The slope of this secant line is $m = \frac{1-3}{2-(-2)} = -\frac{1}{2}$. Hence we estimate $f'(0) \approx -\frac{1}{2}$.

(c) $x = -2$, $x = 2$, and the interval $(3, 5)$

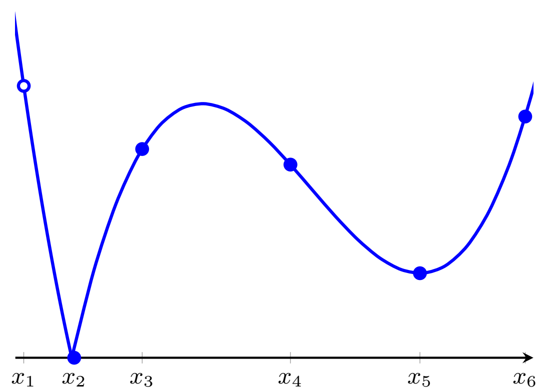
(d) $(-2, 2) \cup (5, 7)$

(e) $(-6, -4) \cup (-4, -2) \cup (2, 3) \cup (7, 10)$

Ex. G-11 3.1/3.2

Fa20 Exam

Consider the graph of $y = f(x)$ below.



(a) For which values of x is $f'(x) \geq 0$? Choose from x_1, x_2, x_3, x_4, x_5, and x_6. Select all that apply.

(b) For which values of x does $f'(x)$ not exist? Choose from x_1, x_2, x_3, x_4, x_5, and x_6. Select all that apply.

(c) Give a brief, one-sentence explanation of your answer to part (b).

exf20

Solution **G-11**

(a) x_3, x_5, x_6

(b) x_1, x_2

G-11

(c) At x_1, f is not continuous. At x_2, the graph of f has a sharp corner.

Ex. G-12 3.1/3.2 Fa20 Exam

Consider the following limit.

$$\lim_{x \to \frac{\pi}{8}} \left(\frac{\tan(2x) - 1}{x - \frac{\pi}{8}} \right)$$

(a) Use the limit definition of derivative to identify this limit as the derivative of some function $f(x)$ at the point $x = a$. You must explicitly identify f and a.

(b) Use your identifications in part (a) to calculate the given limit. Show all work.

exf20

Solution G-12

(a) The limit definition of derivative is

$$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

Comparing this to the given limit, we find that $a = \frac{\pi}{8}$ and $f(x) = \tan(2x)$.

(b) By part (a), the given limit is $f'(\frac{\pi}{8})$. Chain rule gives $f'(x) = 2\sec(2x)^2$, whence the value of the limit is $f'(\frac{\pi}{8}) = 2 \cdot (\sqrt{2})^2 = 4$.

Ex. G-13 3.1/3.2 Sp21 Exam

The following limit represents the derivative of a function f at a point a.

$$f'(a) = \lim_{h \to 0} \left(\frac{5\ln(e^4 + h) - 20}{h} \right)$$

(a) Find a possible function $f(x)$.

(b) For your choice of f in part (a), find a possible value of a.

(c) Calculate the value of the limit. Explain your calculation briefly in one sentence.

exs21

Solution G-13

We compare the limit to the definition of the derivative.

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

(a) $f(x) = 5\ln(x)$

(b) $a = e^4$ (note that $f(e^4) = 5 \cdot 4 = 20$)

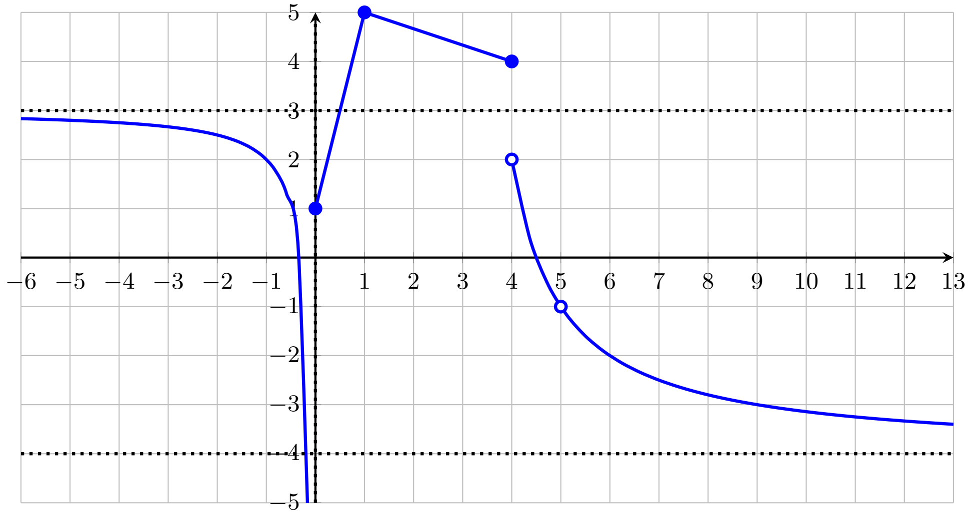
(c) We have $f'(e^4) = \left.\frac{5}{x}\right|_{x=e^4} = \frac{5}{e^4}$.

Ex. E-8 2.5, 2.6, 3.1/3.2 Sp21 Exam

Use the graph of f below to answer the following questions. Dashed lines indicate the location of asymptotes.

exs21

2.5, 2.6, 3.1/3.2 Sp21 **Exam**



(a) Calculate $\lim_{x\to\infty} f(x)$.

(b) Calculate $\lim_{x\to-\infty} f(x)$.

(c) List the values of x where f is not continuous.

(d) List the values of x where f is not differentiable.

(e) What is the sign of $f'(-1)$? (choices: positive, negative, zero, does not exist)

(f) What is the sign of $f'(0.5)$? (choices: positive, negative, zero, does not exist)

exs21

Solution **E-8**

(a) $\lim_{x\to\infty} f(x) = -4$

(b) $\lim_{x\to-\infty} f(x) = 3$

(c) $x = 0$, $x = 4$, $x = 5$

(d) $x = 0$, $x = 1$, $x = 4$, $x = 5$

(e) negative

(f) positive

Ex. G-14 **3.1/3.2, 3.3/3.4/3.5/3.9** Fa21 **Exam**

The following limit represents the derivative of a function f at a point a.

$$f'(a) = \lim_{h\to 0}\left(\frac{9\tan\left(\frac{\pi}{6}+h\right) - \frac{9}{\sqrt{3}}}{h}\right)$$

(a) Find a possible pair for f and a.

(b) Calculate the value of the limit.

exf21

Solution **G-14**

(a) Recall that the definition of the derivative is:

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

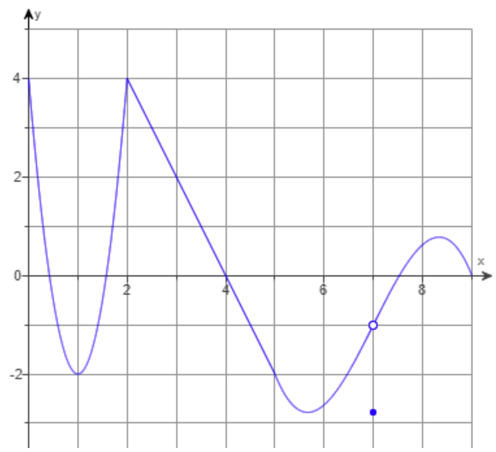
Let $f(x) = 9\tan(x)$ and let $a = \frac{\pi}{6}$. Then the given limit is $f'(a)$.

(b) Observe that $f'(x) = 9\sec(x)^2$, and so the given limit is $9\sec(\frac{\pi}{6})^2 = 9\cdot\frac{4}{3} = 12$.

Ex. G-15 3.1/3.2 Fa21 Exam

For each part, use the graph of $y = f(x)$ to determine whether the value exists. If the value exists, state its sign (negative, positive, or zero).

(a) $f'(1)$ (b) $f'(2)$ (c) $f'(3.5)$ (d) $f'(7)$



exf21

Solution G-15

(a) zero

(b) $f'(2)$ does not exist (the graph of f has a sharp corner at $x = 2$)

(c) negative

(d) $f'(7)$ does not exist (f is not continuous at $x = 7$)

Ex. G-16 3.1/3.2 Fa21 Exam

Let $f(x)$ and $g(x)$ be functions such that $f'(-8) = g'(-8)$ and the line tangent to the graph of f at $x = -8$ is $y = -7x + 6$. For each part, compute the desired value, if possible.

(a) $f(-8)$ (b) $f'(-8)$ (c) $g(-8)$ (d) $g'(-8)$

exf21

Solution G-16

(a) The tangent line to f at a point passes through the graph of f at the point of tangency. So $f(-8)$ is equal to the y-coordinate of the tangent line at $x = -8$. Thus $f(-8) = -7 \cdot (-8) + 6 = 62$.

(b) The slope of the tangent line to f is the derivative of f at the point of tangency. Hence $f'(-8)$ is -7, the slope of the line $y = -7x + 6$.

(c) We are not given enough information to determine $g(8)$. (In particular, the slope of the tangent line to g at $x = -8$ is -7 also, but the y-intercept need not be 6. In other words, the point of tangency need not be the same for both f and g.)

(d) We are given that $f'(-8) = g'(-8)$, whence $g'(-8) = -7$.

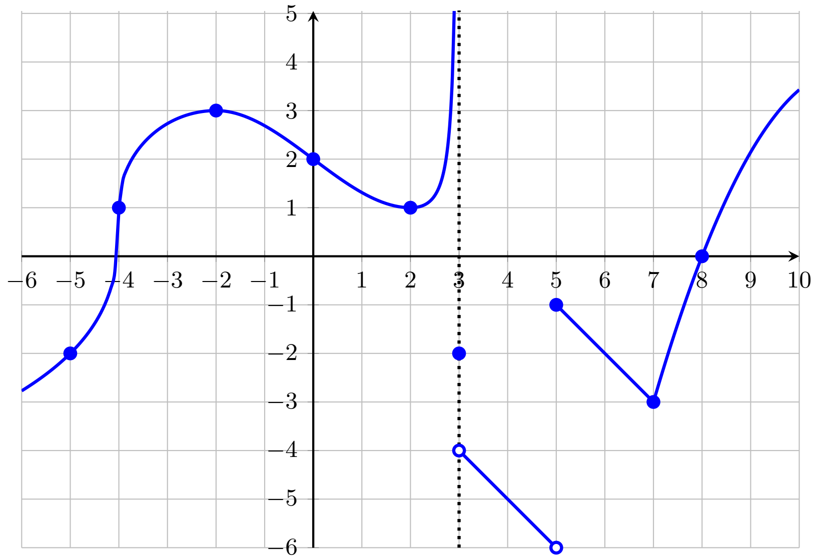
Ex. G-17 3.1/3.2 Sp22 Exam

For each part, use the graph of $y = f(x)$ to determine whether the value exists. If the value exists, state its sign (negative, positive, or zero).

(a) $f'(-4)$ (b) $f'(-2)$ (c) $f'(0)$ (d) $f'(5)$ (e) $f'(8)$

exs22

3.1/3.2 Sp22 **Exam**



exs22

Solution **G-17**

We have the following:

(a) $f'(-4)$ does not exist (there is a vertical tangent at $x = -4$)
(b) $f'(-2) = 0$ (there is a horizontal tangent at $x = -2$)
(c) $f'(0) < 0$ (the tangent line at $x = 0$ has negative slope)
(d) $f'(5)$ does not exist ($f(x)$ is discontinuous at $x = 5$)
(e) $f'(8) > 0$ (the tangent line at $x = 8$ has positive slope)

Ex. G-18 **3.1/3.2** Sp22 **Exam**

For both parts below, $f(x) = \sqrt{2x+1}$.

(a) Use the limit definition of the derivative to calculate $f'(4)$.
(b) Find an equation for the line tangent to the graph of $y = f(x)$ at $x = 4$.

exs22

Solution **G-18**

(a) We start with the limit definition of the derivative and compute the given limit by rationalizing the numerator.

$$\begin{aligned} f'(4) &= \lim_{x\to 4}\left(\frac{f(x)-f(4)}{x-4}\right) = \lim_{x\to 4}\left(\frac{\sqrt{2x+1}-3}{x-4}\right) = \lim_{x\to 4}\left(\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}\right) \\ &= \lim_{x\to 4}\left(\frac{2(x-4)}{(x-4)\left(\sqrt{2x+1}+3\right)}\right) = \lim_{x\to 4}\left(\frac{2}{\sqrt{2x+1}+3}\right) = \frac{1}{3} \end{aligned}$$

(b) Observe that $f(4) = 3$ and $f'(4) = \frac{1}{3}$. Hence the desired tangent line is:

$$y - 3 = \frac{1}{3}(x-4)$$

Ex. G-19 **3.1/3.2** Su22 **Exam**

Find the x-coordinate of each point on the graph of $y = 6x^3 - 9x^2 - 16x + 5$ at which the tangent line is perpendicular to the line $x + 20y = 10$.

exsu22

Solution **G-19**

The equation $x + 20y = 10$ can be written as $y = -\frac{1}{20}x + \frac{1}{2}$, whence the slope of the given line is $-\frac{1}{20}$. The desired tangent line is perpendicular to the given line, and thus has slope 20. So we must solve the equation $\frac{dy}{dx} = 20$, where

G-19

$y = 6x^3 - 9x^2 - 16x + 5$.

$$\frac{dy}{dx} = 20 \Longrightarrow 18x^2 - 18x - 16 = 20 \Longrightarrow 18(x-2)(x+1) = 0$$

Thus the desired x-coordinates are $x = 2$ and $x = -1$.

Ex. G-20 **3.1/3.2, 3.3/3.4/3.5/3.9** Su22 Exam

Suppose that an equation to the tangent line to $y = f(x)$ at $x = 9$ is $y = 3x - 20$. Let $g(x) = xf(x^2)$.

(a) Calculate $f(9)$ and $f'(9)$. Explain.
(b) Calculate $g'(x)$.
(c) Find the tangent line to $y = g(x)$ at $x = -3$.

exsu22

Solution G-20

(a) The tangent line to f at $x = 9$ is the line that passes through $(9, f(9))$ with slope $f'(9)$. The line $y = 3x - 20$ passes through $(9, 7)$ and has slope 3. Thus $f(9) = 7$ and $f'(9) = 3$.

(b) Use product rule, then chain rule.

$$g'(x) = 1 \cdot f(x^2) + x \cdot f'(x^2) \cdot 2x = f(x^2) + 2x^2 f'(x^2)$$

(c) We have the following (use the results of parts (a) and (b)):

$$g(-3) = \left(xf(x^2)\right)\Big|_{x=-3} = -3 \cdot f(9) = -3 \cdot 7 = -21$$
$$g'(-3) = \left(f(x^2) + 2x^2 f'(x^2)\right)\Big|_{x=-3} = f(9) + 18 \cdot f'(9) = 7 + 18 \cdot 3 = 61$$

Thus the tangent line to g at $x = -3$ has the equation:

$$y = -21 + 61(x + 3)$$

Ex. G-21 **3.1/3.2** Su22 Exam

Let $f(x) = \dfrac{4}{x-6} + 3$. Use the limit definition of derivative to calculate $f'(8)$. *If you simply quote a rule, you will receive no credit. You must use the definition of derivative.*

exsu22

Solution G-21

Use the definition of derivative, find a common denominator, and then cancel common factors.

$$f'(8) = \lim_{h \to 0} \left(\frac{f(8+h) - f(8)}{h} \right) = \lim_{h \to 0} \left(\frac{\frac{4}{8+h-6} + 3 - 5}{h} \right) = \lim_{h \to 0} \left(\frac{\frac{4}{2+h} - 2}{h} \right)$$
$$= \lim_{h \to 0} \left(\frac{4 - 2(2+h)}{h(2+h)} \right) = \lim_{h \to 0} \left(\frac{-2h}{h(2+h)} \right) = \lim_{h \to 0} \left(\frac{-2}{2+h} \right) = \frac{-2}{2+0} = -1$$

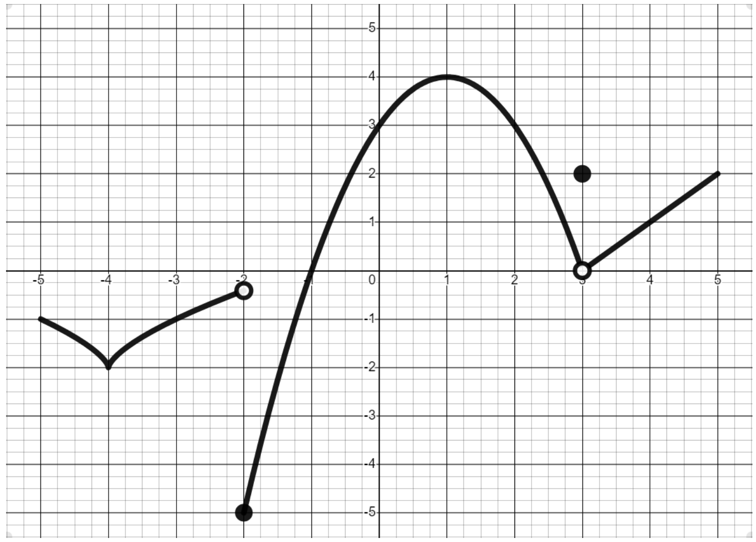
Ex. G-22 **3.1/3.2** Fa22 Exam

For each part, use the graph of $y = f(x)$ to determine whether the value exists. If the value exists, state its sign (negative, positive, or zero).

exf22

3.1/3.2 Fa22 Exam

(a) $f'(-3)$
(b) $f'(-2)$
(c) $f'(-1)$
(d) $f'(1)$
(e) $f'(3)$



exf22

Solution **G-22**

(a) does not exist
(b) does not exist
(c) zero
(d) positive
(e) negative

Ex. G-23 3.1/3.2 Fa22 Exam

Let $f(x) = \dfrac{8x}{x+5}$.

(a) Calculate $f'(x)$ by any method.
(b) Use the limit definition of derivative to calculate $f'(3)$. ***Hint:*** Use your answer from part (a) to check your final answer.

exf22

Solution **G-23**

(a) Use quotient rule.

$$f'(x) = \frac{8(x+5) - 8x \cdot 1}{(x+5)^2} = \frac{40}{(x+5)^2}$$

(b) Observe that $f(3) = 3$, whence by the limit definition of derivative we have:

$$f'(3) = \lim_{x\to 3}\left(\frac{f(x)-f(3)}{x-3}\right) = \lim_{x\to 3}\left(\frac{\frac{8x}{x+5}-3}{x-3}\right) = \lim_{x\to 3}\left(\frac{8x-3(x+5)}{(x-3)(x+5)}\right)$$
$$= \lim_{x\to 3}\left(\frac{5(x-3)}{(x-3)(x+5)}\right) = \lim_{x\to 3}\left(\frac{5}{x+5}\right) = \frac{5}{3+5} = \frac{5}{8}$$

Ex. G-24 3.1/3.2 Sp18 Quiz

Let $f(x) = \dfrac{3-x}{1+x}$. Use the limit definition of derivative to calculate $f'(1)$.

If you simply quote a rule, you will receive zero credit. You must use the definition of derivative.

qs18

Solution **G-24**

We start with the definition of derivative, then simplify and cancel.

$$f'(1) = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{3-(1+h)}{1+(1+h)}-1}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{2-h}{2+h}-1}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{(2-h)-(2+h)}{h(2+h)}\right) = \lim_{h\to 0}\left(\frac{-2h}{h(2+h)}\right) = \lim_{h\to 0}\left(\frac{-2}{2+h}\right) = \frac{-2}{2+0} = -1$$

Hence $f'(1) = -1$.

Ex. G-25 **3.1/3.2** Sp20 **Quiz**

Let $f(x) = x^{-1} - 3x^{-2}$. Use the limit definition of the derivative to calculate $f'(1)$. *(If you simply use shortcut rules, you will receive no credit.)*

qs20

Solution **G-25**

We start with the definition of the derivative, multiply all terms by the common denominator $(1+h)^2$, cancel common factors, and then substitute $h = 0$.

$$f'(1) = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{1+h}-\frac{3}{(1+h)^2}-(-2)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{1+h-3+2(1+h)^2}{h(1+h)^2}\right) = \lim_{h\to 0}\left(\frac{1+h-3+2+4h+2h^2}{h(1+h)^2}\right)$$
$$= \lim_{h\to 0}\left(\frac{h(5+2h)}{h(1+h)^2}\right) = \lim_{h\to 0}\left(\frac{5+2h}{(1+h)^2}\right) = \frac{5+0}{(1+0)^2} = 5$$

Ex. G-26 **3.1/3.2** Sp20 **Quiz**

Let $f(x) = 3x^2 - 5$. Use the limit definition of derivative to find $f'(x)$.

qs20

Solution **G-26**

We start with the definition of the derivative, expand all terms, cancel common factors, and then substitute $h = 0$.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{3(x+h)^2-5-(3x^2-5)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{3x^2+6xh+3h^2-5-3x^2+5}{h}\right) = \lim_{h\to 0}\left(\frac{h(6x+3h)}{h}\right)$$
$$= \lim_{h\to 0}(6x+3h) = 6x$$

Ex. G-27 **3.1/3.2** Su22 **Quiz**

Let $f(x) = \dfrac{x^2-3}{x-1}$. Use the limit definition of derivative to calculate $f'(2)$.

qsu22

Solution **G-27**

We have the following:

$$f'(2) = \lim_{x\to 2}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{x^2-3}{x-1}-1}{x-2}\right) = \lim_{x\to 2}\left(\frac{x^2-3-(x-1)}{(x-1)(x-2)}\right)$$
$$= \lim_{x\to 2}\left(\frac{x^2-x+2}{(x-1)(x-2)}\right) = \lim_{x\to 2}\left(\frac{(x-2)(x+1)}{(x-1)(x-2)}\right) = \lim_{x\to 2}\left(\frac{x+1}{x-1}\right) = \frac{2+1}{2-1} = 3$$

Ex. G-28 **3.1/3.2** Su22 **Quiz**

Use the limit definition of derivative to find an equation of the tangent line to $f(x) = 2x^2 + x + 5$ at $x = -1$.

qsu22

Solution **G-28**

The point of tangency is $(-1, f(-1)) = (-1, 6)$. The slope is given by the definition of derivative.

$$f'(-1) = \lim_{x\to-1}\left(\frac{f(x)-f(-1)}{x+1}\right) = \lim_{x\to-1}\left(\frac{2x^2+x+5-6}{x+1}\right)$$
$$= \lim_{x\to-1}\left(\frac{(2x-1)(x+1)}{x+1}\right)\lim_{x\to-1}(2x-1) = -2-1 = -3$$

Thus an equation of the tangent line is $y = 6 - 3(x+1)$.

Ex. G-29 **3.1/3.2** Fa22 **Quiz**

The limit below is equal to the derivative of some function $f(x)$ at some point $x = a$. Identify both the function f and the value of a. No work is required.

$$f'(a) = \lim_{h\to 0}\left(\frac{\frac{1}{(3+h)^2+1} - \frac{1}{10}}{h}\right)$$

qf22

Solution **G-29**

Compare the given limit to the definition of the derivative.

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

Put $f(x) = \dfrac{1}{x^2+1}$ and $a = 3$. Then the given limit is equal to $f'(3)$.

Ex. G-30 **3.1/3.2** Fa22 **Quiz**

Let $f(x) = 2x^2 - 6x + 10$.

(a) Use the limit definition of derivative to calculate $f'(-1)$.

(b) Find the tangent line to $y = f(x)$ at $x = -1$.

qf22

Solution **G-30**

(a) Observe that $f(-1) = 18$. So then we have the following.

$$f'(-1) = \lim_{x\to-1}\left(\frac{f(x)-f(-1)}{x+1}\right) = \lim_{x\to-1}\left(\frac{2x^2-6x+10-18}{x+1}\right)$$
$$= \lim_{x\to-1}\left(\frac{2(x-4)(x+1)}{x+1}\right) = \lim_{x\to-1}(2(x-4)) = -10$$

(b) The point of tangency is $(-1, 18)$ and the slope of the tangent line is $f'(-1) = -10$. Hence an equation of the tangent line $y = 18 - 10(x+1)$.

Ex. G-31 **3.1/3.2**

Suppose the line described by $y = 5x - 9$ is tangent to the graph of $y = f(x)$ at $x = 4$. For each part, calculate the value or explain why there is not enough information to do so.

Note: The function $f(x)$ is unknown. You can't assume that $f(x) = 5x - 9$.

(a) $f(4)$ (b) $f(3)$ (c) $f'(4)$ (d) $f'(3)$

c3s1

Solution **G-31**

(a) The tangent line at $x = a$ is defined to be the line to pass through the point $(a, f(a))$ with slope $f'(a)$. The line $y = 5x - 9$ passes through $(4, 11)$ and is tangent to the graph of $y = f(x)$ at $x = 4$. Hence $f(4) = 11$.

(b) The tangent line at $x = 4$ has no relation to the function $f(x)$ at any other value of x. So there is not enough information to tell the value of $f(3)$.

G-31

(c) See solution for part (a). The slope of the line $y = 5x - 9$ is 5, whence $f'(4) = 5$.

(d) See solution for part (b). There is not enough information to tell the value of $f'(3)$.

Ex. G-32 **3.1/3.2**

For each part, use the limit definition of the derivative to calculate the derivative of f at $x = 5$. Then find an equation for the line tangent to the graph of $y = f(x)$ at $x = 5$.

(a) $f(x) = 2x - 1$

(b) $f(x) = (2x-1)^2$

(c) $f(x) = \sqrt{2x-1}$

(d) $f(x) = \dfrac{1}{2x-1}$

(e) $f(x) = \dfrac{1}{\sqrt{2x-1}}$

c3s1

Solution G-32

(a) Observe that $f(5) = 9$. Then, by definition, we have the following.

$$f'(5) = \lim_{h\to 0}\left(\frac{f(5+h)-f(5)}{h}\right) = \lim_{h\to 0}\left(\frac{2(5+h)-1-9}{h}\right) = \lim_{h\to 0}\left(\frac{2h}{h}\right) = \lim_{h\to 0}(2) = 2$$

Hence the tangent line has equation $y - 9 = 2(x-5)$.

(b) Observe that $f(5) = 81$. Then, by definition, we have the following.

$$\begin{aligned} f'(5) &= \lim_{h\to 0}\left(\frac{f(5+h)-f(5)}{h}\right) = \lim_{h\to 0}\left(\frac{(2(5+h)-1)^2-81}{h}\right) = \lim_{h\to 0}\left(\frac{(2h+9)^2-81}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{4h^2+36h}{h}\right) = \lim_{h\to 0}(4h+36) = 36 \end{aligned}$$

Hence the tangent line has equation $y - 81 = 36(x-5)$.

(c) Observe that $f(5) = 3$. Then, by definition, we have the following.

$$\begin{aligned} f'(5) &= \lim_{h\to 0}\left(\frac{f(5+h)-f(5)}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{2(5+h)-1}-3}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{2h+9}-3}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{2h+9-9}{h\left(\sqrt{2h+9}+3\right)}\right) = \lim_{h\to 0}\left(\frac{2}{\sqrt{2h+9}+3}\right) = \frac{2}{\sqrt{9}+3} = \frac{1}{3} \end{aligned}$$

Hence the tangent line has equation $y - 3 = \frac{1}{3}(x-5)$.

(d) Observe that $f(5) = \frac{1}{9}$. Then, by definition, we have the following.

$$\begin{aligned} f'(5) &= \lim_{h\to 0}\left(\frac{f(5+h)-f(5)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{2(5+h)-1}-\frac{1}{9}}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{2h+9}-\frac{1}{9}}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{9-(2h+9)}{9h(2h+9)}\right) = \lim_{h\to 0}\left(\frac{-2}{9(2h+9)}\right) = \frac{-2}{9(0+9)} = -\frac{2}{81} \end{aligned}$$

Hence the tangent line has equation $y - \frac{1}{9} = -\frac{2}{81}(x-5)$.

(e) Observe that $f(5) = \frac{1}{3}$. Then, by definition, we have the following.

$$\begin{aligned} f'(5) &= \lim_{h\to 0}\left(\frac{f(5+h)-f(5)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{2(5+h)-1}}-\frac{1}{3}}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{2h+9}}-\frac{1}{3}}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{3-\sqrt{2h+9}}{3h\sqrt{2h+9}}\right) = \lim_{h\to 0}\left(\frac{9-(2h+9)}{3h\sqrt{2h+9}\left(3+\sqrt{2h+9}\right)}\right) = \lim_{h\to 0}\left(\frac{-2}{3\sqrt{2h+9}\left(3+\sqrt{2h+9}\right)}\right) = -\frac{1}{27} \end{aligned}$$

Hence the tangent line has equation $y - \frac{1}{3} = -\frac{1}{27}(x-5)$.

Ex. G-33 **3.1/3.2**

Let $f(x) = 3\sqrt{x}$. Use the limit definition of the derivative to find $f'(x)$. Show all work.

unit3

Solution **G-33**

We have the following work.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{3\sqrt{x+h}-3\sqrt{x}}{h}\right) = \lim_{h\to 0}\left(\frac{9(x+h)-9x}{h\left(3\sqrt{x+h}+3\sqrt{x}\right)}\right)$$
$$= \lim_{h\to 0}\left(\frac{9h}{h\left(3\sqrt{x+h}+3\sqrt{x}\right)}\right) = \lim_{h\to 0}\left(\frac{9}{3\sqrt{x+h}+3\sqrt{x}}\right) = \frac{9}{3\sqrt{x+0}+3\sqrt{x}} = \frac{3}{2\sqrt{x}}$$

Ex. G-34 **3.1/3.2**

Let $f(x) = \dfrac{x+2}{x-3}$. Use the limit definition of derivative to find $f'(2)$.

unit3

Solution **G-34**

Start with the definition of the derivative and then use algebra to eliminate the indeterminate form.

$$f'(2) = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{h+4}{h-1}-(-4)}{h}\right) = \lim_{h\to 0}\left(\frac{h+4+4(h-1)}{h(h-1)}\right)$$
$$= \lim_{h\to 0}\left(\frac{h+4+4h-4}{h(h-1)}\right) = \lim_{h\to 0}\left(\frac{5h}{h(h-1)}\right) = \lim_{h\to 0}\left(\frac{5}{h-1}\right) = \frac{5}{0-1} = -5$$

Ex. G-35 **3.1/3.2**

Let $f(x) = \dfrac{3x+12}{x^2-1}$. Calculate $f'(2)$ directly from the definition of the derivative. You are not allowed to use any shortcut rules.

other

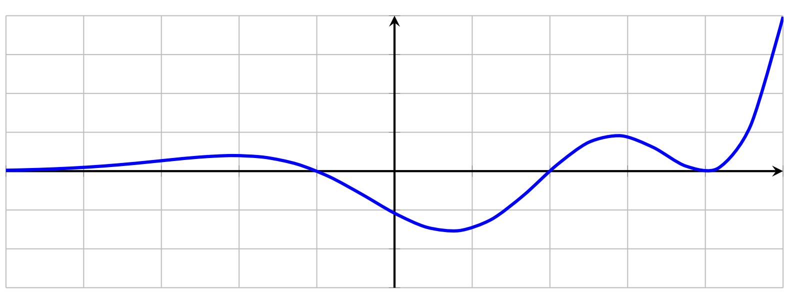
Solution **G-35**

We use the definition of derivative and compute the limit using algebra.

$$f'(2) = \lim_{x\to 2}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{3x+12}{x^2-1}-6}{x-2}\right) = \lim_{x\to 2}\left(\frac{3x+12-6(x^2-1)}{(x-2)(x^2-1)}\right) = \lim_{x\to 2}\left(\frac{-3(2x+3)}{x^2-1}\right) = \frac{-3\cdot 7}{3} = -7$$

Ex. G-36 **3.1/3.2** **⋆Challenge**

The graph of $y = f(x)$ is given below. Sketch a graph of $y = f'(x)$. *Only the general shape is important. The graph does not have to be to scale.*

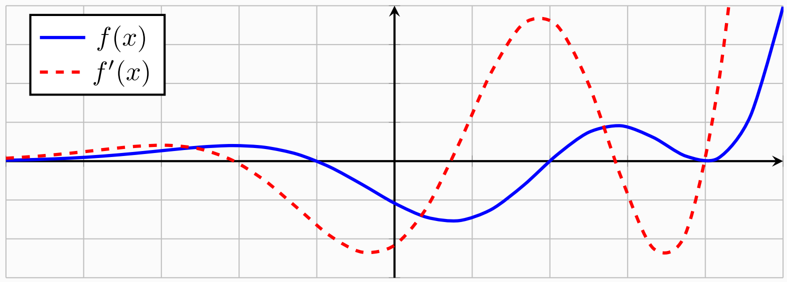


challenge

Solution **G-36**

First identify the points where $f'(x) = 0$ (local minimum or local maximum of $f(x)$). These points cut the number line into several subintervals. Identify the sign (negative or positive) of $f'(x)$ on each subinterval, then smooth out the curve on each subinterval.

G-36



Ex. G-37 3.1/3.2 ⋆Challenge

Consider the following function, where c is an unspecified constant

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 + 2x & \text{if } 0 \leq x < 1 \\ 6x - x^2 + c & \text{if } x \geq 1 \end{cases}$$

(a) Show precisely that $f'(0)$ does not exist.

(b) Find the value of c that makes f differentiable at $x = 1$ or show that no such value exists.

challenge

Solution G-37

Note: A commonly proposed but invalid solution is to compute $f'(x)$ for each separate piece and then check whether the one-sided limits of $f'(x)$ are equal at $x = 0$ and $x = 1$. That would check whether $\lim\limits_{x\to 0} f'(x)$ or $\lim\limits_{x\to 1} f'(x)$ exists, not whether $f'(0)$ or $f'(1)$ exists.

(a) Observe that $f(0) = 0$. Then, by definition, we have the following.

$$f'(0) = \lim_{x\to 0}\left(\frac{f(x) - f(0)}{x}\right) = \lim_{x\to 0}\left(\frac{f(x)}{x}\right)$$

Since $f(x)$ is piecewise defined and changes definition at $x = 0$, we must compute the left- and right-limits.

$$\lim_{x\to 0^-}\left(\frac{f(x)}{x}\right) = \lim_{x\to 0^-}\left(\frac{-x^2}{x}\right) = \lim_{x\to 0^-}(-x) = 0$$
$$\lim_{x\to 0^+}\left(\frac{f(x)}{x}\right) = \lim_{x\to 0^+}\left(\frac{x^2 + 2x}{x}\right) = \lim_{x\to 0^-}(x + 2) = 2$$

The one-sided limits are not equal, whence $f'(0)$ does not exist.

(b) Recall that continuity is a necessary (but not sufficient) condition for differentiability. That is, if f is to be differentiable at $x = 1$, then f must also be continuous at $x = 1$. So first we determine the value of c that makes f continuous at $x = 1$.

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(x^2 + 2x) = 3 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}(6x - x^2 + c) = 5 + c \\ f(1) &= \left(6x - x^2 + c\right)\Big|_{x=1} = 5 + c \end{aligned}$$

So we must have that $3 = 5 + c$, or $c = -2$, and our function is:

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 + 2x & \text{if } 0 \leq x < 1 \\ 6x - x^2 - 2 & \text{if } x \geq 1 \end{cases}$$

Now we check whether f differentiable at $x = 1$. Observe that $f(1) = 3$. So, by definition, we have:

$$f'(1) = \lim_{x\to 1}\left(\frac{f(x) - f(1)}{x - 1}\right) = \lim_{x\to 1}\left(\frac{f(x) - 3}{x - 1}\right)$$

G-37

Since $f(x)$ is piecewise defined and changes definition at $x = 1$, we must compute the left- and right-limits.

$$\lim_{x\to 1^-}\left(\frac{f(x)-3}{x-1}\right) = \lim_{x\to 1^-}\left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x\to 1^-}\left(\frac{(x+3)(x-1)}{x-1}\right) = \lim_{x\to 1^-}(x+3) = 4$$
$$\lim_{x\to 1^+}\left(\frac{f(x)-3}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{6x-x^2-5}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{(5-x)(x-1)}{x-1}\right) = \lim_{x\to 1^+}(5-x) = 4$$

The one-sided limits are equal, whence $f'(1) = 4$. So $c = -2$ does, indeed, make f differentiable at $x = 1$.

Ex. G-38 **3.1/3.2** ⋆Challenge

Use the limit definition of derivative to find the derivative of $f(x) = x^{2/3}$.

challenge

Solution G-38

By definition of derivative,

$$f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{x^{2/3}-a^{2/3}}{x-a}\right)$$

We now change the variable from x to $u = x^{1/3}$ (which implies $x = u^3$). To make the algebra clearer, we also let $w = a^{1/3}$, which effectively just changes the label of the constant a. Observe that if $x = a$, then $u = w$. So the limit "$x \to a$" is equivalent to "$u \to w$". Hence our limit can be written as:

$$f'(a) = \lim_{x\to a}\left(\frac{x^{2/3}-a^{2/3}}{x-a}\right) = \lim_{u\to w}\left(\frac{u^2-w^2}{u^3-w^3}\right)$$

This is a standard limit we have encountered in this course. Factor numerator and denominator, cancel common factors, and substitute $u = w$.

$$f'(a) = \lim_{u\to w}\left(\frac{(u-w)(u+w)}{(u-w)(u^2+uw+w^2)}\right) = \lim_{u\to w}\left(\frac{u+w}{u^2+uw+w^2}\right) = \frac{2w}{3w^2} = \frac{2}{3w} = \frac{2}{3a^{1/3}}$$

Hence we have shown $f'(a) = \frac{2}{3}a^{-1/3}$, as expected from the power rule.

§3.3, 3.4, 3.5, 3.9: Rules for Computing Derivatives

Ex. H-1 **3.3/3.4/3.5/3.9, 3.7** Sp18 **Exam**

For each part, calculate $f'(x)$. *After calculating the derivative, do not simplify your answer.*

(a) $f(x) = \dfrac{x^{-1}x^{8/3}}{4\sqrt[3]{x^2}}$ (b) $f(x) = \left(x + \sqrt{5x-6}\right)^{1/4}$ (c) $f(x) = \dfrac{x^2e^x}{\ln(x) - \cos(x)}$

exs18

Solution **H-1**

(a) Simplifying the exponents, we observe that $f(x) = \frac{1}{4}x$. Hence $f'(x) = \frac{1}{4}$.

(b) Use power rule, then chain rule twice.

$$f'(x) = \frac{1}{4}\left(x + \sqrt{5x-6}\right)^{-3/4} \cdot \left(1 + \frac{1}{2}(5x-6)^{-1/2} \cdot 5\right)$$

(c) Use quotient rule. When differentiating the numerator, use product rule.

$$f'(x) = \frac{\left(x^2e^x + 2xe^x\right) \cdot (\ln(x) - \cos(x)) - \left(x^2e^x\right) \cdot \left(\frac{1}{x} + \sin(x)\right)}{(\ln(x) - \cos(x))^2}$$

Ex. H-2 **3.3/3.4/3.5/3.9** Fa18 **Exam**

Calculate $f'(x)$ where f is the function below.

$$f(x) = \left(\frac{x^8\sin(3x)}{\ln(x) - \ln(11)}\right)^{2/3}$$

After calculating the derivative, do not simplify your answer.

exf18

Solution **H-2**

Use power rule, followed by chain rule. The derivative of the expression inside the power "$\frac{2}{3}$" is given by quotient rule.

$$f'(x) = \frac{2}{3}\left(\frac{x^8\sin(3x)}{\ln(x) - \ln(11)}\right)^{-1/3} \cdot \frac{(\ln(x) - \ln(11))(8x^7\sin(3x) + 3x^8\cos(3x)) - x^8\sin(3x) \cdot \frac{1}{x}}{(\ln(x) - \ln(11))^2}$$

Ex. H-3 **3.3/3.4/3.5/3.9, 3.7** Fa18 **Exam**

Suppose f and g are differentiable for all x. For each part, use the table below or explain why there is not enough information.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	-1	-4	4	2
1	-1	-3	2	-4
2	-4	3	1	-1

(a) Let $F(x) = \dfrac{f(x)}{g(x)}$. Calculate $F'(0)$. (b) Let $G(x) = f\Big(xg(x)\Big)$. Calculate $G'(1)$.

exf18

Solution **H-3**

(a) First use quotient rule.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Now substitute $x = 0$ and use the table of values.

$$F'(0) = \frac{(-4)(4) - (-1)(2)}{4^2} = \frac{-16+2}{16} = -\frac{7}{8}$$

H-3

(b) First use chain rule, then product rule.

$$G'(x) = \frac{d}{dx} f\Big(xg(x)\Big) = f'\Big(xg(x)\Big) \cdot \frac{d}{dx}(xg(x)) = f'\Big(xg(x)\Big) \cdot (1 \cdot g(x) + xg'(x))$$

Now substitute $x = 1$ and use the table of values.

$$G'(1) = f'(1 \cdot g(1)) \cdot (g(1) + 1 \cdot g'(1)) = f'(2) \cdot (g(1) + g'(1)) = 3 \cdot (2 + (-4)) = -6$$

Ex. H-4 **3.3/3.4/3.5/3.9** Sp19 Exam

For each part, calculate $f'(x)$. *Do not simplify your answers.*

(a) $f(x) = e^x \sin(x)$

(b) $f(x) = \dfrac{\ln\left(e^{4x} + 6\right)}{9\tan(x) - \pi^9}$

exs19

Solution H-4

(a) Use product rule.

$$f'(x) = e^x \sin(x) + e^x \cos(x)$$

(b) Start with quotient rule. To differentiate the numerator, use chain rule twice.

$$f'(x) = \frac{\left(\dfrac{1}{e^{4x}+6} \cdot e^{4x} \cdot 4\right)\left(9\tan(x) - \pi^9\right) - \ln(e^{4x}+6) \cdot 9\sec(x)^2}{\left(9\tan(x) - \pi^9\right)^2}$$

Ex. H-5 **3.3/3.4/3.5/3.9** Sp19 Exam

Find the slope of the line tangent to the graph of $y = 3\ln(x) - 6\sqrt{x}$ at $x = 3$.

exs19

Solution H-5

Observe that

$$\frac{dy}{dx} = 3 \cdot \frac{1}{x} - 6 \cdot \frac{1}{2} x^{-1/2} = \frac{3}{x} - \frac{3}{\sqrt{x}}$$

Hence the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{3}{3} - \frac{3}{\sqrt{3}} = 1 - \sqrt{3}$$

Ex. H-6 **3.3/3.4/3.5/3.9** Fa19 Exam

For each part, calculate $f'(x)$. *Do not simplify your answers.*

(a) $f(x) = \dfrac{\ln(x)}{10 - x^3}$

(b) $f(x) = \sqrt{\cos(3 + x^5)}$

exf19

Solution H-6

(a) Use quotient rule.

$$f'(x) = \frac{\frac{1}{x} \cdot (10 - x^3) - \ln(x) \cdot (-3x^2)}{(10 - x^3)^2}$$

(b) Use chain rule twice.

$$f'(x) = \frac{1}{2}\left(\cos(3 + x^5)\right)^{-1/2} \cdot \left(-\sin(3 + x^5)\right) \cdot 5x^4$$

Ex. H-7 **3.3/3.4/3.5/3.9** Fa19 Exam

Find all points on the graph of $f(x) = x\ln(x)$ where the tangent line is horizontal.

exf19

Solution **H-7**

A horizontal line has slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation $f'(x) = 0$.

$$f'(x) = 1 + \ln(x) = 0 \Longrightarrow x = e^{-1}$$

Hence the point on the graph with a horizontal tangent is $\left(e^{-1}, f(e^{-1})\right) = \left(e^{-1}, -e^{-1}\right)$.

Ex. H-8 **3.3/3.4/3.5/3.9** Sp20 **Exam**

For each part, calculate $f'(x)$. *Do not simplify your answers.*

(a) $f(x) = 2x^2 - \dfrac{1}{5x} - 8\sqrt{x} + 14\pi^{3/2}$

(b) $f(x) = \left(\dfrac{x^4 - 20x}{x^3 + 20}\right)^{2/3}$

(c) $f(x) = \sin\left(12x - x^9\right)\ln(x)$

(d) $f(x) = \dfrac{e^{5\sec(6x)+1}}{7}$

exs20

Solution **H-8**

(a) Write the function using exponents.

$$f(x) = 2x^2 - \frac{1}{5}x^{-1} - 8x^{1/2} + 14\pi^{3/2}$$

Differentiate using power rule, noting that $14\pi^{3/2}$ is a constant.

$$f'(x) = 4x - \frac{1}{5}x^{-2} - 4x^{-1/2}$$

(b) Use power rule first, then use chain rule (using quotient rule to find the derivative of the "inside" function).

$$f'(x) = \frac{2}{3}\left(\frac{x^4 - 20x}{x^3 + 20}\right)^{-1/3} \cdot \frac{(4x^3 - 20)(x^3 + 20) - (x^4 + 20x)(3x^2)}{(x^3 + 20)^2}$$

(c) Use product rule. When differentiating the first term, use chain rule.

$$f'(x) = \cos(12x - x^9) \cdot (12 - 9x^8) \cdot \ln(x) + \sin(12x - x^9) \cdot \frac{1}{x}$$

(d) Use chain rule twice. (Do not use quotient rule. The factor of $\frac{1}{7}$ is a constant coefficient.)

$$f'(x) = \frac{1}{7}e^{5\sec(6x)+1} \cdot 5\sec(6x)\tan(6x) \cdot 6$$

Ex. H-9 **3.3/3.4/3.5/3.9** Sp20 **Exam**

Find the x-coordinate of each point on the graph of $f(x) = 3x + \dfrac{10}{x}$ where the tangent line is parallel to the line $y = 20 - 2x$.

exs20

Solution **H-9**

The slope of the line $y = 20 - 2x$ is -2 and parallel lines have equal slopes. Hence we seek all values of x that solve the equation $f'(x) = -2$.

$$f'(x) = -2 \Longrightarrow 3 - \frac{10}{x^2} = -2$$

Solving for x gives $x = -\sqrt{2}$ or $x = \sqrt{2}$.

Ex. H-10 **3.3/3.4/3.5/3.9** Su20 **Exam**

Let $f(x) = x^{15}e^{2-5x}$. Find the x-coordinate of each point where the tangent line to f is horizontal.

exsu20

Solution **H-10**

The tangent line is horizontal wherever $f'(x) = 0$. We find the derivative using the product rule and chain rule.

$$f'(x) = 15x^{14}e^{2-5x} - 5x^{15}e^{2-5x} = 5x^{14}(3 - x)e^{2-5x}$$

Solving $f'(x) = 0$, we find that there is a horizontal tangent line at $x = 0$ and $x = 3$.

Ex. H-11 **3.3/3.4/3.5/3.9** Su20 **Exam**

Let $f(x) = 3x^5 - 2x^3 + 7x - 16$. Find an equation of the tangent line to f at $x = -1$.

exsu20

Solution **H-11**

The tangent line passes through the point $(-1, f(-1)) = (-1, -24)$. Now observe that $f'(x) = 15x^4 - 6x^2 + 7$, whence the slope of the tangent line is $f'(-1) = 16$. So an equation of the tangent line is:

$$y = -24 + 16(x + 1)$$

Ex. H-12 **3.3/3.4/3.5/3.9** Fa20 **Exam**

Consider the function $f(x) = x^3 - 6x + c$, where c is an unspecified constant. Suppose the line $102x - y = 609$ is tangent to the graph of $y = f(x)$ at the point P in the first quadrant.

(a) What is the value of $f'(x)$ at the point P? Give a brief, one-sentence explanation.

(b) Find the x-coordinate of P.

(c) Find the y-coordinate of P.

(d) Find the value of c.

exf20

Solution **H-12**

(a) The slope of the tangent line at P is 102, hence $f'(x) = 102$ at P.

(b) We solve the equation $f'(x) = 102$.

$$3x^2 - 6 = 102 \Longrightarrow x^2 = 36 \Longrightarrow x = 6$$

(We reject the solution $x = -6$ since P is in the first quadrant.)

(c) The tangent line and graph of f coincide at the point of tangency. So substituting $x = 6$ into the equation of the tangent line gives $y = 102 \cdot 6 - 609 = 3$.

(d) We have $f(6) = 6^3 - 6 \cdot 6 + c = 180 + c$. On the other hand, from part (c), $f(6) = 3$. Hence $180 + c = 3$, and so $c = -177$.

Ex. H-13 **3.3/3.4/3.5/3.9** Fa20 **Exam**

Let $f(x) = \dfrac{8e^x}{x-3}$. Find the equation of each horizontal tangent line of f.

exf20

Solution **H-13**

A horizontal tangent line occurs at points where $f'(x) = 0$.

$$f'(x) = \frac{8e^x(x-3) - 8e^x \cdot 1}{(x-3)^3} = \frac{8e^x(x-4)}{(x-3)^2}$$

Solving $f'(x) = 0$ gives $x = 4$ (whence $f(4) = 8e^4$). Hence the only horizontal tangent line is $y = 8e^4$.

Ex. H-14 **3.3/3.4/3.5/3.9** Fa20 **Exam**

Suppose $f(1) = -8$ and $f'(1) = 12$. Let $F(x) = x^3 f(x) + 10$. Find an equation of the tangent line to F at $x = 1$.

exf20

Solution **H-14**

Observe that $F(1) = f(1) + 10 = 2$. Hence the point of tangency is $(1, 2)$. Using product rule, we have

$$F'(x) = 3x^2 f(x) + x^3 f'(x)$$

Hence the slope of the tangent line is $F'(1) = 3f(1) + f'(1) = -12$. So the equation of the desired tangent line is

$$y = 2 - 12(x - 1)$$

Ex. H-15 **3.3/3.4/3.5/3.9** Sp21 **Exam**

Suppose that an equation of the tangent line to f at $x = 5$ is $y = 3x - 8$. Let $g(x) = \dfrac{f(x)}{x^2 + 10}$.

(a) Calculate $f(5)$ and $f'(5)$.
(b) Calculate $g(5)$ and $g'(5)$.
(c) Write down an equation of the tangent line to g at $x = 5$.

exs21

Solution **H-15**

(a) The tangent line to f at $x = a$ has slope $f'(a)$ and passes through $(a, f(a))$. The line $y = 3x-8$, which is tangent to f at $x = 5$ passes through the point $(5, 7)$, whence $f(5) = 7$. the same line has slope 3, whence $f'(5) = 3$.
(b) We have $g(5) = \frac{f(5)}{35} = \frac{1}{5}$. We use quotient rule to find $g'(x)$.

$$g'(x) = \frac{f'(x) \cdot (x^2 + 10) - f(x) \cdot 2x}{(x^2 + 10)^2}$$

Hence $g'(5) = \frac{3 \cdot 35 - 7 \cdot 10}{35^2} = \frac{1}{35}$.
(c) The tangent line to g at $x = 5$ is $y = \frac{1}{5} + \frac{1}{35}(x - 5)$.

Ex. H-16 **3.3/3.4/3.5/3.9, 3.7** Sp21 **Exam**

Suppose $f(2) = -7$ and $f'(2) = 3$.

(a) Let $g(x) = \cos(x)f(x)$. Calculate $g'(2)$.
(b) Let $h(x) = e^{2f(x)+3}$. Calculate $h'(2)$.

exs21

Solution **H-16**

(a) We use product rule.

$$g'(x) = -\sin(x)f(x) + \cos(x)f'(x)$$

Hence $g'(2) = 7\sin(2) + 3\cos(2)$.
(b) We use chain rule.

$$h'(x) = e^{2f(x)+3} \cdot 2f'(x)$$

Hence $h'(2) = 6e^{-11}$.

Ex. H-17 **3.3/3.4/3.5/3.9** Sp21 **Exam**

Let $f(x) = x^2 + bx + c$, where b and c are unspecified constants. An equation of the tangent line to f at $x = 3$ is $12x + y = 10$.

(a) Calculate $f(3)$ and $f'(3)$. Your answers must not contain the letters b or c.
(b) Calculate the value of b.
(c) Calculate the value of c.

exs21

Solution **H-17**

(a) The tangent line to f at $x = 3$ is $12x + y = 10$, which passes through the point $(3, -26)$, whence $f(3) = -26$. The same line has slope -12, whence $f'(3) = -12$.
(b) We have $f'(x) = 2x + b$, whence $f'(3) = 6 + b$. From part (a), we must have $6 + b = -12$, whence $b = -18$.

H-17

(c) We have $f(x) = x^2 - 18x + c$, whence $f(3) = -45 + c$. From part (a), we must have $-45 + c = -26$, whence $c = 19$.

Ex. G-14 **3.1/3.2, 3.3/3.4/3.5/3.9** Fa21 **Exam**

The following limit represents the derivative of a function f at a point a.

$$f'(a) = \lim_{h \to 0} \left(\frac{9\tan\left(\frac{\pi}{6} + h\right) - \frac{9}{\sqrt{3}}}{h} \right)$$

(a) Find a possible pair for f and a.

(b) Calculate the value of the limit.

exf21

Solution **G-14**

(a) Recall that the definition of the derivative is:

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

Let $f(x) = 9\tan(x)$ and let $a = \frac{\pi}{6}$. Then the given limit is $f'(a)$.

(b) Observe that $f'(x) = 9\sec(x)^2$, and so the given limit is $9\sec(\frac{\pi}{6})^2 = 9 \cdot \frac{4}{3} = 12$.

Ex. H-18 **3.3/3.4/3.5/3.9, 3.7** Sp22 **Exam**

For each part, calculate $f'(x)$. After calculating the derivative, do not simplify your answer.

(a) $f(x) = 3x^{13} + 7\sqrt{x} - \dfrac{5}{x^3} + 12$ (b) $f(x) = \dfrac{e^x - 2\sin(x)}{\ln(x) + x^3}$ (c) $f(x) = 2x^4\cos(3e^x)$

exs22

Solution **H-18**

(a) We use power rule several times.

$$\frac{d}{dx}\left(3x^{13} + 7\sqrt{x} - \frac{5}{x^3} + 12\right) = 39x^{12} + \frac{7}{2}x^{-1/2} + 15x^{-4}$$

(b) We use quotient rule.

$$\frac{d}{dx}\left(\frac{e^x - 2\sin(x)}{\ln(x) + x^3}\right) = \frac{(e^x - 2\cos(x))\left(\ln(x) + x^3\right) - (e^x - 2\sin(x))\left(\frac{1}{x} + 3x^2\right)}{\left(\ln(x) + x^3\right)^2}$$

(c) We use product rule, then chain rule.

$$\frac{d}{dx}\left(2x^4\cos(3e^x)\right) = 8x^3\cos(3e^x) + 2x^4 \cdot (-\sin(3e^x)) \cdot 3e^x$$

Ex. H-19 **3.3/3.4/3.5/3.9** Sp22 **Exam**

For both parts below, suppose the line tangent to the graph of $y = f(x)$ at $x = 5$ is $y = 2x - 3$.

(a) Calculate $f(5)$ and $f'(5)$.

(b) Let $g(x) = xf(x) + 14$. Find an equation of the line tangent to the graph of $y = g(x)$ at $x = 5$.

exs22

Solution **H-19**

(a) The tangent line at $x = 5$ intersects the graph of $y = f(x)$ at $x = 5$, whence $f(5) = 2 \cdot 5 - 3 = 7$. The slope of the tangent line at $x = 5$ is $f'(5)$, whence $f'(5) = 2$.

(b) First observe that, by part (a), $g(5) = 5f(5) + 14 = 49$. Then using product rule, we obtain $g'(x) = f(x) + xf'(x)$. Putting $x = 5$ and using part (a) again, we now have $g'(5) = f(5) + 5f'(5) = 17$. Hence the desired tangent line is:

$$y - 49 = 17(x - 5)$$

Ex. H-20 **3.3/3.4/3.5/3.9, 3.7** Su22 **Exam**

For each part, calculate the derivative. After calculating the derivative, do not simplify your answer.

(a) $\dfrac{d}{dx}\left(\tan\left(\dfrac{\ln(x)}{2x-5}\right)\right)$ (b) $\dfrac{d}{dx}\left(3x^7\cos(x) - 8e^{3x}\right)$ (c) $\dfrac{d}{dx}\left(10x^{12} - \dfrac{3}{x^3} + \sqrt[4]{x}\right)$

exsu22

Solution **H-20**

(a) Use chain rule, then use quotient rule.

$$\frac{d}{dx}\left(\tan\left(\frac{\ln(x)}{2x-5}\right)\right) = \sec^2\left(\frac{\ln(x)}{2x-5}\right)\cdot\frac{\frac{1}{x}\cdot(2x-5) - \ln(x)\cdot 2}{(2x-5)^2}$$

(b) Use product rule on the first term and chain rule on the second term.

$$\frac{d}{dx}\left(3x^7\cos(x) - 8e^{3x}\right) = 21x^6\cos(x) - 3x^7\sin(x) - 24e^{3x}$$

(c) Use power rule on each term.

$$\frac{d}{dx}\left(10x^{12} - 3x^{-3} + x^{1/4}\right) = 120x^{11} + 9x^{-4} + \frac{1}{4}x^{-3/4}$$

Ex. G-20 **3.1/3.2, 3.3/3.4/3.5/3.9** Su22 **Exam**

Suppose that an equation to the tangent line to $y = f(x)$ at $x = 9$ is $y = 3x - 20$. Let $g(x) = xf(x^2)$.

(a) Calculate $f(9)$ and $f'(9)$. Explain.
(b) Calculate $g'(x)$.
(c) Find the tangent line to $y = g(x)$ at $x = -3$.

exsu22

Solution **G-20**

(a) The tangent line to f at $x = 9$ is the line that passes through $(9, f(9))$ with slope $f'(9)$. The line $y = 3x - 20$ passes through $(9, 7)$ and has slope 3. Thus $f(9) = 7$ and $f'(9) = 3$.

(b) Use product rule, then chain rule.

$$g'(x) = 1\cdot f(x^2) + x\cdot f'(x^2)\cdot 2x = f(x^2) + 2x^2 f'(x^2)$$

(c) We have the following (use the results of parts (a) and (b)):

$$\begin{aligned} g(-3) &= \left(xf(x^2)\right)\Big|_{x=-3} = -3\cdot f(9) = -3\cdot 7 = -21 \\ g'(-3) &= \left(f(x^2) + 2x^2 f'(x^2)\right)\Big|_{x=-3} = f(9) + 18\cdot f'(9) = 7 + 18\cdot 3 = 61 \end{aligned}$$

Thus the tangent line to g at $x = -3$ has the equation:

$$y = -21 + 61(x+3)$$

Ex. H-21 **3.3/3.4/3.5/3.9** Sp18 **Quiz**

Let $g(x) = x^2\ln(x)$. Find an equation of the tangent line at $x = e$.

qs18

Solution **H-21**

First we compute $g'(x)$ by product rule.

$$g'(x) = x^2\cdot\frac{1}{x} + 2x\cdot\ln(x) = x + 2x\ln(x)$$

Now observe that $g(e) = e^2\ln(e) = e^2$ and $g'(e) = e + 2e\ln(e) = 3e$. So an equation of the tangent line is

$$y - e^2 = 3e(x-e)$$

Ex. H-22 **3.3/3.4/3.5/3.9** Sp20 **Quiz**

Let $f(x) = \dfrac{50e^x}{x^2+1}$. Find an equation of the line tangent to the graph of $y = f(x)$ at $x = 3$.

qs20

Solution **H-22**

The tangent line passes through the point $(3, f(3)) = (3, 5e^3)$, and its slope is $f'(3)$. Quotient rule gives:

$$f'(x) = \frac{50e^x(x^2+1) - 50e^x(2x)}{(x^2+1)^2} = \frac{50e^x(x^2-2x+1)}{(x^2+1)^2}$$

Hence $f'(3) = 2e^3$, and so an equation of the tangent line is

$$y - 5e^3 = 2e^3(x-3)$$

Ex. H-23 **3.3/3.4/3.5/3.9, 3.7** Su22 **Quiz**

Calculate each derivative below. Do not simplify your answer.

(a) $\dfrac{d}{dx}\left(\dfrac{x\sin(x)}{\pi^3+\ln(x)}\right)$

(b) $\dfrac{d}{dx}\left(\left(\sqrt{5x-8}+x^2\right)^{1/3}\right)$

qsu22

Solution **H-23**

(a) Use quotient rule, then product rule.

$$\frac{d}{dx}\left(\frac{x\sin(x)}{\pi^3+\ln(x)}\right) = \frac{(1\cdot\sin(x)+x\cos(x))\left(\pi^3+\ln(x)\right) - x\sin(x)\cdot\frac{1}{x}}{\left(\pi^3+\ln(x)\right)^2}$$

(b) Use chain rule twice.

$$\frac{d}{dx}\left(\left(\sqrt{5x-8}+x^2\right)^{1/3}\right) = \frac{1}{3}\left(\sqrt{5x-8}+x^2\right)^{-2/3}\cdot\left(\frac{1}{2}(5x-8)^{-1/2}\cdot 5 + 2x\right)$$

Ex. H-24 **3.3/3.4/3.5/3.9** Fa22 **Quiz**

For each part, calculate the derivative. You do not have to show work and there is no partial credit.

(a) $\dfrac{d}{dx}\left(\cos(x) - \dfrac{5}{x^7}\right)$

(b) $\dfrac{d}{dx}\left(8\sin(x)\ln(x)\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{2x^4}{10-3x}\right)$

qf22

Solution **H-24**

(a) Use power rule on the second term.

$$\frac{d}{dx}\left(\cos(x) - 5x^{-7}\right) = -\sin(x) + 35x^{-8}$$

(b) Use product rule.

$$\frac{d}{dx}\left(8\sin(x)\ln(x)\right) = 8\cos(x)\ln(x) + 8\sin(x)\cdot\frac{1}{x}$$

(c) Use quotient rule.

$$\frac{d}{dx}\left(\frac{2x^4}{10-3x}\right) = \frac{8x^3(10-3x) - 2x^4(-3)}{(10-3x)^3}$$

Ex. H-25 **3.3/3.4/3.5/3.9** Fa22 **Quiz**

For each part, calculate the derivative. Do not simplify your answer.

(a) $\dfrac{d}{dx}\left(\sqrt[5]{4\sin(x)+e^{3x-7}}\right)$

(b) $\dfrac{d}{dx}\left(\dfrac{2x^4\tan(x)}{3x+10}\right)$

qf22

Solution **H-25**

(a) Use chain rule twice.

$$\frac{d}{dx}\left(\sqrt[5]{4\sin(x)+e^{3x-7}}\right) = \frac{1}{5}\left(4\sin(x)+e^{3x-7}\right)^{-4/5} \cdot \left(4\cos(x)+3e^{3x-7}\right)$$

(b) Use quotient rule, then product rule.

$$\frac{d}{dx}\left(\frac{2x^4\tan(x)}{3x+10}\right) = \frac{\left(8x^3\tan(x)+2x^4\sec^2(x)\right)\cdot(3x+10)-2x^4\tan(x)\cdot 3}{(3x+10)^2}$$

Ex. H-26 **3.3/3.4/3.5/3.9** Fa22 **Quiz**

Find the x-coordinate of each point on the graph of $y = 3x^2 + \dfrac{60}{x}$ where the tangent line is horizontal.

qf22

Solution **H-26**

First write $f(x) = 3x^2 + 60x^{-1}$ and use the power rule.

$$f'(x) = 6x - 60x^{-2}$$

Now we solve the equation $f'(x) = 0$.

$$6x - \frac{60}{x^2} = 0$$
$$6x^3 - 60 = 0$$
$$x^3 = 10 \implies x = 10^{1/3}$$

Thus the graph $y = f(x)$ has a horizontal tangent at $x = 10^{1/3}$ only.

Ex. H-27 **3.3/3.4/3.5/3.9**

For each part, calculate $f'(x)$. *Do not simplify your answer.*

(a) $f(x) = \sqrt{2x} + 3x^2 + e^4$

(b) $f(x) = \dfrac{4}{x} + \ln(4)$

(c) $f(x) = \dfrac{8x^4 - 5x^{1/3} + 1}{x^2}$

(d) $f(x) = \dfrac{x^2+3}{x-1}$

(e) $f(x) = x^3 e^x$

(f) $f(x) = \sqrt{x}\cos(x) - e^x\sin(x)$

(g) $f(x) = \dfrac{\tan(x) + 9x^2}{\ln(x) - 4x}$

(h) $f(x) = \dfrac{x\sin(x)}{1 - e^x\cos(x)}$

c3s3

Solution **H-27**

(a) First write $f(x)$ as $f(x) = \sqrt{2}x^{1/2} + 3x^2 + e^4$. Now use power rule repeatedly. Note that e^4 is a constant.

$$f'(x) = \sqrt{2}\cdot\frac{1}{2}x^{-1/2} + 3\cdot 2x + 0$$

(b) First write $f(x)$ as $f(x) = 4x^{-1} + \ln(4)$. Note that $\ln(4)$ is a constant.

$$f'(x) = 4\cdot(-1)x^{-2} + 0$$

(c) First write $f(x)$ as $f(x) = 8x^2 - 5x^{-5/3} + x^{-2}$. Now use power rule repeatedly.

$$f'(x) = 16x + \frac{25}{3}x^{-8/3} - 2x^{-3}$$

(d) Use quotient rule.

$$f'(x) = \frac{(x-1)(2x) - (x^2+3)(1)}{(x-1)^2}$$

H-27

(e) Use product rule.

$$f'(x) = x^3e^x + 3x^2e^x$$

(f) Use product rule on each term.

$$f'(x) = \left(x^{1/2}(-\sin(x)) + \tfrac{1}{2}x^{-1/2}\cos(x)\right) - (e^x\cos(x) + e^x\sin(x))$$

(g) Use quotient rule.

$$f'(x) = \frac{(\ln(x) - 4x)(\sec(x)^2 + 18x) - (\tan(x) + 9x^2)(\frac{1}{x} - 4)}{(\ln(x) - 4x)^2}$$

(h) Use quotient rule. When differentiating the numerator and denominator individually, use product rule.

$$f'(x) = \frac{(1 - e^x\cos(x))(x\cos(x) + \sin(x)) - (x\sin(x))(e^x\sin(x) - e^x\cos(x))}{(1 - e^x\cos(x))^2}$$

Ex. H-28 **3.3/3.4/3.5/3.9**

Use the quotient rule to prove that $\dfrac{d}{dx}(\cot(x)) = -\csc(x)^2$.

c3s3

Solution H-28

Let $f(x) = \cot(x)$ and write $f(x)$ as $f(x) = \frac{\cos(x)}{\sin(x)}$. Now use quotient rule. Recall the identity $\cos(x)^2 + \sin(x)^2 = 1$ and the definition $\csc(x) = \frac{1}{\sin(x)}$.

$$f'(x) = \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin(x)^2} = \frac{-\left(\sin(x)^2 + \cos(x)^2\right)}{\sin(x)^2} = \frac{-1}{\sin(x)^2} = -\csc(x)^2$$

Ex. H-29 **3.3/3.4/3.5/3.9**

Find the x-coordinate of each point on the graph of the given function where the tangent line is horizontal.

(a) $f(x) = \dfrac{1}{x^2} - \dfrac{1}{x^3}$

(b) $f(x) = (x^2 - 8)e^x$

(c) $f(x) = \dfrac{1}{\sqrt{x}}(x + 9)$

(d) $f(x) = (1 - \sin(x))\sin(x)$

c3s3

Solution H-29

(a) Horizontal lines have zero slope and the derivative gives the slope of the tangent line at x. Hence we must solve the equation $f'(x) = 0$. First we write $f(x)$ as $f(x) = x^{-2} - x^{-3}$. Now use power rule.

$$f'(x) = -2x^{-3} + 3x^{-4} = \frac{-2x + 3}{x^4}$$

. Hence $f(x)$ has a horizontal tangent line at $x = \frac{3}{2}$ (the solution to $f'(x) = 0$).

(b) Horizontal lines have zero slope and the derivative gives the slope of the tangent line at x. Hence we must solve the equation $f'(x) = 0$. Now we compute the derivative using product rule and simplify.

$$f'(x) = (x^2 - 8)e^x + (2x)e^x = (x^2 + 2x - 8)e^x = (x + 4)(x - 2)e^x$$

Hence $f(x)$ has a horizontal tangent line at $x = -4$ and $x = 2$ (the solutions to $f'(x) = 0$).

(c) Horizontal lines have zero slope and the derivative gives the slope of the tangent line at x. Hence we must solve the equation $f'(x) = 0$. First we write $f(x)$ as $f(x) = x^{1/2} + 9x^{-1/2}$. Now use power rule and simplify.

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{9}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x - 9) = \frac{x - 9}{2x^{3/2}}$$

Hence $f(x)$ has a horizontal tangent line at $x = 9$ (the solutions to $f'(x) = 0$).

(d) Horizontal lines have zero slope and the derivative gives the slope of the tangent line at x. Hence we must solve the equation $f'(x) = 0$. Now we compute the derivative using product rule and simplify.

$$f'(x) = (1 - \sin(x))\cos(x) + -\cos(x)\sin(x) = \cos(x) - 2\sin(x)\cos(x) = \cos(x)(1 - 2\sin(x))$$

H-29

Now we solve the equation $f'(x) = 0$. Hence we have $\cos(x) = 0$ or $\sin(x) = 1/2$.

- The equation $\cos(x) = 0$ has two infinite sets of solutions: $x = \frac{\pi}{2} + 2\pi n$ (where n is any integer) or $x = \frac{3\pi}{2} + 2\pi n$ (where n is any integer).
- The equation $\sin(x) = \frac{1}{2}$ also has two infinite sets of solutions: $x = \frac{\pi}{6} + 2\pi n$ (where n is any integer) or $x = \frac{5\pi}{6} + 2\pi n$ (where n is any integer).

Hence the graph of $y = f(x)$ has a horizontal tangent line at any value of x in any of these four sets of solutions.

Ex. H-30 **3.3/3.4/3.5/3.9**

Find an equation for each line tangent to the graph of $f(x) = \dfrac{3x+5}{x+1}$ that is perpendicular to the line $2x - y = 1$.

c3s3

Solution H-30

The line $2x - y = 1$ has slope 2, whence the slope of our desired tangent lines is $-\frac{1}{2}$. The slope of the tangent line at x is given by $f'(x)$, so the desired tangent line occurs at values of x for which $f'(x) = -\frac{1}{2}$. We use quotient rule to compute $f'(x)$ and simplify.

$$f'(x) = \frac{(x+1)(3) - (3x+5)(1)}{(x+1)^2} = \frac{-2}{(x+1)^2}$$

Now we solve the equation $f'(x) = -\frac{1}{2}$.

$$-\frac{2}{(x+1)^2} = -\frac{1}{2} \implies (x+1)^2 = 4 \implies x = -3 \text{ or } x = 1$$

Thus there are two desired tangent lines.

- For $x = -3$, we have $f(-3) = 2$. So the tangent line is $y = 2 - \frac{1}{2}(x+3)$.
- For $x = 1$, we have $f(1) = 4$. So the tangent line is $y = 4 - \frac{1}{2}(x+3)$.

Ex. H-31 **3.3/3.4/3.5/3.9**

For each part, calculate $f'(x)$. Do not simplify your answer after computing the derivative.

(a) $f(x) = \dfrac{\tan(x)}{\pi - \sec(x)}$

(b) $f(x) = \cos\left(e^{-3x}\right)$

(c) $f(x) = \sqrt{\ln(x^2+4) + x\sin(2x)}$

(d) $f(x) = \dfrac{e^{1/x}}{x^{2/3} + x^{1/3}}$

unit3

Solution H-31

(a) Use quotient rule.

$$f'(x) = \frac{(\pi - \sec x)(\sec^2 x) - (\tan x)(-\sec x \tan x)}{(\pi - \sec x)^2}$$

(b) Use the chain rule twice.

$$f'(x) = -\sin\left(e^{-3x}\right) \cdot e^{-3x} \cdot (-3)$$

(c) Use chain rule first on the outermost square root function. Then use chain rule and product rule to compute the derivative of the inner function.

$$f'(x) = \frac{1}{2}\left(\ln(x^2+4) + x\sin(2x)\right)^{-1/2} \cdot \left(\frac{2x}{x^2+4} + \sin(2x) + 2x\cos(2x)\right)$$

(d) Start with quotient rule. When differentiating the numerator, use chain rule.

$$f'(x) = \frac{e^{1/x} \cdot \left(\frac{-1}{x^2}\right) \cdot (x^{2/3} + x^{1/3}) - e^{1/x} \cdot \left(\frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3}\right)}{(x^{2/3} + x^{1/3})^2}$$

Ex. H-32 **3.3/3.4/3.5/3.9, 3.7**

Some values of g, h, g', and h' are given below. Use this table to answer parts (a) and (b).

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
0	1	7	2	3
2	-3	-9	1	5
4	5	-1	1	-6

(a) Let $f(x) = 3g(x)h(x)$. Calculate $f'(2)$.
(b) Let $F(x) = g\left(\sqrt{x}\right)$. Calculate $F'(4)$.

unit3

Solution **H-32**

(a) Use product rule.
$$f'(x) = 3g'(x)h(x) + 3g(x)h'(x)$$
Then substitute $x = 2$ and use the table of values.
$$f'(2) = 3(-9)(1) + 3(-3)(5) = -72$$

(b) Use chain rule.
$$F'(x) = g'\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$
Then substitute $x = 4$ and use the table of values.
$$F'(4) = g'(2) \cdot \frac{1}{2 \cdot 2} = -\frac{9}{4}$$

Ex. H-33 **3.3/3.4/3.5/3.9**

Find an equation of the line normal to the graph of $f(x) = 2x^2 - \ln(x) + 3$ at $x = 1$. (Recall that the normal line is perpendicular to the tangent line.)

unit3

Solution **H-33**

The derivative at a general point is
$$f'(x) = 4x - \frac{1}{x}$$
Henece $f'(1) = 3$, and so the slope of the normal line is $-\frac{1}{3}$. The normal line must pass through $(1, f(1)) = (1, 5)$. Hence the equation of the normal line is
$$y - 5 = -\frac{1}{3}(x - 1)$$

Ex. H-34 **3.3/3.4/3.5/3.9**

Find the x-coordinate of each point on the graph of $y = \dfrac{1}{\sqrt{x}}(x^3 + 15)$ where the tangent line is perpendicular to the line $x + 5y = 1$.

unit3

Solution **H-34**

The slope of the given line $x + 5y = 1$ is $-\frac{1}{5}$, and so we want to find all tangent lines with slope 5. First we write $f(x) = x^{5/2} + 15x^{-1/2}$ and find $f'(x)$.
$$f'(x) = \frac{5}{2}x^{3/2} - \frac{15}{2}x^{-3/2} = \frac{5}{2}x^{-3/2}\left(x^3 - 3\right)$$
Now we solve the equation $f'(x) = 5$.
$$\frac{5}{2}x^{-3/2}\left(x^3 - 3\right) = 5 \implies x^3 - 3 = 2x^{3/2} \implies x^3 - 2x^{3/2} - 3 = 0$$

H-34

This last equation is a quadratic in $u = x^{3/2}$, which we can factor.

$$\left(x^{3/2}+1\right)\left(x^{3/2}-3\right)=0$$

The equation $x^{3/2}+1=0$ has no solution since $x^{3/2} \geq 0$ for all x. The equation $x^{3/2}-3=0$ has the unique solution $x = 3^{2/3}$ (or $x = 9^{1/3}$). Hence the only x-coordinate at which the tangent line is perpendicular to $x + 5y = 1$ is $x = 9^{1/3}$.

Ex. H-35 **3.3/3.4/3.5/3.9, 3.7**

Suppose $f(4) = 7$, $f'(4) = -5$, $g(4) = 4$, and $g'(4) = -3$. Let $F(x) = f\left(\dfrac{x^2}{g(x)}\right)$. Calculate $F'(4)$.

other

Solution H-35

By quotient rule and chain rule, we have:

$$F'(x) = f'\left(\frac{x^2}{g(x)}\right)\cdot\frac{2xg(x)-x^2g'(x)}{g(x)^2}$$

Substituting $x = 4$ and using the given information gives:

$$F'(4) = f'\left(\frac{16}{4}\right)\cdot\frac{8\cdot 4-16\cdot(-3)}{16} = f'(4)\cdot 5 = -25$$

Ex. H-36 **3.3/3.4/3.5/3.9, 3.7**

Find an equation of the tangent line to $f(x) = 4x\cos(\pi x)$ at $x = \frac{1}{4}$.

other

Solution H-36

The point of tangency is $(\frac{1}{4}, f(\frac{1}{4})) = (\frac{1}{4}, \frac{1}{\sqrt{2}})$. The derivative of $f(x)$ is:

$$f'(x) = 4\cos(\pi x) - 4\pi x\sin(\pi x)$$

So the slope of the tangent line is $f'(\frac{1}{4}) = 2\sqrt{2} - \frac{\pi}{\sqrt{2}}$. Hence an equation of the tangent line is:

$$y = \frac{1}{\sqrt{2}} + \left(2\sqrt{2}-\frac{\pi}{\sqrt{2}}\right)\left(x-\frac{\pi}{4}\right)$$

Ex. H-37 **3.3/3.4/3.5/3.9**

Find the x-coordinate of each point on the graph of $y = x^3 - 7x^2 + x + 4$ such that the tangent line there is parallel to the line $6x - y = 1$.

other

Solution H-37

The slope of the given line is 6, and we want to solve $\frac{dy}{dx} = 6$ for the given curve.

$$\frac{dy}{dx} = 3x^2-14x+1 = 6 \Longrightarrow 3x^2-14x-5 = (3x+1)(x-5) = 0 \Longrightarrow x = -\frac{1}{3} \text{ or } x = 5$$

Ex. H-38 **3.3/3.4/3.5/3.9** ⋆Challenge

Find all points on the graph of $y = \dfrac{2}{x} + 3x$ such that the tangent line there passes through $(6, 17)$.

challenge

Solution H-38

Let (a, b) be the unknown point of tangency and consider the tangent line to the given curve at (a, b). Since (a, b) lies on the curve, we have $b = \frac{2}{a} + 3a$. The derivative at a general point is:

$$\frac{dy}{dx} = -\frac{2}{x^2}+3$$

H-38

So the slope of the tangent line is $-\frac{2}{a^2}+3$, and the desired tangent line has the following form:

$$y = \frac{2}{a} + 3a + \left(-\frac{2}{a^2}+3\right)(x-a)$$

The point $(6,17)$ must lie on this tangent line, so substitution of $x=6$ and $y=17$ must give an equation that a satisfies.

$$17 = \frac{2}{a} + 3a + \left(-\frac{2}{a^2}+3\right)(6-a)$$

Clearing all denominators and rearranging gives the equation:

$$a^2+4a-12=0 \implies (a+6)(a-2)=0 \implies a=-6 \text{ or } a=2$$

Hence the tangent line to the graph passes through $(6,17)$ if the point of tangency is $(-6,-\frac{55}{3})$ or $(2,7)$.

Ex. H-39 **3.3/3.4/3.5/3.9** **⋆Challenge**

Find all points P on the graph of $y=4x^2$ with the property that the tangent line at P passes through the point $(2,0)$.

challenge

Solution **H-39**

Let $f(x)=4x^2$. Denote the unknown point P by (a,b). We require two equations to solve for the two unknowns a and b. These equations are derived from the two following conditions.

(i) The point P lies on the curve $y=f(x)$.

(ii) The point $(2,0)$ lies on the line tangent to f at P.

Condition (i) simply gives us $f(a)=b$, or $4a^2=b$. For condition (ii), we first find a general equation for the line tangent to f at P. This tangent line has point of tangency $P=(a,4a^2)$ and slope $f'(a)=8a$. Hence an equation of the tangent line is

$$y=4a^2+8a(x-a) \implies y=8ax-4a^2$$

Condition (ii) states that the point $(2,0)$ lies on this line, whence we have $0=16a-4a^2$. This equation has solutions $a=0$ and $a=4$. Thus there are two such points P: $(0,0)$ and $(4,64)$.

§3.7: The Chain Rule

Ex. H-1 **3.3/3.4/3.5/3.9, 3.7** Sp18 Exam

For each part, calculate $f'(x)$. *After calculating the derivative, do not simplify your answer.*

(a) $f(x) = \dfrac{x^{-1}x^{8/3}}{4\sqrt[3]{x^2}}$ (b) $f(x) = \left(x + \sqrt{5x-6}\right)^{1/4}$ (c) $f(x) = \dfrac{x^2e^x}{\ln(x) - \cos(x)}$

exs18

Solution **H-1**

(a) Simplifying the exponents, we observe that $f(x) = \frac{1}{4}x$. Hence $f'(x) = \frac{1}{4}$.

(b) Use power rule, then chain rule twice.

$$f'(x) = \frac{1}{4}\left(x + \sqrt{5x-6}\right)^{-3/4} \cdot \left(1 + \frac{1}{2}(5x-6)^{-1/2} \cdot 5\right)$$

(c) Use quotient rule. When differentiating the numerator, use product rule.

$$f'(x) = \frac{\left(x^2e^x + 2xe^x\right) \cdot (\ln(x) - \cos(x)) - \left(x^2e^x\right) \cdot \left(\frac{1}{x} + \sin(x)\right)}{(\ln(x) - \cos(x))^2}$$

Ex. H-3 **3.3/3.4/3.5/3.9, 3.7** Fa18 Exam

Suppose f and g are differentiable for all x. For each part, use the table below or explain why there is not enough information.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	−1	−4	4	2
1	−1	−3	2	−4
2	−4	3	1	−1

(a) Let $F(x) = \dfrac{f(x)}{g(x)}$. Calculate $F'(0)$. (b) Let $G(x) = f\Big(xg(x)\Big)$. Calculate $G'(1)$.

exf18

Solution **H-3**

(a) First use quotient rule.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Now substitute $x = 0$ and use the table of values.

$$F'(0) = \frac{(-4)(4) - (-1)(2)}{4^2} = \frac{-16+2}{16} = -\frac{7}{8}$$

(b) First use chain rule, then product rule.

$$G'(x) = \frac{d}{dx}f\Big(xg(x)\Big) = f'\Big(xg(x)\Big) \cdot \frac{d}{dx}(xg(x)) = f'\Big(xg(x)\Big) \cdot (1 \cdot g(x) + xg'(x))$$

Now substitute $x = 1$ and use the table of values.

$$G'(1) = f'(1 \cdot g(1)) \cdot (g(1) + 1 \cdot g'(1)) = f'(2) \cdot (g(1) + g'(1)) = 3 \cdot (2 + (-4)) = -6$$

Ex. I-1 **3.7** Sp19 Exam

Suppose $f(4) = -8$ and $f'(4) = 3$. Let $g(x) = f\left(\frac{1}{4}x^2\right)$. Find $g'(4)$ or explain why it is impossible to do so with the given information.

exs19

Solution **I-1**

First use chain rule: $g'(x) = f'\left(\frac{1}{4}x^2\right) \cdot \frac{1}{2}x$. Hence $g'(4) = f'(4) \cdot 2 = 6$.

Ex. I-2 **3.7** Fa19 **Exam**

Find an equation of the line tangent to the graph of $f(x) = \tan(2x)$ at $x = \frac{\pi}{8}$.

exf19

Solution **I-2**

The tangent line must pass through the point $\left(\frac{\pi}{8}, f(\frac{\pi}{8})\right) = \left(\frac{\pi}{8}, \tan(\frac{\pi}{4})\right) = \left(\frac{\pi}{8}, 1\right)$. Now we find the derivative using chain rule.

$$f'(x) = \sec(2x)^2 \cdot 2$$

Hence the slope of the tangent line is $f'(\frac{\pi}{8}) = 2\sec(\frac{\pi}{4})^2 = 4$. An equation of the tangent line is:

$$y - 1 = 4\left(x - \frac{\pi}{8}\right)$$

Ex. I-3 **3.7** Sp20 **Exam**

Find an equation of the line tangent to the graph of $f(x) = 5e^{2\cos(x)}$ at $x = 3\pi/2$.

exs20

Solution **I-3**

The point of tangency is $\left(\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, 5\right)$. Observe that $f'(x) = 5e^{2\cos(x)} \cdot (-2\sin(x))$. Hence the slope of the tangent line is $f'\left(\frac{3\pi}{2}\right) = 10$. Thus an equation of the tangent line is

$$y - 5 = 10\left(x - \frac{3\pi}{2}\right)$$

Ex. I-4 **3.7** Su20 **Exam**

For each part, calculate the derivative by any valid method.

(a) $f(x) = x^2\cos(3x) + \frac{1}{5x}$

(b) $f(x) = \sqrt{\sin\left(\frac{e^x}{x+1}\right)}$

exsu20

Solution **I-4**

(a) Write the second term as $\frac{1}{5x} = \frac{1}{5}x^{-1}$. Then use product rule and power rule.

$$f'(x) = 2x\cos(3x) - 3x^2\sin(3x) - \frac{1}{5}x^{-2}$$

(b) Use chain rule twice. For the second application of chain rule, use quotient rule.

$$f'(x) = \frac{1}{2}\left(\sin\left(\frac{e^x}{x+1}\right)\right)^{-1/2}\cos\left(\frac{e^x}{x+1}\right) \cdot \frac{e^x(x+1) - e^x \cdot 1}{(x+1)^2}$$

Ex. H-16 **3.3/3.4/3.5/3.9, 3.7** Sp21 **Exam**

Suppose $f(2) = -7$ and $f'(2) = 3$.

(a) Let $g(x) = \cos(x)f(x)$. Calculate $g'(2)$.

(b) Let $h(x) = e^{2f(x)+3}$. Calculate $h'(2)$.

exs21

Solution **H-16**

(a) We use product rule.

$$g'(x) = -\sin(x)f(x) + \cos(x)f'(x)$$

Hence $g'(2) = 7\sin(2) + 3\cos(2)$.

(b) We use chain rule.

$$h'(x) = e^{2f(x)+3} \cdot 2f'(x)$$

Hence $h'(2) = 6e^{-11}$.

Ex. I-5 3.7 Fa21 Exam

Let $f(x) = x^9 e^{4x}$.

(a) Find $f'(x)$.

(b) Explain how to find where the tangent line to the graph of f is horizontal.

(c) Find where the graph of f has a horizontal tangent line.

exf21

Solution I-5

(a) Use product rule and chain rule.

$$f'(x) = 9x^8 e^{4x} + x^9 \cdot 4e^{4x} = x^8 e^{4x}(9 + 4x)$$

(b) We must solve the equation $f'(x) = 0$ for x.

(c) The solutions to $f'(x) = 0$ are $x = 0$ and $x = -\frac{9}{4}$, thus these are the x-values where f has a horizontal tangent line.

Ex. I-6 3.7 Fa21 Exam

Selected values of the functions f and g and their derivatives are given in the table below. Use these values to complete the questions.

x	1	2	3	4
$f(x)$	4	3	2	1
$f'(x)$	-4	-1	-9	-3
$g(x)$	2	1	3	4
$g'(x)$	1	2	4	5

(a) Suppose $h(x) = 5f(x) - 8g(x)$. Find $h'(1)$.

(b) Suppose $p(x) = x^2 f(x)$. Find $p'(2)$.

(c) Suppose $q(x) = f(x^2)$. Find $q'(2)$.

exf21

Solution I-6

(a) We have $h'(x) = 5f'(x) - 8g'(x)$. Thus

$$h'(1) = 5f'(1) - 8g'(1) = 5 \cdot (-4) - 8 \cdot 1 = -28$$

(b) By product rule we have $p'(x) = 2xf(x) + x^2 f'(x)$. Thus

$$p'(2) = 2 \cdot 2 \cdot f(2) + 4 \cdot f'(2) = 4 \cdot 3 + 4 \cdot (-1) = 8$$

(c) By chain rule we have $q'(x) = f'(x^2) \cdot 2x$. Thus

$$q'(2) = f'(4) \cdot 2 \cdot 2 = (-3) \cdot 4 = -12$$

Ex. I-7 3.7 Fa21 Exam

Suppose f is differentiable at x and $g(x) = \dfrac{16\ln(15x)}{6f(x) - \sqrt{x+17}}$. Find $g'(x)$.

exf21

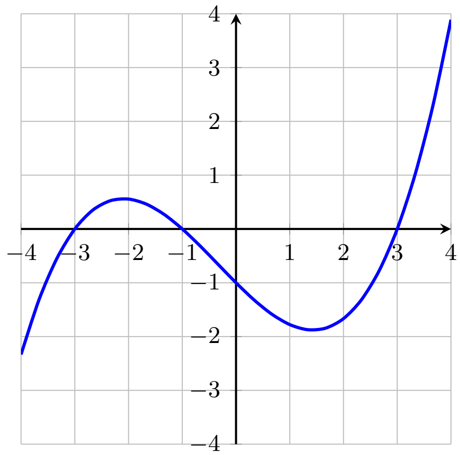
Solution I-7

We start with quotient rule since the expression for $g(x)$ is a quotient. When we differentiate the numerator we must use chain rule.

$$g'(x) = \frac{\left(16 \cdot \frac{1}{15x} \cdot 15\right) \cdot \left(6f(x) - \sqrt{x+17}\right) - (16\ln(15x) \cdot \left(6f'(x) - \frac{1}{2\sqrt{x+7}}\right)}{\left(6f(x) - \sqrt{x+7}\right)^2}$$

Ex. B-8 **2.1/2.2, 3.7, 4.3/4.4** Fa21 **Exam**

For each part, use the graph of $y = g(x)$.



(a) How many solutions does the equation $g'(x) = 0$ have?

(b) Order the following quantities from least to greatest: $g'(-2.5)$, $g'(-2)$, $g'(0)$, and $g'(4)$. In your answer, write these quantities symbolically; do not give a numerical estimate.

(c) What is the sign of $g''(-3)$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(d) Let $h(x) = g(x)^2$. What is the sign of $h'(-4)$(negative, positive, or zero)? If there is not enough information to determine the value, explain why.

exf21

Solution **B-8**

(a) The function g is differentiable for all x and has two local extrema (one local min and one local max). So $g'(x) = 0$ has two solutions.

(b) We note the following: $g'(-2.5)$ is small and positive, $g'(-2) = 0$, $g'(0)$ is small and negative, and $g'(4)$ is large and positive. Thus the correct order is: $g'(0)$, $g'(-2)$, $g'(-2.5)$, $g'(4)$.

(c) The function g is concave down in an interval containing $x = -3$. Thus $g''(-3)$ is positive.

(d) We have $h'(x) = 2g(x)g'(x)$, whence $h'(-4) = 2g(-4)g'(-4)$. Observe that $g(-4) < 0$ and $g'(-4) > 0$. Thus $h'(-4) < 0$.

Ex. H-18 **3.3/3.4/3.5/3.9, 3.7** Sp22 **Exam**

For each part, calculate $f'(x)$. After calculating the derivative, do not simplify your answer.

(a) $f(x) = 3x^{13} + 7\sqrt{x} - \dfrac{5}{x^3} + 12$ (b) $f(x) = \dfrac{e^x - 2\sin(x)}{\ln(x) + x^3}$ (c) $f(x) = 2x^4 \cos(3e^x)$

exs22

Solution **H-18**

(a) We use power rule several times.

$$\frac{d}{dx}\left(3x^{13} + 7\sqrt{x} - \frac{5}{x^3} + 12\right) = 39x^{12} + \frac{7}{2}x^{-1/2} + 15x^{-4}$$

(b) We use quotient rule.

$$\frac{d}{dx}\left(\frac{e^x - 2\sin(x)}{\ln(x) + x^3}\right) = \frac{(e^x - 2\cos(x))\left(\ln(x) + x^3\right) - (e^x - 2\sin(x))\left(\frac{1}{x} + 3x^2\right)}{\left(\ln(x) + x^3\right)^2}$$

(c) We use product rule, then chain rule.

$$\frac{d}{dx}\left(2x^4\cos(3e^x)\right) = 8x^3\cos(3e^x) + 2x^4 \cdot (-\sin(3e^x)) \cdot 3e^x$$

Ex. I-8 **3.7** Sp22 **Exam**

Let $h(x) = \dfrac{f(x^2)}{g(x)}$. Use the table of values below to calculate $h'(1)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-4	6	2	3
2	5	-2	-1	9

exs22

Solution **I-8**

We first calculate $h'(x)$ using quotient rule, then chain rule.

$$h'(x) = \frac{f'(x^2) \cdot 2x \cdot g(x) - f(x^2)g'(x)}{g(x)^2}$$

Now we substitute $x = 1$ and use the table of values.

$$h'(1) = \frac{2f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{2 \cdot 6 \cdot 2 - (-4) \cdot 3}{2^2} = 9$$

Ex. H-20 **3.3/3.4/3.5/3.9, 3.7** Su22 **Exam**

For each part, calculate the derivative. After calculating the derivative, do not simplify your answer.

(a) $\dfrac{d}{dx}\left(\tan\left(\dfrac{\ln(x)}{2x-5}\right)\right)$ (b) $\dfrac{d}{dx}\left(3x^7\cos(x) - 8e^{3x}\right)$ (c) $\dfrac{d}{dx}\left(10x^{12} - \dfrac{3}{x^3} + \sqrt[4]{x}\right)$

exsu22

Solution **H-20**

(a) Use chain rule, then use quotient rule.

$$\frac{d}{dx}\left(\tan\left(\frac{\ln(x)}{2x-5}\right)\right) = \sec^2\left(\frac{\ln(x)}{2x-5}\right) \cdot \frac{\frac{1}{x}\cdot(2x-5) - \ln(x)\cdot 2}{(2x-5)^2}$$

(b) Use product rule on the first term and chain rule on the second term.

$$\frac{d}{dx}\left(3x^7\cos(x) - 8e^{3x}\right) = 21x^6\cos(x) - 3x^7\sin(x) - 24e^{3x}$$

(c) Use power rule on each term.

$$\frac{d}{dx}\left(10x^{12} - 3x^{-3} + x^{1/4}\right) = 120x^{11} + 9x^{-4} + \frac{1}{4}x^{-3/4}$$

Ex. I-9 **3.7** Fa22 **Exam**

For each part, calculate the indicated derivative. Do not simplify your answer.

(a) $\dfrac{d}{dx}\left(7x^{10} + \sqrt[3]{x} - \dfrac{8}{x^{20}} + \sec(8x)\right)$ (b) $\dfrac{d}{dx}\left(\dfrac{\ln\left(x^3+30\right)}{8x}\right)$ (c) $\dfrac{d}{dx}\left(\sin\left(xe^{-5x}\right)\right)$

exf22

Solution **I-9**

(a) Use power rule on the first three terms and chain rule on the last term.

$$\frac{d}{dx}\left(7x^{10} + x^{1/3} - 8x^{-20} + \sec(8x)\right) = 70x^9 + \frac{1}{3}x^{-2/3} + 160x^{-21} + 8\sec(8x)\tan(8x)$$

(b) Use quotient rule, then chain rule.

$$\frac{d}{dx}\left(\frac{\ln(x^3+30)}{8x}\right) = \frac{\frac{3x^2}{x^3+30}\cdot 8x - 8\ln(x^3+30)}{(8x)^2}$$

I-9

(c) Use chain rule, then product rule and chain rule.

$$\frac{d}{dx}\left(\sin\left(xe^{-5x}\right)\right) = \cos\left(xe^{-5x}\right)\cdot\left(e^{-5x} - 5xe^{-5x}\right)$$

Ex. I-10 3.7 Fa22 Exam

Find the coordinates of all points on the graph of $f(x) = x\sqrt{14-x^2}$ where the tangent line is horizontal. You must give both the x- and y-coordinate of each such point.

exf22

Solution I-10

We first find $f'(x)$ using product rule, then chain rule.

$$f'(x) = 1\cdot(14-x^2)^{1/2} + x\cdot\frac{1}{2}(14-x^2)^{-1/2}\cdot(-2x) = \sqrt{14-x^2} - \frac{x^2}{\sqrt{14-x^2}}$$

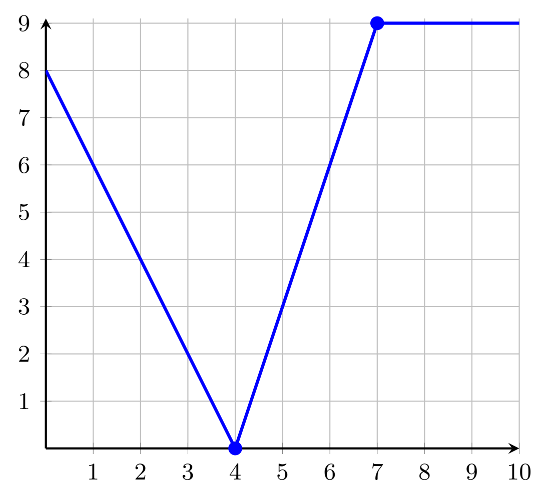
The tangent line to the graph of $f(x)$ is horizontal at points where $f'(x)=0$. To solve $f'(x)=0$, multiply both sides by $\sqrt{14-x^2}$, then solve for x.

$$\sqrt{14-x^2}\cdot\left(\sqrt{14-x^2} - \frac{x^2}{\sqrt{14-x^2}}\right) = 0$$
$$14 - x^2 - x^2 = 0 \Longrightarrow 14 - 2x^2 = 0 \Longrightarrow x^2 = 7 \Longrightarrow x = -\sqrt{7} \text{ or } x = \sqrt{7}$$

Hence the graph has horizontal tangent lines at $x = -\sqrt{7}$ and $x = \sqrt{7}$.

Ex. I-11 3.7 Fa22 Exam

The graph of $y = f(x)$ is given below.



(a) Calculate $f'(6)$. Briefly explain how you found your answer.

(b) Let $g(x) = 9xf(2x)$. Find an equation of the line tangent to the graph of $y = g(x)$ at $x = 3$.

exf22

Solution I-11

(a) The value $f'(6)$ is the slope of the tangent line to $y = f(x)$ at $x = 6$. The graph of $y = f(x)$ is a line with slope 3 on the interval $[4, 7]$. Thus $f'(6) = 3$.

(b) We find $g'(x)$ with product rule and chain rule.

$$g'(x) = 9f(2x) + 9xf'(2x)\cdot 2 = 9f(2x) + 18xf'(2x)$$

Now observe the following:

$$g(3) = 9\cdot 3\cdot f(6) = 9\cdot 3\cdot 6 = 162$$
$$g'(3) = 9\cdot f(6) + 18\cdot 3\cdot f'(6) = 9\cdot 6 + 18\cdot 3\cdot 3 = 216$$

I-11

Thus the desired tangent line is $y = 162 + 216(x - 3)$.

Ex. I-12 **3.7** Sp18 **Quiz**

Calculate $\frac{d}{dx}\left(4x^3 e^{\sin(2x)}\right)$. *After computing the derivative, do not simplify your answer.*

qs18

Solution **I-12**

Use product rule. When differentiating the second factor, use chain rule twice.

$$\frac{d}{dx}\left(4x^3 e^{\sin(2x)}\right) = 4x^3 \cdot e^{\sin(2x)} \cdot \cos(2x) \cdot 2 + 12x^2 \cdot e^{\sin(2x)}$$

Ex. I-13 **3.7** Sp20 **Quiz**

For each part, find $f'(x)$. After computing the derivative, do not simplify your answer.

(a) $f(x) = \sqrt{\tan(x^3)}$

(b) $f(x) = x^{3/4} \ln\left(\sin(x) + x + e^3\right)$

qs20

Solution **I-13**

(a) Use chain rule twice.

$$f'(x) = \frac{1}{2}\left(\tan(x^3)\right)^{-1/2} \cdot \sec(x^3)^2 \cdot 3x^2$$

(b) Use product rule. On the second factor, use chain rule.

$$f'(x) = \frac{3}{4}x^{-1/4} \ln\left(\sin(x) + x + e^3\right) + x^{3/4} \cdot \frac{1}{\sin(x) + x + e^3} \cdot (\cos(x) + 1)$$

Ex. I-14 **3.7** Sp20 **Quiz**

Let $f(x) = x^{12} e^{5-3x}$. Find the x-coordinate of each point at which the graph of $y = f(x)$ has a horizontal tangent line.

qs20

Solution **I-14**

We seek all solutions to the equation $f'(x) = 0$. Using product rule and chain rule, we have

$$f'(x) = 12x^{11} e^{5-3x} + x^{12} e^{5-3x} \cdot (-3) = x^{11} e^{5-3x}(12 - 3x)$$

Recall that $e^z > 0$ for all z. So the solutions to $f'(x) = 0$ are $x = 0$ and $x = 4$.

Ex. I-15 **3.7** Su22 **Quiz**

Find the x-coordinate of each point on the graph of $y = x^3 e^{-5x}$ where the tangent line is horizontal.

qsu22

Solution **I-15**

First we find the derivative using product rule and chain rule.

$$f'(x) = 3x^2 e^{-5x} + x^3 e^{-5x} \cdot (-5) = x^2 e^{-5x}(3 - 5x)$$

The tangent line is horizontal where the derivative is equal to 0.

$$x^2 e^{-5x}(3 - 5x) = 0 \Longrightarrow x = 0 \quad \text{or} \quad x = \frac{3}{5}$$

Ex. H-23 **3.3/3.4/3.5/3.9, 3.7** Su22 **Quiz**

Calculate each derivative below. Do not simplify your answer.

(a) $\frac{d}{dx}\left(\frac{x \sin(x)}{\pi^3 + \ln(x)}\right)$

(b) $\frac{d}{dx}\left(\left(\sqrt{5x - 8} + x^2\right)^{1/3}\right)$

qsu22

Solution **H-23**

(a) Use quotient rule, then product rule.

$$\frac{d}{dx}\left(\frac{x\sin(x)}{\pi^3+\ln(x)}\right) = \frac{(1\cdot\sin(x)+x\cos(x))\left(\pi^3+\ln(x)\right)-x\sin(x)\cdot\frac{1}{x}}{\left(\pi^3+\ln(x)\right)^2}$$

(b) Use chain rule twice.

$$\frac{d}{dx}\left(\left(\sqrt{5x-8}+x^2\right)^{1/3}\right) = \frac{1}{3}\left(\sqrt{5x-8}+x^2\right)^{-2/3}\cdot\left(\frac{1}{2}(5x-8)^{-1/2}\cdot 5+2x\right)$$

Ex. I-16 **3.7** Fa22 **Quiz**

Find the x-coordinate of each point on the graph $y=\left(x^2+x-1\right)e^{3x}$ where the tangent line is horizontal.

qf22

Solution **I-16**

First use product rule (and chain rule) to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = (2x+1)e^{3x}+(x^2+x-1)e^{3x}\cdot 3 = (3x^2+5x-2)e^{3x} = (3x-1)(x+2)e^{3x}$$

The graph has horizontal tangent lines where $\frac{dy}{dx}=0$, that is, at $x=\frac{1}{3}$ and $x=-2$.

Ex. I-17 **3.7**

For each part, calculate $f'(x)$ *Do not simplify your answer.*

(a) $f(x)=\sqrt{\sin(x)}$

(b) $f(x)=\sin(\sqrt{x})$

(c) $f(x)=\sqrt{\sin(\sqrt{x})}$

(d) $f(x)=(x^3-3x+2)^2$

(e) $f(x)=\dfrac{1}{(3x+1)^2}$

(f) $f(x)=(2x+\sec(x))^2$

(g) $f(x)=e^{-2x}\sin(x)$

(h) $f(x)=\dfrac{\ln(2x+1)}{(2x+1)^2}$

(i) $f(x)=(\tan(x)+1)^4\cos(2x)$

(j) $f(x)=\left(\dfrac{6}{9-2x}\right)^8$

(k) $f(x)=\left(\sin\left((4x-5)^2\right)\right)^4$
(*Some authors write this function as* $f(x)=\sin^4(4x-5)^2$.)

(l) $f(x)=\sqrt[3]{\sin(x)\cos(x)}$

(m) $f(x)=\sqrt{\dfrac{x^2-1}{x^3+x}}$

(n) $f(x)=\ln(\ln(x))$

(o) $f(x)=\sin(\sin(\sin(x)))$

(p) $f(x)=\left(x+\left(x+\sin(x)^2\right)^3\right)^4$

(q) $f(x)=|x|$

(*Hint: Recall* $|x|=\sqrt{x^2}$.)

c3s7

Solution **I-17**

(a) $f'(x)=\dfrac{1}{2}(\sin(x))^{-1/2}\cos(x)$

(b) $f'(x)=\cos(\sqrt{x})\cdot\dfrac{1}{2}x^{-1/2}$

(c) $f'(x)=\dfrac{1}{2}\left(\sin(\sqrt{x})\right)^{-1/2}\cos(\sqrt{x})\cdot\dfrac{1}{2}x^{-1/2}$

(d) $f'(x)=2(x^3-3x+2)(3x^2-3)$

(e) $f'(x)=-2(3x+1)^{-3}\cdot 3$

(f) $f'(x)=2\left(2x+\sec(x)\right)\left(2+\sec(x)\tan(x)\right)$

(g) $f'(x)=e^{-2x}\cos(x)-2e^{-2x}\sin(x)$

(h) $f'(x)=\dfrac{(2x+1)^2\cdot\frac{1}{2x+1}\cdot 2-\ln(2x+1)\cdot 2(2x+1)\cdot 2}{(2x+1)^4}$

(i) $f'(x)=(\tan(x)+1)^4(-\sin(2x))\cdot 2+\cos(2x)\cdot 4\left(\tan(x)+1\right)^3\cdot\sec(x)^2$

(j) $f'(x)=6^8\cdot(-8)\cdot(9-2x)^{-9}\cdot(-2)$

(k) $f'(x)=4\left(\sin\left((4x-5)^2\right)\right)^3\cdot\cos\left((4x-5)^2\right)\cdot 2(4x-5)\cdot 4$

(l) $f'(x)=\dfrac{1}{3}\left(\sin(x)\cos(x)\right)^{-2/3}\cdot\left(\sin(x)(-\sin(x))+\cos(x)\cos(x)\right)$

I-17

(m) $f'(x) = \frac{1}{2}\left(\frac{x^2-1}{x^3+x}\right)^{-1/2} \cdot \frac{(x^3+x)(2x)-(x^2-1)(3x^2+1)}{(x^3+x)^2}$

(n) $f'(x) = \frac{1}{\ln(x)} \cdot \frac{1}{x}$

(o) $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$

(p) $f'(x) = 4\left(x + \left(x+\sin(x)^2\right)^3\right)^3 \cdot \left(1 + 3\left(x+\sin(x)^2\right)^2 \cdot (1+2\sin(x)\cos(x))\right)$

(q) $f'(x) = \frac{1}{2}(x^2)^{-1/2} \cdot (2x) = \frac{x}{|x|}$

Ex. I-18 3.7

Find the x-coordinate of each point at which the graph of $y = f(x)$ has a horizontal tangent line.

(a) $f(x) = (2x^2-7)^3$

(b) $f(x) = x^2e^{1-3x}$

(c) $f(x) = \ln(3x^4+6x^2-4x^3-12x+6)$

(d) $f(x) = \frac{(e^{3x}+e^{-3x})^2}{e^{3x}}$

c3s7

Solution I-18

(a) Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation $f'(x) = 0$. Computing the derivative requires chain rule.

$$f'(x) = 3(2x^2-7)^2 \cdot (4x) = 12x(2x^2-7)^2$$

Hence either $12x = 0$ (whence $x = 0$) or $2x^2 - 7 = 0$ (whence $x = -\sqrt{\frac{7}{2}}$ or $x = \sqrt{\frac{7}{2}}$).

(b) Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation $f'(x) = 0$. Computing the derivative requires chain rule and product rule.

$$f'(x) = x^2e^{1-3x} \cdot (-3) + 2x \cdot e^{1-3x} = e^{1-3x}(2x-3x^2) = x(2-3x)e^{1-3x}$$

Hence either $x = 0$ or $-3x+2 = 0$ (whence $x = \frac{2}{3}$). Note that $e^{1-3x} > 0$ for all x.

(c) Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation $f'(x) = 0$. Computing the derivative requires chain rule.

$$f'(x) = \frac{1}{3x^4+6x^2-4x^3-12x+6} \cdot (12x^3+12x-12x^2-12) = \frac{12(x-1)(x^2+1)}{3x^4+6x^2-4x^3-12x+6}$$

Hence $x - 1 = 0$ (whence $x = 1$). (The equation $x^2+1 = 0$ has no solutions.) However, we see that $x = 1$ is not in the domain of f since the argument of a logarithm must be a strictly positive number. (Attempting to substitute $x = 1$ into $f(x)$ gives the expression $\ln(-1)$.) So there are no points where the tangent line is horizontal.

(d) Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation $f'(x) = 0$. Before computing the derivative, we will simplify the function a bit. Combining all terms under one squaring operation gives the following.

$$f(x) = \frac{(e^{3x}+e^{-3x})^2}{e^{3x}} = \frac{(e^{3x}+e^{-3x})^2}{\left(e^{3x/2}\right)^2} = \left(\frac{e^{3x}+e^{-3x}}{e^{3x/2}}\right)^2 = \left(e^{3x/2}+e^{-9x/2}\right)^2$$

Computing the derivative now requires just chain rule.

$$f'(x) = 2\left(e^{3x/2}+e^{-9x/2}\right) \cdot \left(\tfrac{3}{2}e^{3x/2} - \tfrac{9}{2}e^{-9x/2}\right) = 3\left(e^{3x/2}+e^{-9x/2}\right) \cdot \left(e^{6x}-3\right)e^{-9x/2}$$

Now we solve $f'(x) = 0$. Note that $\left(e^{3x/2}+e^{-9x/2}\right)$ and $e^{-9x/2}$ are both strictly positive, so the equation $f'(x) = 0$ reduces to $\left(e^{6x}-3\right) = 0$, whence $x = \frac{1}{6}\ln(3)$.

Ex. I-19 3.7

Suppose g and h are differentiable functions. Selected values of g, h, and their derivatives are given below.

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
2	1	7	2	3
4	-3	-9	1	5
16	5	-1	1	-6

Define the function f by the formula

$$f(x) = g\left(\sqrt{x}\right) h\left(x^2\right)$$

(a) Calculate $f(4)$ or explain why there is not enough information to do so.

(b) Calculate $f'(4)$ or explain why there is not enough information to do so.

c3s7

Solution I-19

(a) $f(4) = g(\sqrt{4})h(4^2) = g(2)h(16) = 1 \cdot 1 = 1$

(b) First we calculate $f'(x)$ using product rule and chain rule (twice!).

$$f'(x) = g'\left(\sqrt{x}\right) \cdot \tfrac{1}{2}x^{-1/2} \cdot h\left(x^2\right) + g\left(\sqrt{x}\right) h'\left(x^2\right) \cdot 2x = \frac{g'(\sqrt{x})h(x^2)}{2\sqrt{x}} + 2x\, g(\sqrt{x})h'(x^2)$$

Now we substitute $x = 4$ and use the table values.

$$f'(4) = \frac{g'(2)h(16)}{4} + 8g(2)h'(16) = \frac{7 \cdot 1}{4} + 8 \cdot 1 \cdot (-6) = -\frac{185}{4}$$

Ex. H-32 3.3/3.4/3.5/3.9, 3.7

Some values of g, h, g', and h' are given below. Use this table to answer parts (a) and (b).

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
0	1	7	2	3
2	-3	-9	1	5
4	5	-1	1	-6

(a) Let $f(x) = 3g(x)h(x)$. Calculate $f'(2)$.

(b) Let $F(x) = g\left(\sqrt{x}\right)$. Calculate $F'(4)$.

unit3

Solution H-32

(a) Use product rule.

$$f'(x) = 3g'(x)h(x) + 3g(x)h'(x)$$

Then substitute $x = 2$ and use the table of values.

$$f'(2) = 3(-9)(1) + 3(-3)(5) = -72$$

(b) Use chain rule.

$$F'(x) = g'\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$

Then substitute $x = 4$ and use the table of values.

$$F'(4) = g'(2) \cdot \frac{1}{2 \cdot 2} = -\frac{9}{4}$$

Ex. I-20 3.7

For each part, calculate $f'(x)$.

(a) $f(x) = \tan(3x^2 + e)$

(b) $f(x) = e^{x/(x+1)}$

unit3

Solution **I-20**

(a) $f'(x) = \sec(3x^2 + e)^2 \cdot 6x$

(b) $f'(x) = e^{x/(x+1)} \cdot \left(\dfrac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}\right) = e^{x/(x+1)} \cdot \dfrac{1}{(x+1)^2}$

Ex. I-21 **3.7**

For each part, calculate $f'(x)$.

(a) $f(x) = \sin\left(7xe^{-3x}\right)$

(b) $f(x) = \sqrt{\dfrac{2\ln(x)}{\tan(3x) - \tan(3)}}$

other

Solution **I-21**

(a) Use chain rule and then product rule.

$$f'(x) = \cos\left(7xe^{-3x}\right) \cdot \left(7x \cdot e^{-3x} \cdot (-3) + e^{-3x} \cdot 7\right)$$

(b) Use chain rule and then quotient rule.

$$f'(x) = \frac{1}{2}\left(\frac{2\ln(x)}{\tan(3x) - \tan(3)}\right)^{-1/2}\left(\frac{(\tan(3x) - \tan(3)) \cdot \frac{2}{x} - 6\ln(x)\sec(3x)^2}{(\tan(3x) - \tan(3))^2}\right)$$

Ex. H-35 **3.3/3.4/3.5/3.9, 3.7**

Suppose $f(4) = 7$, $f'(4) = -5$, $g(4) = 4$, and $g'(4) = -3$. Let $F(x) = f\left(\dfrac{x^2}{g(x)}\right)$. Calculate $F'(4)$.

other

Solution **H-35**

By quotient rule and chain rule, we have:

$$F'(x) = f'\left(\frac{x^2}{g(x)}\right) \cdot \frac{2xg(x) - x^2g'(x)}{g(x)^2}$$

Substituting $x = 4$ and using the given information gives:

$$F'(4) = f'\left(\frac{16}{4}\right) \cdot \frac{8 \cdot 4 - 16 \cdot (-3)}{16} = f'(4) \cdot 5 = -25$$

Ex. H-36 **3.3/3.4/3.5/3.9, 3.7**

Find an equation of the tangent line to $f(x) = 4x\cos(\pi x)$ at $x = \frac{1}{4}$.

other

Solution **H-36**

The point of tangency is $(\frac{1}{4}, f(\frac{1}{4})) = (\frac{1}{4}, \frac{1}{\sqrt{2}})$. The derivative of $f(x)$ is:

$$f'(x) = 4\cos(\pi x) - 4\pi x\sin(\pi x)$$

So the slope of the tangent line is $f'(\frac{1}{4}) = 2\sqrt{2} - \frac{\pi}{\sqrt{2}}$. Hence an equation of the tangent line is:

$$y = \frac{1}{\sqrt{2}} + \left(2\sqrt{2} - \frac{\pi}{\sqrt{2}}\right)\left(x - \frac{\pi}{4}\right)$$

§3.8: Implicit Differentiation

Ex. J-1 3.8 Fa17 Exam

Find all points on the following curve at which the tangent line is horizontal.

$$2x^2 - 4xy + 7y^2 = 45$$

Hint: Find a second equation that such points must satisfy. Then solve a system of two equations for x and y.

exf17

Solution J-1

The tangent line is horizontal at points where $\frac{dy}{dx} = 0$. Using implicit differentiation we have

$$4x - 4x\frac{dy}{dx} - 4y + 14y\frac{dy}{dx} = 0$$

Setting $\frac{dy}{dx} = 0$ gives the equation $4x - 4y = 0$, or $x = y$. Hence the desired points must satisfy both $x = y$ and the original equation. Substituting $x = y$ into the original equation gives

$$2x^2 - 4x^2 + 7x^2 = 45$$

Hence $5x^2 = 45$, or $x = \pm 3$. The points on the graph where the tangent line is horizontal are $(-3, -3)$ and $(3, 3)$.

Ex. J-2 3.8 Sp18 Exam

Find an equation of the line tangent to the following curve at the point $(2, 0)$.

$$x^3 + e^{xy} = 3y + 9$$

exs18

Solution J-2

Implicitly differentiating the equation with respect to x gives

$$3x^2 + e^{xy}\left(x\frac{dy}{dx} + y\right) = 3\frac{dy}{dx}$$

Substituting $x = 2$ and $y = 0$ gives

$$12 + 1 \cdot \left(2\frac{dy}{dx} + 0\right) = 3\frac{dy}{dx} \implies \frac{dy}{dx} = 12$$

Hence the equation of the tangent line is

$$y - 0 = 12(x - 2)$$

Ex. J-3 3.8 Fa18 Exam

Find an equation of the line tangent to the following curve at $(8, 1)$.

$$\sin\left(\frac{\pi x}{y}\right) = x - 8y$$

exf18

Solution J-3

We implicitly differentiate each side of the equation with respect to x.

$$\cos\left(\frac{\pi x}{y}\right) \cdot \left(\frac{y \cdot \pi - \pi x \cdot \frac{dy}{dx}}{y^2}\right) = 1 - 8\frac{dy}{dx}$$

Now we substitute the point $(x, y) = (8, 1)$.

$$1 \cdot \left(\frac{\pi - 8\pi\frac{dy}{dx}}{1}\right) = 1 - 8\frac{dy}{dx}$$

J-3

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \frac{1}{8}$, the slope of the desired tangent line. Hence an equation of the tangent line is

$$y - 1 = \frac{1}{8}(x - 8)$$

Ex. J-4 3.8 Sp19 Exam

Find an equation of the line tangent to the following curve at the point $(1, 1)$.

$$\frac{5x}{y} = 4x + y^3$$

exs19

Solution J-4

Differentiate each side of the equation with respect to x using implicit differentiation.

$$\frac{5 \cdot y - 5x \cdot \frac{dy}{dx}}{y^2} = 4 + 3y^2 \cdot \frac{dy}{dx}$$

Substituting the point $(x, y) = (1, 1)$ gives $5 - 5\frac{dy}{dx} = 4 + 3\frac{dy}{dx}$, whence $\frac{dy}{dx} = \frac{1}{8}$. Hence the tangent line has equation

$$y - 1 = \frac{1}{8}(x - 1)$$

Ex. J-5 3.8 Fa19 Exam

Find an equation of the line tangent to the following curve at the origin.

$$\sin(x + 2y) + 9x + 1 = e^y$$

Solution J-5

Differentiating both sides with respect to x gives

$$\cos(x + 2y) \cdot \left(1 + 2\frac{dy}{dx}\right) + 9 = e^y \cdot \frac{dy}{dx}$$

Now substitute $x = 0$ and $y = 0$.

$$1 + 2\frac{dy}{dx} + 9 = \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = -10$, and so an equation of the tangent line is $y = -10x$.

Ex. J-6 3.8 Fa19 Exam

Find $\dfrac{dy}{dx}$ for a general point on the curve described by the following equation. *Do not simplify your answer.*

$$x^3y^2 + (x + y)^2 = 100$$

Solution J-6

Differentiating both sides with respect to x gives:

$$3x^2y^2 + x^3 2y \cdot \frac{dy}{dx} + 2(x + y) \cdot \left(1 + \frac{dy}{dx}\right) = 0$$

We now algebraically solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-3x^2y^2 - 2(x + y)}{2x^3y + 2(x + y)}$$

Ex. J-7 **3.8** Sp20 **Exam**

A particle in the fourth quadrant is moving along a path described by the equation

$$x^2 + xy + 2y^2 = 16$$

such that at the moment its x-coordinate is 2, its y-coordinate is decreasing at a rate of 5 cm/sec. At what rate is its x-coordinate changing at that time?

exs20

Solution **J-7**

Our table below defines the relevant variables and lists the information described in the problem.

Variables	x y	x-coordinate of particle y-coordinate of particle
Specific Time	$x = 2$ $\frac{dy}{dt} = -3$	"at the moment its x-coordinate is 2" "its y-coordinate is decreasing at a rate of 5 cm/sec"
General Time	(1) $x^2 + xy + 2y^2 = 16$ (2) $2x\frac{dx}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} + 4y\frac{dy}{dt} = 0$	given equation derivative of equation (1)
Unknown	$\frac{dx}{dt}$	"[a]t what rate is its x-coordinate changing"

Putting the specific-time info into equations (1) and (2) gives:

(i) $4 + 2y + 2y^2 = 16$

(ii) $4\frac{dx}{dt} + \frac{dx}{dt}y - 10 - 20y = 0$

Equation (i) gives $y = -3$ (the solution $y = 2$ is rejected since the particle is in the fourth quadrant). Putting $y = -3$ into equation (ii) gives

$$4\frac{dx}{dt} - 3\frac{dx}{dt} - 10 + 60 = 0 \implies \frac{dx}{dt} = -50$$

Thus the x-coordinate is changing at a rate of -50 cm/sec.

Ex. J-8 **3.8** Sp20 **Exam**

Find an equation of line tangent to the following curve at the origin.

$$\sin(x + 3y) + 9x + 1 = e^y$$

exs20

Solution **J-8**

Differentiating both sides with respect to x gives

$$\cos(x + 3y) \cdot \left(1 + 3\frac{dy}{dx}\right) + 9 = e^y \cdot \frac{dy}{dx}$$

Substituting $x = 0$ and $y = 0$ gives

$$1 + 3\frac{dy}{dx} + 9 = \frac{dy}{dx}$$

Hence the slope of the tangent line is $\frac{dy}{dx} = -5$, whence an equation of the tangent line is $y = -5x$.

Ex. J-9 **3.8** Sp20 **Exam**

Consider the curve described by the equation

$$3x^2 + 2xy + 4y^2 = 132$$

exs20

3.8 Sp20 Exam

At any point on this curve, we have

$$\frac{dy}{dx} = \frac{-3x - y}{x + 4y}$$

(a) Describe in two or three sentences the steps you should take to find the points on the curve where the tangent line is horizontal. *Your answer may contain either English, mathematical symbols, or both.*

(b) What is the rightmost (i.e., greatest x-coordinate) point on the curve where the tangent line is horizontal?

(c) Describe in one or two sentences how parts (a) and (b) would change if instead you wanted to find the points where the tangent line is vertical. You do not have to solve the problem again, but only describe generally what you would do differently. *Your answer may contain either English, mathematical symbols, or both.*

exs20

Solution **J-9**

(a) The unknown point (x, y) must lie on the curve and the tangent line is horizontal (i.e., $\frac{dy}{dx} = 0$). So we must solve the following simultaneous set of equations for x and y.

$$3x^2 + 2xy + 4y^2 = 132$$
$$\frac{-3x - y}{x + 4y} = 0$$

(Note that the second equation is equivalent to $y = -3x$.)

(b) Substituting $y = -3x$ into the original equation gives $3x^2 - 6x^2 + 36x^2 = 132$, or $33x^2 = 132$. Hence $x = -2$ or $x = 2$. The x-coordinate of the rightmost point with a horizontal tangent is thus $x = 2$. Since we also have $y = -3x$, the y-coordinate is $y = -6$.

(c) A vertical tangent line has an undefined slope, so we replace the equation $\frac{dy}{dx} = 0$ with "denominator of $\frac{dy}{dx}$ is 0". That is, we must solve the following simultaneous set of equations:

$$3x^2 + 2xy + 4y^2 = 132$$
$$x + 4y = 0$$

Ex. J-10 3.8 Sp20 Exam

Find an equation of the line tangent to the following curve at $(1, 7)$.

$$\ln(xy + x - 7) = 2x + 4y - 30$$

exs20

Solution **J-10**

Differentiating both sides with respect to x gives:

$$\frac{1}{xy + x - 7} \cdot \left(x\frac{dy}{dx} + y + 1\right) = 2 + 4\frac{dy}{dx}$$

Substituting $x = 1$ and $y = 7$ gives

$$\frac{dy}{dx} + 8 = 2 + 4\frac{dy}{dx}$$

Hence the slope of the tangent line is $\frac{dy}{dx} = 2$, whence an equation of the tangent line is

$$y = 7 + 2(x - 1)$$

Ex. J-11 3.8 Sp20 Exam

Consider the curve described by the equation

$$5x^2 - 4xy + y^2 = 8$$

At any point on this curve, we have

$$\frac{dy}{dx} = \frac{-5x + 2y}{-2x + y}$$

(a) Describe in two or three sentences the steps you should take to find each point on the curve where the tangent line is parallel to the line $y = x$. *Your answer may contain either English, mathematical symbols, or both.*

exs20

3.8 Sp20 Exam

(b) What is the leftmost (i.e., least x-coordinate) point on the curve where the tangent line is parallel to $y = x$?

(c) Describe in one or two sentences how parts (a) and (b) would change if instead you wanted to find the points where the tangent line is perpendicular to the line $y = 4$. You do not have to solve the problem again, but only describe generally what you would do differently. *Your answer may contain either English, mathematical symbols, or both.*

exs20

Solution **J-11**

(a) The point must lie on the curve and the tangent line has slope 1 (i.e., $\frac{dy}{dx} = 1$). So we must solve the following simultaneous set of equations for x and y.

$$5x^2 - 4xy + y^2 = 8$$
$$\frac{-5x + 2y}{-2x + y} = 1$$

(Note that the second equation is equivalent to $y = 3x$.)

(b) Substituting $y = 3x$ into the original equation gives $5x^2 - 4x(3x) + (3x)^2 = 8$, or $2x^2 = 8$. Hence $x = -2$ or $x = 2$. The x-coordinate of the leftmost point with a tangent line parallel to $y = x$ is $x = -2$. We have $y = 3x$, whence the y-coordinate is $y = -6$.

(c) The line $y = 4$ is horizontal, so a perpendicular line is vertical. A vertical tangent line has an undefined slope, so we replace the equation $\frac{dy}{dx} = 0$ with "denominator of $\frac{dy}{dx}$ is 0". That is, we must solve the following simultaneous set of equations:

$$5x^2 - 4xy + y^2 = 8$$
$$-2x + y = 0$$

Ex. J-12 3.8 Sp20 Exam

Consider the curve described by the following equation.

$$e^{12x+2y} = 6y - 3xy + 1$$

(a) Find $\dfrac{dy}{dx}$ at a general point on this curve.

(b) Calculate the slope of the line tangent to the curve at $(2, -12)$.

(c) There is a point on the curve close to the origin with coordinates $(0.07, b)$, and the line tangent to the curve at the origin is $y = 3x$. Use linear approximation to estimate the value of b.

exs20

Solution **J-12**

(a) Differentiating both sides with respect to x gives:

$$e^{12x+2y} \cdot \left(12 + 2\frac{dy}{dx}\right) = 6\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y$$

Solving algebraically for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{12e^{12x+2y} + 3y}{6 - 3x - 2e^{12x+2y}}$$

(b) Substituting $x = 2$ and $y = -12$ into the expression above gives $\frac{dy}{dx} = 12$.

(c) The tangent line at the origin is a linear approximation of the curve near the origin. Hence the point $(0.07, b)$ lies approximately on this tangent line. Hence $b \approx 3(0.07) = 0.21$.

Ex. J-13 3.8 Su20 Exam

Consider the curve described by the equation

$$x^4 - x^2y + y^4 = 1$$

(a) Find $\dfrac{dy}{dx}$ at a general point on the curve.

exsu20

3.8 Su20 Exam

(b) Find an equation of the line tangent to the curve at the point $(-1, 1)$.

exsu20

Solution J-13

(a) Use implicit differentiation.

$$4x^3 - 2xy - x^2 \cdot \frac{dy}{dx} + 4y^3 \cdot \frac{dy}{dx} = 0$$

Solving algebraically for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2xy - 4x^3}{4y^3 - x^2}$$

(b) The slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(-1,1)} = \left.\left(\frac{xy - 2x^3}{4y^3 - x^2}\right)\right|_{(x,y)=(-1,1)} = \frac{2}{3}$$

Hence an equation of the tangent line is

$$y = 1 + \frac{2}{3}(x+1)$$

Ex. J-14 3.8 Su20 Exam

On an online exam, a student uses logarithmic differentiation to find the first derivative of

$$f(x) = (3 + \sin(x))^{2+x^2}$$

They type the following two lines for their work.

$$y = (3 + \sin(x))^{2+x^2}$$
$$\ln(y) = \ln(\cdots$$

Unfortunately, the student runs out of time and is unable to submit the rest of their work. Oh no! Find $f'(x)$ by completing the student's work.

exsu20

Solution J-14

We take logs of both sides, use logarithm laws, and then use implicit differentiation.

$$\begin{aligned} y &= (3 + \sin(x))^{2+x^2} \\ \ln(y) &= \ln\left((3 + \sin(x))^{2+x^2}\right) \\ \ln(y) &= (2 + x^2)\ln(3 + \sin(x)) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 2x \cdot \ln(3 + \sin(x)) + (2 + x^2) \cdot \frac{1}{3 + \sin(x)} \cdot \cos(x) \end{aligned}$$

Now solve for $\frac{dy}{dx}$ and replace y with $f(x)$.

$$f'(x) = (3 + \sin(x))^{2+x^2} \cdot \left(2x\ln(3 + \sin(x)) + \frac{(2 + x^2)\cos(x)}{3 + \sin(x)}\right)$$

Ex. J-15 3.8 Fa20 Exam

Consider the following curve, where a and b are unspecified constants.

$$ax^2y - 3xy^2 + 4x = b$$

(a) Show that $\frac{dy}{dx} = \frac{3y^2 - 2axy - 4}{ax^2 - 6xy}$.

(b) Suppose the tangent line to the curve at the point $(1, 1)$ is $y = 1 + 5(x - 1)$. Use part (a) to find the value of a.

exf20

3.8 Fa20 Exam

(c) Use your answer to part (b) to find the value of b.

exf20

Solution J-15

(a) Differentiate both sides of the equation with respect to x, using product rule and chain rule on each of the first two terms.

$$2axy + ax^2\frac{dy}{dx} - 3y^2 - 6xy\frac{dy}{dx} + 4 = 0$$

Collecting like terms and factoring gives:

$$\left(ax^2 - 6xy\right)\frac{dy}{dx} + \left(2axy - 3y^2 + 4\right) = 0$$

Elementary algebra then gives the desired result.

(b) The slope of the tangent line at $(1, 1)$ is 5, whence

$$5 = \left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \left.\left(\frac{3y^2 - 2axy - 4}{ax^2 - 6xy}\right)\right|_{(x,y)=(1,1)} = \frac{-2a - 1}{a - 6}$$

Solving for a gives $a = \frac{29}{7}$.

(c) The point $(1, 1)$ lies on the curve, i.e., the point $(1, 1)$ satisfies the original equation. This implies $a + 1 = b$, and so $b = \frac{36}{7}$.

Ex. J-16 3.8 Sp21 Exam

Consider the curve defined by the equation below, where a and b are unspecified constants.

$$\sqrt{xy} = ay^3 + b$$

Suppose the equation of the tangent line to the curve at the point $(3, 3)$ is $y = 3 + 4(x - 3)$.

(a) What is the value of $\frac{dy}{dx}$ at $(3, 3)$?

(b) Calculate a and b.

exs21

Solution J-16

(a) The slope of the tangent is line is 4, hence $\frac{dy}{dx} = 4$ at $(3, 3)$.

(b) We first use implicit differentiation on the equation of the curve.

$$\frac{1}{2}(xy)^{-1/2}\cdot\left(x\frac{dy}{dx} + y\right) = 3ay^2\cdot\frac{dy}{dx}$$

We now substitute $x = 3$, $y = 3$, and $\frac{dy}{dx} = 4$, which gives us $\frac{15}{6} = 108a$, whence $a = \frac{5}{216}$. We now substitute $x = 3$, $y = 3$, and $a = \frac{5}{216}$ into the equation for the curve, which gives us $3 = \frac{135}{216} + b$, whence $b = \frac{19}{8}$.

Ex. J-17 3.8 Fa21 Exam

Consider the curve defined by the following equation, where A and B are unspecified constants.

$$Ax^2 - 8xy = B\cos(y) + 3$$

(a) Find a formula for $\frac{dy}{dx}$.

(b) Suppose the point $(8, 0)$ is on the curve. Find an equation that A and B must satisfy.

(c) Suppose the tangent line to the curve at the point $(8, 0)$ is $y = 6x - 48$. Find the values of A and B.

exf21

Solution J-17

(a) Using implicit differentation, we obtain:

$$2Ax - 8y - 8x\frac{dy}{dx} = -B\sin(y)\frac{dy}{dx}$$

J-17

Solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2Ax - 8y}{8x - B\sin(y)}$$

(b) The point $(8, 0)$ must satisfy the equation that defines the curve, whence:

$$64A = B + 3$$

(c) We have that $\frac{dy}{dx} = 6$ (the slope of the tangent line) when $x = 8$ and $y = 0$. Hence by part (a) we have:

$$7 = \frac{16A - 0}{64 - 0} = \frac{A}{4}$$

Hence $A = 28$. From part (b) we then have $B = 64A - 3 = 1533$.

Ex. J-18 3.8 Sp22 Exam

Consider the curve described by the following equation:

$$12x^2 + 6xy + y^2 = 20$$

Find all points on the curve where the tangent line is horizontal. Write your answer as a comma-separated list of coordinate pairs.

Hint: Find a second equation that such points must satisfy.

exs22

Solution J-18

We first differentiate each side of the given equation to find an equation for $\frac{dy}{dx}$.

$$24x + 6y + 6x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

At a point where the tangent line is horizontal we have $\frac{dy}{dx} = 0$, and so putting $\frac{dy}{dx} = 0$ in the above equation gives

$$24x + 6y = 0 \Longrightarrow y = -4x$$

Hence any point where the tangent is horizontal must satisfy both the equation for the curve and the equation $y = -4x$. Combining these two equations gives

$$12x^2 + 6x(-4x) + (-4x)^2 = 20$$

This equation is equivalent to $4x^2 = 20$, whence $x = -\sqrt{5}$ or $x = \sqrt{5}$. Recalling that $y = -4x$ at the desired points, we find two points where the tangent line is horizontal: $(-\sqrt{5}, 4\sqrt{5})$ and $(\sqrt{5}, -4\sqrt{5})$.

Ex. J-19 3.8 Su22 Exam

Find all points on the graph of the following equation where the tangent line is vertical.

$$x^2 - 2xy + 10y^2 = 450$$

exsu22

Solution J-19

We first find $\frac{dy}{dx}$ using implicit differentiation.

$$2x - 2y - 2x\frac{dy}{dx} + 20y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ algebraically gives

$$\frac{dy}{dx} = \frac{2y - 2x}{20y - 2x}$$

The slope of a vertical line is undefined (infinite), thus we seek points for which $\frac{dy}{dx}$ is undefined (infinite). Thus vertical tangent lines occur at points where $20y - 2x = 0$, or where $x = 10y$. These points also lie on the curve itself.

J-19

Substituting $x = 10y$ into the equation for the curve gives:

$$(10y)^2 - 2(10y)y + 10y^2 = 450 \Longrightarrow 90y^2 = 450 \Longrightarrow y = \pm\sqrt{5}$$

Hence the points where the curve has a vertical tangent are $(10\sqrt{5}, \sqrt{5})$ and $(-10\sqrt{5}, -\sqrt{5})$.

Alternatively... we can observe that a vertical tangent line occurs where $\frac{dx}{dy} = 0$. Then implicitly differentiate the equation of the curve *with respect to* x and then set $\frac{dx}{dy}$ to 0.

Ex. J-20 3.8 Fa22 Exam

Consider the following curve.

$$\cos(5x + y - 5) = 8xe^y + y - 7$$

(a) Calculate $\dfrac{dy}{dx}$ for a general point on the curve.

(b) Find an equation of the line tangent to the curve at the point $(1, 0)$.

exf22

Solution J-20

(a) Differentiate both sides of the equation with respect to x, using chain rule on the left side and product rule on the right side.

$$-\sin(5x + y - 5) \cdot \left(5 + \frac{dy}{dx}\right) = 8e^y + 8xe^y \frac{dy}{dx} + \frac{dy}{dx}$$

Now algebraically solve for $\frac{dy}{dx}$. First expand the left side, then collect terms multiplying $\frac{dy}{dx}$ on one side.

$$-5\sin(5x + y - 5) - \sin(5x + y - 5)\frac{dy}{dx} = 8e^y + 8xe^y\frac{dy}{dx} + \frac{dy}{dx}$$
$$\left(-\sin(5x + y - 5) - 8xe^y - 1\right)\frac{dy}{dx} = 5\sin(5x + y - 5) + 8e^y$$
$$\frac{dy}{dx} = \frac{5\sin(5x + y - 5) + 8e^y}{-\sin(5x + y - 5) - 8xe^y - 1}$$

(b) We substitute $x = 1$ and $y = 0$ into our formula for $\frac{dy}{dx}$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \frac{5\sin(0) + 8e^0}{-\sin(0) - 8e^0 - 1} = -\frac{8}{9}$$

This is the slope of the desired tangent line. Hence the desired tangent line is

$$y = -\frac{8}{9}(x - 1)$$

Ex. J-21 3.8 Sp20 Quiz

Find an equation of the line tangent to the graph of $xe^y = x^3 + (y-1)^2 - 1$ at the point $(0, 2)$.

qs20

Solution J-21

Implicitly differentiate with respect to x.

$$1 \cdot e^y + x \cdot e^y \cdot \frac{dy}{dx} = 3x^2 + 2(y-1) \cdot \frac{dy}{dx}$$

Now substitute the given point (i.e., $x = 0$ and $y = 2$), and solve for $\frac{dy}{dx}$ to find the slope of the tangent line.

$$e^2 + 0 = 0 + 2 \cdot 1 \cdot \frac{dy}{dx}$$

Hence the slope of the tangent line is $e^2/2$, and an equation of the tangent line is

$$y - 2 = \frac{e^2}{2}x$$

Ex. J-22 **3.8** Su22 **Quiz**

Suppose x and y are implicitly related by the following equation.

$$5 + xy^2 = \frac{y}{2 - x^3}$$

Find $\dfrac{dy}{dx}$ for a general point on the curve.

qsu22

Solution **J-22**

First multiply both sides of the equation by $2 - x^3$ and expand the left side.

$$10 - 5x^3 + 2xy^2 - x^4y^2 = y$$

Now differentiate with respect to x, using product rule twice on the left side.

$$-15x^2 + 2y^2 + 4xy\frac{dy}{dx} - 4x^3y^2 - 2x^4y\frac{dy}{dx} = \frac{dy}{dx}$$

Now solve algebraically for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{15x^2 - 2y^2 + 4x^3y^2}{4xy - 2x^4y - 1}$$

Ex. J-23 **3.8** Su22 **Quiz**

Suppose x and y are implicitly related by the following equation.

$$6x^2 - 3xy + 2y^2 = 52$$

Find all points (both x- and y-coordinates) on the curve where the tangent line is horizontal.

qsu22

Solution **J-23**

First differentiate both sides with respect to x.

$$12x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

At a point where the tangent line is horizontal, we have $\frac{dy}{dx} = 0$. So putting $\frac{dy}{dx} = 0$ in the above equation gives

$$12x - 3y = 0 \Longrightarrow y = 4x$$

Thus the point must both satisfy the equation $y = 4x$ and lie on the curve. So we substitute $y = 4x$ into the original equation that describes the curve.

$$6x^2 - 3x(4x) + 2(4x)^2 = 52 \Longrightarrow 26x^2 = 52 \Longrightarrow x = -\sqrt{2} \quad \text{or} \quad x = \sqrt{2}$$

Thus there are two points where the tangent line is horizontal: $(-\sqrt{2}, -4\sqrt{2})$ and $(\sqrt{2}, 4\sqrt{2})$.

Ex. J-24 **3.8** Fa22 **Quiz**

Find $\dfrac{dy}{dx}$ for a general point on the following curve.

$$x\sin(y) + 10 = \ln(y^2 + x)$$

qf22

Solution **J-24**

Differentiate each side of the equation with respect to x. Use product rule on the left side and chain rule twice on the right side.

$$1 \cdot \sin(y) + x\cos(y)\frac{dy}{dx} = \frac{1}{y^2 + x} \cdot \left(2y\frac{dy}{dx} + 1\right)$$

J-24

Now we algebraically solve for $\frac{dy}{dx}$. Multiply both sides by y^2+x, then solve for $\frac{dy}{dx}$.

$$(y^2+x)\sin(y)+x(y^2+x)\cos(y)\frac{dy}{dx}=2y\frac{dy}{dx}+1$$
$$\left(x(y^2+x)\cos(y)-2y\right)\frac{dy}{dx}=1-(y^2+x)\sin(y)$$
$$\frac{dy}{dx}=\frac{1-(y^2+x)\sin(y)}{x(y^2+x)\cos(y)-2y}$$

Ex. J-25 3.8 Fa22 Quiz

Find the slope of the line tangent to the given curve at the point $(1,\frac{1}{4})$.

$$x\tan(\pi y)=16y^2+3\ln(x)$$

qf22

Solution J-25

Differentiate each side of the equation with respect to x.

$$1\tan(\pi y)+x\sec^2(\pi y)\cdot\pi\frac{dy}{dx}=32y\frac{dy}{dx}+\frac{3}{x}$$

Now substitute the point, i.e., $x=1$ and $y=\frac{1}{4}$. Recall that $\tan(\frac{\pi}{4})=1$ and $\sec(\frac{\pi}{4})=\sqrt{2}$. Hence we obtain:

$$1+2\pi\frac{dy}{dx}=8\frac{dy}{dx}+3$$

Solving for $\frac{dy}{dx}$ gives the slope of the tangent line:

$$\frac{dy}{dx}=\frac{2}{2\pi-8}=\frac{1}{\pi-4}$$

Ex. J-26 3.8

For each part, find $\dfrac{dy}{dx}$ for a general point on the curve described by the given equation.

(a) $x^2+y^4=12x+y$

(b) $y+\dfrac{1}{xy}=x^2$

(c) $\sin(x+y)=x+\cos(y)$

(d) $\ln\left(\dfrac{x-y}{xy}\right)=\dfrac{1}{y}$

(e) $6x^2+3xy+2y^2+17y=6$

c3s8

Solution J-26

(a) Differentiating each side with respect to x gives:

$$2x+4y^3\frac{dy}{dx}=12+\frac{dy}{dx}$$

Algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx}=\frac{12-2x}{4y^3-1}$$

(b) Use negative exponents to rewrite the equation.

$$y+x^{-1}y^{-1}=x^2$$

Now differentiating each side with respect to x, using product rule on $x^{-1}y^{-1}$.

$$\frac{dy}{dx}-x^{-2}y^{-1}-x^{-1}y^{-2}\frac{dy}{dx}=2x$$

> Alternatively, we can multiply the original equation by xy to obtain:
>
> $$xy^2+1=x^3y$$
>
> Differentiating these terms requires one more use of product rule but we can avoid negative exponents.

J-26

Algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2x + x^{-2}y^{-1}}{1 - x^{-1}y^{-2}}$$

(c) Differentiating each side with respect to x gives:

$$\cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right) = 1 - \sin(y)\frac{dy}{dx}$$

Algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{1 - \cos(x+y)}{\sin(y) + \cos(x+y)}$$

(d) Differentiating each side with respect to x gives:

$$\frac{xy}{x-y} \cdot \frac{\left(1 - \frac{dy}{dx}\right) xy - (x-y)\left(y + x\frac{dy}{dx}\right)}{x^2y^2} = \frac{-1}{y^2}\frac{dy}{dx}$$

To solve for $\frac{dy}{dx}$ we simplify the left side of the equation.

$$\frac{y^2 - x^2\frac{dy}{dx}}{xy(x-y)} = \frac{-1}{y^2}\frac{dy}{dx}$$

So now algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{y^3}{x^2y - x^2 + xy}$$

(e) Differentiating each side with respect to x gives:

$$12x + 3y + 3x\frac{dy}{dx} + 6y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

Algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{-12x - 3y}{3x + 6y + 17}$$

Ex. J-27 3.8

Find an equation of the line tangent to the following curve at $\left(\frac{1}{e-2}, 1\right)$.

$$xe^y = 2xy + y^3$$

c3s8

Solution J-27

Differentiating both sides with respect to x gives

$$xe^y \cdot \frac{dy}{dx} + e^y = 2x\frac{dy}{dx} + 2y + 3y^2 \cdot \frac{dy}{dx}$$

Alternatively, we can solve for x in the original equation first:

$$x = \frac{y^3}{e^y - 2y}$$

Then we find $\frac{dx}{dy}$ using normal derivative rules with no implicit differentiation, and use the identity $\frac{dy}{dx} = 1/(\frac{dx}{dy})$.

J-27

Now substitute $x = \frac{1}{e-2}$ and $y = 1$.

$$\frac{e}{e-2} \cdot \frac{dy}{dx} + e = \frac{2}{e-2} \cdot \frac{dy}{dx} + 2 + 3\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \frac{e-2}{2}$, and so the equation of the tangent line is:

$$y = 1 + \frac{e-2}{2}\left(x - \frac{1}{e-2}\right)$$

Ex. J-28 3.8

Find an equation of the line tangent to the following curve at $(0, \pi)$.

$$\sin(x-y) = xy$$

c3s8

Solution J-28

Differentiating both sides with respect to x gives

$$\cos(x-y) \cdot \left(1 - \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

Now substitute $x = 0$ and $y = \pi$.

$$(-1) \cdot \left(1 - \frac{dy}{dx}\right) = \pi$$

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \pi + 1$, and so the equation of the tangent line is:

$$y = \pi + (\pi + 1)x$$

Ex. J-29 3.8

Consider the curve given by the following equation.

$$x^2 + xy + 3y^2 = 99$$

(a) Find all points on the graph of the curve where the tangent line is horizontal.

(b) Find all points on the graph of the curve where the tangent line is vertical.

c3s8

Solution J-29

For both parts of the question, we need $\frac{dy}{dx}$. So we differentiate each side of our equation.

$$2x + y + x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

Algebraically solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{-2x - y}{x + 6y}$$

(a) Let $P = (x, y)$ be the unknown point of tangency. Then P satisfies two conditions:

- The point P lies on the curve, equivalent to $x^2 + xy + 3y^2 = 99$.
- $\frac{dy}{dx} = 0$, equivalent to $-2x - y = 0$ or $y = -2x$.

This is a simultaneous system of two equations for x and y. Putting $y = -2x$ into the first equation gives:

$$x^2 + x(-2x) + 3(-2x)^2 = 99 \implies 11x^2 = 99 \implies x = -3 \text{ or } x = 3$$

Hence there are two points where the tangent line is horizontal: $(-3, 6)$ and $(3, -6)$.

(b) Let $P = (x, y)$ be the unknown point of tangency. Then P satisfies two conditions:

- The point P lies on the curve, equivalent to $x^2 + xy + 3y^2 = 99$.

J-29

- $\frac{dy}{dx}$ is undefined, equivalent to $x + 6y = 0$ or $x = -6y$.

This is a simultaneous system of two equations for x and y. Putting $x = -6y$ into the first equation gives:

$$(-6y)^2 + (-6y)y + 3y^2 = 99 \implies 33y^2 = 99 \implies y = -\sqrt{3} \text{ or } y = \sqrt{3}$$

Hence there are two points where the tangent line is vertical: $(6\sqrt{3}, -\sqrt{3})$ and $(-6\sqrt{3}, \sqrt{3})$.

Ex. J-30 3.8

Find the slope of the tangent line to the curve $x^3 - y^3 = y - 1$ at the point $(1, 1)$.

unit3

Solution J-30

Implicitly differentiate the equation with respect to x to obtain

$$3x^2 - 3y^2 \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

Substituting the point $(x, y) = (1, 1)$ gives

$$3 - 3\frac{dy}{dx} = \frac{dy}{dx}$$

Finally solving for
$ddyx$ gives the slope of the desired tangent line: $\frac{dy}{dx} = \frac{3}{4}$.

Ex. J-31 3.8

Find the slope of the tangent line to the curve $x^3 + xy + y^2 = 7$ at $(1, 2)$.

unit3

Solution J-31

Implicitly differentiating with respect to x gives

$$3x^2 + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

Substituting $(x, y) = (1, 2)$ gives $5 + 5\frac{dy}{dx} = 0$, whence $\frac{dy}{dx} = -1$.

Ex. J-32 3.8

Find an equation of the line normal to the curve $5x^2y + 2y^3 = 22$ at the point $(2, 1)$.

unit3

Solution J-32

Implicitly differentiating with respect to x gives us

$$5x^2\frac{dy}{dx} + 10xy + 6y^2\frac{dy}{dx} = 0$$

Substituting $(x, y) = (2, 1)$ gives $26\frac{dy}{dx} + 20 = 0$, whence $\frac{dy}{dx} = -\frac{10}{13}$. Hence the normal line has a slope of $\frac{13}{10}$ and passes through the point $(2, 1)$. An equation for the normal line is thus

$$y = 1 + \frac{13}{10}(x - 2)$$

Ex. J-33 3.8

Find an equation of the line tangent to the curve $2x^2 - xy + 5y^2 = 24$ at the point $(-1, 2)$.

other

Solution J-33

Implicitly differentiating with respect to x gives

$$4x - x\frac{dy}{dx} - y + 10y\frac{dy}{dx} = 0$$

J-33

Substituting $x = -1$ and $y = 2$ gives $-4 + \frac{dy}{dx} - 2 + 20\frac{dy}{dx} = 0$, whence $\frac{dy}{dx} = \frac{2}{7}$. Hence an equation of the tangent line is

$$y - 2 = \frac{2}{7}(x - (-1))$$

Ex. J-34 3.8

Find an equation of the line tangent to the curve $\sin(x - y) = 4e^{xy} - 4e^9$ at the point $(3, 3)$.

other

Solution J-34

Implicitly differentiating each side of the equation with respect to x gives

$$\cos(x - y) \cdot \left(1 - \frac{dy}{dx}\right) = 4e^{xy}\left(x\frac{dy}{dx} + y\right)$$

Substituting $x = 3$ and $y = 3$ gives

$$1 \cdot \left(1 - \frac{dy}{dx}\right) = 4e^9\left(3\frac{dy}{dx} + 3\right) \implies \frac{dy}{dx} = \frac{1 - 12e^9}{1 + 12e^9}$$

Hence the equation of the tangent line is

$$y - 3 = \frac{1 - 12e^9}{1 + 12e^9}(x - 3)$$

Ex. J-35 3.8 ⋆Challenge

Find all tangent lines to the graph of $9x^2 - 18xy + y^2 = 1800$ that are perpendicular to the line $x + 3y = 10$.

challenge

Solution J-35

First we use implicit differentiation to find an equation for $\frac{dy}{dx}$.

$$18x - 18y - 18x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

The given line has slope $-\frac{1}{3}$, and so the desired tangent line as slope 3. Let (a, b) be the unknown point of tangency. Then $\frac{dy}{dx} = 3$ at that point, whence we have:

$$18a - 18b - 54a + 6b = 0 \implies b = -3a$$

The point (a, b) also lies on the curve, and so (a, b) satisfies the original equation for the curve. Substituting $b = -3a$ gives:

$$9a^2 - 18a(-3a) + (-3a)^2 = 1800 \implies 72a^2 = 1800 \implies a = -5 \text{ or } a = 5$$

Thus there are two such tangent lines: one at $(5, -15)$ (equation of the line is $y = -15 + 3(x - 5)$) and another at $(-5, 15)$ (equation of the line is $y = 15 + 3(x + 5)$).

Ex. J-36 3.8, 4.6 ⋆Challenge

Consider the curve described by the equation

$$\frac{x - y^3}{y + x^2} = x - 12$$

(a) Find an equation for the line tangent to this curve at $(-1, 4)$.

(b) There is a point on the curve with coordinates $(-1.1, b)$. Use linear approximation to estimate b. Round to three decimal places.

(c) There is a point on the curve with coordinates $(a, 4.2)$. Use linear approximation to estimate a. Round to three decimal places.

challenge

Solution **J-36**

(a) We write the equation as follows to make differentiation easier:

$$x - y^3 = xy + x^3 - 12y - 12x^2$$

Differentiating each side with respect to x gives:

$$1 - 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx} + 3x^2 - 12\frac{dy}{dx} - 24x$$

We now substitute $x = -1$ and $y = 4$:

$$1 - 48\frac{dy}{dx} = 4 - \frac{dy}{dx} + 3 - 12\frac{dy}{dx} + 24 \Longrightarrow \frac{dy}{dx} = -\frac{6}{7}$$

So an equation of the tangent line is:

$$y - 4 = -\frac{6}{7}(x+1)$$

(b) Since $(-1.1, b)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate b. That is, the point $(-1.1, b)$ approximately satisfies the equation of the tangent line:

$$b - 4 \approx -\frac{6}{7}(-1.1+1) \Longrightarrow b \approx \frac{28}{6.6} \approx 4.242$$

(c) Since $(a, 4.2)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate a. That is, the point $(a, 4.2)$ approximately satisfies the equation of the tangent line:

$$4.2 - 4 \approx -\frac{6}{7}(a+1) \Longrightarrow a \approx -\frac{7.4}{6.6} \approx -1.233$$

Ex. J-37 3.8 ⋆Challenge

Suppose $x^2 + y^2 = R^2$, where R is a constant. Find $\frac{d^2y}{dx^2}$ and fully simplify your answer as much as possible.

challenge

Solution **J-37**

Differentiating both sides with respect to x, then solving for $\frac{dy}{dx}$ gives:

$$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Differentiating $\frac{dy}{dx}$ with respect to x gives:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substituting and simplifying (using $x^2 + y^2 = R^2$) gives our final answer.

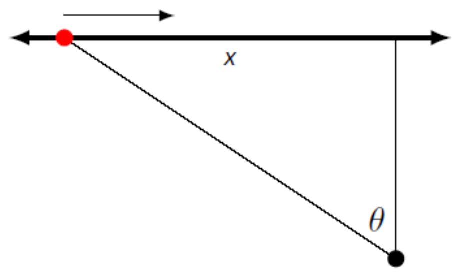
$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{R^2}{y^3}$$

§3.11: Related Rates

Ex. K-1 3.11 Fa17 Exam

A camera is located 5 feet from a straight wire along which a bead is moving at 6 feet per second. The camera automatically turns so that it is pointed at the bead at all times. How fast is the camera turning when the bead is 12 feet from passing closest to the camera?

You must give correct units as part of your answer.



exf17

Solution K-1

Our table below defines the relevant variables and lists the information described in the problem.

Variables	x θ	distance from bead to point closest to the camera viewing angle of camera
Specific Time	$\frac{dx}{dt} = -6$ $x = 12$	"a bead is moving at 6 feet per second" (negative since x is decreasing) "when the bead is 12 feet from passing closest to the camera"
General Time	(1) $x = 5\tan(\theta)$ (2) $\frac{dx}{dt} = 5\sec^2(\theta)\frac{d\theta}{dt}$	right-triangle trigonometry derivative of equation (1)
Unknown	$\frac{d\theta}{dt}$	"[h]ow fast is the camera turning"

Putting the specific-time info into equations (1) and (2) gives:

(i) $12 = 5\tan(\theta)$ (ii) $-6 = 5\sec^2(\theta)\frac{d\theta}{dt}$

Equation (i) can be solved for θ, but we only need the value of $\sec^2(\theta)$. At the specific time, we have a 5-12-13 right triangle. So $\tan(\theta) = \frac{12}{5}$ and $\sec(\theta) = \frac{13}{5}$. So equation (ii) then gives:

$$\frac{d\theta}{dt} = \frac{-6}{5\sec^2(\theta)} = \frac{-6}{5 \cdot \frac{169}{25}} = -\frac{30}{169}$$

So the camera is turning at a rate of $-\frac{30}{169}$ radians per second.

Ex. K-2 3.11 Sp18 Exam

The total surface area of a cube is changing at a rate of 12 in^2/s when the length of one of the sides is 10 in. At what rate is the volume of the cube changing at that time?

exs18

Solution K-2

Our table below defines the relevant variables and lists the information described in the problem.

K-2

Variables	x A V	edge length of the cube total surface area of the cube volume of the of cube
Specific Time	$\frac{dA}{dt} = 12$ $x = 10$	"surface area... is changing at a rate of 12 in^2/s" "when the length of one of the sides is 10 in'
General Time	(1) $A = 6x^2$ (2) $V = x^3$ (3) $\frac{dA}{dt} = 12x\frac{dx}{dt}$ (4) $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$	basic geometry basic geometry derivative of equation (1) derivative of equation (2)
Unknown	$\frac{dV}{dt}$	"[a]t what rate is the volume... changing at that time"

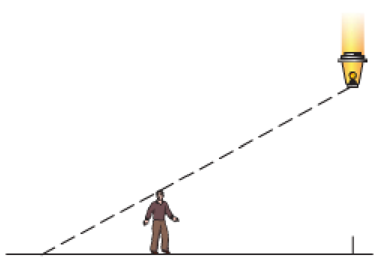
Putting the specific-time info into equations (1) and (2) gives:

(i) $A = 600$ (ii) $V = 1000$ (iii) $12 = 120\frac{dx}{dt}$ (iv) $\frac{dV}{dt} = 300\frac{dx}{dt}$

Equation (iii) gives $\frac{dx}{dt} = \frac{1}{10}$, and so equation (iv) gives $\frac{dV}{dt} = 30$. The volume of the cube is changing at 30 in^3/s.

Ex. K-3 3.11 Fa18 Exam

A person 6 feet tall stands 10 feet from point P, which is directly beneath a lantern hanging 30 feet above the ground. At the moment when the lantern is 9 feet above the ground, the lantern is falling at a rate of 4 ft/sec. At what rate is the length of the person's shadow changing at that moment?



exf18

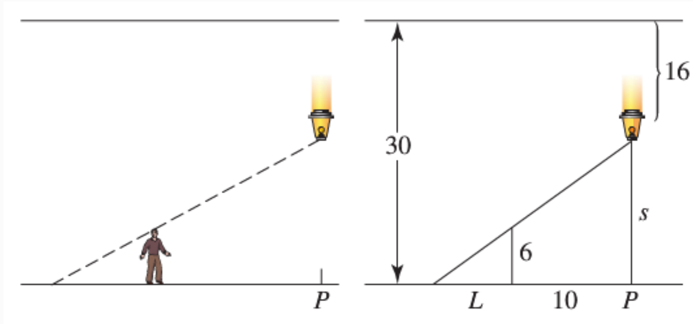
Solution K-3

Our table below defines the relevant variables and lists the information described in the problem.

Variables	L s	length of person's shadow height of lantern
Specific Time	$s = 9$ $\frac{ds}{dt} = -4$	"when the lantern is 9 feet above the ground" "the lantern is falling at a rate of 4 ft/sec"
General Time	(1) $\frac{s}{6} = \frac{L+10}{L} = 1 + \frac{10}{L}$ (2) $\frac{1}{6}\frac{ds}{dt} = -\frac{10}{L^2}\frac{dL}{dt}$	similar triangles (see figure below) derivative of equation (1)
Unknown	$\frac{dL}{dt}$	"at what rate is the length of the person's shadow changing"

The figure below summarizes our variables and specific-time info.

K-3



Putting the specific-time info into equations (1) and (2) gives:

(i) $\frac{9}{6} = 1 + \frac{10}{L}$ (ii) $-\frac{4}{6} = -\frac{10}{L^2}\frac{dL}{dt}$

Equation (i) gives $L = 20$, whence Equation (ii) gives

$$\frac{dL}{dt} = \frac{4L^2}{60} = \frac{80}{3}$$

Thus the shadow's length is increasing by $\frac{80}{3}$ ft/sec.

Ex. K-4 3.11 Sp19 Exam

A child flies a kite at a constant height of 30 feet and the wind is carrying the kite horizontally away from the child at a rate of 5 ft/sec. At what rate must the child let out the string when the kite is 50 feet away from the child?

You must give correct units as part of your answer.

exs19

Solution K-4

Our table below defines the relevant variables and lists the information described in the problem.

Variables	x L	horizontal distance from the child to kite length of kite string, or distance from child to kite
Specific Time	$\frac{dx}{dt} = 5$ $L = 50$	"wind is carrying the kite... at a rate of 5 ft/sec" "when the kite is 50 feet away from the child"
General Time	(1) $x^2 + 900 = L^2$ (2) $2x\frac{dx}{dt} = 2L\frac{dL}{dt}$	Pythagorean theorem derivative of equation (1)
Unknown	$\frac{dL}{dt}$	"at what rate must the child let out the string"

Putting the specific-time info into equations (1) and (2) gives:

(i) $x^2 + 900 = 2500$ (ii) $10x = 100\frac{dL}{dt}$

Equation (i) gives $x = 40$, whence equation (ii) gives $\frac{dL}{dt} = 4$. The string must be let out at a rate of 4 ft/sec.

Ex. K-5 3.11 Fa19 Exam

A spherical snowball melts in such a way that it always remains a sphere, and its volume decreases at 8 cm^3/sec. At what rate is the surface area of the snowball changing when its surface area is 40π cm^2? *You must give correct units as part of your answer.*

Solution K-5

Our table below defines the relevant variables and lists the information described in the problem.

Variables	r A V	radius of snowball surface area of snowball volume of snowball
Specific Time	$\frac{dV}{dt} = -8$ $A = 40\pi$	"its volume decreases at 8 cm^3/sec" "when its surface area is 40π cm^2"
General Time	(1) $A = 4\pi r^2$ (2) $V = \frac{4}{3}\pi r^3$ (3) $\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$ (4) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$	basic geometry basic geometry derivative of equation (1) derivative of equation (2)
Unknown	$\frac{dA}{dt}$	"at what rate is the surface area of the snowball changing"

Putting the specific-time info into equations (1)–(4) gives:

(i) $40\pi = 4\pi r^2$

(ii) $V = \dfrac{4}{3}\pi r^3$

(iii) $\dfrac{dA}{dt} = 8\pi r \dfrac{dr}{dt}$

(iv) $-8 = 4\pi r^2 \dfrac{dr}{dt}$

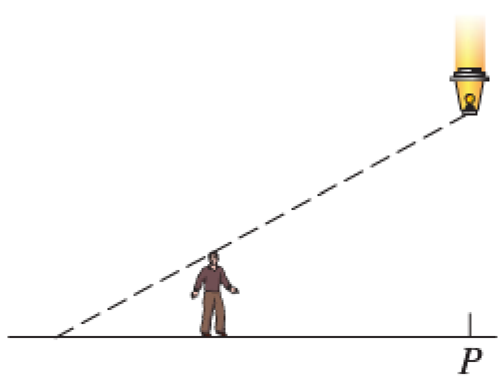
Equation (i) gives $r = \sqrt{10}$, and substituting this value into equation (iv) gives $\frac{dr}{dt} = -\frac{1}{5\pi}$. So now substituting both of these values into equation (iii) gives:

$$\frac{dA}{dt} = 8\pi \cdot \sqrt{10} \cdot \frac{-1}{5\pi} = -\frac{8\sqrt{10}}{5}$$

Thus the surface area is changing at a rate of $-\frac{8\sqrt{10}}{5}$ cm^2/sec.

Ex. K-6 3.11 Sp20 Exam

A 6-ft tall person is initially standing 12 ft from point P directly beneath a lantern hanging 42 ft above the ground, as shown in the diagram below. The person then begins to walk towards point P at 5 ft/sec. Let D denote the distance between the person's feet and the point P. Let S denote the length of the person's shadow.



(a) Write an equation that relates D and S.

(b) Write an equation that expresses the English sentence "*The person then begins to walk towards point P at 5 ft/sec.*"

(c) Is the length of the person's shadow increasing, decreasing or remaining constant?

exs20

3.11 Sp20 Exam

(d) At what rate is the length of the person's shadow changing when the person is 8 ft from point P? Include correct units as part of your answer.

exs20

Solution **K-6**

(a) Use similar triangles to obtain $\frac{D+S}{S} = \frac{42}{6}$. (We may simplify this to $D = 6S$.)

(b) $\frac{dD}{dt} = -5$. (The equation $D = 12 - 5t$ is also acceptable.)

(c) The length of the shadow is decreasing.

(d) The equation $D = 6S$ gives $\frac{dD}{dt} = 6\frac{dS}{dt}$ and we have $\frac{dD}{dt} = -5$, whence $\frac{dS}{dt} = -\frac{5}{6}$ ft/sec.

Ex. K-7 3.11 Sp20 Exam

The volume of a cube is decreasing at the rate of 300 cm^3/sec at the moment its total surface area is 150 cm^2. What is the rate of change of the length of one edge of the cube at this moment?

exs20

Solution **K-7**

Our table below defines the relevant variables and lists the information described in the problem.

Variables	x A V	edge length of cube surface area of cube volume of cube
Specific Time	$\dfrac{dV}{dt} = -300$ $A = 150$	"the volume of a cube is decreasing at the rate of 300 cm^3/sec" "when its total surface area is 150 cm^2"
General Time	(1) $A = 6x^2$ (2) $V = x^3$ (3) $\dfrac{dA}{dt} = 12x\dfrac{dx}{dt}$ (4) $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$	basic geometry basic geometry derivative of equation (1) derivative of equation (2)
Unknown	$\dfrac{dx}{dt}$	"what is the rate of change of the length of one edge of the cube"

Putting the specific-time info into equations (1)–(4) gives:

(i) $150 = 6x^2$ (ii) $V = x^3$ (iii) $\dfrac{dA}{dt} = 12x\dfrac{dx}{dt}$ (iv) $-300 = 3x^2\dfrac{dx}{dt}$

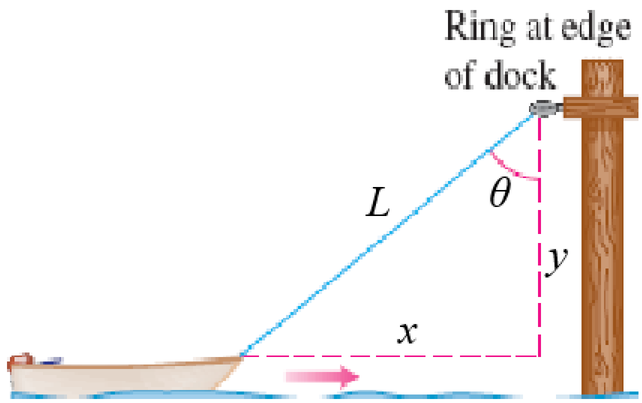
Equation (i) gives $x = 5$, and substituting $x = 5$ into equation (iv) gives $\frac{dx}{dt} = -4$. Thus the edge length of the cube is changing at a rate of -4 cm/sec.

Ex. K-8 3.11 Sp20 Exam

A boat is pulled toward a dock by a rope through a ring on the dock 4 ft above the front of the boat. The rope is hauled in at the rate of 12 ft/sec.

exs20

3.11 Sp20 Exam



(a) Which of the marked variables (x, y, L, and θ) are changing over time?

(b) Write a mathematical equation that expresses the English sentence *"The rope is hauled in at the rate of 12 ft/sec"*.

(c) Is $\cos(\theta)$ increasing, decreasing, or constant?

(d) Write a mathematical expression for *"the rate at which the boat approaches the dock"*.

(e) How fast in ft/sec is the boat approaching the dock when the rope is 5 ft long?

exs20

Solution K-8

(a) The variables x, L, and θ are changing over time.

(b) $\dfrac{dL}{dt} = -12$

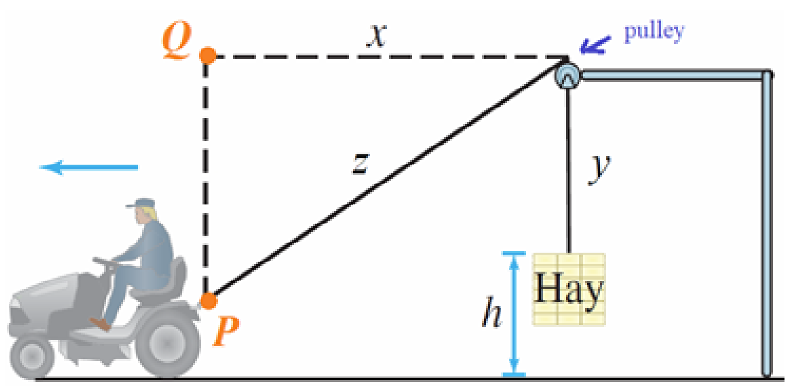
(c) Since θ is decreasing (to 0), $\cos(\theta)$ is increasing (to 1). Alternatively, note that $\cos(\theta) = \frac{4}{L}$. Since L is decreasing, the fraction $\frac{4}{L}$ is increasing.

(d) $\left|\dfrac{dx}{dt}\right|$ is correct, but $\dfrac{dx}{dt}$ or $-\dfrac{dx}{dt}$ is also acceptable.

(e) The Pythagorean theorem gives $x^2 + 16 = L^2$, whence $2x\frac{dx}{dt} = 2L\frac{dL}{dt}$. At the moment when $L = 5$, we have $x = 3$. Substituting these values and $\frac{dL}{dt} = -12$ into the second equation then gives $\frac{dx}{dt} = -20$ ft/sec. So the boat approaches the dock at a rate of 20 ft/sec.

Ex. K-9 3.11 Sp20 Exam

A farmer's tractor pulls a rope of length 12 m attached to a bale of hay through a pulley is 8 m above the ground. The vertical distance between the tractor and the pulley (the distance from P to Q) is 7 m. The tractor is moving to the left at rate of 2 m/sec, which causes the bale of hay to rise off the ground.



(a) The rate of change (with respect to time) of which variable is equal to the speed of the tractor?

(b) Use the Pythagorean theorem to find an equation that holds for all time and involves only the variables x and z.

(c) Use the fact that the length of the rope is constant to find an equation that holds for all time and involves only the variables z and y.

exs20

3.11 Sp20 Exam

(d) Use the fact that the height of the pulley is constant to find an equation that holds for all time and involves only the variables h and y.

(e) Combine the equations from parts (b), (c), and (d) to find an equation that holds for all time and involves only the variables x and h.

(f) The rate of change (with respect to time) of which variable is equal to the rate at which the bale of hay is rising?

(g) Find the rate at which the bale of hay is rising off the ground when the horizontal distance between the tractor and the bale of hay is 8 m.

exs20

Solution K-9

(a) x

(b) $x^2 + 7^2 = z^2$, or $x^2 + 49 = x^2$

(c) $y + z = 12$

(d) $y + h = 8$

(e) Subtracting the last two equations gives $z - h = 4$, or $z = h + 4$. Substituting this expression for z in the first equation gives $x^2 + 49 = (h + 4)^2$. We will write this equation as:

$$h = \sqrt{x^2 + 49} - 4$$

(f) h

(g) Differentiating the equation in part (e) gives:

$$\frac{dh}{dt} = \frac{x\frac{dx}{dt}}{\sqrt{x^2 + 49}}$$

We are given that $\frac{dx}{dt} = 2$ (speed of the tractor) and that $x = 8$ (tractor is 8 m horizontally away from pulley). Hence we have:

$$\frac{dh}{dt} = \frac{16}{\sqrt{113}} \approx 1.51$$

So the bale of hay is rising at approximately 1.51 m/sec.

Ex. K-10 3.11 Su20 Exam

In a right triangle, the base is decreasing in length by 3 cm/sec and the area is increasing by 15 cm^2/sec. (The triangle always remains a right triangle.) At the time when the base is 15 cm in length and the height is 20 cm in length...

(a) ...at what rate is the height changing? (Give a number only.)

(b) ...at what rate is the length of the hypotenuse changing? (Give a number only.)

(c) What are the units of your answer in part (a)?

(d) In part (b), is the length of the hypotenuse increasing, decreasing, or staying constant?

exsu20

Solution K-10

(a) Let b, h, and A be the base, height, and the area of the triangle, respectively. Then we have $A = \frac{1}{2}bh$. Differentiating with respect to t gives:

$$\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt}h + \frac{1}{2}b\frac{dh}{dt}$$

Now we substitute the given information: $\dfrac{db}{dt} = -3$, $\dfrac{dA}{dt} = 15$, $b = 15$, and $h = 20$.

$$15 = \frac{1}{2} \cdot (-3) \cdot 20 + \frac{1}{2} \cdot 15 \cdot \frac{dh}{dt}$$

Solving for $\dfrac{dh}{dt}$ gives $\dfrac{dh}{dt} = 6$.

(b) Let the length of the hypotenuse be L. Then $b^2 + h^2 = L^2$. Differentiating with respect to t gives:

$$2b\frac{db}{dt} + 2h\frac{dh}{dt} = 2L\frac{dL}{dt}$$

K-10

When $b = 15$ and $h = 20$, we have $L = 25$. Now we also substitute the given information and our work from part (a).

$$2 \cdot 15 \cdot (-3) + 2 \cdot 20 \cdot 6 = 2 \cdot 25 \cdot \frac{dL}{dt}$$

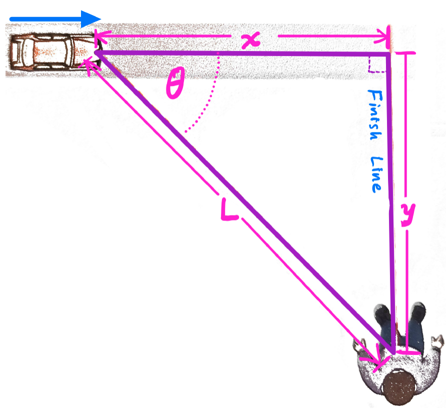
Solving for $\frac{dL}{dt}$ gives $\frac{dL}{dt} = 3$.

(c) The units of $\frac{dh}{dt}$ are cm/sec.

(d) Since $\frac{dL}{dt} > 0$, the length of the hypotenuse is increasing.

Ex. K-11 3.11 Fa20 Exam

At a certain moment, a race official is watching a race car approach the finish line along a straight track at some constant, positive speed. Suppose the official is sitting still at the finish line, 20 m from the point where the car will cross.



For parts (a)–(e), the allowed answers are "positive", "negative", "zero", or "not enough information".

(a) At the moment described, what is the sign of $\frac{dx}{dt}$?

(b) At the moment described, what is the sign of $\frac{dy}{dt}$?

(c) At the moment described, what is the sign of $\frac{dL}{dt}$?

(d) At the moment described, what is the sign of $\frac{d(\cos(\theta))}{dt}$?

(e) At the moment described, what is the sign of $\frac{d^2x}{dt^2}$?

(f) Suppose the speed of the car is 70 m/sec. At what rate is the distance between the car and the race official changing when the car is 60 m from the finish line? *Your answer must have the correct units. Your answer must be exact. No decimal approximations.*

exf20

Solution K-11

(a) negative (x is decreasing)

(b) zero (y is constant)

(c) negative (L is decreasing)

(d) negative (θ is increasing to 90-degrees, whence $\cos(\theta)$ is decreasing to 0)

(e) zero (the speed of the car is constant, whence $\frac{dx}{dt}$ is constant)

(f) Observe that $x^2 + 400 = L^2$, and so $x\frac{dx}{dt} = L\frac{dL}{dt}$. Substituting $\frac{dx}{dt} = -70$ and $x = 60$ gives the equations:

$$3600 + 400 = L^2 \quad , \quad -4200 = L\frac{dL}{dt}$$

The first equation gives $L = \sqrt{4000}$, whence the second equation gives

$$\frac{dL}{dt} = \frac{-4200}{\sqrt{4000}} = -21\sqrt{10}$$

K-11

The distance between the car and the official decreases at a rate of $21\sqrt{10}$ m/sec.

Ex. K-12 3.11 Sp21 Exam

A local gym has two cylindrical swimming pools. The larger pool has radius 20 meters and is filled with water. The smaller pool has radius 12 meters and is empty. Water is drained from the large pool and immediately emptied into the small pool. The height of the water in the small pool increases at a rate of 0.2 m/min.
Let V_L, V_S, h_L, and h_S refer to the volume of the large pool, volume of the small pool, height of the large pool, and height of the small pool, respectively.

(a) How are $\dfrac{dV_L}{dt}$ and $\dfrac{dV_S}{dt}$ related?

(b) What is the sign of $\dfrac{dh_L}{dt}$?

(c) Find $\dfrac{dV_S}{dt}$.

(d) Find $\dfrac{dh_L}{dt}$.

exs21

Solution K-12

(a) The water in the two pools change at the same absolute rate. But the large pool drains while the small pool fills. Hence $\dfrac{dV_L}{dt} = -\dfrac{dV_S}{dt}$.

(b) Water drains from the larger pool, whence $\dfrac{dh_L}{dt}$ is negative.

(c) We have $V_S = 144\pi h_S$, whence $\dfrac{dV_S}{dt} = 144\pi\dfrac{dh_S}{dt}$. Given that $\dfrac{dh_S}{dt} = 0.3$, we find $\dfrac{dV_S}{dt} = 28.8\pi$ m^3/min.

(d) We have $V_L = 400\pi h_L$, whence $\dfrac{dV_L}{dt} = 400\pi\dfrac{dh_L}{dt}$. Using parts (a) and (c), we have:

$$-28.8\pi = -\frac{dV_S}{dt} = \frac{dV_L}{dt} = 400\pi\frac{dh_L}{dt}$$

Hence $\dfrac{dh_L}{dt} = -0.072$ m/min.

Ex. K-13 3.11 Fa21 Exam

The base of a right triangle is decreasing at a constant rate of 10 cm/sec and in such a way that the triangle always remains a right triangle. At the time when the base is 15 cm and the height is 22 cm, the area of the triangle is increasing by 25 cm^2/sec. Use this information to answer the questions below. Let B denote the base of the triangle.

(a) At the described time, what is the sign of $\frac{dB}{dt}$?

(b) At the described time, what is the sign of $\frac{d^2B}{dt^2}$?

(c) At the described time, at what rate is the height changing?

(d) What are the units of the answer to part (c)?

exf21

Solution K-13

(a) We are given that the base is decreasing at the given time, so $\frac{dB}{dt}$ is negative.

(b) We are given that $\frac{dB}{dt}$, the rate at which the base is changing, is constant. Thus $\frac{d^2B}{dt^2}$ is zero.

(c) At any time we have $A = \frac{1}{2}BH$, where A, B, and H are the area, base, and height of the triangle, respectively. Differentiating with respect to time gives us a total of two equations that hold for any time.

$$A = \frac{1}{2}BH$$
$$\frac{dA}{dt} = \frac{1}{2}\frac{dB}{dt}H + \frac{1}{2}B\frac{dH}{dt}$$

At the given time, we have: $\frac{dB}{dt} = -10$, $B = 15$, $H = 22$, and $\frac{dA}{dt} = 25$. Substituting this information into the

K-13

previous two equations gives us two equations that hold only at the described time.

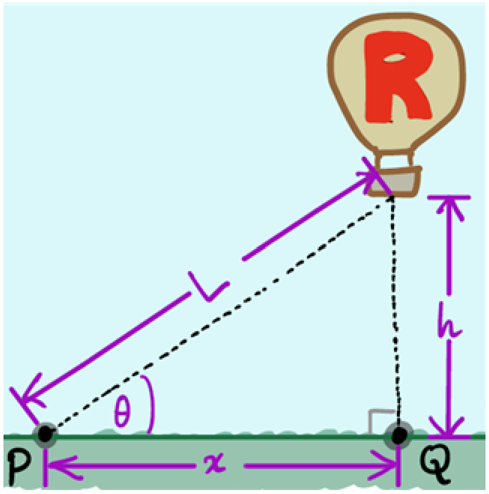
$$A = 165$$
$$25 = -110 + 7.5\frac{dH}{dt}$$

Solving for $\frac{dH}{dt}$ gives $\frac{dH}{dt} = 18$.

(d) The units of $\frac{dH}{dt}$ are cm/sec.

Ex. K-14 3.11 Fa21 Exam

A hot-air balloon is floating directly above the point Q on the ground and is descending at a constant rate of 10 ft/sec. A camera is on the ground at point P, which is 500 feet from point Q. See the figure below.



(a) What is the sign of $\dfrac{dh}{dt}$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(b) How is $\cos(\theta)$ changing over time? Circle your answer below.

(i) increasing over time
(ii) decreasing over time
(iii) constant over time
(iv) sometimes increasing and sometimes decreasing
(v) not enough information to determine

(c) What is the rate of change of the distance between the camera and the balloon when the balloon is 600 feet above the ground? *You must give correct units as part of your answer.*

exf21

Solution K-14

(a) The balloon is descending, whence h is decreasing. So $\frac{dh}{dt}$ is negative.

(b) Note that $\cos(\theta) = \frac{x}{L}$ and x is a fixed number. As the balloon descends, L decreases, whence the fraction $\frac{x}{L}$ must increase. So $\cos(\theta)$ is increasing.

(c) We have $500^2 + h^2 = L^2$ for all t. Differentiating with respect to t (and canceling a factor of 2) gives $h\frac{dh}{dt} = L\frac{dL}{dt}$. At the specified time, we have $h = 600$ and $\frac{dh}{dt} = -10$. So our two equations at the specified time give:

$$500^2 + 600^2 = L^2 \qquad\qquad -6000 = L\frac{dL}{dt}$$

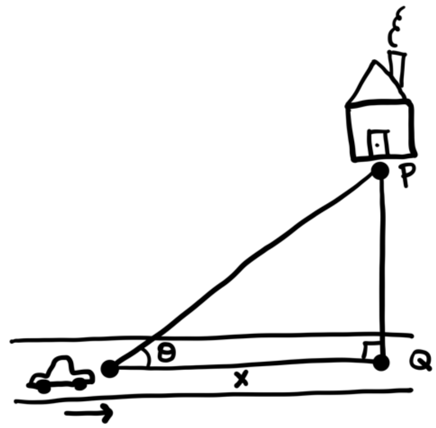
The first equation gives $L = 100\sqrt{41}$, and substituting this value into the second equation gives

$$\frac{dL}{dt} = \frac{-60}{\sqrt{41}}$$

The units are "ft/sec".

Ex. K-15 **3.11** Sp22 **Exam**

A house sits at point P, which is 20 m from point Q on a straight road. A car travels along the road toward the point Q at 19 m/s. Let x be the distance between the car and point Q, and let θ be the angle between the road and the line of sight from the car to the house. See the figure below.



(a) What is the sign of $\dfrac{dx}{dt}$?

(b) What is the sign of $\dfrac{d\theta}{dt}$?

(c) Find the rate of change of the distance between the car and the house when the car is 45 m from point Q. You must include correct units in your answer. You may leave unsimplified radicals in your answer.

exs22

Solution **K-15**

(a) Since x is decreasing, $\dfrac{dx}{dt}$ is negative.

(b) Since θ is increasing, $\dfrac{d\theta}{dt}$ is positive.

(c) Let L be the distance between the car and the house. Observe that we seek the value of $\frac{dL}{dt}$ at the time when $x = 45$.

By Pythagorean Theorem, we have $x^2 + 20^2 = L^2$, and differentiating this equation gives $2x\frac{dx}{dt} = 2L\frac{dL}{dt}$. We are given that $\frac{dx}{dt} = -19$ when $x = 45$, and so substituting this information into our two equations gives us the following:

$$45^2 + 20^2 = L^2$$
$$-90 \cdot 19 = 2L\frac{dL}{dt}$$

The first of these last two equations gives $L = \sqrt{45^2 + 20^2}$, whence the second equation then gives

$$\frac{dL}{dt} = \frac{-45 \cdot 19}{\sqrt{45^2 + 20^2}}$$

The units of our answer are "m/s".

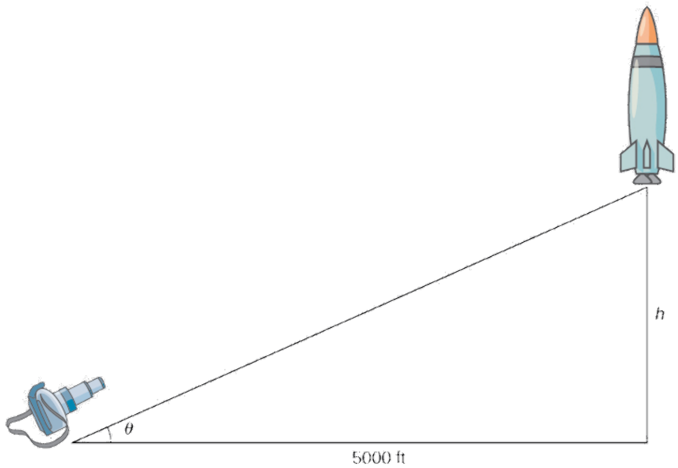
Ex. K-16 **3.11** Su22 **Exam**

A rocket is launched so that it rises vertically. A camera is positioned 5000 feet from the launch pad and turns so that it stays forces on the rocket. At the moment when the rocket is 12,000 feet above the launch pad, its velocity is 600 feet/sec. Let h be the height of the rocket above the launch pad and let θ be the viewing angle of the camera. See the figure below.

exsu22

3.11

Su22 Exam



(a) Determine the sign of $\dfrac{d}{dt}(\cos(\theta))$ at the moment described or determine that there is not enough information to do so.

(b) Determine the sign of $\dfrac{d^2h}{dt^2}$ at the moment described or determine that there is not enough information to do so.

(c) At the moment described, what is the rate at which the camera is turning? That is, what is the rate at which θ is changing over time? *You must include proper units as part of your answer.*

exsu22

Solution

K-16

(a) Note that $\cos(\theta) = \frac{5000}{L}$, where L is the distance from the camera to the rocket (the length of the hypotenuse of the right triangle). Since L is increasing as the rocket rises, the fraction $\frac{5000}{L}$ is decreasing. Thus the sign of $\frac{d}{dt}(\cos(\theta))$ is negative.

(b) Note that $\frac{dh}{dt}$ is the velocity v of the rocket. Thus $\frac{d^2h}{dt^2} = \frac{dv}{dt}$. We are not given any information about $\frac{dv}{dt}$, and so the answer is "not enough information". (For instance, if we were given that the rocket rises at a constant speed, we could conclude $\frac{dv}{dt} = 0$.)

(c) We seek the rate at which the camera is turning, or $\frac{d\theta}{dt}$. Since we are given information about the variables θ and h only, our equation relating them is:

$$\tan(\theta) = \frac{h}{5000}$$

Differentiating this equation with respect to t gives

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{5000}\frac{dh}{dt}$$

At the moment described, we have $h = 12000$ and $\frac{dh}{dt} = 600$. Substituting these values into our previous equations gives:

$$\tan(\theta) = \frac{12}{5} \qquad \sec^2(\theta)\frac{d\theta}{dt} = \frac{3}{25}$$

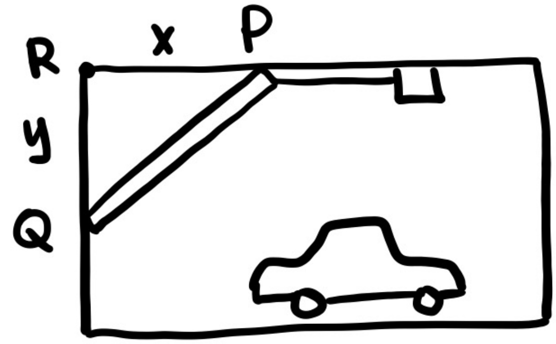
At the moment described, the right triangle is a 5-12-13 right triangle. Thus $\sec(\theta) = \frac{13}{5}$ at the moment described. So our second equation gives:

$$\left(\frac{13}{5}\right)^2 \frac{d\theta}{dt} = \frac{3}{25} \Longrightarrow \frac{d\theta}{dt} = \frac{3}{169}$$

The units are "radians per second".

Ex. K-17 3.11 Fa22 Exam

A solid 14-foot tall garage door opens via a pulley mechanism. As the pulley opens the garage door, the top of the garage door (point P in the figure) moves to the right at 5 ft/s. At the same time, the bottom of the garage door (point Q in the figure) moves straight up.
As shown in the figure, the point R is the fixed point at the top of the garage door frame, x represents the distance between P and R, and y represents the distance between Q and R.



(a) What is the sign of $\dfrac{dx}{dt}$?

(b) What is the sign of $\dfrac{dy}{dt}$?

(c) What is the rate of change of the distance between the points Q and R when the distance between them is 9 feet? You must include correct units in your answer. You may leave unsimplified radicals in your answer.

exf22

Solution K-17

(a) Since x is increasing, $\frac{dx}{dt}$ is positive.

(b) Since y is decreasing, $\frac{dy}{dt}$ is negative.

(c) We have $x^2 + y^2 = 14^2$, and differentiating with respect to time gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At the described time we have $y = 6$ and $\frac{dx}{dt} = 5$. So substituting these values gives:

$$x^2 + 6^2 = 14^2 \qquad 10x + 12\frac{dy}{dt} = 0$$

The first equation gives $x = \sqrt{14^2 - 6^2} = \sqrt{(14-6)(14+6)} = \sqrt{8 \cdot 20} = \sqrt{160}$, whence we obtain

$$\frac{dy}{dt} = -\frac{10x}{12} = -\frac{5\sqrt{160}}{6}$$

The units of $\frac{dy}{dt}$ are ft/sec.

Ex. K-18 3.11 Sp20 Quiz

The image of a certain rectangle of area 30 cm^2 is changing in such a way that its length is decreasing at a rate of 2 cm/sec. and its area remains constant. At what rate is its width changing when its length is 6 cm?

qs20

Solution K-18

Let L and W be the length and width of the rectangle, respectively. Then the equation $LW = 30$ holds for all time, and, upon implicitly differentiating with respect to time, we also have

$$\frac{dL}{dt}W + L\frac{dW}{dt} = 0$$

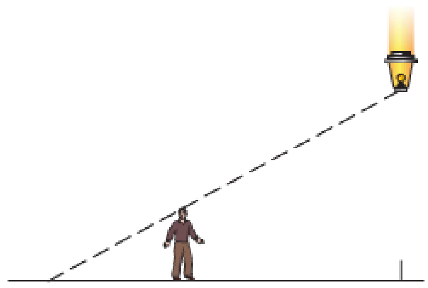
At the specific time of interest, we have $\dfrac{dL}{dt} = -2$ and $L = 6$, and we want to find $\dfrac{dW}{dt}$. Substituting these values into our equations gives:

$$6W = 30$$
$$-2W + 6\frac{dW}{dt} = 0$$

From the first equation we get $W = 5$ and putting this into the second equation gives $\dfrac{dW}{dt} = \dfrac{5}{3}$. Hence the width is increasing at a rate of 5/3 cm/sec.

Ex. K-19 **3.11** Su22 **Quiz**

A 2-meter tall person is initially standing 4 meters from point P directly beneath a lantern hanging 14 meters above the ground, as shown in the figure below. The person then begins to walk towards point P at 1.5 m/sec. Let x denote the distance between the person's feet and the point P. Let y denote the length of the person's shadow.



(a) Write an equation that relates x and y.

(b) Write an equation that expresses the English sentence *"The person begins to walk towards point P at 1.5 m/sec."*

(c) At what rate is the length of the person's shadow changing when the person is 3 meters from point P? *You must include correct units as part of your answer.*

qsu22

Solution **K-19**

(a) Use the principle of similar triangles. The smaller triangle has legs y (length of the shadow) and 2 (height of person). The larger triangle has legs $x+y$ (sum of length of shadow and distance from person to point P) and 14 (height of lantern). Thus we have:
$$\frac{x+y}{14} = \frac{y}{2}$$
Rearranging this equation gives the simpler relation $x = 6y$.

(b) $\dfrac{dx}{dt} = -1.5$

(c) We seek the value of $\frac{dy}{dt}$ at the desired time. Differentiating the equation $x = 6y$ with respect to y gives $\frac{dx}{dt} = 6\frac{dy}{dt}$. Putting $\frac{dx}{dt} = -1.5$ gives $-1.5 = 6\frac{dy}{dt}$, or $\frac{dy}{dt} = -0.25$. The units are m/sec.

Ex. K-20 **3.11** Fa22 **Quiz**

A pebble is dropped into a lake and an expanding circular ripple results. When the radius of the ripple is 8 inches, the area enclosed by the ripple is changing at a rate of 48π in^2/sec. What is the rate at which the radius is changing at this time? *You must include correct units as part of your answer.*

qf22

Solution **K-20**

Let r and A be the radius and area enclosed by the circular ripple, respectively. We seek the value of $\frac{dr}{dt}$ at the time when $r = 8$ and $\frac{dA}{dt} = 48\pi$. We have that $A = \pi r^2$, and differentiating with respect to t gives:
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$
Substituting $r = 8$ and $\frac{dA}{dt} = 48\pi$ gives $48\pi = 16\pi\frac{dr}{dt}$. Hence $\frac{dr}{dt} = 3$ in/sec.

Ex. K-21 **3.11**

A rock is dropped into a lake to create an expanding, circular ripple. When the radius of the ripple is 8 inches, the radius is increasing at a rate of 3 in/sec. At what rate is the area enclosed by the ripple changing at this time?

c3s11

Solution **K-21**

Our table below defines the relevant variables and lists the information described in the problem.

K-21

Variables	r A	radius of the ripple area enclosed by the ripple
Specific Time	$r = 8$ $\dfrac{dr}{dt} = 3$	"when the radius of the ripple is 8 inches" "the radius is increasing at a rate of 3 in/sec"
General Time	(1) $A = \pi r^2$ (2) $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$	basic geometry derivative of equation (1)
Unknown	$\dfrac{dA}{dt}$	"at what rate is the area enclosed by the ripple changing"

Putting the specific-time info into equations (1) and (2) gives:

(i) $A = 64\pi$

(ii) $\dfrac{dA}{dt} = 48\pi$

Thus the area is changing at a rate of 48π in^2/sec.

Ex. K-22 **3.11**

Every day, a flight to Los Angeles flies directly over a man's home at a constant altitude of 4 miles and at a constant speed of 400 miles per hour. At what rate is the angle of elevation of the man's line of sight changing with respect to time when the horizontal distance between the approaching plane and the man's location is exactly 3 miles?

c3s11

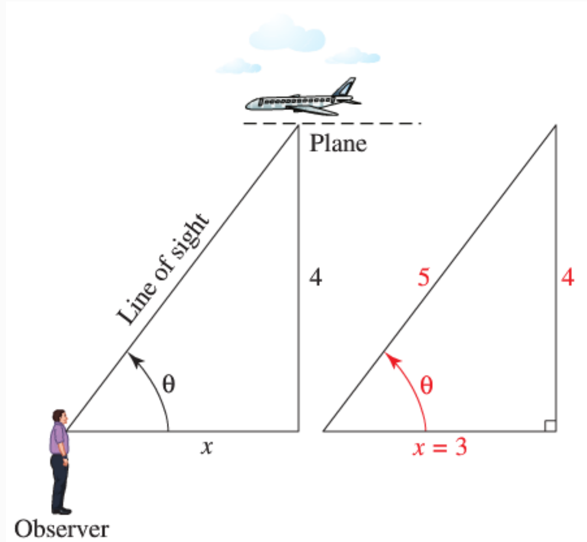
Solution **K-22**

Our table below defines the relevant variables and lists the information described in the problem.

Variables	x θ	horizontal distance from the man to the plane angle of elevation
Specific Time	$\dfrac{dx}{dt} = -400$ $x = 3$	"constant speed of 400 miles per hour" "when the horizontal distance... is exactly 3 miles"
General Time	(1) $\tan(\theta) = \dfrac{4}{x}$ (2) $\sec^2(\theta)\dfrac{d\theta}{dt} = -\dfrac{4}{x^2}\dfrac{dx}{dt}$	right-triangle trigonometry derivative of equation (1)
Unknown	$\dfrac{d\theta}{dt}$	"at what rate is the angle of elevation changing"

The figure below summarizes our variables and specific-time info.

K-22



Putting the specific-time info into equations (1) and (2) gives:

(i) $\tan(\theta) = \dfrac{4}{3}$ (ii) $\sec^2(\theta)\dfrac{d\theta}{dt} = \dfrac{1600}{9}$

From the figure, we see that when $\tan(\theta) = \frac{4}{3}$, we have $\sec(\theta) = \frac{5}{3}$ (use SOHCAHTOA). Thus Equation (ii) gives

$$\frac{d\theta}{dt} = \frac{1600}{9\sec^2(\theta)} = 64$$

The units are radians per hour.

Ex. K-23 3.11

The volume of a spherical balloon is increasing at constant rate of 3 in^3/s. At what rate is the radius of the balloon changing when the radius is 2 in.?

c3s11

Solution K-23

Our table below defines the relevant variables and lists the information described in the problem.

Variables	r V	radius of balloon volume of balloon
Specific Time	$\dfrac{dV}{dt} = 3$ $r = 2$	"the volume... is increasing at a constant rate of 3 in^3/s" "when the radius is 2 in"
General Time	(1) $V = \dfrac{4}{3}\pi r^3$ (2) $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$	basic geometry derivative of equation (1)
Unknown	$\dfrac{dr}{dt}$	"at what rate is the radius of the balloon changing"

Putting the specific-time info into equations (1) and (2) gives:

(i) $V = \dfrac{32}{3}\pi$ (ii) $3 = 16\pi\dfrac{dr}{dt}$

Equation (ii) immediately gives $\frac{dr}{dt} = \frac{3}{16\pi}$ ft/sec.

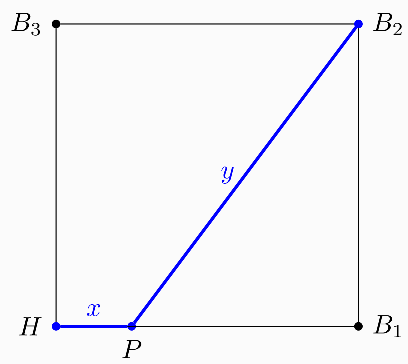
Ex. K-24 **3.11**

Recall that a baseball diamond is a square of side length 90 ft. The corners of the diamond are labeled, in anti-clockwise order, home plate, first base, second base, and third base. A player runs from home plate to first base at a speed of 20 ft/s. How fast is the player's distance from second base changing when the player is halfway to first base?

c3s11

Solution **K-24**

See the figure below. The bases are labeled H, B_1, B_2, and B_3, in order. The player is at point P.



Our table below defines the relevant variables and lists the information described in the problem.

Variables	x y	$\|HP\|$, distance from home plate (H) to player (P) $\|PB_2\|$, distance from player (P) to second base (B_2)
Specific Time	$\frac{dx}{dt} = 20$ $x = 45$	"A player runs... at a speed of 20 ft/s" "when the player is halfway to first base'
General Time	(1) $(90-x)^2 + 90^2 = y^2$ (2) $-2(90-x)\frac{dx}{dt} = 2y\frac{dy}{dt}$	Pythagorean Theorem for ΔPB_1B_2 derivative of equation (1)
Unknown	$\frac{dy}{dt}$	"[h]ow fast is the player's distance from second base changing"

Putting the specific-time info into equations (1) and (2) gives:

(i) $45^2 + 90^2 = y^2$

(ii) $-1800 = 2y\frac{dy}{dt}$

Equation (i) gives $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, whence Equation (ii) gives

$$\frac{dy}{dt} = \frac{-1800}{2y} = -4\sqrt{5}$$

Thus the distance from the player to second base is changing at a rate of $-4\sqrt{5}$ ft/sec.

Ex. K-25 **3.11**

A particle moves along the elliptical path given by $x^2 + 9y^2 = 13$ in such a way that when it is at the point $(-2, 1)$, its x-coordinate is decreasing at the rate of 7 units per second. How fast is the y-coordinate changing at that instant?

c3s11

Solution **K-25**

Our table below defines the relevant variables and lists the information described in the problem.

K-25

Variables	x y	x-coordinate of particle y-coordinate of particle
Specific Time	$x = -2$, $y = 1$ $\dfrac{dx}{dt} = -7$	"when [the particle] is at the point $(-2, 1)$" "its x-coordinate is decreasing at a rate of 7 units per second"
General Time	(1) $x^2 + 9y^2 = 13$ (2) $2x\dfrac{dx}{dt} + 18y\dfrac{dy}{dt} = 0$	equation that describes path derivative of equation (1)
Unknown	$\dfrac{dy}{dt}$	"[h]ow fast is the y-coordinate changing"

Putting the specific-time info into equations (1) and (2) gives:

(i) $4 + 9 \cdot 1 = 13$

(ii) $28 + 18\dfrac{dy}{dt} = 0$

Equation (ii) immediately gives $\frac{dy}{dt} = -\frac{14}{9}$ ft/sec.

Ex. K-26 3.11

The surface area of a sphere is changing at a rate of 16π in^2/s when its radius is 3 in. At what rate is the volume of the sphere changing at that time?

unit4

Solution K-26

Our table below defines the relevant variables and lists the information described in the problem.

Variables	r A V	radius of the sphere surface area of the sphere volume of sphere
Specific Time	$\dfrac{dA}{dt} = 16\pi$ $r = 3$	"the surface area... is changing at a rate of 16π in^2/s" "when its radius is 3 in"
General Time	(1) $A = 4\pi r^2$ (2) $V = \frac{4}{3}\pi r^3$ (3) $\dfrac{dA}{dt} = 8\pi r\dfrac{dr}{dt}$ (4) $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$	basic geometry basic geometry derivative of equation (1) derivative of equation (2)
Unknown	$\dfrac{dV}{dt}$	"at what rate is the volume... changing at that time"

Putting the specific-time info into equations (1)–(4) gives:

(i) $A = 36\pi$

(ii) $V = 36\pi$

(iii) $16\pi = 24\pi\dfrac{dr}{dt}$

(iv) $\dfrac{dV}{dt} = 36\pi\dfrac{dr}{dt}$

Equation (iii) gives $\frac{dr}{dt} = \frac{2}{3}$. Then equation (iv) gives $\frac{dV}{dt} = 24\pi$. The volume is changing at a rate of 24π in^3/sec.

Ex. K-27 3.11

A car traveling north at 40 mi/hr and a truck traveling east at 30 mi/hr leave an intersection at the same time. At what rate will the distance between them be changing 4 hours later?

unit4

Solution **K-27**

Let x be the truck's distance from the intersection and let y be the car's distance from the intersection. If L is the distance between the truck and the car, then $x^2+y^2=L^2$. Differentiating with respect to time t gives

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2L\frac{dL}{dt}$$

We are given that $\frac{dx}{dt}=30$ and $\frac{dy}{dt}=40$. This implies that 4 hours after leaving the intersection, $x=30\cdot 4=120$ and $y=40\cdot 4=160$. At that time, $L=\sqrt{x^2+y^2}=\sqrt{120^2+160^2}=200$. Putting this altogether in the equation displayed above gives us

$$2\cdot 120\cdot 30+2\cdot 160\cdot 40=2\cdot 200\cdot\frac{dL}{dt}$$

So we find that $\frac{dL}{dt}=50$. That is, the distance between the truck and car is increasing at a rate of 50 mi/hr.

Ex. K-28 **3.11**

The altitude of a triangle is increasing at a rate of 1 ft/min. while the area is increasing at a rate of 2 ft/min. At what rate is the base of the triangle changing when the altitude is 10 ft. and the area is 100 ft^2?

unit4

Solution **K-28**

Let b, h, and A denote the base, altitude, and area of the triangle, respectively. Then $2A=bh$, and, after differentiating with respect to time t, we have the following.

$$2\frac{dA}{dt}=b\frac{dh}{dt}+\frac{db}{dt}h$$

We are given that $\frac{dh}{dt}=1$ and $\frac{dA}{dt}=2$. When $h=10$ and $A=100$, we find that $b=20$. Putting this altogether in the equation displayed above gives us

$$2\cdot 2=20\cdot 1+\frac{db}{dt}\cdot 10$$

So we find that $\frac{db}{dt}=-1.6$. That is, the base of the triangle is decreasing at a rate of 1.6 ft/min.

Ex. K-29 **3.11** **⋆Challenge** **!!!**

A water tank in the shape of an inverted cone has height 10 meters and base radius 8 meters. Water flows into the tank at the rate of 32π m^3/min. At what rate is the depth of the water in the tank changing when the water is 5 meters deep?

challenge

4 Chapter 4: Applications of the Derivative

§4.1: Maxima and Minima

Ex. L-1 4.1 Sp18 Exam

Find the minimum and maximum values of $f(x) = 2x^3 - 3x^2 - 12x + 18$ on the interval $[-3, 3]$.

Hint: You may use the factorization $f(x) = (x^2 - 6)(2x - 3)$ to make any required arithmetic easier.

exs18

Solution L-1

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$$

Hence the critical points are $x = -1$ and $x = 2$. Checking the critical values and the endpoint values gives the following. (We may use the factored form of f to make the arithmetic easier.)

x	$f(x)$	reason for check
-3	-27	endpoint
-1	25	critical point ($f'(x) = 0$)
2	-2	critical point ($f'(x) = 0$)
3	9	endpoint

The maximum value of f on $[-3, 3]$ is 25 and the minimum value is -27.

Ex. L-2 4.1 Fa18 Exam

Let $f(x) = 4(x-3)^{1/3} - \frac{1}{3}x + 1$. *Note:* The domain of f is $(-\infty, \infty)$.

(a) Calculate all critical points of f. *For each number you find, you must clearly indicate in your work why it is a critical point.*

(b) What are the absolute extreme values of f on the interval $[2, 30]$?

exf18

Solution L-2

(a) Note that f is continuous for all x. So the critical points of f are those values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Our derivative is:

$$f'(x) = \frac{4}{3}(x-3)^{-2/3} - \frac{1}{3} = \frac{4 - (x-3)^{2/3}}{3(x-3)^{2/3}}$$

Hence $f'(x)$ does not exist when $(x-3)^{2/3} = 0$ (or $x = 3$). For solutions to $f'(x) = 0$, we have:

$$f'(x) = 0 \to 4 = (x-3)^{2/3} \implies 64 = (x-3)^2 \implies x = -5 \text{ or } x = 11$$

So, in summary, f has three critical points: $x = -5$, $x = 3$, and $x = 11$.

(b) Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
2	$-\frac{11}{3}$	endpoint
3	0	critical point ($f'(x)$ DNE)
11	$\frac{16}{3}$	critical point ($f'(x) = 0$)
30	3	endpoint

(We do not check $x = -5$ since that critical point lies outside the interval.) Hence the absolute minimum is $-\frac{11}{3}$ and the absolute maximum is $\frac{16}{3}$.

Ex. L-3 4.1 Sp19 Exam

Find all critical points of $f(x) = x - \frac{3}{2}(x-8)^{2/3}$ or explain why f has no critical points.

exs19

Solution L-3

The first derivative is

$$f'(x) = 1 - (x-8)^{-1/3} = 1 - \frac{1}{(x-8)^{1/3}}$$

Critical points are values of x at which $f'(x)$ does not exist ($x = 8$ only) or where $f'(x) = 0$ ($x = 9$ only, see below).

$$1 - \frac{1}{(x-8)^{1/3}} = 0 \implies 1 = (x-8)^3 \implies 1 = x - 8 \implies x = 9$$

Ex. L-4 4.1 Sp19 Exam

Find the absolute extreme values of $f(x) = \frac{20x}{x^2+4}$ on $[-4, 0]$.

exs19

Solution L-4

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = \frac{(x^2+4)(20) - (20x)(2x)}{(x^2+4)^2} = \frac{-20(x-2)(x+2)}{(x^2+4)^2}$$

Hence the only critical point in $[-4, 0]$ is $x = -2$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-4	-4	endpoint
-2	-5	critical point ($f'(x) = 0$)
0	0	endpoint

The maximum value of f on $[-4, 0]$ is 0 and the minimum value is -5.

Ex. L-5 4.1 Fa19 Exam

Find all critical points of $f(x) = 2 - \left(x^2 - 2x\right)^{1/3}$ or explain why f has no critical points.

Note: The domain of f is $(-\infty, \infty)$.

exf19

Solution L-5

The first derivative of f is:

$$f'(x) = \frac{-(2x-2)}{3(x^2-2x)^{2/3}}$$

Critical points come in two types: where $f'(x)$ does not exist or where $f'(x) = 0$. Note that $f'(x)$ does not exist if $x^2 - 2x = 0$ (i.e., $x = 0$ or $x = 2$) and $f'(x) = 0$ if $x = 1$. Hence f has three critical points: $x = 0$, $x = 1$, and $x = 2$.

Ex. L-6 4.1 Fa19 Exam

For each part, find the absolute extreme values of $f(x)$ on the given interval.

(a) $f(x) = x + \frac{9}{x}$ on $[1, 18]$.

(b) $f(x) = (6 - x)e^x$ on $[0, 6]$.
(***Hint:*** $2 < e < 3$.)

exf19

Solution L-6

(a) The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 1 - \frac{9}{x^2} = \frac{(x+3)(x-3)}{x^2}$$

L-6

Hence the only critical point in the interval $[1, 18]$ is $x = 3$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
1	10	endpoint
3	6	critical point ($f'(x) = 0$)
18	18.5	endpoint

The maximum value of f on $[1, 18]$ is 18.5 and the minimum value is 6.

(b) The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = (-1)e^x + (6 - x)e^x = (5 - x)e^x$$

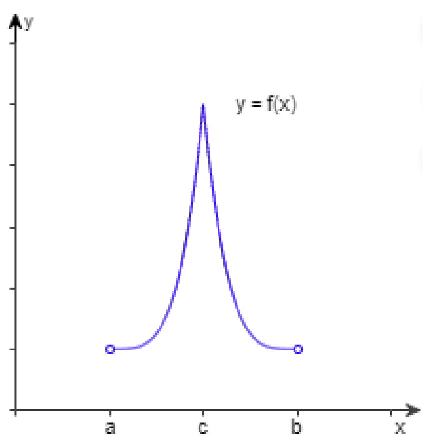
Hence the only critical point is $x = 5$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	6	endpoint
5	e^5	critical point ($f'(x) = 0$)
6	0	endpoint

The maximum value of f on $[0, 6]$ is e^5 and the minimum value is 0.

Ex. L-7 4.1 Sp20 Exam

Determine from the given graph whether the function has any absolute extreme values on (a, b).



exs20

Solution L-7

The function has an absolute maximum value at $x = c$ but does not have an absolute minimum value on (a, b).

Ex. L-8 4.1, 4.3/4.4 Sp20 Exam

Consider the following function

$$g(x) = \frac{3}{2}x^4 + 8x^3 - 36x^2$$

(a) Where does g have a local minimum on $(-7, 3)$? local maximum?

(b) Where does g have a global minimum on $[-7, 3]$? global maximum?

exs20

Solution L-8

(a) We solve $g'(x) = 0$ to find the critical points of g.

$$g'(x) = 6x^3 + 24x^2 - 72x = 6x(x - 2)(x + 6) = 0$$

L-8

Thus the critical points are $x = -6$, $x = 0$, and $x = 2$ (all of which are in $(-7, 3)$). We will use the second derivative test to classify these critical points.

$$g''(x) = 6(3x^2 + 8x - 12)$$

x	$g''(x)$	conclusion
-6	288	local minimum
0	-72	local maximum
2	96	local minimum

(b) We check the critical and endpoint values.

x	$g(x)$	reason for check
-7	-906	endpoint
-6	-1080	critical point ($f'(x) = 0$)
0	0	critical point ($f'(x) = 0$)
2	-56	critical point ($f'(x) = 0$)
3	13.5	endpoint

The maximum value of g on $[-7, 3]$ occurs at $x = 3$ and the minimum value occurs at $x = -6$.

Ex. L-9 4.1 Sp20 Exam

Find all critical points of the function

$$f(x) = 2x^{4/3} - 16x^{2/3} + 24$$

Note: The function f is continuous on the interval $(-\infty, \infty)$.

exs20

Solution L-9

The first derivative of f is

$$f'(x) = \frac{8}{3}x^{1/3} - \frac{32}{3}x^{-1/3} = \frac{8\left(x^{2/3} - 4\right)}{3x^{1/3}}$$

We immediately find that $x = 0$ is a critical point since $f'(0)$ does not exist. The remaining critical points are solutions of $f'(x) = 0$.

$$f'(x) = 0 \implies x^{2/3} - 4 = 0 \implies x^2 = 64 \implies x = -8 \text{ or } x = 8$$

Hence the critical points of f are $x = -8$, $x = 0$, and $x = 8$.

Ex. G-6 3.1/3.2, 4.1, 4.9 Sp20 Exam

Suppose the derivative of f is $f'(x) = 3x^2 - 6x - 9$ and that $f(1) = 10$.

(a) Find an equation of the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Find the critical points of f.

(c) Where does f have a local minimum value? local maximum value?

(d) Calculate $f(0)$.

(e) Calculate the absolute maximum value of f on the interval $[0, 6]$. At what x-value does it occur?

exs20

Solution G-6

(a) We have $f'(1) = 3 - 6 - 9 = -12$, whence an equation of the tangent line is $y = 10 - 12(x - 1)$.

(b) Solving $f'(x) = 0$, we find that the critical points of f are $x = -1$ and $x = 3$.

(c) A sign chart for $f'(x)$ reveals that $f'(x)$ is positive on the intervals $(-\infty, -1)$ and $(3, \infty)$; and $f'(x)$ is negative on the interval $(-1, 3)$. Since f' changes from positive to negative at $x = -1$, a local maximum occurs at $x = -1$. Since f' changes from negative to positive to $x = 3$, a local minimum occurs at $x = 3$.

G-6

(d) We find $f(x)$ by finding the most general antiderivative of $f'(x)$.

$$f(x) = \int f'(x)\,dx = x^3 - 3x^2 - 9x + C$$

The initial condition $f(1) = 10$ implies $1 - 3 - 9 + C = 10$, or $C = 21$. Hence

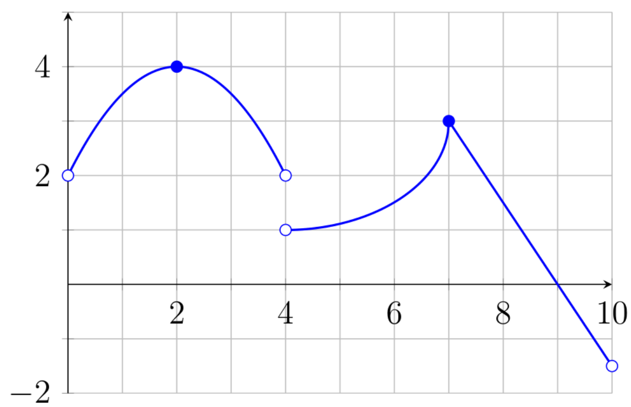
$$f(x) = x^3 - 3x^2 - 9x + 21$$

So $f(0) = 21$.

(e) The absolute maximum of f on $[0, 6]$ can occur only at an endpoint (0 or 6) or a critical number (-1 or 3). Calculating the values of f at these x-values gives: $f(0) = 21$, $f(-1) = 26$, $f(3) = -6$, and $f(6) = 75$. Hence the absolute maximum of f on $[0, 6]$ is 75, occurring at $x = 6$.

Ex. L-10 4.1 Su20 Exam

Suppose $f(x)$ is continuous on $[0, 10]$. The figure below shows the graph of $y = f'(x)$ on $[0, 10]$.
Note: The figure does not show a graph of $f(x)$ but rather its derivative.)



Use the graph to answer the following questions. Read each question carefully. Some questions ask about f and others ask about the derivative f'.

(a) Find the absolute maximum of $f'(x)$ on $(0, 10)$ or determine that it does not exist.
(b) Find the absolute minimum of $f'(x)$ on $(0, 10)$ or determine that it does not exist.
(c) Find all critical points of $f(x)$ in $(0, 10)$.

exsu20

Solution L-10

(a) Since $f'(x)$ takes on the value 4 and no value larger, 4 is the absolute maximum.
(b) The range of $f'(x)$ is $(10, 4]$, and so there is no absolute minimum.
(c) The critical points of f are $x = 4$ (because $f'(4)$ does not exist) and $x = 9$ (because $f'(9) = 0$).

Ex. L-11 4.1 Su20 Exam

Let $f(x) = \dfrac{1 - 2x}{6 + x^2}$. Find the absolute extrema of f on $[-3, 2]$ and where they occur.

exsu20

Solution L-11

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = \frac{(-2)(6 + x^2) - 2x(1 - 2x)}{(6 + x^2)^2} = \frac{2(x - 3)(x + 2)}{(6 + x^2)^2}$$

Hence the only critical point in the interval $[-3, 2]$ is $x = -2$. Checking the critical values and the endpoint values gives the following.

L-11

x	$f(x)$	reason for check
-3	$\frac{7}{15}$	endpoint
-2	$\frac{1}{2}$	critical point ($f'(x) = 0$)
2	$-\frac{3}{10}$	endpoint

The maximum value of f on $[-3, 2]$ is $\frac{1}{2}$ at $x = -2$ and the minimum value is $-\frac{3}{10}$ at $x = 2$.

Ex. L-12 4.1 Su20 Exam

Let $f(x) = x^{1/3}(x-16)^{1/5}$. Find all critical points of f. You must be clear why each of your answers really is a critical point. ***Note:*** The domain of f is $(-\infty, \infty)$.

exsu20

Solution L-12

The first derivative of f is

$$f'(x) = \frac{1}{3}x^{-2/3}(x-16)^{1/5} + x^{1/3} \cdot \frac{1}{5}(x-16)^{-4/5} = \frac{8(x-10)}{15x^{2/3}(x-16)^{4/5}}$$

The critical points of f are where $f'(x)$ does not exist ($x = 0$ and $x = 16$) or where $f'(x) = 0$ ($x = 10$).

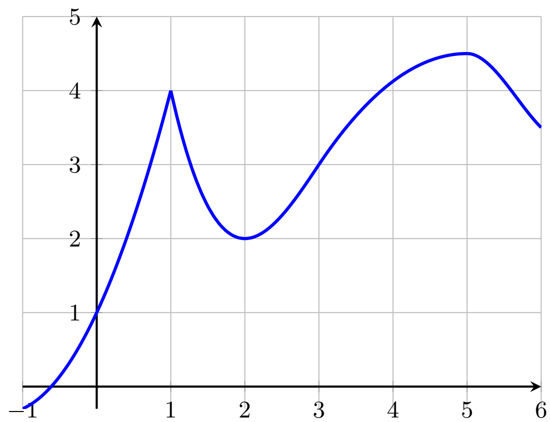
Ex. L-13 4.1 Fa20 Exam

For each part, use the graph of $y = f(x)$. Assume that the domain of f is $(-\infty, \infty)$.

(a) Where does f have a local minimum on the interval $(-1, 6)$?

(b) List all of the critical points of f.

(c) Estimate the absolute maximum of f on $[0, 3]$ or explain why f has no such maximum.



exf20

Solution L-13

(a) There is a local minimum at $x = 2$ only.

(b) The critical points are $x = 1$ (since $f'(1)$ does not exist), $x = 2$ (since $f'(2) = 0$), and $x = 5$ (since $f'(5) = 0$).

(c) The maximum of $f(x)$ on $[0, 3]$ is $f(1) = 4$.

Ex. L-14 4.1 Fa20 Exam

(You may need a basic calculator for this problem.)

Consider the function

$$f(t) = \frac{a}{t^2 - 3t + 25}$$

where a is an unspecified **positive** constant. Suppose the absolute minimum of f on $[0, 6]$ is 3.

(a) Find the value of a. ***Hint:*** First find the absolute minimum of f on $[0, 6]$ in terms of a.

(b) Calculate the absolute maximum of f on $[0, 6]$.

exf20

Solution **L-14**

(a) We first find the absolute extrema of f on $[0,6]$ in terms of a. Since f is differentiable for all t, the only critical points are solutions to $f'(t) = 0$.

$$f'(t) = \frac{a(2t-3)}{(t^2-3t+25)^2} = 0 \Longrightarrow t = 1.5$$

We now make a table that includes any critical values and endpoint values. Observe:

$$f(0) = \frac{a}{25} \quad , \quad f(1.5) = \frac{4a}{91} \quad , \quad f(6) = \frac{a}{43}$$

Since a is positive, we see that the largest of these values is $f(1.5)$ and the smallest of these values is $f(6)$. We are given that the absolute minimum is 3, and so $f(6) = \frac{a}{43} = 3$, whence $a = 129$.

(b) From our previous work, the absolute maximum is $f(1.5) = \frac{4a}{91}$. With $a = 129$, we see that the absolute maximum is $\frac{516}{91}$.

Ex. L-15 4.1 Sp21 Exam

Consider the function below, where A is an unspecified, **positive** constant.

$$f(x) = \frac{A}{x - 8\sqrt{x} + 60}$$

For parts (c) and (d) only, assume the absolute minimum of f on $[0,21]$ is 8.

(a) List all x-values that must be tested to find the absolute extrema of f on $[0,21]$.

(b) At which x-value does the absolute minimum of f occur on $[0,21]$?

(c) Find the value of A.

(d) Find the absolute maximum of f on $[0,21]$ and all x-values at which it occurs.

exs21

Solution **L-15**

(a) We must test the endpoints of the interval ($x = 0$ and $x = 21$), as well as any critical points. Note that f is differentiable on $(0,21)$, so the only critical points are solutions to $f'(x) = 0$.

$$f'(x) = \frac{-A\left(1 - \frac{4}{\sqrt{x}}\right)}{(x - 8\sqrt{x} + 60)^2}$$

Hence the only critical point (and only other number we must test) is $x = 16$.

(b) We test the x-values in part (a). Observe the following: $f(0) = \frac{A}{60}$, $f(16) = \frac{A}{44}$, and $f(21) = \frac{A}{81-8\sqrt{21}} \approx \frac{A}{44.3}$. Hence the minimum of f on $[0,21]$ occurs at $x = 0$.

(c) We are given that the minimum is 8, and so part (b) implies $f(0) = \frac{A}{60} = 8$. Hence $A = 480$.

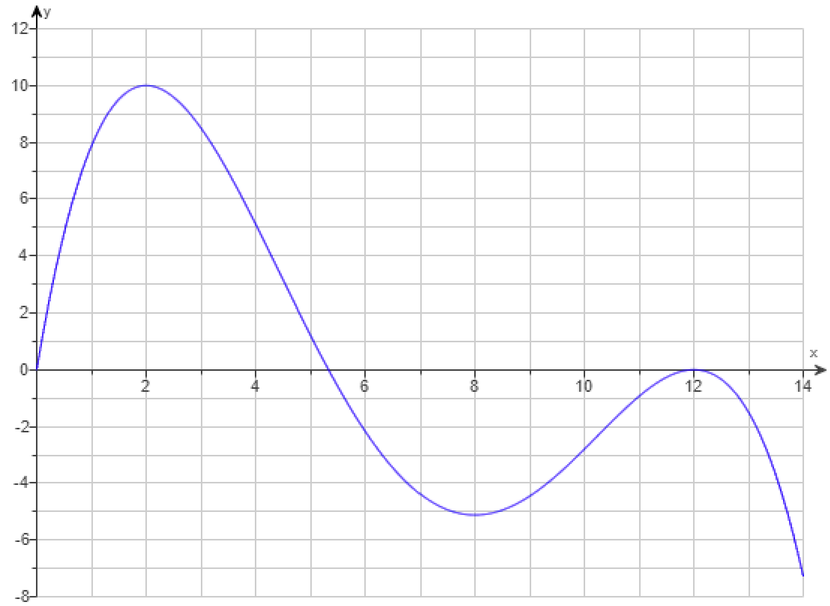
(d) From part (b), the absolute maximum is $f(16) = \frac{A}{44} = \frac{480}{44} = \frac{120}{11}$ (occurring only at $x = 16$).

Ex. L-16 4.1 Sp21 Exam

Use the graph of $y = f(x)$ on $[0,14]$ below to answer the questions.

exs21

4.1 Sp21 Exam



(a) List the critical points of f in $(0, 14)$.

(b) How many local extrema does f have on $(0, 14)$?

(c) Find the absolute maximum of f and the x-value at which it occurs.

(d) Find the absolute minimum of f and the x-value at which it occurs.

exs21

Solution L-16

(a) The critical points are $x = 2$ (since $f'(2) = 0$), $x = 8$ (since $f'(8) = 0$), and $x = 12$ (since $f'(12) = 0$).

(b) There are three local extrema (at the three critical points in part (a)).

(c) The absolute maximum of f is 10 at $x = 2$.

(d) The absolute minimum of f is -7.3 at $x = 14$. (Any reasonable estimate of -7.3 is acceptable.)

Ex. L-17 4.1 Fa21 Exam

Find the absolute extreme values of $f(x) = x^3 - 6x^2 + 9x + 20$ on $[-3, 2]$ and the x-value(s) at which they occur.

exf21

Solution L-17

Since f is differentiable for all x, the only critical points are solutions to $f'(x) = 0$. We have

$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$$

Hence the only critical point is $x = 1$. (We reject the solution $x = 3$ since it is not in the given interval.) We now check the critical values and the endpoint values: $f(-3) = -88$, $f(1) = 24$, and $f(2) = 22$. Hence the absolute minimum is -88 (occurring at $x = -3$) and the absolute maximum is 24 (occurring at $x = 1$).

Ex. L-18 4.1, 4.3/4.4 Fa21 Exam

Let $f(x) = x^3(3x - 4)$.

(a) Find where relative extrema of f occur (if any). Classify each as a local minimum or a local maximum.

(b) Find the absolute extrema of f on $[-1, 2]$ and the x-values at which they occur.

exf21

Solution L-18

(a) We have $f(x) = 3x^4 - 4x^3$, whence $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. The critical points of f are $x = 0$ and $x = 1$. The derivative $f'(x)$ does not change sign at $x = 0$, whence there is no local extremum at $x = 0$. However, $f'(x)$ changes sign from negative to positive at $x = 1$, whence there is a local minimum at $x = 1$. (Alternatively, note that $f''(x) = 36x^2 - 24x$ and $f''(1) = 12 > 0$.)

L-18

(b) We need only compare the endpoint values and critical values: $f(-1) = 7$, $f(0) = 0$, $f(1) = -1$, and $f(2) = 16$. Hence the absolute minimum is -1 at $x = 1$, and the absolute maximum is 16 at $x = 2$.

Ex. L-19 **4.1, 4.6** Fa21 **Exam**

The parts of this problem are not related.

(a) Suppose that when x units are produced, the total cost is $C(x) = 2x^2 + 10x + 18$ and the selling price per unit is $p(x) = 46 - x$. Find the level of production that maximizes total profit.

(b) Suppose the total cost of producing q units is $C(q) = q^3 + 20q^2 + 200q + 2000$. Use marginal analysis to estimate the cost of the 3rd unit.

exf21

Solution **L-19**

(a) The total revenue is $R(x) = xp(x) = 46x - x^2$, and so the total profit is $P(x) = R(x) - C(x) = -3x^2 + 36x - 18$. Profit is maximized when $P'(x) = 0$.

$$0 = P'(x) = -6x + 36 \Longrightarrow x = 6$$

(b) By marginal analysis, the cost of the 3rd unit is approximately:

$$C'(2) = \left(3q^2 + 40q + 200\right)\Big|_{q=2} = 12 + 80 + 200 = 292$$

Ex. L-20 **4.1** Sp22 **Exam**

Find the absolute extreme values of $f(x) = x\,(x-8)^{5/3}$ on the interval $[0, 9]$ and the x-values at which they occur.

exs22

Solution **L-20**

We first compute $f'(x)$.

$$f'(x) = 1 \cdot (x-8)^{5/3} + x \cdot \frac{5}{3}(x-8)^{2/3} = \frac{8}{3}(x-8)^{2/3}(x-3)$$

Hence the critical points are $x = 3$ and $x = 8$ only (solutions to $f'(x) = 0$). Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
3	$-3 \cdot 5^{5/3}$	critical point ($f'(x) = 0$)
8	0	critical point ($f'(x) = 0$)
9	9	endpoint

The maximum value of f on $[0, 9]$ is 9 at $x = 9$ and the minimum value is $-3 \cdot 5^{5/3}$ at $x = 3$.

Ex. L-21 **4.1, 4.3/4.4** Su22 **Exam**

Let $f(x) = Ax^B \ln(x)$, where A and B are unspecified constants. Suppose that $(e^5, 10)$ is a point of local extremum for $f(x)$.

(a) Calculate the values of A and B.

(b) Determine whether $(e^5, 10)$ is a point of local minimum or a point of local maximum for $f(x)$. Explain your answer.

exsu22

Solution **L-21**

(a) Since the point $(e^5, 10)$ lies on the graph of f, we must have $f(e^5) = 10$. Since the point $(e^5, 10)$ is a point of local extremum for f, we must have that $x = e^5$ is a critical point of f, whence $f'(e^5) = 0$. So A and B must simultaneously satisfy the equations:

$$f(e^5) = 10 \qquad f'(e^5) = 0$$

L-21

The derivative of f is:

$$f'(x) = ABx^{B-1}\ln(x) + Ax^B \cdot \frac{1}{x} = ABx^{B-1}\ln(x) + Ax^{B-1} = Ax^{B-1}\left(B\ln(x)+1\right)$$

So our system of equations is:

$$5Ae^{5B} = 10 \qquad Ae^{5(B-1)}\left(5B+1\right) = 0$$

The second equation above gives either $A = 0$ (which can't satisfy the first equation, and thus is not a valid solution) or $5B+1 = 0$. Thus $B = -\frac{1}{5}$. Substituting $B = -\frac{1}{5}$ and solving for A gives:

$$5Ae^{5B} = 10 \Longrightarrow 5Ae^{-1} = 10 \Longrightarrow A = 2e$$

(b) From part (a), we now have f and f':

$$f(x) = 2ex^{-1/5}\ln(x) \qquad f'(x) = 2ex^{-6/5}\left(-\frac{1}{5}\ln(x)+1\right)$$

To determine the nature of the local extremum, we use the first derivative test. The only critical point of f is $x = e^5$, so our sign chart for $f'(x)$ has two intervals to test: $(0, e^5)$, for which we can choose e^4 as a test point; and (e^5, ∞), for which we can choose e^6 as a test point. We have the following:

$$f'(e^4) = 2e \cdot e^{-24/5}\left(-\frac{1}{5}\cdot 4 + 1\right) = \oplus \cdot \left(\frac{1}{5}\right) = \oplus$$
$$f'(e^6) = 2e \cdot e^{-26/5}\left(-\frac{1}{5}\cdot 6 + 1\right) = \oplus \cdot \left(-\frac{1}{5}\right) = \ominus$$

Thus we see that f is increasing on the interval $(0, e^5]$ and decreasing on the interval $[e^5, \infty)$. Thus $x = e^5$ gives rise to a local maximum of f.

Ex. L-22 **4.1, 4.3/4.4** Su22 Exam

For each part, find the absolute extreme values of the given function on the given interval. If a particular extreme value does not exist, write "DNE" as your answer, and explain why that extreme value does not exist.

(a) $f(x) = \dfrac{e}{x} + \ln(x)$ on $[1, e^3]$

(b) $g(x) = 12x - x^3$ on $[0, \infty)$

exsu22

Solution L-22

(a) We first find the critical points by solving $f'(x) = 0$.

$$f'(x) = -\frac{e}{x^2} + \frac{1}{x} = 0 \Longrightarrow -e + x = 0 \Longrightarrow x = e$$

Now we compare the endpoint values and critical value.

$$f(1) = \frac{e}{1} + 0 = e \qquad f(e) = \frac{e}{e} + 1 = 2 \qquad f(e^3) = \frac{e}{e^3} + 3 = \frac{1}{e^2} + 3$$

(Recall that $2 < e < 3$.) Thus the absolute minimum of f is 2 and the absolute maximum of f is $\frac{1}{e^2} + 3$.

(b) We first find the critical points by solving $f'(x) = 0$.

$$f'(x) = 12 - 3x^2 = 0 \Longrightarrow x^2 = 4 \Longrightarrow x = 2$$

(Note that we reject the solution $x = -2$ since it's not in the given interval.) We can't use the extreme value theorem here because the given interval is not bounded.

Observe that $f''(x) = -6x$, whence $f''(2) < 0$. So $x = 2$ gives a local maximum of f on $[0, \infty)$. Since $x = 2$ is the only critical point on this interval, $x = 2$ gives an absolute maximum, and so the absolute maximum of f is $f(2) = 24 - 8 = 16$. However, since $\lim\limits_{x\to\infty} f(x) = -\infty$, there is no absolute minimum.

Ex. L-23 4.1 Sp18 Quiz

Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{10x}{x^2+1}$ on the interval $[0, 2]$.

qs18

Solution L-23

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = \frac{(x^2+1)(10) - (10x)(2x)}{(x^2+1)^2} = \frac{10(1-x^2)}{(x^2+1)^2}$$

Hence the only critical point in the interval $[0, 2]$ is $x = 1$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
1	5	critical point ($f'(x) = 0$)
2	4	endpoint

The maximum value of f on $[0, 2]$ is 5 and the minimum value is 0.

Ex. L-24 4.1 Su22 Quiz

Find the absolute extrema of $f(x) = 10 + 8x^2 - x^4$ on the interval $[-1, 3]$.

qsu22

Solution L-24

Since f is differentiable on its domain, the absolute extrema occur either where $f'(x) = 0$ or at the endpoints of the interval $[-1, 3]$. We have:

$$f'(x) = 16x - 4x^3 = 4x(4 - x^2) = 4x(2 - x)(2 + x) = 0$$

Solutions to $f'(x) = 0$ are $x = 0$ and $x = 2$. (We ignore the solution $x = -2$ since it lies outside the interval $[-1, 3]$.) Now we compare the critical values and the endpoint values: $f(-1) = 17$, $f(0) = 10$, $f(2) = 26$, and $f(3) = 1$. Hence the absolute minimum value is 1 and the absolute maximum value is 26.

Ex. L-25 4.1 Su22 Quiz

Let $f(x) = x^2(5x + 9)^{1/5}$. Observe that the domain of f is $(-\infty, \infty)$. Calculate the critical numbers of f. For each critical number you find, explain precisely why your answer is a critical number.

qsu22

Solution L-25

We first calculate $f'(x)$.

$$f'(x) = 2x(5x + 9)^{1/5} + x^2 \cdot \frac{1}{5}(5x + 9)^{-4/5} \cdot 5 = (5x + 9)^{-4/5}\left(2x(5x + 9) + x^2\right) = \frac{x(11x + 18)}{(5x + 9)^{4/5}}$$

The critical numbers of f are where $f'(x) = 0$ (or $x = 0$ and $x = -\frac{18}{11}$) or where $f'(x)$ does not exist (or $x = -\frac{9}{5}$).

Ex. L-26 4.1 Fa22 Quiz

Let $f(x) = 3x^{4/3} - 300x^{1/3}$. Find all critical points of f. You must make clear why each of your answers is a critical point.

qf22

Solution L-26

First we find $f'(x)$ using the power rule.

$$f'(x) = 4x^{1/3} - 100x^{-2/3} = \frac{4(x - 25)}{x^{2/3}}$$

The domain of $f(x)$ is $(-\infty, \infty)$. Hence critical points of f are solutions to the equation $f'(x) = 0$ (i.e., $x = 25$ only) or where $f'(x)$ does not exist (i.e., $x = 0$ only).

Ex. L-27 4.1 Fa22 Quiz

Find the absolute extrema of $f(x) = \sqrt{2}\sin(x) + \cos^2(x)$ on the interval $[0, \pi]$. *Hint:* You will need the approximation $\sqrt{2} \approx 1.4$.

qf22

Solution L-27

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = \sqrt{2}\cos(x) + 2\cos(x)\cdot(-\sin(x)) = \cos(x)\left(\sqrt{2} - 2\sin(x)\right)$$

The solutions in the interval $[0, \pi]$ to the equation $f'(x) = 0$ are a solution to $\cos(x) = 0$ (i.e., $x = \frac{\pi}{2}$ only) or a solution to $\sin(x) = \frac{\sqrt{2}}{2}$ (i.e., $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ only).

Now we compare the critical values and the endpoint values.

x	$f(x)$	reason for check
0	1	endpoint
$\frac{\pi}{4}$	1.5	critical point ($f'(x) = 0$)
$\frac{\pi}{2}$	$\sqrt{2} \approx 1.4$	critical point ($f'(x) = 0$)
$\frac{3\pi}{4}$	1.5	critical point ($f'(x) = 0$)
π	1	endpoint

The maximum value of f on $[0, \pi]$ is 1 and the minimum value is 1.5.

Ex. L-28 4.1

For each part, find the absolute extreme values of $f(x)$ on the given interval. *You may use a scientific calculator for parts (j) and (k) only.*

(a) $f(x) = x^4 - 8x^2$ on $[-3, 3]$
(b) $f(x) = x^3 + 3x^2 - 24x - 72$ on $[-4, 4]$
(c) $f(x) = \sqrt{x}(x-5)^{1/3}$ on $[0, 6]$
(d) $f(x) = e^{-x}\sin(x)$ on $[0, 2\pi]$
(e) $f(x) = x\left(\ln(x) - 5\right)^2$ on $[e^{-4}, e^4]$
(f) $f(x) = 2x^3 - 9x^2 + 12x$ on $[0, 3]$
(g) $f(x) = \dfrac{1-x}{x^2+3x}$ on $[1, 4]$
(h) $f(x) = x - 2\sin(x)$ on $[0, 2\pi]$
(i) $f(x) = (x - x^2)^{1/3}$ on $[-1, 2]$
(j) $f(x) = x^3 - 24\ln(x)$ on $[\frac{1}{2}, 3]$
(k) $f(x) = 3e^x - e^{2x}$ on $[-\frac{1}{2}, 1]$

c4s1

Solution L-28

(a) The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$$

Hence the critical points are $x = -2$, $x = 0$, and $x = 2$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-3	9	endpoint
-2	-16	critical point ($f'(x) = 0$)
0	0	critical point ($f'(x) = 0$)
2	-16	critical point ($f'(x) = 0$)
3	9	endpoint

The maximum value of f on $[-3, 3]$ is 9 and the minimum value is -16.

(b) The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 3x^2 + 6x - 24 = 3(x-2)(x+4)$$

Hence the critical points are $x = -4$ and $x = 2$. Checking the critical values and the endpoint values gives the following.

L-28

x	$f(x)$	reason for check
-4	8	endpoint
2	-100	critical point ($f'(x)=0$)
4	-56	endpoint

The maximum value of f on $[-4,4]$ is 8 and the minimum value is -100.

(c) We calculate $f'(x)$ to find the critical points.

$$f'(x) = x^{1/2}\cdot\frac{1}{3}(x-5)^{-2/3} + \frac{1}{2}x^{-1/2}(x-5)^{1/3} = \frac{5x-15}{6x^{1/2}(x-5)^{2/3}}$$

Solving $f'(x)=0$ gives gives $5x-15=0$, or $x=3$. Also note that $f'(x)$ DNE if $x=5$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
3	$-3^{1/2}\cdot 2^{1/3}$	critical point ($f'(x)=0$)
5	0	critical point ($f'(x)$ DNE)
6	$6^{1/2}$	endpoint

The maximum value of f on $[0,6]$ is $6^{1/2}$ and the minimum value is $-3^{1/2}\cdot 2^{1/3}$.

(d) The function f is differentiable everywhere. So we solve $f'(x)=0$.

$$f'(x) = e^{-x}\cos(x) - e^{-x}\sin(x) = e^{-x}\left(\cos(x)-\sin(x)\right)$$

Solving this equation thus gives $\cos(x)-\sin(x)=0$ (that is, $\tan(x)=1$). In the interval $[0,2\pi]$ the equation $\tan(x)=1$ has solutions $x=\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
$\frac{\pi}{4}$	$e^{-\pi/4}\cdot\frac{1}{\sqrt{2}}$	critical point ($f'(x)=0$)
$\frac{5\pi}{4}$	$-e^{-5\pi/4}\cdot\frac{1}{\sqrt{2}}$	critical point ($f'(x)$ DNE)
2π	0	endpoint

The maximum value of f on $[0,2\pi]$ is $\frac{e^{-\pi/4}}{\sqrt{2}}$ and the minimum value is $-\frac{e^{-5\pi/4}}{\sqrt{2}}$.

(e) The function f is differentiable on its domain. So we solve $f'(x)=0$.

$$f'(x) = x\cdot 2\left(\ln(x)-5\right)\cdot\frac{1}{x} + \left(\ln(x)-5\right)^2 = \left(\ln(x)-5\right)\left(\ln(x)-3\right)$$

Solving this equation thus gives $\ln(x)-5=0$ (i.e., $x=e^5$) or $\ln(x)-3=0$ (i.e., $x=e^3$). The only critical point is thus $x=e^3$ (e^5 is not in the interval $[e^{-4},e^4]$). Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
e^{-4}	$\frac{81}{e^4}$	endpoint
e^3	$4e^3$	critical point ($f'(x)=0$)
e^4	e^4	endpoint

We can determine which values are least and greatest without a calculator by estimating their ratios and recalling that $2<e<3$:

$$\frac{f(e^3)}{f(e^4)} = \frac{4e^3}{e^4} = \frac{4}{e} > 1$$
$$\frac{f(e^4)}{f(e^{-4}} = \frac{e^4}{\frac{81}{e^4}} = \frac{e^8}{81} > \frac{2^8}{31} = \frac{256}{81} > 1$$

Thus we have $4e^3 > e^4 > \frac{81}{e^4}$.

L-28

The maximum value of f on $[e^{-4}, e^4]$ is $4e^3$ and the minimum value is $\frac{81}{e^4}$.

(f) The function f is differentiable on its domain. So we solve $f'(x) = 0$.

$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$$

Hence the only critical points in $[0, 3]$ are $x = 1$ and $x = 2$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
1	5	critical point ($f'(x) = 0$)
2	4	critical point ($f'(x) = 0$)
3	9	endpoint

Hence the minimum value of f is 0 and the maximum value of f is 9.

(g) The function f is differentiable on its domain. So we solve $f'(x) = 0$.

$$f'(x) = \frac{(x^2+3x)(-1) - (1-x)(2x+3)}{(x^2+3x)^2} = \frac{(x-3)(x+1)}{(x^2+3x)^2}$$

Hence the only critical point in $[1, 4]$ is $x = 3$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
1	0	endpoint
3	$-\frac{1}{9}$	critical point ($f'(x) = 0$)
4	$-\frac{3}{28}$	endpoint

Hence the minimum value of f is $-\frac{1}{9}$ and the maximum value of f is 0.

(h) The function f is differentiable on its domain. So we solve $f'(x) = 0$.

$$f'(x) = 1 - 2\cos(x)$$

Hence the only critical points in $[0, 2\pi]$ are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
$\frac{\pi}{3}$	$\frac{\pi}{3} - \sqrt{3}$	critical point ($f'(x) = 0$)
$\frac{5\pi}{3}$	$\frac{5\pi}{3} + \sqrt{3}$	critical point ($f'(x) = 0$)
2π	2π	endpoint

We can determine which values are least and greatest without a calculator as follows. First note that $\pi < \sqrt{27} = 3\sqrt{3}$, whence $\frac{\pi}{3} - \sqrt{3} < 0$. Then we have

$$\frac{5\pi}{3} + \sqrt{3} = 2\pi - \underbrace{\left(\frac{\pi}{3} - \sqrt{3}\right)}_{\ominus} > 2\pi$$

Thus we find $f(\frac{\pi}{3}) < 0 < 2\pi < f(\frac{5\pi}{3})$.

Hence the minimum value of f is $\frac{\pi}{3} - \sqrt{3}$ and the maximum value of f is $\frac{5\pi}{3} + \sqrt{3}$.

(i) We calculate $f'(x)$ to find the critical points.

$$f'(x) = \frac{1}{3}(x - x^2)^{-2/3}(1 - 2x) = \frac{1 - 2x}{3(x - x^2)^{2/3}}$$

Solving $f'(x) = 0$ gives $1 - 2x = 0$, or $x = \frac{1}{2}$. Also note that $f'(x)$ DNE if $x - x^2 = 0$ (i.e., $x = 0$ or $x = 1$). Checking the critical values and the endpoint values gives the following.

L-28

x	$f(x)$	reason for check
-1	$-2^{1/3}$	endpoint
0	0	critical point ($f'(x)$ DNE)
$\frac{1}{2}$	$4^{-1/3}$	critical point ($f'(x) = 0$)
1	0	critical point ($f'(x)$ DNE)
2	$-2^{1/3}$	endpoint

Hence the minimum value of f is $-2^{1/3}$ and the maximum value of f is $4^{-1/3}$.

(j) The function f is differentiable on its domain. So we solve $f'(x) = 0$.

$$f'(x) = 3x^2 - \frac{24}{x} = \frac{3(x^3 - 8)}{x}$$

Hence the only critical point in $[\frac{1}{2}, 3]$ is $x = 2$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
$\frac{1}{2}$	$\frac{1}{8} + 24\ln(2) \approx 16.8$	endpoint
2	$8 - 24\ln(2) \approx -8.6$	critical point ($f'(x) = 0$)
3	$27 - 24\ln(3) \approx 0.6$	endpoint

Hence the minimum value of f is $8 - 24\ln(2)$ and the maximum value of f is $\frac{1}{8} + 24\ln(2)$.

(k) The function f is differentiable on its domain. So we solve $f'(x) = 0$.

$$f'(x) = 3e^x - 2e^{2x} = e^x(3 - 2e^x)$$

Hence the only critical point in $[-\frac{1}{2}, 1]$ is $x = \ln(\frac{3}{2})$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
$-\frac{1}{2}$	$3e^{-1/2} - e^{-1} \approx 1.5$	endpoint
$\ln(\frac{3}{2})$	$\frac{9}{4} = 2.25$	critical point ($f'(x) = 0$)
1	$3e - e^2 \approx 0.8$	endpoint

Hence the minimum value of f is $3e - e^2$ and the maximum value of f is $\frac{9}{4}$.

Ex. L-29 4.1

Find the absolute extreme values of $f(x) = 3x^4 - 4x^3 - 12x^2$ on $[-2, 1]$.

unit4

Solution L-29

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

Hence the critical points in $[-2, 1]$ are $x = -1$ and $x = 0$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-2	32	endpoint
-1	-5	critical point ($f'(x) = 0$)
0	0	critical point ($f'(x) = 0$)
1	-13	endpoint

The maximum value of f on $[-2, 1]$ is 32 and the minimum value is -13.

Ex. L-30 **4.1**

Find the absolute extreme values of $f(x) = x^2(x+5)^3$ on $[-6, 0]$.

unit4

Solution **L-30**

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 2x(x+5)^3 + x^2 \cdot 3(x+5)^2 = x(x+5)^2(2(x+5)+3x) = 5x(x+2)(x+5)^2$$

Hence the critical points are $x = -5$, $x = -2$, and $x = 0$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-6	-36	endpoint
-5	0	critical point ($f'(x) = 0$)
-2	108	critical point ($f'(x) = 0$)
0	0	endpoint

The maximum value of f on $[-6, 0]$ is 108 and the minimum value is -36.

Ex. L-31 **4.1**

Find the absolute minimum and maximum of $f(x) = (6x+1)e^{3x}$ on the interval $[-1000, 1000]$.

unit4

Solution **L-31**

The function f is differentiable everywhere. So we solve $f'(x) = 0$.

$$f'(x) = 6e^{3x} + (6x+1) \cdot e^{3x} \cdot 3 = 9e^{3x}(2x+1)$$

Hence the only critical point is $x = -\frac{1}{2}$. Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-1000	$-5999e^{-3000}$	endpoint
$-\frac{1}{2}$	$-2e^{-3/2}$	critical point ($f'(x) = 0$)
1000	$6001e^{3000}$	endpoint

The maximum value of f on $[-1000, 1000]$ is $6001e^{3000}$ and the minimum value is $-2e^{-3/2}$.

Ex. L-32 **4.1, 4.9**

The marginal revenue of a certain product is $R'(x) = -9x^2 + 17x + 30$, where x is the level of production. Assume $R(0) = 0$. Find the market price that maximizes revenue.

unit4

Solution **L-32**

Revenue is maximized if $R'(x) = -(9x+10)(x-3) = 0$, or if $x = 3$. (We ignore the solution $x = -\frac{10}{9}$ since x must be positive since it represents level of production.)

Antidifferentiating $R'(x)$, we find that the revenue is $R(x) = -3x^3 + \frac{17}{2}x^2 + 30x + K$, for some unknown constant K. The assumption that $R(0) = 0$ implies that $K = 0$, whence $R(x) = -3x^3 + \frac{17}{2}x^2 + 30x$. Since $R(x) = xp(x)$, the market price is $p(x) = -3x^2 + \frac{17}{2}x + 30$. Hence the market price when revenue is maximized is $p(3) = 28.5$.

Ex. L-33 **4.1**

Calculate the absolute extreme values of $f(x) = \dfrac{225 - 75x^2}{5x + x^3}$ on $[-5, -1]$.

other

Solution **L-33**

The function f is differentiable everywhere in $[-5,-1]$. So we solve $f'(x) = 0$.

$$f'(x) = \frac{-150x - (225 - 75x^2)(5 + 3x^2)}{(5x + x^3)^2} = \frac{75(x^2 + 1)(x^2 - 15)}{(5x + x^3)^2}$$

Hence the only critical point is $x = -\sqrt{15}$ (we reject $x = \sqrt{15}$ since it does not lie in $[-5,-1]$). Checking the critical values and the endpoint values gives the following.

x	$f(x)$	reason for check
-5	$11 = \sqrt{121}$	endpoint
$-\sqrt{15}$	$3\sqrt{15} = \sqrt{135}$	critical point ($f'(x) = 0$)
-1	-25	endpoint

The maximum value of f on $[-5,-1]$ is $\sqrt{135}$ and the minimum value is -25.

§4.3, 4.4: What Derivatives Tell Us and Graphing Functions

Ex. M-1 **4.3/4.4** Fa17 **Exam**

Consider the function $f(x) = (x-5)(x+10)^2 = x^3 + 15x^2 - 500$.

(a) Calculate all x- and y-intercepts of f.

(b) Find where f is increasing and find where f is decreasing. Then calculate the x- and y-coordinates of all local extrema, classifying each as either a local minimum or a local maximum.

(c) Find where f is concave up and find where f is concave down. Then calculate the x- and y-coordinates of all inflection points.

(d) Sketch the graph of $y = f(x)$ on the provided grid. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

exf17

Solution **M-1**

(a) The equation $f(x) = 0$ has solutions $x = 5$ and $x = -10$, whence the x-intercepts are $(5, 0)$ and $(-10, 0)$. The y-intercept is $(0, -500)$.

(b) We calculate a sign chart for the first derivative:

$$f'(x) = 3x^2 + 30x = 3x(x+10)$$

The cut points are the solutions to $f'(x) = 0$: $x = 0$ and $x = -10$.

interval	test point	sign	shape of f
$(-\infty, -10)$	$f'(-21)$	$\ominus\ominus = \oplus$	increasing
$(-10, 0)$	$f'(-5)$	$\ominus\oplus = \ominus$	decreasing
$(0, \infty)$	$f'(1)$	$\oplus\oplus = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[-10, 0]$
f is increasing on:	$(-\infty, -10]$, $[0, \infty)$
f has a local min at:	$x = 0$
f has a local max at:	$x = -10$

(c) We calculate a sign chart for the second derivative:

$$f''(x) = 6x + 30 = 6(x+5)$$

The cut points are the solutions to $f''(x) = 0$: $x = -5$ only.

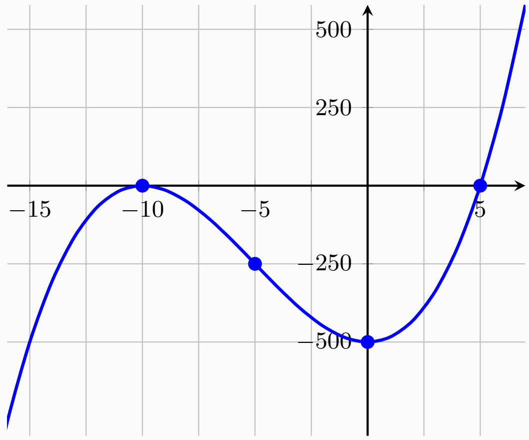
interval	test point	sign	shape of f
$(-\infty, -5)$	$f''(-6)$	$\ominus$	concave down
$(-5, \infty)$	$f''(0)$	$\oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, -5]$
f is concave up on:	$[-5, \infty)$
f has an infl. point at:	$x = -5$

(d) Using the previous solutions, we have the following sketch.

M-1



Ex. M-2 **4.3/4.4** Sp18 **Exam**

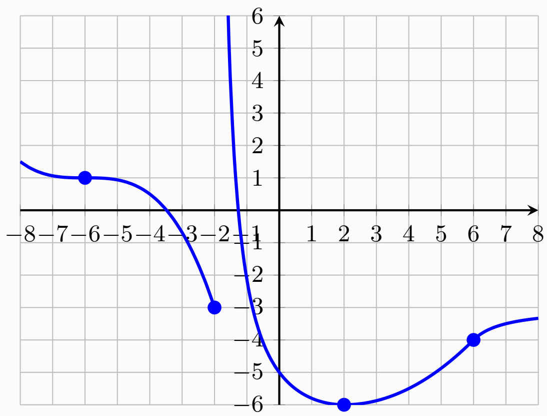
Suppose $f(x)$ satisfies all of the following properties. Sketch a possible graph of $y = f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

domain of f:	$[-8, 8]$
specific points on graph:	$f(-2) = -3$ and $f'(-6) = 0$
asymptotes of f:	$x = -2$ and $y = -3$
f is decreasing on:	$[-8, -2)$, $(-2, 2)$
f is increasing on:	$(2, 8]$
f is concave down on:	$(-1, 1)$
f is concave up on:	$[-8, -1)$, $(1, 8]$

exs18

Solution M-2

There are many such solutions. Here is one.



Ex. M-3 **4.3/4.4** Sp18 **Exam**

Consider the function f and its derivatives below.

$$f(x) = \frac{x^2}{x^2 - 1} \quad , \quad f'(x) = \frac{-2x}{(x^2-1)^2} \quad , \quad f''(x) = \frac{6x^2+2}{(x^2-1)^3}$$

(a) Find all horizontal asymptotes of f.

(b) Find all vertical asymptotes of f. Then at each vertical asymptote you find, calculate the corresponding one-sided limits of f.

(c) Find where f is decreasing and find where f is increasing. Then calculate all points of local extrema, classifying each as either a local minimum, a local maximum, or neither.

(d) Find where f is concave down and find where f is concave up. Then calculate all points of inflection.

exs18

Solution **M-3**

(a) Horizontal asymptotes are found by computing the limits of f at infinity.

$$\lim_{x\to\pm\infty}\left(\frac{x^2}{x^2-1}\right)=\lim_{x\to\pm\infty}\left(\frac{1}{1-\frac{1}{x^2}}\right)=\frac{1}{1-0}=1$$

Hence the only horizontal asymptote is the line $y=1$.

(b) Since f is continuous on its domain, the only candidate vertical asymptotes are the lines $x=-1$ and $x=1$ (since there are the only x-values not in the domain of f). Direct substitution of either $x=-1$ or $x=1$ into $f(x)$ gives the expression "$\frac{1}{0}$", which is undefined but indicates that all of the corresponding one-sided limits at both $x=-1$ and $x=1$ are infinite. Hence $x=-1$ and $x=1$ are vertical asymptotes. Now we may compute the limits using sign analysis.

$$\begin{aligned}\lim_{x\to-1^-}\left(\frac{x^2}{x^2-1}\right)&=\frac{1}{0^+}=+\infty\\ \lim_{x\to-1^+}\left(\frac{x^2}{x^2-1}\right)&=\frac{1}{0^-}=-\infty\\ \lim_{x\to1^-}\left(\frac{x^2}{x^2-1}\right)&=\frac{1}{0^-}=-\infty\\ \lim_{x\to1^+}\left(\frac{x^2}{x^2-1}\right)&=\frac{1}{0^+}=+\infty\end{aligned}$$

(c) We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x)=0$ ($x=0$) and the vertical asymptotes ($x=-1$ and $x=1$).

interval	test point	sign of f'	shape of f
$(-\infty,-1)$	$f'(-2)$	$\frac{\oplus}{\oplus}=\oplus$	increasing
$(-1,0)$	$f'(-0.5)$	$\frac{\oplus}{\oplus}=\oplus$	increasing
$(0,1)$	$f'(0.5)$	$\frac{\ominus}{\oplus}=\ominus$	decreasing
$(1,\infty)$	$f'(2)$	$\frac{\ominus}{\oplus}=\ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$[0,1)$, $(1,\infty)$
f is increasing on:	$(-\infty,-1)$, $(-1,0]$
f has a local min at:	*none*
f has a local max at:	$x=0$

(d) We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x)=0$ (none) and the vertical asymptotes ($x=-1$ and $x=1$).

interval	test point	sign of f''	shape of f
$(-\infty,-1)$	$f''(-2)$	$\frac{\oplus}{\oplus}=\oplus$	concave up
$(-1,1)$	$f''(0)$	$\frac{\oplus}{\ominus}=\ominus$	concave down
$(1,\infty)$	$f''(2)$	$\frac{\oplus}{\oplus}=\oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-1,1)$
f is concave up on:	$(-\infty,-1)$, $(1,\infty)$
f has an infl. point at:	*none*

Ex. M-4 **4.3/4.4** Fa18 **Exam**

Consider the function f and its derivatives below.

$$f(x) = \frac{2x^3 + 3x^2 - 1}{x^3} \quad , \quad f'(x) = \frac{3 - 3x^2}{x^4} \quad , \quad f''(x) = \frac{6x^2 - 12}{x^5}$$

For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

(a) Find all horizontal asymptotes of f.

(b) Find all vertical asymptotes of f. Then for each asymptote, find the corresponding one-sided limits of f.

(c) Find where f is decreasing, where f is increasing, and where f has a local extremum.

(d) Find where f is concave down, where f is concave up, and where f has an inflection point.

exf18

Solution **M-4**

(a) Horizontal asymptotes are found by computing the limit of f as $x \to \pm\infty$.

$$\lim_{x \to \pm\infty} \left(\frac{2x^3 + 3x^2 - 1}{x^3} \right) = \lim_{x \to \pm\infty} \left(2 + \frac{3}{x} - \frac{1}{x^3} \right) = 2 + 0 - 0 = 2$$

Hence the only horizontal asymptote is the line $y = 2$.

(b) Since f is continuous on its domain, the only candidate vertical asymptote is the line $x = 0$ (found by setting the denominator of f equal to 0). Direct substitution of $x = 0$ into $f(x)$ gives the expression $\frac{-1}{0}$, which indicates that the corresponding one-sided limits at $x = 0$ are infinite. Hence the line $x = 0$ is a true vertical asymptote. Now we may compute the limits using sign analysis.

$$\lim_{x \to 0^-} \left(\frac{2x^3 + 3x^2 - 1}{x^3} \right) = \frac{-1}{0^-} = +\infty$$

$$\lim_{x \to 0^+} \left(\frac{2x^3 + 3x^2 - 1}{x^3} \right) = \frac{-1}{0^+} = -\infty$$

(c) We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = -1$ and $x = 1$) and the vertical asymptotes ($x = 0$).

interval	test point	sign of f'	shape of f
$(-\infty, -1)$	$f'(-2)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(-1, 0)$	$f'(-0.5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(0, 1)$	$f'(0.5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(1, \infty)$	$f'(2)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, -1]$, $[1, \infty)$
f is increasing on:	$[-1, 0)$, $(0, 1]$
f has a local min at:	$x = -1$
f has a local max at:	$x = 1$

(d) We calculate a sign chart for the second derivative. The cut points are the solutions to $f''(x) = 0$ ($x = -\sqrt{2}$ and $x = \sqrt{2}$) and the vertical asymptotes ($x = 0$).

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt{2})$	$f'(-2)$	$\frac{\oplus}{\ominus} = \ominus$	concave down
$(-\sqrt{2}, 0)$	$f'(-1)$	$\frac{\ominus}{\ominus} = \oplus$	concave up
$(0, \sqrt{2})$	$f'(1)$	$\frac{\ominus}{\oplus} = \ominus$	concave down
$(\sqrt{2}, \infty)$	$f'(2)$	$\frac{\oplus}{\oplus} = \oplus$	concave up

M-4

Hence we deduce the following about f:

f is concave down on:	$(-\infty, -\sqrt{2}]$, $(0, \sqrt{2}]$
f is concave up on:	$[-\sqrt{2}, 0)$, $[\sqrt{2}, \infty)$
f has an infl. point at:	$x = -\sqrt{2}$, $x = \sqrt{2}$

Ex. M-5 **4.3/4.4** Sp19 **Exam**

Consider the function f and its derivatives below.

$$f(x) = 2x + \frac{8}{x^2} \quad , \quad f'(x) = \frac{2\left(x^3 - 8\right)}{x^3} \quad , \quad f''(x) = \frac{48}{x^4}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

exs19

Solution **M-5**

equation(s) of vertical asymptote(s) of f	$x = 0$
equation(s) of horizontal asymptote(s) of f	NONE
where f is decreasing	$(0, 2]$
where f is increasing	$(-\infty, 0)$, $[2, \infty)$
x-coordinate(s) of local minima of f	$x = 2$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	NONE
where f is concave up	$(-\infty, 0)$, $(0, \infty)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f(x) = 2x + \frac{8}{x^2}} \qquad \boxed{f'(x) = \frac{2(x^3 - 8)}{x^3}} \qquad \boxed{f''(x) = \frac{48}{x^4}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 0$. Hence our candidate vertical asymptote is the line $x = 0$. Indeed, direct substitution of $x = 0$ into the term $\frac{8}{x^2}$ gives the expression $\frac{8}{0}$, which indicates that both one-sided limits are infinite. Hence the line $x = 0$ is a true vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x \pm \infty} \left(2x + \frac{8}{x^2}\right) = \pm\infty + 0 = \pm\infty$$

Since neither limit (as either $x \to -\infty$ or $x \to \infty$) is finite, there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 2$) and the vertical asymptotes ($x = 0$).

M-5

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{2\ominus}{\ominus} = \oplus$	increasing
$(0, 2)$	$f'(1)$	$\frac{2\ominus}{\oplus} = \ominus$	decreasing
$(2, \infty)$	$f'(3)$	$\frac{2\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(0, 2]$
f is increasing on:	$(-\infty, 0)$, $[2, \infty)$
f has a local min at:	$x = 2$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes ($x = 0$).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\frac{48}{\oplus} = \oplus$	concave up
$(0, \infty)$	$f''(1)$	$\frac{48}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	*no interval*
f is concave up on:	$(-\infty, 0)$, $(0, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

Ex. M-6 4.3/4.4 Fa19 Exam

Find the x-coordinate of each inflection point, if any, of $f(x) = x^3 - 12x^2 + 5x - 10$.

Solution M-6

Observe that $f''(x) = 6x - 24 = 6(x - 4)$, which changes sign (from negative to positive) at $x = 4$. Since f is also continuous at $x = 4$, f has an inflection point at $x = 4$.

Ex. M-7 4.3/4.4 Fa19 Exam

Consider the function f and its derivatives below.

$$f(x) = \frac{3x^3 - 2x + 48}{x} \quad , \quad f'(x) = \frac{6(x^3 - 8)}{x^2} \quad , \quad f''(x) = \frac{6(x^3 + 16)}{x^3}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

Solution **M-7**

equation(s) of vertical asymptote(s) of f	$x = 0$
equation(s) of horizontal asymptote(s) of f	NONE
where f is decreasing	$(-\infty, 0)$, $(0, 2]$
where f is increasing	$[2, \infty)$
x-coordinate(s) of local minima of f	$x = 2$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	$[-\sqrt[3]{16}, 0)$
where f is concave up	$(-\infty, -\sqrt[3]{16}]$, $(0, \infty)$
x-coordinate(s) of inflection point(s) of f	$x = -\sqrt[3]{16}$

The first two derivatives of $f(x)$ are

$$\boxed{f(x) = \frac{3x^3 - 2x + 48}{x}} \qquad \boxed{f'(x) = \frac{6(x^3 - 8)}{x^2}} \qquad \boxed{f''(x) = \frac{6(x^3 + 16)}{x^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 0$. Hence our candidate vertical asymptotes is the line $x = 0$. Indeed, direct substitution of $x = 0$ into $f(x)$ gives the expression "$\frac{48}{0}$", which indicates that both one-sided limits are infinite. Hence the line $x = 0$ is a true vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x \pm\infty} f(x) = \lim_{x \pm\infty} \left(3x^2 - 2 + \frac{48}{x}\right) = \infty - 2 + 0 = \infty$$

Hence there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 2$) and the vertical asymptotes ($x = 0$).

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{6\ominus}{\oplus} = \ominus$	decreasing
$(0, 2)$	$f'(1)$	$\frac{6\ominus}{\oplus} = \ominus$	decreasing
$(2, \infty)$	$f'(3)$	$\frac{6\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 0)$, $(0, 2]$
f is increasing on:	$[2, \infty)$
f has a local min at:	$x = 2$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ ($x = -\sqrt[3]{16}$) and the vertical asymptotes ($x = 0$).

M-7

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt[3]{16})$	$f''(-3)$	$\frac{6\ominus}{\ominus} = \oplus$	concave up
$(-\sqrt[3]{16}, 0)$	$f''(-1)$	$\frac{6\oplus}{\ominus} = \ominus$	concave down
$(0, \infty)$	$f''(1)$	$\frac{6\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $[-\sqrt[3]{16}, 0)$
f is concave up on: $(-\infty, -\sqrt[3]{16}]$, $(0, \infty)$
f has an infl. point at: $x = -\sqrt[3]{16}$

(iv) Sketch of graph.
Not required.

Ex. M-8 4.3/4.4 Sp20 Exam

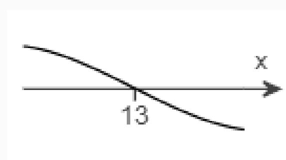
For each part, sketch the graph of a function that satisfies the given properties.

(a) $f(x)$ is decreasing for all x; $f''(x) < 0$ for $x < 13$; $f''(x) > 0$ for $x > 13$.
(b) $f(x)$ has a local minimum at $x = a$ where $f'(a) = 0$.
(c) $f(x)$ has a local maximum at $x = b$ where $f'(b)$ is undefined.

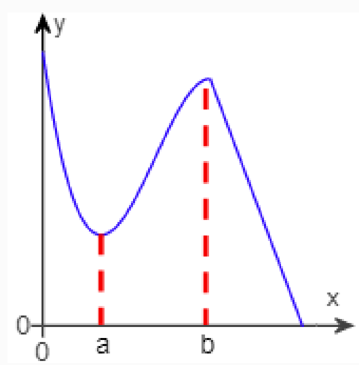
exs20

Solution M-8

(a) Here is one possibility.

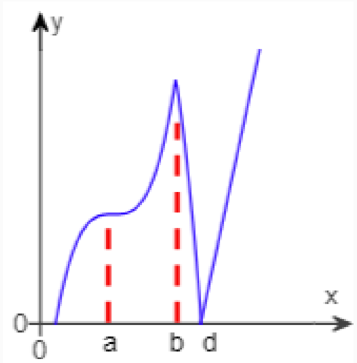


(b) Here is one possibility.



(c) Here is one possibility.

M-8



Ex. M-9 4.3/4.4 Sp20 Exam

The first two derivatives of the function f are given below.

$$f'(x) = \frac{x}{(x-6)^2(x+48)} \quad , \quad f''(x) = \frac{-2(x+12)^2}{(x-6)^3(x+48)^2}$$

(Do not attempt to find a formula for $f(x)$.)

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

exs20

Solution M-9

where f is decreasing	$(-48, 0]$
where f is increasing	$(-\infty, -48)$, $[0, 6)$, $(6, \infty)$
x-coordinate(s) of local minima of f	$x = 0$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	$(6, \infty)$
where f is concave up	$(-\infty, -48)$, $(-48, 6)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{x}{(x-6)^2(x+48)}} \qquad \boxed{f''(x) = \frac{-2(x+12)^2}{(x-6)^3(x+48)^2}}$$

(i) **Vertical asymptotes and horizontal asymptotes.**

Not required since $f(x)$ is not given.

(ii) **Intervals of increase and local extrema.**

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 0$) and where $f'(x)$ is undefined ($x = -48$ and $x = 6$).

M-9

interval	test point	sign of f'	shape of f
$(-\infty, -48)$	$f'(-50)$	$\frac{\ominus}{\oplus\ominus} = \oplus$	increasing
$(-48, 0)$	$f'(-1)$	$\frac{\ominus}{\oplus\oplus} = \ominus$	decreasing
$(0, 6)$	$f'(1)$	$\frac{\oplus}{\oplus\oplus} = \oplus$	increasing
$(6, \infty)$	$f'(7)$	$\frac{\oplus}{\oplus\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-48, 0]$
f is increasing on:	$(-\infty, -48)$, $[0, 6)$, $(6, \infty)$
f has a local min at:	$x = 0$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ ($x = -12$) and where $f''(x)$ is undefined ($x = -48$ and $x = 6$).

interval	test point	sign of f''	shape of f
$(-\infty, -48)$	$f''(-50)$	$\frac{-2\oplus}{\ominus\oplus} = \oplus$	concave up
$(-48, -12)$	$f''(-20)$	$\frac{-2\oplus}{\ominus\oplus} = \oplus$	concave up
$(-12, 6)$	$f''(0)$	$\frac{-2\oplus}{\ominus\oplus} = \oplus$	concave up
$(6, \infty)$	$f''(7)$	$\frac{-2\oplus}{\oplus\oplus} = \ominus$	concave down

Hence we deduce the following about f:

f is concave down on:	$(6, \infty)$
f is concave up on:	$(-\infty, -48)$, $(-48, 6)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

Ex. L-8 **4.1, 4.3/4.4** Sp20 Exam

Consider the following function

$$g(x) = \frac{3}{2}x^4 + 8x^3 - 36x^2$$

(a) Where does g have a local minimum on $(-7, 3)$? local maximum?

(b) Where does g have a global minimum on $[-7, 3]$? global maximum?

exs20

Solution L-8

(a) We solve $g'(x) = 0$ to find the critical points of g.

$$g'(x) = 6x^3 + 24x^2 - 72x = 6x(x-2)(x+6) = 0$$

Thus the critical points are $x = -6$, $x = 0$, and $x = 2$ (all of which are in $(-7, 3)$). We will use the second derivative test to classify these critical points.

$$g''(x) = 6(3x^2 + 8x - 12)$$

L-8

x	$g''(x)$	conclusion
-6	288	local minimum
0	-72	local maximum
2	96	local minimum

(b) We check the critical and endpoint values.

x	$g(x)$	reason for check
-7	-906	endpoint
-6	-1080	critical point ($f'(x) = 0$)
0	0	critical point ($f'(x) = 0$)
2	-56	critical point ($f'(x) = 0$)
3	13.5	endpoint

The maximum value of g on $[-7, 3]$ occurs at $x = 3$ and the minimum value occurs at $x = -6$.

Ex. M-10 **4.3/4.4** Sp20 **Exam**

Consider the function f and its first two derivatives below.

$$f(x) = \frac{99e^x}{(x-25)^{47}} + 98 \quad , \quad f'(x) = \frac{99e^x(x-72)}{(x-25)^{48}} \quad , \quad f''(x) = \frac{99e^x\left((x-72)^2+47\right)}{(x-25)^{49}}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

exs20

Solution **M-10**

equation(s) of vertical asymptote(s) of f	$x = 25$
equation(s) of horizontal asymptote(s) of f	$y = 98$
where f is decreasing	$(-\infty, 25)$, $(25, 72]$
where f is increasing	$[72, \infty)$
x-coordinate(s) of local minima of f	$x = 72$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	$(-\infty, 25)$
where f is concave up	$(25, \infty)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f(x) = \frac{99e^x}{(x-25)^{47}} + 98} \qquad \boxed{f'(x) = \frac{99e^x(x-72)}{(x-25)^{48}}} \qquad \boxed{f''(x) = \frac{99e^x\left((x-72)^2+47\right)}{(x-25)^{49}}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 25$. Hence our candidate vertical asymptote is the line $x = 25$. Indeed, direct substitution of $x = 25$ into the firs term of f gives the expression $\frac{99e^{25}}{0}$, which indicates that both one-sided limits are infinite. Hence the line $x = 25$ is a true vertical asymptote.

M-10

As for the horizontal asymptotes we compute the limits at infinite. For $x \to -\infty$, we have:

$$\lim_{x\to-\infty}\left(\frac{99e^x}{(x-25)^{47}}+98\right)=\frac{0}{-\infty}+98=98$$

For $x \to +\infty$, we first observe the following for any $n > 0$:

$$\lim_{x\to\infty}\underbrace{\left(\frac{e^x}{x^n}\right)}_{\frac{\infty}{\infty}}\overset{H}{=}\lim_{x\to\infty}\underbrace{\left(\frac{e^x}{nx^{n-1}}\right)}_{\frac{\infty}{\infty}}\overset{H}{=}\underbrace{\cdots}_{n\text{ uses of LR}}\overset{H}{=}\lim_{x\to\infty}\left(\frac{e^x}{n!}\right)=\frac{\infty}{n!}=\infty$$

Hence we now have the following:

$$\lim_{x\to+\infty}\left(\frac{99e^x}{(x-25)^{47}}+98\right)=\infty+98=\infty$$

So the only horizontal asymptote is $y = 98$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 72$) and the vertical asymptotes ($x = 25$).

interval	test point	sign of f'	shape of f
$(-\infty, 25)$	$f'(0)$	$\frac{99\oplus\ominus}{\oplus}=\ominus$	decreasing
$(25, 72)$	$f'(26)$	$\frac{99\oplus\ominus}{\oplus}=\ominus$	decreasing
$(72, \infty)$	$f'(73)$	$\frac{99\oplus\oplus}{\oplus}=\oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 25)$, $(25, 72]$
f is increasing on:	$[72, \infty)$
f has a local min at:	$x = 72$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes ($x = 25$).

interval	test point	sign of f''	shape of f
$(-\infty, 25)$	$f''(0)$	$\frac{99\oplus\oplus}{\ominus}=\oplus$	concave down
$(25, \infty)$	$f''(26)$	$\frac{99\oplus\oplus}{\oplus}=\oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 25)$
f is concave up on:	$(25, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

Ex. M-11 **4.3/4.4** Su20 **Exam**

Suppose f is continuous for all x and its first derivative is given by $f'(x) = (x-4)^2(x+2)$.

(a) Where is f decreasing?

(b) A student writes "since $f'(4) = 0$, there is a local extremum (either min or max) at $x = 4$". Is the student correct? Explain.

(c) Where is f concave up?

(d) Find the x-coordinate of each inflection point of f.

exsu20

Solution **M-11**

(a) We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = -2$ and $x = 4$) and where $f'(x)$ does not exist (none).

interval	test point	sign of f'	shape of f
$(-\infty, -2)$	$f'(-3)$	$\oplus\ominus = \ominus$	decreasing
$(-2, 4)$	$f'(0)$	$\oplus\oplus = \oplus$	increasing
$(4, \infty)$	$f'(5)$	$\oplus\oplus = \oplus$	increasing

Thus $f(x)$ is decreasing on $(-\infty, -2]$.

(b) The student is incorrect. In general, the vanishing of the derivative at $x = a$ is not sufficient for there to be a local extremum at $x = a$. There must also be a sign change in the derivative at $x = a$. Indeed, in this case we see that f is increasing on the interval $[-2, \infty)$, whence there is no local extremum at $x = 4$.

(c) We calculate a sign chart for the second derivative:

$$f''(x) = 2(x-4) \cdot 1 \cdot (x+2) + (x-4)^2 \cdot 1 = 3x(x-4)$$

The cut points are the solutions to $f''(x) = 0$ ($x = 0$ and $x = 4$) and where $f''(x)$ does not exist (nowhere).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\ominus\ominus = \oplus$	concave up
$(0, 4)$	$f''(1)$	$\oplus\ominus = \ominus$	concave down
$(4, \infty)$	$f''(5)$	$\oplus\oplus = \oplus$	concave up

Thus $f(x)$ is concave up on $(-\infty, 0]$ and $[4, \infty)$.

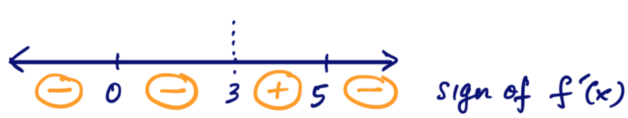
(d) There is an inflection point at both $x = 0$ and $x = 4$ (f is continuous and changes concavity at each of these points).

Ex. M-12 **4.3/4.4** Su20 **Exam**

Suppose $f(x)$ satisfies all of the following properties.

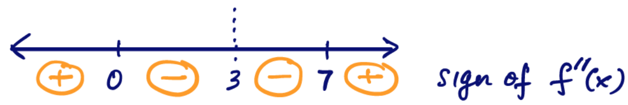
- $f(x)$ is continuous and differentiable on $(-\infty, 3) \cup (3, \infty)$
- $x = 3$ is a vertical asymptote of $f(x)$
- $\lim_{x\to\infty} f(x) = 1$
- the only x-values for which $f'(x) = 0$ are $x = 0$ and $x = 5$
- the only x-values for which $f''(x) = 0$ are $x = 0$ and $x = 7$

A sign chart for the first and second derivatives of f are given below.



exsu20

4.3/4.4 Su20 **Exam**



Use this information to answer the following questions about $f(x)$. ***Note:*** Do not attempt to find a formula for $f(x)$.

(a) Where is f increasing?

(b) Where is f concave down?

(c) At which x-value(s) does f have a local minimum?

(d) At which x-value(s) does f have a local maximum?

(e) Calculate $\lim_{x\to 3^+} f(x)$ or determine there is not enough information to do so.

(f) Calculate $\lim_{x\to -\infty} f(x)$ or determine there is not enough information to do so.

(g) Sketch a possible graph of $y = f(x)$. Clearly mark and label all of the following: local minima, local maxima, inflection points, vertical asymptotes, horizontal asymptotes. *Your graph does not have to be to scale, but the shape must be correct.*

exsu20

Solution **M-12**

(a) On the sign chart for f' we look for intervals where f' is non-negative. Hence f is increasing on $(3, 5]$.

(b) On the sign chart for f'' we look for intervals where f'' is non-positive. Hence f is concave down on $[0, 3)$ and $(3, 7]$.

(c) The first derivative of f never transitions from negative to positive at a point of continuity (f is discontinuous at $x = 3$). So there is no local minimum.

(d) The first derivative of f transitions from positive to negative at $x = 5$ (and f is continuous there). So there is a local maximum at $x = 5$.

(e) Since $x = 3$ is a vertical asymptote, we know that $\lim_{x\to 3^+} f(x)$ is infinite. Since f is increasing on $(3, 5]$, we must have $\lim_{x\to 3^+} f(x) = -\infty$. (This is also consistent with the negative concavity of f on $(3, 7]$.)

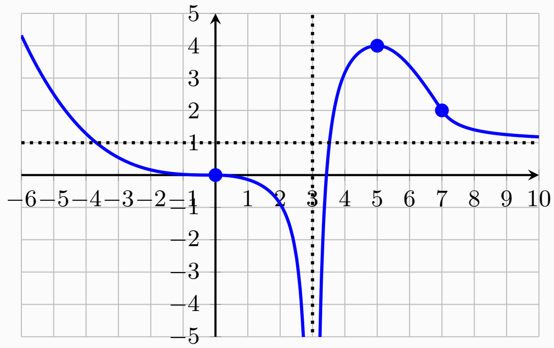
(f) If $\lim_{x\to -\infty} f(x) = L$ for some finite L, then there are three possibilities, all of which are inconsistent with the given information:

- **The graph of f approaches the asymptote $y = L$ from above.** Since f is differentiable this would imply that f would be increasing on an interval of the form $(-\infty, a]$. But f is decreasing on $(-\infty, 0]$.
- **The graph of f approaches the asymptote $y = L$ from below.** Since f is differentiable this would imply that f would have negative concavity on an interval of the form $(-\infty, a]$. But f is concave up on $(-\infty, 0]$.
- **The graph of f oscillates about the asymptote $y = L$.** Since f is differentiable, this would imply that f would have infinitely many local extrema in the interval $(-\infty, 0]$. But the only local extremum is at $x = 5$.

Since f is decreasing on $(-\infty, 0]$, it is also not possible that $\lim_{x\to -\infty} f(x) = -\infty$. Thus the only possibility left that is consistent with all of the given information is $\lim_{x\to -\infty} f(x) = \infty$.

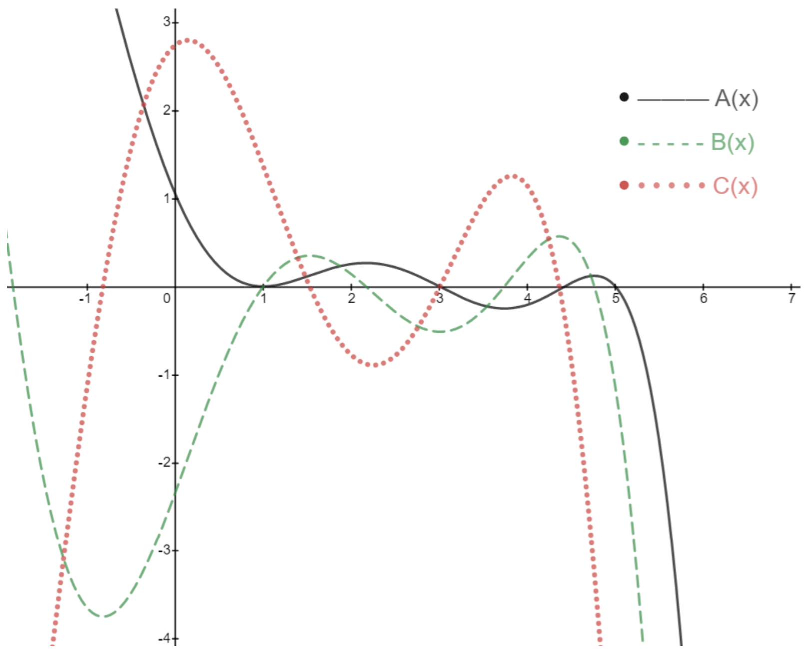
(g) One possibility is shown below.

M-12



Ex. M-13 4.3/4.4 Fa20 Exam

The figure below shows the graphs of f, f', and f''. Identify which graph is that of f''.



exf20

Solution M-13

$f''(x) = C(x)$. *Proof below.*

We can see that $A'(x) = B(x)$ and $B'(x) = C(x)$ by observing the locations of relative extrema and zeros. For instance, $B(x)$ has a zero wherever $A(x)$ has a relative extremum, and $C(x)$ has a zero wherever $B(x)$ has a relative extremum. (Strictly speaking, this is not enough to conlcude $A'(x) = B(x)$ and $B'(x) = C(x)$. However, we can also observe intervals of increase. For instance, $A(x)$ is decreasing wherever $B(x)$ is negative and $A(x)$ is increasing wherever $B(x)$ is positive. The same observation holds for $B(x)$ and $C(x)$.)

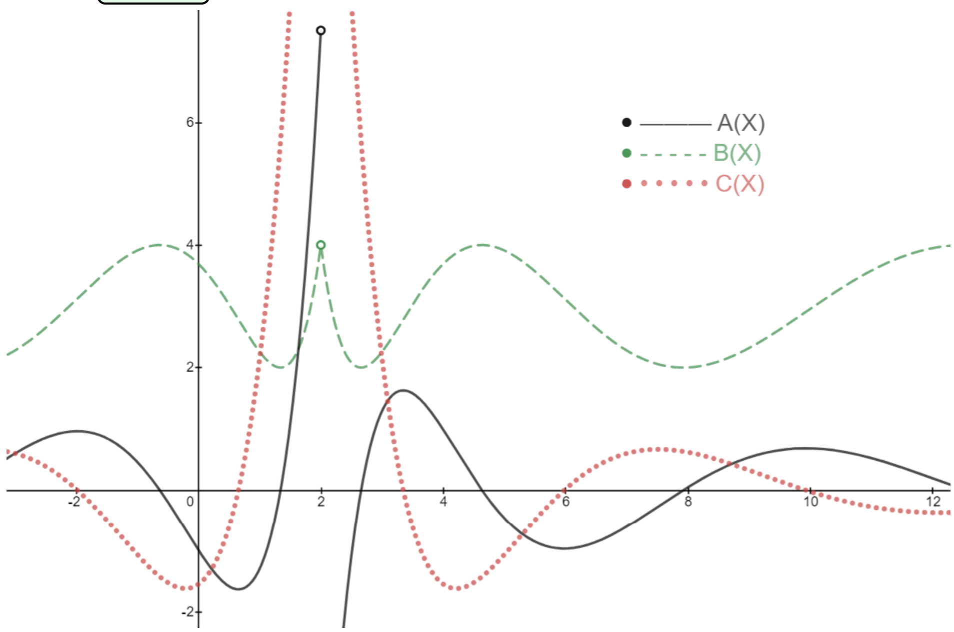
It follows that $A''(x) = C(x)$, and so $f'' = C$.

Ex. M-14 4.3/4.4 Fa20 Exam

The figure below shows the graphs of f, f', and f''. Identify which graph is which.

exf20

4.3/4.4 Fa20 Exam



exf20

Solution **M-14**

The only choice for $B(x)$ is $f(x)$ since B has a removable discontinuity at $x = 2$ but $A(x)$ and $C(x)$ do not. Now we simply observe the behavior near $x = 2$. Note that $B(x)$ is increasing on $(2-\epsilon, 2)$ and decreasing on $(2, 2+\epsilon)$ for some small $\epsilon > 0$. Hence $B'(x) > 0$ on $(2-\epsilon, 2)$ and $B'(x) < 0$ on $(2, 2+\epsilon)$. The only function with these signs is $A(x)$, whence $B'(x) = A(x)$. That leaves only $A'(x) = C(x)$, which we can again verify by a similar argument.

Hence $f = B$, $f' = A$, and $f'' = C$.

Ex. M-15 4.3/4.4 Fa20 Exam

Suppose $f(x)$ satisfies all of the following properties. Sketch a possible graph of $y = f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

Information from $f(x)$:

- $\lim_{x\to-\infty} f(x) = 1$
- $\lim_{x\to\infty} f(x) = 6$
- $x = -3$ is a vertical asymptote for f

Information from $f'(x)$:

- $f'(x) > 0$ on $(2, \infty)$
- $f'(x) < 0$ on $(-\infty, -3)$ and $(-3, 2)$
- $f'(2) = 0$

Information from $f''(x)$:

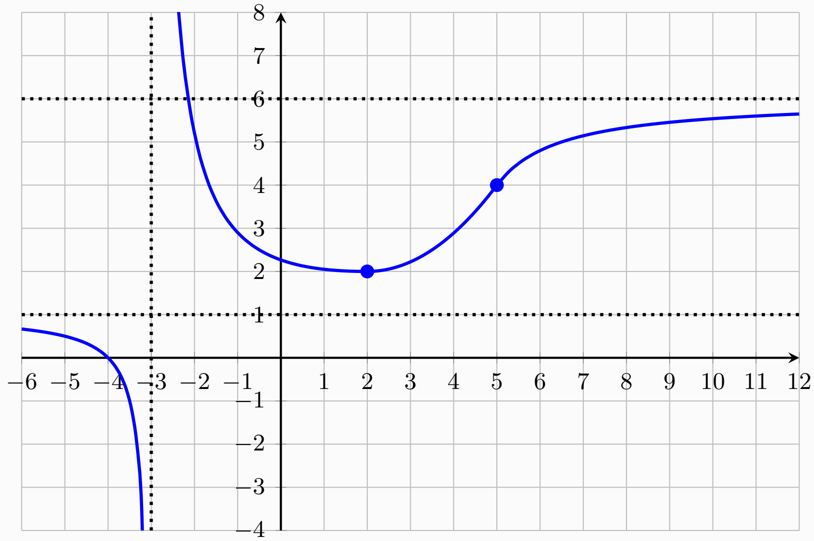
- $f''(x) > 0$ on $(-3, 5)$
- $f''(x) < 0$ on $(-\infty, -3)$ and $(5, \infty)$
- $f''(5) = 0$

exf20

Solution **M-15**

There is one relative minimum at $x = 2$ and one inflection point at $x = 5$. The lines $y = 1$ and $y = 6$ are both horizontal asymptotes. Here is one possibility for the graph.

M-15



Ex. M-16 4.3/4.4 Fa20 Exam

The first and second derivative of f are given below. You may assume that $f(x)$ has a vertical asymptote at $x = 25$ only, but do not attempt to calculate $f(x)$ explicitly.

$$f'(x) = \frac{(x+2)^{1/5}}{(x-25)^2} \quad , \quad f''(x) = \frac{-9(x+5)}{5(x-25)^3(x+2)^{4/5}}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

exf20

Solution M-16

where f is decreasing	$(-\infty, -2]$
where f is increasing	$[-2, 25)$, $(25, \infty)$
x-coordinate(s) of local minima of f	$x = -2$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	$(-\infty, -5]$, $(25, \infty)$
where f is concave up	$[-5, 25)$
x-coordinate(s) of inflection point(s) of f	$x = -5$

The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{(x+2)^{1/5}}{(x-25)^2}} \qquad \boxed{f''(x) = \frac{-9(x+5)}{5(x-25)^3(x+2)^{4/5}}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Not required since $f(x)$ is not given, but we are given that $x = 25$ is the only vertical asymptote of $f(x)$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = -2$) and where $f'(x)$ is undefined ($x = 25$).

M-16

interval	test point	sign of f'	shape of f
$(-\infty,-2)$	$f'(-3)$	$\frac{\ominus}{\oplus\oplus} = \ominus$	decreasing
$(-2,25)$	$f'(0)$	$\frac{\oplus}{\oplus\oplus} = \oplus$	increasing
$(25,\infty)$	$f'(30)$	$\frac{\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty,-2]$
f is increasing on:	$[-2,25)$, $(25,\infty)$
f has a local min at:	$x=-2$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x)=0$ ($x=-5$) and where $f''(x)$ is undefined ($x=-2$ and $x=25$).

interval	test point	sign of f''	shape of f
$(-\infty,-5)$	$f''(-6)$	$\frac{-9\ominus}{5\ominus\oplus} = \ominus$	concave down
$(-5,-2)$	$f''(-4)$	$\frac{-9\oplus}{5\ominus\oplus} = \oplus$	concave up
$(-2,25)$	$f''(0)$	$\frac{-9\oplus}{5\ominus\oplus} = \oplus$	concave up
$(25,\infty)$	$f''(30)$	$\frac{-9\oplus}{5\oplus\oplus} = \ominus$	concave down

Hence we deduce the following about f:

f is concave down on:	$(-\infty,-5]$, $(25,\infty)$
f is concave up on:	$[-5,25)$
f has an infl. point at:	$x=-5$

(iv) Sketch of graph.

Not required.

Ex. M-17 **4.3/4.4** Sp21 **Exam**

Consider the function $f(x)$ whose second derivative is given.

$$f''(x) = \frac{(x-2)^2(x-5)^3}{(x-9)^5}$$

You may assume the domain of $f(x)$ is $(-\infty,9)\cup(9,\infty)$.

Find where $f(x)$ is concave down, where $f(x)$ is concave up, and where $f(x)$ has an inflection point. Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

exs21

Solution M-17

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x)=0$ ($x=2$ and $x=5$) and where $f''(x)$ is undefined ($x=9$).

M-17

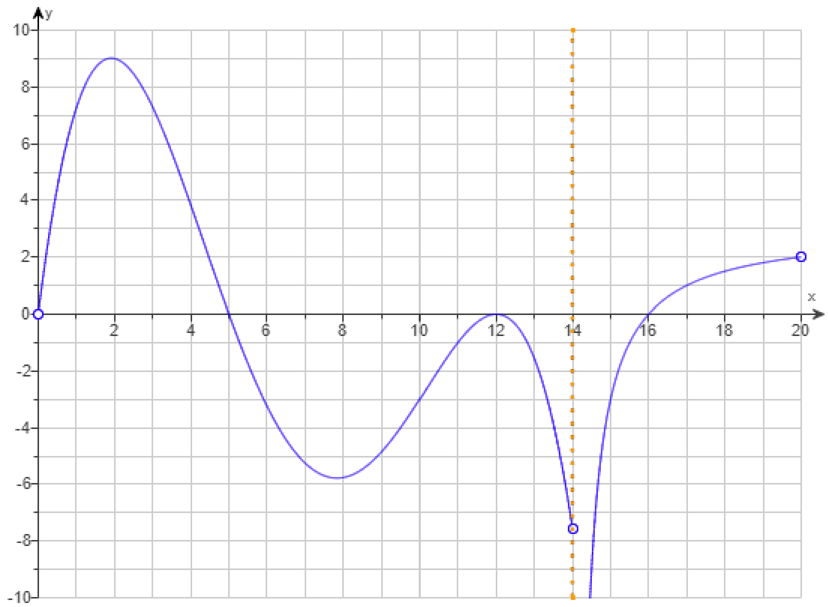
interval	test point	sign of f''	shape of f
$(-\infty, 2)$	$f''(0)$	$\frac{\oplus\ominus}{\ominus} = \oplus$	concave up
$(2, 5)$	$f''(3)$	$\frac{\oplus\ominus}{\ominus} = \oplus$	concave up
$(5, 9)$	$f''(6)$	$\frac{\oplus\oplus}{\ominus} = \ominus$	concave down
$(9, \infty)$	$f''(10)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $[5, 9)$
f is concave up on: $(-\infty, 5]$, $(9, \infty)$
f has an infl. point at: $x = 5$

Ex. M-18 4.3/4.4 Sp21 Exam

Use the graph of $y = f'(x)$ below to answer the questions. You may assume that $f'(x)$ has a vertical asymptote at $x = 14$ and that the domain of f is $(0, 14) \cup (14, 20)$.



Note: You are given a graph of the first derivative of f, not a graph of f.

(a) Find the critical points of f.

(b) Find where f is decreasing, where f is increasing, where f has a local minimum, and where f has a local maximum. Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

exs21

Solution M-18

(a) The critical points of f are $x = 5$ (since $f'(5) = 0$), $x = 12$ (since $f'(12) = 0$), and $x = 16$ (since $f'(16) = 0$).

(b) We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 5$, $x = 12$, and $x = 16$) and the vertical asymptotes ($x = 14$).

M-18

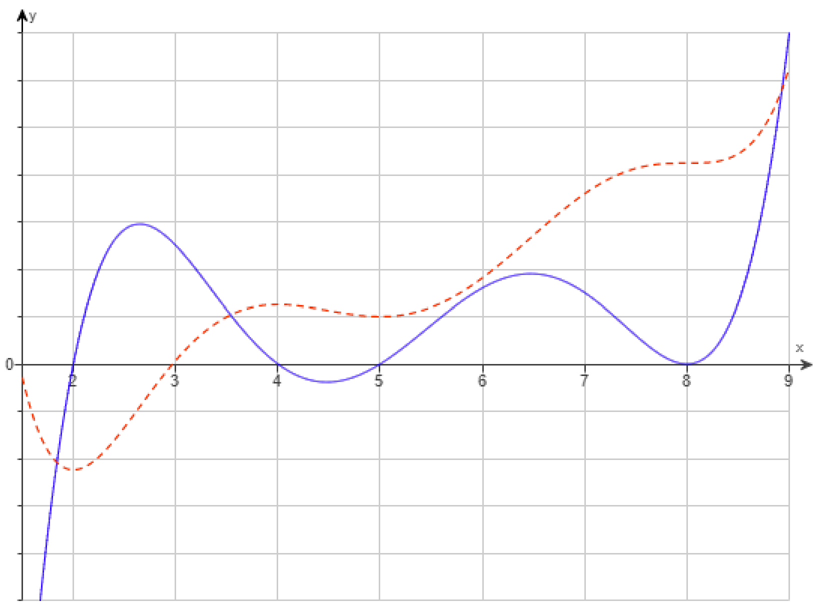
interval	test point	sign of f'	shape of f
$(0,5)$	$f'(1)$	$\oplus$	increasing
$(5,12)$	$f'(6)$	$\ominus$	decreasing
$(12,14)$	$f'(13)$	$\ominus$	decreasing
$(14,16)$	$f'(15)$	$\ominus$	decreasing
$(16,20)$	$f'(17)$	$\oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[5,14)$, $(14,16]$
f is increasing on:	$(0,5]$, $[16,20)$
f has a local min at:	$x=16$
f has a local max at:	$x=5$

Ex. M-19 **4.3/4.4** Sp21 **Exam**

The figure below shows the graphs of two functions. One function is $f(x)$ and the other is $f'(x)$, but you are not told which is which.



(a) Which graph is that of $y=f(x)$?

(b) Explain your answer to part (a) based on the behavior of the graphs at $x=4$ only.

(c) Explain your answer to part (a) based on the behavior of the graphs near $x=3.5$ only.

exs21

Solution M-19

(a) The dashed orange curve is the graph of $y=f(x)$.

(b) The dashed orange curve has a local maximum at $x=4$, whereas the blue solid graph crosses the x-axis from above to below (positive to negative values) at $x=4$. This is consistent only if the dashed orange curve is the graph of $y=f(x)$.

(c) At $x=3.5$, the dashed orange curve is increasing (so its derivative should be positive) and concave down (so its derivative should be decreasing). This is consistent only if the blue solid graph is, indeed, the graph of $y=f'(x)$.

Ex. M-20 **4.3/4.4** Fa21 **Exam**

Consider the function f and its derivatives below.

$$f(x) = \frac{x-3}{x^2-6x-16} \quad , \quad f'(x) = \frac{-(x-3)^2-25}{\left(x^2-6x-16\right)^2} \quad , \quad f''(x) = \frac{2(x-3)\left((x-3)^2+75\right)}{\left(x^2-6x-16\right)^3}$$

Find where f is concave down and where f is concave up; write your answers using interval notation. Also find the x-coordinate of each inflection point of f.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

exf21

Solution **M-20**

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ ($x = 3$) and the vertical asymptotes (solutions to $x^2 - 6x - 16 = 0$, or $x = -2$ and $x = 8$).

interval	test point	sign of f''	shape of f
$(-\infty, -2)$	$f''(-3)$	$\frac{2\ominus\oplus}{\oplus} = \ominus$	concave down
$(-2, 3)$	$f''(0)$	$\frac{2\ominus\oplus}{\ominus} = \oplus$	concave up
$(3, 8)$	$f''(4)$	$\frac{2\oplus\oplus}{\ominus} = \ominus$	concave down
$(8, \infty)$	$f''(9)$	$\frac{2\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 2)$, $[3, 8)$
f is concave up on:	$(-2, 3]$, $(8, \infty)$
f has an infl. point at:	$x = 3$

Ex. M-21 **4.3/4.4** Fa21 **Exam**

Suppose f is differentiable on $(-\infty, 1) \cup (1, \infty)$ and satisfies all of the following properties. Sketch a possible graph of $y = f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

(i) $\lim_{x\to-\infty} f(x) = -3$; $\lim_{x\to\infty} f(x) = \infty$; $\lim_{x\to 1^-} f(x) = -\infty$; $\lim_{x\to 1^+} f(x) = \infty$;

(ii) $f'(x) > 0$ on $(-\infty, -2)$ and $(5, \infty)$; $f'(x) < 0$ on $(-2, 1)$ and $(1, 5)$; $f'(-2) = f'(5) = 0$

(iii) $f''(x) > 0$ on $(-\infty, -7)$ and $(1, \infty)$; $f''(x) < 0$ on $(-7, 1)$; $f''(-7) = 0$

exf21

Solution **M-21**

The conditions can also be summarized as follows:

(i) The lines $y = -3$ and $x = 1$ are horizontal and vertical asymptotes for f, respectively. There is no horizontal asymptote at positive infinity.

(ii) f is increasing on $(-\infty, -2)$ and $(5, \infty)$; f is decreasing on $(-2, 1)$ and $(1, 5)$; there is a local minimum at $x = 5$; there is a local maximum at $x = -2$.

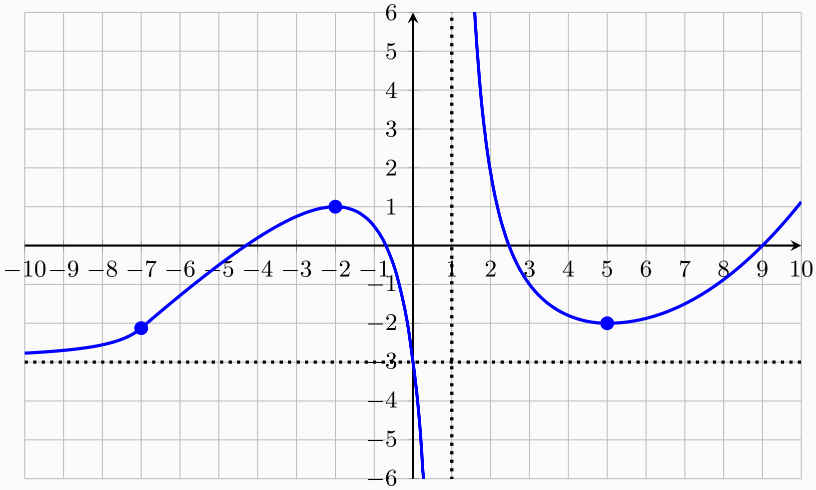
(iii) f is concave up on $(-\infty, -7)$ and $(1, \infty)$; f is concave down on $(-7, 1)$; there is an inflection point at $x = -7$.

The table below summarizes the behavior of f on each subinterval.

interval	behavior of f	notes
$(-\infty, -7)$	increasing, concave up	inflection point at $x = -7$
$(-7, -2)$	increasing, concave down	local maximum at $x = -2$
$(-2, 1)$	decreasing, concave down	vertical asymptote at $x = 1$
$(1, 5)$	decreasing, concave up	local minimum at $x = 6$
$(5, \infty)$	increasing, concave up	$f \to \infty$ as $x \to \infty$

There are many possible functions that satisfy these properties. Here is one.

M-21



Ex. M-22 4.3/4.4 Fa21 Exam

Let $f(x) = -e^{-x}\left(x^2 - 5x - 23\right)$. Find all critical points of f. Then find where f is decreasing and where f is increasing; write your answers using interval notation. Also find where relative extrema of f occur.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

exf21

Solution M-22

Since f is differentiable for all x, the only critical points are solutions to $f'(x) = 0$. Using product rule and chain rule gives

$$f'(x) = (-e^{-x} \cdot (-1))(x^2 - 5x - 23) + (-e^{-x})(2x - 5) = e^{-x}(x^2 - 7x - 18) = e^{-x}(x - 9)(x + 2)$$

Thus the critical points of f are $x = -2$ and $x = 9$. We now construct a sign chart to find the intervals of increase. (Recall that $e^{-x} > 0$ for all x.)

interval	test point	sign of f'	shape of f
$(-\infty, -2)$	$f'(-3) = \oplus\ominus\ominus$	$\oplus$	increasing
$(-2, 9)$	$f'(0) = \oplus\ominus\oplus$	$\ominus$	decreasing
$(9, \infty)$	$f'(10 = \oplus\oplus\oplus$	$\oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[-2, 9]$
f is increasing on:	$(-\infty, -2]$, $[9, \infty)$
f has a local min at:	$x = 9$
f has a local max at:	$x = -2$

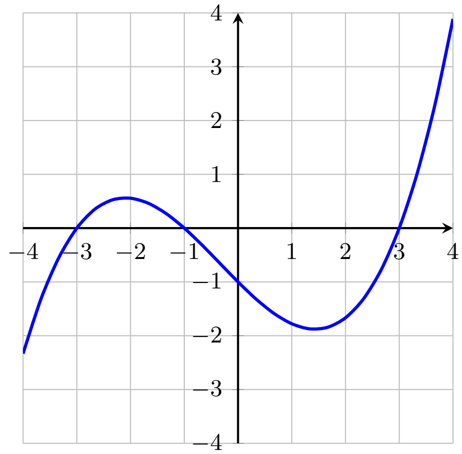
Ex. B-8 2.1/2.2, 3.7, 4.3/4.4 Fa21 Exam

For each part, use the graph of $y = g(x)$.

exf21

2.1/2.2, 3.7, 4.3/4.4

Fa21 **Exam**



(a) How many solutions does the equation $g'(x) = 0$ have?

(b) Order the following quantities from least to greatest: $g'(-2.5)$, $g'(-2)$, $g'(0)$, and $g'(4)$. In your answer, write these quantities symbolically; do not give a numerical estimate.

(c) What is the sign of $g''(-3)$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(d) Let $h(x) = g(x)^2$. What is the sign of $h'(-4)$(negative, positive, or zero)? If there is not enough information to determine the value, explain why.

exf21

Solution **B-8**

(a) The function g is differentiable for all x and has two local extrema (one local min and one local max). So $g'(x) = 0$ has two solutions.

(b) We note the following: $g'(-2.5)$ is small and positive, $g'(-2) = 0$, $g'(0)$ is small and negative, and $g'(4)$ is large and positive. Thus the correct order is: $g'(0)$, $g'(-2)$, $g'(-2.5)$, $g'(4)$.

(c) The function g is concave down in an interval containing $x = -3$. Thus $g''(-3)$ is positive.

(d) We have $h'(x) = 2g(x)g'(x)$, whence $h'(-4) = 2g(-4)g'(-4)$. Observe that $g(-4) < 0$ and $g'(-4) > 0$. Thus $h'(-4) < 0$.

Ex. L-18 **4.1, 4.3/4.4** Fa21 **Exam**

Let $f(x) = x^3(3x - 4)$.

(a) Find where relative extrema of f occur (if any). Classify each as a local minimum or a local maximum.

(b) Find the absolute extrema of f on $[-1, 2]$ and the x-values at which they occur.

exf21

Solution **L-18**

(a) We have $f(x) = 3x^4 - 4x^3$, whence $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. The critical points of f are $x = 0$ and $x = 1$. The derivative $f'(x)$ does not change sign at $x = 0$, whence there is no local extremum at $x = 0$. However, $f'(x)$ changes sign from negative to positive at $x = 1$, whence there is a local minimum at $x = 1$. (Alternatively, note that $f''(x) = 36x^2 - 24x$ and $f''(1) = 12 > 0$.)

(b) We need only compare the endpoint values and critical values: $f(-1) = 7$, $f(0) = 0$, $f(1) = -1$, and $f(2) = 16$. Hence the absolute minimum is -1 at $x = 1$, and the absolute maximum is 16 at $x = 2$.

Ex. M-23 **4.3/4.4** Fa21 **Exam**

Consider the function $g(x)$, whose first two derivatives are given below. ***Note:*** Do not attempt to calculate $g(x)$. Also assume that $g(x)$ has the same domain as $g'(x)$.

$$g'(x) = \frac{8x^{17}}{x - 32} \qquad g''(x) = \frac{128x^{16}(x - 34)}{(x - 32)^2}$$

Fill in the table below with information about the graph of $y = g(x)$. *For each part, write "NONE" as your answer*

exf21

4.3/4.4

Fa21 Exam

if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

You do not have to show work, and each table item will be graded with no partial credit.

exf21

Solution **M-23**

g is increasing on:	$(-\infty, 0]$, $(32, \infty)$
g is decreasing on:	$[0, 32)$
g is concave up on:	$[34, \infty)$
g is concave down on:	$(-\infty, 32)$, $(32, 34]$
x-coordinate(s) of relative maxima	$x = 0$
x-coordinate(s) of relative minima	NONE
x-coordinate(s) of inflection point(s)	$x = 34$

The first two derivatives of $f(x)$ are

$$\boxed{g'(x) = \frac{8x^{17}}{x-32}} \qquad \boxed{g''(x) = \frac{128x^{16}(x-34)}{(x-32)^2}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Not required since $g(x)$ is not given, but we note that the domain of $g'(x)$ is the same as that of $g(x)$, i.e., $(-\infty, 32) \cup (32, \infty)$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $g'(x)$. The cut points are the solutions to $g'(x) = 0$ ($x = 0$) and points not in the domain of $g(x)$ ($x = 32$).

interval	test point	sign of g'	shape of g
$(-\infty, 0)$	$g'(-1)$	$\frac{8\ominus}{\ominus} = \oplus$	increasing
$(0, 32)$	$g'(1)$	$\frac{8\oplus}{\ominus} = \ominus$	decreasing
$(32, \infty)$	$g'(33)$	$\frac{8\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about g:

g is decreasing on:	$[0, 32)$
g is increasing on:	$(-\infty, 0]$, $(32, \infty)$
g has a local min at:	*none*
g has a local max at:	$x = 0$

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $g''(x)$. The cut points are the solutions to $g''(x) = 0$ ($x = 0$ and $x = 34$) and points not in the domain of $g(x)$ ($x = 32$).

M-23

interval	test point	sign of g''	shape of g
$(-\infty, 0)$	$g''(-1)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(0, 32)$	$g'(1)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(32, 34)$	$g'(33)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(34, \infty)$	$g''(35)$	$\frac{128\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about g:

g is concave down on:	$(-\infty, 32)$, $(32, 34]$
g is concave up on:	$[34, \infty)$
g has an infl. point at:	$x = 34$

(iv) Sketch of graph.

Not required.

Ex. M-24 4.3/4.4 Sp22 Exam

Let $f(x) = 4x^5 - 20x^4 + 7x + 32$. Find where f is concave down and where f is concave up; write your answer using interval notation. Also find where inflection points of f occur.

exs22

Solution M-24

We first compute the second derivative of f.

$$f'(x) = 20x^4 - 80x^3 + 7$$
$$f''(x) = 80x^3 - 240x^2 = 80x^2(x-3)$$

We now calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ ($x = 0$ and $x = 3$).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\oplus\ominus = \ominus$	concave down
$(0, 3)$	$f''(1)$	$\oplus\ominus = \ominus$	concave down
$(3, \infty)$	$f''(4)$	$\oplus\oplus = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 3]$
f is concave up on:	$[3, \infty)$
f has an infl. point at:	$x = 3$

Ex. M-25 4.3/4.4 Sp22 Exam

Suppose $f(x)$ satisfies all of the following properties. Sign charts for f' and f'' are also given below. Sketch a possible graph of $y = f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

(i) f is continuous and differentiable on $(-\infty, 2) \cup (2, \infty)$

(ii) $\lim_{x\to-\infty} f(x) = \infty$; $\lim_{x\to\infty} f(x) = \infty$; $\lim_{x\to 2^-} f(x) = -\infty$; $\lim_{x\to 2^+} f(x) = \infty$

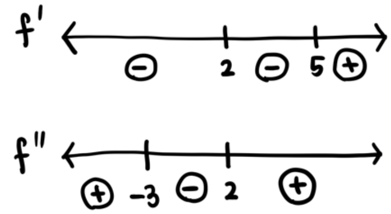
(iii) the only x-value for which $f'(x) = 0$ is $x = 5$

(iv) the only x-value for which $f''(x) = 0$ is $x = -3$

exs22

4.3/4.4

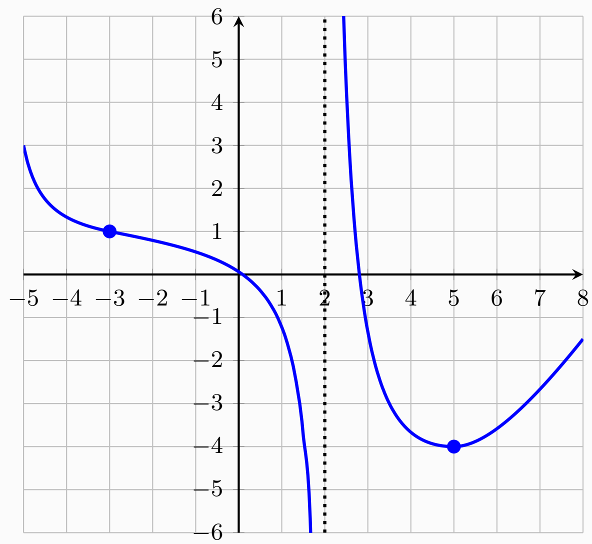
Sp22 Exam



exs22

Solution **M-25**

There are many possibilities. Here is one.



Ex. M-26 4.3/4.4 Sp22 Exam

Let $f(x) = \dfrac{x^2+21}{x-2}$. Find where f is decreasing and where f is increasing; write your answer using interval notation. Also find where the local extrema of f occur.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

exs22

Solution **M-26**

We first compute the first derivative of f.

$$f'(x) = \frac{2x(x-2) - (x^2+21)\cdot 1}{(x-2)^2} = \frac{x^2-4x-21}{(x-2)^2} = \frac{(x+3)(x-7)}{(x-2)^2}$$

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = -3$ and $x = 7$) and the vertical asymptotes ($x = 2$).

interval	test point	sign of f'	shape of f
$(-\infty, -3)$	$f'(-4)$	$\frac{\ominus\ominus}{\oplus} = \oplus$	increasing
$(-3, 2)$	$f'(0)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(2, 7)$	$f'(3)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(7, \infty)$	$f'(8)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[-3, 2)$, $(2, 7]$
f is increasing on:	$(-\infty, -3]$, $[7, \infty)$
f has a local min at:	$x = 7$
f has a local max at:	$x = -3$

Ex. L-21 **4.1, 4.3/4.4** Su22 **Exam**

Let $f(x) = Ax^B \ln(x)$, where A and B are unspecified constants. Suppose that $(e^5, 10)$ is a point of local extremum for $f(x)$.

(a) Calculate the values of A and B.

(b) Determine whether $(e^5, 10)$ is a point of local minimum or a point of local maximum for $f(x)$. Explain your answer.

exsu22

Solution **L-21**

(a) Since the point $(e^5, 10)$ lies on the graph of f, we must have $f(e^5) = 10$. Since the point $(e^5, 10)$ is a point of local extremum for f, we must have that $x = e^5$ is a critical point of f, whence $f'(e^5) = 0$. So A and B must simultaneously satisfy the equations:

$$f(e^5) = 10 \qquad f'(e^5) = 0$$

The derivative of f is:

$$f'(x) = ABx^{B-1}\ln(x) + Ax^B \cdot \frac{1}{x} = ABx^{B-1}\ln(x) + Ax^{B-1} = Ax^{B-1}\left(B\ln(x) + 1\right)$$

So our system of equations is:

$$5Ae^{5B} = 10 \qquad Ae^{5(B-1)}\left(5B + 1\right) = 0$$

The second equation above gives either $A = 0$ (which can't satisfy the first equation, and thus is not a valid solution) or $5B + 1 = 0$. Thus $B = -\frac{1}{5}$. Substituting $B = -\frac{1}{5}$ and solving for A gives:

$$5Ae^{5B} = 10 \implies 5Ae^{-1} = 10 \implies A = 2e$$

(b) From part (a), we now have f and f':

$$f(x) = 2ex^{-1/5}\ln(x) \qquad f'(x) = 2ex^{-6/5}\left(-\frac{1}{5}\ln(x) + 1\right)$$

To determine the nature of the local extremum, we use the first derivative test. The only critical point of f is $x = e^5$, so our sign chart for $f'(x)$ has two intervals to test: $(0, e^5)$, for which we can choose e^4 as a test point; and (e^5, ∞), for which we can choose e^6 as a test point. We have the following:

$$f'(e^4) = 2e \cdot e^{-24/5}\left(-\frac{1}{5} \cdot 4 + 1\right) = \oplus \cdot \left(\frac{1}{5}\right) = \oplus$$

$$f'(e^6) = 2e \cdot e^{-26/5}\left(-\frac{1}{5} \cdot 6 + 1\right) = \oplus \cdot \left(-\frac{1}{5}\right) = \ominus$$

Thus we see that f is increasing on the interval $(0, e^5]$ and decreasing on the interval $[e^5, \infty)$. Thus $x = e^5$ gives rise to a local maximum of f.

Ex. L-22 **4.1, 4.3/4.4** Su22 **Exam**

For each part, find the absolute extreme values of the given function on the given interval. If a particular extreme value does not exist, write "DNE" as your answer, and explain why that extreme value does not exist.

(a) $f(x) = \dfrac{e}{x} + \ln(x)$ on $[1, e^3]$

(b) $g(x) = 12x - x^3$ on $[0, \infty)$

exsu22

Solution **L-22**

(a) We first find the critical points by solving $f'(x) = 0$.

$$f'(x) = -\frac{e}{x^2} + \frac{1}{x} = 0 \implies -e + x = 0 \implies x = e$$

Now we compare the endpoint values and critical value.

$$f(1) = \frac{e}{1} + 0 = e \qquad f(e) = \frac{e}{e} + 1 = 2 \qquad f(e^3) = \frac{e}{e^3} + 3 = \frac{1}{e^2} + 3$$

L-22

(Recall that $2 < e < 3$.) Thus the absolute minimum of f is 2 and the absolute maximum of f is $\frac{1}{e^2} + 3$.

(b) We first find the critical points by solving $f'(x) = 0$.

$$f'(x) = 12 - 3x^2 = 0 \Longrightarrow x^2 = 4 \Longrightarrow x = 2$$

(Note that we reject the solution $x = -2$ since it's not in the given interval.) We can't use the extreme value theorem here because the given interval is not bounded.

Observe that $f''(x) = -6x$, whence $f''(2) < 0$. So $x = 2$ gives a local maximum of f on $[0, \infty)$. Since $x = 2$ is the only critical point on this interval, $x = 2$ gives an absolute maximum, and so the absolute maximum of f is $f(2) = 24 - 8 = 16$. However, since $\lim\limits_{x\to\infty} f(x) = -\infty$, there is no absolute minimum.

Ex. M-27 **4.3/4.4** Su22 **Exam**

Consider the function f and its derivatives below.

$$f(x) = \frac{x^2}{x-7} \qquad f'(x) = \frac{x(x-14)}{(x-7)^2} \qquad f''(x) = \frac{98}{(x-7)^3}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

exsu22

Solution M-27

equation(s) of vertical asymptote(s) of f	$x = 7$
equation(s) of horizontal asymptote(s) of f	NONE
where f is decreasing	$[0, 7)$, $(7, 14]$
where f is increasing	$(-\infty, 0]$, $[14, \infty)$
x-coordinate(s) of local minima of f	$x = 14$
x-coordinate(s) of local maxima of f	$x = 0$
where f is concave down	$(-\infty, 7)$
where f is concave up	$(7, \infty)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$f(x) = \frac{x^2}{x-7} \qquad f'(x) = \frac{x(x-14)}{(x-7)^2} \qquad f''(x) = \frac{98}{(x-7)^3}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 7$. Hence our candidate vertical asymptote is the line $x = 7$. Indeed, direct substitution of $x = 7$ into $f(x)$ gives the expression $\frac{49}{0}$, which indicates that both one-sided limits are infinite. Hence the line $x = 7$ is a true vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x\pm\infty} \left(\frac{x^2}{x-7}\right) = \lim_{x\pm\infty} \left(\frac{x}{1 - \frac{7}{x}}\right) = \frac{\pm\infty}{1-0} = \pm\infty$$

Since neither limit (as either $x \to -\infty$ or $x \to \infty$) is finite, there are no horizontal asymptotes.

M-27

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 0$ and $x = 14$) and the vertical asymptotes ($x = 7$).

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{\ominus\ominus}{\oplus} = \oplus$	increasing
$(0, 7)$	$f'(1)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(7, 14)$	$f'(8)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(14, \infty)$	$f'(15)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[0, 7)$, $(7, 14]$
f is increasing on:	$(-\infty, 0]$, $[14, \infty)$
f has a local min at:	$x = 14$
f has a local max at:	$x = 0$

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes ($x = 7$).

interval	test point	sign of f''	shape of f
$(-\infty, 7)$	$f''(0)$	$\frac{\oplus}{\ominus} = \ominus$	concave down
$(7, \infty)$	$f''(8)$	$\frac{\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 7)$
f is concave up on:	$(7, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

Ex. M-28 4.3/4.4, 4.7 Su22 Exam

Let $f(x) = x^2 e^x$.

(a) Calculate the vertical and horizontal asymptotes of f.

(b) Calculate the critical points of f. Then use the Second Derivative Test to classify each critical point of f as a local minimum or a local maximum. Show your work and label your answers clearly. ***Hint:*** The second derivative of f is $f''(x) = (x^2 + 4x + 2)e^x$.

exsu22

Solution M-28

(a) Since f is a product of functions that are continuous for all x, f is also continuous for all x, and thus f has no vertical asymptotes. For horizontal asymptotes, we have the following (use l'Hospital's rule on the limit at negative infinity):

$$\lim_{x\to\infty} (x^2 e^x) = (+\infty) \cdot (+\infty) = +\infty$$

$$\lim_{x\to-\infty} (x^2 e^x) = \lim_{x\to-\infty} \left(\frac{x^2}{e^{-x}}\right) \stackrel{H}{=} \lim_{x\to-\infty} \left(\frac{2x}{-e^{-x}}\right) \stackrel{H}{=} \lim_{x\to-\infty} \left(\frac{2}{e^{-x}}\right) = \frac{2}{\infty} = 0$$

Thus the only horizontal asymptote of f is $y = 0$.

M-28

(b) We first compute $f'(x)$.

$$f'(x) = 2xe^x + x^2e^x = xe^x(2+x)$$

Thus the critical points (solutions to $f'(x) = 0$) are $x = 0$ and $x = -2$. Now we use the Second Derivative Test.

$$f''(0) = (x^2 + 4x + 2)e^x\Big|_{x=0} = 2$$
$$f''(-2) = (x^2 + 4x + 2)e^x\Big|_{x=-2} = -2e^{-2}$$

Since $f''(0) > 0$, $x = 0$ gives a local minimum of f. Since $f''(-2) < 0$, $x = -2$ gives a local maximum of f.

Ex. M-29 **4.3/4.4** Su22 **Quiz**

Consider the function g and its derivatives below.

$$g(x) = x^2 - \frac{27}{x} \qquad g'(x) = 2x + \frac{27}{x^2} \qquad g''(x) = 2 - \frac{54}{x^3}$$

Fill in the table below with information about the graph of $y = g(x)$. For each part, write "NONE" as your answer if appropriate. (You may use the bottom or back of this page for scratch work.) ***You do not have to show work, and each part of the table will be graded with no partial credit.***

qsu22

Solution M-29

vertical asymptote(s) of g:	$x = 0$
horizontal asymptote(s) of g:	NONE
g is increasing on:	$[-\sqrt[3]{13.5}, 0)$, $(0, \infty)$
g is decreasing on:	$(-\infty, -\sqrt[3]{13.5})$
g is concave up on:	$(-\infty, 0)$, $[3, \infty)$
g is concave down on:	$(0, 3]$
x-coordinate(s) of relative maxima	NONE
x-coordinate(s) of relative minima	$x = -\sqrt[3]{13.5}$
x-coordinate(s) of inflection point(s)	$x = 3$

The first two derivatives of $f(x)$ are

$$\boxed{g(x) = \frac{x^3 - 27}{x}} \qquad \boxed{g'(x) = \frac{2(x^3 + 13.5)}{x^2}} \qquad \boxed{g''(x) = \frac{2(x^3 - 27)}{x^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that g is continuous on its domain, but is undefined for $x = 0$. Hence our candidate vertical asymptote is the line $x = 0$. Indeed, direct substitution of $x = 0$ into the term $\frac{27}{x}$ gives the expression $\frac{27}{0}$, which indicates that both one-sided limits are infinite, whence $x = 0$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x \pm \infty} \left(x^2 - \frac{27}{x} \right) = \infty - 0 = \infty$$

M-29

Since neither limit (as either $x \to -\infty$ or $x \to \infty$) is finite, there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $g'(x)$. The cut points are the solutions to $g'(x) = 0$ ($x = -\sqrt[3]{13.5}$) and vertical asymptotes of $g(x)$ ($x = 0$).

interval	test point	sign of g'	shape of g
$(-\infty, -\sqrt[3]{13.5})$	$g'(-3)$	$\frac{2\ominus}{\oplus} = \ominus$	decreasing
$(-\sqrt[3]{13.5}, 0)$	$g'(-1)$	$\frac{2\oplus}{\oplus} = \ominus$	increasing
$(0, \infty)$	$g'(1)$	$\frac{2\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about g:

g is decreasing on:	$(-\infty, -\sqrt[3]{13.5})$
g is increasing on:	$[-\sqrt[3]{13.5}, 0)$, $(0, \infty)$
g has a local min at:	$x = -\sqrt[3]{13.5}$
g has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $g''(x)$. The cut points are the solutions to $g''(x) = 0$ ($x = 3$) and vertical asymptotes of $g(x)$ ($x = 0$).

interval	test point	sign of g''	shape of g
$(-\infty, 0)$	$g''(-1)$	$\frac{2\ominus}{\ominus} = \oplus$	concave up
$(0, 3)$	$g'(1)$	$\frac{2\ominus}{\oplus} = \ominus$	concave down
$(3, \infty)$	$g''(4)$	$\frac{2\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about g:

g is concave down on:	$(0, 3]$
g is concave up on:	$(-\infty, 0)$, $[3, \infty)$
g has an infl. point at:	$x = 3$

(iv) Sketch of graph.

Not required.

Ex. M-30 4.3/4.4 Fa22 Quiz

Consider the function f and its derivatives below.

$$f(x) = \frac{x^4}{3-x} \qquad f'(x) = \frac{x^3(12-3x)}{(3-x)^2} \qquad f''(x) = \frac{6x^2((x-4)^2+2)}{(3-x)^3}$$

Fill in the table below with information about the graph of $y = f(x)$. Write your answers using interval notation if appropriate. For each part, write "NONE" as your answer if appropriate.

qf22

Solution **M-30**

vertical asymptote(s) of f	$x = 3$
horizontal asymptote(s) of f	NONE
where f is decreasing	$(-\infty, 0]$, $[4, \infty)$
where f is increasing	$[0, 3)$,$(3, 4]$
x-coordinate(s) of local minima of f	$x = 0$
x-coordinate(s) of local maxima of f	$x = 4$
where f is concave down	$(3, \infty)$
where f is concave up	$(-\infty, 3)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f(x) = \frac{x^4}{3-x}} \qquad \boxed{f'(x) = \frac{x^3(12-3x)}{(3-x)^2}} \qquad \boxed{f''(x) = \frac{6x^2((x-4)^2+2)}{(3-x)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 3$. Hence our candidate vertical asymptote is the line $x = 3$. Indeed, direct substitution of $x = 3$ into f gives the expression $\frac{81}{0}$, which indicates that both one-sided limits are infinite, whence $x = 3$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x \pm\infty} \left(\frac{x^4}{3-x}\right) = \lim_{x \pm\infty} \left(\frac{x^3}{\frac{3}{x} - 1}\right) = \frac{\pm\infty}{-1} = \mp\infty$$

Since neither limit (as either $x \to -\infty$ or $x \to \infty$) is finite, there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = 0$ and $x = 4$) and the vertical asymptotes of $f(x)$ ($x = 3$).

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{\ominus\oplus}{\oplus} = \ominus$	decreasing
$(0, 3)$	$f'(1)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing
$(3, 4)$	$f'(3.5)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing
$(4, \infty)$	$f'(5)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 0]$, $[4, \infty)$
f is increasing on:	$[0, 3)$, $(3, 4]$
f has a local min at:	$x = 0$
f has a local max at:	$x = 4$

M-30

(iii) **Intervals of concavity and inflection points.**

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 0$) and the vertical asymptotes of $f(x)$ ($x = 3$).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\frac{6\oplus\oplus}{\oplus} = \oplus$	concave up
$(0, 3)$	$f'(1)$	$\frac{6\oplus\oplus}{\oplus} = \oplus$	concave up
$(3, \infty)$	$f''(4)$	$\frac{6\oplus\oplus}{\ominus} = \ominus$	concave down

Hence we deduce the following about f:

f is concave down on:	$(3, \infty)$
f is concave up on:	$(-\infty, 3)$
f has an infl. point at:	$x = 0$

(iv) **Sketch of graph.**

Not required.

Ex. M-31 4.3/4.4

For each part, do all of the following.

(i) Find all vertical asymptotes and horizontal asymptotes of $f(x)$.

(ii) Find where $f(x)$ is decreasing and where $f(x)$ is increasing. Also find and classify all local extrema of $f(x)$.

(iii) Find where $f(x)$ is concave down and where $f(x)$ is concave up. Also find all inflection points of $f(x)$.

(iv) Sketch a graph of $y = f(x)$.

(a) $f(x) = \frac{1}{3}x^3 - 9x + 2$

(b) $f(x) = (x+1)^2(x-5)$

(c) $f(x) = \dfrac{x}{x^2+1}$

(d) $f(x) = x - \sin(2x)$ (on $[0, \pi]$ only)

(e) $f(x) = 1 + 2x + 18x^{-1}$

(f) $f(x) = 1 - \dfrac{x}{4-x}$

(g) $f(x) = 10x^3 - x^5$

(h) $f(x) = \dfrac{1}{x^3+8}$

(i) $f(x) = \dfrac{x^3}{x-1}$

(j) $f(x) = \dfrac{1}{x^3-3x}$

c4s3

Solution M-31

(a) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = x^2 - 9 = (x-3)(x+3)} \qquad \boxed{f''(x) = 2x}$$

(i) **Vertical asymptotes and horizontal asymptotes.**

Since $f(x)$ is a polynomial, there are no vertical or horizontal asymptotes.

(ii) **Intervals of increase and local extrema.**

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -3$ and $x = 3$).

interval	test point	sign of f'	shape of f
$(-\infty, -3)$	$f'(-4)$	$\ominus\ominus = \oplus$	increasing
$(-3, 3)$	$f'(0)$	$\ominus\oplus = \ominus$	decreasing
$(3, \infty)$	$f'(4)$	$\oplus\oplus = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[-3, 3]$
f is increasing on:	$(-\infty, -3]$, $[3, \infty)$
f has a local min at:	$x = 3$
f has a local max at:	$x = -3$

M-31

(iii) Intervals of concavity and inflection points.

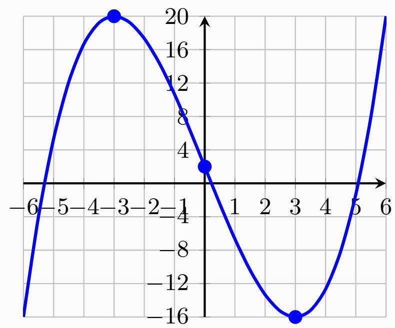
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 0$ only).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\ominus$	concave down
$(0, \infty)$	$f''(1)$	$\oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $(-\infty, 0]$
f is concave up on: $[0, \infty)$
f has an infl. point at: $x = 0$

(iv) Sketch of graph.



(b) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = 3(x+1)(x-3)} \qquad \boxed{f''(x) = 6(x-1)}$$

(i) Vertical asymptotes and horizontal asymptotes.

Since $f(x)$ is a polynomial, there are no vertical or horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -1$ and $x = 3$).

interval	test point	sign of f'	shape of f
$(-\infty, -1)$	$f'(-2)$	$3\ominus\ominus = \oplus$	increasing
$(-1, 3)$	$f'(0)$	$3\ominus\oplus = \ominus$	decreasing
$(3, \infty)$	$f'(4)$	$3\oplus\oplus = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on: $[-1, 3]$
f is increasing on: $(-\infty, -1]$, $[3, \infty)$
f has a local min at: $x = 3$
f has a local max at: $x = -1$

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 1$ only).

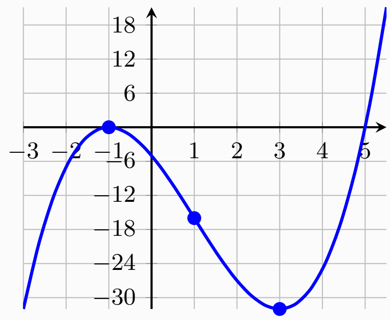
interval	test point	sign of f''	shape of f
$(-\infty, 1)$	$f''(0)$	$6\ominus = \ominus$	concave down
$(1, \infty)$	$f''(2)$	$6\oplus = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $(-\infty, 1]$
f is concave up on: $[1, \infty)$
f has an infl. point at: $x = 1$

M-31

(iv) Sketch of graph.



(c) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{1-x^2}{(x^2+1)^2}} \qquad \boxed{f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Since $f(x)$ is continuous for all x, there are no vertical asymptotes. For the horizontal asymptotes, we have:

$$\lim_{x\to\pm\infty}\left(\frac{x}{x^2+1}\right) = \lim_{x\to\pm\infty}\left(\frac{\frac{1}{x}}{1+\frac{1}{x^2}}\right) = \frac{0}{1+0} = 0$$

So the only horizontal asymptote is $y = 0$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -1$ and $x = 1$).

interval	test point	sign of f'	shape of f
$(-\infty,-1)$	$f'(-2)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(-1,1)$	$f'(0)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(1,\infty)$	$f'(2)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on: $(-\infty,-1]$, $[1,\infty)$
f is increasing on: $[-1,1]$
f has a local min at: $x=-1$
f has a local max at: $x=1$

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 0$, $x = -\sqrt{3}$, and $x = \sqrt{3}$).

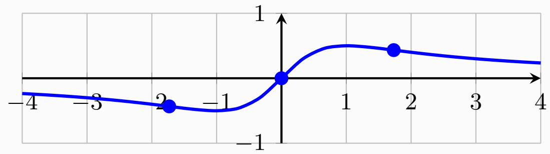
interval	test point	sign of f''	shape of f
$(-\infty,-\sqrt{3})$	$f''(-2)$	$\frac{\ominus\oplus}{\oplus} = \ominus$	concave down
$(-\sqrt{3},0)$	$f''(-1)$	$\frac{\ominus\ominus}{\oplus} = \oplus$	concave up
$(0,\sqrt{3})$	$f''(1)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	concave down
$(\sqrt{3},\infty)$	$f''(2)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $(-\infty,-\sqrt{3}]$, $[0,\sqrt{3}]$
f is concave up on: $[-\sqrt{3},0]$, $[\sqrt{3},\infty)$
f has an infl. point at: $x=-\sqrt{3}$, $x=0$, $x=\sqrt{3}$

M-31

(iv) Sketch of graph.



(d) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = 1 - 2\cos(2x)} \qquad \boxed{f''(x) = 4\sin(2x)}$$

(i) Vertical asymptotes and horizontal asymptotes.

Since $f(x)$ is continuous for all x, there are no vertical asymptotes. Since the domain of f is bounded, it makes no sense to compute the limit of f as $x \to \pm\infty$, so there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ in $[0, \pi]$ ($x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$).

interval	test point	sign of f'	shape of f
$[0, \frac{\pi}{6})$	$f'(0)$	$1 - 2 = -1\ominus$	decreasing
$(\frac{\pi}{6}, \frac{5\pi}{6})$	$f'(\frac{\pi}{2})$	$1 - (-2) = 3 = \oplus$	increasing
$(\frac{5\pi}{6}, \pi]$	$f'(\pi)$	$1 - 2 = -1 = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$[0, \frac{\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$
f is increasing on:	$[\frac{\pi}{6}, \frac{5\pi}{6}]$
f has a local min at:	$x = \frac{\pi}{6}$
f has a local max at:	$x = \frac{5\pi}{6}$

(iii) Intervals of concavity and inflection points.

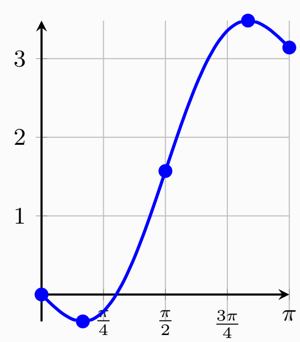
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = \frac{\pi}{2}$ only).

interval	test point	sign of f''	shape of f
$[0, \frac{\pi}{2})$	$f''(\frac{\pi}{4})$	$4 \cdot 1 = \oplus$	concave up
$(\frac{\pi}{2}, \pi]$	$f''(\frac{3\pi}{4})$	$4 \cdot (-1) = \ominus$	concave down

Hence we deduce the following about f:

f is concave down on:	$[\frac{\pi}{2}, \pi]$
f is concave up on:	$[0, \frac{\pi}{2}]$
f has an infl. point at:	$x = \frac{\pi}{2}$

(iv) Sketch of graph.



M-31

(e) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = 2 - \frac{18}{x^2} = \frac{2(x-3)(x+3)}{x^2}} \qquad \boxed{f''(x) = \frac{36}{x^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 0$. Hence our candidate vertical asymptote is the line $x = 0$. Indeed, direct substitution of $x = 0$ into the term $\frac{18}{x}$ gives the expression $\frac{18}{0}$, which indicates that both one-sided limits are infinite, whence $x = 0$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x \pm\infty}\left(1 + 2x + \frac{18}{x}\right) = 1 + 2(\pm\infty) + 0 = \pm\infty$$

Since neither limit (as either $x \to -\infty$ or $x \to \infty$) is finite, there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -3$ and $x = 3$) and the vertical asymptotes of $f(x)$ ($x = 0$).

interval	test point	sign of f'	shape of f
$(-\infty, -3)$	$f'(-4)$	$\frac{2\ominus\ominus}{\oplus} = \oplus$	increasing
$(-3, 0)$	$f'(-1)$	$\frac{2\ominus\oplus}{\oplus} = \ominus$	decreasing
$(0, 3)$	$f'(1)$	$\frac{2\ominus\oplus}{\oplus} = \ominus$	decreasing
$(3, \infty)$	$f'(4)$	$\frac{2\oplus\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on: $[-3, 0)$, $(0, 3]$
f is increasing on: $(-\infty, -3]$, $[3, \infty)$
f has a local min at: $x = 3$
f has a local max at: $x = -3$

(iii) Intervals of concavity and inflection points.

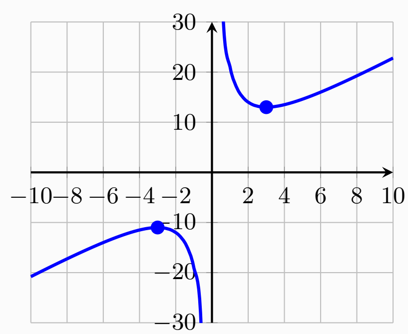
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = 0$).

interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\frac{36}{\ominus} = \ominus$	concave down
$(0, \infty)$	$f''(1)$	$\frac{36}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $(-\infty, 0)$
f is concave up on: $(0, \infty)$
f has an infl. point at: none

(iv) Sketch of graph.



M-31

(f) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{-4}{(x-4)^2} = \frac{2(x-3)(x+3)}{x^2}} \qquad \boxed{f''(x) = \frac{8}{(x-4)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 4$. Hence our candidate vertical asymptote is the line $x = 4$. Indeed, direct substitution of $x = 4$ into the second term of f gives the expression $\frac{4}{0}$, which indicates that both one-sided limits are infinite, whence $x = 4$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x\to\pm\infty}\left(1 - \frac{x}{4-x}\right) = \lim_{x\to\pm\infty}\left(\frac{4-2x}{4-x}\right) \overset{H}{=} \lim_{x\to\pm\infty}\left(\frac{-2}{-1}\right) = 2$$

Hence the only horizontal asymptote is $y = 2$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = 4$).

interval	test point	sign of f'	shape of f
$(-\infty, 4)$	$f'(0)$	$\frac{-4}{\oplus} = \ominus$	decreasing
$(4, \infty)$	$f'(5)$	$\frac{-4}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 4)$, $(4, \infty)$
f is increasing on:	no interval
f has a local min at:	none
f has a local max at:	none

(iii) Intervals of concavity and inflection points.

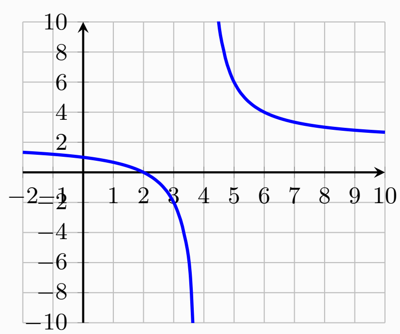
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = 4$).

interval	test point	sign of f''	shape of f
$(-\infty, 4)$	$f''(0)$	$\frac{8}{\ominus} = \ominus$	concave down
$(4, \infty)$	$f''(5)$	$\frac{8}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 4)$
f is concave up on:	$(4, \infty)$
f has an infl. point at:	none

(iv) Sketch of graph.



M-31

(g) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = 5x^2(6 - x^2)} \qquad \boxed{f''(x) = 20x(3 - x^2)}$$

(i) Vertical asymptotes and horizontal asymptotes.

Since $f(x)$ is a polynomial, there are no vertical or horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -\sqrt{6}$, $x = 0$, and $x = \sqrt{6}$).

interval	test point	sign of f'	shape of f
$(-\infty, -\sqrt{6})$	$f'(-3)$	$5\oplus\ominus = \ominus$	decreasing
$(-\sqrt{6}, 0)$	$f'(-1)$	$5\oplus\oplus = \oplus$	increasing
$(0, \sqrt{6})$	$f'(1)$	$5\oplus\oplus = \oplus$	increasing
$(\sqrt{6}, \infty)$	$f'(3)$	$5\oplus\ominus = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on: $(-\infty, -\sqrt{6}]$, $[\sqrt{6}, \infty)$
f is increasing on: $[-\sqrt{6}, \sqrt{6}]$
f has a local min at: $x = -\sqrt{6}$
f has a local max at: $x = \sqrt{6}$

(iii) Intervals of concavity and inflection points.

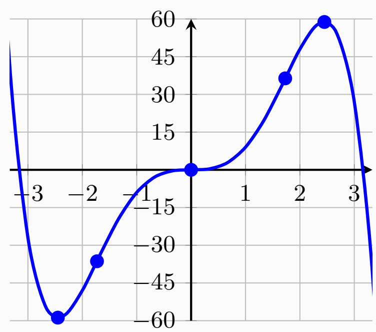
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$).

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt{3})$	$f''(-3)$	$20\ominus\ominus = \oplus$	concave up
$(-\sqrt{3}, 0)$	$f''(-1)$	$20\ominus\oplus = \ominus$	concave down
$(0, \sqrt{3})$	$f''(1)$	$20\oplus\oplus = \oplus$	concave up
$(\sqrt{3}, \infty)$	$f''(3)$	$20\oplus\ominus = \ominus$	concave down

Hence we deduce the following about f:

f is concave down on: $[-\sqrt{3}, 0]$, $[\sqrt{3}, \infty)$
f is concave up on: $(-\infty, -\sqrt{3}]$, $[0, \sqrt{3}]$
f has an infl. point at: $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$

(iv) Sketch of graph.



(h) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{-3x^2}{(x^3 + 8)^2}} \qquad \boxed{f''(x) = \frac{12x(x^3 - 4)}{(x^3 + 8)^3}}$$

M-31

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = -2$. Hence our candidate vertical asymptote is the line $x = -2$. Indeed, direct substitution of $x = -2$ into f gives the expression $\frac{1}{0}$, which indicates that both one-sided limits are infinite, whence $x = -2$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x\to\pm\infty}\left(\frac{1}{x^3+8}\right) = \frac{1}{\pm\infty} = 0$$

Hence the only horizontal asymptote is $y = 0$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = 0$) and the vertical asymptotes of $f(x)$ ($x = -2$).

interval	test point	sign of f'	shape of f
$(-\infty, -2)$	$f'(-3)$	$\frac{-3\oplus}{\oplus} = \ominus$	decreasing
$(-2, 0)$	$f'(-1)$	$\frac{-3\oplus}{\oplus} = \ominus$	decreasing
$(0, \infty)$	$f'(1)$	$\frac{-3\oplus}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, -2)$, $(-2, \infty)$
f is increasing on:	no interval
f has a local min at:	none
f has a local max at:	none

(iii) Intervals of concavity and inflection points.

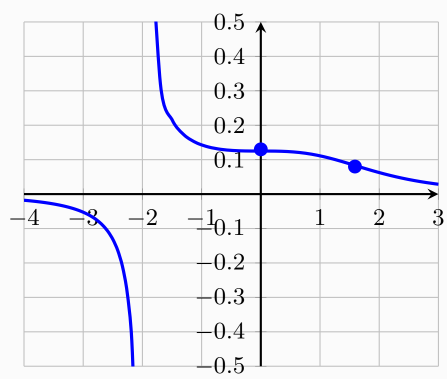
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 0$ and $x = \sqrt[3]{4}$) and the vertical asymptotes of $f(x)$ ($x = -2$).

interval	test point	sign of f''	shape of f
$(-\infty, -2)$	$f''(-3)$	$\frac{12\ominus\ominus}{\ominus} = \ominus$	concave down
$(-2, 0)$	$f''(-1)$	$\frac{12\ominus\ominus}{\oplus} = \oplus$	concave up
$(0, \sqrt[3]{4})$	$f''(1)$	$\frac{12\oplus\ominus}{\oplus} = \ominus$	concave down
$(\sqrt[3]{4}, \infty)$	$f''(2)$	$\frac{12\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, -2)$, $[0, \sqrt[3]{4}]$
f is concave up on:	$(-2, 0]$, $[\sqrt[3]{4}, \infty)$
f has an infl. point at:	$x = 0$ and $x = \sqrt[3]{4}$

(iv) Sketch of graph.



M-31

(i) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{x^2(2x-3)}{(x-1)^2}} \qquad \boxed{f''(x) = \frac{2x\left((x-\frac{3}{2})^2+\frac{3}{4}\right)}{(x-1)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 1$. Hence our candidate vertical asymptote is the line $x = 1$. Indeed, direct substitution of $x = 1$ into f gives the expression $\frac{1}{0}$, which indicates that both one-sided limits are infinite, whence $x = 1$ is, indeed, a vertical asymptote.

As for the horizontal asymptotes we have the following.

$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x-1}\right) = \lim_{x\to\pm\infty}\left(\frac{x^2}{1-\frac{1}{x}}\right) = \frac{\infty}{1-0} = \infty$$

Hence there are no horizontal asymptotes.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = 0$ and $x = \frac{3}{2}$) and the vertical asymptotes of $f(x)$ ($x = 1$).

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(0, 1)$	$f'(0.5)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(1, \frac{3}{2})$	$f'(1.25)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(\frac{3}{2}, \infty)$	$f'(2)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 1)$, $(1, \frac{3}{2}]$
f is increasing on:	$[\frac{3}{2}, \infty)$
f has a local min at:	$x = \frac{3}{2}$
f has a local max at:	none

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = 0$ only) and the vertical asymptotes of $f(x)$ ($x = 1$).

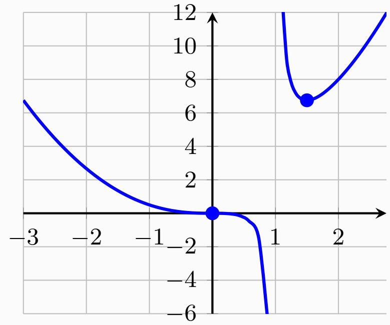
interval	test point	sign of f''	shape of f
$(-\infty, 0)$	$f''(-1)$	$\frac{2\ominus\oplus}{\ominus} = \oplus$	concave up
$(0, 1)$	$f''(0.5)$	$\frac{2\oplus\oplus}{\ominus} = \ominus$	concave down
$(1, \infty)$	$f''(2)$	$\frac{2\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$[0, 1)$
f is concave up on:	$(-\infty, 0]$, $(1, \infty)$
f has an infl. point at:	$x = 0$

(iv) Sketch of graph.

M-31



(j) The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{3(1-x^2)}{(x^3-3x)^2}} \qquad \boxed{f''(x) = \frac{6(2x^4-3x^2+3)}{(x^3-3x)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined if $x^3 - 3x = 0$, or for $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$. Hence our candidate vertical asymptotes are the lines $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$. Indeed, direct substitution of any of these x-values into f gives the expression $\frac{1}{0}$, which indicates that both one-sided limits for each x-value are infinite. Hence all three lines $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$ are true vertical asymptotes.

As for the horizontal asymptotes we have the following.

$$\lim_{x\to\pm\infty}\left(\frac{1}{x^3-3x}\right) = \frac{1}{\infty} = 0$$

Hence the only horizontal asymptote is the line $y = 0$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -1$ and $x = 1$) and the vertical asymptotes of $f(x)$ ($x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$).

interval	test point	sign of f'	shape of f
$(-\infty, -\sqrt{3})$	$f'(-2)$	$\frac{3\ominus}{\oplus} = \ominus$	decreasing
$(-\sqrt{3}, -1)$	$f'(-1.5)$	$\frac{3\ominus}{\oplus} = \ominus$	decreasing
$(-1, 0)$	$f'(-0.5)$	$\frac{3\oplus}{\oplus} = \oplus$	increasing
$(0, 1)$	$f'(0.5)$	$\frac{3\oplus}{\oplus} = \oplus$	increasing
$(1, \sqrt{3})$	$f'(1.5)$	$\frac{3\ominus}{\oplus} = \ominus$	decreasing
$(\sqrt{3}, \infty)$	$f'(2)$	$\frac{3\ominus}{\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, -\sqrt{3})$, $(-\sqrt{3}, -1]$, $[1, \sqrt{3})$, $(\sqrt{3}, \infty)$
f is increasing on:	$[-1, 0)$, $(0, 1]$
f has a local min at:	$x = -1$
f has a local max at:	$x = 1$

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$).

The equation $f''(x) = 0$ is equivalent to $2x^4 - 3x^2 + 3 = 0$, or $2u^2 - 3u + 3 = 0$ with $u = x^2$. The discriminant of this quadratic is negative ($\Delta = (-3)^2 - 4 \cdot 2 \cdot 3 = -15 < 0$), whence there are no solutions to the quadratic equation.

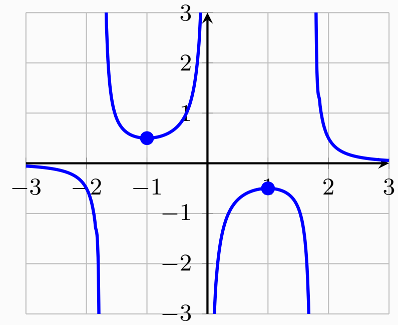
M-31

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt{3})$	$f''(-2)$	$\frac{6\oplus}{\ominus} = \ominus$	concave down
$(-\sqrt{3}, 0)$	$f''(-1)$	$\frac{6\oplus}{\oplus} = \oplus$	concave up
$(0, \sqrt{3})$	$f''(1)$	$\frac{6\oplus}{\ominus} = \ominus$	concave down
$(\sqrt{3}, \infty)$	$f''(2)$	$\frac{6\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on: $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$
f is concave up on: $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$
f has an infl. point at: $x = 0$

(iv) **Sketch of graph.**



Ex. M-32 **4.3/4.4**

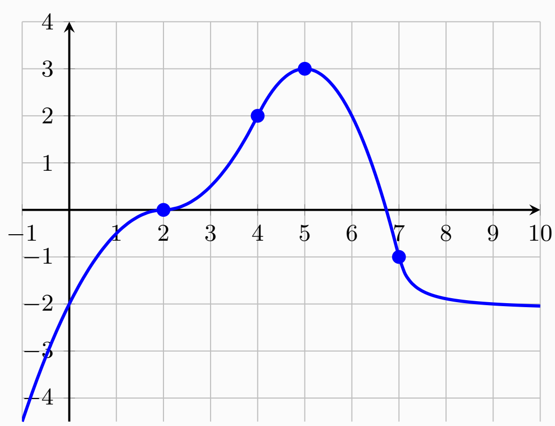
Sketch the graph of a function f that satisfies all of the following conditions.

- $f'(x) > 0$ when $x < 2$ and when $2 < x < 5$
- $f'(x) < 0$ when $x > 5$
- $f'(2) = 0$
- $f''(x) < 0$ when $x < 2$ and when $4 < x < 7$
- $f''(x) > 0$ when $2 < x < 4$ and when $x > 7$

c4s3

Solution **M-32**

There are many possible solutions. Here is one.



Ex. M-33 **4.3/4.4**

Sketch the graph of a function f that satisfies all of the following conditions.

- the lines $y = 1$ and $x = 3$ are asymptotes
- f is increasing for $x < 3$ and $3 < x < 5$, and f is decreasing elsewhere
- the graph of $y = f(x)$ is concave up for $x < 3$ and for $x > 7$
- the graph of $y = f(x)$ is concave down for $3 < x < 7$
- $f(0) = f(5) = 4$ and $f(7) = 2$

c4s3

Solution **M-33**

There are many possible solutions. Here is one.

Ex. M-34 **4.3/4.4**

Consider the function

$$f(x) = e^{-x^2/2}$$

Find where f is concave down and find where f is concave up. Then find all inflection points (x- and y- coordinates). *Write "NONE" for your answer if appropriate.*

unit4

Solution **M-34**

The first two derivatives of f are

$$\boxed{f'(x) = -xe^{-x^2/2}} \qquad \boxed{f''(x) = (x^2 - 1)e^{-x^2/2}}$$

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ ($x = -1$ and $x = 1$).

interval	test point	sign of f''	shape of f
$(-\infty, -1)$	$f''(-2)$	$\oplus\oplus = \oplus$	concave up
$(-1, 1)$	$f'(0)$	$\ominus\oplus = \ominus$	concave down
$(1, \infty)$	$f''(2)$	$\oplus\oplus = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$[-1, 1]$
f is concave up on:	$(-\infty, -1]$, $[1, \infty)$
f has an infl. point at:	$x = -1$, $x = 1$

Ex. M-35 **4.3/4.4**

Consider the function

$$f(x) = \frac{1}{x^2 - 6x}$$

unit4

4.3/4.4

Find all vertical asymptotes of f. Then find where f is decreasing and find where f is increasing. Finally determine the x-coordinates of all local extrema of f (and classify them as either a local minimum or a local maximum). *Write "NONE" for your answer if appropriate.*

unit4

Solution **M-35**

The first derivative of $f(x)$ is given by:

$$\boxed{f(x) = \frac{1}{x(x-6)}} \qquad \boxed{f'(x) = \frac{2(3-x)}{x^2(x-6)^2}}$$

Observe that f is continuous on its domain, but is undefined for $x = 0$ and $x = 6$. Hence our candidate vertical asymptotes are the lines $x = 0$ and $x = 6$. Indeed, direct substitution of $x = 0$ or $x = 6$ into $f(x)$ gives the expression $\frac{1}{0}$, which indicates that both one-sided limits at each x-is value are infinite, whence $x = 0$ and $x = 6$ are, indeed, vertical asymptotes.

Now we calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = 3$) and the vertical asymptotes of $f(x)$ ($x = 0$ and $x = 6$).

interval	test point	sign of f'	shape of f
$(-\infty, 0)$	$f'(-1)$	$\frac{2\oplus}{\oplus\oplus} = \oplus$	increasing
$(0, 3)$	$f'(1)$	$\frac{2\oplus}{\oplus\oplus} = \oplus$	increasing
$(3, 6)$	$f'(4)$	$\frac{2\ominus}{\oplus\oplus} = \ominus$	decreasing
$(6, \infty)$	$f'(7)$	$\frac{2\ominus}{\oplus\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on: $[3, 6)$, $(6, \infty)$
f is increasing on: $(-\infty, 0)$, $(0, 3]$
f has a local min at: *none*
f has a local max at: $x = 3$

Ex. M-36 **4.3/4.4**

Consider the function f and its derivatives below.

$$f(x) = \frac{(x-1)^2}{(x+2)(x-4)} \quad , \quad f'(x) = \frac{-18(x-1)}{(x+2)^2(x-4)^2} \quad , \quad f''(x) = \frac{54((x-1)^2+3)}{(x+2)^3(x-4)^3}$$

Find the vertical and horizontal asymptotes of f. Then find where f is decreasing, where f is increasing, where f is concave down, and where f is concave up. Calculate the x-coordinates of all local minima, local maxima, and points of inflection.

unit4

Solution **M-36**

vertical asymptote(s)	$x=-2$, $x=4$
horizontal asymptote(s)	$y=1$
where f is decreasing	$[1,4)$, $(4,\infty)$
where f is increasing	$(-\infty,-2)$, $(-2,1]$
x-coordinate(s) of local minima	NONE
x-coordinate(s) of local maxima	$x=1$
where f is concave down	$(-2,4)$
where f is concave up	$(-\infty,-2)$, $(4,\infty)$
x-coordinate(s) of inflection point(s)	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f(x)=\frac{(x-1)^2}{(x+2)(x-4)}} \qquad \boxed{f'(x)=\frac{-18(x-1)}{(x+2)^2(x-4)^2}} \qquad \boxed{f''(x)=\frac{54((x-1)^2+3)}{(x+2)^3(x-4)^3}}$$

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x=-2$ and $x=4$. Hence our candidate vertical asymptotes are the lines $x=-2$ and $x=4$. Indeed, direct substitution of either $x=-2$ or $x=4$ into $f(x)$ gives the expression "$\frac{\text{non-zero }\#}{0}$", which indicates that both one-sided limits are infinite. Hence the lines $x=-2$ and $x=4$ are true vertical asymptotes.

As for the horizontal asymptotes we have the following. (After factoring out x^2 from numerator and denominator of $f(x)$.)

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\left(\frac{\left(1-\frac{1}{x}\right)^2}{\left(1+\frac{2}{x}\right)\left(1-\frac{4}{x}\right)}\right)=\frac{(1-0)^2}{(1+0)(1-0)}=1$$

Hence the only horizontal asymptote is the line $y=1$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x)=0$ ($x=1$) and the vertical asymptotes of $f(x)$ ($x=-2$ and $x=4$).

interval	test point	sign of f'	shape of f
$(-\infty,-2)$	$f'(-3)$	$\frac{-18\ominus}{\oplus\oplus}=\oplus$	increasing
$(-2,1)$	$f'(0)$	$\frac{-18\ominus}{\oplus\oplus}=\oplus$	increasing
$(1,4)$	$f'(2)$	$\frac{-18\oplus}{\oplus\oplus}=\ominus$	decreasing
$(4,\infty)$	$f'(5)$	$\frac{-18\oplus}{\oplus\oplus}=\ominus$	decreasing

Hence we deduce the following about f:

f is decreasing on:	$[1,4)$, $(4,\infty)$
f is increasing on:	$(-\infty,-2)$, $(-2,1]$
f has a local min at:	*none*
f has a local max at:	$x=1$

M-36

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = -2$ and $x = 4$).

interval	test point	sign of f''	shape of f
$(-\infty, -2)$	$f''(-3)$	$\frac{54\oplus}{\ominus\ominus} = \oplus$	concave up
$(-2, 4)$	$f'(0)$	$\frac{54\oplus}{\oplus\ominus} = \ominus$	concave down
$(4, \infty)$	$f''(5)$	$\frac{54\oplus}{\oplus\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-2, 4)$
f is concave up on:	$(-\infty, -2)$, $(4, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

Ex. M-37 4.3/4.4

Consider the function f and its derivatives given below.

$$\boxed{f(x) = \frac{1}{(x+4)^2(x-6)^2}} \qquad \boxed{f'(x) = \frac{-4(x-1)}{(x+4)^3(x-6)^3}} \qquad \boxed{f''(x) = \frac{20((x-1)^2+5)}{(x+4)^4(x-6)^4}}$$

(i) Find all vertical asymptotes and horizontal asymptotes of $f(x)$.

(ii) Find where $f(x)$ is decreasing and where $f(x)$ is increasing. Also find and classify all local extrema of $f(x)$.

(iii) Find where $f(x)$ is concave down and where $f(x)$ is concave up. Also find all inflection points of $f(x)$.

(iv) Sketch a graph of $y = f(x)$.

other

Solution M-37

(i) Vertical asymptotes and horizontal asymptotes.

Note that f is continuous on its domain. Hence the only candidate vertical asymptotes are $x = -4$ and $x = 6$. Substitution of either $x = -4$ or $x = 6$ into $f(x)$ gives the expression $\frac{1}{0}$ (i.e., a non-zero number divided by zero), whence $x = -4$ and $x = 6$ are, indeed, both vertical asymptotes.

Note that $\lim_{x\to\pm\infty} f(x) = 0$, hence the only horizontal asymptote is $y = 0$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = 1$) and the vertical asymptotes of $f(x)$ ($x = -4$ and $x = 6$).

interval	test point	sign of f'	shape of f
$(-\infty, -4)$	$f'(-5)$	$\frac{-4\ominus}{\ominus\ominus} = \oplus$	increasing
$(-4, 1)$	$f'(0)$	$\frac{-4\ominus}{\oplus\ominus} = \ominus$	decreasing
$(1, 6)$	$f'(2)$	$\frac{-4\oplus}{\oplus\ominus} = \oplus$	increasing
$(6, \infty)$	$f'(7)$	$\frac{-4\oplus}{\oplus\oplus} = \ominus$	decreasing

Hence we deduce the following about f:

M-37

f is decreasing on:	$(-4, 1]$, $[6, \infty)$
f is increasing on:	$(-\infty, -4)$, $[1, 6)$
f has a local min at:	$x = 1$
f has a local max at:	none

(iii) Intervals of concavity and inflection points.

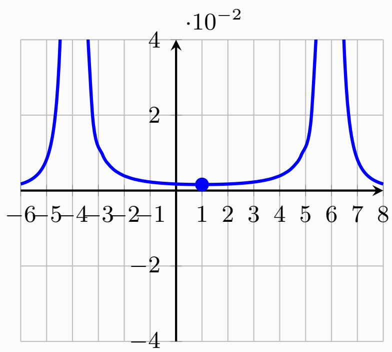
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes of $f(x)$ ($x = -4$ and $x = 6$).

interval	test point	sign of f''	shape of f
$(-\infty, -4)$	$f''(-5)$	$\frac{20\oplus}{\oplus\oplus} = \oplus$	concave up
$(-4, 6)$	$f''(0)$	$\frac{20\oplus}{\oplus\oplus} = \oplus$	concave up
$(6, \infty)$	$f''(7)$	$\frac{20\oplus}{\oplus\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	*no interval*
f is concave up on:	$(-\infty, -4)$, $(-4, 6)$, $(6, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.



Ex. M-38 **4.3/4.4** **⋆Challenge**

Consider the function $f(x) = ax^6e^{-bx}$, where a and b are unspecified constants. Suppose f has a point of local maximum at $(2, 64e^{-2})$. Find the values of a and b.

challenge

Solution **M-38**

The derivative of $f(x)$ is given by:

$$f'(x) = 6ax^5e^{-bx} + ax^6 \cdot e^{-bx} \cdot (-b) = ax^5(6 - bx)e^{-bx}$$

We are given two conditions: (1) $f(2) = 64e^{-2}$ and (2) $f'(2) = 0$ (since $x = 2$ gives a local maximum). So we have the following simultaneous set of equations for a and b:

$$64ae^{-2b} = 64e^{-2}$$
$$32a(6 - 2b)e^{-2b} = 0$$

The second equation implies $a = 0$ or $b = 3$. However, the solution $a = 0$ does not satisfy the first equation, so we must have $b = 3$. So the first equation now gives $64ae^{-6} = 64e^{-2}$, whence $a = e^4$.

Ex. M-39 4.3/4.4 ⋆Challenge !!!

Consider the function $f(x) = (x-3a)(x+2a)^4$, where a is an unspecified **positive** constant. Answer all of the following in terms of a.

(a) where is f decreasing?
(b) where is f increasing?
(c) where does f have a local minimum?
(d) where does f have a local maximum?
(e) where is f concave down?
(f) where is f concave up?
(g) where does f have an inflection point?

Finally, sketch a graph of $y = f(x)$. Your horizontal scale should be in terms of a and your vertical scale should be in terms of a^5.

challenge

Ex. M-40 4.3/4.4 ⋆Challenge !!!

Let $f(x) = \dfrac{e^x}{4+x^3}$. Answer all of the following.

(a) what are the vertical asymptotes of f?
(b) what are the horizontal asymptotes of f?
(c) where is f decreasing?
(d) where is f increasing?
(e) where does f have a local minimum?
(f) where does f have a local maximum?

challenge

Ex. M-41 4.3/4.4 ⋆Challenge

Let $f(x) = \sqrt[3]{x^3 - 48x}$.

(i) Find all vertical asymptotes and horizontal asymptotes of $f(x)$.
(ii) Find where $f(x)$ is decreasing and where $f(x)$ is increasing. Also find and classify all local extrema of $f(x)$.
(iii) Find where $f(x)$ is concave down and where $f(x)$ is concave up. Also find all inflection points of $f(x)$.
(iv) Sketch a graph of $y = f(x)$.

challenge

Solution M-41

The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{x^2-16}{(x^3-48x)^{2/3}}} \qquad \boxed{f''(x) = \frac{-32(x^2+16)}{(x^3-48x)^{5/3}}}$$

(i) **<u>Vertical asymptotes and horizontal asymptotes.</u>**

Since f is continuous for all real numbers, there are no vertical asymptotes. As for the horizontal asymptotes, we have

$$\lim_{x\to\pm\infty}(x^3-48x)^{1/3} = \lim_{x\to\pm\infty}\left(x\cdot\left(1-\frac{48}{x^2}\right)^{1/3}\right) = \pm\infty\cdot(1-0)^{1/3} = \pm\infty$$

Hence there are no horizontal asymptotes.

(ii) **<u>Intervals of increase and local extrema.</u>**

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -4$ and $x = 4$) and where $f'(x)$ DNE ($x = 0$, $x = -\sqrt{48}$, and $x = \sqrt{48}$).

M-41

interval	test point	sign of f'	shape of f
$(-\infty, -\sqrt{48})$	$f'(-7)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(-\sqrt{48}, -4)$	$f'(-5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(-4, 0)$	$f'(-3)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(0, 4)$	$f'(3)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(4, \sqrt{48})$	$f'(5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(\sqrt{48}, \infty)$	$f'(7)$	$\frac{\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on: $[-4, 4]$
f is increasing on: $(-\infty, -4]$, $[4, \infty)$
f has a local min at: $x = 4$
f has a local max at: $x = -4$

(iii) Intervals of concavity and inflection points.

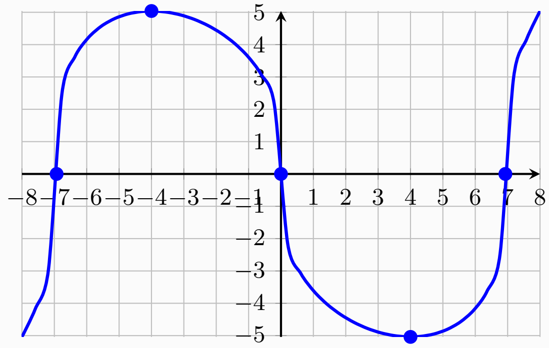
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and where $f''(x)$ DNE ($x = 0$, $x = -\sqrt{48}$, and $x = \sqrt{48}$).

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt{48})$	$f'(-7)$	$\frac{-32\oplus}{\oplus} = \ominus$	concave up
$(-\sqrt{48}, 0)$	$f'(-1)$	$\frac{-32\oplus}{\ominus} = \oplus$	concave down
$(0, \sqrt{48})$	$f'(1)$	$\frac{-32\oplus}{\oplus} = \ominus$	concave up
$(\sqrt{48}, \infty)$	$f'(7)$	$\frac{-32\oplus}{\ominus} = \oplus$	concave down

Hence we deduce the following about f:

f is concave down on: $[-\sqrt{48}, 0]$, $[\sqrt{48}, \infty)$
f is concave up on: $(-\infty, -\sqrt{48}]$, $[0, \sqrt{48}]$
f has an infl. point at: $x = -\sqrt{48}$, $x = 0$, and $x = \sqrt{48}$

(iv) Sketch of graph.



Precise examination of cusps and vertical tangents is beyond the scope of this course. For the sake of completeness, note the following:

$$\lim_{x\to 0^-} f'(x) = -\infty \quad , \quad \lim_{x\to 0^+} f'(x) = -\infty$$

Since the derivative has an infinite limit and it is the same sign of infinity for both one-sided limits, there is a vertical tangent at $x = 0$. Similarly, there is a vertical tangent at both $x = -\sqrt{48}$ and $x = \sqrt{48}$ also.

§4.5: Optimization Problems

Ex. N-1 4.5 Fa17 Exam

A wire of length 51 cm is cut into two pieces. One piece is bent into a square. The other piece is bent into a rectangle whose length is two times its width. How should the wire be cut and the pieces assembled so that the total area enclosed by both pieces is a minimum?

You must use calculus-based methods in your work. You must also justify that your answer really does give the minimum.

exf17

Solution N-1

Let x be the side length of the square and let y be the width of the rectangle (so that the length of the rectangle is $2y$). Our objective function is $F(x, y) = x^2 + 2y^2$ (total area of the square and the rectangle).

The perimeter of the square is $4x$ and the perimeter of the rectangle is $6y$, whence our objective is subject to the constraint $4x + 6y = 51$. The constraint then gives $y = \frac{51-4x}{6}$, and so our objective in one variable is:

$$f(x) = x^2 + 2\left(\frac{51-4x}{6}\right)^2 = x^2 + \frac{1}{18}(51-4x)^2$$

The interval of interest (allowed values of x) is $[0, \frac{51}{4}]$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 2x - \frac{4}{9}(51-4x) = \frac{34}{9}(x-6) = 0 \Longrightarrow x = 6$$

Observe that $f''(x) = \frac{34}{6}$, whence $f''(6) > 0$. So by the Second Derivative Test, $f(6)$ is a local minimum of f. Since $x = 6$ is the only critical point of f, $f(6)$ is also the global minimum of f.

The wire should be cut into a piece 24 cm long (bent into a square) and a piece 27 cm long (bent into the described rectangle).

Ex. N-2 4.5 Sp18 Exam

You are constructing a rectangular box with a total surface area (six sides) of 450 in^2. The length of the box is three times its width. Find the dimensions of the box, measured in inches, with the largest possible volume.

You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.

exs18

Solution N-2

Let L, W, and H denote the length, width, and height of the box, respectively. Our objective function is $F(L, W, H) = LWH$ (the volume of the box).

There are two constraints. First, the length is three times the width, or $L = 3W$. Second, the total surface area is 450, or $2LW + 2LH + 2WH = 450$. Combining the constraints gives:

$$6W^2 + 8WH = 450 \Longrightarrow H = \frac{450-6W^2}{8W} = \frac{225-3W^2}{4W}$$

Substituting for L and H in F gives our objective function in one variable.

$$f(W) = F\left(3W, W, \frac{225-3W^2}{4W}\right) = 3W \cdot W \cdot \frac{225-3W^2}{4W} = \frac{9}{4}\left(75W - W^3\right)$$

We find our interval of interest by considering the restriction that the dimensions of the box must be non-negative. Since $L = 3W$, the inequalities $L \geq 0$ and $W \geq 0$ are equivalent. We must exclude the case $W = 0$ since otherwise the constraint for the surface area would be violated. So now considering H, we have:

$$H \geq 0 \Longrightarrow \frac{225-3W^2}{4W} \geq 0 \Longrightarrow 225 - 3W^2 \geq 0 \Longrightarrow W \leq \sqrt{75}$$

Hence the interval of interest is $(0, \sqrt{75}]$.

N-2

The critical points of f are solutions to $f'(W) = 0$.

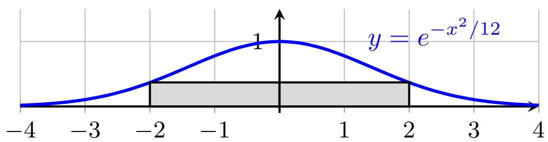
$$f'(W) = \frac{9}{4}\left(75 - 3W^2\right) = 0 \implies W = 5$$

(We reject the solution $W = -5$ since it lies outside the interval of interest.) Observe that $f''(W) = \frac{9}{4}(-6W)$, whence $f''(5) = -\frac{270}{4} < 0$. So by the Second Derivative Test, $f(5)$ is a local maximum of f. Since $W = 5$ is the only critical point of f, $f(5)$ is also the global maximum of f.

The box with the largest possible volume has dimensions $L = 15$, $W = 5$, and $H = 7.5$ (all measured in inches.)

Ex. N-3 4.5 Fa18 Exam

Find the maximum possible area of a rectangle inscribed in the region between the graph of $f(x) = e^{-x^2/12}$ and the x-axis. *You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.*



exf18

Solution N-3

Let (a, b) be the upper right vertex of the rectangle (so by symmetry, $(-a, b)$ is the upper left vertex). Then the width of the rectangle is $2a$ and the height is b.

Our objective function is $F(a, b) = 2ab$ (the area of the rectangle). The upper right vertex lies on the given curve, so our constraint equation is $b = e^{-a^2/12}$, and our objective function in one variable is $f(a) = 2ae^{-a^2/12}$. The upper right vertex must lie in the first quadrant, so our interval of interest is $[0, \infty)$.

The critical poitns of f are solutions to $f'(a) = 0$.

$$f'(a) = 2e^{-a^2/12} + 2ae^{-a^2/12} \cdot \left(-\frac{a}{6}\right) = 2e^{-a^2/12}\left(1 - \frac{a^2}{6}\right) = 0 \implies a^2 = 6 \implies a = \sqrt{6}$$

(We reject the solution $a = -\sqrt{6}$ since it lies outside the interval of interest.)

Now we examine the nature of this critical point using the First Derivative Test.

interval	test point	sign of $f'(a)$	shape of f
$[0, \sqrt{6})$	$f'(1)$	$2 \oplus \oplus = \oplus$	increasing
$(\sqrt{6}, \infty)$	$f'(3)$	$2 \oplus \ominus = \ominus$	decreasing

Hence f is increasing on $[0, \sqrt{6}]$ and decreasing on $[\sqrt{6}, \infty)$, whence a local maximum of f occurs at $a = \sqrt{6}$. Since $a = \sqrt{6}$ is the only critical point, this local maximum must be a global maximum.

Hence the maximum area is $f(\sqrt{6}) = 2\sqrt{\frac{6}{e}}$.

Ex. N-4 4.5 Sp19 Exam

The cost of producing x units is $C(x) = 2x^2 + 5x + 8$. Find the level of production (value of x) that minimizes the average cost. ***Hint:*** Average cost is $AC(x) = \frac{C(x)}{x}$.

exs19

Solution N-4

The average cost is $AC(x) = 2x + 5 + \dfrac{8}{x}$. The critical points are solutions to $AC'(x) = 0$.

$$AC'(x) = 2 - \frac{8}{x^2} = 0 \implies x = 2$$

(We have rejected the solution $x = -2$ since level of production must be non-negative.) Observe that $AC''(x) = \frac{16}{x^3}$,

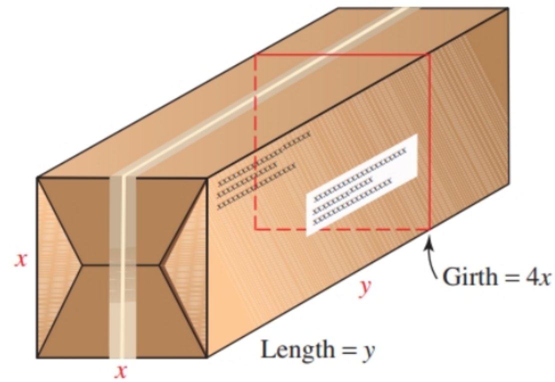
N-4

which is positive for all $x > 0$. So by the Second Derivative Test, $AC(2)$ is a local minimum of AC on $(0, \infty)$. Since $x = 2$ is the only critical point, $AC(2)$ is the global minimum of AC.

Ex. N-5 4.5 Sp19 Exam

According to postal regulations, the sum of the girth and length of a parcel may not exceed 90 inches. What are the dimensions (in inches) of the parcel with the largest possible volume that can be sent, if the parcel is a rectangular box with two square sides?

You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.



exs19

Solution N-5

Let x be the width of the square base and let y be the length of the parcel, as shown in the figure.

Our objective function is $F(x, y) = x^2y$ (total volume of the parcel) subject to the constraint $4x + y = 90$ (sum of girth and length must be 90). The constraint gives $y = 90 - 4x$, whence our objective in one variable is $f(x) = x^2(90 - 4x) = 90x^2 - 4x^3$. Since we must have both $x \geq 0$ and $y \geq 0$ (equivalent to $90 - 4x \geq 0$, or $x \leq 22.5$), we see that the interval of interest is $[0, 22.5]$.

The critical poitns of f are solutions to $f'(x) = 0$.

$$f'(x) = 180x - 12x^2 = 12x(15 - x) \implies x = 0 \text{ or } x = 15$$

Since the interval is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
15	6750	critical point ($f'(x) = 0$)
22.5	0	endpoint

The dimensions of the parcel with the largest volume are 15 in (width of base) and 30 (length of parcel).

Ex. N-6 4.5 Fa19 Exam

When x units of a certain product are produced, the total cost is $C(x) = 5x^2 + 104x + 80$. Find the level of production which minimizes the average cost per unit.

Solution N-6

The average cost per unit is

$$AC(x) = \frac{C(x)}{x} = 5x + 104 + \frac{80}{x}$$

The minimum value of $AC(x)$ occurs at the value of x such that $AC'(x) = 0$. Observe that

$$AC'(x) = 5 - \frac{80}{x^2}$$

and $AC'(x) = 0$ has solutions $x = 4$ and $x = -4$. Since production must be non-negative, average cost is minimized when $x = 4$.

Ex. N-7 4.5 Fa19 Exam

A rectangular container with a closed top and a square base is to be constructed. The top and all four sides of the container are to be made of material that costs $2/ft^2, and the bottom is to be made of material that costs $3/ft^2. Find the container with the largest volume that can be constructed for a total cost of $60.

You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.

Solution N-7

Let L, W, and H denote the length, width, and height of the container. Our objective function is $f(L, W, H) = LWH$, the volume of the container.

There are two constraints. First, the length and width of the base are equal since the base is a square, whence $L = W$. Second, the total cost is $60. The cost of the top base is $2LW$, the cost of the bottom base is $3LW$, and the total cost of the four sides is $2 \cdot (2LH + 2WH)$. Hence the second constraint is $2LW + 3LW + 2(2LH + 2WH) = 60$. Putting $L = W$ into the second constraint gives:

$$5W^2 + 8WH = 60 \Longrightarrow H = \frac{5(12 - W^2)}{8W}$$

So our objective function in one variable is:

$$f(W) = F\left(W, W, \frac{5(12 - W^2)}{8W}\right) = W \cdot W \cdot \frac{5(12 - W^2)}{8W} = \frac{5}{8}\left(12W - W^3\right)$$

Since our variables are lengths, we must have $L \geq 0$ (equivalent to $W \geq 0$), $W \geq 0$, and $H \geq 0$. The last of these is equivalent to the following:

$$H \geq 0 \Longrightarrow \frac{5(12 - W^2)}{8W} \geq 0 \Longrightarrow W \leq \sqrt{12}$$

(Of course, we must also have $W \neq 0$ since otherwise the constraint for the cost is violated.) Putting this together shows that interval of interest is $(0, \sqrt{12}]$.

The critical points of f are solutions to $f'(W) = 0$.

$$f'(W) = \frac{5}{8}\left(12 - 3W^2\right) = \frac{15}{8}(4 - W^2) = 0 \Longrightarrow W = 2$$

Observe that $f''(W) = -\frac{30}{8}W$, whence $f''(2) = -7.5 < 0$. So by the Second Derivative Test, $f(2)$ is a local minimum of f. Since $W = 2$ is the only critical point of f, $f(2)$ is also a global minimum of f.

The container with the largest volume has dimensions $L = 2$, $W = 2$, and $H = 2.5$ (all measured in feet).

Ex. N-8 4.5 Sp20 Exam

Let x be the level of production for a certain commodity. The marginal cost is modeled by the function

$$\frac{dC}{dx} = 3x^2 + 2x$$

and the market price is modeled by the function

$$p(x) = 144 - 2x$$

Suppose that the cost of producing the 1st unit of the commodity is 70.

(a) What is the cost of producing the first 3 units of the commodity?

(b) What is the level of production that maximizes the total profit?

exs20

Solution N-8

(a) The total cost must have the following form:

$$C(x) = \int \frac{dC}{dx}\, dx = \int (3x^2 + 2x)\, dx = x^3 + x^2 + K$$

N-8

where K is some constant. The condition $C(1) = 70$ gives $1 + 1 + K = 70$, whence $K = 68$. So the total cost function is $C(x) = x^3 + x^2 + 68$. Hence the cost of the first 3 units is $C(3) = 104$.

(b) The total revenue is $R(x) = xp(x) = 144x - 2x^2$, and so the total profit is

$$P(x) = R(x) - C(x) = -x^3 - 3x^2 + 144x - 68$$

Total profit is maximized when $P'(x) = 0$, or $-3x^2 - 6x + 144 = -3(x + 8)(x - 6) = 0$. The solutions to this equation are $x = -8$ and $x = 6$. Hence the total profit is maximized when $x = 6$ (production cannot be negative).

Ex. N-9 4.5 Sp20 Exam

Suppose the local post office has a policy that all packages must be shaped like a rectangular box with a sum of length, width, and height not exceeding 144 inches. You plan to construct such a package whose length is 2 times its width. Find the dimensions of the package with the largest volume. For this problem, let L, W, and H be the length, width, and height of the package, respectively.

(a) What is the objective function for this problem in terms of L, W, and H?

(b) There are two constraints for this problem. In terms of L, W, and H, give the constraint equation which corresponds to...

  (i) ...the policy set by the post office.

  (ii) ...your specific plan to construct such a package.

(c) Find the objective function in terms of W only.

(d) What is the interval of interest for the objective function?

(e) Find the values of L, W, and H that give the largest volume.

(f) Suppose the post office adds the additional requirement that the width W of the package must be no smaller than 36 inches and no larger than 40 inches. With this additional policy, what is the width of the package with the largest volume?

exs20

Solution N-9

(a) We seek to maximize the volume of the package, so our objective is $g(L, W, H) = LWH$.

(b) (i) $L + W + H = 144$

  (ii) $L = 2W$

(c) We already have $L = 2W$. From the first constraint, we get $3W + H = 144$, whence $H = 144 - 3W$. Hence the objective function in terms of W only is

$$f(W) = g(2W, W, 144 - 3W) = 2W^2(144 - 3W) = 288W^2 - 6W^3$$

(d) Each of L, W, and H must be non-negative numbers. (We allow them to be 0, since this would correspond to a degenerate package with no volume. That is okay.) The condition $L \geq 0$ is equivalent to $W \geq 0$ since $L = 2W$. The condition $H \geq 0$ is equivalent to $144 - 3W \geq 0$, or $W \leq 48$. Hence the interval of interest (possible values of W) is $[0, 48]$.

(e) The critical points of f are solutions to $f'(W) = 576W - 18W^2 = 18W(32 - W) = 0$. Hence the two critical points are $W = 0$ (already included as an endpoint) and $W = 32$. Since we are working on a closed interval, we may verify that $W = 32$ is the global maximum simply by checking the endpoint and critical values. Since $f(0) = f(48) = 0$ and $f(32) > 0$, it is clear that $W = 32$ gives the global maximum.

Hence the dimensions of the package with the largest volume are $L = 64$, $W = 32$, and $H = 48$.

(f) None of our previous work has changed except that the interval of interest is now $[36, 40]$. We have already determined that $f(W)$ has a global maximum on $[0, 48]$ at $W = 32$. Hence f is decreasing on the interval $[36, 40]$. Hence $f(36) > f(40)$, and so the package with the largest volume now has $W = 36$.

Ex. N-10 4.5 Sp20 Exam

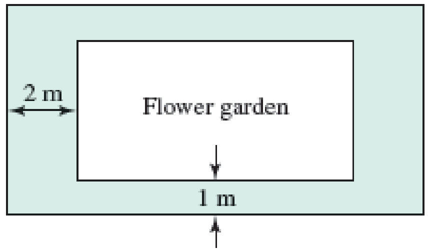
A local park has hired you to construct a rectangular flower garden surrounded by a grass border that is 1 m wide on two sides and 2 m wide on the other two sides. (See the figure below.) The area of the garden only (the small

exs20

4.5 Sp20 Exam

rectangle) must be 126 m^2.

Your primary task is to find the dimensions of the garden that give the smallest possible combined area of the garden and the grass border. For this problem, let W be the horizontal width of the garden and let H be the vertical height of the garden.



(a) What is the objective function for this problem in terms of W and H?

(b) What is the constraint equation for this problem in terms of W and H?

(c) Find the objective function in terms of W only.

(d) What is the interval of interest for the objective function?

(e) Find the values of W and H that minimize the total combined area.

(f) What horizontal width W of the garden will *maximize* the total area?

exs20

Solution N-10

(a) The width of the combined area is $W + 4$ and the height of the combined area if $H + 2$. We seek to minimize the combined area, and so the objective function is

$$g(w, H) = (W + 4)(H + 2)$$

(b) The garden must have an area of 126, and so the constraint equation is $WH = 126$.

(c) Solving for H in the constraint gives $H = \frac{126}{W}$, and substituting this into the objective gives:

$$f(W) = g\left(W, \frac{126}{W}\right) = (W + 4)\left(\frac{126}{W} + 2\right) = 134 + 2W + \frac{504}{W}$$

(d) The width W can be any positive length (note that a length of 0 is not allowed since the garden area must be positive). So the interval of interest is $(0, \infty)$.

(e) We solve $f'(W) = 0$ to find the critical numbers.

$$f'(W) = 2 - \frac{504}{w^2} = 0 \Longrightarrow W = \sqrt{252} = 6\sqrt{7}$$

Observe that $f''(w) = \frac{1108}{W^3}$, which is positive for all $W > 0$. So by the second derivative test, $W = 6\sqrt{7}$ gives a local minimum. Since it gives the only local extreme value on $(0, \infty)$, f has a global minimum value on $(0, \infty)$ at $W = 6\sqrt{7}$. The corresponding height is $H = \frac{126}{6\sqrt{7}} = 3\sqrt{7}$.

(f) None of our work above changes. However, we now note that $f(W) \to \infty$ as $W \to 0^+$ or as $W \to \infty$. Hence there is no maximum combined area. We may obtain an arbitrarily large combined area by simply taking the width W to be either arbitrarily small or arbitrarily large.

Ex. N-11 4.5 Su20 Exam

Farmer Brown wants to create a rectangular pen that must enclose exactly 1800 ft^2. The fencing along the north and south sides of the fence costs \$10/ft and the fencing along the east and west sides costs \$5/ft. (The cost is different because some parts of the fence have to be taller than other parts.) Let x denote the length of the north side and let y denote the length of the east side.

(a) What are the dimensions and total cost of the cheapest pen?

exsu20

4.5 Su20 Exam

(b) Justify that your answer really does give the cheapest pen.

exsu20

Solution N-11

(a) The total cost of the fence is $F(x, y) = 20x + 10y$. We wish to maximize F subject to the constraint $xy = 1800$. Hence our objective function is $f(x) = 20x + \dfrac{18000}{x}$, and our interval of interest is $(0, \infty)$. Observe that

$$f'(x) = 20 - \frac{18000}{x^2}$$

Solving $f'(x) = 0$ gives us the only critical point in our interval: $x = 30$. Hence the optimal dimensions of the fence are $x = 30$ ft and $y = \frac{1800}{30} = 60$ ft. The cost of the cheapest pen is $F(30, 60) = 20 \cdot 30 + 10 \cdot 60 = 1200$ dollars.

(b) Observe that $f''(x) = \dfrac{36000}{x^3}$, and so $f''(30) > 0$. Hence by the second derivative test, $x = 30$ gives a local minimum of $f(x)$. Since $x = 30$ is the only critical point of f on $(0, 30)$, we conclude that this local minimum is also an absolute minimum.

Ex. N-12 4.5 Su20 Exam

In a certain video game, the player may adjust the values of their character's *Intelligence* (denoted by x) and *Dexterity* (denoted by y). These power values must be non-negative but can be any real number (they need not be whole numbers). The player cannot arbitrarily adjust their power, but rather these values must satisfy the equation $x^2+y^2 = 100$. The total damage done (denoted by D) by the spell *Thunderbolt* is given by $D = x + 3y$.

(a) How should the player adjust their power so that *Thunderbolt* does the most possible damage?

(b) What is the minimum possible damage that *Thunderbolt* will do, regardless of how the player adjusts their character's power? How should a player adjust these power values to achieve the minimum possible damage?

exsu20

Solution N-12

(a) We seek to maximize the function $D(x, y) = x + 3y$ subject to the constraint $x^2 + y^2 = 100$ (with x and y non-negative). Solving for y in terms of x gives $y = \sqrt{100 - x^2}$, whence our objective function is

$$f(x) = x + 3\sqrt{100 - x^2}$$

and our interval of interest is $[0, 10]$. Observe that

$$f'(x) = 1 - \frac{3x}{\sqrt{100 - x^2}}$$

Solving $f'(x) = 0$ gives us the only critical point in our interval: $x = \sqrt{10}$.

The extreme values of f must occur at a critical point or an endpoint of $[0, 10]$.

x	$f(x)$	reason for check
0	$30 = \sqrt{900}$	endpoint
$\sqrt{10}$	$10\sqrt{10} = \sqrt{1000}$	critical point ($f'(x) = 0$)
10	$10 = \sqrt{100}$	endpoint

Hence $x = \sqrt{10}$ gives the maximum possible value of f, corresponding to *Intelligence* of $\sqrt{10}$ and *Dexterity* of $y = \sqrt{100 - x^2} = 3\sqrt{10}$.

(b) From our previous work, we see that the absolute minimum of D is 10, occurring when $x = 10$ (and $y = 0$).

Ex. N-13 4.5 Fa20 Exam

A rectangular box with a square base and no top is being constructed to hold a volume of 150 cm^3. The material for the base of the container costs \$6/cm^2 and the material for the sides of the container costs \$2/cm^2. Find the dimensions of the cheapest possible container.

exf20

4.5 Fa20 Exam

You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.

exf20

Solution **N-13**

Let x be the length of the square base and let y be the height of the box, both measured in cm. Our objective function is the total cost of the box, which is given by:

$$C(x,y) = \underbrace{6x^2}_{\text{cost of base}} + \underbrace{8xy}_{\text{cost of sides}}$$

Our constraint is that the volume must be 150 cm^3, whence $x^2y = 150$, or $y = 150/x^2$. Hence our objective function in terms of x only is

$$f(x) = C\left(x, \frac{150}{x^2}\right) = 6x^2 + \frac{1200}{x}$$

We seek an absolute minimum of f on the interval of interest $(0, \infty)$. We have:

$$f'(x) = 12x - \frac{1200}{x^2}$$

The only positive solution to $f'(x) = 0$, and thus our only critical point, is $x = 100^{1/3}$. Observe that $f''(x) = 12 + \frac{2400}{x^3} > 0$ for all $x > 0$. Hence f is concave up on $(0, \infty)$, whence $x = 100^{1/3}$ gives a local minimum of f. Since this is the only critical point, it must also give the absolute minimum.

The dimensions of the cheapest box are $x = 100^{1/3}$ and $y = \frac{150}{100^{2/3}}$.

Ex. N-14 4.5 Sp21 Exam

An airline policy states that all baggage must be shaped like a rectangular box with the sum of the length, width, and height not exceeding 122 inches. You plan to purchase a bag from a company that makes customized bagged whose height must be 3 times its width. Find the dimensions of the baggage with the largest volume. (Let L, W, and H be the length, width, and height of the baggage, respectively.)

(a) Before using any constraints particular to this problem, find the objective function in terms of L, W, and H.

(b) There are two constraints for this problem. One constraint is from the airline and the other is from the baggage company. Find these constraints.

(c) Write the objective function in terms of W only.

(d) Find the interval of interest for the objective function in part (c).

(e) Find the dimensions of the baggage with the largest volume.

exs21

Solution **N-14**

(a) We seek the largest volume, whence the objective is $F(L, W, H) = LWH$.

(b) The airline gives the constraint $L + W + H = 122$ and the baggage company gives the constraint $H = 3W$.

(c) From part (b), we have $L = 122 - W - H = 122 - 4W$, and so the objective in terms of W only is

$$f(W) = f\,(122 - 4W, W, 3W) = 366W^2 - 12W^3$$

(d) All measurements must be non-negative. So we must have $L \geq 0$ (equivalent to $W \leq \frac{122}{4} = \frac{61}{2}$), $W \geq 0$, and $H \geq 0$ (equivalent to $W \geq 0$). Hence the interval of interest for W is $[0, \frac{61}{2}]$.

(e) Observe that $f'(W) = 732W - 36W^2 = 12W(61 - 3W)$, hence the only critical point of f is $W = \frac{61}{3}$. To verify this gives us a maximum volume, we note that $f(0) = f(\frac{61}{2}) = 0$ (testing endpoints). Since $f(\frac{61}{3})$ is clearly positive, we must have an absolute maximum of f on the interval at $W = \frac{61}{3}$. The desired dimensions are thus:

$$L = \frac{122}{3} \quad , \quad W = \frac{61}{3} \quad , \quad H = 61$$

Ex. N-15 4.5 Fa21 Exam

A storage shed with a volume of 1500 ft^3 is to be built in the shape of a rectangular box with a square base. The material for the base costs \$ 6/ft^2, the material for the roof costs \$ 9/ft^2, and the material for the sides costs \$2.50/ft^2. Find the dimensions of the cheapest shed. As you work, fill in the answer boxes below. Let x represent the length of the base of the shed.

objective function in terms of x:	
interval of interest:	
dimensions of cheapest shed (in ft):	______ × ______ × ______ length of base, width of base, height of shed

exf21

Solution N-15

Since we asked to find the cheapest shed, the objective function is the total cost of the shed. Let x be the length of the base of the shed and let h be the height of the shed. Since the base of the shed is a square, the total cost of the shed is

$$C = C_{\text{base}} + C_{\text{roof}} + C_{\text{sides}} = 6x^2 + 9x^2 + 2.5 \cdot 4xh = 15x^2 + 10xh$$

The volume of the shed must be 1500, whence the constraint equation is $x^2h = 1500$, and thus the height is given by $h = \frac{1500}{x^2}$. Substituting the expression for h into C gives the objective in terms of x only.

$$C(x) = 15x^2 + \frac{15000}{x}$$

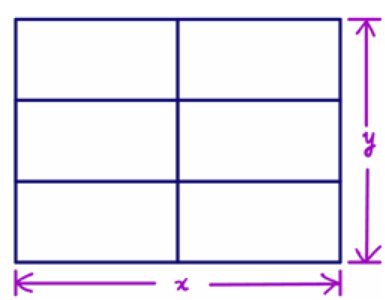
Since x is a length, we must have $x \geq 0$. However, the case $x = 0$ would violate the volume constraint $x^2h = 1500$. There are no further restrictions on the allowed values of x. So the interval of interest for $C(x)$ is $(0, \infty)$. Our goal is to minimize $C(x)$ on this interval.

Since $C(x)$ is differentiable on $(0, \infty)$, the only critical points are solutions to $C'(x) = 0$. We have that $C'(x) = 30x - \frac{15000}{x^2}$, and thus the only solution to $C'(x) = 0$ is $x = 500^{1/3}$. Now observe that $C''(x) = 30 + \frac{30000}{x^3}$, which is positive for all x in $(0, \infty)$. Hence $C(x)$ is concave up on this interval, and we conclude that $x = 500^{1/3}$ does, in fact, give the absolute minimum value of $C(x)$ on $(0, \infty)$.

The dimensions of the cheapest shed are $x = 500^{1/3}$ (length of base and width of base) and $h = \frac{1500}{x^2} = 3 \cdot 500^{1/3}$ (height of shed).

Ex. N-16 4.5 Fa21 Exam

Farmer Green is building an enclosure that must have a total area of 48 m^2. The pen will also be subdivided into 6 pens of equal area, as shown on the right. Find the dimensions of the enclosure that will require the least amount of fencing. *As you work, fill in the answer boxes below. You must use calculus-based methods in your work. You must also justify that your answer really does give the least fencing.*



exf21

4.5 Fa21 Exam

constraint equation in terms of x and y:	
objective function in terms of x only:	
interval of interest:	
dimensions of desired enclosure (in meters):	____ $\times$ ____ total length (x) total width (y)

exf21

Solution N-16

We seek to minimize the total length of fencing, whence our objective function is $F(x, y) = 4x + 3y$. The total area must be 48, whence our constraint equation is $xy = 48$. Solving for y gives $y = \frac{48}{x}$, and substituting this expression into F gives our objective function in terms of x only:

$$f(x) = 4x + \frac{144}{x}$$

The length x can't be negative, but x also can't equal 0 since that would violation the constraint equation. Hence the interval of interest is $(0, \infty)$. We now find the critical points of f on this interval.

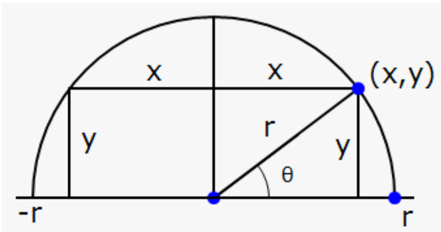
$$f'(x) = 4 - \frac{144}{x^2}$$

Solving $f'(x) = 0$ on the interval $(0, \infty)$ gives $x = 6$. Observe that $f''(x) = \frac{288}{x^3}$, whence $f''(6) > 0$. This means f has a local minimum at $x = 6$. Since $x = 6$ is the only critical point of f, $x = 6$ must also give an absolute minimum. Hence the dimensions of the pen should be $x = 6$ and $y = \frac{48}{6} = 8$.

Ex. N-17 4.5 Sp22 Exam

A rectangle (with base $2x$ and height y) is constructed with its base on the diameter of a semicircle with radius 5 and with its two other vertices on the semicircle. Find the dimensions of the rectangle with the maximum possible area. *As you work, fill in the answer boxes below. You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.*

constraint equation in terms of x and y:	
objective function in terms of x only:	
interval of interest:	
dimensions of rectangle:	____ $\times$ ____ $2x$ (base) y (height)



exs22

Solution **N-17**

Let x be the half-length of the rectangle and let y be the height. Our objective function is $A(x, y) = 2xy$. See the figure. By Pythagorean theorem, $x^2 + y^2 = r^2$ (with $r = 5$), whence our constraint $y = \sqrt{25 - x^2}$. So the objective in one variable is $f(x) = 2x\sqrt{25 - x^2}$. Our interval of interest is $[0, 5]$ (allowed values of x).

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 2x \cdot \frac{-2x}{2\sqrt{25 - x^2}} + 2\sqrt{25 - x^2} = \frac{50 - 4x^2}{\sqrt{25 - x^2}} = 0 \implies x = \frac{5}{\sqrt{2}}$$

(We have rejected the solution $x = -\frac{5}{\sqrt{2}}$ since $x \geq 0$.) Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

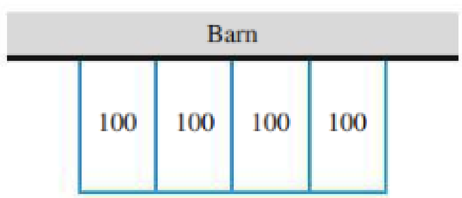
x	$f(x)$	reason for check
0	0	endpoint
$\frac{5}{\sqrt{2}}$	25	critical point ($f'(x) = 0$)
5	0	endpoint

Hence the maximum of $f(x)$ occurs at $x = \frac{5}{\sqrt{2}}$. The dimensions of the rectangle are $2x = 5\sqrt{2}$ (length) and $y = \frac{5}{\sqrt{2}}$ (height).

Ex. N-18 **4.5** Su22 **Exam**

A rancher plans to make four identical and adjacent rectangular pens against a barn, each with an area of 100 m^2 (see the figure below). What are the dimensions of each pen that minimize the amount of fence that must be used? ***Note:*** No fencing is needed on the side of the pen that borders the barn (the north side of the pen).

As you work, fill in the answer boxes below. You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.



constraint equation(s):	
objective function in one variable only:	
interval of interest:	
dimensions of one pen:	______ × ______ horizontal dimension vertical dimension

exsu22

Solution **N-18**

Let x and y be the horizontal and vertical dimensions of one individual pen, respectively. We seek to minimize the total length of fencing, thus our objective function is

$$F(x, y) = 4x + 5y$$

Each individual pen must have area 100, and so our constraint equation is $xy = 100$. Solving for y gives $y = \frac{100}{x}$.

N-18

Thus the objective function can be written in terms of x only as

$$f(x) = 4x + \frac{500}{x}$$

We seek the value of x that gives the absolute minimum of f on the interval $(0, \infty)$. We now find the critical points.

$$f'(x) = 4 - \frac{500}{x^2} = 0 \implies x^2 = 125 \implies x = \sqrt{125}$$

(We reject the solution $x = -\sqrt{125}$ since the length can't be negative.) Since the interval of interest is not closed, we can't use the extreme value theorem to verify the nature of the critical point $x = \sqrt{125}$.

Observe that $f''(x) = \frac{1000}{x^3}$, whence $f''(\sqrt{125}) > 0$, and so $x = \sqrt{125}$ gives a local minimum. Since $x = \sqrt{125}$ is the only critical point in the interval, it must also give an absolute minimum. Thus the desired dimensions of an individual pen are $x = \sqrt{125}$ and $y = \frac{100}{x} = \frac{4}{5}\sqrt{125}$.

Note: In reality, the interval of interest is $(0, \frac{L}{4}]$ where L is the length of the barn, but since we are not given L, we can just assume the barn is sufficiently large. Mathematically, we can assume L is large enough so the critical point of f lies in the interval $(0, \frac{L}{4}]$.

Ex. N-19 4.5 Su22 Quiz

An airline policy states that all carry-on baggage must be box-shaped with a sum of length, width, and height not exceeding 60 in. Suppose the length of a particular carry-on is three times its width. Under the airline's policy, what are the dimensions of such a carry-on with the greatest volume?

You must use calculus-based methods to solve this problem, and you must demonstrate that your answer really does give the greatest volume.

qsu22

Solution N-19

Let L, W, and H be the length, width, and height of the box, respectively. Our goal is to find the dimensions that maximize the value of the objective function $V(L, W, H) = LWH$.

The airline policy gives the constraint $L + W + H = 60$ and the shape of the carry-on gives the constraint $L = 3W$. Hence we also have $3W + W + H = 60$, or $H = 60 - 4W$. Thus our objective function in terms of the one variable W is:

$$f(W) = (3W) \cdot W \cdot (60 - 4W) = 180W^2 - 12W^3$$

The length W must be non-negative, but so must the height, whence $W \geq 0$ and $60 - 4W \geq 0$. So we have $0 \leq W \leq 15$. Thus the interval of interest is $[0, 15]$.

To find the maximum of $f(W)$ on $[0, 15]$, we solve $f'(W) = 0$.

$$f'(W) = 360W - 36W^2 = 36W(10 - W) = 0 \implies W = 0 \quad \text{or} \quad W = 10$$

We now check the critical values and the endpoint values: $f(0) = f(15) = 0$ and $f(10) = 6000$. Hence the absolute maximum value of $f(W)$ occurs when $W = 10$. The other dimensions of the box are $L = 3W = 30$ and $H = 60 - 4W = 20$.

Ex. N-20 4.5 Fa22 Quiz

A rectangle is constructed with its lower two vertices on the x-axis and its upper two vertices on the parabola $y = 75 - 3x^2$. Find the dimensions of the rectangle with the greatest area.

In your work, you must clearly define your variables, identify any constraint equations, and identify your objective function (in terms of one variable). You must also verify that your answer really does give a maximum.

qf22

Solution N-20

Let (x, y) be the coordinates of the upper right vertex of the rectangle. Then the base of the rectangle is $2x$ and the height is y, whence our objective function is $F(x, y) = 2xy$ (area of the rectangle). Our constraint is $y = 75 - 3x^2$. So our objective in one variable is $f(x) = 2x(75 - 3x^2) = 150x - 6x^3$.

We require that $x \geq 0$ and $y \geq 0$ since x and y are lengths. The second inequality is equivalent to $-5 \leq x \leq 5$, and so

N-20

our interval of interest is $[0, 5]$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 150 - 18x^2 = 6(25 - 3x^2) = 0 \implies x = \frac{5}{\sqrt{3}}$$

(We have rejected the solution $x = -\frac{5}{\sqrt{3}}$ since $x \geq 0$. Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
$\frac{5}{\sqrt{3}}$	$\frac{500}{\sqrt{3}}$	critical point ($f'(x) = 0$)
5	0	endpoint

Hence the maximum of f occurs at $x = \frac{5}{\sqrt{3}}$. The dimensions of the rectangle are $\frac{10}{\sqrt{3}}$ (base) and 50 (height).

Ex. N-21 4.5

Find the largest possible product of two numbers whose sum is 180.

c4s5

Solution N-21

Let x and y be the two numbers. Our objective function is $P(x, y) = xy$ subject to the constraint $x + y = 80$. The constraint gives $y = 80 - x$, whence the objective in one variable is $f(x) = x(80 - x) = 80x - x^2$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 80 - 2x = 0 \implies x = 40$$

Observe that $f''(x) = -2$, whence $f''(40) < 0$. So by the Second Derivative Test, $f(40)$ is a local maximum of f. Since $x = 40$ is the only critical point of f, $f(40)$ is also the global maximum of f.

The largest possible product of the numbers is $f(40) = 1600$.

Ex. N-22 4.5

The sum of two numbers is 10. Find the smallest possible value for the sum of their squares.

c4s5

Solution N-22

Let x and y be the two numbers. Our objective function is $S(x, y) = x^2 + y^2$ subject to the constraint $x + y = 10$. The constraint gives $y = 10 - x$, whence the objective in one variable is $f(x) = x^2 + (10 - x)^2$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 2x + 2(10 - x) \cdot (-1) = 4x - 20 = 0 \implies x = 5$$

Observe that $f''(x) = 4$, whence $f''(5) > 0$. So by the Second Derivative Test, $f(5)$ is a local minimum of f. Since $x = 5$ is the only critical point of f, $f(5)$ is also global minimum of f.

The smallest possible sum of squares of the numbers is $f(5) = 50$.

Ex. N-23 4.5

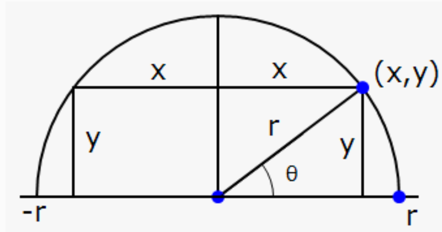
Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 4, assuming that one side of the rectangle lies on the diameter of the semicircle.

c4s5

Solution N-23

Let x be the half-length of the rectangle and let y be the height. Our objective function is $A(x, y) = 2xy$. See the figure below.

N-23



By Pythagorean theorem, $x^2 + y^2 = r^2$ (with $r = 4$), whence our constraint $y = \sqrt{16 - x^2}$. So the objective in one variable is $f(x) = 2x\sqrt{16 - x^2}$. Our interval of interest is $[0, 4]$ (allowed values of x).

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 2x \cdot \frac{-2x}{2\sqrt{16 - x^2}} + 2\sqrt{16 - x^2} = \frac{32 - 4x^2}{\sqrt{16 - x^2}} = 0 \Longrightarrow x = \sqrt{8}$$

(We have rejected the solution $x = -\sqrt{8}$ since $x \geq 0$.) Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
$\sqrt{8}$	16	critical point ($f'(x) = 0$)
4	0	endpoint

Hence the maximum of $f(x)$ occurs at $x = \sqrt{8}$. The dimensions of the rectangle are $2\sqrt{8}$ (length) and $\sqrt{8}$ (height).

Alternatively, let θ be the central angle subtended by a radius and diagonal of the rectangle. (See the figure.) Then $x = 4\cos(\theta)$ and $y = 4\sin(\theta)$. The area of the rectangle is $2xy = 32\cos(\theta)\sin(\theta) = 16\sin(2\theta)$.

Thus we want to find the maximum of the objective function $g(\theta) = 16\sin(2\theta)$ on the interval $[0, \frac{\pi}{2}]$. The maximum of $g(\theta)$ is 16 (the amplitude) and the maximum occurs when $2\theta = \frac{\pi}{2}$, or $\theta = \frac{\pi}{4}$ (corresponding to $x = \sqrt{8}$ and $y = \sqrt{8}$).

Note: This alternative solution does not use calculus at all. No derivatives or limits are used in this solution. The objective function is found using geometry and trigonometric identities. The maximum is found using basic properties of trigonometric functions (e.g., amplitude and maximum values of sine).

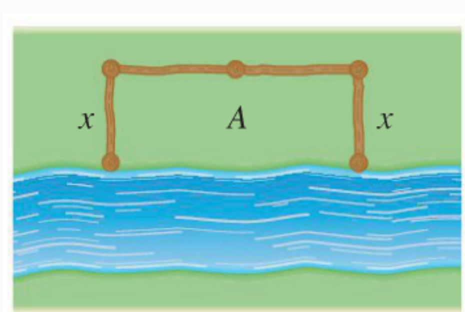
Ex. N-24 **4.5**

A farmer is constructing a rectangular fence along a straight river. The side of the rectangle bordering the river does not need any fencing. If the farmer has 1000 feet of fencing, what is the largest possible area he may enclose?

c4s5

Solution **N-24**

Let x be the length of the plot perpendicular to the river and let y be the length parallel to the river. See the figure below.



Our objective function is $F(x, y) = xy$ subject to the constraint $2x + y = 1000$. The constraint gives $y = 1000 - 2x$, whence our objective in one variable is $f(x) = x(1000 - 2x) = 1000x - 2x^2$. Since we must have both $x \geq 0$ and $y \geq 0$ (equivalent to $1000 - 2x \geq 0$, or $x \leq 500$), we see that the interval of interest is $[0, 500]$.

N-24

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 1000 - 4x = 0 \implies x = 250$$

Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
250	$125,000$	critical point ($f'(x) = 0$)
500	0	endpoint

The largest possible area is $f(250) = 125,000$ ft^2.

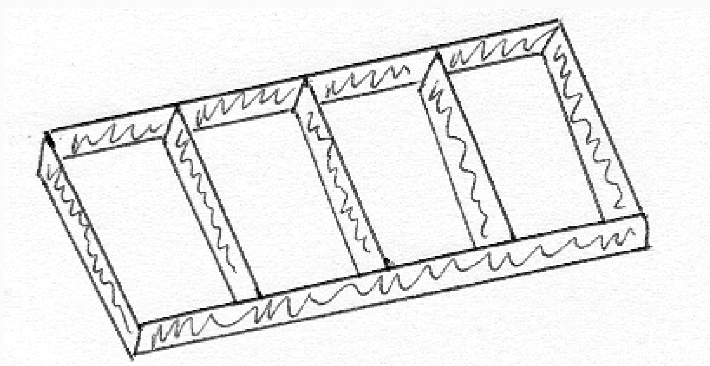
Ex. N-25 4.5

A farmer with 1600 feet of fencing wants to enclose a rectangular area and then divide it into four equal-area pens with fencing parallel to one side of the rectangle. What is the largest possible area that a single pen can enclose?

c4s5

Solution N-25

Let x be the length of each pen (so the length of the entire enclosure is $4x$) and let y be the width of each pen (the widths are parallel to each other, so the width of the entire enclosure is also y). See the figure below.



Our objective function is $F(x, y) = xy$ (the area of one pen), subject to the constraint $8x + 5y = 1600$. The constraint gives $y = \frac{1}{5}(1600 - 8x)$, whence our objective in one variable is $f(x) = \frac{1}{5}x(1600 - 8x) = \frac{8}{5}(200x - x^2)$. Since we must have both $x \geq 0$ and $y \geq 0$ (equivalent to $1600 - 8x \geq 0$, or $x \leq 200$), we see that the interval of interest is $[0, 200]$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = \frac{8}{5}(200 - 2x) = 0 \implies x = 100$$

Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
100	$16,000$	critical point ($f'(x) = 0$)
200	0	endpoint

The largest possible area is $f(100) = 16,000$ ft^2.

Ex. N-26 4.5

Consider the construction of a rectangular aquarium that must hold a volume of 4000 in^3. The length of the base must be twice the width of the base. The top and bottom bases of the tank cost \$1.50/in^2. Each of the sides of the tank costs \$3/in^2. Find the dimensions (length, width, height) of the tank with the least cost.

c4s5

Solution **N-26**

Let L, W, and H denote the length, width, and height of the aquarium. Our objective function is the total cost of the tank. The total cost of the top and bottom bases is $1.5(2LW) = 3LW$, and the total cost of the sides is $3(2LH + 2WH) = 6LH + 6WH$. Hence our objective function is $F(L, W, H) = 3LW + 6LH + 6WH$.

There are two constraints. First, the length of the base is twice the width, or $L = 2W$. Second, the volume is 4000, or $LWH = 4000$. Combining the constraints gives:

$$H = \frac{4000}{LW} = \frac{4000}{2W^2} = \frac{2000}{W^2}$$

Substituting $L = 2W$ and $H = \frac{2000}{W^2}$ gives the objective function in one variable.

$$f(W) = f\left(2W, W, \frac{2000}{W^2}\right) = 6W^2 + \frac{36,000}{W}$$

Our interval of interest is $(0, \infty)$. (The degenerate case $W = 0$ is not allowed since that case violates the constraint $LWH = 4000$.)

The critical points of f are solutions to $f'(W) = 0$.

$$f'(W) = 12W - \frac{36,000}{W^2} = 0 \implies W = \sqrt[3]{3000} = 10\sqrt[3]{3}$$

Observe that $f''(W) = 12 + \frac{72,000}{W^3}$, whence $f''(10\sqrt[3]{3}) = 36 > 0$. So by the Second Derivative Test, $f(10\sqrt[3]{3})$ is a local minimum of f. Since $W = 10\sqrt[3]{3}$ is the only critical point of f, $f(10\sqrt[3]{3})$ is also a global minimum of f.

The tank with the least cost has dimensions $L = 20\sqrt[3]{3}$, $W = 10\sqrt[3]{3}$, and $H = \frac{20}{\sqrt[3]{9}}$ (all measured in inches).

Ex. N-27 **4.5**

Suppose that the total cost of producing s widgets is $C(x) = x^3 + 9x^2 + 18x + 200$ and the selling price per unit is $p(x) = 45 - 2x^2$. At what price should the widgets be sold to maximize total profit?

c4s5

Solution **N-27**

The total revenue is $R(x) = xp(x)$, whence the total profit is:

$$P(x) = R(x) - C(x) = (45x - 2x^3) - (x^3 + 9x^2 + 18x + 200) = -3x^3 - 9x^2 + 27x - 200$$

We seek the value of x in $[0, \infty)$ that gives the maximum of $P(x)$. The critical points of $P(x)$ are solutions to $P'(x) = 0$.

$$P'(x) = -9x^2 - 18x + 27 = -9(x + 3)(x - 1) = 0 \implies x = 1$$

(We have rejected the solution $x = -3$, since $x \geq 0$.) Observe that $P''(x) = -18x - 18$, whence $P''(1) = -36 < 0$. So by the Second Derivative Test, $P(1)$ is a local maximum of P. Since $x = 1$ is the only critical point of P, $P(1)$ is also the global maximum of P.

The level of production that gives the maximum profit is $x = 1$. The corresponding price per unit is $p(1) = 43$.

Ex. N-28 **4.5**

Suppose the total cost of manufacturing x widgets is $C(x) = 3x^2 + 5x + 75$. What level of production minimizes the average cost per unit?

c4s5

Solution **N-28**

The average cost per unit is

$$AC(x) = \frac{C(x)}{x} = 3x + 5 + \frac{75}{x}$$

We seek the value of x in $(0, \infty)$ that gives the minimum of $AC(x)$. The critical points of $AC(x)$ are solutions to $AC'(x) = 0$.

$$AC'(x) = 3 - \frac{75}{x^2} = 0 \implies x = 5$$

N-28

(We reject the solution $x = -5$ since $x \geq 0$.) Observe that $AC''(x) = \frac{150}{x^3}$, whence $AC''(5) = \frac{150}{125} > 0$. So by the Second Derivative Test, $AC(5)$ is a local minimum of AC. Since $x = 5$ is the only critical point of AC, $AC(5)$ is also the global minimum of AC.

The level of production that gives the minimum average cost per unit is $x = 5$.

Ex. N-29 4.5

The total cost of producing x widgets is

$$C(x) = x^3 - 6x^2 + 15x$$

and the selling price per unit is fixed at $p(x) = 6$. Show that if you want to set a level of production to maximize total profit, the best you can do is break even.

c4s5

Solution N-29

The total revenue from selling x widgets is $R(x) = 6x$, and so the total profit is

$$P(x) = R(x) - C(x) = -9x + 6x^2 - x^3$$

If the profit P has a local maximum at x, then $P'(x) = 0$.

$$P'(x) = -9 + 12x - 3x^2 = -3(x-1)(x-3) = 0 \implies x = 1 \text{ or } x = 3$$

Note that $P(1) = -4$ (so we have negative profit if we produce 1 unit) and $P(3) = 0$ (so we break even if we produce 3 units). Since $P(x) \to -\infty$ as $x \to \infty$ and $P(0) = 0$, we see that the maximum value of $P(x)$ on the interval $[0, \infty)$ is 0. So the best we can do is break even.

Ex. N-30 4.5

If x units are produced, then the total cost is $C(x) = x^3+4x^2+60x+200$ and the selling price per unit is $p(x) = 100-3x$. Find the level of production that maximizes the total profit.

unit4

Solution N-30

The total revenue is $R(x) = xp(x) = 100x - 3x^2$, and so the total profit is

$$P(x) = R(x) - C(x) = -x^3 - 7x^2 + 40x - 200$$

Profit is maximized when $P'(x) = 0$.

$$P'(x) = -3x^2 - 14x + 40 = -(3x + 20)(x - 2) = 0 \implies x = 2$$

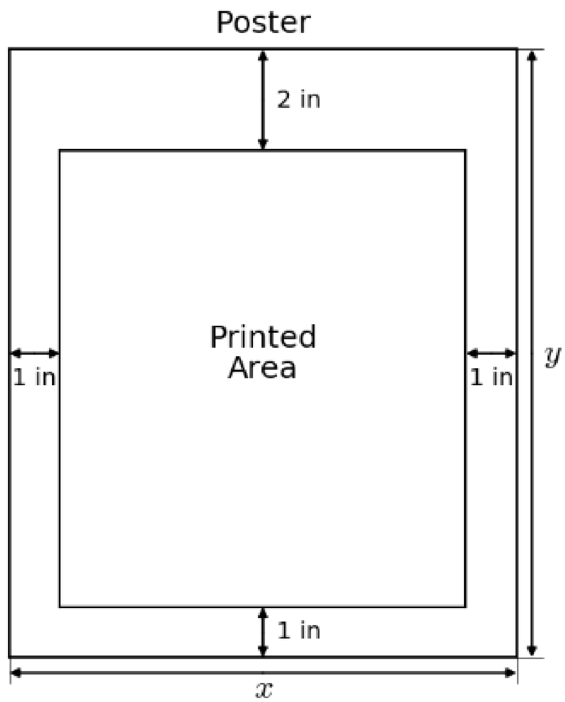
(We reject the solution $x = -\frac{20}{3}$ since production must be non-negative.) We then observe that $P''(x) = -6x - 14$, whence $P''(2) = -28$. Since $P''(2) < 0$, $x = 2$ gives a local maximum of P and thus a global maximum on $[0, \infty)$ since $P(x)$ has only one critical point on $[0, \infty)$.

Ex. N-31 4.5

A poster is to have a total area of 150 in^2, which includes a central printed area, 1-inch margins at the bottom and sides, and a 2-inch margin at the top. What poster dimensions (in inches) will give the largest printed area? Use calculus to justify your answer.
You must demonstrate that your answers really are the optimal dimensions.

unit4

4.5



unit4

Solution **N-31**

Let x and y be the width and height of the poster, as shown in the figure. Our objective function is $F(x,y) = (x-2)(y-3)$, the total area of the printed area only. Our constraint is $xy = 150$, whence $y = \frac{150}{x}$. So our objective in terms of one variable is:

$$f(x) = (x-2)\left(\frac{150}{x} - 3\right) = 156 - 3x - \frac{300}{x}$$

We must have that the dimensions of the printed area are non-negative, i.e., $x - 2 \geq 0$ (or $x \geq 2$) and $y - 3 \geq 0$. The condition $y - 3 \geq 0$ is equivalent to $\frac{150}{x} - 3 \geq 0$, or $x \leq 50$. Thus our interval of interest is $[2, 50]$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = -3 + \frac{300}{x^2} = 0 \implies x = 10$$

(We reject the solution $x = -10$ since it does lie in the interval $[2, 50]$.) Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
2	0	endpoint
10	150	critical point ($f'(x) = 0$)
50	0	endpoint

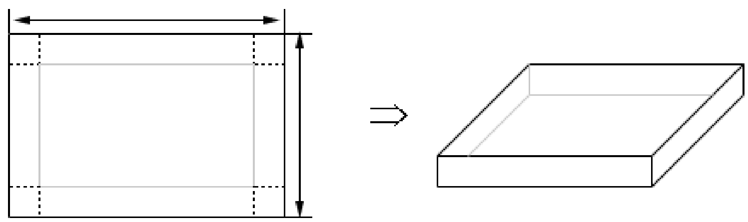
Thus, the poster with the maximum printed area has width $x = 10$ and height $y = 15$ (measured in inches).

Ex. N-32 4.5

A piece of cardboard that is 24 inches wide and 15 inches long is to be used to construct a box with an open top. To do this, congruent squares are cut from each corner of the cardboard, and the flaps are folded up and taped to form the sides of the box. What is the largest possible volume of such a box? Use calculus to justify your answer.
You must demonstrate that your answers really are the optimal dimensions.

unit4

4.5



unit4

Solution **N-32**

Let x be the length of the square that is cut out of each corner. Then the length of each of the top and bottom flaps is $24 - 2x$ and the length of each of the left and right flaps is $15 - 2x$. Note that the lengths of these flaps are the lengths of the base of the box. Since x is the width of each flap, x is also the height of the box.

Our objective function is the total volume of the box (product of length, width, and height), given by:

$$f(x) = (24 - 2x)(15 - 2x)x = 4x^3 - 78x^2 + 360x$$

Note that our objective is already in terms of one variable only.

> Alternatively, we can write the objective as $F(L, W, H) = LWH$, where L, W, and H are the length, width, and height of the box. Then there are three constraints: one equation for each of L, W, and H in terms of x as described above. The presentation above does not make these constraints explicit.

We require that the dimensions of the box be non-negative, whence $x \geq 0$, $24 - 2x \geq 0$ (or $x \leq 12$), and $15 - 2x \geq 0$ (or $x \leq 7.5$). Thus the interval of interest is $[0, 7.5]$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 12x^2 - 156x + 360 = 12(x - 3)(x - 10) = 0 \Longrightarrow x = 3$$

(We reject the solution $x = 10$ since it lies outside the interval $[0, 7.5]$.) Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
3	486	critical point ($f'(x) = 0$)
7.5	0	endpoint

Thus, the largest possible volume of the box is 486 in^3.

Ex. N-33 4.5

A cylindrical can must have a volume of 32π cm^3. The cost of each of the top and bottom is \$6/cm^2 and the cost of the curved side surface is \$3/cm^2. Find the radius and height of the least expensive can. Justify that your answer does, in fact, give the minimum cost.

unit4

Solution **N-33**

Let r and h denote the radius and height of the can, respectively. The objective function is the total cost of the can. The area of each of the top and bottom is πr^2, and so the cost of each is $6\pi r^2$. The area of the sides is $2\pi rh$, and so the cost of the sides is $6\pi rh$. Thus our objective function is:

$$F(r, h) = 12\pi r^2 + 6\pi rh$$

Our constraint is $\pi r^2 h = 32\pi$, whence $h = \frac{32}{r^2}$. So our objective in terms of one variable is

$$f(r) = F\left(r, \frac{32}{r^2}\right) = 12\pi r^2 + 6\pi r \cdot \frac{192}{r^2} = 6\pi\left(2r^2 + \frac{32}{r}\right)$$

N-33

We require both $r \geq 0$ and $h \geq 0$ (or $\frac{32}{r} \geq 0$). The second condition is equivalent to $r > 0$ (given that $r \geq 0$). Hence our interval of interest is $(0, \infty)$.

The critical points of f are solutions to $f'(r) = 0$.

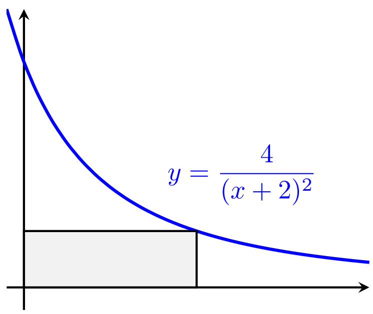
$$f'(r) = 6\pi \left(4r - \frac{32}{r^2}\right) = 0 \Longrightarrow r = 2$$

Observe that $f''(r) = 6\pi \left(4 + \frac{64}{r^3}\right)$, whence $f''(2) = 72\pi > 0$. So by the Second Derivative Test, $f(2)$ is a local minimum of f. Since $r = 2$ is the only critical point of f, $f(2)$ is also the global minimum of f on $(0, \infty)$.

The least expensive can has dimensions $r = 2$ (radius) and $h = 8$ (height).

Ex. N-34 **4.5**

Find the maximum possible area of a rectangle inscribed in the region below the graph of $y = \dfrac{4}{(x+2)^2}$ and in the first quadrant.



other

Solution **N-34**

Let the coordinates of the upper right vertex of the rectangle be (x, y). Our objective function is $F(x, y) = xy$ (the area of the rectangle), subject to the constraint $y = \frac{4}{(x+2)^2}$. Thus our objective in one variable is $f(x) = \frac{4x}{(x+2)^2}$, and our interval of interest is $[0, \infty)$.

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = \frac{4(x+2)^2 - 4x \cdot 2(x+2)}{(x+2)^4} = \frac{-4(x-2)}{(x+2)^3} = 0 \Longrightarrow x = 2$$

By way of the First Derivative Test, we observe that $f'(1) = \frac{4}{27} > 0$ and $f'(3) = \frac{-4}{125} < 0$. Hence $f(x)$ is increasing on $[0, 2]$ and decreasing on $[2, \infty)$. This implies $f(2)$ is, indeed, the maximum value of f. The maximum area is thus $f(2) = \frac{1}{2}$.

Ex. N-35 **4.5** **⋆Challenge** **!!!**

Find the equation of the line through $(2, 4)$ that cuts off the least area from the first quadrant. (Observe that this cut off region is a triangle.)

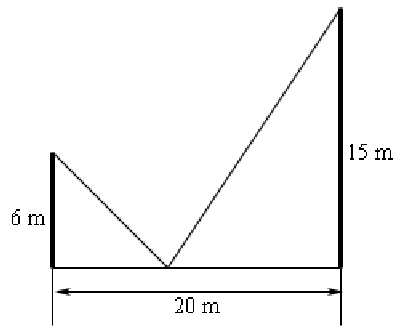
challenge

Ex. N-36 **4.5** **⋆Challenge** **!!!**

Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?

challenge

4.5 ⋆Challenge !!!



challenge

§4.6: Linear Approximation and Differentials

Ex. O-1 4.6 Fa17 Exam

Use a linear approximation to estimate $\sqrt{33}$.

exf17

Solution O-1

We use the tangent line to $f(x) = \sqrt{x}$ at $x = 36$.

Point of Tangency:	$(36, f(36)) = (36, 6)$
Slope of Line:	$f'(x) = \frac{1}{2}x^{-1/2}; \qquad f'(36) = \frac{1}{12}$
Equation of Line:	$y = 6 + \frac{1}{12}(x - 36)$

This means that $\sqrt{x} \approx 6 + \frac{1}{12}(x - 36)$ if x is near 36. Hence we have the estimate:

$$\sqrt{33} \approx 6 + \frac{1}{12}(33 - 36) = 5.75$$

Ex. O-2 4.6 Sp18 Exam

At a certain factory, the daily output is

$$Q(L) = 1500L^{2/3}$$

where L denotes the size of the labor force measured in worker-hours. Currently $1,000$ worker-hours of labor are used each day. Use a linear approximation to estimate the effect on the daily output if the labor force is cut to 975 worker-hours.

exs18

Solution O-2

We seek an estimate of the change in output: $\Delta Q = Q(975) - Q(1000)$. We use the tangent line to $Q(L)$ at $L = 1000$.

Point of Tangency:	$(1000, C(1000))$
Slope of Line:	$Q'(L) = 1000L^{-1/3}; \qquad Q'(1000) = 100$
Equation of Line:	$y - Q(1000) = 100(L - 1000)$

This means that $Q(L) - Q(1000) \approx 100(L - 1000)$ if L is near 1000. Hence we have the estimate:

$$\Delta Q = Q(975) - Q(1000) \approx 100(975 - 1000) = -2500$$

So the output decreases by approximately 2500 units.

Ex. O-3 4.6 Fa18 Exam

The concentration of a certain drug in the bloodstream t hours after the drug is injected is modeled by the following formula.

$$C(t) = \frac{100t}{t^2 + 1}$$

(The concentration is measured in micrograms per milliliter.) Use a linear approximation to estimate the change in the concentration over the time period from 2 to 2.1 hours after injection. Also indicate whether the concentration increases or decreases.

exf18

Solution O-3

We seek an estimate of the change in concentration: $\Delta C = C(2.1) - C(2)$. We use the tangent line to $C(t)$ at $t = 2$.

Point of Tangency:	$(2, C(2))$
Slope of Line:	$C'(t) = \frac{100(-t^2+1)}{(t^2+1)^2}; \qquad C'(2) = -12$
Equation of Line:	$y - C(2) = -12(t - 2)$

This means that $C(t) - C(2) \approx -12(t - 2)$ if t is near 2. Hence we have the estimate:

$$\Delta C = C(2.1) - C(2) \approx -12(2.1 - 2) = -1.2$$

O-3

The total concentration decreases by approximately 1.2 μg/mL.

Ex. O-4 4.6 Sp19 Exam

Use a linear approximation to estimate $\sqrt{35.9}$. Do not simplify your answer.

exs19

Solution O-4

We use the tangent line to $f(x) = \sqrt{x}$ at $x = 36$.

Point of Tangency:	$(36, f(36)) = (36, 6)$
Slope of Line:	$f'(x) = \frac{1}{2}x^{-1/2}$; $\quad f'(6) = \frac{1}{12}$
Equation of Line:	$y = 6 + \frac{1}{12}(x - 36)$

This means that $\sqrt{x} \approx 6 + \frac{1}{12}(x - 36)$ if x is near 6. Hence we have the estimate:

$$\sqrt{35.9} \approx 6 + \frac{1}{12}(35.9 - 36) = 6 - \frac{1}{120}$$

Ex. O-5 4.6 Sp19 Exam

The cost of producing x units is $C(x) = 3x^2 + 4x + 1000$. Use marginal analysis to estimate the cost of producing the 41st unit.

exs19

Solution O-5

The approximate cost of the 41st unit is given by $C'(40)$.

$$C'(40) = (6x + 4)\big|_{x=40} = 6 \cdot 40 + 4 = 244$$

Ex. O-6 4.6 Fa19 Exam

Note: The parts of this problem are not related!

(a) Use linear approximation to estimate the value of $\sqrt{79}$.

(b) A manufacturer's total cost to produce x units is $C(x) = 25\ln(x^2 + 16)$. Use marginal analysis to estimate the cost of the 4th unit.

Solution O-6

(a) We use the tangent line to $f(x) = x^{1/2}$ at $x = 81$.

Point of Tangency:	$(81, f(81)) = (81, 9)$
Slope of Line:	$f'(x) = \frac{1}{2}x^{-1/2}$; $\quad f'(81) = \frac{1}{18}$
Equation of Line:	$y = 9 + \frac{1}{18}(x - 81)$

This means that $\sqrt{x} \approx 9 + \frac{1}{18}(x - 81)$ if x is near 81. Hence we have the estimate:

$$\sqrt{79} \approx 9 + \frac{1}{18}(79 - 81) = \frac{80}{9}$$

(b) Marginal analysis tells us that the approximate cost of the 4th unit is $C'(3)$. So we have:

$$C'(3) = 25 \cdot \frac{2x}{x^2 + 16}\bigg|_{x=3} = \frac{50 \cdot 3}{9 + 16} = \frac{150}{25} = 6$$

Ex. O-7 4.6 Sp20 Exam

Use linear approximation or differentials to estimate the value of $\dfrac{1}{\sqrt[3]{8.48}}$.

exs20

Solution **O-7**

We use the tangent line to $f(x) = x^{-1/3}$ at $x = 8$.

Point of Tangency: $(8, f(8)) = (8, \frac{1}{2})$

Slope of Line: $f'(x) = -\frac{1}{3}x^{-4/3}$; $f'(8) = -\frac{1}{48}$

Equation of Line: $y = \frac{1}{2} - \frac{1}{48}(x-8)$

This means that $x^{-1/3} \approx \frac{1}{2} - \frac{1}{48}(x-8)$ if x is near 8. Hence we have the estimate:

$$(8.48)^{-1/3} \approx \frac{1}{2} - \frac{1}{48}(8.48-8) = 0.5 - 0.01 = 0.49$$

Ex. O-8 4.6 Sp20 Exam

Suppose the cost of manufacturing x units is given by $C(x) = x^3 + 5x^2 + 12x + 50$.

(a) What is the exact cost of producing the 3rd unit?

(b) Using marginal analysis, estimate the cost of producing the 3rd unit.

exs20

Solution **O-8**

(a) $C(3) - C(2) = 56$

(b) $C'(2) = \left(3x^2 + 10x + 12\right)\Big|_{x=2} = 44$

Ex. O-9 4.6 Sp20 Exam

Use linear approximation to estimate the value of $(0.98)^3 - 5(0.98)^2 + 4(0.98) + 10$.

exs20

Solution **O-9**

We use the tangent line to $f(x) = x^3 - 5x^2 + 4x + 10$ at $x = 1$.

Point of Tangency: $(1, f(1)) = (1, 10)$

Slope of Line: $f'(x) = 3x^2 - 10x + 4$; $f'(1) = -2$

Equation of Line: $y = 10 - 2(x-1)$

This means that $x^3 - 5x^2 + 4x + 10 \approx 10 - 2(x-1)$ if x is near 1. Hence we have the estimate:

$$(0.98)^3 - 5(0.98)^2 + 4(0.98) + 10 \approx 10 - 2(0.98-1) = 10.04$$

Ex. O-10 4.6 Sp20 Exam

If x units are produced, the total cost is $C(x) = x^2 + 15x + 24$ and the selling price per unit is

$$p(x) = \frac{156}{x^2 - 4x + 16}$$

(a) What is the exact cost of producing the 3rd unit?

(b) Using marginal analysis, estimate the revenue from the 3rd unit sold.

exs20

Solution **O-10**

(a) $C(3) - C(2) = 20$

(b) The revenue is

$$R(x) = xp(x) = \frac{156x}{x^2 - 4x + 16}$$

So by marginal analysis, the revenue from the 3rd unit is approximately

$$R'(2) = \left(\frac{156(16 - x^2)}{(x^2 - 4x + 16)^2}\right)\Bigg|_{x=2} = 13$$

Ex. O-11 4.6 Sp20 Exam

Suppose the cost (in dollars) of manufacturing q units is given by

$$C(q) = 6q^2 + 34q + 112$$

Use marginal analysis to estimate the cost of producing the 5th unit.

exs20

Solution O-11

The exact cost of the 5th unit is $\Delta C = C(5) - C(4)$, which is approximately $C'(4)$ by linear approximation. Hence

$$\Delta C \approx C'(4) = (12q + 34)\big|_{q=4} = 82$$

Ex. O-12 4.6 Su20 Exam

Given that x units of a commodity are sold, the selling price per unit is $p(x) = \dfrac{5000}{x^2 + 64}$.

(a) Calculate the revenue function.

(b) Calculate the exact revenue derived from the 7th unit.

(c) Using marginal analysis, estimate the revenue derived from the 7th unit.

exsu20

Solution O-12

(a) $R(x) = xp(x) = \dfrac{5000x}{x^2 + 64}$

(b) The exact revenue is

$$R(7) - R(6) = \frac{35000}{113} - \frac{30000}{100} = \frac{1100}{113} \approx 9.735$$

(c) The approximate revenue is

$$R'(6) = \left(\frac{5000(64 - x^2)}{(x^2 + 64)^2}\right)\Bigg|_{x=6} = \frac{5000 \cdot 28}{100^2} = 14$$

Ex. O-13 4.6 Su20 Exam

The total number of gallons in a water tank at t hours is given by $N(t) = 40t^{2/5}$. Use a linear approximation to estimate the number of gallons added to the water between $t = 32$ and $t = 35$.

exsu20

Solution O-13

We seek an estimate of the change in water: $\Delta N = N(35) - N(32)$. We use the tangent line to $N(t)$ at $t = 32$.

Point of Tangency:	$(32, N(32))$
Slope of Line:	$N'(t) = 16t^{-3/5}$; $\quad N'(32) = 2$
Equation of Line:	$y - N(32) = 2(t - 32)$

This means that $N(t) - N(32) \approx 2(t - 32)$ if t is near 32. Hence we have the estimate:

$$\Delta N = N(35) - N(32) \approx 2(35 - 32) = 6$$

The number of gallons increases by approximately 6.

Ex. O-14 4.6 Fa20 Exam

Suppose f is differentiable on $(-\infty, \infty)$, $f(5) = 3$, and $f'(5) = -7$. Use linear approximation to estimate $f(5.1)$.

exf20

Solution O-14

The tangent line to f at $x = 5$ is $y = 3 - 7(x - 5)$. Hence $f(5.1) \approx 3 - 7(5.1 - 5) = 2.3$.

Ex. O-15 4.6 Fa20 Exam

Use linear approximation to estimate $\sqrt[3]{29} - \sqrt[3]{27}$. *Your final answer must be exact and may not contain any radicals.*

exf20

Solution O-15

We use the tangent line to $f(x) = x^{1/3}$ at $x = 27$.

$$\begin{array}{ll} \text{Point of Tangency:} & (27, f(27)) = (27, 27^{1/3}) \\ \text{Slope of Line:} & f'(x) = \frac{1}{3}x^{-2/3}; \qquad f'(27) = \frac{1}{27} \\ \text{Equation of Line:} & y - 271/3 = \frac{1}{27}(x - 27) \end{array}$$

This means that $x^{1/3} - 271/3 \approx \frac{1}{27}(x - 27)$ if x is near 27. Hence we have the estimate:

$$29^{1/3} - 271/3 \approx \frac{1}{27}(29 - 27) = \frac{2}{27}$$

Ex. O-16 4.6 Sp21 Exam

Use the identity $4^2 + \sqrt{4} = 18$ and linear approximation to estimate $(3.81)^2 + \sqrt{3.81}$.

exs21

Solution O-16

We use the tangent line to $f(x) = x^2 + x^{1/2}$ at $x = 4$.

$$\begin{array}{ll} \text{Point of Tangency:} & (4, f(4)) = (4, 18) \\ \text{Slope of Line:} & f'(x) = 2x + \frac{1}{2}x^{-1/2}; \qquad f'(4) = \frac{35}{4} \\ \text{Equation of Line:} & y = 18 + \frac{35}{4}(x - 4) \end{array}$$

This means that $x^2 + \sqrt{x} \approx 18 + \frac{35}{4}(x - 4)$ if x is near 4. Hence we have the estimate:

$$(3.81)^2 + \sqrt{3.81} \approx 18 + \frac{35}{4}(3.81 - 4) = 16.3375$$

Ex. O-17 4.6 Sp21 Exam

The total cost (in dollars) of producing x items is modeled by the function $C(x) = x^2 + 4x + 3$, and the price per item (in dollars) is $p(x) = \dfrac{98x + 49}{x + 3}$.

(a) Calculate the exact cost of producing the 5th item.

(b) Using marginal analysis, estimate the revenue derived from producing the 5th item.

exs21

Solution O-17

(a) $C(5) - C(4) = 48 - 35 = 13$.

(b) The revenue is $R(x) = xp(x) = \dfrac{98x^2 + 49x}{x + 3}$. Hence the desired marginal revenue is

$$R'(4) = \left(\frac{49(2x^2 + 12x + 3)}{(x + 3)^2}\right)\Bigg|_{x=4} = 83$$

Ex. O-18 4.6 Fa21 Exam

Use linear approximation to estimate $\tan\left(\frac{\pi}{4} + 0.12\right) - \tan\left(\frac{\pi}{4}\right)$.

exf21

Solution O-18

We use the tangent line to $f(x) = \tan(x)$ at $x = \frac{\pi}{4}$.

O-18

Point of Tangency:	$(\frac{\pi}{4}, f(\frac{\pi}{4})) = (\frac{\pi}{4}, \tan(\frac{\pi}{4}))$
Slope of Line:	$f'(x) = \sec(x)^2$; $f'(\frac{\pi}{4}) = 2$
Equation of Line:	$y - \tan(\frac{\pi}{4}) = 2(x - \frac{\pi}{4})$

This means that $\tan(x) - \tan(\frac{\pi}{4}) \approx 2(x - \frac{\pi}{4})$ if x is near $\frac{\pi}{4}$. Hence we have the estimate:

$$\tan\left(\frac{\pi}{4} + 0.12\right) - \tan\left(\frac{\pi}{4}\right) \approx 2\left(\frac{\pi}{4} + 0.12 - \frac{\pi}{4}\right) = 0.24$$

Ex. L-19 **4.1, 4.6** Fa21 **Exam**

The parts of this problem are not related.

(a) Suppose that when x units are produced, the total cost is $C(x) = 2x^2 + 10x + 18$ and the selling price per unit is $p(x) = 46 - x$. Find the level of production that maximizes total profit.

(b) Suppose the total cost of producing q units is $C(q) = q^3 + 20q^2 + 200q + 2000$. Use marginal analysis to estimate the cost of the 3rd unit.

exf21

Solution L-19

(a) The total revenue is $R(x) = xp(x) = 46x - x^2$, and so the total profit is $P(x) = R(x) - C(x) = -3x^2 + 36x - 18$. Profit is maximized when $P'(x) = 0$.

$$0 = P'(x) = -6x + 36 \Longrightarrow x = 6$$

(b) By marginal analysis, the cost of the 3rd unit is approximately:

$$C'(2) = \left(3q^2 + 40q + 200\right)\Big|_{q=2} = 12 + 80 + 200 = 292$$

Ex. O-19 **4.6** Sp18 **Quiz**

Use a linear approximation to estimate the value of $\dfrac{1}{\sqrt[4]{0.96}}$.

qs18

Solution O-19

We use the tangent line to $f(x) = x^{-1/4}$ at $x = 1$.

Point of Tangency:	$(1, f(1)) = (1, 1)$
Slope of Line:	$f'(x) = -\frac{1}{4}x^{-5/4}$; $f'(1) = -\frac{1}{4}$
Equation of Line:	$y = 1 - \frac{1}{4}(x - 1)$

This means that $x^{-1/4} \approx 1 - \frac{1}{4}(x - 1)$ if x is near 1. Hence we have the estimate:

$$(0.96)^{-1/4} \approx 1 - \frac{1}{4}(0.96 - 1) = 1.01$$

Ex. O-20 **4.6** Sp20 **Quiz**

Use a linear approximation to estimate the value of $(2.01)^5 - 5 \cdot (2.01)^3 + 9$.

qs20

Solution O-20

We use the tangent line to $f(x) = x^5 - 5x^3 + 9$ at $x = 2$.

Point of Tangency:	$(2, f(2)) = (2, 1)$
Slope of Line:	$f'(x) = 5x^4 - 15x^2$; $f'(2) = 20$
Equation of Line:	$y = 1 + 20(x - 2)$

This means that $x^5 - 5x^3 + 9 \approx 1 + 20(x - 2)$ if x is near 2. Hence we have the estimate:

$$(2.01)^5 - 5 \cdot (2.01)^3 + 9 \approx 1 + 20(2.01 - 2) = 1.2$$

Ex. O-21 4.6 Sp20 Quiz

The total cost of producing x units is $C(x) = 10x^3 + 500x^2 + 1000x + 24000$.

(a) Write a numerical expression equal to the exact cost of the 11th unit.

(b) Use marginal analysis to estimate the cost of the 11th unit. Write your answer as an exact decimal or as a fraction of integers.

qs20

Solution O-21

(a) $C(11) - C(10)$

(b) We use the standard approximation $MC(x) = C(x+1) - C(x) \approx C'(x)$, and so we estimate $MC(10) \approx C'(10)$.

$$C'(x) = 30x^2 + 1000x + 1000 \Longrightarrow MC(10) \approx C'(10) = 14,000$$

Ex. O-22 4.6 Su22 Quiz

Use linear approximation to estimate the number $\dfrac{1}{(2.9)^2}$. Do not simplify your answer.

qsu22

Solution O-22

We use the tangent line to $f(x) = x^{-2}$ at $x = 3$.

Point of Tangency:	$(3, f(3)) = (3, \frac{1}{9})$	
Slope of Line:	$f'(x) = -2x^{-3}$;	$f'(3) = \frac{2}{27}$
Equation of Line:	$y = \frac{1}{9} - \frac{2}{27}(x-3)$	

This means that $\frac{1}{x^2} \approx \frac{1}{9} - \frac{2}{27}(x-3)$ if x is near 3. Hence we have the estimate:

$$\frac{1}{(2.9)^2} \approx \frac{1}{9} - \frac{2}{27}(2.9-3) = \frac{28}{270}$$

Ex. O-23 4.6 Su22 Quiz

The position of a particle at time t is given by $x(t) = 5 + 20t^{3/5} + t$. Use linear approximation to estimate the change in the particle's position between $t = 32$ and $t = 35$. Do not simplify your answer.

qsu22

Solution O-23

We seek an estimate of the change in cost: $\Delta x = x(35) - x(32)$. We use the tangent line to $x(t)$ at $t = 32$.

Point of Tangency:	$(32, x(32))$	
Slope of Line:	$x'(t) = 12t^{-2/5} + 1$;	$x'(32) = 4$
Equation of Line:	$y - x(32) = 4(t-32)$	

This means that $x(t) - x(32) \approx 4(t-32)$ if t is near 32. Hence we have the estimate:

$$\Delta x = x(35) - x(32) \approx 4(35-32) = 12$$

The position will increase by approximately 12.

Ex. O-24 4.6 Su22 Quiz

If x units of a certain product are being produced, the marginal cost is

$$\frac{dC}{dx} = 5 + 12x + 20x^{3/2}$$

Suppose the total cost of producing 1 unit is 100 (measured in thousands of dollars). Calculate the total cost of producing 4 units.

qsu22

Solution **O-24**

We compute the antiderivative to obtain the total cost.

$$C(x) = \int (5 + 12x + 20x^{3/2})\, dx = 5x + 6x^2 + 8x^{5/2} + K$$

where K is some constant. We are given the condition $C(1) = 100$, whence $19 + K = 100$, or $K = 81$. The total cost function is thus:

$$C(x) = 5x + 6x^2 + 8x^{5/2} + 81$$

So the total cost of producing 4 units is

$$C(4) = 20 + 24 + 8 \cdot 32 + 81 = 381$$

Ex. O-25 **4.6**

For each part, use a linear approximation to estimate the given value. *Each answer should be an exact rational number.*

(a) $e^{0.1}$

(b) $\ln(1.04)$

(c) $\dfrac{1}{\sqrt[3]{25}}$

(d) $\left(\sec\left(\frac{\pi}{4} - 0.02\right)\right)^2$

(e) $\sqrt{96}$

(f) $(5.01)^3 - 2(5.01) + 3$

c4s6

Solution **O-25**

(a) We use the tangent line to $f(x) = e^x$ at $x = 0$.

$$\begin{aligned} &\text{Point of Tangency:} && (0, f(0)) = (0, 1) \\ &\text{Slope of Line:} && f'(x) = e^x; \qquad f'(0) = 1 \\ &\text{Equation of Line:} && y = 1 + x \end{aligned}$$

This means that $e^x \approx 1 + x$ if x is near 0. Hence we have the estimate:

$$e^{0.1} \approx 1 + 0.1 = 1.1$$

(b) We use the tangent line to $f(x) = \ln(x)$ at $x = 1$.

$$\begin{aligned} &\text{Point of Tangency:} && (1, f(1)) = (1, 0) \\ &\text{Slope of Line:} && f'(x) = \tfrac{1}{x}; \qquad f'(1) = 1 \\ &\text{Equation of Line:} && y = x - 1 \end{aligned}$$

This means that $\ln(x) \approx x - 1$ if x is near 1. Hence we have the estimate:

$$\ln(1.04) \approx 1.04 - 1 = 0.04$$

(c) We use the tangent line to $f(x) = x^{-1/3}$ at $x = 27$.

$$\begin{aligned} &\text{Point of Tangency:} && (27, f(27)) = (27, \tfrac{1}{3}) \\ &\text{Slope of Line:} && f'(x) = -\tfrac{1}{3}x^{-4/3}; \qquad f'(27) = -\tfrac{1}{243} \\ &\text{Equation of Line:} && y = \tfrac{1}{3} - \tfrac{1}{243}(x - 27) \end{aligned}$$

This means that $x^{-1/3} \approx \frac{1}{3} - \frac{1}{243}(x - 27)$ if x is near 27. Hence we have the estimate:

$$25^{-1/3} \approx \frac{1}{3} - \frac{1}{243}(25 - 27) = \frac{83}{243}$$

(d) We use the tangent line to $f(x) = \sec^2(x)$ at $x = \frac{\pi}{4}$.

O-25

Point of Tangency: $(\frac{\pi}{4}, f(\frac{\pi}{4})) = (\frac{\pi}{4}, 2)$

Slope of Line: $f'(x) = 2\sec(x)\sec(x)\tan(x)$; $f'(\frac{\pi}{4}) = 2 \cdot \sqrt{2} \cdot \sqrt{2} \cdot 1 = 4$

Equation of Line: $y = 2 + 4(x - \frac{\pi}{4})$

This means that $\sec^2(x) \approx 2 + 4(x - \frac{\pi}{4})$ if x is near $\frac{\pi}{4}$. Hence we have the estimate:

$$\sec^2\left(\frac{\pi}{4} - 0.02\right) \approx 2 + 4\left(\frac{\pi}{4} - 0.02 - \frac{\pi}{4}\right) = 1.92$$

(e) We use the tangent line to $f(x) = x^{1/2}$ at $x = 100$.

Point of Tangency: $(100, f(100)) = (100, 10)$

Slope of Line: $f'(x) = \frac{1}{2}x^{-1/2}$; $f'(100) = \frac{1}{20}$

Equation of Line: $y = 10 + \frac{1}{20}(x - 100)$

This means that $\sqrt{x} \approx 10 + \frac{1}{20}(x - 100)$ if x is near 100. Hence we have the estimate:

$$\sqrt{96} \approx 10 + \frac{1}{20}(96 - 100) = 9.8$$

(f) We use the tangent line to $f(x) = x^3 - 2x + 3$ at $x = 5$.

Point of Tangency: $(5, f(5)) = (5, 118)$

Slope of Line: $f'(x) = 3x^2 - 2$; $f'(5) = 73$

Equation of Line: $y = 118 + 73(x - 5)$

This means that $x^3 - 2x + 3 \approx 118 + 73(x - 5)$ if x is near 5. Hence we have the estimate:

$$(5.01)^3 - 2(5.01) + 3 \approx 118 + 73(5.01 - 5) = 118.73$$

Ex. O-26 4.6

When the level of production is q units, the total cost (in dollars) is $C(q) = q^5 - 2q^3 + 3q^2 - 2$. The current level of production is 3 units, and the manufacturer is planning to increase this to 3.01 units. Use a linear approximation to estimate how the total cost will change as a result.

c4s6

Solution O-26

We seek an estimate of the change in cost: $\Delta C = C(3.01) - C(3)$. We use the tangent line to $C(q)$ at $q = 3$.

Point of Tangency: $(3, C(3))$

Slope of Line: $C'(q) = 5q^4 - 6q^2 + 6q$; $C'(3) = 369$

Equation of Line: $y - C(3) = 369(q - 3)$

This means that $C(q) - C(3) \approx 369(q - 3)$ if q is near 3. Hence we have the estimate:

$$\Delta C = C(3.01) - C(3) \approx 369(3.01 - 3) = 3.69$$

The total cost will increase by approximately 3.69 dollars.

Ex. O-27 4.6

When the level of production is q units, the total cost (in dollars) is $C(q) = 3q^2 + q + 500$.

(a) What is the exact cost of manufacturing the 41st unit?

(b) Use marginal analysis to estimate the cost of manufacturing the 41st unit.

c4s6

Solution **O-27**

(a) The exact cost (in dollars) is

$$MC(40) = C(41) - C(40) = 1984 - 1740 = 244$$

(b) We have the standard estimate for marginal cost: $MC(q) \approx C'(q)$.

$$C'(q) = 6q + 1 \implies MC(40) \approx C'(40) = 241$$

Ex. O-28 4.6

The total revenue from selling x units of a certain product is $R(x) = 40 - \frac{200}{x+5}$. Using marginal analysis, estimate the revenue from selling the 6th unit.

unit4

Solution **O-28**

The revenue from the 6th item is $MR(5) \approx R'(5)$.

$$R'(x) = \frac{200}{(x+5)^2} \implies MR(5) \approx R'(5) = \frac{200}{(5+5)^2} = 2$$

Ex. O-29 4.6

Use a linear approximation to estimate the value of $(16.32)^{1/4}$.

unit4

Solution **O-29**

We use the tangent line to $f(x) = x^{1/4}$ at $x = 16$.

Point of Tangency:	$(16, f(16)) = (16, 2)$
Slope of Line:	$f'(x) = \frac{1}{4}x^{-3/4}$; $\quad f'(16) = \frac{1}{32}$
Equation of Line:	$y = 2 + \frac{1}{32}(x - 16)$

This means that $x^{1/4} \approx 2 + \frac{1}{32}(x - 16)$ if x is near 16. Hence we have the estimate:

$$(16.32)^{1/4} \approx 2 + \frac{1}{32}(16.32 - 16) = 2.01$$

Ex. O-30 4.6

Use linear approximation to estimate $(33.6)^{1/5}$.

unit4

Solution **O-30**

We use the tangent line to $f(x) = x^{1/5}$ at $x = 32$.

Point of Tangency:	$(32, f(32)) = (32, 2)$
Slope of Line:	$f'(x) = \frac{1}{5}x^{-4/5}$; $\quad f'(2) = \frac{1}{80}$
Equation of Line:	$y = 2 + \frac{1}{80}(x - 32)$

This means that $x^{1/5} \approx 2 + \frac{1}{80}(x - 32)$ if x is near 32. Hence we have the estimate:

$$(33.6)^{1/5} \approx 2 + \frac{1}{80}(33.6 - 32) = 2.02$$

Ex. O-31 4.6

Use linear approximation to estimate $\sec\left(\frac{\pi}{6} + 0.12\right) - \sec\left(\frac{\pi}{6}\right)$.

other

Solution **O-31**

We use the tangent line to $f(x) = \sec(x)$ at $x = \frac{\pi}{6}$.

Point of Tangency: $(\frac{\pi}{6}, f(\frac{\pi}{6})) = (\frac{\pi}{6}, \sec(\frac{\pi}{6}))$

Slope of Line: $f'(x) = \sec(x)\tan(x)$; $\quad f'(\frac{\pi}{6}) = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3}$

Equation of Line: $y - \sec(\frac{\pi}{6}) = \frac{2}{3}(x - \frac{\pi}{6})$

This means that $\sec(x) - \sec(\frac{\pi}{6}) \approx \frac{2}{3}(x - \frac{\pi}{6})$ if x is near $\frac{\pi}{6}$. Hence we have the estimate:

$$\sec\left(\frac{\pi}{6} + 0.12\right) - \sec\left(\frac{\pi}{6}\right) \approx \frac{2}{3}\left(\frac{\pi}{6} + 0.12 - \frac{\pi}{6}\right) = 0.08$$

Ex. J-36 **3.8, 4.6** **⋆Challenge**

Consider the curve described by the equation

$$\frac{x - y^3}{y + x^2} = x - 12$$

(a) Find an equation for the line tangent to this curve at $(-1, 4)$.

(b) There is a point on the curve with coordinates $(-1.1, b)$. Use linear approximation to estimate b. Round to three decimal places.

(c) There is a point on the curve with coordinates $(a, 4.2)$. Use linear approximation to estimate a. Round to three decimal places.

challenge

Solution **J-36**

(a) We write the equation as follows to make differentiation easier:

$$x - y^3 = xy + x^3 - 12y - 12x^2$$

Differentiating each side with respect to x gives:

$$1 - 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx} + 3x^2 - 12\frac{dy}{dx} - 24x$$

We now substitute $x = -1$ and $y = 4$:

$$1 - 48\frac{dy}{dx} = 4 - \frac{dy}{dx} + 3 - 12\frac{dy}{dx} + 24 \Longrightarrow \frac{dy}{dx} = -\frac{6}{7}$$

So an equation of the tangent line is:

$$y - 4 = -\frac{6}{7}(x + 1)$$

(b) Since $(-1.1, b)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate b. That is, the point $(-1.1, b)$ approximately satisfies the equation of the tangent line:

$$b - 4 \approx -\frac{6}{7}(-1.1 + 1) \Longrightarrow b \approx \frac{28}{6.6} \approx 4.242$$

(c) Since $(a, 4.2)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate a. That is, the point $(a, 4.2)$ approximately satisfies the equation of the tangent line:

$$4.2 - 4 \approx -\frac{6}{7}(a + 1) \Longrightarrow a \approx -\frac{7.4}{6.6} \approx -1.233$$

Ex. O-32 **4.6** **⋆Challenge**

The acceleration (measured in m/s^2) of a particle moving along the x-axis is given by

$$a(t) = 14t^{3/4} - 6t^2 + 1$$

and the particle is at rest (zero velocity) when $t = 1$. Use a linear approximation to estimate the particle's change in

challenge

4.6 ⋆Challenge

position between $t = 16$ and $t = 16.02$.

challenge

Solution **O-32**

We seek an estimate of the change in position: $\Delta x = x(16.02) - x(16)$. We use the tangent line to $x(t)$ at $t = 16$. Note that the slope of this tangent line is given by $v(16)$, so we first find the velocity function by antidifferentiating the acceleration $a(t)$.

$$v(t) = \int (14t^{3/4} - 6t^2 + 1)\, dt = 8t^{7/4} - 2t^3 + t + C$$

The particle is at rest when $t = 1$, or $v(1) = 0$. So $8 - 2 + 1 + C = 0$, whence $C = -7$ and our velocity function is

$$v(t) = 8t^{7/4} - 2t^3 + t - 7$$

Now we return to finding the tangent line to $x(t)$ at $t = 16$.

Point of Tangency:	$(16, x(16))$
Slope of Line:	$x'(t) = v(t) = 8t^{7/4} - 2t^3 + t - 7; \qquad x'(16) = v(16) = -7159$
Equation of Line:	$y - x(16) = -7159(t - 16)$

This means that $x(t) - x(16) \approx -7159(t - 16)$ if t is near 16. Hence we have the estimate:

$$\Delta x = x(16.02) - x(16) \approx -7159(16.02 - 16) = -143.18$$

The particle's position decreases by approximately 143.18.

§4.7: L'Hôpital's Rule

Ex. P-1 4.7 Fa17 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 0}(1-\sin(4x))^{6/x}$

(b) $\lim_{x\to 1}\left(\frac{xe^{4x}+4e^4-5e^4x}{(x-1)^2}\right)$

exf17

Solution P-1

(a) First determine the indeterminate exponent.

$$\lim_{x\to 0}\underbrace{(1-\sin(4x))^{6/x}}_{1^\infty} := L$$

Now consider $\ln(L)$ and use l'Hospital's Rule.

$$\ln(L) = \lim_{x\to 0}\underbrace{\left(\frac{6\ln(1-\sin(4x))}{x}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\left(\frac{6\cdot\frac{1}{1-\sin(4x)}\cdot(-4\cos(4x))}{1}\right) = -24$$

So $\ln(L)=-24$, whence $L=e^{-24}$.

(b) Use l'Hospital's Rule twice.

$$\lim_{x\to 1}\underbrace{\left(\frac{xe^{4x}+4e^4-5e^4x}{(x-1)^2}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 1}\underbrace{\left(\frac{4xe^{4x}+e^{4x}-5e^4}{2(x-1)}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 1}\left(\frac{16xe^{4x}+8e^{4x}}{2}\right) = 12e^4$$

Ex. P-2 4.7 Sp18 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 1}\left(\frac{x^{1/4}-1}{e^{2x}-e^2}\right)$

(b) $\lim_{x\to 1}\left((x-1)\tan\left(\frac{\pi x}{2}\right)\right)$

exs18

Solution P-2

(a) Use l'Hospital's Rule.

$$\lim_{x\to 1}\underbrace{\left(\frac{x^{1/4}-1}{e^{2x}-e^2}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 1}\left(\frac{\frac{1}{4}x^{-3/4}}{2e^{2x}}\right) = \frac{\frac{1}{4}}{2e^2} = \frac{1}{8e^2}$$

(b) Write the product as a quotient and then use l'Hospital's Rule.

$$\lim_{x\to 1}\underbrace{\left((x-1)\tan\left(\frac{\pi x}{2}\right)\right)}_{0\cdot\pm\infty} = \lim_{x\to 1}\underbrace{\left(\frac{(x-1)\sin\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 1}\left(\frac{\sin\left(\frac{\pi x}{2}\right)+(x-1)\cos\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}{-\sin\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}\right) = -\frac{2}{\pi}$$

Ex. P-3 4.7 Fa18 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 0}\left(\frac{1-\cos(9x)}{x^2}\right)$

(b) $\lim_{x\to 0}(1-3x)^{5/x}$

exf18

Solution **P-3**

(a) Direct substitution of $x = 0$ gives the indeterminate form $\frac{0}{0}$, whence we may use L'Hospital's Rule (twice).

$$\lim_{x\to 0}\left(\frac{1-\cos(9x)}{x^2}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{9\sin(9x)}{2x}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{81\cos(9x)}{2}\right) = \frac{81}{2}$$

(b) Direct substitution of $x = 0$ gives the indeterminate form $1^{\pm\infty}$, whence we let L be the desired limit and consider $\ln(L)$.

$$\ln(L) = \ln\left(\lim_{x\to 0}(1-3x)^{5/x}\right) = \lim_{x\to 0}\ln\left((1-3x)^{5/x}\right) = \lim_{x\to 0}\left(\frac{5\ln(1-3x)}{x}\right)$$

Direct substitution of $x = 0$ now gives the indeterminate form $\frac{0}{0}$, whence we may use L'Hospital's Rule.

$$\lim_{x\to 0}\left(\frac{5\ln(1-3x)}{x}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{5\cdot\frac{1}{1-3x}\cdot(-3)}{1}\right) = -15$$

Hence $\ln(L) = -15$, and so $L = e^{-15}$.

Ex. E-2 **2.5, 4.7** Sp19 **Exam**

The parts of this problem *are* related!

(a) Show that $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-3}\right) = 1$.

(b) Calculate the following limit or show it does not exist.

$$\lim_{x\to\infty}\left(\frac{x}{x-3}\right)^x$$

Hint: First use part (a) to identify the appropriate indeterminate form.

exs19

Solution **E-2**

(a) We have the following.

$$\lim_{x\to\infty}\left(\frac{x}{x-3}\right) = \lim_{x\to\infty}\left(\frac{1}{1-\frac{3}{x}}\right) = \frac{1}{1-0} = 1$$

(b) The result of part (a) implies that as $x \to \infty$, our limit has the indeterminate form 1^∞. Let L be the desired limit. Then we have the following.

$$\ln(L) = \lim_{x\to\infty}\ln\left[\left(\frac{x}{x-3}\right)^x\right] = \lim_{x\to\infty}\left[x\ln\left(\frac{x}{x-3}\right)\right] = \lim_{x\to\infty}\left[\frac{\ln\left(\frac{x}{x-3}\right)}{\frac{1}{x}}\right]$$

As $x \to \infty$, we now have the indeterminate form $\frac{0}{0}$, so we may use L'Hospital's Rule.

$$\ln(L) \overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{x-3}{x}\cdot\frac{(x-3)\cdot 1 - x\cdot 1}{(x-3)^2}}{\frac{-1}{x^2}}\right) = \lim_{x\to\infty}\left(\frac{3x}{x-3}\right) = \lim_{x\to\infty}\left(\frac{3}{1-\frac{3}{x}}\right) = 3$$

We have found that $\ln(L) = 3$, whence $L = e^3$.

Ex. P-4 **4.7** Fa19 **Exam**

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

4.7 Fa19 Exam

(a) $\lim_{x\to\pi}\left(\dfrac{1+\cos(x)}{(x-\pi)^2}\right)$

(b) $\lim_{x\to\infty}\left(1-\dfrac{12}{x}\right)^{5x}$

(c) $\lim_{x\to 1}\left(\dfrac{x^3-2x^2-5x+6}{x^3+x^2+x-3}\right)$

(d) $\lim_{x\to 4^+}\left(\dfrac{2x-x^2}{x-4}\right)$

Solution P-4

(a) Use LR twice.

$$\lim_{x\to 0}\underbrace{\left(\frac{1+\cos(x)}{(x-\pi)^2}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\underbrace{\left(\frac{-\sin(x)}{2(x-\pi)}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\left(\frac{-\cos(x)}{2}\right) = -\frac{1}{2}$$

(b) First determine the indeterminate exponent.

$$\lim_{x\to\infty}\underbrace{\left(1-\frac{12}{x}\right)^{5x}}_{1^\infty} := L$$

Now consider $\ln(L)$ and use LR where appropriate.

$$\ln(L) = \lim_{x\to\infty}\underbrace{\left(5x\ln\left(1-\frac{12}{x}\right)\right)}_{\infty\cdot 0} = \lim_{x\to\infty}\underbrace{\left(\frac{5\ln\left(1-\frac{12}{x}\right)}{1/x}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{5}{1-\frac{12}{x}}\cdot\frac{12}{x^2}}{-1/x^2}\right) = \lim_{x\to\infty}\left(\frac{-60}{1-\frac{12}{x}}\right) = -60$$

So $\ln(L) = -60$, whence $L = e^{-60}$.

(c) Standard application of LR.

$$\lim_{x\to 1}\underbrace{\left(\frac{x^3-2x^2-5x+6}{x^3+x^2+x-3}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 1}\left(\frac{3x^2-4x-5}{3x^2+2x+1}\right) = \frac{3-4+5}{3+2+1} = -1$$

(d) Substitution of $x=4$ gives $-\frac{8}{0}$, which indicates that the one-sided limit is infinite. Now we do sign analysis to determine the sign of infinity. If $x\to 4^+$, we may assume x is slightly greater than 4. This means $x-4$ is slightly greater than 0, so that $x-4>0$. So now we have

$$\lim_{x\to 4+}\left(\frac{2x-x^2}{x-4}\right) = \frac{-8}{0^+} = -\infty$$

Ex. P-5 4.7 Sp20 Exam

Calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

$$\lim_{x\to 0^+}\left(\sqrt{12x+9}-\sqrt{2x+4}\right)^{1/x}$$

exs20

Solution P-5

Direct substitution of $x=0$ gives the indeterminate form 1^∞. So we use logarithms to write the limit as a quotient,

P-5

and we use L'Hospital's Rule.

$$\begin{aligned} L &= \lim_{x\to 0^+} \left(\sqrt{12x+9}-\sqrt{2x+4}\right)^{1/x} \\ \ln(L) &= \lim_{x\to 0^+} \ln\left(\left(\sqrt{12x+9}-\sqrt{2x+4}\right)^{1/x}\right) \\ \ln(L) &= \lim_{x\to 0^+} \frac{\ln\left(\sqrt{12x+9}-\sqrt{2x+4}\right)}{x} \\ \ln(L) &\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sqrt{12x+9}-\sqrt{2x+4}}\cdot\left(\frac{6}{\sqrt{12x+9}}-\frac{1}{\sqrt{2x+4}}\right)}{1} \\ \ln(L) &= \frac{\frac{1}{1}\cdot\left(\frac{6}{3}-\frac{1}{2}\right)}{1} = \frac{3}{2} \end{aligned}$$

So $\ln(L) = \frac{3}{2}$, whence $L = e^{3/2}$.

Ex. P-6 4.7 Sp20 Exam

Suppose you want to compute a limit that is in the form of a quotient, i.e., a limit of the form:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$$

Suppose you have already determined that L'Hospital's Rule is applicable. Explain the next step in your calculation, i.e., how do you apply L'Hospital's Rule? *Your answer may contain either English, mathematical symbols, or both.*

exs20

Solution P-6

Compute the limit $\lim\limits_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$.

Ex. P-7 4.7 Sp20 Exam

Each of the following limits is written in the form of a quotient. Which limits can be calculated using L'Hospital's Rule directly, i.e., by applying L'Hospital's Rule as the immediately next step without any other algebra or modification? Select all that apply.

(a) $\lim\limits_{x\to\pi}\left(\frac{\sin(7x)}{x}\right)$

(b) $\lim\limits_{x\to 2}\left(\frac{x^3+3x-14}{x^2-5x+6}\right)$

(c) $\lim\limits_{x\to\infty}\left(\frac{x^{-1}+5}{x^{-2}+8}\right)$

(d) $\lim\limits_{x\to 9^-}\left(\frac{x^{3/2}+x-36}{x-\sqrt{x}-6}\right)$

(e) $\lim\limits_{x\to\infty}\left(\frac{e^x+10}{e^x-3}\right)$

(f) $\lim\limits_{x\to-\infty}\left(\frac{e^x+10}{e^x-3}\right)$

exs20

Solution P-7

The only indeterminate quotients (for which L'Hospital's Rule is directly applicable) are $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Hence the only limits above that can be computed with L'Hospital's Rule are: (b), (d), and (e).

Ex. D-2 2.4, 4.7 Sp20 Exam

Which of the following limits are equal to $+\infty$? Select all that apply.

(a) $\lim\limits_{x\to 5^-}\left(\frac{x^2+25}{5-x}\right)$

(b) $\lim\limits_{x\to 5^+}\left(\frac{x^2+25}{5-x}\right)$

(c) $\lim\limits_{x\to -3^-}\left(\frac{x^3}{|x+3|}\right)$

(d) $\lim\limits_{x\to 0^-}\left(\frac{x^4-2x-5}{\sin(x)}\right)$

(e) $\lim\limits_{x\to 1^+}\left(\frac{x^6-x^2}{x-1}\right)$

exs20

Solution D-2

Direct substitution of each x-value gives $\frac{\text{non-zero }\#}{0}$ only for (a)–(d). A sign analysis of numerator and denominator

D-2

then shows that only (a) and (d) are equal to $+\infty$. As for (e), we apply L'Hospital's Rule and find

$$\lim_{x\to 1^+}\left(\frac{x^6-x^2}{x-1}\right) \stackrel{H}{=} \lim_{x\to 1^+}\left(\frac{6x^5-2x}{1}\right) = 4$$

Hence only (a) and (d) are correct choices.

Ex. P-8 4.7 Sp20 Exam

Calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

$$\lim_{x\to\infty}\left(5x^3+2x^2+8\right)^{1/\ln(x)}$$

exs20

Solution P-8

Direct substitution of "$x \to \infty$" gives the indeterminate form ∞^0. So we use logarithms to write the limit as a quotient, and we use L'Hospital's Rule.

$$\begin{aligned}
L &= \lim_{x\to\infty}\left(5x^3+2x^2+8\right)^{1/\ln(x)} \\
\ln(L) &= \lim_{x\to\infty}\ln\left(\left(5x^3+2x^2+8\right)^{1/\ln(x)}\right) \\
\ln(L) &= \lim_{x\to\infty}\left(\frac{\ln(5x^3+2x^2+8)}{\ln(x)}\right) \\
\ln(L) &\stackrel{H}{=} \lim_{x\to\infty}\left(\frac{\frac{1}{5x^3+2x^2+8}\cdot(15x^2+4x)}{\frac{1}{x}}\right) \\
\ln(L) &= \lim_{x\to\infty}\left(\frac{15x^3+4x^2}{5x^3+2x^2+8}\right) \\
\ln(L) &= \lim_{x\to\infty}\left(\frac{15+\frac{4}{x}}{5+\frac{2}{x}+\frac{8}{x^3}}\right) = \frac{15+0+0}{5+0+0} = 3
\end{aligned}$$

So $\ln(L) = 3$, whence $L = e^3$.

Ex. P-9 4.7 Sp20 Exam

Suppose you have determined

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = \infty$$

and you want to calculate the following limit:

$$L = \lim_{x\to a}\left(f(x)g(x)\right)$$

You recall that to calculate L, you have to use L'Hospital's Rule. What is the next step you must take before you are able to apply L'Hospital's Rule directly to the limit L? *Your answer may contain either English, mathematical symbols, or both.*

exs20

Solution P-9

Write the product $f(x)g(x)$ as a quotient instead. For example, $\dfrac{g(x)}{1/f(x)}$.

Ex. P-10 4.7 Sp20 Exam

Which of the following are indeterminate forms? Recall that in this course, we have learned that limits with indeterminate forms may often be computed using L'Hospital's Rule.

exs20

4.7 Sp20 Exam

(a) $\frac{0}{0}$

(b) $0 \cdot \infty$

(c) $\frac{\infty}{-\infty}$

(d) $\frac{0}{\infty}$

(e) 2^∞

(f) $3 \cdot (-\infty)$

(g) $\infty \cdot (-\infty)$

(h) ∞^0

(i) ∞^∞

exs20

Solution P-10

The only indeterminate forms are (a), (b), (c), and (h). The other choices are equivalent to, respectively: (d) 0, (e) ∞, (f) $-\infty$, (g) $-\infty$, and (i) ∞.

Ex. P-11 4.7 Sp20 Exam

A student is asked to calculate the following limit using l'Hospital's Rule and to show all their work.

$$L = \lim_{x\to 0}\left(\frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)}\right)$$

The student decides to cheat, so they find the solution online (shown below) and they submit the work as their own!

$$\begin{aligned}
L &= \lim_{x\to 0}\left(\frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)}\right) && (1)\\
&= \lim_{x\to 0}\left(\frac{2\cos(2x) + 34x + 2}{8x + \sec(x)^2}\right) && (2)\\
&= \lim_{x\to 0}\left(\frac{-4\sin(2x) + 34}{8 + 2\sec(x)^2\tan(x)}\right) && (3)\\
&= \frac{-4\sin(0) + 34}{8 + 2\sec(0)^2\tan(0)} && (4)\\
&= \frac{0 + 34}{8 + 0} && (5)\\
&= \frac{17}{4} && (6)
\end{aligned}$$

Unfortunately, this solution contains an error, and so the student lost all credit for the problem. The student was also later determined to be responsible for cheating, and so they earned a grade of 0 on the entire exam!

Your task is to find and correct the error(s). Answer the following questions.

(a) There may be several errors in this solution. Which line is the first incorrect line?

(b) Explain the error in the first incorrect line in your own words.

(c) Calculate the correct value of L (the original limit).

exs20

Solution P-11

(a) The first incorrect line is line (3).

(b) In the transition from line (2) to line (3), the student has differentiated the numerator and denominator separately, presumably to use l'Hospital's Rule. However, this is an incorrect application as the limit in line (2) does not have an indeterminate form. L'Hospital's Rule cannot be used there.

(c) Substitution of $x = 0$ in line (2) gives the correct value: $L = 4$.

Ex. P-12 4.7 Su20 Exam

Calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

$$\lim_{x\to 1}\left(\frac{\tan(\pi x)}{\sqrt{2+x^3} - \sqrt{2+x}}\right)$$

exsu20

Solution **P-12**

Direct substitution of $x = 1$ gives the indeterminate form "$\frac{0}{0}$". So we use l'Hospital's rule.

$$\lim_{x\to 1}\left(\frac{\tan(\pi x)}{\sqrt{2+x^3}-\sqrt{2+x}}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{\pi\sec(\pi x)^2}{\frac{3x^2}{2\sqrt{2+x^3}}-\frac{1}{2\sqrt{2+x}}}\right) = \frac{\pi\cdot 1^2}{\frac{3}{2\sqrt{3}}-\frac{1}{2\sqrt{3}}} = \sqrt{3}\pi$$

Ex. P-13 4.7 Su20 Exam

Consider the following limit.

$$L = \lim_{x\to -3}(4+x)^{7/(6+2x)}$$

(a) What indeterminate form does this limit have?

(b) Explain why l'Hospital's rule cannot be used on this limit in its current form.

(c) Calculate the value of L.

exsu20

Solution **P-13**

(a) Direct substitution of $x = -3$ gives "$1^{1/0}$", equivalent to the indeterminate form "1^{∞}".

(b) L'Hospital's rule cannot be used because the limit is not in the form of a quotient.

(c) We take logarithms and then use l'Hospital's rule. Let L be the given limit. Then we have:

$$\ln(L) = \lim_{x\to -3}\ln\left((4+x)^{7/(6+2x)}\right) = \lim_{x\to -3}\left(\frac{7}{6+2x}\cdot\ln(4+x)\right) = \lim_{x\to -3}\left(\frac{7\ln(4+x)}{6+2x}\right) = \lim_{x\to -3}\left(\frac{7\cdot\frac{1}{4+x}}{2}\right) = \frac{7}{2}$$

So $\ln(L) = \frac{7}{2}$, whence $L = e^{7/2}$.

Ex. P-14 4.7 Fa20 Exam

Consider the limit $L = \lim_{x\to 2^-}((x-2)\ln(2-x))$.

(a) Does this limit have an indeterminate form? If so, which indeterminate form?

(b) Explain why l'Hospital's rule cannot be used on this limit in its current form.

(c) Write the limit in an equivalent form to which l'Hospital's rule may be applied.

Note: You are not required to calculate the limit; do not attempt to do so.

exf20

Solution **P-14**

(a) Yes, the form $0\cdot(-\infty)$.

(b) The expression is not written as an indeterminate quotient.

(c) One possibility is $\lim_{x\to 2^-}\left(\frac{\ln(2-x)}{\frac{1}{x-2}}\right)$.

Ex. P-15 4.7 Fa20 Exam

Suppose $f'(x)$ is continuous with $f(3) = 2$ and $f'(3) = -8$. Calculate the following limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

$$\lim_{x\to 1}\left(\frac{2x^4 - f(3x^{1/4})}{x^2-4x+3}\right)$$

exf20

Solution **P-15**

Direct substitution of $x = 1$ gives "$\frac{0}{0}$", and so we use l'Hospital's rule, followed by direct substitution.

$$\lim_{x\to 1}\left(\frac{2x^4 - f(3x^{1/4})}{x^2-4x+3}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{8x^3 - f'(3x^{1/4})\cdot\frac{3}{4}x^{-3/4}}{2x-4}\right) = \frac{8-(-8)\cdot\frac{3}{4}}{-2} = -7$$

Ex. P-16 4.7 Sp21 Exam

Suppose $f''(x)$ is continuous. You are also given the following values:

$$f\left(\frac{1}{8}\right) = 20 \quad , \quad f'\left(\frac{1}{8}\right) = -22$$

Calculate the following limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

$$\lim_{x\to 8}\left(\frac{20 - f\left(\frac{1}{x}\right)}{x^2 + x - 72}\right)$$

exs21

Solution P-16

Since f is continuous, we may substitute $x = 8$ to obtain the indeterminate form "$\frac{0}{0}$". So we may use L'Hospital's Rule.

$$\lim_{x\to 8}\left(\frac{20 - f\left(\frac{1}{x}\right)}{x^2 + x - 72}\right) \overset{H}{=} \lim_{x\to 8}\left(\frac{-f'(\frac{1}{x}) \cdot (-\frac{1}{x^2})}{2x+1}\right)$$

Since f' is continuous, we substitute $x = 8$, and we find the limit is $\frac{-(-22)\cdot(-\frac{1}{8^2})}{17} = -\frac{11}{544}$.

Ex. D-13 2.4, 2.5, 4.7 Fa21 Exam

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to 1}\left(\frac{x^4 - x}{\ln(77x - 76)}\right)$

(b) $\displaystyle\lim_{x\to -\infty}\left(\frac{\sqrt{36x^2 + 63}}{31x}\right)$

(c) $\displaystyle\lim_{x\to 2^+} f(x)$, with $f(x) = \begin{cases} 1 + 4x & x \le 2 \\ \dfrac{x^2 - 4}{x - 2} & x > 2 \end{cases}$

(d) $\displaystyle\lim_{x\to 5^-}\left(\frac{\cos(\pi x)}{x^2 - 25}\right)$

exf21

Solution D-13

(a) Direct substitution gives "$\frac{0}{0}$", and so we use L'Hospital's Rule.

$$\lim_{x\to 1}\left(\frac{x^4 - x}{\ln(77x - 76)}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{4x^3 - 1}{\frac{1}{77x-76}\cdot 77}\right) = \frac{3}{77}$$

(b) We factor out x^2 from inside the square root in the numerator. Observe that since x goes to *negative* infinity, we have $\sqrt{x^2} = |x| = -x$.

$$\lim_{x\to -\infty}\left(\frac{\sqrt{36x^2 + 63}}{31x}\right) = \lim_{x\to -\infty}\left(\frac{-x\sqrt{36 + \frac{63}{x^2}}}{31x}\right) = \lim_{x\to -\infty}\left(\frac{-\sqrt{36 + \frac{63}{x^2}}}{31}\right) = \frac{-6}{31}$$

(c) We factor and cancel.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x\to 2^+}\left(\frac{(x-2)(x+2)}{x - 2}\right) = \lim_{x\to 2^+}(x + 2) = 4$$

(d) Direct substitution gives "$\frac{-1}{0}$", whence the one-sided limit must be infinite. Observe that the numerator is negative (goes to -1) as $x \to 5^-$, and the denominator goes to 0 but remains negative as $x \to 5^-$. (For instance, use test points such as $x = 4.99$.) Hence the desired limit is $\frac{-1}{0^-} = +\infty$.

Ex. D-14 2.4, 4.7 Fa21 Exam

For each part, find all vertical asymptotes of the given function.

(a) $f(x) = \dfrac{x^2 - 8x + 15}{x^2 - 9}$

(b) $g(x) = \dfrac{e^{x+3} - 1}{x^2 - 9}$

exf21

Solution **D-14**

(a) First factor and cancel.

$$f(x) = \frac{x^2 - 8x + 15}{x^2 - 9} = \frac{(x-3)(x-5)}{(x-3)(x+3)} = \frac{x-5}{x+3}$$

Hence $f(x)$ has a vertical asymptote at $x = -3$ only.

(b) We note that the denominator of $g(x)$ equals 0 only when $x = -3$ or $x = 3$. Direct substitution of $x = 3$ gives the expression "$\frac{e^6-1}{0}$" (nonzero number divided by 0), and so $x = 3$ is a vertical asymptote of $g(x)$. However, we have the following for $x = -3$ after using L'Hospital's Rule:

$$\lim_{x\to-3} g(x) = \lim_{x\to-3} \left(\frac{e^{x+3} - 1}{x^2 - 9}\right) \stackrel{H}{=} \lim_{x\to-3} \left(\frac{e^{x+3}}{2x}\right) = -\frac{1}{6}$$

Since this limit is not infinite, there is no vertical asymptote at $x = -3$.

Ex. P-17 **4.7** Sp22 **Exam**

For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to\pi} \left(\frac{\cos(6x) - 1}{(x-\pi)^2}\right)$

(b) $\displaystyle\lim_{x\to 0} \left(e^{2x} + 3x\right)^{1/x}$

exs22

Solution **P-17**

(a) Direct substitution of $x = \pi$ gives "$\frac{0}{0}$". So we use l'Hospital's Rule (twice).

$$\lim_{x\to\pi} \left(\frac{\cos(6x) - 1}{(x-\pi)^2}\right) \stackrel{H}{=} \lim_{x\to\pi} \left(\frac{-6\sin(6x)}{2(x-\pi)}\right) \stackrel{H}{=} \lim_{x\to\pi} \left(\frac{-36\cos(6x)}{2}\right) = \frac{-36\cdot 1}{2} = -18$$

(b) Direction substitution of $x = 0$ gives "1^∞". We let L be the desired limit, take logarithms, and use l'Hospital's Rule.

$$\ln(L) = \lim_{x\to 0} \ln\left(\left(e^{2x} + 3x\right)^{1/x}\right) = \lim_{x\to 0} \left(\frac{\ln(e^{2x} + 3x)}{x}\right) \stackrel{H}{=} \lim_{x\to 0} \left(\frac{\frac{1}{e^{2x}+3x}\cdot(2e^{2x} + 3)}{1}\right) = \frac{2+3}{1+0} = 5$$

We find that $\ln(L) = 5$, whence $L = e^5$.

Ex. M-28 **4.3/4.4, 4.7** Su22 **Exam**

Let $f(x) = x^2 e^x$.

(a) Calculate the vertical and horizontal asymptotes of f.

(b) Calculate the critical points of f. Then use the Second Derivative Test to classify each critical point of f as a local minimum or a local maximum. Show your work and label your answers clearly. ***Hint:*** The second derivative of f is $f''(x) = (x^2 + 4x + 2)e^x$.

exsu22

Solution **M-28**

(a) Since f is a product of functions that are continuous for all x, f is also continuous for all x, and thus f has no vertical asymptotes. For horizontal asymptotes, we have the following (use l'Hospital's rule on the limit at negative infinity):

$$\lim_{x\to\infty} (x^2 e^x) = (+\infty)\cdot(+\infty) = +\infty$$

$$\lim_{x\to-\infty} (x^2 e^x) = \lim_{x\to-\infty} \left(\frac{x^2}{e^{-x}}\right) \stackrel{H}{=} \lim_{x\to-\infty} \left(\frac{2x}{-e^{-x}}\right) \stackrel{H}{=} \lim_{x\to-\infty} \left(\frac{2}{e^{-x}}\right) = \frac{2}{\infty} = 0$$

Thus the only horizontal asymptote of f is $y = 0$.

(b) We first compute $f'(x)$.

$$f'(x) = 2xe^x + x^2 e^x = xe^x(2+x)$$

M-28

Thus the critical points (solutions to $f'(x) = 0$) are $x = 0$ and $x = -2$. Now we use the Second Derivative Test.

$$f''(0) = (x^2 + 4x + 2)e^x\big|_{x=0} = 2$$
$$f''(-2) = (x^2 + 4x + 2)e^x\big|_{x=-2} = -2e^{-2}$$

Since $f''(0) > 0$, $x = 0$ gives a local minimum of f. Since $f''(-2) < 0$, $x = -2$ gives a local maximum of f.

Ex. P-18 4.7 Su22 Exam

Let $f(x) = \dfrac{x\sin(Ax)}{\sin^2(2x)}$, where A is a constant. Suppose $\lim_{x\to 0} f(x) = -6$. Calculate A.

exsu22

Solution P-18

We first compute the given limit in terms of A. Substitution of $x = 0$ gives "$\frac{0}{0}$", so we use l'Hospital's rule (twice), then use direct substitution.

$$\lim_{x\to 0}\left(\frac{x\sin(Ax)}{\sin^2(2x)}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{\sin(Ax) + Ax\cos(Ax)}{4\sin(2x)\cos(2x)}\right)$$
$$\overset{H}{=} \lim_{x\to 0}\left(\frac{A\cos(Ax) + A\cos(Ax) - A^2x\sin(Ax)}{8\cos(2x)\cos(2x) - 8\sin(2x)\sin(2x)}\right) = \frac{A + A - 0}{2A(A-0)} = \frac{2A}{8-0} = \frac{A}{4}$$

Alternatively... we can use special trigonometric limits instead of l'Hospital's rule.

$$\lim_{x\to 0}\left(\frac{x\sin(Ax)}{\sin^2(2x)}\right) = \lim_{x\to 0}\left(\frac{2x}{\sin(2x)}\cdot\frac{2x}{\sin(2x)}\cdot\frac{\sin(Ax)}{Ax}\cdot\frac{x\cdot Ax}{2x\cdot 2x}\right) = 1\cdot 1\cdot 1\cdot\frac{A}{4} = \frac{A}{4}$$

We are given that the limit is -6, whence $\frac{A}{4} = -6$, and so $A = -24$.

Ex. P-19 4.7 Sp18 Quiz

Calculate the following limit or show it does not exist.

$$\lim_{x\to 0}\left(\frac{x - \ln(1+x)}{1 - \cos(2x)}\right)$$

qs18

Solution P-19

Substitution of $x = 0$ gives $\frac{0}{0}$, so we use L'Hospital's Rule (twice).

$$\lim_{x\to 0}\left(\frac{x - \ln(1+x)}{1 - \cos(2x)}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{1 - \frac{1}{1+x}}{2\sin(2x)}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{\frac{1}{(1+x)^2}}{4\cos(2x)}\right) = \frac{1}{4}$$

Ex. P-20 4.7 Sp20 Quiz

For each part, calculate the limit or show that it does not exist. If the limit is infinite, your answer should be "$+\infty$" or "$-\infty$".

(a) $\displaystyle\lim_{x\to e}\left(\frac{1 - \ln(x)}{x^2\ln(x) - e^2}\right)$

(b) $\displaystyle\lim_{x\to 2^+}\left(\frac{\cos(\pi x)}{x^2 - 6x + 8}\right)$

qs20

Solution P-20

(a) Direct substitution of $x = e$ gives $\frac{0}{0}$, so we may use L'Hospital's Rule.

$$\lim_{x\to e}\left(\frac{1 - \ln(x)}{x^2\ln(x) - e^2}\right) \overset{H}{=} \lim_{x\to e}\left(\frac{-\frac{1}{x}}{2x\ln(x) + x}\right) = \frac{-\frac{1}{e}}{2e\cdot 1 + e} = -\frac{1}{3e^2}$$

(b) Direct substitution of $x = 2$ gives $\frac{1}{0}$, and so the one-sided limit is infinite. The problem is reduced to a sign analysis. Note that if $x \to 2^+$, the numerator remains positive since $\cos(\pi x) \to 1 > 0$. The denominator may be factored as $x^2 - 6x + 8 = (x-2)(x-4)$. We note that as $x \to 2^+$, we may assume x is close to 2 and $x > 2$.

P-20

Hence $x-2>0$ and $x-4<0$. Putting this altogether, we have

$$\lim_{x\to 2^+}\left(\frac{\cos(\pi x)}{(x-2)(x-4)}\right)=\frac{\oplus}{\oplus\ominus}\infty=-\infty$$

Ex. P-21 4.7 Su22 Quiz

Calculate the limit or show that it does not exist. If the limit is infinite, write "$+\infty$" or "$-\infty$" as your answer, instead of "does not exist", as appropriate.

$$\lim_{x\to 0}\left(\frac{xe^{-2x}+\cos(x)-1-x}{x^2}\right)$$

qsu22

Solution P-21

Direct substitution of $x=0$ gives the indeterminate form "$\frac{0}{0}$". So we use L'Hospital's Rule (and again each time we encounter the same indeterminate form).

$$\lim_{x\to 0}\left(\frac{xe^{-2x}+\cos(x)-1-x}{x^2}\right)\overset{H}{=}\lim_{x\to 0}\left(\frac{e^{-2x}-2xe^{-2x}-\sin(x)-1}{2x}\right)$$
$$\overset{H}{=}\lim_{x\to 0}\left(\frac{-2e^{-2x}-2e^{-2x}+4xe^{-2x}-\cos(x)}{2}\right)=\frac{-2-2+0-1}{2}=-\frac{5}{2}$$

Ex. P-22 4.7 Fa22 Quiz

Compute $\lim_{x\to 0}\left(\frac{e^{-5x}-1}{\ln(1+13x)}\right)$. If the limit is infinite, write "$+\infty$" or "$-\infty$" instead of "DNE".

qf22

Solution P-22

Direct substitution of $x=0$ gives the indeterminate form "$\frac{0}{0}$", whence we can use l'Hospital's Rule.

$$\lim_{x\to 0}\left(\frac{e^{-5x}-1}{\ln(1+13x)}\right)\overset{H}{=}\lim_{x\to 0}\left(\frac{-5e^{-5x}}{\frac{1}{1+13x}\cdot 13}\right)=\frac{-5\cdot 1}{\frac{1}{1+0}\cdot 13}=-\frac{5}{13}$$

Ex. P-23 4.7 Fa22 Quiz

Calculate the limit below or determine it does not exist.

$$\lim_{x\to 0}\left(\frac{\cos(5x)-\cos(4x)}{3x^2}\right)$$

qf22

Solution P-23

Direct of substitution of $x=0$ gives the indeterminate expression "$\frac{0}{0}$". So we use l'Hospital's Rule (twice).

$$\lim_{x\to 0}\left(\frac{\cos(5x)-\cos(4x)}{3x^2}\right)\overset{H}{=}\lim_{x\to 0}\left(\frac{-5\sin(5x)+4\sin(4x)}{6x}\right)\overset{H}{=}\lim_{x\to 0}\left(\frac{-25\cos(5x)+16\cos(4x)}{6}\right)=\frac{-25+16}{6}=-\frac{3}{2}$$

Ex. P-24 4.7

For each part, calculate the limit or show that it does not exist.

c4s7

4.7

(a) $\lim_{x\to 0}\left(\frac{e^{2x}-1-2x-2x^2}{x^3}\right)$

(b) $\lim_{x\to 1}\left(\frac{x^3-1}{x^4-x}\right)$

(c) $\lim_{x\to 2}\left(\frac{x^3-8}{x^4-x}\right)$

(d) $\lim_{x\to\infty}\left(\frac{x-1}{x+2}\right)$

(e) $\lim_{x\to\infty}\left(\frac{x-1}{x+2}\right)^x$

(f) $\lim_{x\to\frac{\pi}{2}}\left(\frac{\sec(x)}{\tan(x)}\right)$

(g) $\lim_{x\to 0^+}(\sin(2x)\ln(x))$

(h) $\lim_{x\to 0^+}\left(x^{-4}\ln(x)\right)$

(i) $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)$

(j) $\lim_{x\to 3}\left(\frac{\sqrt{x+1}-2}{x^3-7x-6}\right)$

(k) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$

(l) $\lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{1}{x}\right)$

(m) $\lim_{x\to 0}(\cos(x))^{3/x^2}$

(n) $\lim_{x\to\infty}\left(\frac{x}{\sqrt{3x^2+4}}\right)$

(o) $\lim_{x\to\infty}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)$

(p) $\lim_{x\to 0}(1-\sin(2x))^{1/\tan(3x)}$

(q) $\lim_{x\to 0}\left(\frac{x\sin(x)}{1-\cos(x)}\right)$

(r) $\lim_{x\to\frac{\pi}{2}}\left(\left(x-\frac{\pi}{2}\right)\tan(x)\right)$

c4s7

Solution **P-24**

Recall that L'Hospital's Rule (LR) can be used only for indeterminate quotients of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. With some algebra, we can transform indeterminate products ($0\cdot\infty$), indeterminate exponents (1^∞, 0^0, or ∞^0), and indeterminate differences ($\infty-\infty$ or $-\infty+\infty$) into indeterminate quotients.

We must justify use of LR by verifying which indeterminate form we have at each step. In the solutions below, the notation "$\overset{H}{=}$" indicates that LR has been used in that step. Consider the following expressions.

$$\lim_{x\to 0^+}\underbrace{(x\ln(x))}_{0\cdot(-\infty)}=\lim_{x\to 0^+}\underbrace{\left(\frac{\ln(x)}{1/x}\right)}_{\frac{0}{0}}\overset{H}{=}\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0$$

Note that the indeterminate form is noted at each step. LR is used in the second step only; all other steps follow from algebra or computing simple limits.

(a) Standard applications of LR.

$$\lim_{x\to 0}\underbrace{\left(\frac{e^{2x}-1-2x-2x^2}{x^3}\right)}_{\frac{0}{0}}\overset{H}{=}\lim_{x\to 0}\underbrace{\left(\frac{2e^{2x}-2-4x}{3x^2}\right)}_{\frac{0}{0}}\overset{H}{=}\lim_{x\to 0}\underbrace{\left(\frac{4e^{2x}-4}{6x}\right)}_{\frac{0}{0}}\overset{H}{=}\lim_{x\to 0}\left(\frac{8e^{2x}}{6}\right)=\frac{8}{6}$$

(b) Factor and cancel.

$$\lim_{x\to 1}\left(\frac{x^3-1}{x^4-x}\right)=\lim_{x\to 1}\left(\frac{x^3-1}{x(x^3-1)}\right)=\lim_{x\to 1}\left(\frac{1}{x}\right)=1$$

(c) Direct substitution.

$$\lim_{x\to 2}\left(\frac{x^3-8}{x^4-x}\right)=\frac{0}{14}=0$$

(d) Standard application of LR.

$$\lim_{x\to\infty}\underbrace{\left(\frac{x-1}{x+2}\right)}_{\frac{\infty}{\infty}}\overset{H}{=}\lim_{x\to\infty}\left(\frac{1}{1}\right)=1$$

(e) We use the result of part (d) to determine the indeterminate form.

$$\lim_{x\to\infty}\underbrace{\left(\frac{x-1}{x+2}\right)^x}_{1^\infty}:=L$$

P-24

Now consider $\ln(L)$.

$$\ln(L) = \lim_{x\to\infty} \underbrace{\left(x \ln\left(\frac{x-1}{x+3}\right)\right)}_{\infty\cdot 0} = \lim_{x\to\infty} \underbrace{\left(\frac{\ln\left(\frac{x-1}{x+2}\right)}{1/x}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to\infty} \left(\frac{\frac{x+2}{x-1}\cdot\frac{(x+2)\cdot 1-(x-1)\cdot 1}{(x+2)^2}}{-1/x^2}\right) = \lim_{x\to\infty}\left(\frac{-3x^2}{(x-1)(x+2)}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^2}{x^2}\cdot\frac{-3}{\left(1-\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}\right) = \lim_{x\to\infty}\left(\frac{-3}{\left(1-\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}\right) = \frac{-3}{(1-0)(1+0)} = -3$$

So $\ln(L) = -3$, whence $L = e^{-3}$.

(f) Simplify and cancel. (LR is applicable, but it leads to an endless loop.)

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\sec(x)}{\tan(x)}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1/\cos(x)}{\sin(x)/\cos(x)}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\sin(x)}\right) = 1$$

(g) Write the product as a quotient, then use LR. Also use the special limit $\lim\limits_{\theta\to 0}\left(\dfrac{\sin(\theta)}{\theta}\right) = 1$.

$$\lim_{x\to 0^+} \underbrace{(\sin(2x)\ln(x))}_{0\cdot(-\infty)} = \lim_{x\to 0^+} \underbrace{\left(\frac{\ln(x)}{\csc(2x)}\right)}_{\frac{-\infty}{\infty}} \stackrel{H}{=} \lim_{x\to 0^+}\left(\frac{1/x}{-2\csc(2x)\cot(2x)}\right) = \lim_{x\to 0^+}\left(-\frac{\sin(2x)}{2x}\cdot\tan(2x)\right) = 0$$

(h) LR is not applicable here.

$$\lim_{x\to 0^+}\left(x^{-4}\ln(x)\right) = (+\infty)(-\infty) = -\infty$$

(i) Rationalize the denominator of the first term, combine terms, then rationalize the numerator. No need for LR.

$$\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right) = \lim_{x\to 4}\left(\frac{\sqrt{x}+2}{x-4} - \frac{4}{x-4}\right) = \lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right)$$
$$= \lim_{x\to 4}\left(\frac{x-4}{(x-4)(\sqrt{x}+2)}\right) = \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right) = \frac{1}{4}$$

(j) Standard application of LR.

$$\lim_{x\to 3} \underbrace{\left(\frac{\sqrt{x+1}-2}{x^3-7x-6}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 3}\left(\frac{\frac{1}{2\sqrt{x+1}}}{3x^2-7}\right) = \frac{\frac{1}{2\cdot 2}}{3\cdot 9-7} = \frac{1}{80}$$

(k) First rationalize the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) = \lim_{x\to\infty}\left(\left(\sqrt{x^2+x}-x\right)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty} \underbrace{\left(\frac{x}{\sqrt{x^2+x}+x}\right)}_{\frac{\infty}{\infty}}$$

At this point, we can use LR, but this leads to an endless chain. Instead, we factor out and cancel the dominant terms. We use the identity $\sqrt{x^2} = |x|$ and that, in turn, $|x| = x$ since we can assume $x > 0$ if $x \to \infty$.

$$\lim_{x\to\infty}\left(\frac{x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty}\left(\frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}+x}\right) = \lim_{x\to\infty}\left(\frac{1}{\sqrt{1+\frac{1}{x}}+1}\right) = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

(l) Find a common denominator, then use LR twice.

$$\lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right) = \lim_{x\to 0} \underbrace{\left(\frac{x-\sin(x)}{x\sin(x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0} \underbrace{\left(\frac{1-\cos(x)}{\sin(x)+x\cos(x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\left(\frac{\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}\right) = 0$$

P-24

(m) First determine the indeterminate exponent.

$$\lim_{x\to 0} \underbrace{(\cos(x))^{3/x^2}}_{1^\infty} := L$$

Now consider $\ln(L)$.

$$\ln(L) = \lim_{x\to 0} \underbrace{\left(\frac{3\ln(\cos(x))}{x^2}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\left(\frac{3\cdot \frac{1}{\cos(x)}\cdot(-\sin(x))}{2x}\right) = \lim_{x\to 0}\underbrace{\left(\frac{-3\tan(x)}{2x}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\left(\frac{-3\sec^2(x)}{2}\right) = -\frac{3}{2}$$

So $\ln(L) = -\frac{3}{2}$, whence $L = e^{-3/2}$.

(n) Factor out dominant terms. (LR is applicable, but it leads to an endless loop.) Since $x\to\infty$, we can assume $x > 0$. Thus $\sqrt{x^2} = |x| = x$.

$$\lim_{x\to\infty}\left(\frac{x}{\sqrt{3x^2+4}}\right) = \lim_{x\to\infty}\left(\frac{x}{\sqrt{x^2}\sqrt{3+\frac{4}{x^2}}}\right) = \lim_{x\to\infty}\left(\frac{x}{x}\cdot\frac{1}{\sqrt{3+\frac{4}{x^2}}}\right) = \lim_{x\to\infty}\left(1\cdot\frac{1}{\sqrt{3+\frac{4}{x^2}}}\right) = 1\cdot\frac{1}{\sqrt{3+0}} = \frac{1}{\sqrt{3}}$$

(o) Factor out dominant terms. (LR is applicable, but it leads to an endless loop.)

$$\lim_{x\to\infty}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \lim_{x\to\infty}\left(\frac{e^x}{e^x}\cdot\frac{1-e^{-2x}}{1+e^{-2x}}\right) = \lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right) = 1\cdot\frac{1-0}{1+0} = 1$$

(p) First determine the indeterminate exponent.

$$\lim_{x\to 0}\underbrace{(1-\sin(2x))^{1/\tan(3x)}}_{1^{\pm\infty}} := L$$

Now consider $\ln(L)$.

$$\ln(L) = \lim_{x\to 0}\underbrace{\left(\frac{\ln(1-\sin(2x))}{\tan(3x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\left(\frac{\frac{1}{1-\sin(2x)}\cdot(-2\cos(2x))}{3\sec(3x)^2}\right) = \frac{\frac{1}{1-0}\cdot(-2)}{3\cdot 1} = -\frac{2}{3}$$

So $\ln(L) = -\frac{2}{3}$, whence $L = e^{-2/3}$.

(q) Standard application of LR.

$$\lim_{x\to 0}\underbrace{\left(\frac{x\sin(x)}{1-\cos(x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\underbrace{\left(\frac{x\cos(x)+\sin(x)}{\sin(x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to 0}\left(\frac{\cos(x)-x\sin(x)+\cos(x)}{\cos(x)}\right) = \frac{1-0+1}{1} = 2$$

(r) Write the product as a quotient, then use LR.

$$\lim_{x\to\frac{\pi}{2}}\underbrace{\left(\left(x-\frac{\pi}{2}\right)\tan(x)\right)}_{0\cdot(\pm\infty)} = \lim_{x\to\frac{\pi}{2}}\underbrace{\left(\frac{\left(x-\frac{\pi}{2}\right)\sin(x)}{\cos(x)}\right)}_{\frac{0}{0}} \stackrel{H}{=} \lim_{x\to\frac{\pi}{2}}\left(\frac{\left(x-\frac{\pi}{2}\right)\cos(x)+\sin(x)}{-\sin(x)}\right) = \frac{0\cdot 0+1}{-1} = -1$$

Ex. P-25 **4.7**

Find the equation of each horizontal asymptote of $f(x) = \dfrac{2e^x-5}{3e^x+2}$.

unit2

Solution **P-25**

We calculate the limits of f at infinity. For the limit $x \to \infty$, we have the form $\frac{\infty}{\infty}$, so we use Ll'Hôpital's Rule.

$$\lim_{x\to\infty}\left(\frac{2e^x-5}{3e^x+2}\right) \overset{H}{=} \lim_{x\to\infty}\left(\frac{2e^x}{3e^x}\right) = \frac{2}{3}$$

For the limit $x \to -\infty$ recall that $e^x \to 0$, and so

$$\lim_{x\to-\infty}\left(\frac{2e^x-5}{3e^x+2}\right) = \frac{0-5}{0+2} = -\frac{5}{2}$$

So the two horizontal asymptotes of $f(x)$ are $y = \frac{2}{3}$ and $y = -\frac{5}{2}$.

Ex. P-26 **4.7**

For each part, calculate the limit or show that it does not exist. If the limit is infinite, write "∞" or "$-\infty$" as your answer, as appropriate.

(a) $\displaystyle\lim_{x\to 3^-}\left(\frac{x^2+6}{3-x}\right)$ (b) $\displaystyle\lim_{x\to 0}(1-\sin(3x))^{1/x}$ (c) $\displaystyle\lim_{x\to -3}\left((x+3)\tan\left(\frac{\pi x}{2}\right)\right)$

unit4

Solution **P-26**

(a) Direct substitution of $x = 3$ gives the expression "$\frac{15}{0}$", which is not indeterminate, but instead indicates that the one-sided limit is infinite. Observe that the denominator $3 - x$ approaches 0 as $x \to 3^-$, but remains positive. (Recall that the notation $x \to 3^-$ implies $x < 3$.) Hence we have

$$\lim_{x\to 3^-}\left(\frac{x^2+6}{3-x}\right) = \frac{\oplus}{\oplus}\infty = +\infty$$

(b) First determine the indeterminate exponent.

$$L = \lim_{x\to 0}\underbrace{(1-\sin(3x))^{1/x}}_{1^{\pm\infty}}$$

Now consider $\ln(L)$ and use l'Hospital's Rule.

$$\ln(L) = \lim_{x\to 0}\underbrace{\left(\frac{\ln(1-\sin(3x))}{x}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\left(\frac{\frac{1}{1-\sin(3x)}\cdot(-3\cos(3x))}{1}\right) = -3$$

So $\ln(L) = -3$, whence $L = e^{-3}$.

(c) We have the indeterminate product $0 \cdot \infty$. Write the product as a quotient, then use LR.

$$\lim_{x\to -3}\left((x+3)\tan\left(\frac{\pi x}{2}\right)\right) = \lim_{x\to -3}\underbrace{\left(\frac{(x+3)\sin(\frac{\pi}{2}x)}{\cos(\frac{\pi}{2}x)}\right)}_{\frac{0}{0}} \overset{H}{=} \lim_{x\to -3}\left(\frac{\sin(\frac{\pi}{2}x)+(x+3)\cos(\frac{\pi}{2}x)\cdot\frac{\pi}{2}}{-\sin(\frac{\pi}{2}x)\cdot\frac{\pi}{2}}\right) = -\frac{2}{\pi}$$

Ex. P-27 **4.7**

Calculate each limit.

(a) $\displaystyle\lim_{x\to 0}\left(\frac{\sin(x)^2}{\sin(2x^2)}\right)$ (b) $\displaystyle\lim_{x\to 1}\left(\frac{\ln(x^2+2)-\ln(3)}{x-1}\right)$

unit4

Solution **P-27**

(a) We use the special limit $\lim\limits_{\theta \to 0} \left(\frac{\sin(\theta)}{\theta}\right) = 1$ several times.

$$\lim_{x \to 0} \left(\frac{\sin(x)^2}{\sin(2x^2)}\right) = \lim_{x \to 0} \left(\frac{\sin(x)}{x} \cdot \frac{\sin(x)}{x} \cdot \frac{2x^2}{\sin(2x^2)} \cdot \frac{x \cdot x}{2x^2}\right)$$
$$= \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right) \cdot \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right) \cdot \lim_{x \to 0} \left(\frac{2x^2}{\sin(2x^2)}\right) \cdot \frac{1}{2} = 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

> Alternatively, we can use l'Hospital's Rule several times. However, this requires multiple uses of product rule and possibly complicated algebra. The solution given above is simpler.

(b) Substitution of $x = 1$ gives the indeterminate form "$\frac{0}{0}$". Using L'Hospital's Rule gives the following.

$$\lim_{x \to 1} \left(\frac{\ln(x^2+2) - \ln(3)}{x-1}\right) \overset{H}{=} \lim_{x \to 1} \left(\frac{\frac{1}{x^2+2} \cdot 2x}{1}\right) = \frac{2}{3}$$

Ex. P-28 **4.7**

For each part, calculate the limit or show it does not exist.

(a) $\lim\limits_{x \to 2} \left(\dfrac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x}\right)$ (b) $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^{3x}$ (c) $\lim\limits_{x \to 0} \left(\dfrac{\sin(5x) - 5x}{x^3}\right)$

unit4

Solution **P-28**

(a) Rationalize the numerator and cancel common factors.

$$\lim_{x \to 2} \left(\frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x}\right) = \lim_{x \to 2} \left(\frac{-x+2}{x(x-2)(\sqrt{x+2} + \sqrt{2x})}\right) = \lim_{x \to 2} \left(\frac{-1}{x(\sqrt{x+2} + \sqrt{2x})}\right) = \frac{-1}{2(2+2)} = -\frac{1}{8}$$

> L'Hospital's Rule is also applicable here, but might be more trouble than it's worth. Application of LR still requires algebraic manipulation similar to that in the solution given above.

(b) Let L be the desired limit and consider $\ln(L)$.

$$\ln(L) = \lim_{x \to \infty} \ln\left(\left(1 + \frac{2}{x}\right)^{3x}\right) = \lim_{x \to \infty} \left(3x \ln\left(1 + \frac{2}{x}\right)\right) = \lim_{x \to \infty} \left(\frac{3\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{x}}\right)$$

We now have the indeterminate form "$\frac{\infty}{\infty}$", whence we may use L'Hospital's Rule.

$$\ln(L) \overset{H}{=} \lim_{x \to \infty} \left(\frac{3 \cdot \frac{1}{1 + \frac{2}{x}} \cdot \frac{-2}{x^2}}{\frac{-1}{x^2}}\right) = \lim_{x \to \infty} \left(\frac{6}{1 + \frac{2}{x}}\right) = \frac{6}{1+0} = 6$$

We have shown $\ln(L) = 6$, whence $L = e^6$.

(c) Use L'Hospital's Rule repeatedly (each time verifying the indeterminate form "$\frac{0}{0}$").

$$\lim_{x \to 0} \left(\frac{\sin(5x) - 5x}{x^3}\right) \overset{H}{=} \lim_{x \to 0} \left(\frac{5\cos(5x) - 5}{3x^2}\right) \overset{H}{=} \lim_{x \to 0} \left(\frac{-25\sin(5x)}{6x}\right) \overset{H}{=} \lim_{x \to 0} \left(\frac{-125\cos(5x)}{6}\right) = -\frac{125}{6}$$

Ex. P-29 4.7 ⋆Challenge

Suppose f'' is continuous for all x. Calculate $\lim_{h\to 0}\left(\dfrac{f(x+5h)+f(x-5h)-2f(x)}{h^2}\right)$.

challenge

Solution P-29

Since f'' is continuous, so are f and f'. This means all of $f(x)$, $f'(x)$, and $f''(x)$ have the direct substitution property of limits for all inputs. Substitution of $h=0$ into the given limit gives "$\frac{0}{0}$", and so we use l'Hospital's Rule (LR). Note that we must differentiate the numerator and denominator with respect to h, not with respect to x. We treat x as a constant.

$$\lim_{h\to 0}\left(\frac{f(x+5h)+f(x-5h)-2f(x)}{h^2}\right) \overset{H}{=} \lim_{h\to 0}\left(\frac{5f'(x+5h)-5f'(x-5h)}{2h}\right)$$

Substitution of $h=0$ again gives "$\frac{0}{0}$", and so we use LR again.

$$\lim_{h\to 0}\left(\frac{5f'(x+5h)-5f'(x-5h)}{2h}\right) \overset{H}{=} \lim_{h\to 0}\left(\frac{25f''(x+5h)+25f''(x-5h)}{2}\right) = \frac{25f''(x)+25f''(x)}{2} = 25f''(x)$$

Ex. P-30 4.7 ⋆Challenge

Suppose f' is continuous for all x and $f(0)=0$. Calculate $\lim_{x\to 0^+}(1+f(2x))^{4/x}$.

challenge

Solution P-30

Since f' is continuous, so is f. This means both $f(x)$ and $f'(x)$ have the direct substitution property for all inputs. Substitution of $x=0$ into the given limit gives "$1^{\pm\infty}$". So we let L denote the given limit and consider $\ln(L)$ instead.

$$\ln(L) = \lim_{x\to 0^+}\left(\ln\left((1+f(2x))^{4/x}\right)\right) = \lim_{x\to 0^+}\left(\frac{4\ln(1+f(2x))}{x}\right) \overset{H}{=} \lim_{x\to 0^+}\left(\frac{4\cdot\frac{1}{1+f(2x)}\cdot f'(2x)\cdot 2}{1}\right) = 8f'(0)$$

So $\ln(L)=8f'(0)$, whence $L=e^{8f'(0)}$.

§4.9: Antiderivatives

Ex. G-6 **3.1/3.2, 4.1, 4.9** Sp20 **Exam**

Suppose the derivative of f is $f'(x) = 3x^2 - 6x - 9$ and that $f(1) = 10$.

(a) Find an equation of the line tangent to the graph of $y = f(x)$ at $x = 1$.
(b) Find the critical points of f.
(c) Where does f have a local minimum value? local maximum value?
(d) Calculate $f(0)$.
(e) Calculate the absolute maximum value of f on the interval $[0, 6]$. At what x-value does it occur?

exs20

Solution **G-6**

(a) We have $f'(1) = 3 - 6 - 9 = -12$, whence an equation of the tangent line is $y = 10 - 12(x - 1)$.
(b) Solving $f'(x) = 0$, we find that the critical points of f are $x = -1$ and $x = 3$.
(c) A sign chart for $f'(x)$ reveals that $f'(x)$ is positive on the intervals $(-\infty, -1)$ and $(3, \infty)$; and $f'(x)$ is negative on the interval $(-1, 3)$. Since f' changes from positive to negative at $x = -1$, a local maximum occurs at $x = -1$. Since f' changes from negative to positive to $x = 3$, a local minimum occurs at $x = 3$.
(d) We find $f(x)$ by finding the most general antiderivative of $f'(x)$.

$$f(x) = \int f'(x)\,dx = x^3 - 3x^2 - 9x + C$$

The initial condition $f(1) = 10$ implies $1 - 3 - 9 + C = 10$, or $C = 21$. Hence

$$f(x) = x^3 - 3x^2 - 9x + 21$$

So $f(0) = 21$.
(e) The absolute maximum of f on $[0, 6]$ can occur only at an endpoint (0 or 6) or a critical number (-1 or 3). Calculating the values of f at these x-values gives: $f(0) = 21$, $f(-1) = 26$, $f(3) = -6$, and $f(6) = 75$. Hence the absolute maximum of f on $[0, 6]$ is 75, occurring at $x = 6$.

Ex. Q-1 **4.9** Su20 **Exam**

Given that x units of a commodity are sold, the marginal cost is

$$\frac{dC}{dx} = 9x^2 + 4x + 15x^{1/4} + 10$$

Suppose the total cost of producing the 1st unit is 100. Calculate the total cost of producing the first 16 units.

exsu20

Solution **Q-1**

Antidifferentiation gives us the total cost function.

$$C(x) = \int \left(9x^2 + 4x + 15x^{1/4} + 10\right)\,dx = 3x^3 + 2x^2 + 12x^{5/4} + 10x + K$$

We are given that $C(1) = 100$, whence $3 + 2 + 12 + 10 + K = 100$, and so $K = 73$. So then the total cost of producing 16 units is

$$C(16) = \left(3x^3 + 2x^2 + 12x^{5/4} + 10x + 73\right)\Big|_{x=16} = 13,417$$

Ex. Q-2 **4.9** Fa20 **Exam**

Let $V(t)$ denote the volume of water, measured in gallons, in a tank at time t. The tank is initially filled with 5 gallons of water. At $t = 0$, water flows in at a rate in gal/min given by $V'(t) = 0.5(196 - t^2)$ for $0 \le t \le 10$. Find the total amount of water in the tank after 4 minutes.

exf20

Solution Q-2

Computing the antiderivative of $V'(t)$ immediately gives $V(t) = 0.5(196t - \frac{1}{3}t^3) + C$ for some constant C. The condition $V(0) = 5$ implies $C = 5$, whence $V(t) = 98t - \frac{1}{6}t^2 + 5$. The volume of water in the tank after 4 minutes is $V(4) = \frac{1183}{3}$ gallons.

Ex. Q-3 4.9 Sp21 Exam

A particle travels along the x-axis with velocity (measured in ft/sec) at any time t (measures in sec) given by

$$v(t) = 4t^3 - 2t + 2$$

The particle is at $x = 3$ when $t = 2$.

(a) Find the position of the particle at any time t.
(b) Find the position of the particle at time $t = 4$.
(c) Find the acceleration of the particle when $t = 4$.

exs21

Solution Q-3

(a) To find the position, we find the antiderivative of $v(t)$ first.

$$x(t) = \int v(t)\,dt = \int (4t^3 - 2t + 2)\,dt = t^4 - t^2 + 2t + C$$

We are given $x = 3$ when $t = 2$, whence $3 = 16 - 4 + 4 + C$, and so $C = -13$. The position of the particle at any time t is

$$x(t) = t^4 - t^2 + 2t - 13$$

(b) We have $x(4) = 256 - 16 + 8 - 13 = 235$.
(c) The acceleration is the derivative of velocity, so $a(4) = v'(4) = \left(12t^2 - 2\right)\big|_{t=4} = 190$.

Ex. Q-4 4.9 Fa21 Exam

For any time $t > 0$, the acceleration of a particle is given by $a(t) = 1 + \dfrac{3}{\sqrt{t}}$, and the particle has velocity $v = -20$ when $t = 1$. Find the velocity of the particle when $t = 16$.

exf21

Solution Q-4

We first obtain the velocity by antidifferentiating the acceleration.

$$v(t) = \int a(t)\,dt = \int \left(1 + 3t^{-1/2}\right)\,dt = t + 6t^{1/2} + C$$

We are given that $v(1) = -20$, whence $-20 = 1 + 6 + C$, and so $C = -27$. Our velocity function is:

$$v(t) = t + 6t^{1/2} - 27$$

Thus $v(16) = 16 + 6 \cdot 4 - 27 = 13$.

Ex. Q-5 4.9 Sp18 Quiz

Calculate the following antiderivatives.

(a) $\displaystyle\int (\cos(w) + 2\sin(w) - 3e^w)\,dw$

(b) $\displaystyle\int \frac{3t^3 - \sqrt[3]{t} + 2t}{t^2}\,dt$

qs18

Solution Q-5

(a) Use familiar derivative rules backwards.

$$\int (\cos(w) + 2\sin(w) - 3e^w)\,dw = \sin(w) - 2\cos(w) - 3e^w + C$$

Q-5

(b) Write the integrand as a sum of power functions, then antidifferentiate.

$$\int \frac{3t^3 - \sqrt[3]{t} + 2t}{t^2}\, dt = \int \left(3t - t^{-5/3} + 2t^{-1}\right)\, dt = \frac{3}{2}t^2 + \frac{3}{2}t^{-2/3} + 2\ln(|t|) + C$$

Ex. Q-6 4.9, 5.3 Su22 Quiz

Calculate each of the following. You do not have to simplify your answers.

(a) $\displaystyle\int \left(\frac{3t^2 - \sqrt{t} + 4}{5t}\right)\, dt$ (b) $\displaystyle\int_{-1}^{3} \left(3x^2 + 2e^x\right)\, dx$

qsu22

Solution Q-6

(a) Divide each term and then antidifferentiate.

$$\int \left(\frac{3t^2 - \sqrt{t} + 4}{5t}\right)\, dt = \int \left(\frac{3}{5}t - \frac{1}{5}t^{-1/2} + \frac{4}{5}t^{-1}\right)\, dt = \frac{3}{10}t^2 - \frac{2}{5}t^{1/2} + \frac{4}{5}\ln(|t|) + C$$

(b) Find the antiderivative, then use the fundamental theorem of calculus.

$$\int_{-1}^{3} \left(3x^2 + 2e^x\right)\, dx = \left(x^3 + 2e^x\right)\Big|_{-1}^{3} = \left(27 + 2e^3\right) - \left(-1 + 2e^{-1}\right) = 28 + 2e^3 - 2e^{-1}$$

Ex. Q-7 4.9 Fa22 Quiz

The total number of rabbits in a certain region t weeks after observations have begun is modeled by the equation $N(t) = 200 + 36t^{2/3}$. Use a linear approximation to estimate the increase in the rabbit population between $t = 64$ and $t = 67$.

qf22

Solution Q-7

We seek an estimate of the change in number of rabbits: $\Delta N = N(67) - N(64)$. We use the tangent line to $N(t)$ at $t = 64$.

Point of Tangency:	$(64, C(64))$
Slope of Line:	$N'(t) = 24t^{-1/3}$; $N'(64) = 6$
Equation of Line:	$y - N(64) = 6(t - 64)$

This means that $N(t) - N(64) \approx 6(t - 64)$ if t is near 64. Hence we have the estimate:

$$\Delta N = N(67) - N(64) \approx 6(67 - 64) = 18$$

The number of rabbits increases by approximately 18.

Ex. Q-8 4.9 Fa22 Quiz

When x units of a product are produced, the derivative of the total cost C (measured in \$) is:

$$\frac{dC}{dx} = 3x^2 + 40x + 100$$

Suppose the total cost of producing 1 unit is \$150. Find the total cost of producing the first 2 units.

qf22

Solution Q-8

We find the total cost by antidifferentiation.

$$C(x) = \int (3x^2 + 40x + 100)\, dx = x^3 + 20x^2 + 100x + K$$

We are given $C(1) = 150$, or $1 + 20 + 100 + K = 150$, whence $K = 29$. So the total cost is

$$C(x) = x^3 + 20x^2 + 100x + 29$$

Q-8

Thus the total cost of the first 2 units is $C(2) = 317$.

Ex. Q-9 4.9

For each part, find the antiderivative.

(a) $\displaystyle\int (4 - 9x + x^2)\, dx$

(b) $\displaystyle\int \left(12e^x + \sin(x) - \frac{\cos(x)}{4}\right) dx$

(c) $\displaystyle\int (6y - y^3)^2\, dy$

(d) $\displaystyle\int (86t^7 - \sqrt[3]{t})\, dt$

(e) $\displaystyle\int \frac{3t^3 - 6\sqrt{t} - \frac{9}{t}}{t}\, dt$

(f) $\displaystyle\int \left(1 - \frac{1}{u}\right)\left(2 + \frac{3}{\sqrt{u}}\right) du$

c4s9

Solution Q-9

(a) Use power rule.

$$\int (4 - 9x + x^2)\, dx = 4x - \frac{9}{2}x^2 + \frac{1}{3}x^3 + C$$

(b) Use exponential and trigonometric derivative rules.

$$\int \left(12e^x + \sin(x) - \frac{\cos(x)}{4}\right) dx = 12e^x - \cos(x) - \frac{\sin(x)}{4} + C$$

(c) Expand the integrand, then antidifferentiate each term using power rule.

$$\int (6y - y^3)^2\, dy = \int (36y^2 - 12y^4 + y^6)\, dy = 12y^3 - \frac{12}{5}y^5 + \frac{1}{7}y^7 + C$$

(d) Use power rule.

$$\int (86t^7 - \sqrt[3]{t})\, dt = \frac{86}{8}t^8 - \frac{3}{4}t^{4/3} + C$$

(e) Write the integrand as a sum of power functions then antidifferentiate. each term using power rule.

$$\int \frac{3t^3 - 6\sqrt{t} - \frac{9}{t}}{t}\, dt = \int \left(3t^2 - 6t^{-1/2} - 9t^{-2}\right) dt = t^3 - 12t^{1/2} + 9t^{-1} + C$$

(f) Expand the integrand, then antidifferentiate each term using power rule.

$$\int \left(1 - \frac{1}{u}\right)\left(2 + \frac{3}{\sqrt{u}}\right) du = \int \left(2 + 3u^{-1/2} - 2u^{-1} - 3u^{-3/2}\right) du = 2u + 6u^{1/2} - 2\ln(|u|) + 6u^{-1/2} + C$$

Ex. Q-10 4.9

The marginal revenue of a certain commodity is $R'(x) = -9x^2 + 24x + 48$. Find the price for which the total revenue is a maximum. (Assume that $R(0) = 0$.)

c4s9

Solution Q-10

Revenue is maximized when $R'(x) = 0$

$$R'(x) = -9(3x + 4)(x - 4) = 0 \implies x = 4$$

So revenue is maximized when $x = 4$. (We reject the solution $x = -\frac{4}{3}$ since level of production is non-negative.) To find the price, we first find the total revenue, which we obtain by antidifferentiation.

$$R(x) = \int R'(x)\, dx = \int (-9x^2 + 24x + 48)\, dx = -3x^3 + 12x^2 + 48x + K$$

Since $R(0) = 0$, we find that $K = 0$. So the total revenue is $R(x) = -3x^3 + 12x^2 + 48x$. Since revenue is $R(x) = xp(x)$, we now have the price:

$$p(x) = \frac{R(x)}{x} = -3x^2 + 12x + 48$$

Hence the price that maximizes the revenue is $p(4) = 48$.

Ex. Q-11 **4.9**

A particle moves along the x-axis in such a way that its acceleration at time $t > 0$ is

$$a(t) = 1 - \frac{1}{t^2}$$

The particle's velocity when $t = 2$ is $v = 5.5$. What is the net distance the particle travels between $t = 3$ and $t = 6$?

c4s9

Solution **Q-11**

First we find the particle's velocity by anti-differentiating $a(t)$.

$$v(t) = \int a(t)\,dt = \int \left(1 - t^{-2}\right)\,dt = t + t^{-1} + C_1$$

Now we find the value of C_1 by using the fact that $v(2) = 5.5$.

$$5.5 = 2 + \frac{1}{2} + C_1 \Longrightarrow C_1 = 3$$

Hence the particle's velocity is $v(t) = t + \frac{1}{t} + 3$. Now we find the particle's position by anti-differentiating $v(t)$.

$$x(t) = \int v(t)\,dt = \int \left(t + \frac{1}{t} + 3\right)\,dt = \frac{1}{2}t^2 + \ln(|t|) + 3t + C_2$$

The value of C_2 is not needed since we are only interested in a difference of position. The net distance traveled between $t = 3$ and $t = 6$ is

$$\Delta x = x(6) - x(3) = \left(\frac{1}{2}\cdot 36 + \ln(6) + 18 + C_2\right) - \left(\frac{1}{2}\cdot 9 + \ln(3) + 9 + C_2\right) = 22.5 + \ln(2)$$

Ex. Q-12 **4.9**

The position of a particle on the x-axis (measured in meters) at time t (measured in seconds) is modeled by the equation $f(t) = 100 + 8t^{3/4} - 5t$. Use a linear approximation to estimate the change in the particle's position between $t = 81$ and $t = 83$.

unit4

Solution **Q-12**

We seek an estimate of the change in position: $\Delta x = f(83) - f(81)$. We use the tangent line to $f(t)$ at $t = 81$.

Point of Tangency:	$(81, f(81))$
Slope of Line:	$f'(t) = 6t^{-1/4} - 5;\qquad f'(81) = -3$
Equation of Line:	$y - f(81) = -3(t - 81)$

This means that $f(t) - f(81) \approx -3(t - 81)$ if t is near 81. Hence we have the estimate:

$$\Delta x = f(83) - f(81) \approx -3(83 - 81) = -6$$

The particle's position decreases by about 6 meters.

Ex. L-32 **4.1, 4.9**

The marginal revenue of a certain product is $R'(x) = -9x^2 + 17x + 30$, where x is the level of production. Assume $R(0) = 0$. Find the market price that maximizes revenue.

unit4

Solution **L-32**

Revenue is maximized if $R'(x) = -(9x + 10)(x - 3) = 0$, or if $x = 3$. (We ignore the solution $x = -\frac{10}{9}$ since x must be positive since it represents level of production.)

Antidifferentiating $R'(x)$, we find that the revenue is $R(x) = -3x^3 + \frac{17}{2}x^2 + 30x + K$, for some unknown constant K. The assumption that $R(0) = 0$ implies that $K = 0$, whence $R(x) = -3x^3 + \frac{17}{2}x^2 + 30x$. Since $R(x) = xp(x)$, the market price is $p(x) = -3x^2 + \frac{17}{2}x + 30$. Hence the market price when revenue is maximized is $p(3) = 28.5$.

Ex. Q-13 **4.9**

The marginal cost (in dollars) of a certain product is $C'(x) = 6x^2 + 30x + 200$. If it costs \$250 to produce 1 unit, how much does it cost to produce 10 units?

unit4

Solution **Q-13**

Antidifferentiating $C'(x)$ shows that $C(x) = 2x^3 + 15x^2 + 200x + K$, for some unknown constant K. The condition $C(1) = 250$ implies that $K = 33$, and so the cost function is $C(x) = 2x^3 + 15x^2 + 200x + 33$. Hence the cost of producing 10 units is $C(10) = 5533$ dollars.

Ex. Q-14 **4.9**

For each part, find the antiderivative or integral.

(a) $\displaystyle\int \frac{2x+\sqrt{x}-1}{x}\,dx$

(b) $\displaystyle\int (2x+3)^{12}\,dx$

(c) $\displaystyle\int_0^1 e^x(1+e^{-2x})\,dx$

(d) $\displaystyle\int_0^{\pi/2} (1+\sin(x))^5\cos(x)\,dx$

unit5

Solution **Q-14**

(a) $\displaystyle\int \frac{2x+\sqrt{x}-1}{x}\,dx = \int \left(2 + x^{-1/2} - x^{-1}\right)\,dx = 2x + 2x^{1/2} - \ln|x| + C$

(b) Substitute $u = 2x+3$ (whence $\frac{1}{2}du = dx$).

$$\int (2x+3)^{12}\,dx = \int \frac{1}{2}u^{12}\,du = \frac{1}{26}u^{13} + C = \frac{1}{26}(2x+3)^{13} + C$$

(c) Expand the integrand and then split into two integrals.

$$\int_0^1 e^x(1+e^{-2x})\,dx = \int_0^1 \left(e^x + e^{-x}\right)\,dx = \int_0^1 e^x\,dx + \int_0^1 e^{-x}\,dx$$

For the first integral, use fundamental theorem of calculus. For the second integral substitute $u = -x$ (whence $-du = dx$) and then use the fundamental theorem of calculus.

$$\int_0^1 e^x\,dx = e^x\Big|_0^1 = e - 1$$
$$\int_0^1 e^{-x}\,dx = \int_0^{-1} (-e^u)\,du = -e^u\Big|_0^{-1} = -e^{-1} + 1$$

Adding the integrals gives a final answer of $e - e^{-1}$.

(d) Substitute $u = 1+\sin(x)$ (whence $du = \cos(x)\,dx$).

$$\int_0^{\pi/2} (1+\sin(x))^5\cos(x)\,dx = \int_1^2 u^5\,du = \frac{1}{6}u^6\Big|_1^2 = \frac{2^6}{6} - \frac{1}{6} = \frac{21}{2}$$

Ex. Q-15 **4.9, 5.3, 5.5**

For each part, find the antiderivative or integral.

(a) $\displaystyle\int t^2\cos(1-t^3)\,dt$

(b) $\displaystyle\int \sqrt{x-1}\,dx$

(c) $\displaystyle\int_2^3 \frac{\ln(x)}{x}\,dx$

(d) $\displaystyle\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x}-1}\,dx$

unit5

Solution **Q-15**

(a) Substitute $u = 1-t^3$ (whence $-\frac{1}{3}du = t^2\,dt$).

$$\int t^2\cos(1-t^3)\,dt = \int \left(-\frac{1}{3}\cos(u)\right)\,du = -\frac{1}{3}\sin(u) + C = -\frac{1}{3}\sin(1-t^3) + C$$

Q-15

(b) Substitute $u = x - 1$ (whence $du = dx$).

$$\int \sqrt{x-1}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x-1)^{3/2} + C$$

(c) Substitute $u = \ln(x)$ (whence $du = \frac{1}{x}\,dx$).

$$\int_2^3 \frac{\ln(x)}{x}\,dx = \int_{\ln(2)}^{\ln(3)} u\,du = \frac{1}{2}u^2\Big|_{\ln(2)}^{\ln(3)} = \frac{\ln(3)^2 - \ln(2)^2}{2}$$

(d) Substitute $u = e^{2x} - 1$ (whence $\frac{1}{2}du = e^{2x}\,dx$).

$$\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x}-1}\,dx = \int_0^8 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_0^8 = \frac{8^{3/2}}{3}$$

5 Chapter 5: Integration

§5.1, 5.2: Introduction to the Integral

Ex. R-1 5.1/5.2 Sp20 Exam

Suppose f is a continuous function such that all of the following hold:

$$\int_{-1}^{6} f(x)\,dx = -15 \quad , \quad \int_{6}^{9} f(x)\,dx = 14 \quad , \quad \int_{0}^{9} f(x)\,dx = 19$$

Calculate the quantities below or determine there is not enough information.

(a) $\displaystyle\int_{-1}^{9} f(x)\,dx$ (c) $\displaystyle\int_{-1}^{6} |f(x)|\,dx$ (e) $\displaystyle\int_{-1}^{0} f(x)\,dx$

(b) $\displaystyle\int_{0}^{6} f(x)\,dx$ (d) $\displaystyle\left|\int_{-1}^{6} f(x)\,dx\right|$ (f) $\displaystyle\int_{6}^{9} (3f(x)+4)\,dx$

exs20

Solution R-1

(a) $\displaystyle\int_{-1}^{9} f(x)\,dx = \int_{-1}^{6} f(x)\,dx + \int_{6}^{9} f(x)\,dx = -15 + 14 = -1$

(b) $\displaystyle\int_{0}^{6} f(x)\,dx = \int_{0}^{9} f(x)\,dx - \int_{6}^{9} f(x)\,dx = 19 - 14 = 5$

(c) not enough information

(d) $\displaystyle\left|\int_{-1}^{6} f(x)\,dx\right| = |-15| = 15$

(e) Use part (b).

$$\int_{-1}^{0} f(x)\,dx = \int_{-1}^{6} f(x)\,dx - \int_{0}^{6} f(x)\,dx = -15 - 5 = -20$$

(f) Use linearity.

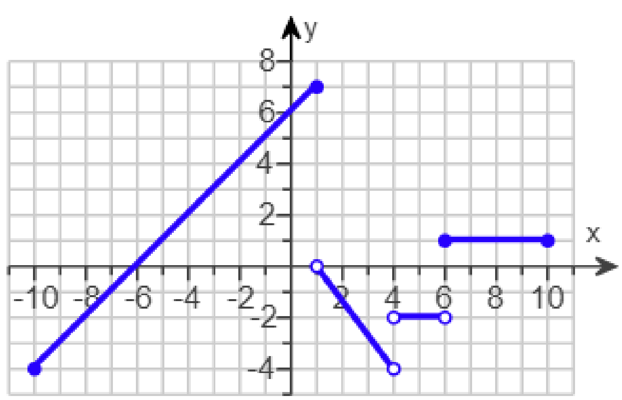
$$\int_{6}^{9} (3f(x)+4)\,dx = 3\int_{6}^{9} f(x)\,dx + \int_{6}^{9} 4\,dx$$

The second integral is the area of a rectangle of height 4 and width 3. So we have:

$$\int_{6}^{9} (3f(x)+4)\,dx = 3\cdot 14 + 3\cdot 4 = 54$$

Ex. R-2 5.1/5.2 Sp20 Exam

Use the graph of $y = f(x)$ to calculate the integrals below.



(a) $\displaystyle\int_{0}^{1} f(x)\,dx$ (b) $\displaystyle\int_{1}^{6} f(x)\,dx$ (c) $\displaystyle\int_{-10}^{10} f(x)\,dx$

exs20

Solution **R-2**

(a) The integral is the area of a trapezoid with parallel bases of length 6 and 7, with height 1. Hence

$$\int_0^1 f(x)\,dx = \frac{1}{2}(6+7)\cdot 1 = 6.5$$

(b) The integral represents the net area of a region that consists of a triangle (base 3, height 4) and a rectangle (base 2, height 2). Note that both are below the x-axis, and so the net area is negative.

$$\int_1^6 f(x)\,dx = -\left(\frac{1}{2}\cdot 3\cdot 4 + 2\cdot 2\right) = -10$$

(c) We have already computed most of this integral in parts (a) and (b). For the remaining parts we have one triangle below the x-axis, one triangle above the x-axis, and one rectangle above the x-axis.

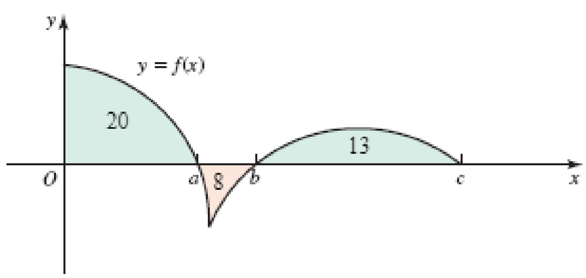
$$\int_{-10}^{-6} f(x)\,dx = -\frac{1}{2}\cdot 4\cdot 4 = -8$$
$$\int_{-6}^{0} f(x)\,dx = \frac{1}{2}\cdot 6\cdot 6 = 18$$
$$\int_{6}^{10} f(x)\,dx = 1\cdot 4 = 4$$

Putting everything together gives:

$$\int_{-10}^{10} f(x)\,dx = -8+18+6.5-10+4 = 10.5$$

Ex. R-3 **5.1/5.2** Sp20 **Exam**

The figure below shows the area of regions bounded by the graph of $y = f(x)$ and the x-axis, where $a = 4$, $b = 6$, and $c = 15$. Evaluate $\displaystyle\int_a^c (11f(x) - 6)\;dx$.



exs20

Solution **R-3**

Split up the integral using linearity properties.

$$\int_a^c (11f(x)-6)\;dx = 11\int_a^c f(x)\,dx - \int_a^c 6\,dx = 11\cdot(13-8) - 6\cdot(15-4) = -11$$

Ex. R-4 **5.1/5.2** Sp20 **Exam**

Consider the integral below.

$$\int_{-2}^{1} \sqrt{9-(x-1)^2}\,dx$$

(a) Explain in your own words how you can calculate this integral without using Riemann sums or the fundamental theorem of calculus. ***Hint:*** Try graphing the integrand!

(b) Find the exact value of the integral.

exs20

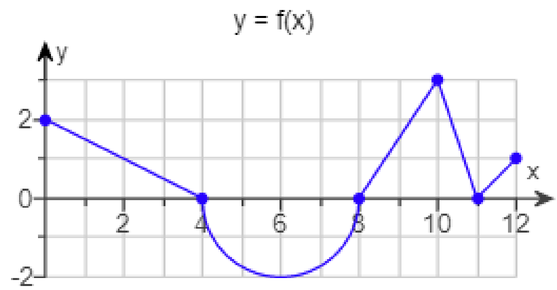
Solution **R-4**

(a) Observe that the graph of $y = \sqrt{9-(x-1)^2}$ is the top half of a circle with center $(1,0)$ and radius 3. The leftmost point on the circle is $(-2,0)$. Thus the integral is equal to the area of the left half of this semi-disc. That is, the region is congruent to a quarter-disc with radius 3.

(b) The area of the region is $\frac{\pi r^2}{4}$ with $r = 3$, hence the area is $\frac{9\pi}{4}$.

Ex. R-5 **5.1/5.2, 5.3** Sp20 **Exam**

Define the function g by $g(x) = \displaystyle\int_0^x f(t)\,dt$, where the graph of $y = f(x)$ is given below. The graph consists of four line segments and one semicricle. ***Note:*** f and g are different functions!



(a) Calculate $f'(9)$.
(b) Calculate $f'(6)$.
(c) Calculate $g'(6)$.
(d) Calculate $g(11) - g(8)$.
(e) Is the statement "$g(4) > g(0)$" true or false?
(f) Find the critical numbers of g in the interval $(0,12)$.

exs20

Solution **R-5**

(a) Observe that $f'(9)$ is simply the slope of given graph at $x = 9$. Hence $f'(9) = \frac{3-0}{10-8} = 1.5$.

(b) Observe that $f'(6)$ is the derivative of the given graph at $x = 6$, and f has a horizontal tangent line at $x = 6$. Hence $f'(6) = 0$.

(c) By the fundamental theorem of calculus, $g'(x) = f(x)$. Hence $g'(6) = f(6) = -2$.

(d) By the additivity property of integrals, $g(11) - g(8) = \int_8^{11} f(t)\,dt$. This is the area of the region below the graph of $y = f(t)$ and above the interval $[8,11]$ on the t-axis. Note that this region is a triangle with base 3 and height 3. Hence $g(11) - g(8) = \frac{1}{2} \cdot 3 \cdot 3 = 4.5$.

(e) Note that $g(0) = 0$ by properties of integrals, and $g(4) > 0$ since $g(4)$ is the area of a triangle that lies above the t-axis. Hence the given statement is true.

(f) The critical numbers of g are those x-values where either $g'(x) = 0$ or $g'(x)$ does not exist. Recall from part (c) that $g'(x) = f(x)$. Clearly $f(x)$ is defined everywhere on $(0,12)$. So the only critical numbers of g are the solutions to $f'(x) = 0$: $x = 4$, $x = 8$, and $x = 11$.

Ex. R-6 **5.1/5.2** Su20 **Exam**

Suppose f is continuous on $[0,8]$ and has the following integrals:

$$\int_0^3 f(x)\,dx = 2 \qquad \int_3^5 f(x)\,dx = 7 \qquad \int_0^8 f(x)\,dx = 15$$

For each part, calculate the integral or determine there is not enough information to do so.

(a) $\displaystyle\int_0^5 f(x)\,dx$ (b) $\displaystyle\int_5^3 f(x)\,dx$ (c) $\displaystyle\int_5^8 f(x)\,dx$ (d) $\displaystyle\int_3^8 (2f(x)-6)\,dx$

exsu20

Solution **R-6**

(a) $\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = 2 + 7 = 9$

(b) $\int_5^3 f(x)\,dx = -\int_3^5 f(x)\,dx = -7$

(c) $\int_5^8 f(x)\,dx = \int_0^8 f(x)\,dx - \int_0^5 f(x)\,dx = 15 - 9 = 6$

(d) First observe:

$$\int_3^8 (2f(x) - 6)\,dx = 2 \cdot \int_3^8 f(x)\,dx - \int_3^8 6\,dx$$

For the second integral on the right side, we note that it gives the area of a rectangle with length $8 - 3 = 5$ and height 6. Hence

$$\int_3^8 6\,dx = 5 \cdot 6 = 30$$

For the other integral, we have the following:

$$\int_3^8 f(x)\,dx = \int_0^8 f(x)\,dx - \int_0^3 f(x)\,dx = 15 - 2 = 13$$

Putting this altogether gives us our final answer:

$$\int_3^8 (2f(x) - 6)\,dx = 2 \cdot \int_3^8 f(x)\,dx - \int_3^8 6\,dx = 2 \cdot 13 - 30 = -4$$

Ex. R-7 **5.1/5.2** Su20 **Exam**

Calculate $\int_0^{\sqrt{10}} \left(x + \sqrt{10 - x^2}\right)\,dx$ using geometry and properties of integrals only. Do not attempt to use the fundamental theorem of calculus.

exsu20

Solution **R-7**

First we split the integral into two separate integrals.

$$\int_0^{\sqrt{10}} \left(x + \sqrt{10 - x^2}\right)\,dx = \underbrace{\int_0^{\sqrt{10}} x\,dx}_{A} + \underbrace{\int_0^{\sqrt{10}} \sqrt{10 - x^2}\,dx}_{B}$$

Now we use geometry to calculate A and B.

Integral A gives the area under the graph of $y = x$ from $x = 0$ to $x = \sqrt{10}$. This region is a triangle with base $\sqrt{10}$ and height $\sqrt{10}$. Thus $A = \frac{1}{2} \cdot \sqrt{10} \cdot \sqrt{10} = 5$.

Integral B gives the area under the graph of $y = \sqrt{10 - x^2}$ from $x = 0$ to $x = \sqrt{10}$. This region is a quarter-disc with center $(0, 0)$ and radius $\sqrt{10}$. Thus $B = \frac{1}{4}\pi \cdot (\sqrt{10})^2 = \frac{5}{2}\pi$.

Hence altogether our desired integral is

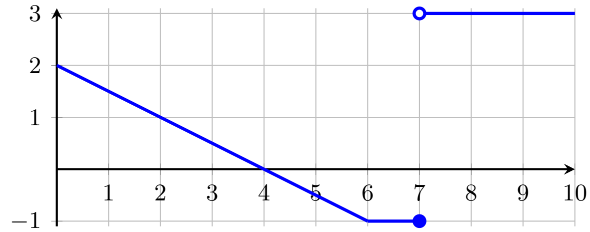
$$\int_0^{\sqrt{10}} \left(x + \sqrt{10 - x^2}\right)\,dx = 5 + \frac{5}{2}\pi$$

Ex. R-8 **5.1/5.2, 5.3** Fa21 **Exam**

Let $F(x) = \int_0^x f(t)\,dt$, where the graph of $y = f(t)$ is given below. For each part, use this information to calculate the indicated item.

exf21

5.1/5.2, 5.3 Fa21 **Exam**



(a) $F(10)$ (b) $F'(6)$ (c) $\int_0^6 |f(t)|\, dt$ (d) $\int_0^4 (f'(t)+5)\, dt$

exf21

Solution **R-8**

(a) The value of $F(10)$ is equal to the (net) area bounded by the graph of $y = f(x)$, the t-axis, and the vertical lines $t = 0$ and $t = 10$.

- The region from $t = 0$ to $t = 4$ consists of a triangle with base 4 and height 2, hence area $\frac{1}{2}(4)(2) = 4$.
- The region from $t = 4$ to $t = 7$ consists of a trapezoid with parallel bases 1 and 3 and height 1, hence area $\frac{1}{2}(3+1)(1) = 2$.
- The region from $t = 7$ to $t = 10$ consists of a square of length 3, hence area 9.

The total net area is $F(10) = 4 - 2 + 9 = 11$.

(b) By the fundamental theorem of calculus, $F'(6) = f(6) = -1$.

(c) Observe that the graph of $y = |f(t)|$ is identical to the graph of $y = f(t)$, except any portion of the graph below the t-axis is reflected across (above) the t-axis. This effectively means that we can compute the desired integral using the graph of $y = f(t)$, but counting any area below the t-axis as positive instead of as negative.

The region from $t = 0$ to $t = 4$ has area 4 and the region from $t = 4$ to $t = 6$ has area 1. Hence the desired integral is $\int_0^6 |f(t)|\, dt = 4 + 1 = 5$.

(d) By the fundamental theorem of calculus, we have:

$$\int_0^4 (f'(t)+5)\, dt = (f(t)+5t)\big|_0^4 = (f(4)+20) - (f(0)+0) = 0 + 20 - 2 = 18$$

Ex. R-9 **5.1/5.2** Fa22 **Quiz**

Let $f(x) = 12 - 3x$. Calculate each of the following integrals using geometry. If you use the Fundamental Theorem of Calculus, you will receive no credit.

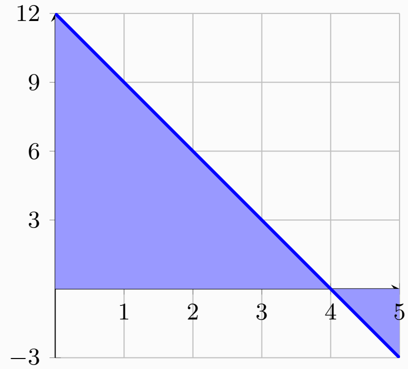
(a) $\int_0^5 f(x)\, dx$ (b) $\int_0^5 |f(x)|\, dx$

qf22

Solution **R-9**

(a) The integral gives the net area of the region bounded by the x-axis, the vertical lines $x = 0$ and $x = 5$, and the line $y = 12 - 3x$. See the figure below.

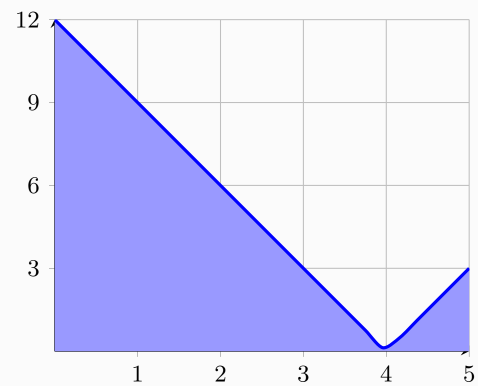
R-9



This region consists of two triangles: (1) larger triangle with base 4 and height 12 with area of $\frac{1}{2} \cdot 4 \cdot 12 = 24$, and (2) smaller triangle with base 1 and height 3 with area of $\frac{1}{2} \cdot 1 \cdot 3 = 1.5$. The net area of a region below the x-axis is negative, and so we have:

$$\int_0^5 f(x)\, dx = 24 - 1.5 = 22.5$$

(b) The integral gives the net area of the region bounded by the x-axis, the vertical lines $x = 0$ and $x = 5$, and the curve $y = |12 - 3x|$. See the figure below.



Note that in general, the graph of $y = |g(x)|$ is identical to the graph of $y = g(x)$ except for values of x for which $g(x) < 0$. For such values of x, the graph of $y = |g(x)|$ is a reflection of the graph of $y = g(x)$ across the x-axis.

This region consists of triangles congruent to those in part (a), except both triangles lie above the x-axis. So we have:

$$\int_0^5 |f(x)|\, dx = 24 + 1.5 = 25.5$$

Ex. R-10 **5.1/5.2, 5.3, 5.5** Fa22 **Quiz**

Calculate each of the following integrals using any valid method taught in this course. You may need to use basic geometry, the Fundamental Theorem of Calculus, substitution rule, or some combination.

(a) $\displaystyle\int_{-5}^{0} \sqrt{25 - x^2}\, dx$ (b) $\displaystyle\int_0^1 6x^2(x^3 + 26)^{1/2}\, dx$ (c) $\displaystyle\int_{-\ln(5)}^{\ln(6)} (2e^x + 3)\, dx$

qf22

Solution R-10

(a) The integral gives the net area of the region bounded by the x-axis, the vertical lines $x = -5$ and $x = 0$, and the curve $y = \sqrt{25 - x^2}$. The curve $y = \sqrt{25 - x^2}$ consists of the top of a semicircle with center $(0, 0)$ and radius 5.

R-10

Thus the region is a quarter-disc with radius 5. So we have:

$$\int_{-5}^{0} \sqrt{25-x^2}\,dx = \frac{1}{4}\pi \cdot 5^2 = \frac{25}{4}\pi$$

(b) We will use substitution rule. Let $u = x^3 + 26$, whence $\frac{du}{dx} = 3x^2$, and so $dx = \frac{du}{3x^2}$. The limits of integration change from $x = 0$ and $x = 1$ to $u = 26$ and $u = 27$, respectively. So now we have:

$$\int_0^1 6x^2(x^3+26)^{1/2}\,dx = \int_{26}^{27} 6x^2u^{1/2} \cdot \frac{du}{3x^2} = \int_{26}^{27} 2u^{1/2}\,du = \left(\frac{4}{3}u^{3/2}\right)\bigg|_{26}^{27} = \frac{4}{3}\left(27^{3/2} - 26^{3/2}\right)$$

(c) We use the fundamental theorem of calculus immediately.

$$\int_{-\ln(5)}^{\ln(6)} (2e^x+3)\,dx = (2e^x+3x)\big|_{-\ln(5)}^{\ln(6)} = \left(2e^{\ln 6} + 3\ln(6)\right) - \left(2e^{-\ln 5} - 3\ln(5)\right) = 11.6 + 3\ln(30)$$

(Note that $e^{-\ln 5} = e^{\ln(1/5)} = \frac{1}{5}$.)

Ex. R-11 **5.1/5.2**

For each part, use geometry to calculate the integral.

(a) $\displaystyle\int_{-1}^{9} (27-3x)\ dx$

(b) $\displaystyle\int_{-2}^{4} (3x+15)\ dx$

(c) $\displaystyle\int_{0}^{12} (2x-10)\ dx$

(d) $\displaystyle\int_{-3}^{5} (|x|-1)\ dx$

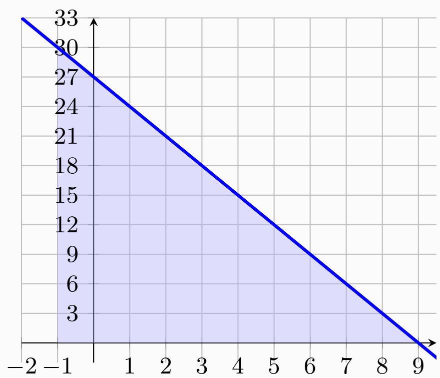
(e) $\displaystyle\int_{-4}^{0} \sqrt{16-x^2}\,dx$

(f) $\displaystyle\int_{2}^{10} \sqrt{64-(x-10)^2}\,dx$

c5s1

Solution R-11

(a) The figure below shows a graph of $y = 27 - 3x$; the given integral is the net area of the shaded region.

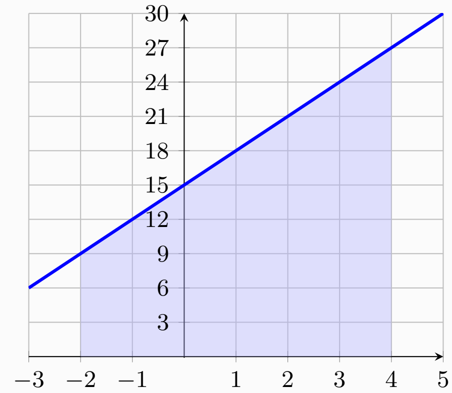


The region is a triangle with base 10 and height 27. Hence

$$\int_{-1}^{9} (27-3x)\ dx = \frac{1}{2}\cdot 10 \cdot 27 = 135$$

(b) The figure below shows a graph of $y = 3x + 15$; the given integral is the net area of the shaded region.

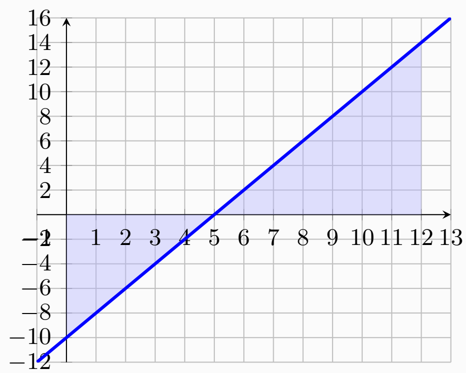
R-11



The region is a trapezoid with bases 9 and 27, and width 6. Hence

$$\int_{-2}^{4} (3x + 15)\ dx = \frac{1}{2} \cdot (9 + 27) \cdot 6 = 108$$

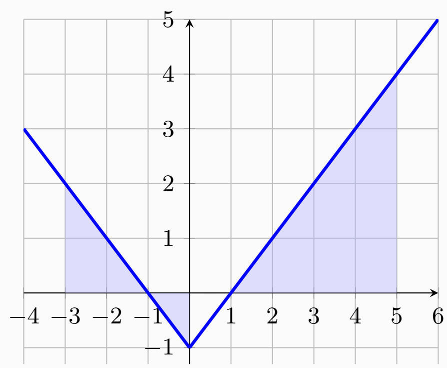
(c) The figure below shows a graph of $y = 2x - 10$; the given integral is the net area of the shaded region.



The region consists of two triangles: (1) above the x-axis with base 7 and height 14, and (2) below the x-axis with base 5 and height 10. Hence

$$\int_{0}^{12} (2x - 10)\ dx = \frac{1}{2} \cdot 7 \cdot 14 - \frac{1}{2} \cdot 5 \cdot 10 = 24$$

(d) The figure below shows a graph of $y = |x| - 1$; the given integral is the net area of the shaded region.

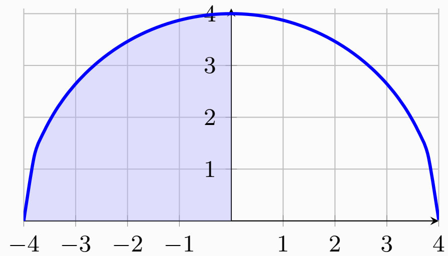


The region consists of three triangles: (1) above the x-axis with base 2 and height 2, (2) above the x-axis with base 4 and height 4, and (3) below the x-axis with base 2 and height 1. Hence

$$\int_{-3}^{5} (|x| - 1)\ dx = \frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot 4 \cdot 4 - \frac{1}{2} \cdot 2 \cdot 1 = 9$$

R-11

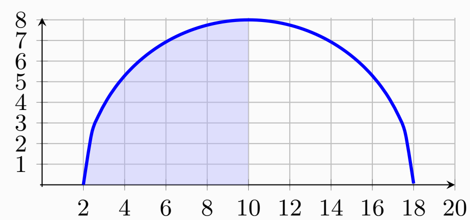
(e) The equation $y = \sqrt{16 - x^2}$ can be written $x^2 + y^2 = 16$, which we recognize as the equation for the circle with center $(0, 0)$ and radius 4. The figure below thus shows a graph of $y = \sqrt{16 - x^2}$; the given integral is the net area of the shaded region.



The region is a quarter disc with radius 4. Hence

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx = \frac{1}{4}\pi \cdot 4^2 = 4\pi$$

(f) the equation $y = \sqrt{64 - (x - 10)^2}$ can be written as $(x - 10)^2 + y^2 = 64$, which we recognize as the equation for the circle with center $(10, 0)$ and radius 8. The figure below thus shows a graph of $y = \sqrt{64 - (x - 10)^2}$; the given integral is the net area of the shaded region.

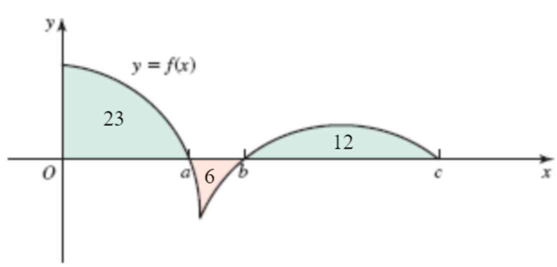


The region is a half disc with radius 8. Hence

$$\int_{2}^{10} \sqrt{64 - (x - 10)^2}\, dx = \frac{1}{2}\pi \cdot 8^2 = 32\pi$$

Ex. R-12 **5.1/5.2**

For each part, use the graph below to calculate the integral. Write your answer in terms of a, b, and c, if necessary. If there is not information to calculate the integral, explain why.



(a) $\displaystyle\int_0^a f(x)\, dx$

(b) $\displaystyle\int_0^b f(x)\, dx$

(c) $\displaystyle\int_a^c f(x)\, dx$

(d) $\displaystyle\int_0^c |f(x)|\, dx$

(e) $\displaystyle\int_0^c (2\,|f(x)| + 3f(x))\, dx$

(f) $\displaystyle\int_a^0 f(x)\, dx$

(g) $\displaystyle\int_c^a |f(x)|\, dx$

(h) $\displaystyle\int_0^c (2f(x) + 3)\, dx$

(i) $\displaystyle\int_0^a f(x)^2\, dx$

c5s1

Solution **R-12**

(a) $\displaystyle\int_0^a f(x)\,dx = 23$

(b) $\displaystyle\int_0^b f(x)\,dx = 23 - 6 = 17$

(c) $\displaystyle\int_a^c f(x)\,dx = -6 + 12 = 6$

(d) $\displaystyle\int_0^c |f(x)|\,dx = 23 + 6 + 12 = 41$

(e) $\displaystyle\int_0^c (2|f(x)| + 3f(x))\,dx = 2\int_0^c |f(x)|\,dx + 3\int_0^c f(x)\,dx = 2\cdot 41 + 3\cdot(23 - 6 + 12) = 169$

(f) $\displaystyle\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx = -23$

(g) $\displaystyle\int_c^a |f(x)|\,dx = -\int_a^c |f(x)|\,dx - (6 + 12) = -18$

(h) $\displaystyle\int_0^c (2f(x) + 3)\,dx = 2\int_0^c f(x)\,dx + \int_0^c 3\,dx = 2\cdot(23 - 6 + 12) + 3c = 58 + 3c$

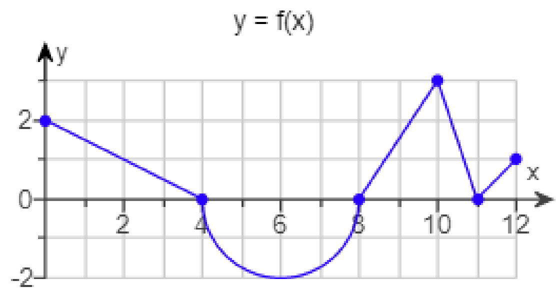
For the last integral we used the fact that $\int_0^c 3\,dx$ is the area of a rectangle with width c and height 3.

(i) There is not enough information to calculate $\int_0^a f(x)^2\,dx$. There is no simple geometric relationship between the graphs of $y = f(x)$ and $y = f(x)^2$ since neither graph is a translation, reflection, or rotation of the other.

§5.3: Fundamental Theorem of Calculus

Ex. R-5 5.1/5.2, 5.3 Sp20 Exam

Define the function g by $g(x) = \int_0^x f(t)\,dt$, where the graph of $y = f(x)$ is given below. The graph consists of four line segments and one semicricle. ***Note:*** f and g are different functions!



(a) Calculate $f'(9)$.
(b) Calculate $f'(6)$.
(c) Calculate $g'(6)$.
(d) Calculate $g(11) - g(8)$.
(e) Is the statement "$g(4) > g(0)$" true or false?
(f) Find the critical numbers of g in the interval $(0, 12)$.

exs20

Solution R-5

(a) Observe that $f'(9)$ is simply the slope of given graph at $x = 9$. Hence $f'(9) = \frac{3-0}{10-8} = 1.5$.
(b) Observe that $f'(6)$ is the derivative of the given graph at $x = 6$, and f has a horizontal tangent line at $x = 6$. Hence $f'(6) = 0$.
(c) By the fundamental theorem of calculus, $g'(x) = f(x)$. Hence $g'(6) = f(6) = -2$.
(d) By the additivity property of integrals, $g(11) - g(8) = \int_8^{11} f(t)\,dt$. This is the area of the region below the graph of $y = f(t)$ and above the interval $[8, 11]$ on the t-axis. Note that this region is a triangle with base 3 and height 3. Hence $g(11) - g(8) = \frac{1}{2} \cdot 3 \cdot 3 = 4.5$.
(e) Note that $g(0) = 0$ by properties of integrals, and $g(4) > 0$ since $g(4)$ is the area of a triangle that lies above the t-axis. Hence the given statement is true.
(f) The critical numbers of g are those x-values where either $g'(x) = 0$ or $g'(x)$ does not exist. Recall from part (c) that $g'(x) = f(x)$. Clearly $f(x)$ is defined everywhere on $(0, 12)$. So the only critical numbers of g are the solutions to $f'(x) = 0$: $x = 4$, $x = 8$, and $x = 11$.

Ex. S-1 5.3 Su20 Exam

Let $f(x) = 5 + \int_{-3}^{x} t^2 e^t\,dt$. Find an equation of the tangent line to f at $x = -3$.

exsu20

Solution S-1

Note that $f(-3) = 5 + 0 = 5$ and, by the fundamental theorem of calculus, $f'(x) = x^2 e^x$. Hence $f'(-3) = 9e^{-3}$, and an equation of our tangent line is

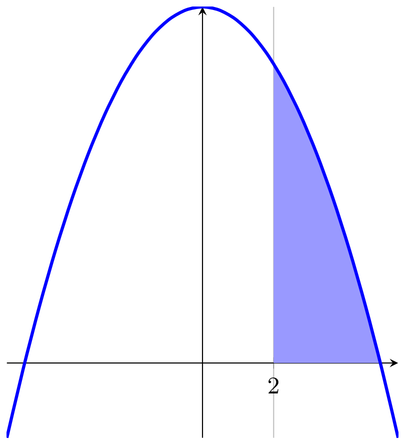
$$y = 5 + 9e^{-3}(x + 3)$$

Ex. S-2 5.3 Su20 Exam

The curve $y = 25 - x^2$ is shown in the figure below. Calculate the area of the shaded region.

exsu20

5.3 Su20 Exam



exsu20

Solution **S-2**

The graph crosses the x-axis where $25 - x^2 = 0$, or at $x = -5$ and $x = 5$. Hence we seek the area under the graph of $y = 25 - x^2$ from $x = 2$ to $x = 5$. That area is given by an integral, which is calculated using the fundamental theorem of calculus below.

$$\int_2^5 (25 - x^2)\,dx = \left(25x - \frac{1}{3}x^3\right)\Big|_2^5 = \left(125 - \frac{1}{3}\cdot 125\right) - \left(50 - \frac{8}{3}\right) = 36$$

Ex. S-3 5.3, 5.5 Fa21 Exam

The parts of this problem are not related.

(a) Calculate the integral $\displaystyle\int_2^4 \frac{18t - 3t^2}{t}\,dt$.

(b) Calculate the area of the region below the curve $y = 23\sin(x)\cos^2(x)$ and above the interval $\left[0, \frac{\pi}{2}\right]$ on the x-axis. (Note that $y \geq 0$ on this interval.)

exf21

Solution **S-3**

(a) Simplify the integrand using basic algebra, then use the fundamental theorem of calculus.

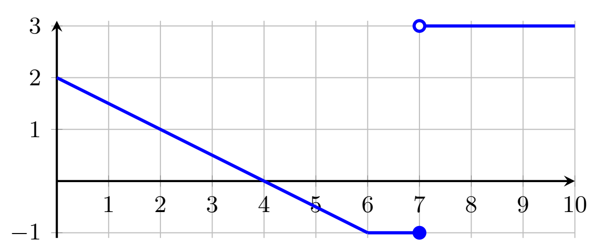
$$\int_2^4 \frac{18t - 3t^2}{t}\,dt = \int_2^4 (18 - 3t)\,dt = \left(18t - \frac{3}{2}t^2\right)\Big|_2^4 = (72 - 24) - (36 - 6) = 18$$

(b) The area of the region is equal to the integral $\displaystyle\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx$. We use substitution rule with $u = \cos(x)$ (whence $-du = \sin(x)\,dx$).

$$\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx = \int_1^0 (-23u^2)\,du = \left(-\frac{23}{3}u^3\right)\Big|_1^0 = 0 - \frac{-23}{3} = \frac{23}{3}$$

Ex. R-8 5.1/5.2, 5.3 Fa21 Exam

Let $F(x) = \displaystyle\int_0^x f(t)\,dt$, where the graph of $y = f(t)$ is given below. For each part, use this information to calculate the indicated item.



exf21

5.1/5.2, 5.3 Fa21 Exam

(a) $F(10)$ (b) $F'(6)$ (c) $\displaystyle\int_0^6 |f(t)|\, dt$ (d) $\displaystyle\int_0^4 (f'(t)+5)\, dt$

exf21

Solution **R-8**

(a) The value of $F(10)$ is equal to the (net) area bounded by the graph of $y = f(x)$, the t-axis, and the vertical lines $t = 0$ and $t = 10$.

- The region from $t = 0$ to $t = 4$ consists of a triangle with base 4 and height 2, hence area $\frac{1}{2}(4)(2) = 4$.
- The region from $t = 4$ to $t = 7$ consists of a trapezoid with parallel bases 1 and 3 and height 1, hence area $\frac{1}{2}(3+1)(1) = 2$.
- The region from $t = 7$ to $t = 10$ consists of a square of length 3, hence area 9.

The total net area is $F(10) = 4 - 2 + 9 = 11$.

(b) By the fundamental theorem of calculus, $F'(6) = f(6) = -1$.

(c) Observe that the graph of $y = |f(t)|$ is identical to the graph of $y = f(t)$, except any portion of the graph below the t-axis is reflected across (above) the t-axis. This effectively means that we can compute the desired integral using the graph of $y = f(t)$, but counting any area below the t-axis as positive instead of as negative.

The region from $t = 0$ to $t = 4$ has area 4 and the region from $t = 4$ to $t = 6$ has area 1. Hence the desired integral is $\displaystyle\int_0^6 |f(t)|\, dt = 4 + 1 = 5$.

(d) By the fundamental theorem of calculus, we have:

$$\int_0^4 (f'(t)+5)\, dt = (f(t)+5t)\big|_0^4 = (f(4)+20) - (f(0)+0) = 0 + 20 - 2 = 18$$

Ex. Q-6 4.9, 5.3 Su22 Quiz

Calculate each of the following. You do not have to simplify your answers.

(a) $\displaystyle\int \left(\frac{3t^2 - \sqrt{t} + 4}{5t}\right) dt$ (b) $\displaystyle\int_{-1}^3 \left(3x^2 + 2e^x\right) dx$

qsu22

Solution **Q-6**

(a) Divide each term and then antidifferentiate.

$$\int \left(\frac{3t^2 - \sqrt{t} + 4}{5t}\right) dt = \int \left(\frac{3}{5}t - \frac{1}{5}t^{-1/2} + \frac{4}{5}t^{-1}\right) dt = \frac{3}{10}t^2 - \frac{2}{5}t^{1/2} + \frac{4}{5}\ln(|t|) + C$$

(b) Find the antiderivative, then use the fundamental theorem of calculus.

$$\int_{-1}^3 \left(3x^2 + 2e^x\right) dx = \left(x^3 + 2e^x\right)\Big|_{-1}^3 = \left(27 + 2e^3\right) - \left(-1 + 2e^{-1}\right) = 28 + 2e^3 - 2e^{-1}$$

Ex. S-4 5.3 Fa22 Quiz

Find the area of the region bounded by the graph of $y = (x^4+1)^2$, the x-axis, and the lines $x = 0$ and $x = 1$.

qf22

Solution **S-4**

The graph of $y = (x^4+1)^2$ lies entirely above the x-axis. So the desired area is given by the integral below:

$$\int_0^1 (x^4+1)^2\, dx = \int_0^1 \left(x^8 + 2x^4 + 1\right) dx = \left(\frac{1}{9}x^9 + \frac{2}{5}x^5 + x\right)\Big|_0^1 = \left(\frac{1}{9} + \frac{2}{5} + 1\right) - (0+0+0) = \frac{23}{45}$$

Ex. R-10 5.1/5.2, 5.3, 5.5 Fa22 Quiz

Calculate each of the following integrals using any valid method taught in this course. You may need to use basic geometry, the Fundamental Theorem of Calculus, substitution rule, or some combination.

qf22

5.1/5.2, 5.3, 5.5

Fa22 **Quiz**

(a) $\displaystyle\int_{-5}^{0} \sqrt{25-x^2}\,dx$ (b) $\displaystyle\int_{0}^{1} 6x^2(x^3+26)^{1/2}\,dx$ (c) $\displaystyle\int_{-\ln(5)}^{\ln(6)} (2e^x+3)\,dx$

qf22

Solution **R-10**

(a) The integral gives the net area of the region bounded by the x-axis, the vertical lines $x=-5$ and $x=0$, and the curve $y=\sqrt{25-x^2}$. The curve $y=\sqrt{25-x^2}$ consists of the top of a semicircle with center $(0,0)$ and radius 5. Thus the region is a quarter-disc with radius 5. So we have:

$$\int_{-5}^{0} \sqrt{25-x^2}\,dx = \frac{1}{4}\pi\cdot 5^2 = \frac{25}{4}\pi$$

(b) We will use substitution rule. Let $u=x^3+26$, whence $\frac{du}{dx}=3x^2$, and so $dx=\frac{du}{3x^2}$. The limits of integration change from $x=0$ and $x=1$ to $u=26$ and $u=27$, respectively. So now we have:

$$\int_0^1 6x^2(x^3+26)^{1/2}\,dx = \int_{26}^{27} 6x^2u^{1/2}\cdot\frac{du}{3x^2} = \int_{26}^{27} 2u^{1/2}\,du = \left(\frac{4}{3}u^{3/2}\right)\Big|_{26}^{27} = \frac{4}{3}\left(27^{3/2}-26^{3/2}\right)$$

(c) We use the fundamental theorem of calculus immediately.

$$\int_{-\ln(5)}^{\ln(6)} (2e^x+3)\,dx = (2e^x+3x)\big|_{-\ln(5)}^{\ln(6)} = \left(2e^{\ln 6}+3\ln(6)\right)-\left(2e^{-\ln 5}-3\ln(5)\right) = 11.6+3\ln(30)$$

(Note that $e^{-\ln 5}=e^{\ln(1/5)}=\frac{1}{5}$.)

Ex. S-5 5.3

For each part, evaluate the integral using geometry, the Fundamental Theorem of Calculus, or a combination.

(a) $\displaystyle\int_{-3}^{5} (-8)\,dx$

(b) $\displaystyle\int_{4}^{36} \sqrt{2x}\,dx$

(c) $\displaystyle\int_{-\ln(3)}^{\ln(8)} 5e^x\,dx$

(d) $\displaystyle\int_0^9 \sqrt{x}(x^2-x+1)\,dx$

(e) $\displaystyle\int_9^{10} \frac{a}{x}\,dx$

(f) $\displaystyle\int_{-4}^{4} \sqrt{16-x^2}\,dx$

(g) $\displaystyle\int_{-2}^{5} (2x-|x|)\,dx$

(h) $\displaystyle\int_{-\pi}^{\pi/2} \sin(x)\,dx$

(i) $\displaystyle\left|\int_{-\pi}^{\pi/2} \sin(x)\,dx\right|$

(j) $\displaystyle\int_{-\pi}^{\pi/2} |\sin(x)|\,dx$

c5s3

Solution **S-5**

(a) Use FTC.

$$\int_{-3}^{5}(-8)\,dx = -8x\Big|_{-3}^{5} = (-40)-(24) = -64$$

(b) Use FTC.

$$\int_4^{36}\sqrt{2x}\,dx = \int_4^{36}\sqrt{2}x^{1/2}\,dx = \sqrt{2}\cdot\frac{2}{3}x^{3/2}\Big|_4^{36} = \left(\frac{2\sqrt{2}}{3}\cdot 216\right)-\left(\frac{2\sqrt{2}}{3}\cdot 8\right) = \frac{416\sqrt{2}}{3}$$

(c) Use FTC.

$$\int_{-\ln(3)}^{\ln(8)} 5e^x\,dx = 5e^x\Big|_{-\ln(3)}^{\ln(8)} = (5\cdot 8)-\left(5\cdot\tfrac{1}{3}\right) = \frac{115}{3}$$

(d) Use FTC.

$$\int_0^9\sqrt{x}(x^2-x+1)\,dx = \int_0^9\left(x^{5/2}-x^{3/2}+x^{1/2}\right)\,dx = \left(\frac{2}{7}x^{7/2}-\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}\right)\Big|_0^9$$
$$= \left(\frac{2}{7}\cdot 3^7-\frac{2}{5}\cdot 3^5+\frac{2}{3}\cdot 3^3\right)-0 = \frac{19,098}{35}$$

S-5

(e) Use FTC.

$$\int_9^{10} \frac{a}{x}\,dx = a\ln(|x|)\Big|_9^{10} = a\ln(10) - a\ln(9) = a\ln\left(\frac{10}{9}\right)$$

(f) The integral represents the area under the curve $y = \sqrt{16 - x^2}$ and above the interval $[-4, 4]$ on the x-axis. This region is a half-disc centered at the origin with radius $r = 4$. Therefore the area (and the integral) is

$$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \frac{1}{2}\pi \cdot 4^2 = 8\pi$$

(g) First we write the integrand $y = 2x - |x|$ as a piecewise function.

$$2x - |x| = \begin{cases} 2x - (-x) & \text{if } x < 0 \\ 2x - x & \text{if } x \geq 0 \end{cases} = \begin{cases} 3x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Now we split the integral into two separate integrals.

$$\begin{aligned} \int_{-2}^{5} (2x - |x|)\,dx &= \int_{-2}^{0} (2x - |x|)\,dx + \int_{0}^{5} (2x - |x|)\,dx = \int_{-2}^{0} 3x\,dx + \int_0^5 x\,dx \\ &= \left(\tfrac{3}{2}x^2\Big|_{-2}^{0}\right) + \left(\tfrac{1}{2}x^2\Big|_0^5\right) = \left(0 - \tfrac{3}{2}(-2)^2\right) + \left(\tfrac{1}{2}\cdot 5^2 - 0\right) = \frac{13}{2} \end{aligned}$$

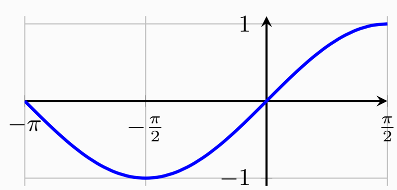
(h) Use FTC.

$$\int_{-\pi}^{\pi/2} \sin(x)\,dx = -\cos(x)\Big|_{-\pi}^{\pi/2} = -\cos\left(\tfrac{\pi}{2}\right) - (-\cos(\pi)) = 0 - 1 = -1$$

(i) Use the previous part.

$$\left|\int_{-\pi}^{\pi/2} \sin(x)\,dx\right| = |-1| = 1$$

(j) The figure below shows a graph of $y = \sin(x)$ on $[-\pi, \frac{\pi}{2}]$.



Observe that $\sin(x) \leq 0$ on $[-\pi, 0]$, and so we have:

$$|\sin(x)| = \begin{cases} -\sin(x) & \text{if } -\pi \leq x < 0 \\ \sin(x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

To compute our integral, we split into two integrals.

$$\int_{-\pi}^{\pi/2} |\sin(x)|\,dx = -\int_{-\pi}^{0} \sin(x)\,dx + \int_0^{\pi/2} \sin(x)\,dx = \left(\cos(x)\Big|_{-\pi}^{0}\right) + \left(-\cos(x)\Big|_0^{\pi/2}\right) = (1-(-1)) + (0-(-1)) = 3$$

Ex. S-6 **5.3**

For each part, calculate $F'(x)$.

(a) $F(x) = \displaystyle\int_{-3}^{x} \frac{t^4 - t^2 + 1}{\sqrt{t^6 + 1}}\,dt$

(b) $F(x) = \displaystyle\int_{-\pi}^{x} \sqrt[3]{w}(w^2 - 2w + 5)\,dw$

c5s3

Solution **S-6**

(a) Use FTC.

$$F'(x) = \frac{x^4 - x^2 + 1}{\sqrt{x^6 + 1}}$$

(b) Use FTC.

$$F'(x) = \sqrt[3]{x}(x^2 - 2x + 5)$$

Ex. S-7 **5.3**

Let $f(x) = \begin{cases} 4x - x^2 & \text{if } x \leq 2 \\ \dfrac{8}{x} & \text{if } x > 2 \end{cases}$.

(a) Show that $f(x)$ is continuous on $[-1, 4]$.

(b) Sketch the region whose net area is given by the integral $\displaystyle\int_{-1}^{4} f(x)\,dx$.

(c) Evaluate $\displaystyle\int_{-1}^{4} f(x)\,dx$.

c5s3

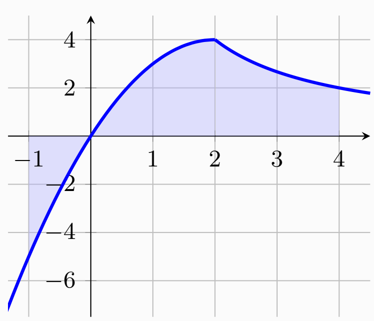
Solution **S-7**

(a) Each "piece" of $f(x)$ is continuous on the respective domain. Hence the only point of possible discontinuity is $x = 2$. Now observe that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - x^2) = 8 - 4 = 4$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{8}{x}\right) = \frac{8}{2} = 4$$
$$f(2) = (4x - x^2)\big|_{x=2} = 4$$

Since these three numbers are all equal, $f(x)$ is continuous at $x = 2$.

(b) We have the following.



(c) We split the integral $x = 2$ into two separate integrals.

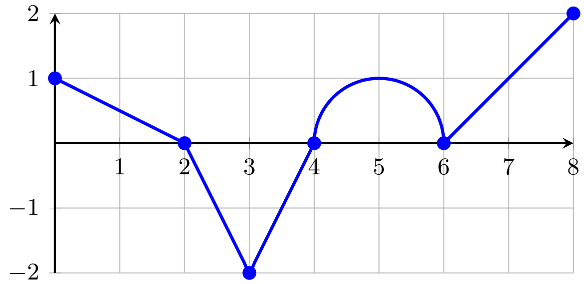
$$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx = \int_{-1}^{2} (4x - x^2)\,dx + \int_{2}^{4} \frac{8}{x}\,dx$$
$$= \left(2x^2 - \tfrac{1}{3}x^3\right)\Big|_{-1}^{2} + 8\ln(|x|)\Big|_{2}^{4} = \left((8 - \tfrac{8}{3}) - (2 + \tfrac{1}{3})\right) + (8\ln(4) - 8\ln(2)) = 3 + 8\ln(2)$$

Ex. S-8 **5.3**

Let $g(x) = \int_0^x f(t)\,dt$, where the graph of $y = f(x)$ is given below. This graph consists of four line segments and one semicircle.

c5s3

5.3



(a) Is the statement "$g(4) > g(2)$" true or false? Explain your answer.

(b) Evaluate $g(8)$.

(c) Where is g decreasing and where is g increasing? Where in $(0, 8)$ does g have a local minimum? local maximum?

(d) Where is g concave down and where is g concave up? Where in $(0, 8)$ does g have an inflection point?

c5s3

Solution **S-8**

When we analyze $g(x)$ we will need its derivatives. So observe that by the FTC, we have

$$\boxed{g'(x) = f(x)} \qquad \boxed{g''(x) = f'(x)}$$

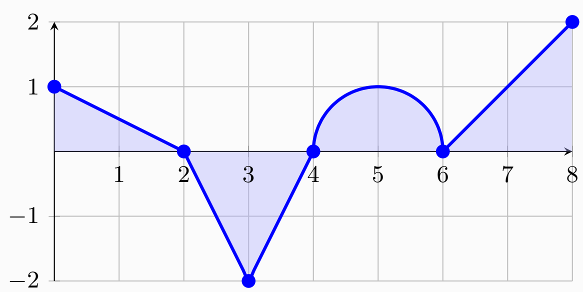
Note that the graph shows $f(x)$, not $g(x)$. The graph, in fact, shows the *derivative* of $g(x)$.

(a) ***False.*** Since the graph of $y = f(x)$ lies below the x-axis for $2 \le x \le 4$, we have $\int_2^4 f(x)\,dx < 0$. So we have:

$$g(4) - g(2) = \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx = \int_2^4 f(x)\,dx < 0$$

That is, $g(4) - g(2) < 0$.

(b) The number $g(8)$ is the net area of the region between the graph of $y = f(x)$ and the interval $[0, 8]$ on the x-axis. (see the shaded region below).



This region consists of three triangles and a half-disc. Net area above the x-axis is positive and net area below the x-axis is negative.

$$g(8) = \frac{1}{2} \cdot 2 \cdot 1 - \frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2}\pi \cdot 1^2 + \frac{1}{2} \cdot 2 \cdot 2 = 1 + \frac{\pi}{2}$$

(c) We calculate a sign chart for $g'(x)$ (i.e., $f(x)$). Note that the figure shows the graph of $f(x)$. The cut points are the solutions to $f(x) = 0$ ($x = 2$, $x = 4$, and $x = 6$) or where $f(x)$ DNE (none).

interval	test point	sign of $g' = f$	shape of g
$[0, 2)$	$f(1)$	$\oplus$	increasing
$(2, 4)$	$f(3)$	$\ominus$	decreasing
$(4, 6)$	$f(5)$	$\oplus$	increasing
$(6, 8]$	$f(7)$	$\oplus$	increasing

Hence we deduce the following about g:

S-8

g is decreasing on:	$[2, 4]$
g is increasing on:	$[0, 2]$, $[4, 8]$
g has a local min at:	$x = 4$
g has a local max at:	$x = 2$

(d) We calculate a sign chart for $g''(x)$ (i.e., $f'(x)$). Note that $f'(x)$ is the *derivative* (i.e., the slope of the tangent line) of the function shown in the figure. The cut points are the solutions to $f'(x) = 0$ ($x = 5$ only) or where $f'(x)$ DNE ($x = 2$, $x = 3$, $x = 4$, $x = 6$).

interval	test point	sign of $g'' = f'$	shape of g
$[0, 2)$	$f'(1)$	$\ominus$	concave down
$(2, 3)$	$f'(2.5)$	$\ominus$	concave down
$(3, 4)$	$f'(3.5)$	$\oplus$	concave up
$(4, 5)$	$f'(4.5)$	$\oplus$	concave up
$(5, 6)$	$f'(5.5)$	$\ominus$	concave down
$(6, 8]$	$f'(7)$	$\oplus$	concave up

Hence we deduce the following about g:

g is concave down on:	$[0, 3]$, $[5, 6]$
g is concave up on:	$[3, 5]$, $[6, 8]$
g has an inflection point at:	$x = 3$, $x = 5$, and $x = 6$

Ex. S-9 5.3

The parts of this question are not related.

(a) Find $F'(x)$ with $F(x) = \displaystyle\int_{-1}^{x} \frac{t^5}{3 + t^6}\, dt$.

(b) Find $\displaystyle\int_0^5 f(t)\, dt$ with $f(x) = \begin{cases} x & \text{if } x < 1 \\ \frac{1}{x} & \text{if } x \geq 1 \end{cases}$.

unit5

Solution S-9

(a) Use FTC: $F'(x) = \dfrac{x^5}{3 + x^6}$.

(b) Split the integral into two integrals using the subdivision property. Then use FTC for each integral.

$$\int_0^5 f(t)\, dt = \int_0^1 t\, dt + \int_1^5 \frac{1}{t}\, dt = \left(\frac{1}{2}t^2\Big|_0^1\right) + \left(\ln(t)\Big|_1^5\right) = \left(\frac{1}{2} - 0\right) + (\ln(5) - \ln(1)) = \frac{1}{2} + \ln(5)$$

Ex. Q-15 4.9, 5.3, 5.5

For each part, find the antiderivative or integral.

(a) $\displaystyle\int t^2 \cos(1 - t^3)\, dt$ (b) $\displaystyle\int \sqrt{x - 1}\, dx$ (c) $\displaystyle\int_2^3 \frac{\ln(x)}{x}\, dx$ (d) $\displaystyle\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x} - 1}\, dx$

unit5

Solution Q-15

(a) Substitute $u = 1 - t^3$ (whence $-\frac{1}{3}du = t^2\, dt$).

$$\int t^2 \cos(1 - t^3)\, dt = \int \left(-\frac{1}{3}\cos(u)\right) du = -\frac{1}{3}\sin(u) + C = -\frac{1}{3}\sin(1 - t^3) + C$$

(b) Substitute $u = x - 1$ (whence $du = dx$).

$$\int \sqrt{x - 1}\, dx = \int u^{1/2}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x - 1)^{3/2} + C$$

Q-15

(c) Substitute $u = \ln(x)$ (whence $du = \frac{1}{x}\,dx$).

$$\int_2^3 \frac{\ln(x)}{x}\,dx = \int_{\ln(2)}^{\ln(3)} u\,du = \frac{1}{2}u^2\Big|_{\ln(2)}^{\ln(3)} = \frac{\ln(3)^2 - \ln(2)^2}{2}$$

(d) Substitute $u = e^{2x} - 1$ (whence $\frac{1}{2}du = e^{2x}\,dx$).

$$\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x}-1}\,dx = \int_0^8 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_0^8 = \frac{8^{3/2}}{3}$$

§5.5: Substitution Rule

Ex. T-1 5.5 Sp20 Exam

Note: The parts of this problem are not related.

(a) Suppose we use the fundamental theorem of calculus to calculate an integral as follows:

$$\int_a^b g(u)\,du = G(b) - G(a)$$

What is the relationship between the functions g and G?

(b) Calculate the following definite integral:

$$\int_{e^{-3}}^{e^2} \frac{2\ln(x) - 3}{5x}\,dx$$

(c) Consider the following indefinite integral:

$$J = \int \frac{\ln(x)}{3x^2}\,dx$$

Use the substitution $u = \ln(x)$ to write J as an equivalent indefinite integral in terms of u. *Do not attempt to calculate J.*

exs20

Solution T-1

(a) The function g is the derivative of G (equivalently, G is an antiderivative of g).

(b) We use the substitution $u = 2\ln(x) - 3$, whence $\frac{du}{dx} = \frac{2}{x}$ (or $dx = \frac{1}{2}xdu$). We find the new limits of integration by substituting the old limits of integration into our relation $u = 2\ln(x) - 3$. Hence the new limits are:

$$x = e^{-3} \implies u = 2\cdot(-3) - 3 = -9$$
$$x = e^2 \implies u = 2\cdot(2) - 3 = 1$$

So the new lower and upper limits of integration are -9 and 1, respectively. So now we have the following:

$$\int_{e^{-3}}^{e^2} \frac{2\ln(x) - 3}{5x}\,dx = \int_{-9}^{1} \frac{u}{5x}\cdot\frac{x}{2}du = \int_{-9}^{1} \frac{u}{10}\,du = \frac{u^2}{20}\bigg|_{-9}^{1} = \frac{1}{20}(1 - 81) = -4$$

(c) We have $u = \ln(x)$, whence $\frac{du}{dx} = \frac{1}{x}$, or $dx = x\,du$. Hence we have:

$$J = \int \frac{u}{3x^2}\cdot(x\,du) = \int \frac{u}{3x}\,du$$

We are still left with a factor of x, but the integrand must be only in terms of u. Since $u = \ln(x)$, we have $x = e^u$. Hence we have:

$$J = \int \frac{u}{3x}\,du = \int \frac{u}{3e^u}\,du$$

Ex. T-2 5.5 Su20 Exam

Find the unique positive value of a such that $\displaystyle\int_0^a \frac{x}{x^2+1}\,dx = 3$.

exsu20

Solution T-2

We use substitution rule with $u = x^2 + 1$ to calculate the integral. Note that with this choice of u, we have $\frac{du}{dx} = 2x$, or $dx = \frac{du}{2x}$. The limits of integration change from $x = 0$ and $x = a$ to $u = 1$ and $u = a^2 + 1$, respectively. Hence we have the following:

$$\int_0^a \frac{x}{x^2+1}\,dx = \int_1^{a^2+1} \frac{1}{2u}\,du = \frac{1}{2}\ln(u)\bigg|_1^{a^2+1} = \frac{1}{2}\ln(a^2+1) - 0 = \frac{1}{2}\ln(a^2+1)$$

We now solve the equation $\frac{1}{2}\ln(a^2+1) = 3$ to find that $a = \sqrt{e^6 - 1}$ (we have kept only the positive root).

Ex. S-3 **5.3, 5.5** Fa21 **Exam**

The parts of this problem are not related.

(a) Calculate the integral $\displaystyle\int_2^4 \frac{18t-3t^2}{t}\,dt$.

(b) Calculate the area of the region below the curve $y = 23\sin(x)\cos^2(x)$ and above the interval $\left[0, \frac{\pi}{2}\right]$ on the x-axis. (Note that $y \geq 0$ on this interval.)

exf21

Solution **S-3**

(a) Simplify the integrand using basic algebra, then use the fundamental theorem of calculus.

$$\int_2^4 \frac{18t-3t^2}{t}\,dt = \int_2^4 (18-3t)\,dt = \left(18t - \frac{3}{2}t^2\right)\Big|_2^4 = (72-24)-(36-6) = 18$$

(b) The area of the region is equal to the integral $\displaystyle\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx$. We use substitution rule with $u = \cos(x)$ (whence $-du = \sin(x)\,dx$).

$$\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx = \int_1^0 (-23u^2)\,du = \left(-\frac{23}{3}u^3\right)\Big|_1^0 = 0 - \frac{-23}{3} = \frac{23}{3}$$

Ex. R-10 **5.1/5.2, 5.3, 5.5** Fa22 **Quiz**

Calculate each of the following integrals using any valid method taught in this course. You may need to use basic geometry, the Fundamental Theorem of Calculus, substitution rule, or some combination.

(a) $\displaystyle\int_{-5}^0 \sqrt{25-x^2}\,dx$ (b) $\displaystyle\int_0^1 6x^2(x^3+26)^{1/2}\,dx$ (c) $\displaystyle\int_{-\ln(5)}^{\ln(6)} (2e^x+3)\,dx$

qf22

Solution **R-10**

(a) The integral gives the net area of the region bounded by the x-axis, the vertical lines $x=-5$ and $x=0$, and the curve $y=\sqrt{25-x^2}$. The curve $y=\sqrt{25-x^2}$ consists of the top of a semicircle with center $(0,0)$ and radius 5. Thus the region is a quarter-disc with radius 5. So we have:

$$\int_{-5}^0 \sqrt{25-x^2}\,dx = \frac{1}{4}\pi\cdot 5^2 = \frac{25}{4}\pi$$

(b) We will use substitution rule. Let $u = x^3+26$, whence $\frac{du}{dx} = 3x^2$, and so $dx = \frac{du}{3x^2}$. The limits of integration change from $x=0$ and $x=1$ to $u=26$ and $u=27$, respectively. So now we have:

$$\int_0^1 6x^2(x^3+26)^{1/2}\,dx = \int_{26}^{27} 6x^2u^{1/2}\cdot\frac{du}{3x^2} = \int_{26}^{27} 2u^{1/2}\,du = \left(\frac{4}{3}u^{3/2}\right)\Big|_{26}^{27} = \frac{4}{3}\left(27^{3/2}-26^{3/2}\right)$$

(c) We use the fundamental theorem of calculus immediately.

$$\int_{-\ln(5)}^{\ln(6)} (2e^x+3)\,dx = (2e^x+3x)\big|_{-\ln(5)}^{\ln(6)} = \left(2e^{\ln 6}+3\ln(6)\right) - \left(2e^{-\ln 5} - 3\ln(5)\right) = 11.6 + 3\ln(30)$$

(Note that $e^{-\ln 5} = e^{\ln(1/5)} = \frac{1}{5}$.)

Ex. T-3 **5.5**

For each part, find the antiderivative.

(a) $\displaystyle\int (5x-7)^{14}\,dx$ (c) $\displaystyle\int \cos(4-x)\,dx$ (e) $\displaystyle\int \frac{1}{x\ln(x)\ln(\ln(x))}\,dx$

(b) $\displaystyle\int \frac{x^3}{\sqrt{9-x^4}}\,dx$ (d) $\displaystyle\int x\sqrt{2x+1}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{w}(\sqrt{w}+7)}\,dw$

c5s5

Solution **T-3**

(a) Substitute $u = 5x - 7$.

$$\boxed{\begin{aligned} u &= 5x-7 \\ du &= 5\,dx \\ dx &= \frac{du}{5} \end{aligned}}$$

$$\int (5x-7)^{14}\,dx = \int \frac{1}{5}u^{14}\,du = \frac{1}{75}u^{15} + C = \frac{1}{75}(5x-7)^{15} + C$$

(b) Substitute $u = 9 - x^4$.

$$\boxed{\begin{aligned} u &= 9-x^4 \\ du &= -4x^3\,dx \\ dx &= \frac{du}{-4x^3} \end{aligned}}$$

$$\int \frac{x^3}{\sqrt{9-x^4}}\,dx = \int \frac{x^3}{\sqrt{u}} \cdot \frac{du}{-4x^3} = \int \left(\frac{-1}{4}u^{-1/2}\right)\,du = \frac{-1}{2}u^{1/2} + C = -\frac{1}{2}\sqrt{9-x^4} + C$$

(c) Substitute $u = 4 - x$.

$$\boxed{\begin{aligned} u &= 4-x \\ du &= -dx \\ dx &= -du \end{aligned}}$$

$$\int \cos(4-x)\,dx = \int (-\cos(u))\,du) = -\sin(u) + C = -\sin(4-x) + C$$

(d) Substitute $u = 2x + 1$.

$$\boxed{\begin{aligned} u &= 2x+1 \\ x &= \frac{u-1}{2} \\ du &= 2\,dx \\ dx &= \frac{du}{2} \end{aligned}}$$

$$\begin{aligned} \int x\sqrt{2x+1}\,dx &= \int \frac{u-1}{2} \cdot \sqrt{u} \cdot \frac{du}{2} = \int \frac{1}{4}(u^{3/2} - u^{1/2})\,du \\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C \end{aligned}$$

(e) Substitute $u = \ln(x)$.

$$\boxed{\begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x}\,dx \\ dx &= x\,du \end{aligned}}$$

T-3

$$\int \frac{1}{x\ln(x)\ln(\ln(x))}\,dx = \int \frac{1}{xu\ln(u)} \cdot x\,du = \int \frac{1}{u\ln(u)}\,du$$

Now make a second substitution of $w = \ln(u)$.

$$\boxed{\begin{aligned} w &= \ln(u) \\ dw &= \frac{1}{u}\,du \\ du &= u\,dw \end{aligned}}$$

$$\int \frac{1}{u\ln(u)}\,du = \int \frac{1}{uw} \cdot u\,dw = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|\ln(u)| + C = \ln|\ln(\ln(x))| + C$$

(f) Substitute $u = \sqrt{w} + 7$.

$$\boxed{\begin{aligned} u &= \sqrt{w} + 7 \\ du &= \frac{1}{2\sqrt{w}}\,dw \\ dw &= 2\sqrt{w}\,du \end{aligned}}$$

$$\int \frac{1}{\sqrt{w}(\sqrt{w}+7)}\,dw = \int \frac{1}{\sqrt{w}u} \cdot 2\sqrt{w}\,du = \int \frac{2}{u}\,du = 2\ln|u| + C = 2\ln\left|\sqrt{w}+7\right| + C$$

Ex. T-4 5.5

For each part, calculate the integral.

(a) $\displaystyle\int_0^1 \frac{5x^2}{3x^3+2}\,dx$

(b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \tan(3\theta)\,d\theta$

(c) $\displaystyle\int_0^2 (e^{3x} - e^{-3x})^2\,dx$

(d) $\displaystyle\int_0^{\ln(2)} \frac{1}{1+e^{-t}}\,dt$

(e) $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x}\,dx$

(f) $\displaystyle\int_{-1}^1 \frac{2x}{2x-9}\,dx$

c5s5

Solution T-4

(a) Substitute $u = 3x^3 + 2$

$$\boxed{\begin{aligned} u &= 3x^3 + 2 \\ du &= 9x^2\,dx \\ dx &= \frac{du}{9x^2} \end{aligned}} \quad \boxed{\begin{aligned} x = 0 &\implies u = 2 \\ x = 1 &\implies u = 5 \end{aligned}}$$

$$\int_0^1 \frac{5x^2}{3x^3+2}\,dx = \int_2^5 \frac{5x^2}{u} \cdot \frac{du}{9x^2} = \int_2^5 \frac{5}{9}\frac{1}{u}\,du = \frac{5}{9}\ln(u)\Big|_2^5 = \frac{5}{9}(\ln(5) - \ln(2))$$

(b) Substitute $u = \cos(3\theta)$ and write $\tan(3\theta) = \frac{\sin(3\theta)}{\cos(3\theta)}$.

$$\boxed{\begin{aligned} u &= \cos(3\theta) \\ du &= -3\sin(3\theta)\,d\theta \\ d\theta &= \frac{du}{-3\sin(3\theta)} \end{aligned}} \quad \boxed{\begin{aligned} \theta = \frac{\pi}{4} &\implies u = -\frac{1}{\sqrt{2}} \\ \theta = \frac{\pi}{3} &\implies u = -1 \end{aligned}}$$

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \tan(3\theta)\,d\theta &= \int_{-\frac{1}{\sqrt{2}}}^{-1} \frac{\sin(3\theta)}{u} \cdot \frac{du}{-3\sin(3\theta)} = \int_{-\frac{1}{\sqrt{2}}}^{-1} \left(-\frac{1}{3}\cdot\frac{1}{u}\right)du \\ &= -\frac{1}{3}\ln|u|\Big|_{-\frac{1}{\sqrt{2}}}^{-1} = -\frac{1}{3}\left(\ln(1) - \ln\left(\frac{1}{\sqrt{2}}\right)\right) = -\frac{\ln(2)}{6} \end{aligned}$$

T-4

(c) First expand the integrand.

$$\int_0^2 (e^{3x} - e^{-3x})^2\,dx = \int_0^2 \left(e^{6x} - 2 + e^{-6x}\right)\,dx$$

Observe the following simple antiderivatives.

$$\int e^{6x}\,dx = \frac{e^{6x}}{6} + C \quad , \quad \int e^{-6x}\,dx = -\frac{e^{-6x}}{6} + C$$

(These antiderivatives can be determined by inspection or by substitution of $u = 6x$ or $u = -6x$.) So now we have the following.

$$\int_0^2 \left(e^{6x} - 2 + e^{-6x}\right)\,dx = \left(\frac{e^{6x}}{6} - 2x - \frac{e^{-6x}}{6}\right)\Big|_0^2 = \frac{e^{12}}{6} - 4 - \frac{e^{-12}}{6}$$

(d) First rewrite the integrand using algebra.

$$\frac{1}{1+e^{-t}} = \frac{e^t}{e^t+1}$$

Now substitute $u = e^t + 1$.

$$\boxed{\begin{aligned} u &= e^t + 1 \\ du &= e^t\,dt \\ dt &= \frac{du}{e^t} \end{aligned}} \quad \boxed{\begin{aligned} t = 0 &\Longrightarrow u = 2 \\ t = \ln(2) &\Longrightarrow u = 3 \end{aligned}}$$

$$\int_0^{\ln(2)} \frac{e^t}{e^t+1}\,dt = \int_2^3 \frac{e^t}{u}\cdot\frac{du}{e^t} = \int_2^3 \frac{1}{u}\,du = \ln(u)\Big|_2^3 = \ln(3) - \ln(2)$$

(e) Substitute $u = \ln(x)$.

$$\boxed{\begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x}\,dx \\ dx &= x\,du \end{aligned}} \quad \boxed{\begin{aligned} x = 1 &\Longrightarrow u = 0 \\ x = e^3 &\Longrightarrow u = 3 \end{aligned}}$$

$$\int_1^{e^3} \frac{\ln(x)}{x}\,dx = \int_0^3 \frac{u}{x}\cdot x\,du = \int_0^3 u\,du = \frac{1}{2}u^2\Big|_0^3 = \frac{9}{2}$$

(f) Substitute $u = 2x - 9$.

$$\boxed{\begin{aligned} u &= 2x - 9 \\ u + 9 &= 2x \\ du &= 2\,dx \\ dx &= \frac{du}{2} \end{aligned}} \quad \boxed{\begin{aligned} x = -1 &\Longrightarrow u = -11 \\ x = 1 &\Longrightarrow u = -7 \end{aligned}}$$

$$\int_{-1}^1 \frac{2x}{2x-9}\,dx = \int_{-11}^{-7} \frac{u+9}{u}\cdot\frac{du}{2} = \int_{-11}^{-7} \frac{1}{2}\left(1 + \frac{9}{u}\right)\,du = \frac{1}{2}\left(u + 9\ln|u|\right)\Big|_{-11}^{-7}$$
$$= \frac{1}{2}\left(-7 + 9\ln(7)\right) - \frac{1}{2}\left(-11 + 9\ln(11)\right) = 2 + \frac{9}{2}\ln\left(\frac{7}{11}\right)$$

Ex. T-5 **5.5**

For each part, find the area of the region under the given curve.

(a) $y = t\sqrt{t^2+9}$ on $[0,4]$

(b) $y = x(x-3)^{1/3}$ on $[3,11]$

(c) $y = \sin(2x)^2\cos(2x)$ on $[0, \frac{\pi}{4}]$

(d) $y = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ on $[1,9]$

c5s5

Solution **T-5**

(a) Substitute $u = t^2+9$.

$$\boxed{\begin{aligned} u &= t^2+9 \\ du &= 2t\,dt \\ dt &= \frac{du}{2t} \end{aligned}} \quad \boxed{\begin{aligned} t = 0 &\implies u = 9 \\ t = 4 &\implies u = 25 \end{aligned}}$$

$$A = \int_0^4 t\sqrt{t^2+9}\,dt = \int_9^{25} t\sqrt{u}\frac{du}{2t} = \int_9^{25} \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_9^{25} = \frac{1}{3}(125-27) = \frac{98}{3}$$

(b) Substitute $u = x-3$.

$$\boxed{\begin{aligned} u &= x-3 \\ x&u+3 \\ du &= dx \end{aligned}} \quad \boxed{\begin{aligned} x = 3 &\implies u = 0 \\ x = 11 &\implies u = 8 \end{aligned}}$$

$$A = \int_3^{11} x(x-3)^{1/3}\,dx = \int_0^8 (u+3)u^{1/3}\,du = \int_0^8 (u^{4/3}+3u^{1/3})\,du = \left(\frac{3}{7}u^{7/3}+\frac{9}{4}u^{4/3}\right)\Big|_0^8 = \frac{636}{7}$$

(c) Substitute $u = \sin(2x)$.

$$\boxed{\begin{aligned} u &= \sin(2x) \\ du &= 2\cos(2x)\,dx \\ dx &= \frac{du}{2\cos(2x)} \end{aligned}} \quad \boxed{\begin{aligned} x = 0 &\implies u = 0 \\ x = \frac{\pi}{4} &\implies u = 1 \end{aligned}}$$

$$A = \int_0^{\frac{\pi}{4}} \sin(2x)^2\cos(2x)\,dx = \int_0^1 u^2\cos(2x)\cdot\frac{du}{2\cos(2x)} = \int_0^1 \frac{1}{2}u^2\,du = \frac{1}{6}u^3\Big|_0^1 = \frac{1}{6}$$

(d) Substitute $u = \sqrt{x}$.

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \\ dx &= 2\sqrt{x}\,du \end{aligned}} \quad \boxed{\begin{aligned} x = 1 &\implies u = 1 \\ x = 9 &\implies u = 3 \end{aligned}}$$

$$A = \int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^3 \frac{e^u}{\sqrt{x}}\cdot 2\sqrt{x}\,du = \int_1^3 2e^u\,du = 2e^u\Big|_1^3 = 2e^3-2e$$

Ex. Q-15 **4.9, 5.3, 5.5**

For each part, find the antiderivative or integral.

(a) $\displaystyle\int t^2\cos(1-t^3)\,dt$

(b) $\displaystyle\int \sqrt{x-1}\,dx$

(c) $\displaystyle\int_2^3 \frac{\ln(x)}{x}\,dx$

(d) $\displaystyle\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x}-1}\,dx$

unit5

Solution **Q-15**

(a) Substitute $u = 1 - t^3$ (whence $-\frac{1}{3}du = t^2\,dt$).

$$\int t^2\cos(1-t^3)\,dt = \int \left(-\frac{1}{3}\cos(u)\right)\,du = -\frac{1}{3}\sin(u) + C = -\frac{1}{3}\sin(1-t^3) + C$$

(b) Substitute $u = x - 1$ (whence $du = dx$).

$$\int \sqrt{x-1}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x-1)^{3/2} + C$$

(c) Substitute $u = \ln(x)$ (whence $du = \frac{1}{x}\,dx$).

$$\int_2^3 \frac{\ln(x)}{x}\,dx = \int_{\ln(2)}^{\ln(3)} u\,du = \frac{1}{2}u^2\Big|_{\ln(2)}^{\ln(3)} = \frac{\ln(3)^2 - \ln(2)^2}{2}$$

(d) Substitute $u = e^{2x} - 1$ (whence $\frac{1}{2}du = e^{2x}\,dx$).

$$\int_0^{\ln(3)} e^{2x}\sqrt{e^{2x}-1}\,dx = \int_0^8 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_0^8 = \frac{8^{3/2}}{3}$$

6 Chapter 6: Additional Exercises

True or False?

Ex. U-1 **True/False** Sp19 **Exam**

For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

(a) T F $\ln(3) - \ln(11) = \dfrac{\ln(3)}{\ln(11)}$

(b) T F The domain of $f(x) = \sqrt[9]{x-4}$ is all real numbers.

(c) T F The lines $9x + y = 1$ and $x - 9y = 4$ are perpendicular to each other.

(d) T F The equations $2\ln(x) = 0$ and $\ln(x^2) = 0$ have the same solutions.

(e) T F $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$

exs19

Solution **U-1**

(a) ***False.*** The correct identity is $\ln(3) - \ln(11) = \ln\left(\frac{3}{11}\right)$.

(b) ***True.*** Every real number has an odd root.

(c) ***True.*** The slope of the line $9x + y = 1$ is $m_1 = -9$ and the slope of the line is $x - 9y = 4$ is $m_2 = \frac{1}{9}$. Since $m_1 m_2 = -1$, the lines are perpendicular.

(d) ***False.*** The equation $2\ln(x) = 0$ has one solution: $x = 1$. The equation $\ln(x^2) = 0$ has two solutions: $x = 1$ and $x = -1$. (The identity $\ln(x^b) = b\ln(x)$ is true only if $x > 0$.)

(e) ***True.*** The reference angle for $\frac{5\pi}{6}$ is $\frac{\pi}{6}$, and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. Since the angle $\frac{5\pi}{6}$ lies in the second quadrant, its cosine is negative.

Ex. U-2 **True/False** Sp20 **Exam**

4 Each of the following statements describes a scenario in which a certain rectangle is changing over time. For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

(a) T F If two opposite sides of the rectangle increase in length and if the area remains constant, then the other two opposite sides must decrease in length.

(b) T F If the area of the rectangle increases, then all sides of the rectangle must also increase in length.

(c) T F If the length of the rectangle remains the same, then the area and the width of the rectangle cannot change in opposite ways (i.e., one cannot increase while the other decreases).

(d) T F If two opposite sides of the rectangle increase in length and the other two opposite sides decrease in length, then the area of the rectangle must remain constant.

exs20

Solution **U-2**

This problem can be answered by physical considerations alone. We may also use the equation $A = LW$, from which it follows:

$$\frac{dA}{dt} = \frac{dL}{dt}W + L\frac{dW}{dt}$$

Note that L and W must be positive numbers since they are lengths.

(a) ***True.*** If $\frac{dL}{dt} > 0$ and $\frac{dA}{dt} = 0$, then $\frac{dW}{dt} = -\frac{W}{L}\frac{dL}{dt} < 0$.

(b) ***False.*** If $\frac{dA}{dt} > 0$, it is possible for at least one of $\frac{dL}{dt}$ and $\frac{dW}{dt}$ to be negative. For instance, consider a rectangle with $L = W = 1$, $\frac{dL}{dt} = 2$, and $\frac{dW}{dt} = -1$.

(c) ***True.*** If $\frac{dL}{dt} = 0$, then we must have $\frac{dA}{dt} = L\frac{dW}{dt}$. Since $L > 0$, we see that $\frac{dA}{dt}$ and $\frac{dW}{dt}$ must have the same sign.

(d) ***False.*** If $\frac{dL}{dt} > 0$ and $\frac{dW}{dt} < 0$, it is possible to have $\frac{dA}{dt} \neq 0$. See part (b) for an example.

Ex. U-3 True/False Sp20 Exam

The numbers a, b, and c (which are not necessarily positive) satisfy the formula $a = \frac{b}{c}$. The choices below describe scenarios in which the numbers a, b, and c are changing over time. For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

Hint: There is at most one true statement.

(a) $\boxed{\text{T}}$ $\boxed{\text{F}}$ Suppose a, b, and c are all positive numbers. If a and b are both increasing, then c must also be increasing.

(b) $\boxed{\text{T}}$ $\boxed{\text{F}}$ Suppose b is a positive number and c is a negative number. If b and c are both increasing, then a must be decreasing.

(c) $\boxed{\text{T}}$ $\boxed{\text{F}}$ Suppose a, b, and c are all positive numbers. If a is constant, then it is possible for b and c to change in opposite ways (i.e., one can increase while the other decreases).

(d) $\boxed{\text{T}}$ $\boxed{\text{F}}$ Suppose c is a positive number. If b is constant and c is increasing, then a must be decreasing.

exs20

Solution U-3

Choice (b) is the only true scenario.

To solve this problem, we first use implicit differentiation with respect to time to obtain

$$a' = \frac{cb' - bc'}{c^2}$$

where the primes denote differentiation with respect to t.

(a) ***False.*** Put $b = c = 1$, $a' = 2$, and $b' = 1$. Then we have $2 = 1 - c'$, whence $c' = -1$. So it is possible for c to be decreasing.

(b) ***True.*** We have $b > 0$, $c < 0$, $b' > 0$, and $c' > 0$. A sign analysis of a' gives:

$$a' = \frac{\ominus\oplus - \oplus\oplus}{\oplus} = \frac{\ominus - \oplus}{\oplus}$$

Note that a negative number minus a positive number is a negative number. So the numerator above is negative, whence a' must be negative.

(c) ***False.*** If a is constant, then $a' = 0$, and we must have $cb' = bc'$, or $b'/c' = b/c$. The right side of this equation is positive, whence b'/c' must also be positive. This means that b' and c' must both have the same sign, i.e., b and c cannot change in opposite ways.

(d) ***False.*** If b is constant, then $b' = 0$, and we must have $a' = -\frac{bc'}{c^2}$. Since c and c' are both positive, we may take $c = c' = 1$ and $b = -1$, whence $a' = 1$. So it is possible for a to be increasing.

Ex. U-4 True/False Sp21 Exam

For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

(a) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $\lim\limits_{x\to a} f(x)$ can be evaluated by direct substitution, then f is continuous at $x = a$.

(b) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The value of $\lim\limits_{x\to a} f(x)$, if it exists, is found by calculating $f(a)$.

(c) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If f is not differentiable at $x = a$, then f is also not continuous at $x = a$.

exs21

Solution U-4

(a) ***True.*** This statement is equivalent to $\lim\limits_{x\to a} f(x) = f(a)$ which is the definition of continuity (of $f(x)$ at $x = a$).

(b) ***False.*** The limit $\lim\limits_{x\to a} f(x) = f(a)$ is independent of $f(a)$. (Indeed, the latter need not even exist for the limit to exist.)

(c) ***False.*** The function $f(x) = |x|$ is not differentiable at $x = 0$ but continuous for all x.

Ex. U-5 True/False Su22 Exam

For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

(a) T F If $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$ both exist, then $\lim_{x\to 1} (f(x)g(x))$ exists.

(b) T F If $f(9)$ is undefined, then $\lim_{x\to 9} f(x)$ does not exist.

(c) T F If $\lim_{x\to 1^+} f(x) = 10$ and $\lim_{x\to 1} f(x)$ exists, then $\lim_{x\to 1} f(x) = 10$.

(d) T F A function is continuous for all x if its domain is $(-\infty, \infty)$.

(e) T F If $f(x)$ is continuous at $x = 3$, then $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

(f) T F If $\lim_{x\to 2} f(x)$ exists, then f is continuous at $x = 2$.

(g) T F If $\lim_{x\to 5^-} f(x) = -\infty$, then $\lim_{x\to 5^+} f(x) = +\infty$.

(h) T F A function can have two different horizontal asymptotes.

exsu22

Solution U-5

(a) ***True.*** This follows by the product law for limits.

(b) ***False.*** Let $f(x) = 0$ for all x except $x = 9$, with $f(9)$ undefined. Then $\lim_{x\to 9} f(x) = 0$. (The value $f(a)$ is completely independent of the limit $\lim_{x\to a} f(x)$.)

(c) ***True.*** If a two-sided limit exists, then it must be equal to the corresponding left- and right-limits.

(d) ***False.*** Let $f(x) = 0$ for all x except $x = 2$, with $f(2) = 1$. Then f has domain $(-\infty, \infty)$ but is discontinuous at $x = 2$.

(e) ***True.*** If f is continuous at $x = 3$, then, in particular, $\lim_{x\to 3} f(x)$ exists, which then implies the corresponding left- and right-limits at $x = 3$ are equal.

(f) ***False.*** Let f be the function in part (d). Then $\lim_{x\to 2} f(x) = 0$ but f is not continuous at $x = 2$.

(g) ***False.*** Let $f(x) = -\frac{1}{(x-5)^2}$. Then $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = -\infty$.

(h) ***True.*** Let $f(x) = 0$ for $x \leq 0$ and let $f(x) = 1$ for $x > 0$. Then f has two horizontal asymptotes: $x = 0$ and $x = 1$.

Ex. U-6 True/False Su22 Exam

For each part, mark "T" if the statement is true or mark "F" if the statement is false. You do not have to explain your answers or show any work.

(a) T F If f is continuous at $x = 3$, then f is differentiable at $x = 3$.

(b) T F If f is differentiable at $x = 3$, then f is continuous at $x = 3$.

(c) T F If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$ for all x.

(d) T F The function $f(x) = |x|$ has two tangent lines at $x = 0$: the lines $y = x$ and $y = -x$.

(e) T F If $f(x) = x^{1/3}$, then $f'(0)$ does not exist.

(f) T F If $f(x) = x^{1/3}$, then there is no tangent line to f at $x = 0$.

(g) T F $\frac{d}{dx}(e^{2x}) = 2xe^{2x-1}$

(h) T F A certain cylindrical tank has a radius of 5 ft. If the height of the water in the tank increases at a constant rate, then the volume of the water in the tank also increases at a constant rate.

exsu22

Solution **U-6**

(a) ***False.*** Let $f(x) = |x-3|$. Then f is continuous at $x = 3$ but not differentiable at $x = 3$.

(b) ***True.*** This is the exact statement of Theorem 3.1 on page 146 of the textbook (Briggs et al., *Pearson 2018*).

(c) ***False.*** Let $f(x) = 0$ and let $g(x) = 1$. Then $f'(x) = g'(x)$ but f and g are not the same function.

(d) ***False*** Since $f(x) = |x|$ is not differentiable at $x = 0$, the tangent line to f at $x = 0$ doesn't exist.

(e) ***True.*** By definition of the derivative, we have:

$$f'(0) = \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right) = \lim_{h\to 0}\left(\frac{h^{1/3}}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h^{2/3}}\right) = +\infty$$

Since this limit is not a finite number, $f'(0)$ does not exist.

(Alternatively, we can observe that the graph of $y = f(x)$ has a vertical tangent line at $x = 0$. Thus $f'(0)$ does not exist.)

(f) ***False.*** From the solution for part (e), we see that $f'(0)$ does not exist but the corresponding limit is $+\infty$. Thus there is a vertical tangent line to f at $x = 0$.

(Alternatively, we can observe that the graph of $y = f(x)$ has a vertical tangent line at $x = 0$.)

(g) ***False.*** Chain rule gives $\dfrac{d}{dx}(e^{2x}) = 2e^{2x}$.

(h) ***True.*** Note that $V = \frac{25\pi}{3}h$, where V and h are the volume and height of the water in the tank, respectively. Taking derivatives gives $\frac{dV}{dt} = \frac{25\pi}{3}\frac{dh}{dt}$. Thus if $\frac{dh}{dt}$ is constant, so is $\frac{dV}{dt}$.

Ex. U-7 **True/False** Su22 **Quiz**

If $f(x)$ is not defined at $x = a$, then which of the following must be true?

(a) $\lim_{x\to a} f(x)$ cannot exist

(b) $\lim_{x\to a^+} f(x)$ must be infinite (either $+\infty$ or $-\infty$)

(c) $\lim_{x\to a} f(x)$ could be 0

(d) none of the above

qsu22

Solution **U-7**

Choice (c).

The function value $f(a)$ is independent of $\lim_{x\to a} f(x)$. The function value is irrelevant when computing the limit. For instance, let $f(x) = \frac{x^2}{x}$. Then $f(x)$ is not defined at $x = 0$, but $\lim_{x\to 0} f(x) = 0$. So this limit could be 0 even if $f(a)$ is undefined.

Ex. U-8 **True/False** Su22 **Quiz**

If $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x) = 0$, then which of the following is true about $\lim_{x\to a^-} \dfrac{f(x)}{g(x)}$?

(a) The limit does not exist, and is not infinite.

(b) The limit is infinite (either $+\infty$ or $-\infty$).

(c) The limit must exist.

(d) There is not enough information to say anything about the limit's value.

qsu22

Solution **U-8**

Choice (d).

The limit $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ has the indeterminate form "$\frac{0}{0}$", which gives no information on the value of the limit or even

U-8

whether the limit exists. For instance, consider each of the following limits:

$$\lim_{x\to 0^-}\left(\frac{x}{x}\right) \qquad \lim_{x\to 0^-}\left(\frac{x^2}{x}\right) \qquad \lim_{x\to 0^-}\left(\frac{x}{x^2}\right)$$

Then each of these limits has the indeterminate form "$\frac{0}{0}$". However, these limits are equal to 1, 0, and $-\infty$, respectively.

Ex. U-9 **True/False**

Zero or more of the following statements are true for all real numbers a, x, and y. Determine which statements are true and determine which statements are false. For each false statement, find values of a, x, and y that make the statement false.

(a) $a(x+y) = ax + ay$

(b) $a(x+y)^2 = (ax+ay)^2$

(c) $a(x+y)^2 = ax^2 + ay^2$

(d) $a\sqrt{x+y} = \sqrt{a^2x + a^2y}$

(e) $\sin(x+y) = \sin(x) + \sin(y)$

(f) $\cos(ax) = a\cos(x)$

(g) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

(h) $\dfrac{a}{x+y} = \dfrac{a}{x} + \dfrac{a}{y}$

unit1

Solution U-9

(a) ***True.***

(b) ***False.*** Let $a = 2$, $x = 1$, and $y = 0$. The left side is 2 and the right side is 4.

(c) ***False.*** Let $a = x = y = 1$. The left side is 4 and the right side is 2.

(d) ***False.*** Let $a = -1$, $x = 1$, and $y = 0$. The left side is -1 and the right side is 1.

(e) ***False.*** Let $x = y = \frac{\pi}{2}$. The left side is 0 and the right side is 2.

(f) ***False.*** Let $a = x = 0$. The left side is 1 and the right side is 0.

(g) ***False.*** Let $x = y = 1$. The left side is $\sqrt{2}$ and the right side is 2.

(h) ***False.*** Let $a = x = y = 1$. The left side is $\frac{1}{2}$ and the right side is 2.

Extra Challenges

Ex. A-68 **Algebra/Precalculus** **⋆Challenge**

Let $f(x) = \dfrac{2}{3-\sqrt{x}}$. Fully simplify the difference quotient $\dfrac{f(4+h)-f(4)}{h}$ for $h \neq 0$ (i.e., simplify the expression all common factors of h have been canceled.)

challenge

Solution **A-68**

We calculate the composition, find a common denominator, rationalize the numerator, and then expand the denominator. Our goal is to cancel the factor of h.

$$\begin{aligned}\frac{f(4+h)-f(4)}{h} &= \frac{\frac{2}{3-\sqrt{4+h}}-2}{h} = \frac{2-2(3-\sqrt{4+h})}{h\left(3-\sqrt{4+h}\right)} = \frac{-4+2\sqrt{4+h}}{h\left(3-\sqrt{4+h}\right)} \cdot \frac{-4-2\sqrt{4+h}}{-4-2\sqrt{4+h}} \\ &= \frac{(16-4(4+h))}{h\left(3-\sqrt{4+h}\right)\left(-4-2\sqrt{4+h}\right)} = \frac{-4h}{-2h\left(2-h+\sqrt{4+h}\right)} = \frac{2}{2-h+\sqrt{4+h}}\end{aligned}$$

Ex. D-23 **2.4, 2.5** **⋆Challenge**

For each function, find all horizontal asymptotes and vertical asymptotes. Then, at each vertical asymptote, calculate both one-sided limits.

(a) $f(x) = \dfrac{4x^3+4x^2-8x}{x^3+3x^2-4}$ (b) $f(x) = \dfrac{4x^3-\sqrt{x^6+17}}{5x^3-40}$

challenge

Solution **D-23**

(a) First we factor the denominator. Let $p(x) = x^3+3x^2-4$ and observe that $p(1)=0$, whence $x-1$ is a factor of $p(x)$. Performing long division of polynomials then gives $p(x) = (x-1)(x^2+4x+4) = (x-1)(x+2)^2$. So for $x \neq 1$ and $x \neq -2$, we have:

$$f(x) = \frac{4x^3+4x^2-8x}{x^3+3x^2-4} = \frac{4x(x+2)(x-1)}{(x-1)(x+2)^2} = \frac{4x}{x+2}$$

Hence the only vertical asymptote of $f(x)$ is the line $x=-2$.

Precisely, we have that $\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{4x}{x+2}\right) = \frac{4}{3}$. Since this limit is finite, $x=1$ is not a vertical asymptote of $f(x)$.

For the one-sided limits we have:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^-} = +\infty$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left(\frac{4x}{x+2}\right) = \frac{-8}{0^+} = -\infty$$

As for the horizontal asymptotes, we have the following:

$$\lim_{x\to\pm\infty}\left(\frac{4x}{x+2}\right) = \lim_{x\to\pm\infty}\left(\frac{4}{1+\frac{2}{x}}\right) = \frac{4}{1+0} = 4$$

Hence the only horizontal asymptote of $f(x)$ is the line $y=4$.

(b) Observe that the only solution to $5x^3-40=0$ is $x=2$, whence the only candidate vertical asymptote of $f(x)$ is $x=2$. Direct substitution of $x=2$ into $f(x)$ gives "$\frac{23}{0}$", which indicates the one-sided limits at $x=2$ are both infinite, and so $x=2$ is, indeed, a true vertical asymptote.

For the one-sided limits we have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^-} = -\infty$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+}\left(\frac{4x^3-\sqrt{x^6+17}}{5x^3-40}\right) = \frac{23}{0^+} = +\infty$$

D-23

As for the horizontal asymptotes, we first perform some algebra to rewrite $f(x)$.

$$\frac{4x^3-\sqrt{x^6+17}}{5x^3-40} = \frac{4x^3-\sqrt{x^6}\sqrt{1+\frac{17}{x^6}}}{5x^3-40} = \frac{4-\frac{|x|^3}{x^3}\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}$$

For $x>0$, we note that $\frac{|x|^3}{x^3} = \frac{x^3}{x^3} = 1$. For $x<0$, we note that $\frac{|x|^3}{x^3} = \frac{-x^3}{x^3} = -1$. So now we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4+\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4+\sqrt{1+0}}{5-0} = 1$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty}\left(\frac{4-\sqrt{1+\frac{17}{x^6}}}{5-\frac{40}{x^3}}\right) = \frac{4-\sqrt{1+0}}{5-0} = \frac{3}{5}$$

Thus the two horizontal asymptotes of $f(x)$ are $y=1$ and $y=\frac{3}{5}$.

Ex. F-42 2.6 ⋆Challenge

Consider $f(x) = \dfrac{\tan(2x)}{|5x|}$.

(a) Where is f not continuous?

(b) Is it possible to redefine f at $x=0$ to make f continuous there? Explain your answer.

Hint: For the limit of f as $x\to 0$, examine the one-sided limits first.

challenge

Solution F-42

(a) The numerator $\tan(2x)$ is continuous precisely on its domain, hence not continuous wherever $\cos(2x)=0$, that is, wherever $2x$ is an odd multiple of $\frac{\pi}{2}$. The denominator $|5x|$ vanishes when $x=0$, and so $f(x)$ is also not continuous when $x=0$. Hence $f(x)$ is not continuous at the following x-values:

$$x = \ldots, \frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

(b) We must compute the limit $\lim\limits_{x\to 0} f(x)$. Observe that $|x|=-x$ for $x<0$ and $|x|=x$ for $x>0$. So we have:

$$\lim_{x\to 0^-}\left(\frac{\tan(2x)}{|5x|}\right) = \lim_{x\to 0^-}\left(\frac{\sin(2x)}{2x}\cdot\frac{2x}{-5x}\right) = 1\cdot\frac{-2}{5} = -\frac{2}{5}$$
$$\lim_{x\to 0^+}\left(\frac{\tan(2x)}{|5x|}\right) = \lim_{x\to 0^+}\left(\frac{\sin(2x)}{2x}\cdot\frac{2x}{5x}\right) = 1\cdot\frac{2}{5} = \frac{2}{5}$$

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist, and so there is no value which we can give to $f(0)$ to ensure continuity of $f(x)$ at $x=0$.

Ex. H-38 3.3/3.4/3.5/3.9 ⋆Challenge

Find all points on the graph of $y = \dfrac{2}{x} + 3x$ such that the tangent line there passes through $(6,17)$.

challenge

Solution H-38

Let (a,b) be the unknown point of tangency and consider the tangent line to the given curve at (a,b). Since (a,b) lies on the curve, we have $b=\frac{2}{a}+3a$. The derivative at a general point is:

$$\frac{dy}{dx} = -\frac{2}{x^2}+3$$

H-38

So the slope of the tangent line is $-\frac{2}{a^2}+3$, and the desired tangent line has the following form:

$$y = \frac{2}{a} + 3a + \left(-\frac{2}{a^2} + 3\right)(x-a)$$

The point $(6, 17)$ must lie on this tangent line, so substitution of $x = 6$ and $y = 17$ must give an equation that a satisfies.

$$17 = \frac{2}{a} + 3a + \left(-\frac{2}{a^2} + 3\right)(6-a)$$

Clearing all denominators and rearranging gives the equation:

$$a^2 + 4a - 12 = 0 \implies (a+6)(a-2) = 0 \implies a = -6 \text{ or } a = 2$$

Hence the tangent line to the graph passes through $(6, 17)$ if the point of tangency is $(-6, -\frac{55}{3})$ or $(2, 7)$.

Ex. J-35 3.8 ⋆Challenge

Find all tangent lines to the graph of $9x^2 - 18xy + y^2 = 1800$ that are perpendicular to the line $x + 3y = 10$.

challenge

Solution J-35

First we use implicit differentiation to find an equation for $\frac{dy}{dx}$.

$$18x - 18y - 18x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

The given line has slope $-\frac{1}{3}$, and so the desired tangent line as slope 3. Let (a, b) be the unknown point of tangency. Then $\frac{dy}{dx} = 3$ at that point, whence we have:

$$18a - 18b - 54a + 6b = 0 \implies b = -3a$$

The point (a, b) also lies on the curve, and so (a, b) satisfies the original equation for the curve. Substituting $b = -3a$ gives:

$$9a^2 - 18a(-3a) + (-3a)^2 = 1800 \implies 72a^2 = 1800 \implies a = -5 \text{ or } a = 5$$

Thus there are two such tangent lines: one at $(5, -15)$ (equation of the line is $y = -15 + 3(x-5)$) and another at $(-5, 15)$ (equation of the line is $y = 15 + 3(x+5)$).

Ex. K-29 3.11 ⋆Challenge !!!

A water tank in the shape of an inverted cone has height 10 meters and base radius 8 meters. Water flows into the tank at the rate of 32π m^3/min. At what rate is the depth of the water in the tank changing when the water is 5 meters deep?

challenge

Ex. J-36 3.8, 4.6 ⋆Challenge

Consider the curve described by the equation

$$\frac{x - y^3}{y + x^2} = x - 12$$

(a) Find an equation for the line tangent to this curve at $(-1, 4)$.

(b) There is a point on the curve with coordinates $(-1.1, b)$. Use linear approximation to estimate b. Round to three decimal places.

(c) There is a point on the curve with coordinates $(a, 4.2)$. Use linear approximation to estimate a. Round to three decimal places.

challenge

Solution J-36

(a) We write the equation as follows to make differentiation easier:

$$x - y^3 = xy + x^3 - 12y - 12x^2$$

J-36

Differentiating each side with respect to x gives:

$$1 - 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx} + 3x^2 - 12\frac{dy}{dx} - 24x$$

We now substitute $x = -1$ and $y = 4$:

$$1 - 48\frac{dy}{dx} = 4 - \frac{dy}{dx} + 3 - 12\frac{dy}{dx} + 24 \Longrightarrow \frac{dy}{dx} = -\frac{6}{7}$$

So an equation of the tangent line is:

$$y - 4 = -\frac{6}{7}(x+1)$$

(b) Since $(-1.1, b)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate b. That is, the point $(-1.1, b)$ approximately satisfies the equation of the tangent line:

$$b - 4 \approx -\frac{6}{7}(-1.1+1) \Longrightarrow b \approx \frac{28}{6.6} \approx 4.242$$

(c) Since $(a, 4.2)$ is near $(-1, 4)$, we can use the tangent line from part (a) to approximate a. That is, the point $(a, 4.2)$ approximately satisfies the equation of the tangent line:

$$4.2 - 4 \approx -\frac{6}{7}(a+1) \Longrightarrow a \approx -\frac{7.4}{6.6} \approx -1.233$$

Ex. O-32 **4.6** **⋆Challenge**

The acceleration (measured in m/s^2) of a particle moving along the x-axis is given by

$$a(t) = 14t^{3/4} - 6t^2 + 1$$

and the particle is at rest (zero velocity) when $t = 1$. Use a linear approximation to estimate the particle's change in position between $t = 16$ and $t = 16.02$.

challenge

Solution **O-32**

We seek an estimate of the change in position: $\Delta x = x(16.02) - x(16)$. We use the tangent line to $x(t)$ at $t = 16$. Note that the slope of this tangent line is given by $v(16)$, so we first find the velocity function by antidifferentiating the acceleration $a(t)$.

$$v(t) = \int (14t^{3/4} - 6t^2 + 1)\,dt = 8t^{7/4} - 2t^3 + t + C$$

The particle is at rest when $t = 1$, or $v(1) = 0$. So $8 - 2 + 1 + C = 0$, whence $C = -7$ and our velocity function is

$$v(t) = 8t^{7/4} - 2t^3 + t - 7$$

Now we return to finding the tangent line to $x(t)$ at $t = 16$.

Point of Tangency:	$(16, x(16))$	
Slope of Line:	$x'(t) = v(t) = 8t^{7/4} - 2t^3 + t - 7;$	$x'(16) = v(16) = -7159$
Equation of Line:	$y - x(16) = -7159(t-16)$	

This means that $x(t) - x(16) \approx -7159(t-16)$ if t is near 16. Hence we have the estimate:

$$\Delta x = x(16.02) - x(16) \approx -7159(16.02 - 16) = -143.18$$

The particle's position decreases by approximately 143.18.

Ex. M-38 **4.3/4.4** **⋆Challenge**

Consider the function $f(x) = ax^6e^{-bx}$, where a and b are unspecified constants. Suppose f has a point of local maximum at $(2, 64e^{-2})$. Find the values of a and b.

challenge

Solution **M-38**

The derivative of $f(x)$ is given by:

$$f'(x) = 6ax^5e^{-bx} + ax^6 \cdot e^{-bx} \cdot (-b) = ax^5(6 - bx)e^{-bx}$$

We are given two conditions: (1) $f(2) = 64e^{-2}$ and (2) $f'(2) = 0$ (since $x = 2$ gives a local maximum). So we have the following simultaneous set of equations for a and b:

$$64ae^{-2b} = 64e^{-2}$$
$$32a(6 - 2b)e^{-2b} = 0$$

The second equation implies $a = 0$ or $b = 3$. However, the solution $a = 0$ does not satisfy the first equation, so we must have $b = 3$. So the first equation now gives $64ae^{-6} = 64e^{-2}$, whence $a = e^4$.

Ex. P-29 4.7 ⋆Challenge

Suppose f'' is continuous for all x. Calculate $\lim_{h\to 0}\left(\dfrac{f(x+5h)+f(x-5h)-2f(x)}{h^2}\right)$.

challenge

Solution **P-29**

Since f'' is continuous, so are f and f'. This means all of $f(x)$, $f'(x)$, and $f''(x)$ have the direct substitution property of limits for all inputs. Substitution of $h = 0$ into the given limit gives "$\frac{0}{0}$", and so we use l'Hospital's Rule (LR). Note that we must differentiate the numerator and denominator with respect to h, not with respect to x. We treat x as a constant.

$$\lim_{h\to 0}\left(\frac{f(x+5h)+f(x-5h)-2f(x)}{h^2}\right) \overset{H}{=} \lim_{h\to 0}\left(\frac{5f'(x+5h)-5f'(x-5h)}{2h}\right)$$

Substitution of $h = 0$ again gives "$\frac{0}{0}$", and so we use LR again.

$$\lim_{h\to 0}\left(\frac{5f'(x+5h)-5f'(x-5h)}{2h}\right) \overset{H}{=} \lim_{h\to 0}\left(\frac{25f''(x+5h)+25f''(x-5h)}{2}\right) = \frac{25f''(x)+25f''(x)}{2} = 25f''(x)$$

Ex. P-30 4.7 ⋆Challenge

Suppose f' is continuous for all x and $f(0) = 0$. Calculate $\lim_{x\to 0^+}(1+f(2x))^{4/x}$.

challenge

Solution **P-30**

Since f' is continuous, so is f. This means both $f(x)$ and $f'(x)$ have the direct substitution property for all inputs. Substitution of $x = 0$ into the given limit gives "$1^{\pm\infty}$". So we let L denote the given limit and consider $\ln(L)$ instead.

$$\ln(L) = \lim_{x\to 0^+}\left(\ln\left((1+f(2x))^{4/x}\right)\right) = \lim_{x\to 0^+}\left(\frac{4\ln(1+f(2x))}{x}\right) \overset{H}{=} \lim_{x\to 0^+}\left(\frac{4\cdot\frac{1}{1+f(2x)}\cdot f'(2x)\cdot 2}{1}\right) = 8f'(0)$$

So $\ln(L) = 8f'(0)$, whence $L = e^{8f'(0)}$.

Ex. M-39 4.3/4.4 ⋆Challenge !!!

Consider the function $f(x) = (x-3a)(x+2a)^4$, where a is an unspecified **positive** constant. Answer all of the following in terms of a.

(a) where is f decreasing?
(b) where is f increasing?
(c) where does f have a local minimum?
(d) where does f have a local maximum?
(e) where is f concave down?
(f) where is f concave up?
(g) where does f have an inflection point?

Finally, sketch a graph of $y = f(x)$. Your horizontal scale should be in terms of a and your vertical scale should be in terms of a^5.

challenge

Ex. M-40 **4.3/4.4** ⋆Challenge !!!

Let $f(x) = \dfrac{e^x}{4+x^3}$. Answer all of the following.

(a) what are the vertical asymptotes of f?

(b) what are the horizontal asymptotes of f?

(c) where is f decreasing?

(d) where is f increasing?

(e) where does f have a local minimum?

(f) where does f have a local maximum?

challenge

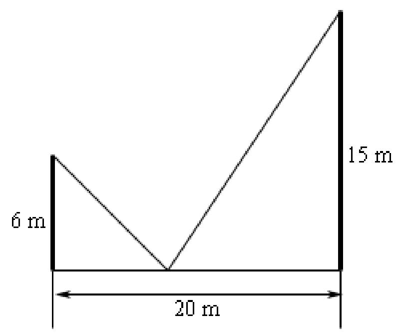
Ex. N-35 **4.5** ⋆Challenge !!!

Find the equation of the line through $(2, 4)$ that cuts off the least area from the first quadrant. (Observe that this cut off region is a triangle.)

challenge

Ex. N-36 **4.5** ⋆Challenge !!!

Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?



challenge

Ex. B-14 **2.1/2.2** ⋆Challenge

Suppose $\lim_{x\to 0}(f(x)+g(x))$ exists. Is it true that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ also exist? Explain your answer.

challenge

Solution **B-14**

No. Let $f(x)$ be *any* function such that $\lim_{x\to 0} f(x)$ does not exist. (For example, $f(x) = \frac{|x|}{x}$.) Let $g(x) = -f(x)$. Then

$$\lim_{x\to 0}(f(x)+g(x)) = \lim_{x\to 0}(f(x)-f(x)) = \lim_{x\to 0}(0) = 0$$

Hence $\lim_{x\to 0}(f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Ex. E-17 **2.5** ⋆Challenge

Find all horizontal asymptotes of $f(x) = \dfrac{2x}{x-\sqrt{x^2+10}}$.

challenge

Solution **E-17**

As $x \to \pm\infty$, we see that $f(x)$ has an "$\frac{\infty}{\infty}$"-form (or equivalent variant). So first we factor out dominant terms from numerator and denominator to write f in an algebraically equivalent way for $x \neq 0$. Recall that $\sqrt{x^2} = |x|$.

$$f(x) = \frac{2x}{x-\sqrt{x^2+10}} = \frac{2x}{x-\sqrt{x^2\left(1+\frac{10}{x^2}\right)}} = \frac{2x}{x-|x|\sqrt{1+\frac{10}{x^2}}} = \frac{2}{1-\frac{|x|}{x}\sqrt{1+\frac{10}{x^2}}}$$

Now we calculate the horizontal asymptotes. For the limit $x \to -\infty$, we may assume x is negative, whence $|x| = -x$

E-17

and $\frac{|x|}{x} = -1$. So we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{2}{1+\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{1+\sqrt{1+0}} = \frac{2}{1+1} = 1$$

So the line $y = 1$ is a horizontal asymptote.

Now for the limit $x \to +\infty$, we may assume x is positive, whence $|x| = x$, and $\frac{|x|}{x} = 1$. So we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{2}{1-\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{1-\sqrt{1+0}} = \frac{2}{1-1} = \frac{2}{0}$$

This is an undefined expression, but recall that a limit of the form "$\frac{c}{0}$" (with $c \neq 0$) indicates that the limit is infinite. So there is no other horizontal asymptote.

Bonus: What is the value of this last limit? The above limit must be either $+\infty$ or $-\infty$. Observe that $1 + \frac{10}{x^2} > 1$ for all $x \neq 0$, which implies that $\sqrt{1+\frac{10}{x^2}} > 1$ for all such x, and so

$$1 - \sqrt{1+\frac{10}{x^2}} < 0$$

Thus as $x \to +\infty$, we see that $1 - \sqrt{1+\frac{10}{x^2}}$ approaches 0 but remains negative. Hence we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{2}{1-\sqrt{1+\frac{10}{x^2}}}\right) = \frac{2}{0^-} = -\infty$$

Ex. F-43 **2.6** ⋆**Challenge**

Find the values of the constants a and b that make f continuous at $x = 0$. *You may assume $a > 0$.*

$$f(x) = \begin{cases} \dfrac{1-\cos(ax)}{x^2} & , \quad x < 0 \\ 2a + b & , \quad x = 0 \\ \dfrac{x^2 - bx}{\sin(x)} & , \quad x > 0 \end{cases}$$

challenge

Solution **F-43**

We need only force continuity at $x = 0$.

$$\begin{aligned}
\lim_{x\to0^-} f(x) &= \lim_{x\to0^-}\left(\frac{1-\cos(ax)}{x^2}\right) = \lim_{x\to0^-}\left(\frac{1-\cos(ax)}{x^2}\cdot\frac{1+\cos(ax)}{1+\cos(ax)}\right) \\
&= \lim_{x\to0^-}\left(\frac{1-\cos(ax)^2}{x^2(1+\cos(ax))}\right) = \lim_{x\to0^-}\left(\frac{\sin(ax)^2}{x^2(1+\cos(ax))}\right) \\
&= \lim_{x\to0^-}\left(\left(\frac{\sin(ax)}{x}\right)^2\cdot\frac{1}{1+\cos(ax)}\right) \\
&= \lim_{x\to0^-}\left(\left(a\cdot\frac{\sin(ax)}{ax}\right)^2\cdot\frac{1}{1+\cos(ax)}\right) = (a\cdot1)^2\cdot\frac{1}{1+1} = \frac{a^2}{2} \\
\lim_{x\to0^+} f(x) &= \lim_{x\to0^+}\left(\frac{x^2-bx}{\sin(x)}\right) = \lim_{x\to0^+}\left(\frac{x}{\sin(x)}\cdot(x-b)\right) = 1\cdot(0-b) = -b \\
f(0) &= 2a + b
\end{aligned}$$

F-43

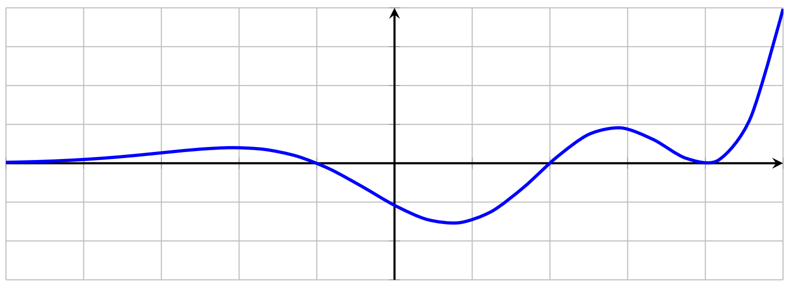
If f is to be continuous at $x = 0$, these three values must be equal. Hence we obtain the following system of equations:

$$\frac{a^2}{2} = 2a + b$$
$$-b = 2a + b$$

The second equation is equivalent to $a = -b$, and substituting into the first equation gives $\frac{a^2}{2} = a$. Dividing by a (which we are told is positive!) gives $\frac{a}{2} = 1$, or $a = 2$. Hence we must have $a = 2$ and $b = -2$.

Ex. G-36 **3.1/3.2** **⋆Challenge**

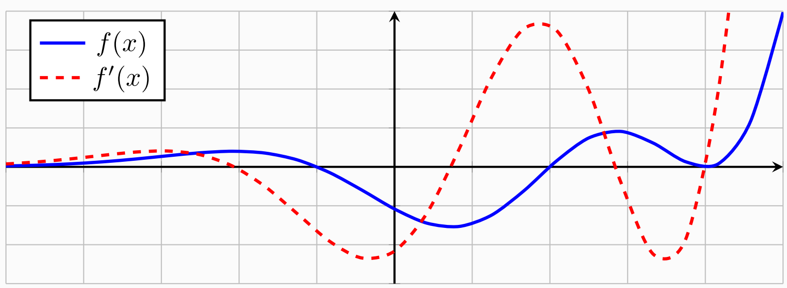
The graph of $y = f(x)$ is given below. Sketch a graph of $y = f'(x)$. *Only the general shape is important. The graph does not have to be to scale.*



challenge

Solution **G-36**

First identify the points where $f'(x) = 0$ (local minimum or local maximum of $f(x)$). These points cut the number line into several subintervals. Identify the sign (negative or positive) of $f'(x)$ on each subinterval, then smooth out the curve on each subinterval.



Ex. G-37 **3.1/3.2** **⋆Challenge**

Consider the following function, where c is an unspecified constant

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 + 2x & \text{if } 0 \le x < 1 \\ 6x - x^2 + c & \text{if } x \ge 1 \end{cases}$$

(a) Show precisely that $f'(0)$ does not exist.

(b) Find the value of c that makes f differentiable at $x = 1$ or show that no such value exists.

challenge

Solution **G-37**

Note: A commonly proposed but invalid solution is to compute $f'(x)$ for each separate piece and then check whether the one-sided limits of $f'(x)$ are equal at $x = 0$ and $x = 1$. That would check whether $\lim_{x\to 0} f'(x)$ or $\lim_{x\to 1} f'(x)$ exists, not whether $f'(0)$ or $f'(1)$ exists.

(a) Observe that $f(0) = 0$. Then, by definition, we have the following.

$$f'(0) = \lim_{x\to 0}\left(\frac{f(x) - f(0)}{x}\right) = \lim_{x\to 0}\left(\frac{f(x)}{x}\right)$$

G-37

Since $f(x)$ is piecewise defined and changes definition at $x = 0$, we must compute the left- and right-limits.

$$\lim_{x\to 0^-}\left(\frac{f(x)}{x}\right) = \lim_{x\to 0^-}\left(\frac{-x^2}{x}\right) = \lim_{x\to 0^-}(-x) = 0$$
$$\lim_{x\to 0^+}\left(\frac{f(x)}{x}\right) = \lim_{x\to 0^+}\left(\frac{x^2+2x}{x}\right) = \lim_{x\to 0^-}(x+2) = 2$$

The one-sided limits are not equal, whence $f'(0)$ does not exist.

(b) Recall that continuity is a necessary (but not sufficient) condition for differentiability. That is, if f is to be differentiable at $x = 1$, then f must also be continuous at $x = 1$. So first we determine the value of c that makes f continuous at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+2x) = 3$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(6x - x^2 + c) = 5 + c$$
$$f(1) = \left(6x - x^2 + c\right)\Big|_{x=1} = 5 + c$$

So we must have that $3 = 5 + c$, or $c = -2$, and our function is:

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 + 2x & \text{if } 0 \le x < 1 \\ 6x - x^2 - 2 & \text{if } x \ge 1 \end{cases}$$

Now we check whether f differentiable at $x = 1$. Observe that $f(1) = 3$. So, by definition, we have:

$$f'(1) = \lim_{x\to 1}\left(\frac{f(x)-f(1)}{x-1}\right) = \lim_{x\to 1}\left(\frac{f(x)-3}{x-1}\right)$$

Since $f(x)$ is piecewise defined and changes definition at $x = 1$, we must compute the left- and right-limits.

$$\lim_{x\to 1^-}\left(\frac{f(x)-3}{x-1}\right) = \lim_{x\to 1^-}\left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x\to 1^-}\left(\frac{(x+3)(x-1)}{x-1}\right) = \lim_{x\to 1^-}(x+3) = 4$$
$$\lim_{x\to 1^+}\left(\frac{f(x)-3}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{6x-x^2-5}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{(5-x)(x-1)}{x-1}\right) = \lim_{x\to 1^+}(5-x) = 4$$

The one-sided limits are equal, whence $f'(1) = 4$. So $c = -2$ does, indeed, make f differentiable at $x = 1$.

Ex. H-39 **3.3/3.4/3.5/3.9** ⋆**Challenge**

Find all points P on the graph of $y = 4x^2$ with the property that the tangent line at P passes through the point $(2, 0)$.

challenge

Solution H-39

Let $f(x) = 4x^2$. Denote the unknown point P by (a, b). We require two equations to solve for the two unknowns a and b. These equations are derived from the two following conditions.

(i) The point P lies on the curve $y = f(x)$.

(ii) The point $(2, 0)$ lies on the line tangent to f at P.

Condition (i) simply gives us $f(a) = b$, or $4a^2 = b$. For condition (ii), we first find a general equation for the line tangent to f at P. This tangent line has point of tangency $P = (a, 4a^2)$ and slope $f'(a) = 8a$. Hence an equation of the tangent line is

$$y = 4a^2 + 8a(x-a) \Longrightarrow y = 8ax - 4a^2$$

Condition (ii) states that the point $(2, 0)$ lies on this line, whence we have $0 = 16a - 4a^2$. This equation has solutions $a = 0$ and $a = 4$. Thus there are two such points P: $(0, 0)$ and $(4, 64)$.

Ex. J-37 **3.8** ⋆**Challenge**

Suppose $x^2 + y^2 = R^2$, where R is a constant. Find $\frac{d^2y}{dx^2}$ and fully simplify your answer as much as possible.

challenge

Solution **J-37**

Differentiating both sides with respect to x, then solving for $\frac{dy}{dx}$ gives:

$$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Differentiating $\frac{dy}{dx}$ with respect to x gives:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substituting and simplifying (using $x^2 + y^2 = R^2$) gives our final answer.

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{R^2}{y^3}$$

Ex. M-41 **4.3/4.4** ⋆**Challenge**

Let $f(x) = \sqrt[3]{x^3 - 48x}$.

(i) Find all vertical asymptotes and horizontal asymptotes of $f(x)$.

(ii) Find where $f(x)$ is decreasing and where $f(x)$ is increasing. Also find and classify all local extrema of $f(x)$.

(iii) Find where $f(x)$ is concave down and where $f(x)$ is concave up. Also find all inflection points of $f(x)$.

(iv) Sketch a graph of $y = f(x)$.

challenge

Solution **M-41**

The first two derivatives of $f(x)$ are

$$\boxed{f'(x) = \frac{x^2 - 16}{(x^3 - 48x)^{2/3}}} \qquad \boxed{f''(x) = \frac{-32(x^2 + 16)}{(x^3 - 48x)^{5/3}}}$$

(i) **Vertical asymptotes and horizontal asymptotes.**

Since f is continuous for all real numbers, there are no vertical asymptotes. As for the horizontal asymptotes, we have

$$\lim_{x\to\pm\infty}(x^3 - 48x)^{1/3} = \lim_{x\to\pm\infty}\left(x \cdot \left(1 - \frac{48}{x^2}\right)^{1/3}\right) = \pm\infty \cdot (1-0)^{1/3} = \pm\infty$$

Hence there are no horizontal asymptotes.

(ii) **Intervals of increase and local extrema.**

We calculate a sign chart for $f'(x)$. The cut points are the solutions to $f'(x) = 0$ ($x = -4$ and $x = 4$) and where $f'(x)$ DNE ($x = 0$, $x = -\sqrt{48}$, and $x = \sqrt{48}$).

interval	test point	sign of f'	shape of f
$(-\infty, -\sqrt{48})$	$f'(-7)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(-\sqrt{48}, -4)$	$f'(-5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(-4, 0)$	$f'(-3)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(0, 4)$	$f'(3)$	$\frac{\ominus}{\oplus} = \ominus$	decreasing
$(4, \sqrt{48})$	$f'(5)$	$\frac{\oplus}{\oplus} = \oplus$	increasing
$(\sqrt{48}, \infty)$	$f'(7)$	$\frac{\oplus}{\oplus} = \oplus$	increasing

M-41

Hence we deduce the following about f:

f is decreasing on:	$[-4, 4]$
f is increasing on:	$(-\infty, -4]$, $[4, \infty)$
f has a local min at:	$x = 4$
f has a local max at:	$x = -4$

(iii) Intervals of concavity and inflection points.

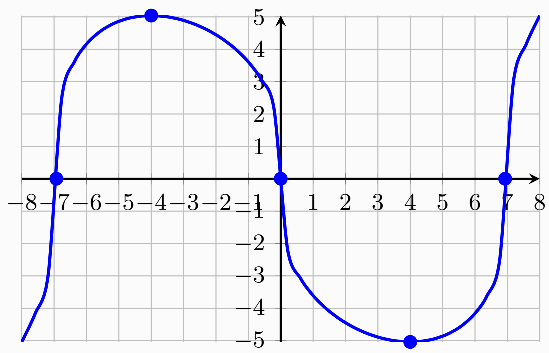
We calculate a sign chart for $f''(x)$. The cut points are the solutions to $f''(x) = 0$ (none) and where $f''(x)$ DNE ($x = 0$, $x = -\sqrt{48}$, and $x = \sqrt{48}$).

interval	test point	sign of f''	shape of f
$(-\infty, -\sqrt{48})$	$f'(-7)$	$\frac{-32\oplus}{\oplus} = \ominus$	concave up
$(-\sqrt{48}, 0)$	$f'(-1)$	$\frac{-32\oplus}{\ominus} = \oplus$	concave down
$(0, \sqrt{48})$	$f'(1)$	$\frac{-32\oplus}{\oplus} = \ominus$	concave up
$(\sqrt{48}, \infty)$	$f'(7)$	$\frac{-32\oplus}{\ominus} = \oplus$	concave down

Hence we deduce the following about f:

f is concave down on:	$[-\sqrt{48}, 0]$, $[\sqrt{48}, \infty)$
f is concave up on:	$(-\infty, -\sqrt{48}]$, $[0, \sqrt{48}]$
f has an infl. point at:	$x = -\sqrt{48}$, $x = 0$, and $x = \sqrt{48}$

(iv) Sketch of graph.



Precise examination of cusps and vertical tangents is beyond the scope of this course. For the sake of completeness, note the following:

$$\lim_{x\to 0^-} f'(x) = -\infty \quad , \quad \lim_{x\to 0^+} f'(x) = -\infty$$

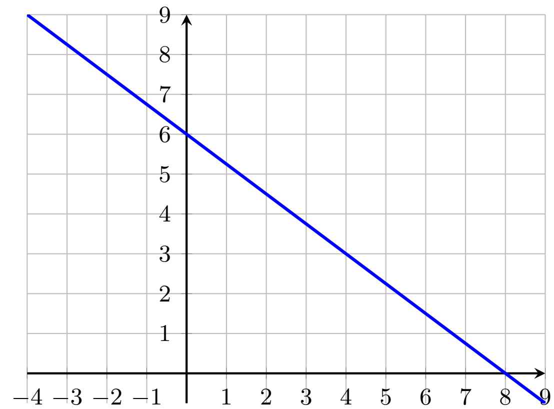
Since the derivative has an infinite limit and it is the same sign of infinity for both one-sided limits, there is a vertical tangent at $x = 0$. Similarly, there is a vertical tangent at both $x = -\sqrt{48}$ and $x = \sqrt{48}$ also.

7 Chapter 7: Sample Exams (Set A)

Sample Precalculus Exam A

A-25. For each part, use the graph of $y = g(x)$ given below and let $f(x) = 8x^2 - 4x + 15$.

(a) Find an expression for $g(x)$.
(b) Calculate the y-intercept of the graph of $y = f(g(x))$.
(c) Calculate $g(f(x))$.



Solution **A-25**

(a) We observe from the figure that the graph of $y = g(x)$ is a line that passes through the points $(0, 6)$ and $(8, 0)$. Hence an equation for this line in point-slope form is

$$g(x) = 0 + \frac{0-6}{8-0}(x-8) = -\frac{3}{4}(x-8)$$

(b) The desired y-intercept is the point $(0, f(g(0)))$. Note that since the y-intercept of g is $(0, 6)$, we have $g(0) = 6$. Hence $f(g(0)) = f(6) = 8 \cdot 6^2 - 4 \cdot 6 + 15 = 264$.

(c) We have

$$g(f(x)) = -\frac{3}{4}(f(x) - 8) = -\frac{3}{4}\left(8x^2 - 4x + 7\right)$$

A-26. A 100-gram sample of a radioactive substance decays to 65% of its initial mass in 15 hours. Recall that the mass of the sample M at time t satisfies $M(t) = M_0 e^{kt}$ for some constants M_0 and k.

(a) Find the growth constant k.
(b) Find the mass of the sample after 22 hours.
(c) Find the time in hours when the sample will have a mass of 41 grams.

Solution **A-26**

(a) We are given that $M(15) = 0.65M(0)$, which is equivalent to $M_0 e^{15k} = 0.65M_0$. Canceling the constant M_0, taking logarithms, and solving for k gives

$$k = \frac{\ln(0.65)}{15}$$

(b) We are given $M_0 = 100$, and so the mass at $t = 22$ is

$$M(22) = M_0 e^{22k} = 100e^{\ln(0.65)/15} = 100 \cdot (0.65)^{15}$$

(c) We must solve the equation $M(t) = 41$, or $100e^{kt} = 41$. Dividing by 100, taking logarithms, and solving for t gives

$$t = \frac{\ln(0.41)}{k} = 15 \cdot \frac{\ln(0.41)}{\ln(0.65)}$$

A-27. A rectangular box is constructed according to the following rules.

- The length of the box is 5 times its width.
- The volume of the box is 110 cubic feet.

Let L, W, and H be the length, width, and height of the box (measured in feet), respectively.

(a) Write an equation in terms of L, W, and H that expresses the first constraint.
(b) Write an equation in terms of L, W, and H that expresses the second constraint.
(c) Write an expression for $S(W)$, the total surface area of the box as a function of W.
(d) Suppose the rules also require that the sum of the box's length and width be less than 78 feet. What is the domain of $S(W)$ in this context?

Solution **A-27**

(a) $L = 5W$
(b) $LWH = 110$
(c) The total surface area in terms of L, and W, and H is

$$S = 2(LW + LH + WH)$$

Putting the first constraint into the second gives $5W^2H = 110$, which then gives $H = \frac{22}{W^2}$. Now substituting our expressions for L and H in terms of W into our expression for S gives

$$S(W) = 2\left(5W \cdot W + 5W \cdot \frac{22}{W^2} + W \cdot \frac{22}{W^2}\right) = 10W^2 + \frac{264}{W}$$

(d) The new rule implies the constraint $L+W < 78$, or $6W < 78$ (given $L = 5W$). Hence $W < 13$. Of course, since W represents a distance, we must also have $W \geq 0$. Hence the domain of $S(W)$ in this context is $0 \leq W < 13$, or the interval $[0, 13)$.

A-28. Suppose $\log_{16}(x) = A$ and $\log_{16}(y) = B$. Rewrite the expression below in terms of A and B. Your final answer may not contain any logarithm symbol.

$$\log_{16}\left(\frac{4x^7}{\sqrt[9]{y}}\right)$$

Solution **A-28**

Using various logarithm rules and the identity $4 = 16^{1/2}$ gives the following.

$$\begin{aligned}
\log_{16}\left(\frac{4x^7}{\sqrt[9]{y}}\right) &= \log_{16}(4x^7) - \log_{16}(\sqrt[9]{y}) \\
&= \log_{16}(4) + \log_{16}(x^7) - \log_{16}(y^{1/9}) \\
&= \log_{16}(16^{1/2}) + 7\log_{16}(x) - \frac{1}{9}\log_{16}(y) \\
&= \frac{1}{2} + 7A - \frac{1}{9}B
\end{aligned}$$

A-29. Let $f(x) = \sqrt{3x}$ and assume $h \neq 0$. Fully simplify each of the following expressions:

(a) $f(x+h)$ (b) $f(x+h) - f(x)$ (c) $\dfrac{f(x+h) - f(x)}{h}$

Solution **A-29**

(a) $f(x+h) = \sqrt{3(x+h)}$
(b) $f(x+h) - f(x) = \sqrt{3(x+h)} - \sqrt{3x}$
(c) Rationalize the numerator, then simplify.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} = \frac{3(x+h) - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)} = \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}}$$

A-30. Consider the function $f(x) = \dfrac{x-6}{x^2 - 9x + 20}$.

(a) Solve the equation $f(x) = 0$.
(b) List all numbers that are not in the domain of $f(x)$.
(c) Solve the inequality $f(x) > 0$ and write your answer using interval notation.

Solution **A-30**

(a) The equation $f(x) = 0$ is equivalent to $x - 6 = 0$, and so the only solution is $x = 6$.

(b) Since $f(x)$ is rational, its domain is the set of all real numbers except where the denominator vanishes. The equation $x^2 - 9x + 20 = 0$ is equivalent to $(x-4)(x-5) = 0$, whence the only numbers not in the domain of $f(x)$ are $x = 4$ and $x = 5$.

(c) We construct a sign chart whose cut points are those x-values where $f(x) = 0$ or where $f(x)$ is undefined. Hence the cut points are $x = 4$, $x = 5$, and $x = 6$. We then examine the sign of $f(x) = \frac{x-6}{(x-4)(x-5)}$ on each of the corresponding sub-intervals.

interval	test point	sign of $f(x)$	truth of inequality
$(-\infty, 4)$	$x = 0$	$\frac{\ominus}{\ominus\ominus} = \ominus$	false
$(4, 5)$	$x = 4.5$	$\frac{\ominus}{\oplus\ominus} = \oplus$	true
$(5, 6)$	$x = 5.5$	$\frac{\ominus}{\oplus\oplus} = \ominus$	false
$(6, \infty)$	$x = 7$	$\frac{\oplus}{\oplus\oplus} = \oplus$	true

None of the cut points satisfy the inequality. Hence the solution to the inequality $f(x) > 0$ is $(4, 5) \cup (6, \infty)$.

A-31. Find all solutions to the following equation in the interval $[0, 2\pi)$.

$$2\sin(\theta)\cos(\theta) - \cos(\theta) = 0$$

Solution **A-31**

Factoring gives $\cos(\theta)(2\sin(\theta) - 1) = 0$, whence solutions to the equation are solutions to $\cos(\theta) = 0$ or $\sin(\theta) = \frac{1}{2}$.

Recall that on the unit circle, a point (x, y) corresponds to the point $(\cos(\theta), \sin(\theta))$. Hence solving the equation $\cos(\theta) = 0$ is equivalent to solving $x = 0$ on the unit circle; we get the two solutions $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Solving the equation $\sin(\theta) = \frac{1}{2}$ is equivalent to solving $y = \frac{1}{2}$ on the unit circle; we get the two solutions $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hence the original equation has 4 solutions in the given interval: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

A-32. Complete each of the following algebra exercises.

(a) Fully factor the polynomial $5x^4 + 25x^3 - 180x^2$.

(b) Solve the rational equation below.

$$\frac{4}{x+5} + \frac{9x}{x^2 - 25} = \frac{6}{x-5}$$

(c) Simplify the complex fraction below by writing it as a simple fraction.

$$\frac{\frac{4}{x} - \frac{2}{xy}}{8 + \frac{7}{y}}$$

Solution **A-32**

(a) $5x^4 + 25x^3 - 180x^2 = 5x^2(x^2 + 5x - 36) = 5x^2(x+9)(x-4)$

(b) Observe that $x^2 - 25 = (x-5)(x+5)$, hence $x^2 - 25$ serves as a common denominator for all terms. Multiplying each side of the equation by $x^2 - 25$ and canceling common factors gives

$$4(x-5) + 9x = 6(x+5)$$

Expanding each side and collecting like terms gives $7x - 50 = 0$, whence the only solution is $x = \frac{50}{7}$.

(c) Observe that the common denominator of the terms $\frac{4}{x}$, $\frac{2}{xy}$, 8, and $\frac{7}{y}$ is xy. We multiply the complex fraction

A-32

by $\frac{xy}{xy}$ and distribute.

$$\frac{\dfrac{4}{x}-\dfrac{2}{xy}}{8+\dfrac{7}{y}}\cdot\frac{xy}{xy}=\frac{4y-2}{8xy+7x}$$

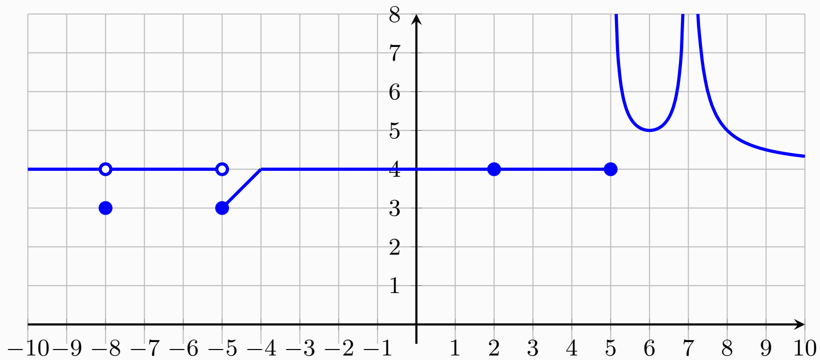
Sample Midterm Exam #1A

F-25. On the axes provided, sketch the graph of a function $f(x)$ that satisfies all of the following properties. ***Note:*** Make sure to read these properties carefully!

- the domain of $f(x)$ is $[-10, 7) \cup (7, 10]$
- $\lim_{x\to -8} f(x)$ exists but f is discontinuous at $x = -8$
- $\lim_{x\to -5^+} f(x) = f(-5)$ but $\lim_{x\to -5} f(x)$ does not exist
- $\lim_{x\to 2^-} f(x) = 4$ and f is continuous at $x = 2$
- the line $x = 5$ is a vertical asymptote for f (***Note:*** $x = 5$ is in the domain of f.)
- $\lim_{x\to 7} f(x) = +\infty$ (***Note:*** $x = 7$ is not in the domain of f.)

Solution **F-25**

There are many such solutions. Here is one.



C-25. For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\lim_{x\to 8} \left(\frac{(x-2)^2 - 36}{x-8} \right)$ (b) $\lim_{x\to 5} \left(\frac{40-8x}{\sqrt{19-3x}-2} \right)$ (c) $\lim_{x\to 2^-} \left(\frac{4+x}{x^2+x-6} \right)$

Solution **C-25**

(a) Expand the numerator. Then cancel common factors.

$$\lim_{x\to 8} \left(\frac{(x-2)^2 - 36}{x-8} \right) = \lim_{x\to 8} \left(\frac{x^2-4x-32}{x-8} \right) = \lim_{x\to 8} \left(\frac{(x-8)(x+4)}{x-8} \right) = \lim_{x\to 8} (x+4) = 12$$

(b) Rationalize the denominator. Then cancel common factors.

$$\begin{aligned} \lim_{x\to 5} \left(\frac{40-8x}{\sqrt{19-3x}-2} \right) &= \lim_{x\to 5} \left(\frac{40-8x}{\sqrt{19-3x}-2} \cdot \frac{\sqrt{19-3x}+2}{\sqrt{19-3x}+2} \right) \\ &= \lim_{x\to 5} \left(\frac{8(5-x)\left(\sqrt{19-3x}+2\right)}{19-3x-4} \right) = \lim_{x\to 5} \left(\frac{8(5-x)\left(\sqrt{19-3x}+2\right)}{3(5-x)} \right) \\ &= \lim_{x\to 5} \left(\frac{8}{3}\left(\sqrt{19-3x}+2\right) \right) = \frac{8}{3}(\sqrt{4}+2) = \frac{32}{3} \end{aligned}$$

(c) Direct substitution of $x = 2$ gives the undefined expression "$\frac{6}{0}$" (i.e., a nonzero number divided by 0). Hence the one-sided limit is infinite. Observe that the denominator is $x^2 + x - 6 = (x+3)(x-2)$. As $x \to 2^-$, the factor $(x+3)$ is positive and the factor $(x-2)$ is negative. Thus the entire fraction has the following sign as $x \to 2^-$: $\frac{6}{\oplus\ominus} = \ominus$. Thus the limit is equal to $-\infty$.

F-26. Consider the function below, where a and b are unspecified constants.

$$f(x) = \begin{cases} \dfrac{\sin(4x)\sin(6x)}{x^2} & x < 0 \\ ax + b & 0 \le x \le 1 \\ \dfrac{5x+2}{x-1} - \dfrac{2x+5}{x^2-x} & x > 1 \end{cases}$$

(a) Calculate $\displaystyle\lim_{x\to 0^-} f(x)$.

(b) Calculate $\displaystyle\lim_{x\to 1^+} f(x)$.

(c) Find the values of a and b for which f is continuous for all x, or determine that no such values exist. *In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.*

Solution **F-26**

(a) Rearrange the terms and use the special trigonometric limit $\displaystyle\lim_{\theta\to 0}\left(\frac{\sin(a\theta)}{a\theta}\right) = 1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin(4x)\sin(6x)}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{\sin(4x)}{4x}\cdot\frac{\sin(6x)}{6x}\cdot 4\cdot 6\right) = 1\cdot 1\cdot 4\cdot 6 = 24$$

(b) Find a common denominator. Then cancel common factors.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\frac{5x+2}{x-1} - \frac{2x+5}{x^2-x}\right) = \lim_{x\to 1^+}\left(\frac{5x^2+2x}{x^2-x} - \frac{2x+5}{x^2-x}\right)$$
$$= \lim_{x\to 1^+}\left(\frac{5x^2-5}{x^2-x}\right) = \lim_{x\to 1^+}\left(\frac{5(x-1)(x+1)}{x(x-1)}\right) = \lim_{x\to 1^+}\left(\frac{5(x+1)}{x}\right) = \frac{5(1+1)}{1} = 10$$

(c) If f is to be continuous at $x = 0$, the left-limit, right-limit, and function value of f at $x = 0$ must be equal.

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= 24 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(ax+b) = b \\ f(0) &= (ax+b)\big|_{x=0} = b \end{aligned}$$

Thus we must have $b = 24$. If f is to be continuous at $x = 1$, the left-limit, right-limit, and function value of f at $x = 1$ must be equal.

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(ax+b) = a+b \\ \lim_{x\to 1^+} f(x) &= 10 \\ f(0) &= (ax+b)\big|_{x=1} = a+b \end{aligned}$$

Thus we must have $a + b = 10$. Given $b = 24$, we find that $a = -14$.

D-17. Find all vertical asymptotes of the function $f(x) = \dfrac{x^3 - 36x}{x^3 - 12x^2 + 36x}$.

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

Solution **D-17**

Since $f(x)$ is a rational function, VA's can occur only where the denominator of $f(x)$ vanishes.

$$x^3 - 12x^2 + 36x = 0 \Longrightarrow x(x^2 - 12x + 36) = x(x-6)^2 = 0$$

Thus $f(x)$ can have a VA at $x = 0$ or $x = 6$ only.

D-17

For $x = 0$, we note the following:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^3-36x}{x^3-12x^2+36x}\right) = \lim_{x\to 0}\left(\frac{x(x-6)(x+6)}{x(x-6)^2}\right) = \lim_{xto0}\left(\frac{x+6}{x-6}\right) = \frac{0+6}{0-6} = -1$$

Since this limit is finite, we find that the line $x = 0$ is not a VA for $f(x)$.

For $x = 6$, we note the following:

$$\lim_{x\to 6} f(x) = \lim_{x\to 6}\left(\frac{x+6}{x-6}\right)$$

At this point, direct substitution of $x = 6$ gives the expression "$\frac{12}{0}$" (i.e., a nonzero number divided by 0). This immediately implies that each corresponding one-sided limit is infinite. Thus the line $x = 6$ is a VA for $f(x)$.

E-11. Find all horizontal asymptotes of the function $h(x) = \dfrac{6x+5}{\sqrt{4x^2-9}}$.

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

Solution E-11

We must compute the limits at infinity. First we complete some algebraic manipulations by first factoring out the largest powers of x in numerator and denominator of $h(x)$, separately. Note that $\sqrt{x^2} = |x|$.

$$\frac{6x+5}{\sqrt{4x^2+9}} = \frac{x\left(6+\frac{5}{x}\right)}{\sqrt{x^2\left(4+\frac{9}{x^2}\right)}} = \frac{x}{\sqrt{x^2}}\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}} = \frac{x}{|x|}\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}$$

Now we compute the necessary limits. Note that as $x\to\infty$, we have $|x| = x$, and so $x/|x| = x/x = 1$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{x}{|x|}\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = \lim_{x\to+\infty}\left(1\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = 1\cdot\frac{6+0}{\sqrt{4+0}} = \frac{6}{2} = 3$$

Now note that as $x\to-\infty$, we have $|x| = -x$, and so $x/|x| = x/(-x) = -1$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{x}{|x|}\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = \lim_{x\to-\infty}\left(-1\cdot\frac{6+\frac{5}{x}}{\sqrt{4+\frac{9}{x^2}}}\right) = -1\cdot\frac{6+0}{\sqrt{4+0}} = \frac{6}{2} = -3$$

Thus the HA's of $h(x)$ are the lines $y = 3$ and $y = -3$.

Sample Midterm Exam #2A

I-9. For each part, calculate the indicated derivative. Do not simplify your answer.

(a) $\dfrac{d}{dx}\left(7x^{10} + \sqrt[3]{x} - \dfrac{8}{x^{20}} + \sec(8x)\right)$ (b) $\dfrac{d}{dx}\left(\dfrac{\ln\left(x^3+30\right)}{8x}\right)$ (c) $\dfrac{d}{dx}\left(\sin\left(xe^{-5x}\right)\right)$

Solution **I-9**

(a) Use power rule on the first three terms and chain rule on the last term.

$$\frac{d}{dx}\left(7x^{10} + x^{1/3} - 8x^{-20} + \sec(8x)\right) = 70x^9 + \frac{1}{3}x^{-2/3} + 160x^{-21} + 8\sec(8x)\tan(8x)$$

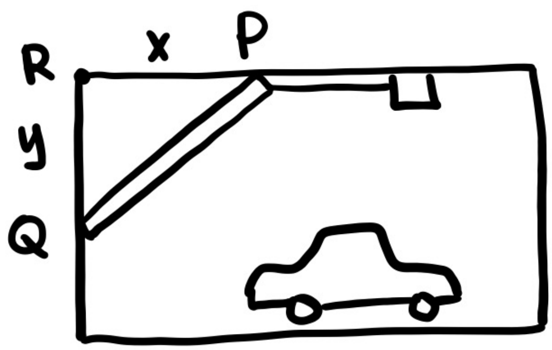
(b) Use quotient rule, then chain rule.

$$\frac{d}{dx}\left(\frac{\ln(x^3+30)}{8x}\right) = \frac{\frac{3x^2}{x^3+30}\cdot 8x - 8\ln(x^3+30)}{(8x)^2}$$

(c) Use chain rule, then product rule and chain rule.

$$\frac{d}{dx}\left(\sin\left(xe^{-5x}\right)\right) = \cos\left(xe^{-5x}\right)\cdot\left(e^{-5x} - 5xe^{-5x}\right)$$

K-17. A solid 14-foot tall garage door opens via a pulley mechanism. As the pulley opens the garage door, the top of the garage door (point P in the figure) moves to the right at 5 ft/s. At the same time, the bottom of the garage door (point Q in the figure) moves straight up.
As shown in the figure, the point R is the fixed point at the top of the garage door frame, x represents the distance between P and R, and y represents the distance between Q and R.



(a) What is the sign of $\dfrac{dx}{dt}$?

(b) What is the sign of $\dfrac{dy}{dt}$?

(c) What is the rate of change of the distance between the points Q and R when the distance between them is 9 feet? You must include correct units in your answer. You may leave unsimplified radicals in your answer.

Solution **K-17**

(a) Since x is increasing, $\frac{dx}{dt}$ is positive.

(b) Since y is decreasing, $\frac{dy}{dt}$ is negative.

(c) We have $x^2 + y^2 = 14^2$, and differentiating with respect to time gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. At the described time we have $y = 6$ and $\frac{dx}{dt} = 5$. So substituting these values gives:

$$x^2 + 6^2 = 14^2 \qquad 10x + 12\frac{dy}{dt} = 0$$

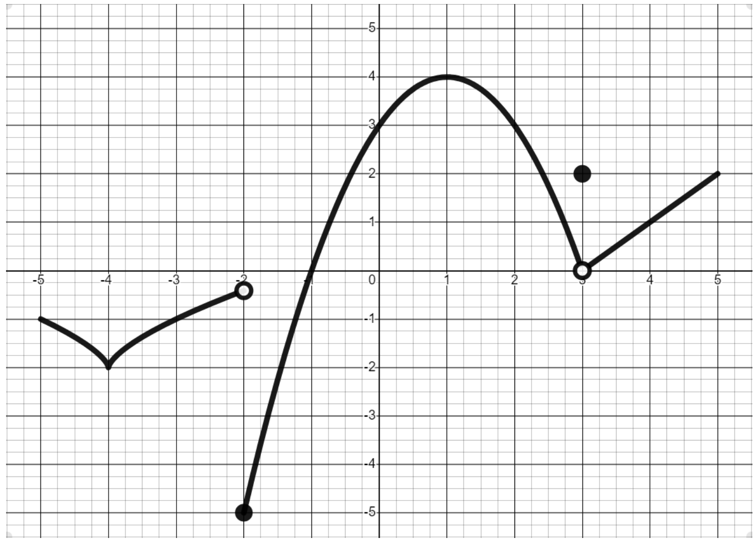
The first equation gives $x = \sqrt{14^2 - 6^2} = \sqrt{(14-6)(14+6)} = \sqrt{8\cdot 20} = \sqrt{160}$, whence we obtain

$$\frac{dy}{dt} = -\frac{10x}{12} = -\frac{5\sqrt{160}}{6}$$

The units of $\frac{dy}{dt}$ are ft/sec.

G-22. For each part, use the graph of $y = f(x)$ to determine whether the value exists. If the value exists, state its sign (negative, positive, or zero).

(a) $f'(-3)$
(b) $f'(-2)$
(c) $f'(-1)$
(d) $f'(1)$
(e) $f'(3)$



Solution **G-22**

(a) does not exist
(b) does not exist
(c) zero
(d) positive
(e) negative

J-20. Consider the following curve.

$$\cos(5x + y - 5) = 8xe^y + y - 7$$

(a) Calculate $\dfrac{dy}{dx}$ for a general point on the curve.
(b) Find an equation of the line tangent to the curve at the point $(1, 0)$.

Solution **J-20**

(a) Differentiate both sides of the equation with respect to x, using chain rule on the left side and product rule on the right side.

$$-\sin(5x + y - 5) \cdot \left(5 + \frac{dy}{dx}\right) = 8e^y + 8xe^y \frac{dy}{dx} + \frac{dy}{dx}$$

Now algebraically solve for $\frac{dy}{dx}$. First expand the left side, then collect terms multiplying $\frac{dy}{dx}$ on one side.

$$-5\sin(5x + y - 5) - \sin(5x + y - 5)\frac{dy}{dx} = 8e^y + 8xe^y\frac{dy}{dx} + \frac{dy}{dx}$$

$$(-\sin(5x + y - 5) - 8xe^y - 1)\frac{dy}{dx} = 5\sin(5x + y - 5) + 8e^y$$

$$\frac{dy}{dx} = \frac{5\sin(5x + y - 5) + 8e^y}{-\sin(5x + y - 5) - 8xe^y - 1}$$

(b) We substitute $x = 1$ and $y = 0$ into our formula for $\frac{dy}{dx}$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \frac{5\sin(0) + 8e^0}{-\sin(0) - 8e^0 - 1} = -\frac{8}{9}$$

This is the slope of the desired tangent line. Hence the desired tangent line is

$$y = -\frac{8}{9}(x - 1)$$

I-10. Find the coordinates of all points on the graph of $f(x) = x\sqrt{14 - x^2}$ where the tangent line is horizontal. You must give both the x- and y-coordinate of each such point.

Solution **I-10**

We first find $f'(x)$ using product rule, then chain rule.

$$f'(x) = 1 \cdot (14-x^2)^{1/2} + x \cdot \frac{1}{2}(14-x^2)^{-1/2} \cdot (-2x) = \sqrt{14-x^2} - \frac{x^2}{\sqrt{14-x^2}}$$

The tangent line to the graph of $f(x)$ is horizontal at points where $f'(x) = 0$. To solve $f'(x) = 0$, multiply both sides by $\sqrt{14-x^2}$, then solve for x.

$$\sqrt{14-x^2} \cdot \left(\sqrt{14-x^2} - \frac{x^2}{\sqrt{14-x^2}}\right) = 0$$
$$14 - x^2 - x^2 = 0 \Longrightarrow 14 - 2x^2 = 0 \Longrightarrow x^2 = 7 \Longrightarrow x = -\sqrt{7} \text{ or } x = \sqrt{7}$$

Hence the graph has horizontal tangent lines at $x = -\sqrt{7}$ and $x = \sqrt{7}$.

G-23. Let $f(x) = \dfrac{8x}{x+5}$.

(a) Calculate $f'(x)$ by any method.

(b) Use the limit definition of derivative to calculate $f'(3)$. ***Hint:*** Use your answer from part (a) to check your final answer.

Solution **G-23**

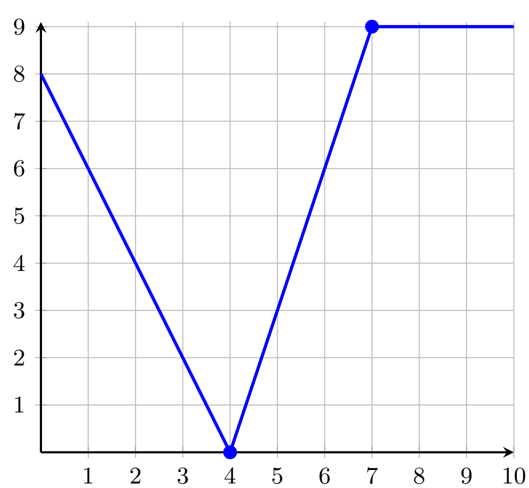
(a) Use quotient rule.

$$f'(x) = \frac{8(x+5) - 8x \cdot 1}{(x+5)^2} = \frac{40}{(x+5)^2}$$

(b) Observe that $f(3) = 3$, whence by the limit definition of derivative we have:

$$f'(3) = \lim_{x\to 3}\left(\frac{f(x)-f(3)}{x-3}\right) = \lim_{x\to 3}\left(\frac{\frac{8x}{x+5} - 3}{x-3}\right) = \lim_{x\to 3}\left(\frac{8x - 3(x+5)}{(x-3)(x+5)}\right)$$
$$= \lim_{x\to 3}\left(\frac{5(x-3)}{(x-3)(x+5)}\right) = \lim_{x\to 3}\left(\frac{5}{x+5}\right) = \frac{5}{3+5} = \frac{5}{8}$$

I-11. The graph of $y = f(x)$ is given below.



(a) Calculate $f'(6)$. Briefly explain how you found your answer.

(b) Let $g(x) = 9xf(2x)$. Find an equation of the line tangent to the graph of $y = g(x)$ at $x = 3$.

Solution **I-11**

(a) The value $f'(6)$ is the slope of the tangent line to $y = f(x)$ at $x = 6$. The graph of $y = f(x)$ is a line with slope 3 on the interval $[4, 7]$. Thus $f'(6) = 3$.

I-11

(b) We find $g'(x)$ with product rule and chain rule.

$$g'(x) = 9f(2x) + 9xf'(2x) \cdot 2 = 9f(2x) + 18xf'(2x)$$

Now observe the following:

$$g(3) = 9 \cdot 3 \cdot f(6) = 9 \cdot 3 \cdot 6 = 162$$
$$g'(3) = 9 \cdot f(6) + 18 \cdot 3 \cdot f'(6) = 9 \cdot 6 + 18 \cdot 3 \cdot 3 = 216$$

Thus the desired tangent line is $y = 162 + 216(x - 3)$.

Sample Midterm Exam #3A

P-17. For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to\pi}\left(\frac{\cos(6x)-1}{(x-\pi)^2}\right)$ (b) $\displaystyle\lim_{x\to 0}\left(e^{2x}+3x\right)^{1/x}$

Solution **P-17**

(a) Direct substitution of $x=\pi$ gives "$\frac{0}{0}$". So we use l'Hospital's Rule (twice).

$$\lim_{x\to\pi}\left(\frac{\cos(6x)-1}{(x-\pi)^2}\right)\overset{H}{=}\lim_{x\to\pi}\left(\frac{-6\sin(6x)}{2(x-\pi)}\right)\overset{H}{=}\lim_{x\to\pi}\left(\frac{-36\cos(6x)}{2}\right)=\frac{-36\cdot 1}{2}=-18$$

(b) Direction substitution of $x=0$ gives "1^∞". We let L be the desired limit, take logarithms, and use l'Hospital's Rule.

$$\ln(L)=\lim_{x\to 0}\ln\left((e^{2x}+3x)^{1/x}\right)=\lim_{x\to 0}\left(\frac{\ln(e^{2x}+3x)}{x}\right)\overset{H}{=}\lim_{x\to 0}\left(\frac{\frac{1}{e^{2x}+3x}\cdot(2e^{2x}+3)}{1}\right)=\frac{2+3}{1+0}=5$$

We find that $\ln(L)=5$, whence $L=e^5$.

M-24. Let $f(x)=4x^5-20x^4+7x+32$. Find where f is concave down and where f is concave up; write your answer using interval notation. Also find where inflection points of f occur.

Solution **M-24**

We first compute the second derivative of f.

$$f'(x)=20x^4-80x^3+7$$
$$f''(x)=80x^3-240x^2=80x^2(x-3)$$

We now calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x)=0$ ($x=0$ and $x=3$).

interval	test point	sign of f''	shape of f
$(-\infty,0)$	$f''(-1)$	$\oplus\ominus=\ominus$	concave down
$(0,3)$	$f''(1)$	$\oplus\ominus=\ominus$	concave down
$(3,\infty)$	$f''(4)$	$\oplus\oplus=\oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty,3]$
f is concave up on:	$[3,\infty)$
f has an infl. point at:	$x=3$

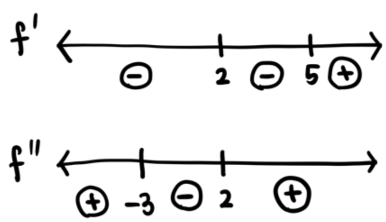
M-25. Suppose $f(x)$ satisfies all of the following properties. Sign charts for f' and f'' are also given below. Sketch a possible graph of $y=f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

(i) f is continuous and differentiable on $(-\infty,2)\cup(2,\infty)$

(ii) $\displaystyle\lim_{x\to-\infty}f(x)=\infty$; $\displaystyle\lim_{x\to\infty}f(x)=\infty$; $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$; $\displaystyle\lim_{x\to 2^+}f(x)=\infty$

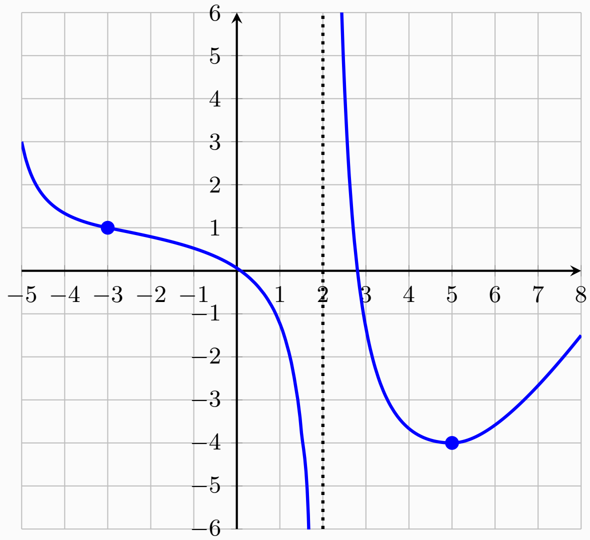
(iii) the only x-value for which $f'(x)=0$ is $x=5$

(iv) the only x-value for which $f''(x)=0$ is $x=-3$



Solution **M-25**

There are many possibilities. Here is one.



M-26. Let $f(x) = \dfrac{x^2+21}{x-2}$. Find where f is decreasing and where f is increasing; write your answer using interval notation. Also find where the local extrema of f occur.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

Solution **M-26**

We first compute the first derivative of f.

$$f'(x) = \frac{2x(x-2) - (x^2+21)\cdot 1}{(x-2)^2} = \frac{x^2-4x-21}{(x-2)^2} = \frac{(x+3)(x-7)}{(x-2)^2}$$

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = -3$ and $x = 7$) and the vertical asymptotes ($x = 2$).

interval	test point	sign of f'	shape of f
$(-\infty, -3)$	$f'(-4)$	$\frac{\ominus\ominus}{\oplus} = \oplus$	increasing
$(-3, 2)$	$f'(0)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(2, 7)$	$f'(3)$	$\frac{\oplus\ominus}{\oplus} = \ominus$	decreasing
$(7, \infty)$	$f'(8)$	$\frac{\oplus\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about f:

f is decreasing on: $[-3, 2)$, $(2, 7]$
f is increasing on: $(-\infty, -3]$, $[7, \infty)$
f has a local min at: $x = 7$
f has a local max at: $x = -3$

L-20. Find the absolute extreme values of $f(x) = x\,(x-8)^{5/3}$ on the interval $[0, 9]$ and the x-values at which they occur.

Solution **L-20**

We first compute $f'(x)$.

$$f'(x) = 1 \cdot (x-8)^{5/3} + x \cdot \frac{5}{3}(x-8)^{2/3} = \frac{8}{3}(x-8)^{2/3}(x-3)$$

Hence the critical points are $x = 3$ and $x = 8$ only (solutions to $f'(x) = 0$). Checking the critical values and the

L-20

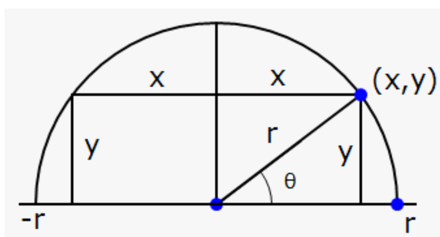
endpoint values gives the following.

x	$f(x)$	reason for check
0	0	endpoint
3	$-3 \cdot 5^{5/3}$	critical point ($f'(x) = 0$)
8	0	critical point ($f'(x) = 0$)
9	9	endpoint

The maximum value of f on $[0, 9]$ is 9 at $x = 9$ and the minimum value is $-3 \cdot 5^{5/3}$ at $x = 3$.

N-17. A rectangle (with base $2x$ and height y) is constructed with its base on the diameter of a semicircle with radius 5 and with its two other vertices on the semicircle. Find the dimensions of the rectangle with the maximum possible area. *As you work, fill in the answer boxes below. You must use calculus-based methods in your work. You must also justify that your answer really does give the maximum.*

constraint equation in terms of x and y:	
objective function in terms of x only:	
interval of interest:	
dimensions of rectangle:	______ $\times$ ______ $2x$ (base) y (height)



Solution N-17

Let x be the half-length of the rectangle and let y be the height. Our objective function is $A(x, y) = 2xy$. See the figure. By Pythagorean theorem, $x^2 + y^2 = r^2$ (with $r = 5$), whence our constraint $y = \sqrt{25 - x^2}$. So the objective in one variable is $f(x) = 2x\sqrt{25 - x^2}$. Our interval of interest is $[0, 5]$ (allowed values of x).

The critical points of f are solutions to $f'(x) = 0$.

$$f'(x) = 2x \cdot \frac{-2x}{2\sqrt{25 - x^2}} + 2\sqrt{25 - x^2} = \frac{50 - 4x^2}{\sqrt{25 - x^2}} = 0 \Longrightarrow x = \frac{5}{\sqrt{2}}$$

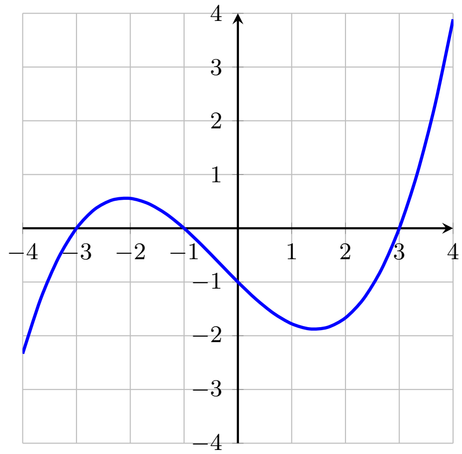
(We have rejected the solution $x = -\frac{5}{\sqrt{2}}$ since $x \geq 0$.) Since the interval of interest is closed and bounded, we need only check the critical values and endpoint values.

x	$f(x)$	reason for check
0	0	endpoint
$\frac{5}{\sqrt{2}}$	25	critical point ($f'(x) = 0$)
5	0	endpoint

Hence the maximum of $f(x)$ occurs at $x = \frac{5}{\sqrt{2}}$. The dimensions of the rectangle are $2x = 5\sqrt{2}$ (length) and $y = \frac{5}{\sqrt{2}}$ (height).

Sample Final Exam A

B-8. For each part, use the graph of $y = g(x)$.



(a) How many solutions does the equation $g'(x) = 0$ have?

(b) Order the following quantities from least to greatest: $g'(-2.5)$, $g'(-2)$, $g'(0)$, and $g'(4)$. In your answer, write these quantities symbolically; do not give a numerical estimate.

(c) What is the sign of $g''(-3)$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(d) Let $h(x) = g(x)^2$. What is the sign of $h'(-4)$(negative, positive, or zero)? If there is not enough information to determine the value, explain why.

Solution **B-8**

(a) The function g is differentiable for all x and has two local extrema (one local min and one local max). So $g'(x) = 0$ has two solutions.

(b) We note the following: $g'(-2.5)$ is small and positive, $g'(-2) = 0$, $g'(0)$ is small and negative, and $g'(4)$ is large and positive. Thus the correct order is: $g'(0)$, $g'(-2)$, $g'(-2.5)$, $g'(4)$.

(c) The function g is concave down in an interval containing $x = -3$. Thus $g''(-3)$ is positive.

(d) We have $h'(x) = 2g(x)g'(x)$, whence $h'(-4) = 2g(-4)g'(-4)$. Observe that $g(-4) < 0$ and $g'(-4) > 0$. Thus $h'(-4) < 0$.

F-19. Let $f(x)$ be the following function, where k is an unspecified constant. Find the value of k that makes f continuous at $x = 2$ or determine that no such value of k exists.

$$f(x) = \begin{cases} 27x - kx^2 & x < 2 \\ -6 & x = 2 \\ 3x^3 + k & x > 2 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

Solution **F-19**

We first compute the left-limit, right-limit, and function value at $x = 2$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (27x - kx^2) = 54 - 4k$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x^3 + k) = 24 + kf(2) = -6$$

If f is to be continuous at $x = 2$, these quantities must all be equal. Hence we must have $54 - 4k = -6$ and $24 + k = -6$. However, this is impossible since the first equation gives $k = 15$ and the second equation gives $k = -30$. There is no value of k that satisfies both equations simultaneously. Hence there is no value of k for which f is continuous at $x = 2$.

1. Consider the curve described by the following equation: $2x^2 - 2xy + 3y^2 = 60$.

(a) Find $\dfrac{dy}{dx}$ for a general point on the curve.

(b) Find the x-coordinate of each point on the curve where the tangent line is horizontal.

Solution **1**

(a) We use implicit differentiation with respect to x.

$$4x - 2y - 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

Solving algebraically for $\frac{dy}{dx}$ then gives:

$$\frac{dy}{dx} = \frac{2y - 4x}{6y - 2x}$$

(b) The tangent line is horizontal at points where $\frac{dy}{dx} = 0$, or where $2y - 4x = 0$, or where $y = 2x$. Such points must also lie on the curve, whence such points must satisfy both the equation $y = 2x$ and the equation $2x^2 - 2xy + 3y^2 = 60$.

Substituting the former into the latter gives $2x^2 - 4x^2 + 12x^2 = 60$, or $10x^2 = 60$, or $x = \pm\sqrt{6}$. Hence the two points where the tangent line is horizontal are $(-\sqrt{6}, -2\sqrt{6})$ and $(\sqrt{6}, 2\sqrt{6})$.

S-3. The parts of this problem are not related.

(a) Calculate the integral $\displaystyle\int_2^4 \frac{18t - 3t^2}{t}\,dt$.

(b) Calculate the area of the region below the curve $y = 23\sin(x)\cos^2(x)$ and above the interval $\left[0, \frac{\pi}{2}\right]$ on the x-axis. (Note that $y \geq 0$ on this interval.)

Solution **S-3**

(a) Simplify the integrand using basic algebra, then use the fundamental theorem of calculus.

$$\int_2^4 \frac{18t - 3t^2}{t}\,dt = \int_2^4 (18 - 3t)\,dt = \left(18t - \frac{3}{2}t^2\right)\Big|_2^4 = (72 - 24) - (36 - 6) = 18$$

(b) The area of the region is equal to the integral $\displaystyle\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx$. We use substitution rule with $u = \cos(x)$ (whence $-du = \sin(x)\,dx$).

$$\int_0^{\pi/2} 23\sin(x)\cos^2(x)\,dx = \int_1^0 (-23u^2)\,du = \left(-\frac{23}{3}u^3\right)\Big|_1^0 = 0 - \frac{-23}{3} = \frac{23}{3}$$

D-13. For each part, calculate the limit or show that it does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

(a) $\displaystyle\lim_{x\to 1}\left(\frac{x^4 - x}{\ln(77x - 76)}\right)$

(b) $\displaystyle\lim_{x\to -\infty}\left(\frac{\sqrt{36x^2 + 63}}{31x}\right)$

(c) $\displaystyle\lim_{x\to 2^+} f(x)$, with $f(x) = \begin{cases} 1 + 4x & x \leq 2 \\ \dfrac{x^2 - 4}{x - 2} & x > 2 \end{cases}$

(d) $\displaystyle\lim_{x\to 5^-}\left(\frac{\cos(\pi x)}{x^2 - 25}\right)$

Solution **D-13**

(a) Direct substitution gives "$\frac{0}{0}$", and so we use L'Hospital's Rule.

$$\lim_{x\to 1}\left(\frac{x^4 - x}{\ln(77x - 76)}\right) \overset{H}{=} \lim_{x\to 1}\left(\frac{4x^3 - 1}{\frac{1}{77x - 76}\cdot 77}\right) = \frac{3}{77}$$

(b) We factor out x^2 from inside the square root in the numerator. Observe that since x goes to *negative* infinity,

D-13

we have $\sqrt{x^2} = |x| = -x$.

$$\lim_{x\to-\infty}\left(\frac{\sqrt{36x^2+63}}{31x}\right) = \lim_{x\to-\infty}\left(\frac{-x\sqrt{36+\frac{63}{x^2}}}{31x}\right) = \lim_{x\to-\infty}\left(\frac{-\sqrt{36+\frac{63}{x^2}}}{31}\right) = \frac{-6}{31}$$

(c) We factor and cancel.

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to2^+}\left(\frac{(x-2)(x+2)}{x-2}\right) = \lim_{x\to2^+}(x+2) = 4$$

(d) Direct substitution gives "$\frac{-1}{0}$", whence the one-sided limit must be infinite. Observe that the numerator is negative (goes to -1) as $x \to 5^-$, and the denominator goes to 0 but remains negative as $x \to 5^-$. (For instance, use test points such as $x = 4.99$.) Hence the desired limit is $\frac{-1}{0^-} = +\infty$.

Q-4. For any time $t > 0$, the acceleration of a particle is given by $a(t) = 1 + \dfrac{3}{\sqrt{t}}$, and the particle has velocity $v = -20$ when $t = 1$. Find the velocity of the particle when $t = 16$.

Solution Q-4

We first obtain the velocity by antidifferentiating the acceleration.

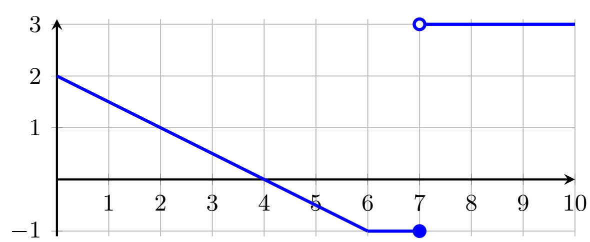
$$v(t) = \int a(t)\,dt = \int \left(1 + 3t^{-1/2}\right)\,dt = t + 6t^{1/2} + C$$

We are given that $v(1) = -20$, whence $-20 = 1 + 6 + C$, and so $C = -27$. Our velocity function is:

$$v(t) = t + 6t^{1/2} - 27$$

Thus $v(16) = 16 + 6 \cdot 4 - 27 = 13$.

R-8. Let $F(x) = \displaystyle\int_0^x f(t)\,dt$, where the graph of $y = f(t)$ is given below. For each part, use this information to calculate the indicated item.



(a) $F(10)$ (b) $F'(6)$ (c) $\displaystyle\int_0^6 |f(t)|\,dt$ (d) $\displaystyle\int_0^4 (f'(t)+5)\,dt$

Solution R-8

(a) The value of $F(10)$ is equal to the (net) area bounded by the graph of $y = f(x)$, the t-axis, and the vertical lines $t = 0$ and $t = 10$.

- The region from $t = 0$ to $t = 4$ consists of a triangle with base 4 and height 2, hence area $\frac{1}{2}(4)(2) = 4$.
- The region from $t = 4$ to $t = 7$ consists of a trapezoid with parallel bases 1 and 3 and height 1, hence area $\frac{1}{2}(3+1)(1) = 2$.
- The region from $t = 7$ to $t = 10$ consists of a square of length 3, hence area 9.

The total net area is $F(10) = 4 - 2 + 9 = 11$.

(b) By the fundamental theorem of calculus, $F'(6) = f(6) = -1$.

(c) Observe that the graph of $y = |f(t)|$ is identical to the graph of $y = f(t)$, except any portion of the graph

R-8

below the t-axis is reflected across (above) the t-axis. This effectively means that we can compute the desired integral using the graph of $y = f(t)$, but counting any area below the t-axis as positive instead of as negative.

The region from $t = 0$ to $t = 4$ has area 4 and the region from $t = 4$ to $t = 6$ has area 1. Hence the desired integral is $\displaystyle\int_0^6 |f(t)|\,dt = 4 + 1 = 5$.

(d) By the fundamental theorem of calculus, we have:

$$\int_0^4 (f'(t) + 5)\ dt = (f(t) + 5t)\big|_0^4 = (f(4) + 20) - (f(0) + 0) = 0 + 20 - 2 = 18$$

O-18. Use linear approximation to estimate $\tan\left(\frac{\pi}{4} + 0.12\right) - \tan\left(\frac{\pi}{4}\right)$.

Solution O-18

We use the tangent line to $f(x) = \tan(x)$ at $x = \frac{\pi}{4}$.

Point of Tangency:	$(\frac{\pi}{4}, f(\frac{\pi}{4})) = (\frac{\pi}{4}, \tan(\frac{\pi}{4}))$
Slope of Line:	$f'(x) = \sec(x)^2; \qquad f'(\frac{\pi}{4}) = 2$
Equation of Line:	$y - \tan(\frac{\pi}{4}) = 2(x - \frac{\pi}{4})$

This means that $\tan(x) - \tan(\frac{\pi}{4}) \approx 2(x - \frac{\pi}{4})$ if x is near $\frac{\pi}{4}$. Hence we have the estimate:

$$\tan\left(\frac{\pi}{4} + 0.12\right) - \tan\left(\frac{\pi}{4}\right) \approx 2\left(\frac{\pi}{4} + 0.12 - \frac{\pi}{4}\right) = 0.24$$

L-18. Let $f(x) = x^3(3x - 4)$.

(a) Find where relative extrema of f occur (if any). Classify each as a local minimum or a local maximum.

(b) Find the absolute extrema of f on $[-1, 2]$ and the x-values at which they occur.

Solution L-18

(a) We have $f(x) = 3x^4 - 4x^3$, whence $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. The critical points of f are $x = 0$ and $x = 1$. The derivative $f'(x)$ does not change sign at $x = 0$, whence there is no local extremum at $x = 0$. However, $f'(x)$ changes sign from negative to positive at $x = 1$, whence there is a local minimum at $x = 1$. (Alternatively, note that $f''(x) = 36x^2 - 24x$ and $f''(1) = 12 > 0$.)

(b) We need only compare the endpoint values and critical values: $f(-1) = 7$, $f(0) = 0$, $f(1) = -1$, and $f(2) = 16$. Hence the absolute minimum is -1 at $x = 1$, and the absolute maximum is 16 at $x = 2$.

D-14. For each part, find all vertical asymptotes of the given function.

(a) $f(x) = \dfrac{x^2 - 8x + 15}{x^2 - 9}$ (b) $g(x) = \dfrac{e^{x+3} - 1}{x^2 - 9}$

Solution D-14

(a) First factor and cancel.

$$f(x) = \frac{x^2 - 8x + 15}{x^2 - 9} = \frac{(x-3)(x-5)}{(x-3)(x+3)} = \frac{x-5}{x+3}$$

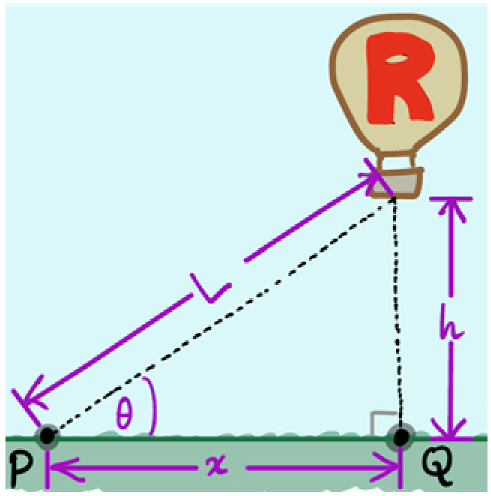
Hence $f(x)$ has a vertical asymptote at $x = -3$ only.

(b) We note that the denominator of $g(x)$ equals 0 only when $x = -3$ or $x = 3$. Direct substitution of $x = 3$ gives the expression "$\frac{e^6-1}{0}$" (nonzero number divided by 0), and so $x = 3$ is a vertical asymptote of $g(x)$. However, we have the following for $x = -3$ after using L'Hospital's Rule:

$$\lim_{x\to-3} g(x) = \lim_{x\to-3}\left(\frac{e^{x+3} - 1}{x^2 - 9}\right) \overset{H}{=} \lim_{x\to-3}\left(\frac{e^{x+3}}{2x}\right) = -\frac{1}{6}$$

Since this limit is not infinite, there is no vertical asymptote at $x = -3$.

K-14. A hot-air balloon is floating directly above the point Q on the ground and is descending at a constant rate of 10 ft/sec. A camera is on the ground at point P, which is 500 feet from point Q. See the figure below.



(a) What is the sign of $\dfrac{dh}{dt}$ (negative, positive, or zero)? If there is not enough information to determine the value, explain why.

(b) How is $\cos(\theta)$ changing over time? Circle your answer below.

(i) increasing over time
(ii) decreasing over time
(iii) constant over time
(iv) sometimes increasing and sometimes decreasing
(v) not enough information to determine

(c) What is the rate of change of the distance between the camera and the balloon when the balloon is 600 feet above the ground? *You must give correct units as part of your answer.*

Solution **K-14**

(a) The balloon is descending, whence h is decreasing. So $\frac{dh}{dt}$ is negative.

(b) Note that $\cos(\theta) = \frac{x}{L}$ and x is a fixed number. As the balloon descends, L decreases, whence the fraction $\frac{x}{L}$ must increase. So $\cos(\theta)$ is increasing.

(c) We have $500^2 + h^2 = L^2$ for all t. Differentiating with respect to t (and canceling a factor of 2) gives $h\frac{dh}{dt} = L\frac{dL}{dt}$. At the specified time, we have $h = 600$ and $\frac{dh}{dt} = -10$. So our two equations at the specified time give:

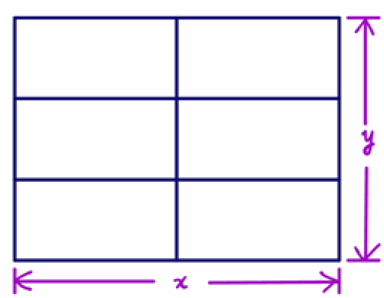
$$500^2 + 600^2 = L^2 \qquad\qquad -6000 = L\frac{dL}{dt}$$

The first equation gives $L = 100\sqrt{41}$, and substituting this value into the second equation gives

$$\frac{dL}{dt} = \frac{-60}{\sqrt{41}}$$

The units are “ft/sec”.

N-16. Farmer Green is building an enclosure that must have a total area of 48 m^2. The pen will also be subdivided into 6 pens of equal area, as shown on the right. Find the dimensions of the enclosure that will require the least amount of fencing. *As you work, fill in the answer boxes below. You must use calculus-based methods in your work. You must also justify that your answer really does give the least fencing.*



constraint equation in terms of x and y:	
objective function in terms of x only:	
interval of interest:	
dimensions of desired enclosure (in meters):	_____ $\times$ _____ total length (x) total width (y)

Solution **N-16**

We seek to minimize the total length of fencing, whence our objective function is $F(x,y) = 4x + 3y$. The total area must be 48, whence our constraint equation is $xy = 48$. Solving for y gives $y = \frac{48}{x}$, and substituting this expression into F gives our objective function in terms of x only:

$$f(x) = 4x + \frac{144}{x}$$

The length x can't be negative, but x also can't equal 0 since that would violation the constraint equation. Hence the interval of interest is $(0, \infty)$. We now find the critical points of f on this interval.

$$f'(x) = 4 - \frac{144}{x^2}$$

Solving $f'(x) = 0$ on the interval $(0, \infty)$ gives $x = 6$. Observe that $f''(x) = \frac{288}{x^3}$, whence $f''(6) > 0$. This means f has a local minimum at $x = 6$. Since $x = 6$ is the only critical point of f, $x = 6$ must also give an absolute minimum. Hence the dimensions of the pen should be $x = 6$ and $y = \frac{48}{6} = 8$.

M-23. Consider the function $g(x)$, whose first two derivatives are given below. ***Note:*** Do not attempt to calculate $g(x)$. Also assume that $g(x)$ has the same domain as $g'(x)$.

$$g'(x) = \frac{8x^{17}}{x-32} \qquad g''(x) = \frac{128x^{16}(x-34)}{(x-32)^2}$$

Fill in the table below with information about the graph of $y = g(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

Solution **M-23**

g is increasing on:	$(-\infty, 0]$, $(32, \infty)$
g is decreasing on:	$[0, 32)$
g is concave up on:	$[34, \infty)$
g is concave down on:	$(-\infty, 32)$, $(32, 34]$
x-coordinate(s) of relative maxima	$x = 0$
x-coordinate(s) of relative minima	NONE
x-coordinate(s) of inflection point(s)	$x = 34$

M-23

The first two derivatives of $f(x)$ are

$$\boxed{g'(x) = \frac{8x^{17}}{x-32}} \qquad \boxed{g''(x) = \frac{128x^{16}(x-34)}{(x-32)^2}}$$

(i) **Vertical asymptotes and horizontal asymptotes.**

Not required since $g(x)$ is not given, but we note that the domain of $g'(x)$ is the same as that of $g(x)$, i.e., $(-\infty, 32) \cup (32, \infty)$.

(ii) **Intervals of increase and local extrema.**

We calculate a sign chart for $g'(x)$. The cut points are the solutions to $g'(x) = 0$ ($x = 0$) and points not in the domain of $g(x)$ ($x = 32$).

interval	test point	sign of g'	shape of g
$(-\infty, 0)$	$g'(-1)$	$\frac{8\ominus}{\ominus} = \oplus$	increasing
$(0, 32)$	$g'(1)$	$\frac{8\oplus}{\ominus} = \ominus$	decreasing
$(32, \infty)$	$g'(33)$	$\frac{8\oplus}{\oplus} = \oplus$	increasing

Hence we deduce the following about g:

g is decreasing on:	$[0, 32)$
g is increasing on:	$(-\infty, 0]$, $(32, \infty)$
g has a local min at:	*none*
g has a local max at:	$x = 0$

(iii) **Intervals of concavity and inflection points.**

We calculate a sign chart for $g''(x)$. The cut points are the solutions to $g''(x) = 0$ ($x = 0$ and $x = 34$) and points not in the domain of $g(x)$ ($x = 32$).

interval	test point	sign of g''	shape of g
$(-\infty, 0)$	$g''(-1)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(0, 32)$	$g'(1)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(32, 34)$	$g'(33)$	$\frac{128\oplus\ominus}{\oplus} = \ominus$	concave down
$(34, \infty)$	$g''(35)$	$\frac{128\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about g:

g is concave down on:	$(-\infty, 32)$, $(32, 34]$
g is concave up on:	$[34, \infty)$
g has an infl. point at:	$x = 34$

(iv) **Sketch of graph.**

Not required.

L-19. The parts of this problem are not related.

(a) Suppose that when x units are produced, the total cost is $C(x) = 2x^2 + 10x + 18$ and the selling price per unit is $p(x) = 46 - x$. Find the level of production that maximizes total profit.

(b) Suppose the total cost of producing q units is $C(q) = q^3 + 20q^2 + 200q + 2000$. Use marginal analysis to estimate the cost of the 3rd unit.

Solution **L-19**

(a) The total revenue is $R(x) = xp(x) = 46x - x^2$, and so the total profit is $P(x) = R(x) - C(x) = -3x^2 + 36x - 18$. Profit is maximized when $P'(x) = 0$.

$$0 = P'(x) = -6x + 36 \Longrightarrow x = 6$$

(b) By marginal analysis, the cost of the 3rd unit is approximately:

$$C'(2) = \left(3q^2 + 40q + 200\right)\Big|_{q=2} = 12 + 80 + 200 = 292$$

8 Chapter 8: Sample Exams (Set B)

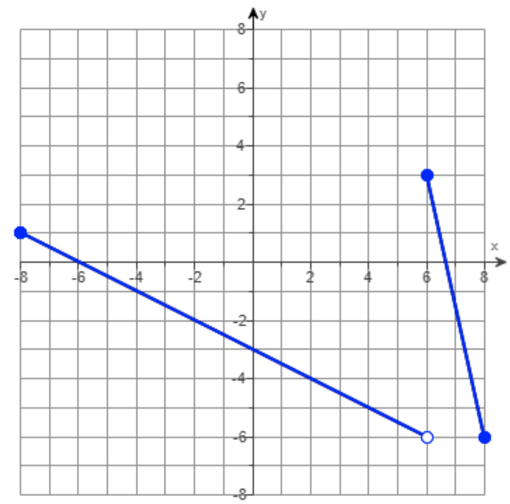
Sample Precalculus Exam B

A-16. The graph of $y = f(x)$ is given below.

Note that f is piecewise linear. An explicit formula for $f(x)$ can be written in the following form, where A and B are constants.

$$f(x) = \begin{cases} y_1(x) & \text{if } -8 \le x < A \\ y_2(x) & \text{if } B \le x \le 8 \end{cases}$$

Calculate each of A, B, $y_1(x)$, and $y_2(x)$.



Solution **A-16**

We see that the graph of f consists of two line segments, one valid for $-8 \le x < 6$ (hence $A = 6$) and the other valid for $6 \le x \le 8$ (hence $B = 6$).

We find $y_1(x)$ by finding the equation of the line through $(-8, 1)$ and $(6, -6)$. We find $y_2(x)$ by finding the equation of the line through $(6, 3)$ and $(8, -6)$. So using point-slope form, we have the following:

$$y_1(x) = 1 + \frac{-6-1}{6-(-8)}(x-(-8)) = 1 - \frac{1}{2}(x+8)$$

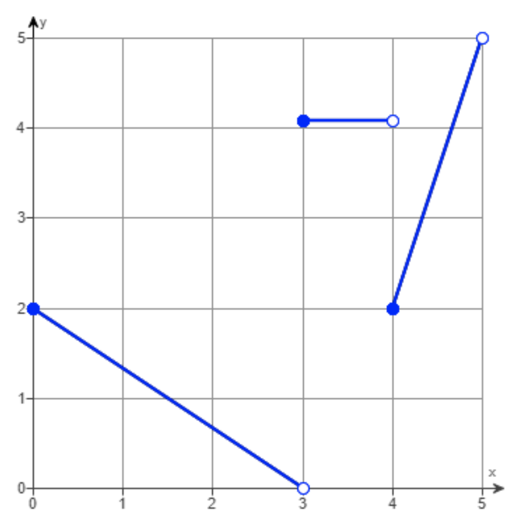
$$y_2(x) = -6 + \frac{-6-3}{8-6}(x-8) = -6 - \frac{9}{2}(x-8)$$

A-17. For each part, use the graph of $y = f(x)$.

(a) Calculate $f(f(2))$.

(b) State the domain of f in interval notation.

(c) State the range of f in interval notation.



Solution **A-17**

(a) Since f is piecewise linear, we can use point-slope form to find an equation for f valid for $0 \le x < 3$.

$$f(x) = 2 + \frac{0-2}{3-0}(x-0) = 2 - \frac{2}{3}x$$

Hence we find $f(2) = 2 - \frac{2}{3} \cdot 2 = \frac{2}{3}$, whence $f(f(2)) = f(\frac{2}{3}) = 2 - \frac{2}{3} \cdot \frac{2}{3} = \frac{14}{9}$.

(b) The domain of f is $[0, 5)$.

(c) The range of f is $(0, 5)$.

A-18. Suppose $\log_3(x) = A$ and $\log_3(y) = B$. Rewrite the expression below in terms of A and B. Your final answer may not contain any logarithm symbol.

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right)$$

Solution **A-18**

We have the following:

$$\log_3\left(\frac{27\sqrt{x}}{y^4}\right) = \log_3(27) + \log_3(\sqrt{x}) - \log_3(y^4) = 3 + \frac{1}{2}\log_3(x) - 4\log_3(y) = 3 + \frac{1}{2}A - 4B$$

A-19. Rewrite the expression below as a single logarithm. Assume x and y are positive.

$$\frac{1}{2}\left(\log_5(x) - 7\log_5(y)\right) + 3\log_5(x-1)$$

Solution **A-19**

We have the following:

$$\frac{1}{2}\left(\log_5(x) - 7\log_5(y)\right) + 3\log_5(x-1) = \frac{1}{2}\log_5\left(\frac{x}{y^7}\right) + \log_5\left((x-1)^3\right)$$
$$= \log_5\left(\frac{x^{1/2}}{y^{7/2}}\right) + \log_5\left((x-1)^3\right) = \log_5\left(\frac{x^{1/2}(x-1)^3}{y^{7/2}}\right)$$

A-20. Suppose $\cos(\theta) = \frac{A}{7}$ with $0 < A < 7$ and $\sin(\theta) < 0$. Find $\sec(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ in terms of A.

Solution **A-20**

By definition of secant,

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{7}{A}$$

Using the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$ and recalling that $\sin(\theta) < 0$, we have

$$\sin(\theta) = -\sqrt{1 - \cos(\theta)^2} = -\sqrt{1 - \frac{A^2}{49}} = -\frac{\sqrt{49 - A^2}}{7}$$

By definition of tangent,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\sqrt{1 - \frac{A^2}{49}}}{\frac{A}{7}} = -\frac{\sqrt{49 - A^2}}{A}$$

A-21. A bacteria colony has an initial population of 3500. The population grows exponentially and triples every 7 hours. Recall that this means the population P at time t satisfies $P(t) = P_0 e^{kt}$ for some constants P_0 and k.

(a) Find the exact value of the growth constant k.

(b) Find the population after 25 hours.

(c) Find the time (in hours) when the population will be 12,600.

Solution **A-21**

(a) We are given that $P(7) = 3P(0)$, or $e^{7k} = 3$. Hence $k = \frac{1}{7}\ln(3)$.

(b) $P(25) = 3500e^{25k} = 3500 \cdot 3^{25/7} \approx 177040$.

(c) We have to solve the equation $12600 = 3500e^{kt}$ for t. Dividing by 3500 and taking logarithms gives $t = 7 \cdot \dfrac{\ln(18/5)}{\ln(3)} \approx 8.16$.

A-22. A rectangular box is constructed according to the following rules.

- the length of the box is twice its width
- the height of the box is 5 feet more than three times the length

Let ℓ, w, and h denote the length, width, and height of the box, respectively, measured in feet.

(a) Write the height of the box in terms of w.

(b) Write an expression for $V(w)$, the volume of the box measured in cubic feet, as a function of its width.

(c) Suppose the rules also require that the sum of the box's width and height to be less than 26 feet. Under this condition, what is the domain of the function $V(w)$?

Solution **A-22**

(a) The first condition gives $\ell = 2w$, and the second condition gives $h = 3\ell + 5$. Hence $h = 3(2w) + 5 = 6w + 5$.

(b) The volume of the box is $V(w) = \ell \cdot w \cdot h = 2w \cdot w \cdot (6w + 5)$.

(c) We are given that $w + h < 26$, or $w + 6w + 5 < 26$. Solving for w gives $w < 3$. Since width must also be non-negative, we find that the domain of $V(w)$ is $0 \le w < 3$, or $w \in [0, 3)$ in interval notation.

A-23. Let $f(x) = \dfrac{2}{3x}$ and assume $h \neq 0$. Fully simplify each of the following expressions:

(a) $f(x + h)$

(b) $f(x + h) - f(x)$

(c) $\dfrac{f(x+h) - f(x)}{h}$

Solution **A-23**

(a) $f(x + h) = \dfrac{2}{3(x + h)}$

(b) $f(x + h) - f(x) = \dfrac{2}{3(x + h)} - \dfrac{2}{3x}$

(c) We have the following.

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{2}{3(x+h)} - \frac{2}{3x}}{h} = \frac{2x - 2(x+h)}{3hx(x+h)} = \frac{-2h}{3hx(x+h)} = \frac{-2}{3x(x+h)}$$

A-24. Find the domain of the function $f(x) = \sqrt{x^2 + x - 6} + \ln(10 - x)$. Write your answer using interval notation.

Solution **A-24**

We examine the square root and the logarithm separately.

The argument of the square root cannot be negative, hence we must have $x^2 + x - 6 \ge 0$. This is equivalent to $(x+3)(x-2) \ge 0$. To solve this inequality, we construct a sign chart and test each of the intervals $(-\infty, -3)$, $(-3, 2)$, and $(2, \infty)$. We find that the solution to the inequality is $(-\infty, -3] \cup [2, \infty)$.

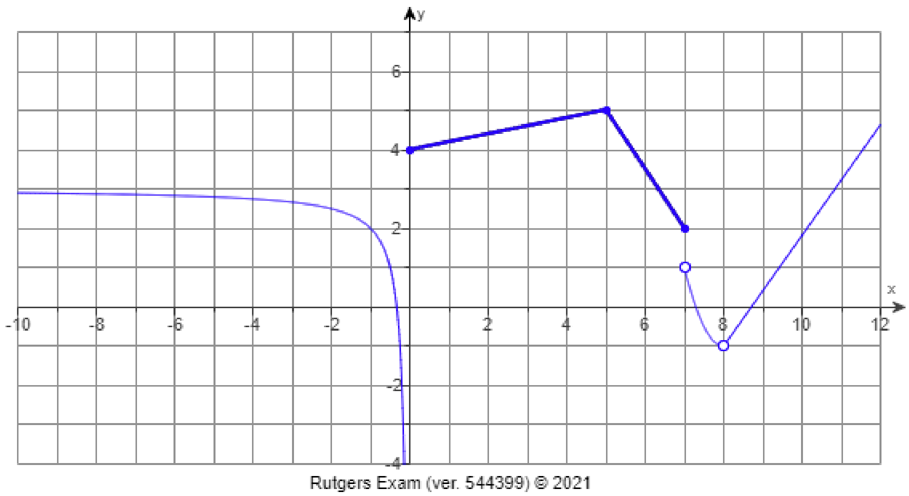
The argument of the logarithm cannot be negative or zero, hence we must have $10 - x > 0$, or $x < 10$ (or $(-\infty, 10)$ in interval notation).

The domain of f is the intersection of the solutions to these two inequalities.

$$(-\infty, -3] \cup [2, 10)$$

Sample Midterm Exam #1B

B-6. For each part, use the graph of $y = f(x)$.



(a) List the x-values where f is not continuous or determine that f is continuous for all x.

(b) List all vertical asymptotes of f.

(c) List all horizontal asymptotes of f.

(d) Calculate $\lim_{x\to 8} f(x)$ or determine that the limit does not exist.

(e) At $x = 7$, which of the one-sided limits of f exist?

Solution **B-6**

(a) $x = 0, 7, 8$ only

(b) $x = 0$ only

(c) $y = 3$ only

(d) $\lim_{x\to 8} f(x) = -1$

(e) Both the left- and right-limits of $f(x)$ at $x = 7$ exist.

F-17. Consider the piecewise-defined function $f(x)$ below; A and B are unspecified constants and $g(x)$ is an unspecified function with domain $[94, \infty)$.

$$f(x) = \begin{cases} Ax^2 + 8 & x < 75 \\ \ln(B) + 6 & x = 75 \\ \dfrac{x - 75}{\sqrt{x+6} - 9} & 75 < x < 94 \\ 19 & x = 94 \\ g(x) & x > 94 \end{cases}$$

(a) Find $\lim_{x\to 75^-} f(x)$ in terms of A and B.

(b) Find $\lim_{x\to 75^+} f(x)$ in terms of A and B.

(c) Find the exact values of A and B for which f is continuous at $x = 75$.

(d) Suppose $g(94) = 19$. What does this imply about $\lim_{x\to 94} f(x)$? Select the best answer.

(i) $\lim_{x\to 94} f(x)$ exists.

(ii) $\lim_{x\to 94} f(x)$ does not exist.

(iii) It gives no information about $\lim_{x\to 94} f(x)$.

Solution **F-17**

(a) $\lim_{x\to 75^-} f(x) = \lim_{x\to 75^-} (Ax^2+8) = A\cdot 75^2 + 8 = 5625A + 8$

(b) We have the following:

$$\begin{aligned}\lim_{x\to 75^+} f(x) &= \lim_{x\to 75^+}\left(\frac{x-75}{\sqrt{x+6}-9}\right) = \lim_{x\to 75^+}\left(\frac{x-75}{\sqrt{x+6}-9}\cdot\frac{\sqrt{x+6}+9}{\sqrt{x+6}+9}\right)\\ &= \lim_{x\to 75^+}\left(\frac{(x-75)(\sqrt{x+6}+9)}{x+6-81}\right) = \lim_{x\to 75^+}\left(\sqrt{x+6}+9\right)\\ &= \sqrt{81}+9 = 18\end{aligned}$$

(c) We need the left-limit, right-limit, and function value of $f(x)$ at $x = 75$ all to be equal. Thus we must have:

$$5625A + 8 = 18 = \ln(B) + 6$$

Thus $A = \frac{10}{5625}$ and $B = e^{12}$.

(d) ***Choice (iii)***. Note that $\lim_{x\to 94^-} f(x) = \lim_{x\to 94^-}\left(\frac{x-75}{\sqrt{x+6}-9}\right) = 19$ (use direct substitution). So for $\lim_{x\to 94} f(x)$ to exist, we require only that $19 = \lim_{x\to 94^+} f(x) = \lim_{x\to 94^+} g(x)$. However, we are given no information at all about this right-limit of g since the function value $g(94)$ is irrelevant to its value.

B-7. The position of a particle (measured in feet) after t seconds is modeled by the following function.

$$h(t) = -16t^2 + 96t + 100$$

(a) Calculate the average velocity of the particle (in feet per second) between $t = 4$ and $t = 5$.

(b) Find an equation of the secant line between $(4, h(4))$ and $(5, h(5))$.

Solution **B-7**

(a) $\overline{v} = \frac{\Delta h}{\Delta t} = \frac{h(5)-h(4)}{5-4} = \frac{-16(25-16)+96(5-4)}{1} = -48$

(b) The slope of the secant line is -48 and the secant line passes through $(4, h(4)) = (4, 228)$. Hence an equation of the secant line is $y = 228 - 48(t-4)$.

C-20. Suppose $\lim_{x\to 6} |f(x)| = 2$. Which of the following statements must be true about $\lim_{x\to 6} f(x)$?

(i) $\lim_{x\to 6} f(x)$ does not exist.

(ii) $\lim_{x\to 6} f(x) = 2$.

(iii) $\lim_{x\to 6} f(x)$ exists and is equal to either 2 or -2, but there is not enough information to determine which of these possibilities must be true.

(iv) There is not enough information about $f(x)$ to determine whether $\lim_{x\to 6} f(x)$ exists.

(v) $\lim_{x\to 6} f(x) = -2$.

Solution **C-20**

Choice (iv). Consider these two examples, both of which satisfy the hypothesis $\lim_{x\to 6} |f(x)| = 2$.

- $f(x) = 2$. Then $\lim_{x\to 6} f(x)$ exists and is equal to 2.
- $f(x) = 2$ for $x < 6$ and $f(x) = -2$ for $x \geq 2$. Then $\lim_{x\to 6} f(x)$ does not exist (the left- and right-limits at $x = 6$ are not equal).

Thus it is not possible to determine whether $\lim_{x\to 6} f(x)$ exists.

C-21. Consider the following function, where k is an unspecified constant.

$$f(x) = \frac{4x^2 - kx}{x^2 + 12x + 32}$$

(a) Find the value of k for which $\lim_{x\to -4} f(x)$ exists.

(b) For the value of k described in part (a), evaluate $\lim_{x\to -4} f(x)$.

Solution **C-21**

(a) Direct substitution of $x=-4$ into $f(x)$ gives the undefined expression "$\frac{64+4k}{0}$". If the number $64+4k$ were non-zero, then we would conclude there is a vertical asymptote for f at $x=-4$. However, since $\lim_{x\to -4} f(x)$ exists, we must have $64+4k=0$, whence $k=-16$.

(b) With $k=-16$, we have the following.

$$\lim_{x\to -4}\left(\frac{4x^2+16x}{x^2+12x+32}\right)=\lim_{x\to -4}\left(\frac{4x(x+4)}{(x+8)(x+4)}\right)=\lim_{x\to -4}\left(\frac{4x}{x+8}\right)=-4$$

C-22. Suppose $\lim_{x\to 0}\left(\frac{f(x)}{x}\right)=8$. Calculate $\lim_{x\to 0}\left(\frac{f(x)}{\sin(6x)}\right)$ or show that the limit does not exist. *If the limit is "$+\infty$" or "$-\infty$", write that as your answer, instead of "does not exist".*

Solution **C-22**

We have the following:

$$\lim_{x\to 0}\left(\frac{f(x)}{\sin(6x)}\right)=\lim_{x\to 0}\left(\frac{1}{6}\cdot\frac{f(x)}{x}\cdot\frac{6x}{\sin(6x)}\right)=\frac{1}{6}\cdot 8\cdot 1=\frac{4}{3}$$

F-18. Consider the following function.

$$f(x)=\frac{x^2-x-6}{x^3-2x^2-3x}$$

(a) Where is f discontinuous?

(b) At the leftmost x-value where f is discontinuous, what type of discontinuity does f have (removable, jump, infinite (vertical asymptote), or other)?

(c) At the rightmost x-value where f is discontinuous, what type of discontinuity does f have (removable, jump, infinite (vertical asymptote), or other)?

Solution **F-18**

First we note the following:

$$f(x)=\frac{x^2-x-6}{x^3-2x^2-3x}=\frac{(x+2)(x-3)}{x(x+1)(x-3)}$$

(a) The function f is continuous on its domain, hence discontinuous at $x=-1,0,3$ only.

(b) ***Choice (iii).*** Direct substitution of $x=-1$ into $f(x)$ gives the undefined expression "$\frac{-6}{0}$", indicating a vertical asymptote at $x=-1$.

(c) ***Choice (i).*** We see that $\lim_{x\to 3} f(x)=\lim_{x\to 3}\left(\frac{x+2}{x(x+1)}\right)=\frac{5}{12}$. Since this limit exists, f has a removable discontinuity at $x=3$.

E-9. Let $f(x)=\frac{8+6e^x}{9e^x-\pi^6}$.

(a) Evaluate $\lim_{x\to\infty} f(x)$.

(b) Evaluate $\lim_{x\to -\infty} f(x)$.

(c) List all vertical asymptotes of f.

Solution **E-9**

(a) Divide each term by e^x and recall that $\lim_{x\to\infty} e^{-x}=0$.

$$\lim_{x\to\infty}\left(\frac{8+6e^x}{9e^x-\pi^6}\right)=\lim_{x\to\infty}\left(\frac{8e^{-x}+6}{9-\pi^6e^{-x}}\right)=\frac{0+6}{9-0}=\frac{2}{3}$$

(b) Recall that $\lim_{x\to -\infty} e^x=0$.

$$\lim_{x\to -\infty}\left(\frac{8+6e^x}{9e^x-\pi^6}\right)=\frac{8+0}{0-\pi^6}=-\frac{8}{\pi^6}$$

E-9

(c) The denominator vanishes if $x = \ln(\frac{\pi^6}{9})$, and the numerator does not vanish at this x-value. Hence the only vertical asymptote of f is the line $x = \ln(\frac{\pi^6}{9})$.

Sample Midterm Exam #2B

G-14. The following limit represents the derivative of a function f at a point a.

$$f'(a) = \lim_{h \to 0} \left(\frac{9\tan\left(\frac{\pi}{6} + h\right) - \frac{9}{\sqrt{3}}}{h} \right)$$

(a) Find a possible pair for f and a.

(b) Calculate the value of the limit.

Solution **G-14**

(a) Recall that the definition of the derivative is:

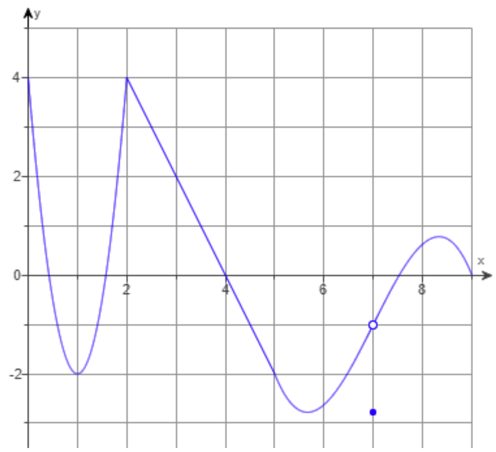
$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

Let $f(x) = 9\tan(x)$ and let $a = \frac{\pi}{6}$. Then the given limit is $f'(a)$.

(b) Observe that $f'(x) = 9\sec(x)^2$, and so the given limit is $9\sec(\frac{\pi}{6})^2 = 9 \cdot \frac{4}{3} = 12$.

G-15. For each part, use the graph of $y = f(x)$ to determine whether the value exists. If the value exists, state its sign (negative, positive, or zero).

(a) $f'(1)$ (b) $f'(2)$ (c) $f'(3.5)$ (d) $f'(7)$



Solution **G-15**

(a) zero

(b) $f'(2)$ does not exist (the graph of f has a sharp corner at $x = 2$)

(c) negative

(d) $f'(7)$ does not exist (f is not continuous at $x = 7$)

I-5. Let $f(x) = x^9 e^{4x}$.

(a) Find $f'(x)$.

(b) Explain how to find where the tangent line to the graph of f is horizontal.

(c) Find where the graph of f has a horizontal tangent line.

Solution **I-5**

(a) Use product rule and chain rule.

$$f'(x) = 9x^8 e^{4x} + x^9 \cdot 4e^{4x} = x^8 e^{4x}(9 + 4x)$$

(b) We must solve the equation $f'(x) = 0$ for x.

(c) The solutions to $f'(x) = 0$ are $x = 0$ and $x = -\frac{9}{4}$, thus these are the x-values where f has a horizontal tangent line.

I-6. Selected values of the functions f and g and their derivatives are given in the table below. Use these values to complete the questions.

x	1	2	3	4
$f(x)$	4	3	2	1
$f'(x)$	-4	-1	-9	-3
$g(x)$	2	1	3	4
$g'(x)$	1	2	4	5

(a) Suppose $h(x) = 5f(x) - 8g(x)$. Find $h'(1)$.
(b) Suppose $p(x) = x^2 f(x)$. Find $p'(2)$.
(c) Suppose $q(x) = f(x^2)$. Find $q'(2)$.

Solution **I-6**

(a) We have $h'(x) = 5f'(x) - 8g'(x)$. Thus

$$h'(1) = 5f'(1) - 8g'(1) = 5 \cdot (-4) - 8 \cdot 1 = -28$$

(b) By product rule we have $p'(x) = 2xf(x) + x^2 f'(x)$. Thus

$$p'(2) = 2 \cdot 2 \cdot f(2) + 4 \cdot f'(2) = 4 \cdot 3 + 4 \cdot (-1) = 8$$

(c) By chain rule we have $q'(x) = f'(x^2) \cdot 2x$. Thus

$$q'(2) = f'(4) \cdot 2 \cdot 2 = (-3) \cdot 4 = -12$$

G-16. Let $f(x)$ and $g(x)$ be functions such that $f'(-8) = g'(-8)$ and the line tangent to the graph of f at $x = -8$ is $y = -7x + 6$. For each part, compute the desired value, if possible.

(a) $f(-8)$ (b) $f'(-8)$ (c) $g(-8)$ (d) $g'(-8)$

Solution **G-16**

(a) The tangent line to f at a point passes through the graph of f at the point of tangency. So $f(-8)$ is equal to the y-coordinate of the tangent line at $x = -8$. Thus $f(-8) = -7 \cdot (-8) + 6 = 62$.

(b) The slope of the tangent line to f is the derivative of f at the point of tangency. Hence $f'(-8)$ is -7, the slope of the line $y = -7x + 6$.

(c) We are not given enough information to determine $g(8)$. (In particular, the slope of the tangent line to g at $x = -8$ is -7 also, but the y-intercept need not be 6. In other words, the point of tangency need not be the same for both f and g.)

(d) We are given that $f'(-8) = g'(-8)$, whence $g'(-8) = -7$.

J-17. Consider the curve defined by the following equation, where A and B are unspecified constants.

$$Ax^2 - 8xy = B\cos(y) + 3$$

(a) Find a formula for $\frac{dy}{dx}$.
(b) Suppose the point $(8, 0)$ is on the curve. Find an equation that A and B must satisfy.
(c) Suppose the tangent line to the curve at the point $(8, 0)$ is $y = 6x - 48$. Find the values of A and B.

Solution **J-17**

(a) Using implicit differentation, we obtain:

$$2Ax - 8y - 8x\frac{dy}{dx} = -B\sin(y)\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2Ax - 8y}{8x - B\sin(y)}$$

(b) The point $(8, 0)$ must satisfy the equation that defines the curve, whence:

$$64A = B + 3$$

J-17

(c) We have that $\frac{dy}{dx} = 6$ (the slope of the tangent line) when $x = 8$ and $y = 0$. Hence by part (a) we have:

$$7 = \frac{16A - 0}{64 - 0} = \frac{A}{4}$$

Hence $A = 28$. From part (b) we then have $B = 64A - 3 = 1533$.

K-13. The base of a right triangle is decreasing at a constant rate of 10 cm/sec and in such a way that the triangle always remains a right triangle. At the time when the base is 15 cm and the height is 22 cm, the area of the triangle is increasing by 25 cm^2/sec. Use this information to answer the questions below. Let B denote the base of the triangle.

(a) At the described time, what is the sign of $\frac{dB}{dt}$?

(b) At the described time, what is the sign of $\frac{d^2B}{dt^2}$?

(c) At the described time, at what rate is the height changing?

(d) What are the units of the answer to part (c)?

Solution K-13

(a) We are given that the base is decreasing at the given time, so $\frac{dB}{dt}$ is negative.

(b) We are given that $\frac{dB}{dt}$, the rate at which the base is changing, is constant. Thus $\frac{d^2B}{dt^2}$ is zero.

(c) At any time we have $A = \frac{1}{2}BH$, where A, B, and H are the area, base, and height of the triangle, respectively. Differentiating with respect to time gives us a total of two equations that hold for any time.

$$A = \frac{1}{2}BH$$
$$\frac{dA}{dt} = \frac{1}{2}\frac{dB}{dt}H + \frac{1}{2}B\frac{dH}{dt}$$

At the given time, we have: $\frac{dB}{dt} = -10$, $B = 15$, $H = 22$, and $\frac{dA}{dt} = 25$. Substituting this information into the previous two equations gives us two equations that hold only at the described time.

$$A = 165$$
$$25 = -110 + 7.5\frac{dH}{dt}$$

Solving for $\frac{dH}{dt}$ gives $\frac{dH}{dt} = 18$.

(d) The units of $\frac{dH}{dt}$ are cm/sec.

I-7. Suppose f is differentiable at x and $g(x) = \dfrac{16\ln(15x)}{6f(x) - \sqrt{x+17}}$. Find $g'(x)$.

Solution I-7

We start with quotient rule since the expression for $g(x)$ is a quotient. When we differentiate the numerator we must use chain rule.

$$g'(x) = \frac{\left(16 \cdot \frac{1}{15x} \cdot 15\right) \cdot \left(6f(x) - \sqrt{x+17}\right) - \left(16\ln(15x) \cdot \left(6f'(x) - \frac{1}{2\sqrt{x+7}}\right)\right.}{\left(6f(x) - \sqrt{x+7}\right)^2}$$

Sample Midterm Exam #3B

L-17. Find the absolute extreme values of $f(x) = x^3 - 6x^2 + 9x + 20$ on $[-3, 2]$ and the x-value(s) at which they occur.

Solution **L-17**

Since f is differentiable for all x, the only critical points are solutions to $f'(x) = 0$. We have

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

Hence the only critical point is $x = 1$. (We reject the solution $x = 3$ since it is not in the given interval.) We now check the critical values and the endpoint values: $f(-3) = -88$, $f(1) = 24$, and $f(2) = 22$. Hence the absolute minimum is -88 (occurring at $x = -3$) and the absolute maximum is 24 (occurring at $x = 1$).

M-20. Consider the function f and its derivatives below.

$$f(x) = \frac{x-3}{x^2 - 6x - 16} \quad , \quad f'(x) = \frac{-(x-3)^2 - 25}{(x^2 - 6x - 16)^2} \quad , \quad f''(x) = \frac{2(x-3)\left((x-3)^2 + 75\right)}{(x^2 - 6x - 16)^3}$$

Find where f is concave down and where f is concave up; write your answers using interval notation. Also find the x-coordinate of each inflection point of f.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

Solution **M-20**

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ ($x = 3$) and the vertical asymptotes (solutions to $x^2 - 6x - 16 = 0$, or $x = -2$ and $x = 8$).

interval	test point	sign of f''	shape of f
$(-\infty, -2)$	$f''(-3)$	$\frac{2\ominus\oplus}{\oplus} = \ominus$	concave down
$(-2, 3)$	$f''(0)$	$\frac{2\ominus\oplus}{\ominus} = \oplus$	concave up
$(3, 8)$	$f''(4)$	$\frac{2\oplus\oplus}{\ominus} = \ominus$	concave down
$(8, \infty)$	$f''(9)$	$\frac{2\oplus\oplus}{\oplus} = \oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 2)$, $[3, 8)$
f is concave up on:	$(-2, 3]$, $(8, \infty)$
f has an infl. point at:	$x = 3$

M-21. Suppose f is differentiable on $(-\infty, 1) \cup (1, \infty)$ and satisfies all of the following properties. Sketch a possible graph of $y = f(x)$ on the axes provided. *Label all asymptotes, local extrema, and inflection points. Your graph need not to be to scale, but it must have the correct shape.*

(i) $\displaystyle\lim_{x\to-\infty} f(x) = -3$; $\displaystyle\lim_{x\to\infty} f(x) = \infty$; $\displaystyle\lim_{x\to1^-} f(x) = -\infty$; $\displaystyle\lim_{x\to1^+} f(x) = \infty$;

(ii) $f'(x) > 0$ on $(-\infty, -2)$ and $(5, \infty)$; $f'(x) < 0$ on $(-2, 1)$ and $(1, 5)$; $f'(-2) = f'(5) = 0$

(iii) $f''(x) > 0$ on $(-\infty, -7)$ and $(1, \infty)$; $f''(x) < 0$ on $(-7, 1)$; $f''(-7) = 0$

Solution **M-21**

The conditions can also be summarized as follows:

(i) The lines $y = -3$ and $x = 1$ are horizontal and vertical asymptotes for f, respectively. There is no horizontal asymptote at positive infinity.

(ii) f is increasing on $(-\infty, -2)$ and $(5, \infty)$; f is decreasing on $(-2, 1)$ and $(1, 5)$; there is a local minimum at $x = 5$; there is a local maximum at $x = -2$.

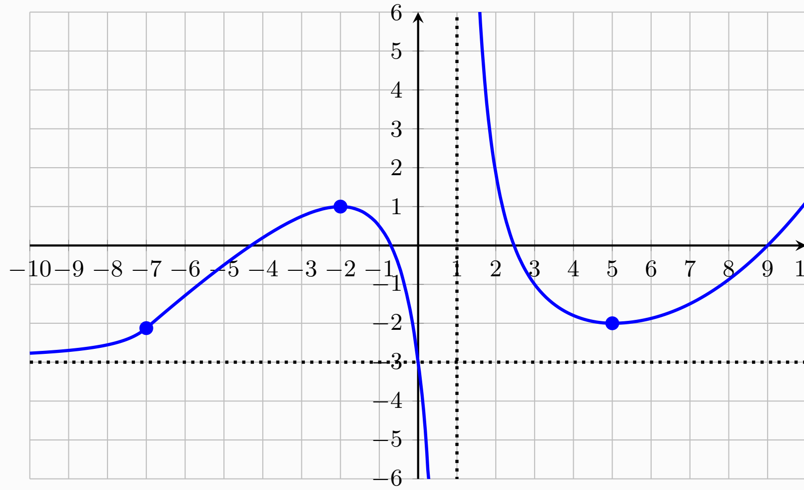
(iii) f is concave up on $(-\infty, -7)$ and $(1, \infty)$; f is concave down on $(-7, 1)$; there is an inflection point at $x = -7$.

The table below summarizes the behavior of f on each subinterval.

M-21

interval	behavior of f	notes
$(-\infty, -7)$	increasing, concave up	inflection point at $x = -7$
$(-7, -2)$	increasing, concave down	local maximum at $x = -2$
$(-2, 1)$	decreasing, concave down	vertical asymptote at $x = 1$
$(1, 5)$	decreasing, concave up	local minimum at $x = 6$
$(5, \infty)$	increasing, concave up	$f \to \infty$ as $x \to \infty$

There are many possible functions that satisfy these properties. Here is one.



N-15. A storage shed with a volume of 1500 ft^3 is to be built in the shape of a rectangular box with a square base. The material for the base costs \$ 6/ft^2, the material for the roof costs \$ 9/ft^2, and the material for the sides costs \$2.50/ft^2. Find the dimensions of the cheapest shed. As you work, fill in the answer boxes below. Let x represent the length of the base of the shed.

objective function in terms of x:	
interval of interest:	
dimensions of cheapest shed (in ft):	______ × ______ × ______ (length of base, width of base, height of shed)

Solution N-15

Since we asked to find the cheapest shed, the objective function is the total cost of the shed. Let x be the length of the base of the shed and let h be the height of the shed. Since the base of the shed is a square, the total cost of the shed is

$$C = C_{\text{base}} + C_{\text{roof}} + C_{\text{sides}} = 6x^2 + 9x^2 + 2.5 \cdot 4xh = 15x^2 + 10xh$$

The volume of the shed must be 1500, whence the constraint equation is $x^2h = 1500$, and thus the height is given by $h = \frac{1500}{x^2}$. Substituting the expression for h into C gives the objective in terms of x only.

$$C(x) = 15x^2 + \frac{15000}{x}$$

Since x is a length, we must have $x \geq 0$. However, the case $x = 0$ would violate the volume constraint $x^2h = 1500$. There are no further restrictions on the allowed values of x. So the interval of interest for $C(x)$ is $(0, \infty)$. Our goal is to minimize $C(x)$ on this interval.

Since $C(x)$ is differentiable on $(0, \infty)$, the only critical points are solutions to $C'(x) = 0$. We have that $C'(x) = 30x - \frac{15000}{x^2}$, and thus the only solution to $C'(x) = 0$ is $x = 500^{1/3}$. Now observe that $C''(x) = 30 + \frac{30000}{x^3}$, which

N-15

is positive for all x in $(0, \infty)$. Hence $C(x)$ is concave up on this interval, and we conclude that $x = 500^{1/3}$ does, in fact, give the absolute minimum value of $C(x)$ on $(0, \infty)$.

The dimensions of the cheapest shed are $x = 500^{1/3}$ (length of base and width of base) and $h = \frac{1500}{x^2} = 3 \cdot 500^{1/3}$ (height of shed).

M-22. Let $f(x) = -e^{-x}\left(x^2 - 5x - 23\right)$. Find all critical points of f. Then find where f is decreasing and where f is increasing; write your answers using interval notation. Also find where relative extrema of f occur.

Write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.

Solution M-22

Since f is differentiable for all x, the only critical points are solutions to $f'(x) = 0$. Using product rule and chain rule gives

$$f'(x) = (-e^{-x} \cdot (-1))(x^2 - 5x - 23) + (-e^{-x})(2x - 5) = e^{-x}(x^2 - 7x - 18) = e^{-x}(x - 9)(x + 2)$$

Thus the critical points of f are $x = -2$ and $x = 9$. We now construct a sign chart to find the intervals of increase. (Recall that $e^{-x} > 0$ for all x.)

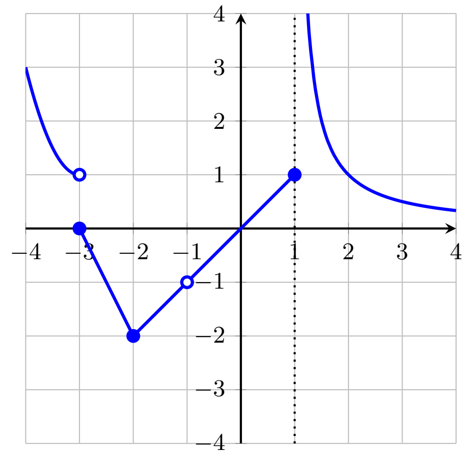
interval	test point	sign of f'	shape of f
$(-\infty, -2)$	$f'(-3) = \oplus\ominus\ominus$	$\oplus$	increasing
$(-2, 9)$	$f'(0) = \oplus\ominus\oplus$	$\ominus$	decreasing
$(9, \infty)$	$f'(10 = \oplus\oplus\oplus$	$\oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$[-2, 9]$
f is increasing on:	$(-\infty, -2]$, $[9, \infty)$
f has a local min at:	$x = 9$
f has a local max at:	$x = -2$

Sample Final Exam B

B-2. The graph of $y = f(x)$ is given below. Find all values of a in $(-4, 4)$ such that $\lim_{x\to a} f(x)$ does not exist.



Solution B-2

$a = -3$ and $a = 1$ only.

G-5. Which statement is true about the graph of $f(x) = |x| + 91$ at the point $(0, 91)$?

(a) The graph has a tangent line at $y = 91$.
(b) The graph has infinitely many tangent lines.
(c) The graph has no tangent line.
(d) The graph has two tangent lines: $y = x + 91$ and $y = -x + 91$.
(e) None of the above statements is true.

Solution G-5

Choice C. Since $f(x)$ is not differentiable at $x = 0$, $f'(0)$ doesn't exist. So there is no tangent line at $x = 0$.

O-11. Suppose the cost (in dollars) of manufacturing q units is given by

$$C(q) = 6q^2 + 34q + 112$$

Use marginal analysis to estimate the cost of producing the 5th unit.

Solution O-11

The exact cost of the 5th unit is $\Delta C = C(5) - C(4)$, which is approximately $C'(4)$ by linear approximation. Hence

$$\Delta C \approx C'(4) = (12q + 34)\big|_{q=4} = 82$$

F-10. Consider the function $f(x)$, where k is an unspecified constant. Find the value of k for which f continuous for all x, or show that no such value of k exists.

$$f(x) = \begin{cases} 38 + kx & x < 3 \\ kx^2 + x - k & x \geq 3 \end{cases}$$

In your work, you must use limit-based methods to solve this problem. Solutions that have work that is not based on limits will not receive full credit.

Solution F-10

First we calculate the left-limit, right-limit, and function value at $x = 3$.

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} (38 + kx) = 38 + 3k \\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} \left(kx^2 + x - k\right) = 8k + 3 \\ f(3) &= 8k + 3 \end{aligned}$$

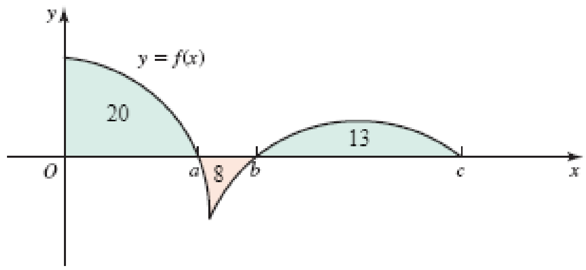
F-10

To make f continuous at $x = 3$, the left-limit, right-limit, and function value at $x = 3$ must all be equal. Hence we must have

$$38 + 3k = 8k + 3$$

Hence $k = 7$.

R-3. The figure below shows the area of regions bounded by the graph of $y = f(x)$ and the x-axis, where $a = 4$, $b = 6$, and $c = 15$. Evaluate $\int_a^c (11f(x) - 6)\, dx$.



Solution R-3

Split up the integral using linearity properties.

$$\int_a^c (11f(x) - 6)\, dx = 11\int_a^c f(x)\, dx - \int_a^c 6\, dx = 11 \cdot (13 - 8) - 6 \cdot (15 - 4) = -11$$

M-10. Consider the function f and its first two derivatives below.

$$f(x) = \frac{99e^x}{(x-25)^{47}} + 98 \quad , \quad f'(x) = \frac{99e^x(x-72)}{(x-25)^{48}} \quad , \quad f''(x) = \frac{99e^x\left((x-72)^2+47\right)}{(x-25)^{49}}$$

Fill in the table below with information about the graph of $y = f(x)$. *For each part, write "NONE" as your answer if appropriate. Where applicable, give a comma-separated list of intervals that are as inclusive as possible.*

You do not have to show work, and each table item will be graded with no partial credit.

Solution M-10

equation(s) of vertical asymptote(s) of f	$x = 25$
equation(s) of horizontal asymptote(s) of f	$y = 98$
where f is decreasing	$(-\infty, 25)$, $(25, 72]$
where f is increasing	$[72, \infty)$
x-coordinate(s) of local minima of f	$x = 72$
x-coordinate(s) of local maxima of f	NONE
where f is concave down	$(-\infty, 25)$
where f is concave up	$(25, \infty)$
x-coordinate(s) of inflection point(s) of f	NONE

The first two derivatives of $f(x)$ are

$$\boxed{f(x) = \frac{99e^x}{(x-25)^{47}} + 98} \quad \boxed{f'(x) = \frac{99e^x(x-72)}{(x-25)^{48}}} \quad \boxed{f''(x) = \frac{99e^x\left((x-72)^2+47\right)}{(x-25)^{49}}}$$

M-10

(i) Vertical asymptotes and horizontal asymptotes.

Observe that f is continuous on its domain, but is undefined for $x = 25$. Hence our candidate vertical asymptote is the line $x = 25$. Indeed, direct substitution of $x = 25$ into the firs term of f gives the expression $\frac{99e^{25}}{0}$, which indicates that both one-sided limits are infinite. Hence the line $x = 25$ is a true vertical asymptote.

As for the horizontal asymptotes we compute the limits at infinite. For $x \to -\infty$, we have:

$$\lim_{x\to-\infty}\left(\frac{99e^x}{(x-25)^{47}}+98\right)=\frac{0}{-\infty}+98=98$$

For $x \to +\infty$, we first observe the following for any $n > 0$:

$$\lim_{x\to\infty}\underbrace{\left(\frac{e^x}{x^n}\right)}_{\frac{\infty}{\infty}} \overset{H}{=} \lim_{x\to\infty}\underbrace{\left(\frac{e^x}{nx^{n-1}}\right)}_{\frac{\infty}{\infty}} \overset{H}{=} \underbrace{\cdots}_{n\text{ uses of LR}} \overset{H}{=} \lim_{x\to\infty}\left(\frac{e^x}{n!}\right)=\frac{\infty}{n!}=\infty$$

Hence we now have the following:

$$\lim_{x\to+\infty}\left(\frac{99e^x}{(x-25)^{47}}+98\right)=\infty+98=\infty$$

So the only horizontal asymptote is $y = 98$.

(ii) Intervals of increase and local extrema.

We calculate a sign chart for the first derivative. The cut points are the solutions to $f'(x) = 0$ ($x = 72$) and the vertical asymptotes ($x = 25$).

interval	test point	sign of f'	shape of f
$(-\infty, 25)$	$f'(0)$	$\frac{99\oplus\ominus}{\oplus}=\ominus$	decreasing
$(25, 72)$	$f'(26)$	$\frac{99\oplus\ominus}{\oplus}=\ominus$	decreasing
$(72, \infty)$	$f'(73)$	$\frac{99\oplus\oplus}{\oplus}=\oplus$	increasing

Hence we deduce the following about f:

f is decreasing on:	$(-\infty, 25)$, $(25, 72]$
f is increasing on:	$[72, \infty)$
f has a local min at:	$x = 72$
f has a local max at:	*none*

(iii) Intervals of concavity and inflection points.

We calculate a sign chart for the second derivative: The cut points are the solutions to $f''(x) = 0$ (none) and the vertical asymptotes ($x = 25$).

interval	test point	sign of f''	shape of f
$(-\infty, 25)$	$f''(0)$	$\frac{99\oplus\oplus}{\ominus}=\oplus$	concave down
$(25, \infty)$	$f''(26)$	$\frac{99\oplus\oplus}{\oplus}=\oplus$	concave up

Hence we deduce the following about f:

f is concave down on:	$(-\infty, 25)$
f is concave up on:	$(25, \infty)$
f has an infl. point at:	*none*

(iv) Sketch of graph.

Not required.

P-11. A student is asked to calculate the following limit using l'Hospital's Rule and to show all their work.

$$L = \lim_{x \to 0} \left(\frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)} \right)$$

The student decides to cheat, so they find the solution online (shown below) and they submit the work as their own!

$$\begin{aligned}
L &= \lim_{x \to 0} \left(\frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)} \right) && (1) \\
&= \lim_{x \to 0} \left(\frac{2\cos(2x) + 34x + 2}{8x + \sec(x)^2} \right) && (2) \\
&= \lim_{x \to 0} \left(\frac{-4\sin(2x) + 34}{8 + 2\sec(x)^2 \tan(x)} \right) && (3) \\
&= \frac{-4\sin(0) + 34}{8 + 2\sec(0)^2 \tan(0)} && (4) \\
&= \frac{0 + 34}{8 + 0} && (5) \\
&= \frac{17}{4} && (6)
\end{aligned}$$

Unfortunately, this solution contains an error, and so the student lost all credit for the problem. The student was also later determined to be responsible for cheating, and so they earned a grade of 0 on the entire exam!

Your task is to find and correct the error(s). Answer the following questions.

(a) There may be several errors in this solution. Which line is the first incorrect line?

(b) Explain the error in the first incorrect line in your own words.

(c) Calculate the correct value of L (the original limit).

Solution **P-11**

(a) The first incorrect line is line (3).

(b) In the transition from line (2) to line (3), the student has differentiated the numerator and denominator separately, presumably to use l'Hospital's Rule. However, this is an incorrect application as the limit in line (2) does not have an indeterminate form. L'Hospital's Rule cannot be used there.

(c) Substitution of $x = 0$ in line (2) gives the correct value: $L = 4$.

R-4. Consider the integral below.

$$\int_{-2}^{1} \sqrt{9 - (x-1)^2}\, dx$$

(a) Explain in your own words how you can calculate this integral without using Riemann sums or the fundamental theorem of calculus. ***Hint:*** Try graphing the integrand!

(b) Find the exact value of the integral.

Solution **R-4**

(a) Observe that the graph of $y = \sqrt{9 - (x-1)^2}$ is the top half of a circle with center $(1, 0)$ and radius 3. The leftmost point on the circle is $(-2, 0)$. Thus the integral is equal to the area of the left half of this semi-disc. That is, the region is congruent to a quarter-disc with radius 3.

(b) The area of the region is $\frac{\pi r^2}{4}$ with $r = 3$, hence the area is $\frac{9\pi}{4}$.

J-12. Consider the curve described by the following equation.

$$e^{12x+2y} = 6y - 3xy + 1$$

(a) Find $\dfrac{dy}{dx}$ at a general point on this curve.

(b) Calculate the slope of the line tangent to the curve at $(2, -12)$.

(c) There is a point on the curve close to the origin with coordinates $(0.07, b)$, and the line tangent to the curve at the origin is $y = 3x$. Use linear approximation to estimate the value of b.

Solution **J-12**

(a) Differentiating both sides with respect to x gives:

$$e^{12x+2y} \cdot \left(12 + 2\frac{dy}{dx}\right) = 6\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y$$

Solving algebraically for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{12e^{12x+2y} + 3y}{6 - 3x - 2e^{12x+2y}}$$

(b) Substituting $x = 2$ and $y = -12$ into the expression above gives $\frac{dy}{dx} = 12$.

(c) The tangent line at the origin is a linear approximation of the curve near the origin. Hence the point $(0.07, b)$ lies approximately on this tangent line. Hence $b \approx 3(0.07) = 0.21$.

G-6. Suppose the derivative of f is $f'(x) = 3x^2 - 6x - 9$ and that $f(1) = 10$.

(a) Find an equation of the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Find the critical points of f.

(c) Where does f have a local minimum value? local maximum value?

(d) Calculate $f(0)$.

(e) Calculate the absolute maximum value of f on the interval $[0, 6]$. At what x-value does it occur?

Solution **G-6**

(a) We have $f'(1) = 3 - 6 - 9 = -12$, whence an equation of the tangent line is $y = 10 - 12(x - 1)$.

(b) Solving $f'(x) = 0$, we find that the critical points of f are $x = -1$ and $x = 3$.

(c) A sign chart for $f'(x)$ reveals that $f'(x)$ is positive on the intervals $(-\infty, -1)$ and $(3, \infty)$; and $f'(x)$ is negative on the interval $(-1, 3)$. Since f' changes from positive to negative at $x = -1$, a local maximum occurs at $x = -1$. Since f' changes from negative to positive to $x = 3$, a local minimum occurs at $x = 3$.

(d) We find $f(x)$ by finding the most general antiderivative of $f'(x)$.

$$f(x) = \int f'(x)\,dx = x^3 - 3x^2 - 9x + C$$

The initial condition $f(1) = 10$ implies $1 - 3 - 9 + C = 10$, or $C = 21$. Hence

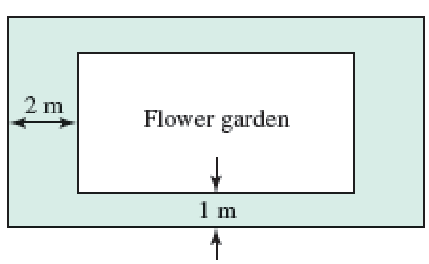
$$f(x) = x^3 - 3x^2 - 9x + 21$$

So $f(0) = 21$.

(e) The absolute maximum of f on $[0, 6]$ can occur only at an endpoint (0 or 6) or a critical number (-1 or 3). Calculating the values of f at these x-values gives: $f(0) = 21$, $f(-1) = 26$, $f(3) = -6$, and $f(6) = 75$. Hence the absolute maximum of f on $[0, 6]$ is 75, occurring at $x = 6$.

N-10. A local park has hired you to construct a rectangular flower garden surrounded by a grass border that is 1 m wide on two sides and 2 m wide on the other two sides. (See the figure below.) The area of the garden only (the small rectangle) must be 126 m^2.

Your primary task is to find the dimensions of the garden that give the smallest possible combined area of the garden and the grass border. For this problem, let W be the horizontal width of the garden and let H be the vertical height of the garden.



(a) What is the objective function for this problem in terms of W and H?
(b) What is the constraint equation for this problem in terms of W and H?
(c) Find the objective function in terms of W only.
(d) What is the interval of interest for the objective function?
(e) Find the values of W and H that minimize the total combined area.
(f) What horizontal width W of the garden will *maximize* the total area?

Solution **N-10**

(a) The width of the combined area is $W+4$ and the height of the combined area if $H+2$. We seek to minimize the combined area, and so the objective function is

$$g(w, H) = (W+4)(H+2)$$

(b) The garden must have an area of 126, and so the constraint equation is $WH = 126$.

(c) Solving for H in the constraint gives $H = \frac{126}{W}$, and substituting this into the objective gives:

$$f(W) = g\left(W, \frac{126}{W}\right) = (W+4)\left(\frac{126}{W}+2\right) = 134 + 2W + \frac{504}{W}$$

(d) The width W can be any positive length (note that a length of 0 is not allowed since the garden area must be positive). So the interval of interest is $(0, \infty)$.

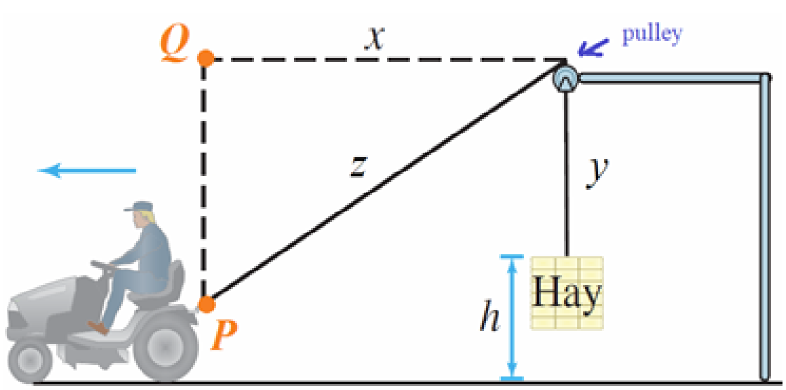
(e) We solve $f'(W) = 0$ to find the critical numbers.

$$f'(W) = 2 - \frac{504}{w^2} = 0 \Longrightarrow W = \sqrt{252} = 6\sqrt{7}$$

Observe that $f''(w) = \frac{1108}{W^3}$, which is positive for all $W > 0$. So by the second derivative test, $W = 6\sqrt{7}$ gives a local minimum. Since it gives the only local extreme value on $(0, \infty)$, f has a global minimum value on $(0, \infty)$ at $W = 6\sqrt{7}$. The corresponding height is $H = \frac{126}{6\sqrt{7}} = 3\sqrt{7}$.

(f) None of our work above changes. However, we now note that $f(W) \to \infty$ as $W \to 0^+$ or as $W \to \infty$. Hence there is no maximum combined area. We may obtain an arbitrarily large combined area by simply taking the width W to be either arbitrarily small or arbitrarily large.

K-9. A farmer's tractor pulls a rope of length 12 m attached to a bale of hay through a pulley is 8 m above the ground. The vertical distance between the tractor and the pulley (the distance from P to Q) is 7 m. The tractor is moving to the left at rate of 2 m/sec, which causes the bale of hay to rise off the ground.



(a) The rate of change (with respect to time) of which variable is equal to the speed of the tractor?
(b) Use the Pythagorean theorem to find an equation that holds for all time and involves only the variables x and z.
(c) Use the fact that the length of the rope is constant to find an equation that holds for all time and involves only the variables z and y.
(d) Use the fact that the height of the pulley is constant to find an equation that holds for all time and involves only the variables h and y.

(e) Combine the equations from parts (b), (c), and (d) to find an equation that holds for all time and involves only the variables x and h.

(f) The rate of change (with respect to time) of which variable is equal to the rate at which the bale of hay is rising?

(g) Find the rate at which the bale of hay is rising off the ground when the horizontal distance between the tractor and the bale of hay is 8 m.

Solution **K-9**

(a) x

(b) $x^2 + 7^2 = z^2$, or $x^2 + 49 = x^2$

(c) $y + z = 12$

(d) $y + h = 8$

(e) Subtracting the last two equations gives $z - h = 4$, or $z = h + 4$. Substituting this expression for z in the first equation gives $x^2 + 49 = (h+4)^2$. We will write this equation as:

$$h = \sqrt{x^2 + 49} - 4$$

(f) h

(g) Differentiating the equation in part (e) gives:

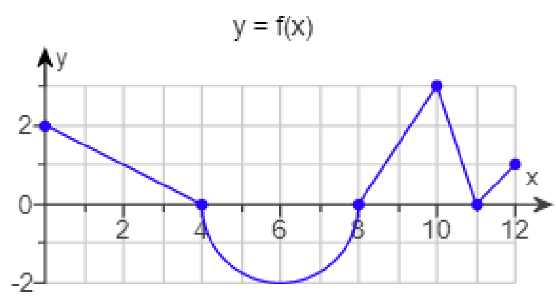
$$\frac{dh}{dt} = \frac{x\frac{dx}{dt}}{\sqrt{x^2 + 49}}$$

We are given that $\frac{dx}{dt} = 2$ (speed of the tractor) and that $x = 8$ (tractor is 8 m horizontally away from pulley). Hence we have:

$$\frac{dh}{dt} = \frac{16}{\sqrt{113}} \approx 1.51$$

So the bale of hay is rising at approximately 1.51 m/sec.

R-5. Define the function g by $g(x) = \int_0^x f(t)\,dt$, where the graph of $y = f(x)$ is given below. The graph consists of four line segments and one semicricle. ***Note:*** f and g are different functions!



(a) Calculate $f'(9)$.

(b) Calculate $f'(6)$.

(c) Calculate $g'(6)$.

(d) Calculate $g(11) - g(8)$.

(e) Is the statement "$g(4) > g(0)$" true or false?

(f) Find the critical numbers of g in the interval $(0, 12)$.

Solution **R-5**

(a) Observe that $f'(9)$ is simply the slope of given graph at $x = 9$. Hence $f'(9) = \frac{3-0}{10-8} = 1.5$.

(b) Observe that $f'(6)$ is the derivative of the given graph at $x = 6$, and f has a horizontal tangent line at $x = 6$. Hence $f'(6) = 0$.

(c) By the fundamental theorem of calculus, $g'(x) = f(x)$. Hence $g'(6) = f(6) = -2$.

R-5

(d) By the additivity property of integrals, $g(11) - g(8) = \int_8^{11} f(t)\,dt$. This is the area of the region below the graph of $y = f(t)$ and above the interval $[8, 11]$ on the t-axis. Note that this region is a triangle with base 3 and height 3. Hence $g(11) - g(8) = \frac{1}{2} \cdot 3 \cdot 3 = 4.5$.

(e) Note that $g(0) = 0$ by properties of integrals, and $g(4) > 0$ since $g(4)$ is the area of a triangle that lies above the t-axis. Hence the given statement is true.

(f) The critical numbers of g are those x-values where either $g'(x) = 0$ or $g'(x)$ does not exist. Recall from part (c) that $g'(x) = f(x)$. Clearly $f(x)$ is defined everywhere on $(0, 12)$. So the only critical numbers of g are the solutions to $f'(x) = 0$: $x = 4$, $x = 8$, and $x = 11$.

T-1. ***Note:*** The parts of this problem are not related.

(a) Suppose we use the fundamental theorem of calculus to calculate an integral as follows:

$$\int_a^b g(u)\,du = G(b) - G(a)$$

What is the relationship between the functions g and G?

(b) Calculate the following definite integral:

$$\int_{e^{-3}}^{e^2} \frac{2\ln(x) - 3}{5x}\,dx$$

(c) Consider the following indefinite integral:

$$J = \int \frac{\ln(x)}{3x^2}\,dx$$

Use the substitution $u = \ln(x)$ to write J as an equivalent indefinite integral in terms of u. *Do not attempt to calculate J.*

Solution T-1

(a) The function g is the derivative of G (equivalently, G is an antiderivative of g).

(b) We use the substitution $u = 2\ln(x) - 3$, whence $\frac{du}{dx} = \frac{2}{x}$ (or $dx = \frac{1}{2}x du$). We find the new limits of integration by substituting the old limits of integration into our relation $u = 2\ln(x) - 3$. Hence the new limits are:

$$x = e^{-3} \implies u = 2 \cdot (-3) - 3 = -9$$
$$x = e^2 \implies u = 2 \cdot (2) - 3 = 1$$

So the new lower and upper limits of integration are -9 and 1, respectively. So now we have the following:

$$\int_{e^{-3}}^{e^2} \frac{2\ln(x) - 3}{5x}\,dx = \int_{-9}^{1} \frac{u}{5x} \cdot \frac{x}{2} du = \int_{-9}^{1} \frac{u}{10}\,du = \left.\frac{u^2}{20}\right|_{-9}^{1} = \frac{1}{20}(1 - 81) = -4$$

(c) We have $u = \ln(x)$, whence $\frac{du}{dx} = \frac{1}{x}$, or $dx = x\,du$. Hence we have:

$$J = \int \frac{u}{3x^2} \cdot (x\,du) = \int \frac{u}{3x}\,du$$

We are still left with a factor of x, but the integrand must be only in terms of u. Since $u = \ln(x)$, we have $x = e^u$. Hence we have:

$$J = \int \frac{u}{3x}\,du = \int \frac{u}{3e^u}\,du$$