Section 10.10

Applications of Taylor Series

Learning Goals

10.10.1 Find a limit of functions using their Maclaurin series	10.10: 29,31,35,36
10.10.2 Use series to estimate values of integrals and transcendental numbers	10.10: 15, 17, 19, 20, 23

① Computing limits

Suppose we want to compute a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ leading to an indeterminate form $\frac{0}{0}$.
We can use L'Hopital's Rule, but that may lead to difficult derivatives computations, especially if we must use it multiple times. Taylor Series give us an alternative.

Examples : 1) $\lim_{x \to 0} \frac{e^{x^3}-1}{x^2 \sin(2x)}$

We find Maclaurin series of numerator and denominator.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

$$= 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots$$

So $$e^{x^3} - 1 = x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots = \sum_{n=1}^{\infty} \frac{x^{3n}}{n!}$$

start at $n=1$ because the term for $n=0$ is now on the left-hand side

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

So $$x^2 \sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+3}}{(2n+1)!} = 2x^3 - \frac{8x^5}{6} + \frac{32x^7}{120} - \cdots$$

Now $$\lim_{x \to 0} \frac{e^{x^3}-1}{x^2 \sin(2x)} = \lim_{x \to 0} \frac{x^3 + \frac{x^6}{2} + \cdots}{2x^3 - \frac{8x^5}{6} + \cdots} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to 0} \frac{1 + \overbrace{\frac{x^3}{2} + \cdots (\text{positive powers of } x) \cdots}^{\to 0 \text{ when } x \to 0}}{2 - \underbrace{\frac{8x^2}{6} + \cdots (\text{positive powers of } x) \cdots}_{\to 0 \text{ when } x \to 0}}$$

$$= \boxed{\frac{1}{2}}.$$

2) $\displaystyle \lim_{x \to 0} \frac{x^3}{\sin(x) - \tan^{-1}(x)}$

We find the first few terms of the Maclaurin series of the denominator.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

So $\sin(x) - \tan^{-1}(x) = (1-1)x + \left(-\frac{1}{6} + \frac{1}{3}\right)x^3 + \left(\frac{1}{120} - \frac{1}{5}\right)x^5 + \cdots$

$$= \frac{x^3}{6} - \frac{23x^5}{120} + \cdots$$

Now $$\lim_{x \to 0} \frac{x^3}{\sin(x) - \tan^{-1}(x)} = \lim_{x \to 0} \frac{x^3}{\frac{x^3}{6} - \frac{23x^5}{120} + \cdots} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to 0} \frac{1}{\frac{1}{6} - \frac{23x^2}{120} + \cdots \text{(positive powers of } x)\cdots}$$

$\longrightarrow 0$ when $x \to 0$

$$= \frac{1}{\frac{1}{6}} = \boxed{6}.$$

3) $\displaystyle \lim_{x \to \infty} x^8\left(2\cos\left(\frac{1}{x^2}\right) - 2 + \frac{1}{x^4}\right)$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$\Rightarrow \cos\left(\frac{1}{x^2}\right) = 1 - \frac{1}{2x^4} + \frac{1}{24x^8} - \frac{1}{720x^{12}} + \cdots$$

$$\Rightarrow 2\cos\left(\frac{1}{x^2}\right) = 2 - \frac{1}{x^4} + \frac{1}{12x^8} - \frac{1}{360x^{12}} + \cdots$$

So $$2\cos\left(\frac{1}{x^2}\right) - 2 + \frac{1}{x^4} = \frac{1}{12x^8} - \frac{1}{360x^{12}} + \cdots$$

Now $$\lim_{x \to \infty} x^8\left(2\cos\left(\frac{1}{x^2}\right) - 2 + \frac{1}{x^4}\right) = \lim_{x \to \infty} x^8\left(\frac{1}{12x^8} - \frac{1}{360x^{12}} + \cdots\right)$$

$$= \lim_{x \to \infty} \frac{1}{12} - \underbrace{\frac{1}{360x^4} + \cdots (\text{negative powers of } x) \cdots}_{\to 0 \text{ when } x \to \infty}$$

$$= \boxed{\frac{1}{12}}.$$

② Approximations of integrals.

Some integrals cannot be computed using elementary functions. We can use power series to express some of these integrals as sums of series using term-by-term integration.

Examples: 1) Write the integral $I = \int_0^{1/2} e^{3x^2}\,dx$ the sum of an infinite series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow \quad e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}$$

Integrate term-by-term: $\displaystyle\int_0^{1/2} e^{3x^2}\,dx = \sum_{n=0}^{\infty} \int_0^{1/2} \frac{3^n x^{2n}}{n!}\,dx$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \int_0^{1/2} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot \left[\frac{x^{2n+1}}{2n+1}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \left(\frac{1}{(2n+1)2^{2n+1}} - 0\right)$$

So we get $\boxed{I = \displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!\,(2n+1)\,2^{2n+1}}} = \dfrac{1}{2} + \dfrac{3}{3 \cdot 2^3} + \dfrac{3^2}{2 \cdot 5 \cdot 2^5} + \cdots$

2) Write the integral $\int_0^1 x^3 \cos(x^3)\,dx$ as the sum of an infinite series.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\Rightarrow \quad \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$\Rightarrow \quad x^3\cos(x^3) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!}$$

Integrate term-by-term:

$$\int_0^1 x^3\cos(x^3)\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{6n+3}}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^1 x^{6n+3}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[\frac{x^{6n+4}}{6n+4}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{6n+4} - 0\right)$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(6n+4)}}.$$

2) Write $I = \int_0^1 \sin(t^2)\,dt$ as the sum of an infinite series. Then estimate the value of I with an error of at most 10^{-6}.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (t^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}$$

Integrate term-by-term: $\int_0^1 \sin(t^2)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 t^{4n+2}\,dt$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{t^{4n+3}}{4n+3} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{4n+3} - 0 \right)$$

So $\boxed{I = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)}}$

We will estimate I using a partial sum S_N. We need to find N such that $\underbrace{|I - S_N|}_{\text{error}} \leq 10^{-6}$.

We can use the Alternating Series Estimation Theorem because:

- $a_n = \frac{1}{(2n+1)!\,(4n+3)} \geq 0$.
- a_n is decreasing since $(2n+1)!\,(4n+3)$ is increasing.
- $\lim_{n \to \infty} \frac{1}{(2n+1)!\,(4n+3)} = 0$.

ASET: $|I - S_N| \leq a_{N+1}$ so we want $a_{N+1} \leq 10^{-6}$.

$$\frac{1}{(2N+3)!\,(4N+7)} \leq 10^{-6}$$

$$(2N+3)!\,(4N+7) \geq 10^{6}$$

N	$(2N+3)!\,(4N+7)$
0	42 ✗
1	1320 ✗
2	75600 ✗
3	6894720 ✓

$\Rightarrow \quad N \geq 3$.

So we can estimate I using S_3.

$$I \approx S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)!\,(4n+3)} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} \approx \boxed{0.310268}.$$

Practice:

1) Compute the following limits.

a) $\lim_{x \to 0} \dfrac{\tan^{-1}(x^2) - x^2}{\cos(x^3) - 1}$ b) $\lim_{x \to 3} \dfrac{x^2 - 9}{\ln(x-2)}$

2) Use Maclaurin series to write each integral as the sum of an infinite series

a) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x^4+1}$ b) $\displaystyle\int_0^2 \frac{\sin(x)}{x}\, dx.$

Solutions:

1) a) $\tan^{-1}(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$

$\Rightarrow \tan^{-1}(x^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^2)^{2n+1}}{2n+1} = x^2 - \dfrac{x^6}{3} + \dfrac{x^{10}}{5} - \cdots$

So $\tan^{-1}(x^2) - x^2 = -\dfrac{x^6}{3} + \dfrac{x^{10}}{5} - \cdots$

$\cos(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^3)^{2n}}{(2n)!} = 1 - \dfrac{x^6}{2} + \dfrac{x^{12}}{24} - \cdots$

So $\cos(x^3) - 1 = -\dfrac{x^6}{2} + \dfrac{x^{12}}{24} - \cdots$

Now $\lim_{x \to 0} \dfrac{\tan^{-1}(x^2) - x^2}{\cos(x^3) - 1} = \lim_{x \to 0} \dfrac{-\frac{x^6}{3} + \frac{x^{10}}{5} - \cdots}{-\frac{x^6}{2} + \frac{x^{12}}{24} - \cdots} \cdot \dfrac{\frac{1}{x^6}}{\frac{1}{x^6}}$

$= \lim_{x \to 0} \dfrac{-\frac{1}{3} + \frac{x^4}{5} - \cdots}{-\frac{1}{2} + \frac{x^6}{24} - \cdots} = \dfrac{-\frac{1}{3}}{-\frac{1}{2}} = \boxed{\dfrac{2}{3}}$

b) $\ln(1+x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$

and $\ln(x-2) = \ln(1+(x-3)) = (x-3) - \frac{(x-3)^2}{2} + \cdots$

Now $\lim_{x\to 3} \frac{x^2-9}{\ln(x-2)} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3) - \frac{(x-3)^2}{2} + \cdots} \cdot \frac{\frac{1}{x-3}}{\frac{1}{x-3}}$

$= \lim_{x\to 3} \frac{x+3}{1 - \underbrace{\frac{(x-3)^2}{2} + \cdots (\text{positive powers of } (x-3)) \cdots}_{\to 0 \text{ } x\to 3}}$

$= \boxed{6}$

2) a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x|<1$

so $\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

Integrate term-by-term: $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n \int_0^{\frac{1}{2}} x^{4n}\,dx$

$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{4n+1}}{4n+1}\right]_0^{\frac{1}{2}}$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)\,2^{4n+1}}}$

b) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ so $\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

Integrate term-by-term: $\int_0^2 \frac{\sin(x)}{x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^2 x^{2n}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{x^{2n+1}}{2n+1}\right]_0^2$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!\,(2n+1)}}$