Section 10.1

Sequences

Learning Goals

Learning Goal	*Homework Problems*
10.1.1 Find the few first terms of a sequence whose terms is defined directly as a function of n or recursively	10.1: 3,11
10.1.2 Find the general term of a sequence given its first few terms	10.1: 17,25
10.1.3 Determine whether a sequence converges or diverges by directly evaluating the limit of its term	10.1: 31,34,35,39,43,72,73
10.1.4 Determine whether a sequence converges or diverges by identifying its term with a function and then evaluating the limit	10.1: 51,53,57,61,74,89,91
10.1.5 Determine whether a sequence diverges or converges and find the limit if converges using the squeeze theorem for sequences	10.1: 49
10.1.6 Using the Bounded Monotonic Sequences theorem to find the limit of a recursive sequence	10.1:101,103

Conceptual introduction: a sequence is a function whose domain is a set of integers. Notation: $\{a_n\}$ or $\{a_n\}_{n=n_0}^{\infty}$

We call n the index (= the input of the function), a_n is the n^{th} term (= the output of the input n).

We think about sequences as ordered lists of numbers:

$$a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots$$

Examples:

1) The sequence of even positive integers: $2, 4, 6, 8, 10, \ldots$

Index	Term
1	$2 = a_1$
2	$4 = a_2$
3	$6 = a_3$
4	$8 = a_4$
5	$10 = a_5$
⋮	⋮

We have a formula for the general term of the sequence: $a_n = 2n$

The sequence is $\{2n\}_{n=1}^{\infty}$.

Remarks: sequences can start at any integer: $\{2n\}_{n=3}^{\infty}$ is the sequence $6, 8, 10, 12, \ldots$

2) Constant sequence: $\{3\}_{n=n_0}^{\infty}$ is the sequence $3, 3, 3, \ldots$

3) The sequence of positive odd numbers: $1, 3, 5, 7, 9, \ldots$

General term: $\{2n-1\}_{n=1}^{\infty}$ or $\{2n+1\}_{n=0}^{\infty}$ both work.

4) Alternating sequence: $-1, 1, -1, 1, -1, \ldots$

General term: $a_n = (-1)^n$

The sequence can be written as $\{(-1)^n\}_{n=1}^{\infty}$ or $\{(-1)^{n+1}\}_{n=0}^{\infty}$.

5) The sequence $8, 2, 8, 2, 8, 2, \ldots$

To find the general term, observe that $\begin{cases} 8 = 5+3 \\ 2 = 5-3 \end{cases}$

General term: $a_n = 5 + 3(-1)^n, \quad n \geq 0$

6) The sequence $3, 6, 12, 24, 48, \ldots$

Observe that the ratio between consecutive terms is always the same: $3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{\times 2} 48, \ldots$ geometric sequence with common ratio $= 2 = \dfrac{a_{n+1}}{a_n}$

General term: $a_n = 3 \cdot 2^n, \quad n \geq 0$

Geometric sequence: $a_n = cr^n$, $\quad c = a_0 =$ first term

$r = \dfrac{a_{n+1}}{a_n} =$ common ratio

7) The factorial sequence $\{n!\}_{n=0}^{\infty}$ where $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

$0! = 1, \quad 1! = 1, \quad 2! = 2 \cdot 1 = 1, \quad 3! = 3 \cdot 2 \cdot 1 = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ etc.

8) The sequence defined recursively by $\begin{cases} a_1 = 4 \\ a_{n+1} = 1 - 2a_n \end{cases}$

first term

RECURSION FORMULA

$a_1 = 4$

$a_2 = 1 - 2a_1 = 1 - 2 \cdot 4 = -7$ (recurtion formula for $n=1$)

$a_3 = 1 - 2a_2 = 1 - 2(-7) = 15$ (recurtion formula for $n=2$)

$a_4 = 1 - 2a_3 = 1 - 2(15) = -29$

etc. This is an example of a recursive sequence.

Geometric sequences can be defined recursively:

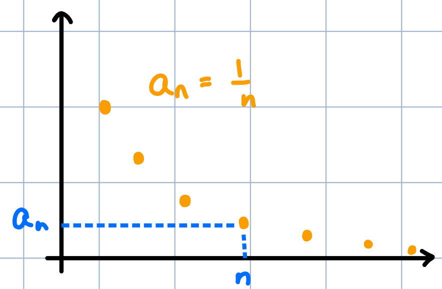
$\begin{cases} a_0 = c \\ a_{n+1} = ra_n \end{cases}$ gives a geometric sequence with first term c and common ratio r.

The factorial sequence can be defined recursively: $\begin{cases} a_0 = 1 \\ a_{n+1} = (n+1)a_n \end{cases}$.

Limits of sequences: the only type of limit that makes sense is for $n \to \infty$.

$\boxed{\begin{array}{l} \lim\limits_{n\to\infty} a_n = L \text{ if } a_n \text{ can be made arbitrarily close to } L \text{ when } n \text{ is arbitrarily large.} \\ \text{We say that } \{a_n\} \text{ \underline{converges} if the limit exists and is finite,} \\ \text{\underline{diverges} otherwise.} \end{array}}$

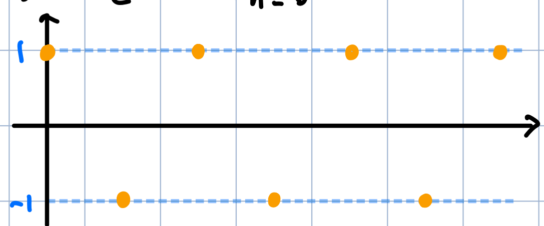
Examples: 1) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$



$\boxed{\lim\limits_{n\to\infty} \frac{1}{n} = 0}$

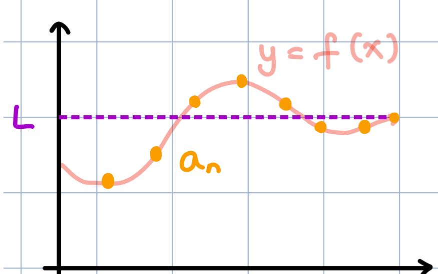
in general, any limit of the form $\frac{\text{constant}}{\infty}$ is 0.

2) $\left\{(-1)^n\right\}_{n=0}^{+\infty} = 1, -1, 1, -1, 1, \ldots$



$\boxed{\lim\limits_{n\to\infty} (-1)^n \text{ DNE}}$

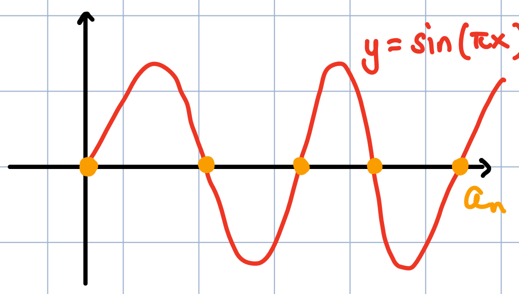
Theorem:



$\boxed{\begin{array}{l} \text{If } a_n = f(n) \text{ for a function } f \text{ and if} \\ \lim\limits_{x\to\infty} f(x) = L, \text{ then } \lim\limits_{n\to\infty} a_n = L \end{array}}$

In practice: we can treat n as a continuous variable and use methods seen in Calc I to compute limits of functions, provided it makes sense.

⚠ If $\lim_{x\to\infty} f(x)$ DNE, we cannot conclude that $\lim_{n\to\infty} a_n$ DNE



For instance, $a_n = \sin(\pi n) \overset{n \text{ integer}}{=} 0$

$f(x) = \sin(\pi x)$

$\lim_{x\to\infty} f(x)$ DNE but $\lim_{n\to\infty} a_n = 0$

<u>Examples</u> : 1) $\displaystyle \lim_{n\to\infty} \frac{2n+1}{1-3n} = \lim_{n\to\infty} \frac{(2n+1)\cdot\frac{1}{n}}{(1-3n)\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\frac{1}{n}-3} = \frac{2+0}{0-3} = \boxed{-\frac{2}{3}}$

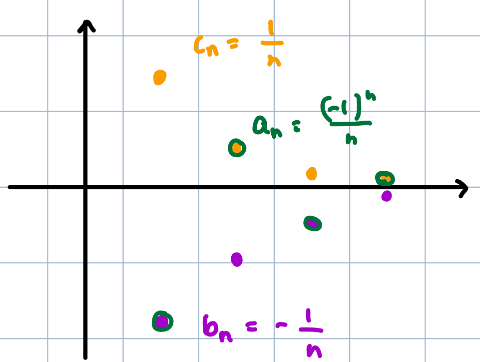
or use L'Hôpital's Rule: $\displaystyle \lim_{n\to\infty} \frac{2n+1}{1-3n} = \lim_{x\to\infty} \frac{2x+1}{1-3x} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{n\to\infty} \frac{2}{-3} = \boxed{-\frac{2}{3}}$.

treat n as a continuous variable

2) $\displaystyle \lim_{n\to\infty} \frac{(-1)^n}{n}$: we cannot use the theorem and treat n as a continuous variable because $f(x) = \frac{(-1)^x}{x}$ is undefined if $x = \frac{1}{\text{even } \#}$.

Instead, we can use the <u>Sandwich Theorem</u>:

$$\boxed{\text{If } b_n \le a_n \le c_n \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} a_n = L.}$$



Here, $-1 \le (-1)^n \le 1$

so $\displaystyle \frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$

and $\displaystyle \lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

So $\displaystyle \boxed{\lim_{n\to\infty} \frac{(-1)^n}{n} = 0}$

3) $\displaystyle \lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{x\to\infty} \frac{2^x}{x^2}$ (treat n as a continuous variable)

$\displaystyle \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\ln(2)2^x}{2x} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\ln(2)^2 2^x}{2} = \boxed{\infty}$, so $\left\{\frac{2^n}{n^2}\right\}_n$ diverges.

4) Limit of geometric sequences $\{cr^n\}_{n=0}^{\infty}$.

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<-1 \end{cases}}$$

So $\lim_{n\to\infty}\left(\frac{-7}{11}\right)^n = 0$, $\lim_{n\to\infty}(-2)^n$ DNE, $\lim_{n\to\infty} 2^n = \infty$.

5) Useful common limits :

a) $\boxed{\lim_{n\to\infty} c^n = 0 \quad \text{if } |c|<1}$

b) $\boxed{\lim_{n\to\infty} c^{1/n} = 1 \quad \text{if } c>0}$

c) $\boxed{\lim_{n\to\infty} n^{1/n} = 1}$

d) $\boxed{\lim_{n\to\infty} \frac{c^n}{n!} = 0 \quad \text{for any } c}$

e) $\boxed{\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c \quad \text{for any } c}$

We saw a) above.

b) $$\lim_{n\to\infty} c^{1/n} \underset{\boxed{a^b = e^{b\ln(a)}}}{=} \lim_{n\to\infty} e^{\overbrace{\frac{1}{n}\ln(c)}^{\to 0 \text{ when } n\to\infty}} \underset{e^x \text{ continuous at } x=0.}{=} e^0 = 1.$$

This is an application of the <u>Continuous Function Theorem</u>:

$$\boxed{\text{If } \lim_{n\to\infty} a_n = L \text{ and } f \text{ is continuous at } x=L, \text{ then } \lim_{n\to\infty} f(a_n) = f(L)}$$

For indeterminate powers $(\infty^0, 1^\infty, 0^0)$ like c), e), we can rewrite the expression with $a^b = e^{b\ln(a)}$ or taking the ln, and then use L'Hôpital's Rule.

c) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{\ln(n)}{n}}$

$\frac{\infty}{\infty}$, use L'H.

$= \lim_{x\to\infty} e^{\frac{\ln(x)}{x}}$ (treat n as a continuous variable)

$= \lim_{x\to\infty} e^{\frac{\frac{1}{x}}{1}} = e^0 = \boxed{1}$.

e) $\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{c}{n}\right)}$ let's calculate the limit of the exponent.

$$\lim_{n\to\infty} n\ln\left(1+\frac{c}{n}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\cancel{-}\frac{c}{\cancel{x^2}}\cdot\frac{1}{1+c/x}}{\cancel{-}\frac{1}{\cancel{x^2}}}$$

$$= \lim_{x\to\infty} \frac{c}{1+\frac{c}{x}} = c.$$

So $\boxed{\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c}$.

6) $\lim_{n\to\infty} a_n$ where a_n is the sequence defined recursively by

$$\begin{cases} a_1 = -4 \\ a_{n+1} = \sqrt{8+2a_n} \end{cases}$$

First few terms of a_n:

$a_1 = -4$

$a_2 = \sqrt{8+2a_1} = \sqrt{8+2(-4)} = 0$

$a_3 = \sqrt{8+2a_2} = \sqrt{8+2\cdot 0} = \sqrt{8} = 2\sqrt{2}$

$a_4 = \sqrt{8+2a_3} = \sqrt{8+4\sqrt{2}} = 2\sqrt{2+\sqrt{2}}$

etc.

Assume we know that a_n converges to L. How can we find L?

Because $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n+1} = L$

same list of numbers, just indexed differently

$$a_{n+1} = \sqrt{8+2a_n}$$
$\Downarrow$ take limit when $n \to \infty$

$L = \sqrt{8+2L}$ we can now solve for L.

$L^2 = 8+2L$

$L^2 - 2L - 8 = 0$

$(L-4)(L+2) = 0 \Rightarrow L = 4$ or $L = -2$

Since $a_n \geq 0$ for $n \geq 2$, we deduce $\boxed{L = 4}$.

We assumed that $\{a_n\}$ converges to do this reasoning. The following theorem helps proving convergence.

Theorem: if $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ is convergent.

Bounded: for some $M > 0$, we have $|a_n| \leq M$ for all n.

Monotonic: either increasing $a_{n+1} \geq a_n$ for $n \geq n_0$

or decreasing $a_{n+1} \leq a_n$ for $n \geq n_0$.

Practice: calculate the limits of the following sequences or show that they diverge.

1) $\lim\limits_{n \to \infty} \left(1 + \sin\left(\frac{3}{n}\right)\right)^n$

2) $\lim\limits_{n \to \infty} \dfrac{3\cos(n^2) + 2n}{n+1}$

3) $\lim\limits_{n \to \infty} \dfrac{5^{n+1} - 3^{2n}}{7^n}$

4) $\lim\limits_{n \to \infty} \left(\dfrac{3n^2 + n + 1}{\sqrt{n^2+1}}\right)^{1/n}$

Solutions :

1) $\lim_{n\to\infty} \left(1+\sin\left(\frac{3}{n}\right)\right)^n = \lim_{n\to\infty} e^{n\ln(1+\sin(\frac{3}{n}))}$ calculate limit of exponent.

$$\lim_{n\to\infty} n\ln\left(1+\sin\left(\frac{3}{n}\right)\right) = \lim_{x\to\infty} \frac{\ln\left(1+\sin\left(\frac{3}{x}\right)\right)}{\frac{1}{x}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\cancel{-}\frac{3}{\cancel{x^2}}\cos\left(\frac{3}{x}\right)\frac{1}{1+\sin(3/x)}}{\cancel{-}\frac{1}{\cancel{x^2}}}$$

$$= \lim_{x\to\infty} \frac{3\cos\left(\frac{3}{x}\right)}{1+\sin\left(\frac{3}{x}\right)} = 3.$$

So $\boxed{\lim_{n\to\infty} \left(1+\sin\left(\frac{3}{n}\right)\right)^n = e^3}$.

2) $\lim_{n\to\infty} \frac{3\cos(n^2)+2n}{n+1}$

Recall :

$$\boxed{\begin{array}{l} -1 \leqslant \cos(\dots) \leqslant 1 \\ -1 \leqslant \sin(\dots) \leqslant 1 \\ -\frac{\pi}{2} \leqslant \tan^{-1}(\dots) \leqslant \frac{\pi}{2} \end{array}}$$

We have $-1 \leqslant \cos(n^2) \leqslant 1$

$$-3 \leqslant 3\cos(n^2) \leqslant 3$$

$$\frac{-3+2n}{n+1} \leqslant \frac{3\cos(n^2)+2n}{n+1} \leqslant \frac{3+2n}{n+1}$$

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{-3+2n}{n+1} = \lim_{n\to\infty} \frac{-3/n+2}{1+1/n} = \frac{0+2}{1+0} = 2 \\ \lim_{n\to\infty} \frac{3+2n}{n+1} = \lim_{n\to\infty} \frac{3/n+2}{1+1/n} = \frac{0+2}{1+0} = 2 \end{array}\right\} \Rightarrow \boxed{\lim_{n\to\infty} \frac{3\cos(n^2)+2n}{n+1} = 2}$$

by the Sandwich Theorem

3) $$\lim_{n\to\infty} \frac{5^{n+1}-3^{2n}}{7^n} = \lim_{n\to\infty} \left(\frac{5\cdot 5^n}{7^n} - \frac{(3^2)^n}{7^n}\right) = \lim_{n\to\infty} \left(5\left(\frac{5}{7}\right)^n - \left(\frac{9}{7}\right)^n\right)$$

$$= \text{“ } 5\cdot 0 - \infty \text{ ''} = \boxed{-\infty}.$$

4) $$\lim_{n\to\infty} \left(\frac{3n^2+n+1}{\sqrt{n^2+1}}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n^2(3+1/n+1/n^2)}{n\sqrt{1+1/n^2}}\right)^{1/n}$$

$$= \lim_{n\to\infty} n^{1/n} \cdot \left(\frac{3+1/n+1/n^2}{\sqrt{1+1/n^2}}\right)^{1/n} = 1\cdot 3^0 = \boxed{1}.$$